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Further results on the metric dimension and spectrum of graphs

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ABSTRACT

The concept of metric dimension in graphs has the aim of finding a set of vertices in a graph with the smallest size that can be used as a reference to identify all vertices in the graph uniquely. Formally, let G be a connected graph, and let $S = \{s_1, \ldots, s_k\} \subseteq V(G)$ be an ordered set. For every $v \in V(G)$, we define $r(v|S) = (d(v, s_1), \ldots, d(v, s_k))$ where d is the distance function of G. We call S a *resolving set* if $r(u|S) \neq r(v|S)$ for every $u, v \in V(G)$, $u \neq v$. The *metric dimension* of G, denoted by $\dim(G)$, is the smallest integer k such that G has a resolving set of size k. Recently, the authors have initiated research on the relation between the metric dimension of a graph and its nullity (that is, the multiplicity of 0 in its adjacency spectrum), and we have obtained several results. In this paper, we present some new relationships between the metric dimension and the spectrum of graphs. In detail, we present an inequality involving the metric dimension and nullity of any bipartite or singular graph. Then, we give an infinite class of graphs having equal metric dimension and nullity using the rooted product of graphs. Finally, for any connected graph G other than a path, we show that a submatrix of the distance matrix of G, associated with a minimal resolving set of G, has the full-rank property.

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RESUMEN

El concepto de dimensión métrica en grafos tiene como propósito encontrar un conjunto de vértices en un grafo con el menor tamaño que puede usarse como referencia para identificar únicamente todos los vértices del grafo. Formalmente, sea G un grafo conexo, y sea $S = \{s_1, \ldots, s_k\} \subseteq V(G)$ un conjunto ordenado. Para todo $v \in V(G)$, definimos $r(v|S) = (d(v, s_1), \ldots, d(v, s_k))$ donde d es la función de distancia de G. Llamamos a S un *conjunto resolvente* si $r(u|S) \neq r(v|S)$ para todo $u, v \in V(G)$, $u \neq v$. La *dimensión métrica* de G, denotada por $\dim(G)$, es el entero más pequeño k tal que G tiene un conjunto resolvente de tamaño k. Recientemente, los autores han comenzado a investigar sobre la relación entre la dimensión métrica de un grafo y su nulidad (es decir, la multiplicidad de 0 en su espectro de adyacencia), y hemos obtenido diversos resultados. En este artículo, presentamos algunas relaciones nuevas entre la dimensión métrica y el espectro de grafos. En detalle, presentamos una desigualdad que involucra la dimensión métrica y la nulidad de cualquier grafo bipartito o singular. Luego, entregamos una clase infinita de grafos con igual dimensión métrica y nulidad usando el producto enraizado de grafos. Finalmente, para todo grafo conexo G distinto de un camino, mostramos que una submatriz de la matriz de distancia de G, asociada a un conjunto resolvente mínimo de G, tiene la propiedad de rango completo.

Keywords and Phrases: Metric dimension, spectrum, nullity, distance matrix.

2020 AMS Mathematics Subject Classification: 05C12, 05C50.

1 Introduction

In the 1960s, Slater [14] and Harary and Melter [10] independently introduced the concept of metric dimension of graphs. They introduced the term *locating set* or *resolving set* which refers to a set of vertices used to identify each vertex in a graph uniquely. A resolving set with the smallest size is called a *basis*, and its cardinality is referred to as the *metric dimension* of the graph. Since the metric dimension of graphs and its variations have direct applicability to several real-world issues like robot navigation [12] and chemistry [3], research on them has grown rapidly in the recent few decades. See, for example, [15] and [13] for surveys on this topic. On the other hand, in 1972, Cvetković, Gutman, and Trinajstić [5], and then Cvetković and Gutman [4], introduced the nullity of a graph as a new invariant; it is the multiplicity of 0 as an eigenvalue of the graph's adjacency matrix. They further investigated the connection between graph nullity and chemical structures. Excellent overviews of graph nullity can be found in [1] and [9].

