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Absolutely continuous spectrum preservation: A new proof for unitary operators under finite-rank multiplicative perturbations

Pablo A. Diaz[1,✉]

[1] *Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Avenida Libertador Bernardo O'Higgins nº 3363, Santiago, Chile.*
pablo.diaz@usach.cl✉

ABSTRACT

We will provide a new proof of the Birman-Krein theorem for unitary operators multiplicatively perturbed by finite-rank operators, which is nothing more than the Kato-Rosenblum theorem, but instead of self-adjoint operators. In other words, U is a unitary operator and X is a unitary operator given by a finite rank perturbation of the identity, *i.e.*, $X = \mathbf{1} + W$ with W finite rank. We show that U and its perturbed version UX (or XU) are unitarily equivalent on their absolutely continuous subspaces.

RESUMEN

Entregamos una nueva demostración del teorema de Birman-Krein para operadores unitarios perturbados multiplicativamente por operadores de rango finito, que no es más que el teorema de Kato-Rosenblum, pero en lugar de operadores autoadjuntos. En otras palabras, U es un operador unitario y X es un operador unitario dado por una perturbación de rango finito de la identidad, *i.e.*, $X = \mathbf{1} + W$ con W de rango finito. Mostramos que U y su versión perturbada UX (o XU) son unitariamente equivalentes en sus subspacios absolutamente continuos.

Keywords and Phrases: Absolutely continuous measure, finite rank perturbations, multiplicative perturbation, unitary operators.

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1 Introduction

One of the great theorems in spectral theory is the famous Kato-Rosenblum theorem [4]:

Theorem 1.1. *If A and T are self-adjoint operators, and A is trace class, then the absolutely continuous parts of T and $T+A$ are unitarily equivalent.*

This theorem tells us that T and $T+A$ have the same absolutely continuous spectrum.

Our motivation is to provide an alternative proof of the Birman-Krein theorem [1], which serves as the unitary counterpart to the Kato-Rosenblum theorem, for the case of unitary operators under multiplicative finite-rank perturbations (hence trace-class operators). Specifically, we are interested in the preservation of absolutely continuous spectrum under transformations of the form $U \mapsto UX$ (or XU), where U and X unitary operator. It is worth mentioning here that X is a unitary operator, but not of finite rank, however, it can be expressed as $X = \mathbf{1} + W$, where $W = X - \mathbf{1}$ is indeed a finite-rank operator. This ensures that X differs from the identity only on a finite-dimensional subspace. While Birman and Krein mention how the proof would proceed if X were of rank 1 or finite rank, they do not provide a detailed demonstration. Our work fills this gap by presenting a novel proof of this theorem.

In this proof, we avoid the use of scattering theory, which has been the traditional approach to this problem. Notably, L. de Branges and L. Shulman previously addressed similar in [5,6] and [2] where they employed scattering theory (wave operator limits). In contrast our approach is more restrictive than the general case, as it applies only when X is a unitary operator perturbed by a finite rank operator.

In Section 2, we introduce the general framework for multiplicatively perturbed unitary operators. In Section 3, to illustrate our general result, we examine the case where the perturbation is of rank 1. Finally, in Section 4, we present our main result.

2 Multiplicative perturbations

It is often convenient to express the unitary operator X as $X = e^{iY}$, where Y is a self-adjoint and bounded operator. If Y is of trace class, it can be written as

$$Y = \sum_{j=1}^{\infty} \omega_j P_{\varphi_j},$$

where $P_{\varphi_j} = \langle \varphi_j, \cdot \rangle \varphi_j$, $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal sequence, and $\sum_j |\omega_j| < \infty$.

