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Weak solutions of a discrete Robin problem involving the anisotropic $\vec{p}$-mean curvature operator

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ABSTRACT

This work investigates the existence and uniqueness of a solution to a discrete Robin boundary value problem involving the anisotropic $\vec{p}$-mean curvature operator. The existence result is established through variational methods, specifically by applying the Mountain Pass Theorem of Ambrosetti and Rabinowitz in combination with Ekeland's Variational Principle. Uniqueness is obtained under the assumption of Lipschitz continuity on the nonlinear term.

RESUMEN

Este trabajo investiga la existencia y unicidad de una solución a un problema discreto de valores en la frontera de Robin que involucra el operador de $\vec{p}$-curvatura media anisotrópico. El resultado de existencia se establece a través de métodos variacionales, específicamente aplicando el Teorema del Paso de la Montaña de Ambrosetti y Rabinowitz en combinación con el Principio Variacional de Ekeland. La unicidad se obtiene bajo la hipótesis de continuidad Lipschitz del término no-lineal.

Keywords and Phrases: Discrete Robin problem, boundary value problems, anisotropic $\vec{p}$-mean curvature operator, critical point, nontrivial solution, mountain pass theorem, Ekeland variational principle.

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1 Introduction

In this article, we study the following nonlinear discrete Robin problem.

$$\begin{cases} -\triangle\left(\left(1+\phi_{p(k-1)}\left(\triangle u(k-1)\right)\right)|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right)=\lambda f(k,u(k)), & k\in\mathbb{Z}\left[1,T\right], \\ \triangle u(0)=u(T+1)=0, \end{cases} \tag{1.1}$$

where $T \geq 2$ is a positive integer.

For fixed integers a, b such that $a < b$, we denote by $\mathbb{Z}[a, b]$ the discrete interval $\{a, a+1, \dots, b-1, b\}$. The parameter λ is positive. The forward difference operator is given by $\triangle u(k-1) = u(k)-u(k-1)$. The function $\phi_{p(k)} : \mathbb{R} \to \mathbb{R}$ is defined by $\phi_{p(k)}(s) = \dfrac{|s|^{p(k)}}{\sqrt{1+|s|^{2p(k)}}}$, for every $s \in \mathbb{R}$. The functions p and f will be defined precisely in the subsequent sections.

In problem (1.1), we consider two boundary conditions: a Neumann boundary condition ($\triangle u(0) = 0$) and a Dirichlet boundary condition ($u(T + 1) = 0$). In the literature, these are referred to as mixed boundary conditions (see [25]).

Difference equations arise in many research fields as the discrete counterpart of partial differential equations and are often studied via numerical analysis. In this context, the operator in problem (1.1),

$$\triangle\left(\left(1+\frac{|\triangle u(k-1)|^{p(k-1)}}{\sqrt{1+|\triangle u(k-1)|^{2p(k-1)}}}\right)|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right)$$

represents the discrete counterpart of the following $\vec{p}$-anisotropic operator

$$\left(\left(1+\frac{|u'(t)|^{p(t)}}{\sqrt{1+|u'(t)|^{2p(t)}}}\right)|u'(t)|^{p(t)-2}u'(t)\right)'.$$

In recent years, equations involving the anisotropic $\vec{p}$-mean curvature operator, under various boundary conditions, have become a significant and captivating research topic. Problem (1.1) has been specifically analyzed in [4], where Dirichlet-type boundary conditions were applied through the use of variational methods and critical point theory. In this framework, problem (1.1) also serves as a discrete analogue of the following problem.

$$\begin{cases} -\left(\left(1+\dfrac{|u'(t)|^{p(t)}}{\sqrt{1+|u'(t)|^{2p(t)}}}\right)|u'(t)|^{p(t)-2}u'(t)\right)'=\lambda f(t,u(t)), & t\in(0,1), \\ u'(0)=u(1)=0. \end{cases} \tag{1.2}$$

Problem (1.2) and its multi-dimensional variants arise in various applications, including elasticity mechanics [38,41], electrorheological fluids [14,20,37,38], and image restoration [11]. In [11], Chen *et al.* studied a functional with a variable exponent $1 \leq p(t) \leq 2$, which serves as a model for image denoising, enhancement and restoration.

The existence of a solution to a nonlinear difference equation can be proved using fixed point theory and the method of upper and lower solution techniques, as seen in [12,21] and the references therein. It is well known that critical point theory is a crucial tool for addressing problems involving differential equations.

Variational methods for difference equations were introduced by Guo and Yu [18]. The variational methods have been employed to study various equations, yielding different results. We refer to recent results involving anisotropic discrete boundary value problems [15–17, 23, 25, 26, 29, 39] and references therein. Discrete problems involving anisotropic exponents were firstly discussed in [24, 32].

In [32], by using the mountain pass theorem and Ekeland variational principle, the authors proved the existence of a continuous spectrum of eigenvalues for the following problem

$$\begin{cases} -\triangle\left(\phi_{p(k-1)}\left(\triangle u(k-1)\right)\right) = \lambda |u(k)|^{q(k)-2}, \quad k \in \mathbb{Z}\left[1,T\right], \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.3}$$

where $\phi_{p(\cdot)}(s) = |s|^{p(\cdot)-2}s$, $p : \mathbb{Z}\left[0,T\right] \to [2,\infty)$, $q : \mathbb{Z}\left[1,T\right] \to [2,\infty)$ and λ is a positive constant.

In [24], Koné and Ouaro showed, by using the minimization method, the existence and uniqueness of weak solutions to the following problem

$$\begin{cases} -\triangle\left(a(k-1, \triangle u(k-1))\right) = f(k), \quad k \in \mathbb{Z}\left[1,T\right], \\ u(0) = u(T+1) = 0. \end{cases} \tag{1.4}$$

We note that problem (1.4) is a generalization of (1.3). Indeed, in the particular case where $a(k,\xi) = |\xi|^{p(k)-2}\xi$ for all $k \in \mathbb{Z}\left[0,T\right]$ and $\xi \in \mathbb{R}$, the operator in (1.4) reduces to the $p(k)$-Laplacian, *i.e.*,

$$\triangle_{p(k-1)}u(k-1) := \phi_{p(k-1)}\left(\triangle u(k-1)\right) = |\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1).$$

In [22], the authors studied the following Robin problem

$$\begin{cases} \triangle^2 u(k-1) = f(k, u(k)), \quad k \in \mathbb{Z}\left[1,T\right], \\ u(0) = \triangle u(T) = 0. \end{cases} \tag{1.5}$$

Using the strongly monotone operator principle and critical point theory, the authors proved the existence of nontrivial solutions for (1.5).

In [10], Chen *et al.* considered the following Robin problem

$$\begin{cases} \nabla\left(\dfrac{\triangle u_k}{\sqrt{1-(\triangle u_k)^2}}\right)+\lambda\mu_k(u_k)^q=0, \quad k\in\mathbb{Z}\left[1,T\right], \\ \triangle u_0=u_{T+1}=0. \end{cases} \tag{1.6}$$

By combining the method of upper and lower solutions with Brouwer degree theory and Szulkin's critical point theory for convex, lower semicontinuous perturbations of C^1 functions, the authors determined the ranges of the parameter λ for which problem (1.6) admits zero, one, or two positive solutions. In [28], by using critical point theory, the authors considered the existence of infinitely many positive solutions of the following discrete Robin problem with ϕ-Laplacian

$$\begin{cases} -\triangle(\varphi_p(\triangle u_{k-1}))+q_k\varphi_p(u_k)=\lambda f(k,u_k), \quad k\in\mathbb{Z}\left[1,T\right], \\ \triangle u_0=u_{T+1}=0, \end{cases} \tag{1.7}$$

where φ_p is a special ϕ-Laplacian operator (see [31]) defined by $\varphi_p(s)=\dfrac{p|s|^{p-2}s}{2\sqrt{1+|s|^p}}$ with $p\geq 2$.