Despite the growth of interest in the metric dimension of graphs, its connection to the graph's spectrum has not been studied further. Recently, the authors [7] have initiated research on the relation between the metric dimension of a graph and its spectrum, and we have obtained several results. This research was motivated by the observation that the equality $\dim(G) = \eta(G)$, where $\dim(G)$ and $\eta(G)$ respectively denote the metric dimension and nullity of the graph G, holds for complete bipartite graphs $K_{r,s}$ where $r \neq s$, paths P_n where n is odd, and cycles C_n where $n \equiv 0 \pmod 4$. This paper aims to provide further connections between the two concepts. In detail, we first give an inequality involving $\dim(G)$ and $\eta(G)$ for any bipartite or singular graphs G, generalizing our previous result for trees. Then, we give an infinite class of graphs G where $\dim(G) = \eta(G)$ using the rooted product of graphs. Finally, we give another relation between the metric dimension of a graph and its distance matrix. We show that for any connected graph G, a submatrix of its distance matrix, associated with a minimal resolving set of G, has the full-rank property.

All the graphs considered in this study are finite, simple, and undirected. We refer to Diestel [6] for the basic definitions related to graphs. An *empty graph* $\emptyset$ is the graph without any vertices and edges. Let $G = (V(G), E(G))$ be a graph. We simply write $V = V(G)$ and $E = E(G)$ if the graph is clear from context. Two vertices $u, v \in V$ are said to be *adjacent* if $uv \in E$. The *open neighborhood* of a vertex $u \in V$ is the set $N_G(u) := \{v \in V : uv \in E\}$, and the *closed neighborhood* of u is $N_G[u] := \{u\} \cup N_G(u)$. The *degree* of a vertex $u \in V$, denoted by $\deg(u)$, is the size of $N_G(u)$. A vertex is called *pendant* if it has degree one, and let $p(G)$ denote the number of pendant vertices of G. For two distinct vertices u, v in a graph G, the *distance* $d(u, v)$ of u and v is the length of a shortest path connecting u and v. We denote by P_n, C_n, $K_{m,n}$, and K_n for paths, cycles, complete bipartite, and complete graphs. For two integers $a \leq b$, we define $[a, b] := \{x \in \mathbb{Z} : a \leq x \leq b\}$.

Let $u, v \in V$, $u \neq v$. We say that a vertex $s \in V$ *resolves* u and v if $d(u, s) \neq d(v, s)$. Let

$S = \{s_1, s_2, \ldots, s_k\} \subseteq V$ be an ordered subset of V. The *representation* of $v \in V$ with respect to S, denoted by $r(v|S)$, is the vector $r(v|S) = (d(v, s_1), d(v, s_2), d(v, s_3), \ldots, d(v, s_k))$. We call S a *resolving set* of G if $r(u|S) \neq r(v|S)$ for every distinct pair $u, v \in V$, that is, if each vertex of G has a unique representation with respect to S. In other words, S is a resolving set if and only if every pair of distinct vertices $u, v \in V$ is resolved by an element of S. A resolving set of G with minimum size is called a *basis* of G. The cardinality of a basis of G is called the *metric dimension* of G which is denoted by $\dim(G)$. A resolving set of G is called minimal if for every $S_0 \subset S$, S_0 is not a resolving set of G, that is, S does not contain a smaller resolving set of G.

Let $G = (V, E)$ be a graph of order n with $V = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of G is the $n \times n$ matrix $\mathbf{A} = \mathbf{A}(G) = (a_{ij})$ whose entry a_{ij} is equal to 1 if v_i and v_j are adjacent, and 0 otherwise. The *distance matrix* of G is the matrix $\mathbf{D} = \mathbf{D}(G) = (d_{ij})$, where $d_{ij} = d(v_i, v_j)$. For $\mathbf{M} \in \{\mathbf{A}, \mathbf{D}\}$, the $\mathbf{M}$-*spectrum* of G, denoted by $\text{spec}_{\mathbf{M}}(G)$, is the set of eigenvalues of $\mathbf{M}(G)$ together with their multiplicities. If the distinct eigenvalues of $\mathbf{M}(G)$ are $\lambda_1 > \lambda_2 > \cdots > \lambda_s$, and their multiplicities are $m_1, m_2, \ldots, m_s$, respectively, then we write $\text{spec}_{\mathbf{M}}(G) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_s^{m_s}\}$. For an eigenvalue λ, we may write $m_{\mathbf{M}}(\lambda)$ to denote the multiplicity of λ in $\text{spec}_{\mathbf{M}}(G)$. The *nullity* of G, denoted by $\eta(G)$, is the multiplicity of eigenvalue 0 in $\text{spec}_{\mathbf{A}}(G)$, that is, $\eta(G) = m_{\mathbf{A}}(0)$. We call a graph G *singular* if $\eta(G) > 0$. For the trivial case, we define $\eta(\emptyset) = 0$.

2 Preliminary Results

In this section, we provide some known results that are useful in our discussions.