Consider the perturbation

$$U \mapsto UX.$$

Then, $UX = U(\mathbf{1} + W) = U + UW$, where $W = \sum_{j=1}^{\infty} \frac{(iY)^j}{j!}$. Since $P_{\varphi_j} P_{\varphi_k} = 0$ for $j \neq k$, we have

$$e^{i(\omega_1 P_{\varphi_1} + \omega_2 P_{\varphi_2})} = e^{i\omega_1 P_{\varphi_1}} e^{i\omega_2 P_{\varphi_2}}.$$

Let $X_n = e^{i \sum_{j=1}^{n} \omega_j P_{\varphi_j}}$. Then, the commutator $[X_n, X_m] = 0$ for all $n, m \in \mathbb{N}$, meaning the operators commute. Thus,

$$X_{n+k} = e^{i \sum_{j=1}^{n+k} \omega_j P_{\varphi_j}} = e^{i \sum_{i=1}^{n} \omega_i P_{\varphi_i}} e^{i \sum_{j=n+1}^{n+k} \omega_j P_{\varphi_j}}.$$

Formally, the unitary operator X can be expressed as

$$e^{iY} = e^{i\omega_1 P_{\varphi_1}} \cdot e^{i\omega_2 P_{\varphi_2}} \cdots e^{i\omega_k P_{\varphi_k}} \cdots = \cdots e^{i\omega_k P_{\varphi_k}} \cdots e^{i\omega_2 P_{\varphi_2}} \cdot e^{i\omega_1 P_{\varphi_1}}. \tag{2.1}$$

Remark 2.1. (1) *If Y is of rank 1 and $\mathbf{1}$ is the identity operator, then*

$$e^{i\omega P_\varphi} = \sum_{j \geq 0} \frac{(i\omega P_\varphi)^j}{j!} = \mathbf{1} + \sum_{j \geq 1} \frac{(i\omega P_\varphi)^j}{j!} = \mathbf{1} + \sum_{j \geq 1} \frac{(i\omega)^j P_\varphi}{j!} = \mathbf{1} + (e^{i\omega} - 1) P_\varphi.$$

(2) *Let $\beta_j = (e^{i\omega_j} - 1) \in \mathbb{C}$. Since P_{φ_j} is a projection operator for all $j \in \mathbb{N}$, we have*

$$(\mathbf{1} + \beta_j P_{\varphi_j})(\mathbf{1} + \beta_k P_{\varphi_k}) = \mathbf{1} + \beta_j P_{\varphi_j} + \beta_k P_{\varphi_k}, \quad \text{for } j \neq k.$$

(3) *For $\omega_j \in \mathbb{R}$,*

$$|\beta_j| = |e^{i\omega_j} - 1| \leq |\omega_j|,$$

for all $j \in \mathbb{N}$.

To justify equality (2.1), we present the following lemma.

Lemma 2.2. *Let $X_n = e^{i \sum_{j=1}^{n} \omega_j P_{\varphi_j}}$, then $\{X_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $B(\mathcal{H})$.*

Proof. Using Remark 2.1, we have:

$$\begin{aligned}
||X_{n+k} - X_n|| &= \left|\left| X_n e^{i \sum_{j=n+1}^{n+k} \omega_j P_{\varphi_j}} - X_n \right|\right| \\
&= \left|\left| X_n \left[e^{i\omega_{n+1} P_{\varphi_{n+1}}} \cdot e^{i\omega_{n+2} P_{\varphi_{n+2}}} \cdots e^{i\omega_{n+k} P_{\varphi_{n+k}}} - \mathbf{1} \right] \right|\right| \\
&= \left|\left| X_n \left[\left(\mathbf{1} + \left(e^{i\omega_{n+1}} - 1\right) P_{\varphi_{n+1}}\right) \cdots \left(\mathbf{1} + \left(e^{i\omega_{n+k}} - 1\right) P_{\varphi_{n+k}}\right) - \mathbf{1} \right] \right|\right| \\
&= \left|\left| X_n \left[\left(\mathbf{1} + \beta_{n+1} P_{\varphi_{n+1}}\right) \cdot \left(\mathbf{1} + \beta_{n+2} P_{\varphi_{n+2}}\right) \cdots \left(\mathbf{1} + \beta_{n+k} P_{\varphi_{n+k}}\right) - \mathbf{1} \right] \right|\right| \\
&\leq ||X_n|| \cdot \left|\left| \left(\mathbf{1} + \beta_{n+1} P_{\varphi_{n+1}}\right) \cdot \left(\mathbf{1} + \beta_{n+2} P_{\varphi_{n+2}}\right) \cdots \left(\mathbf{1} + \beta_{n+k} P_{\varphi_{n+k}}\right) - \mathbf{1} \right|\right| \\
&= \left|\left| \left(\mathbf{1} + \beta_{n+1} P_{\varphi_{n+1}}\right) \cdot \left(\mathbf{1} + \beta_{n+2} P_{\varphi_{n+2}}\right) \cdots \left(\mathbf{1} + \beta_{n+k} P_{\varphi_{n+k}}\right) - \mathbf{1} \right|\right| \\
&= \left|\left| \left(\mathbf{1} + \beta_{n+1} P_{\varphi_{n+1}} + \beta_{n+2} P_{\varphi_{n+2}}\right) \cdots \left(\mathbf{1} + \beta_{n+k-1} P_{\varphi_{n+k-1}} + \beta_{n+k} P_{\varphi_{n+k}}\right) - \mathbf{1} \right|\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq |\beta_{n+1}| \cdot \left|\left|P_{\varphi_{n+1}}\right|\right| + |\beta_{n+2}| \cdot \left|\left|P_{\varphi_{n+2}}\right|\right| + \cdots + \ |\beta_{n+k}| \cdot \left|\left|P_{\varphi_{n+k}}\right|\right| \\
&= \left|\left|\mathbf{1} + \beta_{n+1}P_{\varphi_{n+1}} + \beta_{n+2}P_{\varphi_{n+2}} + \cdots + \beta_{n+k}P_{\varphi_{n+k}} - \mathbf{1}\right|\right| \\
&= \left|\left|\beta_{n+1}P_{\varphi_{n+1}} + \beta_{n+2}P_{\varphi_{n+2}} + \cdots + \beta_{n+k}P_{\varphi_{n+k}}\right|\right| \\
&= |\beta_{n+1}| + |\beta_{n+2}| + \cdots + |\beta_{n+k}| \\
&= \left|e^{i\omega_{n+1}} - 1\right| + \left|e^{i\omega_{n+2}} - 1\right| + \cdots + \left|e^{i\omega_{n+k}} - 1\right| \\
&\leq |\omega_{n+1}| + |\omega_{n+2}| + \cdots + |\omega_{n+k}| \\
&= \sum_{j=n+1}^{n+k} |\omega_j| \to 0,
\end{aligned}
$$

for $n \to \infty$, since it is of trace class, that is, $\{\omega_j\}_{j\in\mathbb{N}} \in l^1$. □

As an immediate result, we have:

Corollary 2.3. *Let U be another unitary operator, then $\{UX_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $B(\mathcal{H})$.*

Lemma 2.4. *$\|UX_n - UX\| \to 0$, when $n \to \infty$.*

Proof. Analogous to Lemma 2.2 and Corollary 2.3, if we have that $Y = \sum_{j=1}^{\infty} \omega_j P_{\varphi_j}$, then

$$
\begin{aligned}
||X_n - X|| = \left|\left|X_n - e^{iY}\right|\right| &= \left|\left|e^{i\sum_{j=1}^n \omega_j P_{\varphi_j}} - e^{i\sum_{j=1}^n \omega_j P_{\varphi_j}} \cdot e^{i\sum_{j>n} \omega_j P_{\varphi_j}}\right|\right| \\
&= \left|\left|e^{i\sum_{i=1}^n \omega_j P_{\varphi_j}} \left(\mathbf{1} - e^{i\sum_{j>n} \omega_j P_{\varphi_j}}\right)\right|\right| \leq \left|\left|\mathbf{1} - e^{i\sum_{j>n} \omega_j P_{\varphi_j}}\right|\right| \leq \sum_{j>n} |\omega_j| \to 0,
\end{aligned}
$$

when $n \to \infty$, and therefore $||UX_n - UX|| \to 0$. □

For unitary operators, the Cauchy and Borel transforms of a Borel measure μ on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ are given by

$$
F_\mu(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t), \quad |z| < 1, \quad R_\mu(z) = \int_0^{2\pi} \frac{d\mu(t)}{e^{it} - z}, \quad |z| < 1,
$$

respectively. Here F_μ and R_μ are related by:

$$
F_\mu(z) = 1 + 2zR_\mu(z). \tag{2.2}
$$

Here we are interested in the properties of F_μ; for this, we have the following theorem [7].