In [19], by using variational methods, Hadjian and Bagheri established the existence of at least one nontrivial solution for the following problem

$$\begin{cases} -\triangle(\phi_c(\triangle u_{k-1}))=\lambda f(k,u_k), \quad k\in\mathbb{Z}\left[1,T\right], \\ u_0=u_{T+1}=0, \end{cases} \tag{1.8}$$

where ϕ_c is a special ϕ-Laplacian operator (see [31]) defined by $\phi_c(s)=\dfrac{s}{\sqrt{1+s^2}}$.

For the study of the following Robin problem involving a second-order nonlinear difference equation

$$\begin{cases} \nabla\left(\dfrac{\triangle u_k}{\sqrt{1-(\triangle u_k)^2}}\right)+\lambda f(k,u_k)=0, \quad k\in\mathbb{Z}\left[1,T\right], \\ \triangle u_0=\alpha u_1=0, \quad u_{T+1}=0, \end{cases} \tag{1.9}$$

we refer to [36]. In the particular case where $f(k,t)=\mu_k t^q$ and $\alpha=1$, we obtain the problem studied by Chen *et al.* [10]. The authors used different methods to obtain the existence and multiplicity of solutions for a discrete boundary value problem in [1, 2, 5, 7, 9, 34, 40].

In this article, we use the Ambrosetti-Rabinowitz mountain pass theorem (see [3]), Ekeland's variational principle and a Lipschitz continuity condition on the nonlinear term. Using these tools, we establish the existence and uniqueness of a nontrivial solution to a discrete Robin problem involving equations with the anisotropic $\vec{p}$-mean curvature operator.

The remainder of this article is organized as follows. In Section 2, we present some auxiliary results related to problem (1.1) and recall the abstract critical point theorem. Section 3 develops the variational framework associated with problem (1.1) and introduces our main results. Finally, we identify conditions under which problem (1.1) admits a unique nontrivial solution.

2 Preliminaries

Throughout this article, we denote

$$p^+ = \max_{k\in\mathbb{Z}[0,T]} p(k), \quad p^- = \min_{k\in\mathbb{Z}[0,T]} p(k), \quad r^+ = \max_{k\in\mathbb{Z}\in[1,T]} r(k) \quad \text{and} \quad r^- = \min_{k\in\mathbb{Z}[1,T]} r(k).$$

We consider the T-dimensional Banach space

$$H = \{u : \mathbb{Z}\,[0, T+1] \to \mathbb{R} \quad \text{such that} \quad \triangle u(0) = u(T+1) = 0\},$$

equipped with the norm

$$\|u\| = \left(\sum_{k=1}^{T} |\triangle u(k)|^{p^-}\right)^{1/p^-}. \tag{2.1}$$

However, we will use the following norm in H at times

$$\|u\|_\infty = \max_{k\in\mathbb{Z}[0,T+1]} |u(k)|, \quad \text{for all} \quad u \in H.$$

The space H will also be equipped with the following Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\left\{\mu > 0 : \sum_{k=1}^{T} \frac{1}{p(k)} \left|\frac{\triangle u(k)}{\mu}\right|^{p(k)} \le 1\right\}.$$

Since on H, all norms are equivalent, then there exist two constants $0 < K_1 < K_2$ such that

$$K_1\|u\|_{p(\cdot)} \le \|u\| \le K_2\|u\|_{p(\cdot)}. \tag{2.2}$$

Next, let $\rho_{p(\cdot)} : H \to \mathbb{R}$ be given by

$$\rho_{p(\cdot)}(u) = \sum_{k=1}^{T} \frac{1}{p(k)} |\triangle u(k)|^{p(k)}.$$

Remark 2.1. *If $u \in H$, then the following properties hold.*

$$\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^+}, \tag{2.3}$$

$$\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^-}. \tag{2.4}$$

To establish our main result, we introduce the following quotient

$$\Lambda_1 = \inf_{u\in H\setminus\{0\}} \frac{\displaystyle\sum_{k=1}^{T} \frac{1}{p(k)} \left(|\triangle u(k)|^{p(k)} + \sqrt{1+|\triangle u(k)|^{2p(k)}} - 1 \right)}{\displaystyle\sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{p(k)}}. \tag{2.5}$$

We say that λ is an eigenvalue of problem (1.1) whenever the problem admits a nontrivial solution.

It should be emphasized that Λ_1 represents the first eigenvalue of problem (1.1) in the particular case where

$$f(k,u(k)) = |u(k)|^{p(k)-2}u(k).$$

In addition, Λ_1 serves as a critical threshold parameter governing the existence of nontrivial solutions to problem (1.1), thus guaranteeing the consistency of the analysis.

Let us also define the function

$$F(k,\xi) = \int_0^{\xi} f(k,s)\, ds, \quad \text{for all} \quad (k,\xi) \in \mathbb{Z}\,[1,T] \times \mathbb{R}.$$

We also make the following assumptions for the study of problem (1.1).

(H_1) For each $k \in \mathbb{Z}[1,T]$, the mapping $f(k,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2) There exist a constant $C_1 > 0$ and a function

$$r(\cdot) : \mathbb{Z}[1,T] \to [2,\infty)$$

such that:

(*i*) $|f(k,\xi)| \le C_1\big(1+|\xi|^{r(k)-1}\big)$, $\forall k \in \mathbb{Z}[1,T]$, $\forall \xi \in \mathbb{R}$.

(*ii*) $\displaystyle\liminf_{|\xi|\to 0} \frac{F(k,\xi)}{|\xi|^{r(k)}} \ge 0$, for all $k \in \mathbb{Z}[1,T]$.

In particular, assumption $(H_2)(i)$ implies that there exists a constant $C_2 > 0$ such that

$$|F(k,\xi)| \le C_2\big(1+|\xi|^{r(k)}\big), \quad \forall k \in \mathbb{Z}[1,T], \quad \forall \xi \in \mathbb{R}.$$

(H_3) $\displaystyle\liminf_{|\xi|\to\infty} \frac{F(k,\xi)}{|\xi|^{r^-}} \ge 0$, for all $k \in \mathbb{Z}[1,T]$.

(H_4) For every $\lambda \in (0,\Lambda_1)$,

$$\limsup_{|\xi|\to 0} \frac{\lambda f(k,\xi)}{|\xi|^{p(k)-2}\xi} < \Lambda_1, \quad \text{for all} \quad k \in \mathbb{Z}[1,T].$$

(H_5) $p(\cdot) : \mathbb{Z}[0,T] \to (2,\infty)$.

Example 2.2. *The function*

$$f(x,t) := \begin{cases} |t|^{p(x)-2}t, & \text{if } |t| < 1 \\ |t|^{r(x)-2}t, & \text{if } |t| \geq 1, \end{cases}$$

with $r^- > p^+$, satisfies assumptions (H_1), (H_2), (H_3) and (H_4).

This example provides a concrete instance of the broader class of functions considered in problem (1.1).

In the sequel, we will use the following auxiliary results.