Theorem 2.1 ([3,12])**.** *A graph G has $\dim(G) = 1$ if and only if G is a path.*

Theorem 2.2 ([15])**.** *For every integer $n \geq 3$, $\dim(C_n) = 2$.*

Let G and H be two graphs. The union $G \cup H$ is the graph where $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The join $G \vee H$ is the graph obtained by taking the two graphs and connecting, by an edge, each vertex in G to each vertex in H. Furthermore, the complement $\overline{G}$ of G has $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : uv \notin E(G), u, v \in V(G)\}$.

Theorem 2.3 ([3])**.** *Let G be a graph of order $n \geq 4$. Then, $\dim(G) = n - 2$ if and only if $G = K_{r,s}$ $(r, s \geq 1)$, $G = K_s \vee \overline{K_t}$ $(s \geq 1, t \geq 2)$, or $G = K_s \vee (K_1 \cup K_t)$ $(s, t \geq 1)$.*

For the case of trees, we need the following definitions. A vertex of degree at least 3 in a graph G is called a *major vertex* of G. A pendant vertex u of G is called a *terminal vertex of a major vertex* v of G if $d(u, v) < d(u, w)$ for every other major vertex w of G. In other words, a pendant vertex u is a terminal vertex of v if v is the closest major vertex from u. The *terminal degree* $\text{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is called an *exterior major vertex* of G if $\operatorname{ter}(v) > 0$. Let $\sigma(G)$ denote the sum of the terminal degrees of all major vertices of G, and let $\operatorname{ex}(G)$ denote the number of exterior major vertices of G. With these definitions, we may calculate the metric dimension of trees other than a path by the following formula.

Theorem 2.4 ([3, 12, 14])**.** *If T is a tree other than a path, then*

$$\dim(T) = \sigma(T) - \operatorname{ex}(T) = \sum_{\substack{v \in V \\ \operatorname{ter}(v) > 1}} (\operatorname{ter}(v) - 1).$$

The proof of Theorem 2.4 utilizes the following general bound for any connected graphs.

Lemma 2.5 ([3])**.** *If G is a connected graph, then* $\dim(G) \geq \sigma(G) - \operatorname{ex}(G)$.

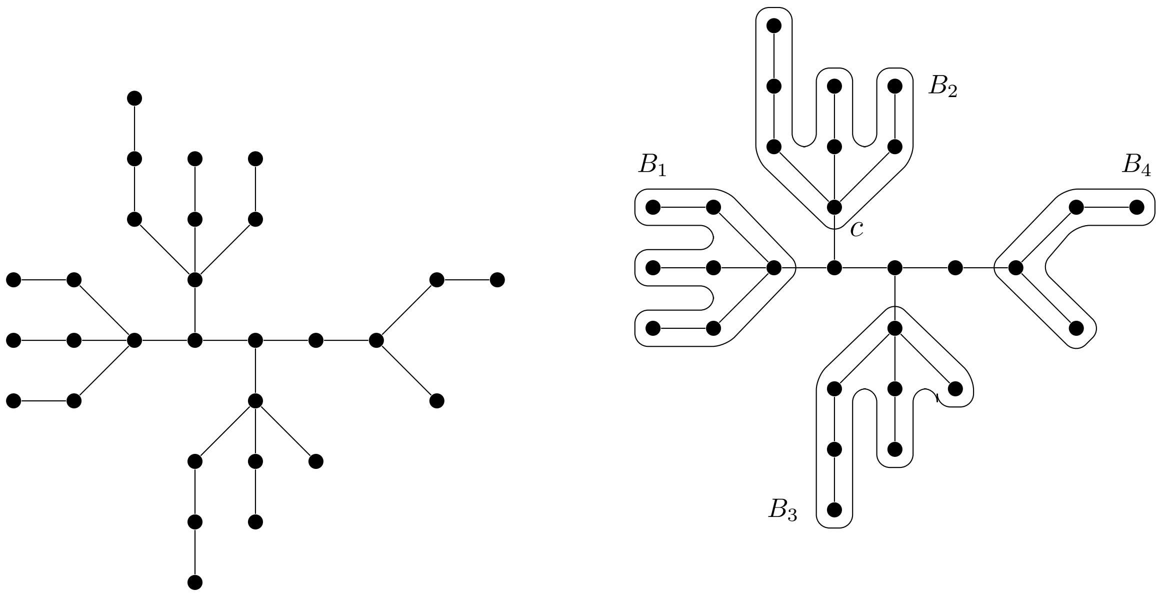
For an exterior major vertex v in G, a *tail of* v is a path connecting v to one of its terminal vertex, excluding v. Thus, an exterior major vertex v has $\operatorname{ter}(v)$ tails. We call a tail *odd* or *even* if it has an odd or even number of vertices, respectively. A *branch* B is a subgraph of G induced by an exterior major vertex v in G and all its tails. In this case, we call v the *stem vertex* of B. Thus, a branch with n tails is a subdivision of the star graph $K_{1,n}$. We say a branch B is of Type I if it has at least one odd tail and Type II otherwise. In Figure 1b, the branches of T in Figure 1a are the blocked subgraphs B_1, B_2, B_3, and B_4. The vertex c is the stem of B_2. The branches B_2, B_3, and B_4 are of Type I, while the branch B_1 is of Type II. With these additional definitions, observe that the second equality in Theorem 2.4 indicates that the metric dimension of a tree depends only on the structure of its branches.