Theorem 2.5. *Let μ be a Borel measure on the unit circle $\mathbb{T}$, then*

(1) $\lim_{r\to 1} F_\mu\left(re^{it}\right)$ *exists for almost every t, and if*

$$d\mu(t) = f(t)\frac{dt}{2\pi} + d\mu_s(t)$$

defines $f(t)$, then $f(t) = \Re F_\mu\left(e^{it}\right)$.

(2) *t_0 is a pure point of μ if and only if $\lim_{r\to 1}(1-r)\Re F_\mu\left(re^{it_0}\right) \neq 0$ and in general*

$$\lim_{r\to 1}(1-r)\Re F_\mu\left(re^{it_0}\right) = \mu\left(\{t_0\}\right).$$

(3) *$d\mu_s$ is supported in $\left\{t \mid \lim_{r\to 1} F_\mu\left(re^{it}\right) = \infty\right\}$.*

Remark 2.6. *This last theorem relates nontangential limits of this transform to the singular $d\mu_s$ and absolutely continuous parts of $d\mu$.*

3 Rank 1 case

Now, let us consider the case of a rank 1 perturbation:

$$U_\omega = UX_\omega = U\left(\mathbf{1} + \left(e^{i\omega} - 1\right)P_\varphi\right).$$

Note that the intensity parameter ω exhibits periodicity, and it suffices to consider $0 \leq \omega < 2\pi$. Here, φ is a normalized vector in the Hilbert space $\mathcal{H}$ that is cyclic for the unitary operator U, meaning that the closure

$$\overline{\text{Lin}\left\{U^j\varphi \mid j \in \mathbb{Z}\right\}} = \mathcal{H},$$

with $U^0 = \mathbf{1}$. Since φ is cyclic for U, it is also cyclic for U_ω, for all $\omega \in \mathbb{R}$.

To simplify the notation, let μ^ω denote the spectral measure of the pair (U_ω, φ), $U_0 = U$, $\mu^0 = \mu$, $F_\omega = F_{\mu^\omega}$ and $R_\omega = R_{\mu^\omega}$. Clearly, $F_0(z) = F_\mu(z)$ and $R_0(z) = R_\mu(z)$, where the Cauchy and Borel transforms are respectively given by

$$F_\omega(z) = \left\langle \varphi, \left(U_\omega + z\mathbf{1}\right)\left(U_\omega - z\mathbf{1}\right)^{-1}\varphi\right\rangle, \quad R_\omega(z) = \left\langle \varphi, R_z\left(U_\omega\right)\varphi\right\rangle,$$

where $R_z(U_\omega) = \left(U_\omega - z\mathbf{1}\right)^{-1}$ is the resolvent operator.

Our goal is to prove that the measures μ^ω_{ac} and μ_{ac} are equivalent, which implies that their Radon-Nikodym derivatives are equal almost everywhere with respect to the Lebesgue measure (up to a non-vanishing factor, which in this case is 1 due to the specific form of the transformation). This equivalence of the measures implies the unitary equivalence of the absolutely continuous parts of U_ω and U.

Lemma 3.1. *$R_z(U)(U\varphi) = (\mathbf{1} + zR_z(U))\varphi$.*

Proof. In fact,

$$\begin{aligned} R_z(U)(U\varphi) - zR_z(U)(\varphi) &= R_z(U)\,[U(\varphi) - z\varphi] \\ &= R_z(U)\,[U - z\mathbf{1}]\,(\varphi) = (U - z\mathbf{1})^{-1}(U - z\mathbf{1})\varphi = \varphi. \end{aligned}$$

□

Remark 3.2. *By the previous lemma, we then have that:*

$$\begin{aligned} \langle \varphi, R_z(U)(U\varphi)\rangle &= \langle \varphi, (\mathbf{1} + zR_z(U))\varphi\rangle \\ &= \langle \varphi, \mathbf{1}\varphi\rangle + z\,\langle \varphi, R_z(U)\varphi\rangle = \langle \varphi, \varphi\rangle + z\,\langle \varphi, R_z(U)\varphi\rangle = 1 + zR_\mu(z). \end{aligned}$$