Lemma 2.3 ([16, 35])**.** *(a) For all $u \in H$ with $\|u\| > 1$,*

$$\sum_{k=1}^{T} \frac{1}{p(k)} |\triangle u(k)|^{p(k)} \geq \frac{1}{p^+}\left(\|u\|^{p^-} - T\right).$$

(b) For all $u \in H$ with $\|u\| < 1$,

$$\sum_{k=1}^{T} \frac{1}{p(k)} |\triangle u(k)|^{p(k)} \geq \frac{1}{p^+ T^{(p^+-p^-)/p^-}} \|u\|^{p^+}.$$

(c) For all $u \in H$ and for any $m \geq 2$,

$$\sum_{k=1}^{T} |u(k)|^m \leq \left(T^{(p^- -1)/p^-}\right)^m T\|u\|^m.$$

(d) For all $u \in H$ and all $p^+ \geq 2$,

$$\sum_{k=1}^{T} |\triangle u(k)|^{p^+} \leq 2^{p^+}\left(T^{(p^- -1)/p^-}\right)^{p^+} T\|u\|^{p^+}.$$

(e) For all $u \in H$ and all $p^+ \geq 2$,

$$\sum_{k=1}^{T} \frac{1}{p(k)} |\triangle u(k)|^{p(k)} \leq \frac{T}{p^-}\left[2^{p^+}\left(T^{(p^- -1)/p^-}\right)^{p^+} \|u\|^{p^+} + 1\right].$$

The energy functional associated with problem (1.1) is defined by $J_\lambda : H \to \mathbb{R}$ as follows

$$J_\lambda(u) = \sum_{k=1}^{T}\left[\frac{1}{p(k)}\left(|\triangle u(k)|^{p(k)} + \sqrt{1 + |\triangle u(k)|^{2p(k)}} - 1\right) - \lambda F(k, u(k))\right]. \tag{2.6}$$

Definition 2.4. *We say that* $u \in H$ *is a weak solution of the problem* (1.1) *if*

$$\sum_{k=1}^{T}\left[\left(1+\phi_{p(k)}(\triangle u(k))\right)|\triangle u(k)|^{p(k)-2}\triangle u(k)\triangle v(k)-\lambda f(k,u(k))v(k)\right]=0, \qquad (2.7)$$

for any $v \in H$ *and*

$$\sum_{k=1}^{T}\left[\left(1+\phi_{p(k)}(\triangle u(k))\right)|\triangle u(k)|^{p(k)}-\lambda f(k,u(k))u(k)\right]=0. \qquad (2.8)$$

We define the functionals $\Phi, \Psi : H \to \mathbb{R}$ by

$$\Phi(u)=\sum_{k=1}^{T}\frac{1}{p(k)}\left(|\triangle u(k)|^{p(k)}+\sqrt{1+|\triangle u(k)|^{2p(k)}}-1\right)$$

and

$$\Psi(u)=\sum_{k=1}^{T}F(k,u(k)).$$

The functional is now written as: $J_\lambda(u) = \Phi(u) - \lambda\Psi(u)$.

Proposition 2.5. *The functional* J_λ *is well-defined on* H *and is of class* $C^1(H,\mathbb{R})$ *with the derivative given by*

$$\langle J_\lambda'(u),v\rangle=\sum_{k=1}^{T}\left[\left(1+\phi_{p(k)}(\triangle u(k))\right)|\triangle u(k)|^{p(k)-2}\triangle u(k)\triangle v(k)-\lambda f(k,u(k))v(k)\right], \qquad (2.9)$$

for all $u, v \in H$.

The proof of Proposition 2.5 is a consequence of the proof of the following lemma.

Lemma 2.6. *The functionals* Φ *and* Ψ *are well-defined on* H*, and both belong to the class* $C^1(H,\mathbb{R})$*. Moreover, their derivatives are given by*

$$\langle\Phi'(u),v\rangle=\sum_{k=1}^{T}\left(1+\phi_{p(k)}(\triangle u(k))\right)|\triangle u(k)|^{p(k)-2}\triangle u(k)\triangle v(k),\quad \langle\Psi'(u),v\rangle=\sum_{k=1}^{T}f(k,u(k))v(k),$$

for all $u, v \in H$.

Furthermore, the critical points of the functional J_λ *in* H *coincide with the weak solutions of problem* (1.1).

Since the proof of Lemma 2.6 is very similar to that of Lemma 3.4 in [17] and Lemma 2.3 in [23], it is omitted.

Owing to the finite-dimensional setting, every weak solution of problem (1.1) is a strong (*i.e.*, classical) solution. Consequently, solving problem (1.1) amounts to finding the critical points of the functional J.

We now introduce the following results, which will be useful in the subsequent analysis.

Proposition 2.7 ([33]). *Assume that the condition (H_5) holds. Then, $\Lambda_1 > 0$.*

Definition 2.8. *Let E be a real Banach space and let $J : E \to \mathbb{R}$ be a functional. We say that J satisfies the* Palais-Smale condition *(abbreviated as (PS) condition) if every sequence $\{u_n\} \subset E$ such that $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, admits a convergent subsequence in E.*

Moreover, a sequence $\{u_n\} \subset E$ is said to satisfy the Palais-Smale condition at level $c \in \mathbb{R}$, *denoted by $(PS)_c$, if*

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Lemma 2.9 ([39]). *Let E be a finite-dimensional Banach space and let $J \in C^1(E, \mathbb{R})$ be an anti-coercive functional. Then, J satisfies the (PS) condition.*

Lemma 2.10 ([30, Mountain pass lemma]). *Let E be a real Banach space. Assume that $J \in C^1(E, \mathbb{R})$ satisfies the (PS) condition. Suppose also that:*

(i) $J(0) = 0$;

(ii) *there exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$;*

(iii) *there exists u_1 in E with $\|u_1\| \geq \rho$ such that $J(u_1) < 0$.*

Then, J has a critical value $c \geq \alpha$ which can be characterized by

$$c = \inf_{h \in \Gamma} \max_{s \in [0,1]} J\left(h(s)\right),$$

where $\Gamma = \{h \in C\left([0,1], E\right) : h(0) = 0,\ h(1) = u_1\}$.

Theorem 2.11 ([30]). *Let E be a real Banach space and $J : E \to \mathbb{R}$. If J is weakly lower semicontinuous and coercive, i.e. $\lim\limits_{\|x\| \to \infty} J(x) = \infty$, then there exists $x_0 \in E$ such that $\inf\limits_{x \in E} J(x) = J(x_0)$.*

Moreover if $J \in C^1(E, \mathbb{R})$, then x_0 is a critical point of J, i.e. $J'(x_0) = 0$.

Theorem 2.12 ([13, Ekeland's variational principle]). *Let X be a complete metric space and $\Phi : X \to \mathbb{R}$ a lower semicontinuous function bounded from below. Then, for every $\epsilon > 0$ and $\overline{u} \in X$ be given such that*

$$\Phi(\overline{u}) \leq \inf_{u \in X} \Phi(u) + \epsilon,$$

there exists $u_\epsilon \in X$ such that

(i) $\Phi(u_\epsilon) \leq \Phi(\overline{u})$,

(ii) $d(u_\epsilon, \overline{u}) < \epsilon$,

(iii) $\Phi(u_\epsilon) < \Phi(u) + \epsilon d(u, u_\epsilon)$ *for each* $u \neq u_\epsilon$.

Corollary 2.13 ([13])**.** *Let X be a complete metric space and $\Phi : X \to \mathbb{R}$ be a lower semicontinuous function bounded below. Assume that $\Phi \in C^1(X, \mathbb{R})$. Then, for every $\varepsilon > 0$, there exists $u_\varepsilon \in X$ such that*

(i) $\Phi(u_\varepsilon) \leq \inf_{u \in X} \Phi(u) + \varepsilon$,

(ii) $\|\Phi'(u_\varepsilon)\| \leq \varepsilon$.

3 Existence and uniqueness of weak nontrivial solutions

This section focuses on the existence and uniqueness of nontrivial weak solutions to problem (1.1). We have the following result.