Lemma 3.3. *For $|z| \neq 1$*

$$R_\omega(z) = \frac{R_0(z)}{e^{i\omega} + z(e^{i\omega} - 1)R_0(z)} \quad \textit{and} \quad F_\omega(z) = \frac{(e^{i\omega} - 1) + (e^{i\omega} + 1)F_0(z)}{(e^{i\omega} + 1) + (e^{i\omega} - 1)F_0(z)}.$$

Proof. By the second resolvent identity, we have that

$$R_z(U) - R_z(U_\omega) = R_z(U)(U_\omega - U)R_z(U_\omega) = R_z(U)((e^{i\omega} - 1)UP_\varphi)R_z(U_\omega),$$

then

$$\begin{aligned} \langle \varphi, R_z(U)\varphi\rangle - \langle \varphi, R_z(U_\omega)\varphi\rangle &= \left\langle \varphi, R_z(U)((e^{i\omega} - 1)UP_\varphi)R_z(U_\omega)\varphi\right\rangle \\ &= (e^{i\omega} - 1)\,\langle \varphi, R_z(U)U\,(\langle \varphi, R_Z(U_\omega)\varphi\rangle\,\varphi)\rangle \\ &= (e^{i\omega} - 1)\,\langle \varphi, R_z(U_\omega)\varphi\rangle\,\langle \varphi, R_z(U)U\varphi\rangle \\ &= (e^{i\omega} - 1)\,\langle \varphi, R_z(U_\omega)\varphi\rangle\,[1 + z\,\langle \varphi, R_z(U)\varphi\rangle]\,, \end{aligned}$$

that is, $R_0(z) - R_\omega(z) = (e^{i\omega} - 1)R_\omega(z)\,[1 + zR_0(z)]$, therefore,

$$R_\omega(z) = \frac{R_0(z)}{e^{i\omega} + z(e^{i\omega} - 1)R_0(z)}.$$

Now, by (2.2), we have

$$\begin{aligned} F_\omega(z) = 2zR_\omega(z) + 1 &= 2z\frac{R_0(z)}{e^{i\omega} + z(e^{i\omega} - 1)R_0(z)} + 1 \\ &= \frac{e^{i\omega} + z(e^{i\omega} - 1)R_0(z) + 2zR_0(z)}{e^{i\omega} + z(e^{i\omega} - 1)R_0(z)} = \frac{e^{i\omega} + z(e^{i\omega} + 1)R_0(z)}{e^{i\omega} + z(e^{i\omega} - 1)R_0(z)} \end{aligned}$$

$$= \frac{2e^{i\omega} + 2ze^{i\omega}R_0(z) + 2zR_0(z)}{2e^{i\omega} + 2ze^{i\omega}R_0(z) - 2zR_0(z)} = \frac{e^{i\omega} - 1 + e^{i\omega} + 2e^{i\omega}zR_0(z) + 1 + 2zR_0(z)}{e^{i\omega} + 1 + e^{i\omega} + 2e^{i\omega}zR_0(z) - 1 - 2zR_0(z)}$$

$$= \frac{(e^{i\omega} - 1) + (e^{i\omega} + 1)(1 + 2zR_0(z))}{(e^{i\omega} + 1) + (e^{i\omega} - 1)(1 + 2zR_0(z))} = \frac{(e^{i\omega} - 1) + (e^{i\omega} + 1)F_0(z)}{(e^{i\omega} + 1) + (e^{i\omega} - 1)F_0(z)}. \qquad \square$$