Theorem 3.1. *Assume that the hypotheses (H_1)-(H_5) hold. If $(r^- > p^+)$ or $(r^+ < p^-)$ or $(r^- < p^-)$, then there exist $\lambda^*, \rho, \Lambda^* > 0$ such that for any $\lambda > \lambda^*$ and $\Lambda_1 - \rho \in (\lambda, \Lambda^*)$, the problem (1.1) has at least one weak nontrivial solution.*

Proof. We can distinguish the following three cases:

Case 1: $r^- > p^+$

In this instance, we will demonstrate that J_λ possesses a "mountain pass geometry."

Lemma 3.2. *Assume that the hypotheses of Theorem 3.1 are satisfied, then.*

(i) *There exist $a, \varrho > 0$ and $\rho, \Lambda^* > 0$ such that for any $\lambda > 0$ and $\Lambda_1 - \rho \in (\lambda, \Lambda^*)$, one has*

$$J_\lambda(u) \geq a > 0 \quad \textit{for all} \quad u \in H \quad \textit{with} \quad \|u\| = \varrho.$$

(ii) *There exists $e \in H$ with $\|e\| > \varrho$ such that*

$$J_\lambda(e) < 0.$$

Proof. (i) Using hypothesis (H_4), for any $\lambda \in (0, \Lambda_1)$, we can find $\rho, \beta > 0$ such that $\lambda \leq \Lambda_1 - \rho$ and

$$\lambda f(k, \xi) \leq (\Lambda_1 - \rho) |\xi|^{p(k)-2}\xi, \quad \text{for all} \quad (k, \xi) \in \mathbb{Z}[1, T] \times \mathbb{R} \quad \text{and} \quad |\xi| \leq \beta.$$

In particular, if f is as in Example 2.2, then $\beta = 1$.

We deduce for $\xi \in (0, \beta]$, that

$$\lambda F(k,\xi) \le (\Lambda_1 - \rho) \int_0^\xi |s|^{p(k)-2} s\, ds = (\Lambda_1 - \rho) \int_0^\xi s^{p(k)-2} s\, ds = \frac{1}{p(k)} (\Lambda_1 - \rho) |\xi|^{p(k)}$$

and for $\xi \in [-\beta, 0)$, we infer that

$$\lambda F(k,\xi) \le (\Lambda_1 - \rho) \int_\xi^0 |s|^{p(k)-2} s\, ds = (\Lambda_1 - \rho) \int_\xi^0 (-s)^{p(k)-2} s\, ds = \frac{1}{p(k)} (\Lambda_1 - \rho) |\xi|^{p(k)}.$$

Then, it follows that

$$\lambda F(k,\xi) \le \frac{1}{p(k)} (\Lambda_1 - \rho) |\xi|^{p(k)}, \quad \text{for all} \quad k \in \mathbb{Z}[1,T] \quad \text{and} \quad |\xi| \le \beta. \tag{3.1}$$

Let $u \in H$ be such that $|u(k)| \le \beta$ for all $k \in \mathbb{Z}[1,T]$. Then, by relation (2.1), we have

$$\|u\| \le 2\beta T^{1/p^-}.$$

Now, let $u \in H$ be fixed such that $\|u\| \le 1$. Define

$$\kappa = \min\left\{2\beta T^{1/p^-},\, 1\right\}.$$

Then, for any $u \in H$ satisfying $\|u\| \le \kappa$, it follows from relations (2.5), (3.1), and assertions (b) and (c) of Lemma 2.3 that

$$\begin{aligned}
J_\lambda(u) &\ge \Phi(u) - (\Lambda_1 - \rho) \sum_{\substack{k=1 \\ |u(k)|>1}}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} - (\Lambda_1 - \rho) \sum_{\substack{k=1 \\ |u(k)|>1}}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} \\
&\ge \Phi(u) - (\Lambda_1 - \rho) \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{r(k)} - (\Lambda_1 - \rho) \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} \\
&\ge \frac{\Lambda_1 - (\Lambda_1 - \rho)}{\Lambda_1} \Phi(u) - \frac{(\Lambda_1 - \rho)}{p^+} \sum_{k=1}^{T} |u(k)|^{r^-} \\
&\ge \frac{\rho}{\Lambda_1 p^+} T^{(p^- - p^+)/p^-} \|u\|^{p^+} - \frac{(\Lambda_1 - \rho)}{p^+} \left(T^{(p^- - 1)/p^-}\right)^{r^-} T \|u\|^{r^-} \\
&= \left(c_1 \varrho^{p^+ - r^-} - (\Lambda_1 - \rho) c_2\right) \varrho^{r^-},
\end{aligned}$$

where c_1 and c_2 are positive constants.

Hence, choosing $\Lambda^* = \dfrac{c_1 \varrho^{p^+ - r^-}}{2c_2}$, then, for any $\Lambda_1 - \rho \in (\lambda, \Lambda^*)$, there exist some positive numbers $0 < \varrho < \kappa$ and $a = \dfrac{c_1 \varrho^{p^+}}{2} > 0$ such that $J_\lambda(u) \ge a > 0$ for all $u \in H$ with $\|u\| = \varrho$.

(*ii*) Fix $\lambda > 0$. By (H_3), for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$F(k,\xi) \geq \varepsilon|\xi|^{r^-}, \quad \text{for all} \quad k \in \mathbb{Z}[1,T] \quad \text{and all} \quad \xi \in \mathbb{R}, \quad \text{with} \quad |\xi| > \eta.$$

Since $\xi \to F(k,\xi) - \varepsilon|\xi|^{r^-}$ is continuous on $[-\eta,\eta]$, there is a constant $C_\eta > 0$ such that

$$F(k,\xi) - \varepsilon|\xi|^{r^-} \geq -C_\eta, \quad \text{for all} \quad k \in \mathbb{Z}[1,T] \quad \text{and all} \quad \xi \in [-\eta,\eta].$$

Hence, we get

$$F(k,\xi) \geq \varepsilon|\xi|^{r^-} - C_\eta, \quad \text{for all } (k,\xi) \in \mathbb{Z}[1,T] \times \mathbb{R}. \tag{3.2}$$

So, from (3.2) and Lemma 2.3 (*e*), we obtain

$$\begin{aligned} J_\lambda(u) &= \sum_{k=1}^{T} \frac{1}{p(k)} \left(|\triangle u(k)|^{p(k)} + \sqrt{1 + |\triangle u(k)|^{2p(k)}} - 1 \right) - \lambda \sum_{k=1}^{T} F(k,u(k)) \\ &\leq \frac{2}{p^-} \sum_{k=1}^{T} |\triangle u(k)|^{p(k)} - \lambda \sum_{k=1}^{T} \left(\varepsilon |u(k)|^{r^-} - C_\eta \right) \\ &\leq \frac{2T}{p^-} \left[2^{p^+} \left(T^{\frac{p^- - 1}{p^-}} \right)^{p^+} \|u\|^{p^+} + 1 \right] - \lambda\varepsilon \sum_{k=1}^{T} |u(k)|^{r^-} + \lambda T C_\eta. \end{aligned} \tag{3.3}$$

As

$$\|u\|^{p^-} \leq 2^{p^- - 1} \sum_{k=1}^{T} \left(|u(k+1)|^{p^-} + |u(k)|^{p^-} \right) \leq$$

$$2^{p^-} \sum_{k=1}^{T} |u(k)|^{p^-} \leq 2^{p^-} T^{\frac{r^- - p^-}{r^-}} \left(\sum_{k=1}^{T} |u(k)|^{r^-} \right)^{\frac{p^-}{r^-}},$$

which means that

$$\sum_{k=1}^{T} |u(k)|^{r^-} \geq 2^{-r^-} T^{\frac{p^- - r^-}{p^-}} \|u\|^{r^-}. \tag{3.4}$$