Remark 3.4. *From the previous lemma, we can express $F_0(z)$ in terms of $F_{\omega_1}(z)$ as follows:*

$$F_0(z) = \frac{(e^{i\omega_1} + 1)F_{\omega_1}(z) - (e^{i\omega_1} - 1)}{-(e^{i\omega_1} - 1)F_{\omega_1}(z) + (e^{i\omega_1} + 1)}, \tag{3.1}$$

for ω_1 in $[0, 2\pi)$. Now, if $\omega_2 \in [0, 2\pi)$ with $\omega_1 \neq \omega_2$, we can express $F_{\omega_2}(z)$ in terms of $F_0(z)$. Using (3.1)*, we can establish a relationship between F_{ω_2} and F_{ω_1} as follows:*

$$F_{\omega_2}(z) = \frac{e^{i\omega_2} - e^{i\omega_1} + \left(e^{i\omega_1} + e^{i\omega_2}\right)F_{\omega_1}(z)}{e^{i\omega_2} + e^{i\omega_1} + \left(e^{i\omega_2} - e^{i\omega_1}\right)F_{\omega_1}(z)} \quad \textit{and} \quad \Re F_{\omega_2}(z) = \frac{(1 + y^2)\Re F_{\omega_1}(z)}{\left|1 + iyF_{\omega_1}(z)\right|^2},$$

with $iy = \dfrac{e^{i\omega_2} - e^{i\omega_1}}{e^{i\omega_2} + e^{i\omega_1}}$.

Remark 3.5. *By Theorem 2.5, we know that the singular part of the measure μ_s is supported on*

$$S = \left\{t \mid \lim_{r\to 1} F(re^{it}) = \infty\right\}.$$

Let us define the sets

$$S_1 = \left\{t \mid \lim_{r\to 1} F_{\omega_1}(re^{it}) = \infty\right\} \quad \textit{and} \quad S_2 = \left\{t \mid \lim_{r\to 1} F_{\omega_2}(re^{it}) = \infty\right\}.$$

These sets are mutually disjoint. Indeed, since $\omega_1 \neq \omega_2$, and using Remark 3.4, it follows that if $t \in S_1$, then

$$\lim_{r\to 1} F_{\omega_2}(re^{it}) = \frac{e^{i\omega_2} + e^{i\omega_1}}{e^{i\omega_2} - e^{i\omega_1}} \neq \infty.$$

Therefore, $t \notin S_2$, which implies $S_1 \cap S_2 = \emptyset$. Thus, the measures $\mu_s^{\omega_1}$ and $\mu_s^{\omega_2}$ are mutually singular. This last result could be considered the equivalent of Donoghue's Theorem, but for unitary operators [3].

Theorem 3.6. *For all $\omega_1 \neq \omega_2$, the absolutely continuous parts of U_{ω_1} and U_{ω_2} are unitarily equivalent.*

Proof. Since $U_{\omega_1} = U_{\omega_2} + \left(e^{i\omega_1} - e^{i\omega_2}\right)UP_\varphi$, let us define the sets

$$L_1 = \left\{t \mid \lim_{r\to 1} F_{\omega_1}(re^{it}) = \infty \quad \text{or} \quad \nexists \lim_{r\to 1} F_{\omega_1}(re^{it})\right\}, \quad L_2 = \left\{t \mid \lim_{r\to 1} F_{\omega_1}(re^{it}) = \frac{e^{i\omega_2} + e^{i\omega_1}}{e^{i\omega_1} - e^{i\omega_2}}\right\},$$

moreover, by Theorem 2.5, the measures of these sets are zero. If we define $G = L_1 \cup L_2$, then the measure of G is also zero. From Remark 3.4, we obtain:

$$\left\{t \in \mathbb{T}\backslash G \mid \lim_{r\to 1} F_{\omega_1}(re^{it}) = 0\right\} = \left\{t \in \mathbb{T}\backslash G \mid \lim_{r\to 1} F_{\omega_2}(re^{it}) = 0\right\},$$

thus, for almost every $t \in \mathbb{T}\backslash G$, we have: $\lim_{r\to 1} \Re_{\omega_1}(re^{it}) \neq 0$ if and only if $\lim_{r\to 1} \Re_{\omega_2}(re^{it}) \neq 0$. By Theorem 2.5, the Radon-Nikodym derivative of the absolutely continuous part of the spectral measure is given by the real part of the boundary value of the Cauchy transform. Therefore, the above equivalence implies that the set of points t where the density of $(d\mu^{\omega_1})_{ac}$ is zero (or non-zero) coincides, up to a set of Lebesgue measure zero, with the set where the density of $(d\mu^{\omega_2})_{ac}$ is zero (or non-zero). This means that the measures $(d\mu^{\omega_1})_{ac}$ and $(d\mu^{\omega_2})_{ac}$ are mutually absolutely continuous with respect to each other (and with respect to Lebesgue measure), hence equivalent. The unitary equivalence of the absolutely continuous parts of U_{ω_1} and U_{ω_2} then follows from the spectral theorem. □