Then, it follows from (3.3) and (3.4) that

$$J_\lambda(u) \leq \frac{T}{p^-} 2^{p^+ + 1} \left(T^{\frac{p^- - 1}{p^-}} \right)^{p^+} \|u\|^{p^+} - \lambda\varepsilon 2^{-r^-} T^{\frac{p^- - r^-}{p^-}} \|u\|^{r^-} + \frac{2T}{p^-} + \lambda T C_\eta. \tag{3.5}$$

Since $r^- > p^+$, $J_\lambda(u) \to -\infty$ as $\|u\| \to \infty$. Thus, J_λ is anti-coercive. Consequently, there exists $e \in H$ with $\|e\| > \varrho$ such that $J_\lambda(e) < 0$. □

Lemma 3.3. *Assume that the hypotheses of Theorem 3.1 hold. Then, for any $\lambda > 0$, the functional J_λ satisfies Palais-Smale condition.*

Proof. By Lemma 3.2 (ii), the functional J_λ is anti-coercive. Therefore, by Lemma 2.3, the functional J_λ satisfies the Palais-Smale condition for any $\lambda > 0$. Thus, our problem (1.1) has at least one nontrivial solution. □

Case 2: $r^+ < p^-$

In the second case, we apply a direct variational approach. We verify that the functional J_λ has a critical point. Let $\lambda > 0$ be fixed, since H is a finite-dimensional space and J_λ is of class $C^1(H, \mathbb{R})$, it is sufficient to prove that J_λ is coercive.

Let $\|u\| > 1$. Then, by (2.5), (2.6), (a) and (c) of Lemma 2.3, one has

$$\begin{aligned} J_\lambda(u) &\geq \Phi(u) - (\Lambda_1 - \rho) \sum_{\substack{k=1 \\ |u(k)|>1}}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} - (\Lambda_1 - \rho) \sum_{\substack{k=1 \\ |u(k)|>1}}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} \\ &\geq \Phi(u) - (\Lambda_1 - \rho) \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{r(k)} - (\Lambda_1 - \rho) \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} \\ &\geq \frac{\Lambda_1 - (\Lambda_1 - \rho)}{\Lambda_1} \Phi(u) - \frac{(\Lambda_1 - \rho)}{p^+} \sum_{k=1}^{T} |u(k)|^{r^+} \\ &\geq \frac{\rho}{\Lambda_1 p^+} \|u\|^{p^-} - \frac{(\Lambda_1 - \rho)}{p^+} \left(T^{(p^- - 1)/p^-} \right)^{r^+} T \|u\|^{r^+} - K(T), \end{aligned}$$

where $K(T)$ is a positive constant. Therefore, choosing $\Lambda^* = \dfrac{\rho}{\Lambda_1 \left(T^{(p^- - 1)/p^-} \right)^{r^+} T}$, since $r^+ < p^-$, one deduces that J_λ is coercive.

Now, let $u_* \in H$ be a global minimum of J_λ, which is a critical point of J_λ and, in turn, a weak solution of the problem (1.1).

We now show that u_* is nontrivial for λ large enough.

Let $d \in (0, 1)$ be a fixed real and $k_0 \in \mathbb{Z}[1, T]$, we define a function $w \in H$ by

$$w(k) = \begin{cases} d & \text{if} \quad k = k_0, \\ 0 & \text{if} \quad k \in \mathbb{Z}[1, T] - \{k_0\}. \end{cases}$$

Then, we deduce by $(H_2)(ii)$ that

$$J_\lambda(w) = \frac{1}{p(k_0)}\left(|d|^{p(k_0)} + \sqrt{1+|d|^{2p(k_0)}} - 1\right) - \lambda F(k_0, w(k_0))$$
$$\leq \frac{2}{p(k_0)}|d|^{p(k_0)} - \lambda F(k_0, d) \leq \frac{2}{p^-}d^{p^-} - \lambda C d^{r^+}.$$

Thus, if we choose λ^* as

$$\lambda^* = \frac{2}{p^- C} d^{p^- - r^+},$$

then for any $\lambda > \lambda^*$ and $r^+ < p^-$, $J_\lambda(w) < 0$. Since u_* is a global minimum of J_λ, it follows that $J_\lambda(u_*) < 0$ for any $\lambda > \lambda^*$; therefore u_* is a weak nontrivial solution of problem (1.1).

Case 3: $r^- < p^-$

In this case, we apply the Ekeland's variational principle.

Lemma 3.4. *Assume that $(H_2)(ii)$ holds and $r^- < p^-$. Then, there is $\overline{v} \in H$ such that $J_\lambda(\overline{v}) < 0$.*

Proof. Take $d \in (0, \kappa)$, where κ is as in the proof of Lemma 3.2 (ii), such that $d < \left(\frac{p^-\lambda C}{2}\right)^{\frac{1}{p^- - r^-}}$. Let $k_0 \in \mathbb{Z}[1, T]$ with $r(k_0) = r^-$. Consider any fixed $\overline{v} \in H$ such that $\overline{v}(k_0) = d$ and $\overline{v}(k) = 0$ for any $k \in \mathbb{Z}[1, T] \setminus \{k_0\}$. Using the condition $(H_2)(ii)$, we have

$$J_\lambda(\overline{v}) \leq \frac{2}{p(k_0)} d^{p(k_0)} - \lambda C d^{r(k_0)} \leq \frac{2}{p^-} d^{p^-} - \lambda C d^{r^-}.$$

Then,

$$J_\lambda(\overline{v}) < 0,$$

for all $d < \left(\frac{p^-\lambda C}{2}\right)^{\frac{1}{p^- - r^-}}$. The proof is thus complete. □

Relation (i) of Lemma 3.2 implies that

$$\inf_{u \in \partial B_\kappa} J_\lambda(u) > 0,$$

where $B_\kappa = \{u \in H \text{ such that } \|u\| \leq \kappa\}$. On the other hand, observe that Lemma 3.3 implies that there exists $\overline{v} \in H$ such that $J_\lambda(\overline{v}) < 0$, for every $d < \left(\frac{p^-\lambda C}{2}\right)^{\frac{1}{p^- - r^-}}$. Recall that $\overline{v} \in int\, B_\kappa$. Thus,

$$\inf_{u \in int\, B_\kappa} J_\lambda(u) < 0.$$

So, it follows

$$\inf_{u\in int\, B_\kappa} J_\lambda(u) < \inf_{u\in\partial B_\kappa} J_\lambda(u).$$

Let $\epsilon > 0$ be fixed, such that

$$0 < \epsilon < \inf_{u\in\partial B_\kappa} J_\lambda(u) - \inf_{u\in int\, B_\kappa} J_\lambda(u).$$

Applying Ekeland's variational principle to the functional $J_\lambda : B_\kappa \to \mathbb{R}$, there exists $u_\epsilon \in B_\kappa$ such that

$$J_\lambda(u_\epsilon) < \inf_{u\in B_\kappa} J_\lambda(u) + \epsilon \quad \text{and} \quad J_\lambda(u_\epsilon) < J_\lambda(u) + \epsilon\|u - u_\epsilon\| \quad \text{for all} \quad u \neq u_\epsilon.$$

Moreover,

$$J_\lambda(u_\epsilon) < \inf_{u\in B_\kappa} J_\lambda(u) + \epsilon \leq \inf_{u\in int\, B_\kappa} J_\lambda(u) + \epsilon < \inf_{u\in\partial B_\kappa} J_\lambda(u),$$

then, we infer that $u_\epsilon \in int\, B_\kappa$. Next, we introduce the function $\psi_\lambda : B_\kappa \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = J_\lambda(u) + \epsilon\|u - u_\epsilon\| \quad \text{for all} \quad u \neq u_\epsilon.$$