Remark 3.7. *From this theorem, under the specific choices $\omega_2 = \omega$ and $\omega_1 = 0$, establishes the equality $\mu^{\omega}_{ac} = \mu_{ac}$ of the absolutely continuous spectral measures. Consequently, the absolutely continuous parts of the operators U_ω and U_0 are unitarily equivalent, proving our original claim.*

4 Finite rank case

We consider the perturbation of the unitary operator U_0 by another unitary operator X, defined as:

$$U = U_0X = U_0(\mathbf{1} + W) = U_0 + U_0W,$$

where W is an operator given by:

$$W = \sum_{j=1}^{n} \beta_j P_{\varphi_j},$$

with $\beta_j = (e^{i\omega_j} - 1)$ and $\omega_j \in [0, 2\pi)$ for $j = 1, 2, \ldots, n$.

Using the second resolvent identity, we have:

$$R_z(U_0) - R_z(U) = R_z(U_0)(U - U_0)R_z(U),$$

substituting $U - U_0 = U_0W$, we obtain:

$$R_z(U_0) - R_z(U) = R_z(U_0)(U_0W)R_z(U).$$

Furthermore, we observe that:

$$R_z(U_0) - R_z(U) = WR_z(U) + zR_z(U_0)WR_z(U).$$

To simplify the notations, as in the rank 1 case, we will use that $R_{U_0} = R_0$, $R_0^{k,m}(z) = \langle \varphi_k, R_z(U_0)\varphi_m\rangle$ and $R_U^{k,m}(z) = \langle \varphi_k, R_z(U)\varphi_m\rangle$ for any $k, m \in \{1, 2, \dots, n\}$, and viewing these as matrix elements, we have

$$R_0^{k,m}(z) - R_U^{k,m}(z) = \beta_k R_U^{k,m}(z) + z\sum_{j=1}^{n} R_0^{k,j}(z)\beta_j R_U^{j,k}(z),$$

which means

$$R_0(z) - R_U(z) = MR_U(z) + zR_0(z)MR_U(z),$$

where

$$M = \begin{bmatrix} \beta_1 & 0 & 0 & \cdots & 0 \\ 0 & \beta_2 & 0 & \cdots & 0 \\ 0 & 0 & \beta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_n \end{bmatrix}, \quad \Omega = \begin{bmatrix} e^{i\omega_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{i\omega_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{i\omega_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{i\omega_n} \end{bmatrix} = M + I,$$

with I the $n \times n$ identity matrix, then

$$R_U(z) = (M + I + zR_0(z)M)^{-1}R_0(z) = (\Omega + zR_0(z)(\Omega - I))^{-1}R_0(z).$$

And since $F_U(z) = I + 2zR_U(z)$, we have that

$$F_U(z) = (2I + M + F_0(z)M)^{-1}(M + F_0(z)(M + 2I)),$$

or

$$F_U(z) = [(\Omega + I) + F_0(z)(\Omega - I)]^{-1}\left((\Omega - I) + F_0(z)(\Omega + I)\right).$$

and if we separate the matrix Ω in the following way

$$\Omega = \begin{bmatrix} \cos(\omega_1) + i\sin(\omega_1) & 0 & 0 & \cdots & 0 \\ 0 & \cos(\omega_2) + i\sin(\omega_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cos(\omega_n) + i\sin(\omega_n) \end{bmatrix} := C + iS,$$

then $\Omega + \Omega^* = 2C$ and $MM^* = 2(I - C)$, with M^* is the conjugate matrix of M, therefore

$$2\Re F_U(z) = F_U(z) + F_U(z) = 2((\Omega + I) + F_0(z)(\Omega - I))^{-1}\Re F_0(z)((\Omega + I) + (\Omega - I)F_0(z))^{-1},$$

$$\Re F_U(z) = ((\Omega + I) + F_0(z)(\Omega - I))^{-1}\Re F_0(z)((\Omega^* + I) + (\Omega^* - I)F_0^*(z))^{-1}, \tag{4.1}$$

or

$$\Re F_0(z) = ((\Omega + I) + F_0(z)(\Omega - I))\Re F_U(z)((\Omega^* + I) + (\Omega^* - I)F_0^*(z)), \tag{4.2}$$

for $|z| < 1$.

Remark 4.1. *By the second resolvent identity,*

$$R_z(U_0) = R_z(U) + R_z(U_0)(U_0W)R_z(U) = (I + R_z(U_0)(U_0W))R_z(U),$$

let

$$A = I + R_z(U_0)(U_0W) = R_z(U_0)(U - z) = I + W + zR_z(U_0)W,$$

and since $R_z(U_0) = \frac{1}{2z}(F_z(U_0) - I)$*, then* $A = I + \frac{W}{2} + \frac{1}{2}F_z(U_0)W$*, and given that* A *is bounded with a bounded inverse, we have* $A = R_z(U_0)(U - z)$ *and* $A^{-1} = R_z(U)(U_0 - z)$*. Then, if* $2A = T = 2I + W + F_z(U_0)W$ *and since* A *is invertible,* T *is invertible*

$$T^{-1} = \frac{1}{2}A^{-1} = \frac{1}{2}R_z(U)(U_0 - z) = \frac{1}{4z}(F_z(U) - I)(U_0 - z).$$

Therefore $(2I + M + F_0(z)M)^{-1}$ *is invertible.*

Let us consider the sets

$$I_{m,k}(U_0) := \left\{ t \in [0, 2\pi) \mid \left| \lim_{r\uparrow 1} F_0^{m,k}(re^{it}) \right| = \infty \text{ or } \nexists \lim_{r\uparrow 1} F_0^{m,k}(re^{it}) \right\}$$
$$N_m(U_0) := \left\{ t \in [0, 2\pi) \mid \lim_{r\uparrow 1} F_0^{m,n}(re^{it})(\omega_m - 1) = -\Omega - I \right\},$$

where $\lim_{r\uparrow 1} \omega_m F_0^{m,n}(re^{it})$ is an element of $\lim_{r\uparrow 1} \Omega F_0^{m,n}(re^{it})$. Then, the measures of $I_{m,k}(U_0)$ and $N_m(U_0)$ are zero, by Theorem 2.5, for all m, k and a.e. $t \in [0, 2\pi)$. Now, let us consider the union of these two sets, this is

$$G := \bigcup_{m,k=1}^{n} \Big(N_m(U_0) \cup N_m(U) \cup I_{m,k}(U_0) \cup I_{m,k}(U) \Big),$$

then the measure of G is also zero a.e. $t \in [0, 2\pi)$ and from the equations (4.1) and (4.2), we have

$$\left\{ t \in [0, 2\pi)\backslash G \mid \lim_{r\uparrow 1} \Re F_U(re^{it}) = 0 \right\} \subset \left\{ t \in [0, 2\pi)\backslash G \mid \lim_{r\uparrow 1} \Re F_0(re^{it}) = 0 \right\}$$

and

$$\left\{ t \in [0, 2\pi)\backslash G \mid \lim_{r\uparrow 1} \Re F_0(re^{it}) = 0 \right\} \subset \left\{ t \in [0, 2\pi)\backslash G \mid \lim_{r\uparrow 1} \Re F_U(re^{it}) = 0 \right\},$$

therefore, for almost every $t \in [0, 2\pi)\backslash G$, we have: $\lim_{r\to 1} \Re F_0(re^{it}) \neq 0$ if and only if $\lim_{r\to 1} \Re F_U(re^{it}) \neq 0$. Applying Theorem 2.5 again, we conclude that the absolutely continuous parts of the spectral measures for U_0 and U are mutually absolutely continuous. This equivalence of measures implies the unitary equivalence of the absolutely continuous parts of the operators U and U_0.

Remark 4.2. *In the same way, we can obtain this result for a perturbation* $U \mapsto XU$.

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