So, it follows that u_ϵ is a minimum point of ψ_λ and thus

$$\frac{\psi_\lambda(u_\epsilon + \theta v) - \psi_\lambda(u_\epsilon)}{\theta} \geq 0, \tag{3.6}$$

for all $v \in B_\kappa$ and all $\theta > 0$ small enough. Therefore, using relation (3.6), we deduce that

$$\frac{J_\lambda(u_\epsilon + \theta v) - J_\lambda(u_\epsilon)}{\theta} + \epsilon\|v\| \geq 0.$$

Letting $\theta \to 0^+$, we obtain

$$J'_\lambda(u_\epsilon, v) + \epsilon\|v\| \geq 0 \quad \text{for all} \quad u \in H, \tag{3.7}$$

where $J'_\lambda(u_\epsilon, v)$ is the directional derivative of the function J_λ at u_ϵ in the direction of v. Since

$$J'_\lambda(u_\epsilon, v) = \langle J'_\lambda(u_\epsilon), v\rangle = J'_\lambda(u_\epsilon)v,$$

we obtain from (3.7),

$$\|J'_\lambda(u_\epsilon)\| \leq \epsilon.$$

Thus, we deduce that there exists a sequence $\{u_n\} \subset int\, B_\kappa$ such that

$$J_\lambda(u_n) \to c = \inf_{u\in B_\kappa} J_\lambda(u) \quad \text{and} \quad J'_\lambda(u_n) \to 0 \quad \text{as} \quad n \to \infty.$$

As the sequence $\{u_n\}$ is bounded in H, then there exists $u_0 \in H$ such that, up to a subsequence, $\{u_n\}$ converges to u_0 in H. Hence, the problem (1.1) has a nontrivial solution. □

Lemma 3.5. *Let $\lambda > 0$. Suppose that conditions (H_1)-(H_5) are satisfied. If $u \in H$ is a solution of problem* (1.1)*, then there exist two positive constants κ_1 and κ_2 such that $\kappa_1 \leq \|u\| \leq \kappa_2$.*

Proof. The proof of this lemma is organized into two steps, as outlined below.

Step 1. Assume that $u \in H$ is a solution of (1.1) with $\|u\|_{p(\cdot)} \leq 1$. Set $\zeta = \dfrac{p^-}{\lambda\left(T^{\frac{p^- -1}{p^-}}\right)^{p^+} TK_2^{p^+}}$.

Since f satisfies (H_4), for any $\lambda \in (0, \Lambda_1)$, there exist $\rho, \beta > 0$ such that $\lambda \leq \Lambda_1 - \rho < \zeta$ and

$$\lambda f(k,\xi) \leq (\Lambda_1 - \rho)\,|\xi|^{p(k)-2}\xi \quad \text{for all} \quad k \in \mathbb{Z}[1,T] \quad \text{and} \quad \xi \in \mathbb{R} \quad \text{with} \quad |\xi| \leq \beta.$$

On the other hand, by $(H_2)(i)$, there exists a positive constant L such that

$$\lambda |f(k,\xi)| \leq L|\xi|^{r(k)-1}, \quad \text{for all} \quad (k,\xi) \in \mathbb{Z}[1,T] \times \mathbb{R} \quad \text{and} \quad |\xi| > \beta,$$

where $L = \lambda\left(\frac{1}{\beta^{r(k)-1}} + 1\right)$. Consequently, we get that

$$\lambda |f(k,\xi)| \leq (\Lambda_1 - \rho)\,|\xi|^{p(k)-1} + L|\xi|^{r(k)-1} \quad \text{for all} \quad k \in \mathbb{Z}[1,T] \quad \text{and} \quad \xi \in \mathbb{R}.$$

Using the above inequality, (2.2), (2.4), (2.8) and Lemma 2.3 (c), we obtain

$$\begin{aligned}
\|u\|_{p(\cdot)}^{p^+} &\leq \rho_{p(\cdot)}(u) = \sum_{k=1}^{T} \frac{1}{p(k)}\,|\triangle u(k)|^{p(k)} \leq \frac{1}{p^-}\sum_{k=1}^{T} |\triangle u(k)|^{p(k)} \\
&\leq \frac{1}{p^-}\sum_{k=1}^{T}\left(1 + \phi_{p(k)}(\triangle u(k))\right)|\triangle u(k)|^{p(k)} = \frac{\lambda}{p^-}\sum_{k=1}^{T} f(k,u(k))u(k) \\
&\leq \frac{\lambda}{p^-}(\Lambda_1 - \rho)\sum_{k=1}^{T}|u(k)|^{p(k)} + \frac{\lambda L}{p^-}\sum_{k=1}^{T}|u(k)|^{r(k)} \\
&\leq \frac{\lambda}{p^-}(\Lambda_1 - \rho)\sum_{k=1}^{T}|u(k)|^{p^+} + \frac{\lambda L}{p^-}\sum_{k=1}^{T}|u(k)|^{r^+} \\
&\leq \frac{\lambda}{p^-}(\Lambda_1 - \rho)\left(T^{\frac{p^- -1}{p^-}}\right)^{p^+} T\|u\|^{p^+} + \frac{\lambda L}{p^-}\left(T^{\frac{p^- -1}{p^-}}\right)^{r^+} T\|u\|^{r^+} \\
&\leq \frac{\lambda}{p^-}(\Lambda_1 - \rho)\left(T^{\frac{p^- -1}{p^-}}\right)^{p^+} TK_2^{p^+}\|u\|_{p(\cdot)}^{p^+} + \frac{\lambda L}{p^-}\left(T^{\frac{p^- -1}{p^-}}\right)^{r^+} TK_2^{r^+}\|u\|_{p(\cdot)}^{r^+}.
\end{aligned}$$

Therefore,

$$\|u\|_{p(\cdot)} \geq \left[\frac{1 - \frac{\lambda}{p^-}(\Lambda_1 - \rho)\left(T^{\frac{p^- -1}{p^-}}\right)^{p^+} TK_2^{p^+}}{\frac{\lambda L}{p^-}\left(T^{\frac{p^- -1}{p^-}}\right)^{r^+} TK_2^{r^+}} \right]^{\frac{1}{r^+ - p^+}}.$$

Set

$$\kappa_1^* = \left[\frac{1 - \frac{\lambda}{p^-}(\Lambda_1 - \rho)\left(T^{\frac{p^- -1}{p^-}}\right)^{p^+} TK_2^{p^+}}{\frac{\lambda L}{p^-}\left(T^{\frac{p^- -1}{p^-}}\right)^{r^+} TK_2^{r^+}} \right]^{\frac{1}{r^+ - p^+}}$$

and note that

$$0 < \kappa_1^* < 1.$$

Indeed, since

$$\lambda \leq \Lambda_1 - \rho < \frac{p^-}{\lambda\left(T^{\frac{p^- -1}{p^-}}\right)^{p^+} TK_2^{p^+}},$$

it follows that

$$0 < 1 - \frac{\lambda}{p^-}(\Lambda_1 - \rho)\left(T^{\frac{p^- -1}{p^-}}\right)^{p^+} TK_2^{p^+} < 1.$$

Clearly, $\lambda L \left(T^{\frac{p^- -1}{p^-}}\right)^{r^+} TK_2^{r^+} > p^-$. Hence, $0 < \kappa_1^* < 1$.

Step 2. Suppose that $u \in H$ is a solution of (1.1) such that $\|u\|_{p(\cdot)} \geq 1$. Then, there exists a constant $\kappa_2^* > 1$ such that $\|u\|_{p(\cdot)} \leq \kappa_2^*$.

According to (2.6) and (2.8), one has

$$\begin{aligned}
&r^- \left(J_\lambda(u) + \lambda \sum_{k=1}^{T} F(k, u(k)) \right) - \lambda \sum_{k=1}^{T} f(k, u(k))u(k) \\
&= r^- \sum_{k=1}^{T} \frac{1}{p(k)} \left(|\triangle u(k)|^{p(k)} + \sqrt{1 + |\triangle u(k)|^{2p(k)}} - 1 \right) - \sum_{k=1}^{T} \left(1 + \phi_{p(k)}(\triangle u(k))\right) |\triangle u(k)|^{p(k)} \\
&\geq \frac{r^-}{p^+} \sum_{k=1}^{T} |\triangle u(k)|^{p(k)} - \sum_{k=1}^{T} |\triangle u(k)|^{p(k)} = \left(\frac{r^-}{p^+} - 1 \right) \sum_{k=1}^{T} |\triangle u(k)|^{p(k)}.
\end{aligned}$$

Recall the Ambrosetti-Rabinowitz condition:

$$\frac{r^-}{\xi} \leq \frac{f(k,\xi)}{F(k,\xi)}, \quad \text{for all} \quad (k,\xi) \in \mathbb{Z}[1,T] \times \mathbb{R} \quad \text{and for some} \quad r^- > p^+. \tag{3.8}$$

Integrating, we obtain that (3.2) holds (see [6, Remark 5.2] or [8, Remark 3.7]).

Combining the above inequalities with (2.2), (3.5), (3.8) and Lemma 2.3, it follows that

$$\begin{aligned}
& r^-\left(J_\lambda(u)+\lambda\sum_{k=1}^{T}F(k,u(k))\right)-\lambda\sum_{k=1}^{T}f(k,u(k))u(k)\\
&\leq r^-J_\lambda(u)+r^-\lambda\sum_{k=1}^{T}\frac{1}{r^-}f(k,u(k))u(k)-\lambda\sum_{k=1}^{T}f(k,u(k))u(k)\\
&=r^-J_\lambda(u)=r^-\inf_{h\in\Gamma}\max_{s\in[0,1]}J_\lambda\left(h(s)\right)\leq r^-\max_{s\in[0,1]}J_\lambda\left(se\right)\leq r^-\max_{s\geq 0}J_\lambda\left(s\frac{e}{\|e\|_{p(\cdot)}}\right)\\
&\leq r^-\max_{s\geq 0}\left(\frac{T}{p^-}2^{p^++1}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+}s^{p^+}\frac{\|e\|^{p^+}}{\|e\|^{p^+}_{p(\cdot)}}-\lambda\varepsilon 2^{-r^-}T^{\frac{p^--r^-}{p^-}}s^{r^-}\frac{\|e\|^{r^-}}{\|e\|^{r^-}_{p(\cdot)}}+\frac{2T}{p^-}+\lambda TC_\eta\right)\\
&\leq r^-\max_{s\geq 0}\left(\frac{T}{p^-}2^{p^++1}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+}s^{p^+}K_2^{p^+}-\lambda\varepsilon 2^{-r^-}T^{\frac{p^--r^-}{p^-}}s^{r^-}K_1^{r^-}+\frac{2T}{p^-}+\lambda TC_\eta\right),
\end{aligned}$$

where $e\in H$ is given by Lemma 3.2 (ii). Hence from (2.3), we infer that

$$\begin{aligned}
&\left(\frac{r^-}{p^+}-1\right)\|u\|^{p^-}_{p(\cdot)}\leq\left(\frac{r^-}{p^+}-1\right)\rho_{p(\cdot)}(u)=\left(\frac{r^-}{p^+}-1\right)\sum_{k=1}^{T}\frac{1}{p(k)}\left|\triangle u(k)\right|^{p(k)}\\
&\leq\frac{r^-}{p^-}\max_{s\geq 0}\left(\frac{T}{p^-}2^{p^++1}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+}s^{p^+}K_2^{p^+}-\lambda\varepsilon 2^{-r^-}T^{\frac{p^--r^-}{p^-}}s^{r^-}K_1^{r^-}+\frac{2T}{p^-}+\lambda TC_\eta\right).
\end{aligned}$$

Let

$$\sigma(s)=\frac{T}{p^-}2^{p^++1}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+}s^{p^+}K_2^{p^+}-\lambda\varepsilon 2^{-r^-}T^{\frac{p^--r^-}{p^-}}s^{r^-}K_1^{r^-}+\frac{2T}{p^-}+\lambda TC_\eta$$

and $\frac{d\sigma}{ds}(s)=0$. Since $r^->p^+$, then $\sigma(s)\to-\infty$ as $s\to\infty$.

Therefore,

$$\frac{d\sigma}{ds}(s)=\frac{T}{p^-}2^{p^++1}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+}p^+K_2^{p^+}s^{p^+-1}-\lambda\varepsilon 2^{-r^-}T^{\frac{p^--r^-}{p^-}}r^-K_1^{r^-}s^{r^--1},$$

which implies that

$$s^{r^--p^+}=\frac{\frac{T}{p^-}2^{p^++1}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+}p^+K_2^{p^+}}{\lambda\varepsilon 2^{-r^-}T^{\frac{p^--r^-}{p^-}}r^-K_1^{r^-}}.$$

So,

$$s=\left[\frac{\frac{T}{p^-}2^{p^++1}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+}p^+K_2^{p^+}}{\lambda\varepsilon 2^{-r^-}T^{\frac{p^--r^-}{p^-}}r^-K_1^{r^-}}\right]^{\frac{1}{r^--p^+}}.$$

Let

$$\kappa_2^* = \left[\frac{\frac{r^-}{p^-}\left(\frac{T}{p^-}2^{p^+ +1}\left(T^{\frac{p^- -1}{p^-}}\right)^{p^+} s^{p^+} K_2^{p^+} - \lambda\varepsilon 2^{-r^-} T^{\frac{p^- - r^-}{p^-}} s^{r^-} K_1^{r^-} + \frac{2T}{p^-} + \lambda T C_\eta\right)}{\frac{r^-}{p^+} - 1} \right]^{1/p^-}.$$

Thus, by the definition of σ, one has

$$\sigma_{max}(s) \geq \frac{2T}{p^-} + \lambda T C_\eta,$$

which is equivalent to saying

$$r^- \sigma_{max}(s) \geq \frac{r^-}{p^-} 2T + r^- \lambda T C_\eta > \frac{r^-}{p^-} 2T > \frac{r^-}{p^-} \geq \frac{r^-}{p^+} > \frac{r^-}{p^+} - 1.$$

Since $r^- > p^+$ and $2 < p^- \leq p(\cdot) < p^+ < \infty$, we infer that $\kappa_2^* > 1$ and by (2.2), there exist some constants $\kappa_1 = K_1 \kappa_1^*$, $\kappa_2 = K_2 \kappa_2^*$ such that $\kappa_1 \leq \|u\| \leq \kappa_2$.

The proof of Lemma 3.5 is then complete. □

Next, we examine conditions under which our problem (1.1) has a unique non trivial solution.

Lemma 3.6. *There exists a constant $c > 0$ such that for all $k \in \mathbb{Z}[1, T]$ and $s > 0$,*

$$\min\left\{ \left(1 + \phi_{p(k)}(s)\right) s^{p(k)-2}, s^{p(k)-1} \frac{\partial \phi_{p(k)}}{\partial s}(s) + (p(k) - 1)\left(1 + \phi_{p(k)}(s)\right) s^{p(k)-2} \right\} \geq c s^{p(k)-2},$$

where $c = \min\{1, p^- - 1\}$.

Proof. For all $s > 0$, we observe that

$$\left(1 + \phi_{p(k)}(s)\right) s^{p(k)-2} \geq s^{p(k)-2} = 1 \times s^{p(k)-2}.$$

One also has

$$\frac{\partial \phi_{p(k)}}{\partial s}(s) = \frac{p(k) s^{p(k)-1}}{\left(1 + s^{2p(k)}\right)^{3/2}}.$$

At more, one has

$$\begin{aligned} & s^{p(k)-1} \frac{\partial \phi_{p(k)}}{\partial s}(s) + (p(k) - 1)\left(1 + \phi_{p(k)}(s)\right) s^{p(k)-2} \\ & \quad = (p(k) - 1) s^{p(k)-2} + \frac{(2p(k) - 1) s^{2p(k)-2} + (p(k) - 1) s^{4p(k)-2}}{\left(1 + s^{2p(k)}\right)^{3/2}} \geq \left(p^- - 1\right) s^{p(k)-2}. \end{aligned}$$

Hence, for all $s > 0$,

$$\min\{\left(1+\phi_{p(k)}(s)\right)s^{p(k)-2}, s^{p(k)-1}\frac{\partial\phi_{p(k)}}{\partial s}(s)+(p(k)-1)\left(1+\phi_{p(k)}(s)\right)s^{p(k)-2}\}\geq \min\{1,p^{-}-1\}s^{p(k)-2}. \quad \square$$

As in [27], one has the following result.

Lemma 3.7. *There exists a positive constant c such that*

$$\left(\left(1+\phi_{p(k)}(\xi)\right)|\xi|^{p(k)-2}\xi-\left(1+\phi_{p(k)}(\eta)\right)|\eta|^{p(k)-2}\eta\right)(\xi-\eta)\geq c4^{2-p(k)}|\xi-\eta|^{p(k)},$$

for all $\xi,\eta\in\mathbb{R}$ with $(\xi,\eta)\neq(0,0)$.

Let us now introduce the following hypothesis.

(H_6) There exist a constant $0<\delta<\dfrac{p^{-}c4^{2-p^{+}}}{\lambda\left(T^{\frac{p^{-}-1}{p^{-}}}\right)^{p^{+}}TK_2^{p^{+}}(2\kappa_2^{*})^{p^{+}-p^{-}}}$ such that

$$|f(k,\xi)-f(k,\eta)|\leq\delta|\xi-\eta|^{p^{+}-1}\quad\text{for all}\quad k\in\mathbb{Z}[1,T]\quad\text{and}\quad\xi,\eta\in\mathbb{R}\quad\text{with}\quad\xi\neq\eta.$$

One has the following result.

Theorem 3.8. *Under assumptions (H_1)-(H_5) and (H_6), there exists a unique nontrivial solution of problem (1.1).*

Proof. Let u and v be two non-trivial solutions to problem (1.1). Then, by (2.7), we have

$$\sum_{k=1}^{T}\left(1+\phi_{p(k)}(\triangle u(k))\right)|\triangle u(k)|^{p(k)-2}\triangle u(k)\triangle(u-v)(k)=\lambda\sum_{k=1}^{T}f(k,u(k))(u-v)(k) \tag{3.9}$$

and

$$\sum_{k=1}^{T}\left(1+\phi_{p(k)}(\triangle v(k))\right)|\triangle v(k)|^{p(k)-2}\triangle v(k)\triangle(u-v)(k)=\lambda\sum_{k=1}^{T}f(k,v(k))(u-v)(k). \tag{3.10}$$

Subtracting (3.9) and (3.10), we obtain

$$\sum_{k=1}^{T}\left[\left(1+\phi_{p(k)}(\triangle u(k))\right)|\triangle u(k)|^{p(k)-2}\triangle u(k)-\left(1+\phi_{p(k)}(\triangle v(k))\right)|\triangle v(k)|^{p(k)-2}\triangle v(k)\right]\triangle(u-v)(k)$$
$$=\lambda\sum_{k=1}^{T}\left[f(k,u(k))-f(k,v(k))\right](u-v)(k). \tag{3.11}$$

If $\|u-v\|_{p(\cdot)} \leq 1$, then using (2.4), Lemma 3.6, (3.11), (H_6) and Lemma 2.3 (c), we deduce from (2.2) that

$$\begin{aligned}
c4^{2-p^+}\|u-v\|_{p(\cdot)}^{p^+} \leq c4^{2-p^+}\rho_{p(\cdot)}(u-v) &\leq \frac{1}{p^-}\sum_{k=1}^{T} c4^{2-p(k)}|\triangle u(k)-\triangle v(k)|^{p(k)} \\
&\leq \frac{1}{p^-}\sum_{k=1}^{T}\Bigg(\left(1+\phi_{p(k)}(\triangle u(k))\right)|\triangle u(k)|^{p(k)-2}\triangle u(k) \\
&- \left(1+\phi_{p(k)}(\triangle v(k))\right)|\triangle v(k)|^{p(k)-2}\triangle v(k)\Bigg)(\triangle u(k)-\triangle v(k)) \\
&= \frac{\lambda}{p^-}\sum_{k=1}^{T}\left[f(k,u(k))-f(k,v(k))\right](u-v)(k) \\
&\leq \frac{\lambda\delta}{p^-}\sum_{k=1}^{T}|u(k)-v(k)|^{p^+} \leq \frac{\lambda\delta}{p^-}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} T\|u-v\|^{p^+} \\
&\leq \frac{\lambda\delta}{p^-}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}\|u-v\|_{p(\cdot)}^{p^+}.
\end{aligned}$$

Therefore,

$$\left[c4^{2-p^+} - \frac{\lambda\delta}{p^-}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}\right]\|u-v\|_{p(\cdot)}^{p^+} \leq 0.$$

Recall that the constant δ is such that $\delta < \dfrac{p^-c4^{2-p^+}}{\lambda\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}}$.

Hence, $\|u-v\|_{p(\cdot)}^{p^+} = 0$, which implies that $u=v$.

Now, let $\|u-v\|_{p(\cdot)} \geq 1$. Similarly, we can deduce that

$$c4^{2-p^+}\|u-v\|_{p(\cdot)}^{p^-} \leq c4^{2-p^+}\rho_{p(\cdot)}(u-v) \leq \frac{\lambda\delta}{p^-}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}\|u-v\|_{p(\cdot)}^{p^+}.$$

Consequently,

$$\|u-v\|_{p(\cdot)}^{p^+-p^-} \geq \frac{p^-c4^{2-p^+}}{\lambda\delta\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}}.$$

Which is equivalent to say

$$\|u-v\|_{p(\cdot)} \geq \left[\frac{p^-c4^{2-p^+}}{\lambda\delta\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}}\right]^{\frac{1}{p^+-p^-}}.$$

It is then clear that if u, v are solutions to problem (1.1) and $\delta < \dfrac{p^- c 4^{2-p^+}}{\lambda \left(T^{\frac{p^- -1}{p^-}}\right)^{p^+} T K_2^{p^+} (2\kappa_2^*)^{p^+ - p^-}}$,

then

$$2\kappa_2^* < \|u-v\|_{p(\cdot)} \leq \|u\|_{p(\cdot)} + \|v\|_{p(\cdot)} \leq 2\kappa_2^*.$$

This contradicts the assumption that $\|u-v\|_{p(\cdot)} \geq 1$. Consequently, it follows that $u = v$. $\square$

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