We now discuss the rooted and corona product of graphs. Let G be a graph where $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $\mathcal{H}$ be a set of n graphs $H_1, H_2, \ldots, H_n$ where a vertex in H_i is chosen as the *root* of H_i, $i \in [1, n]$. The *rooted product* of G by $\mathcal{H}$, denoted by $G(\mathcal{H})$, is the graph obtained by identifying the root of H_i and v_i for every $i \in [1, n]$ [8]. A special case of rooted product of graphs is the caterpillar graph. A caterpillar is a tree such that the removal of its pendants produces a path. For positive integers k and $n_1, n_2, \ldots, n_k$, a caterpillar $CP(n_1, n_2, \ldots, n_k)$ is the graph $P_k(\{K_{1,n_1}, \ldots, K_{1,n_k}\})$ by taking the center vertex of each K_{1,n_i} as its root.

Let G and H be two graphs with $|G| = n$. The *corona product* $G \odot H$ is defined as the graph obtained by taking one copy of G and n copies of H, and we connect (by an edge) every vertex in the ith copy of H with the ith vertex of G [16]. For the case where $H = \overline{K_m}$ for some positive integer m, we have $G \odot \overline{K_m} = G(\mathcal{H})$ where $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$, $H_i = K_{1,m}$ for every $i \in [1, n]$.

Theorem 2.6 ([11])**.** *If G is a connected graph of order n, and $t \in \mathbb{N}$, $t \geq 2$, then* $\dim(G \odot \overline{K_t}) = n(t-1)$

Theorem 2.7 ([11])**.** *If G is a connected graph of order n, and $\mathcal{H} = \{K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_n}\}$ where $m_i \geq 2$ for every $i \in [1, n]$, then* $\dim(G(\mathcal{H})) = \sum_{i=1}^{n} (m_i - 1)$.



(a) The tree T (b) The branches of T

Figure 1: A tree and its branches

We now discuss the results related to the spectrum and nullity of graphs.

Theorem 2.8 ([2])**.**

(1) *For every positive integers* r, s, $\text{spec}_{\mathbf{A}}(K_{r,s}) = \{\pm\sqrt{rs}, 0^{r+s-2}\}$.

(2) *For every integer* $n \geq 2$, $\text{spec}_{\mathbf{A}}(C_n) = \{2\cos(2\pi k/n) : k \in [1, n]\}$.

(3) *For every integer* $n \geq 1$, $\text{spec}_{\mathbf{A}}(P_n) = \{2\cos(\pi k/(n+1)) : k \in [1, n]\}$.

We can see from Theorem 2.8 that $\eta(K_{r,s}) = r + s - 2$; $\eta(C_n) = 2$ if $n \equiv 0 \pmod 4$, and 0 if otherwise; and $\eta(P_n) = 1$ if n is odd, and 0 if n is even. The following observation is immediate from Theorems 2.8, 2.1, 2.2, and 2.3.

Observation 2.9. *The condition* $\dim(G) = \eta(G)$ *holds if* G *is one of the following graphs:*

(1) $K_{r,s}$ *where* $r \neq s$, *or*

(2) C_n *where* $n \equiv 0 \pmod 4$, *or*

(3) P_n *where* n *is odd.*

Lemma 2.10 ([9])**.** *Let* G *be a graph order* n. *Then,* $\eta(G) = n$ *if and only if* $G = \overline{K_n}$.

The following lemmas are very useful in many parts of our discussion.

Lemma 2.11 ([4]). *Let G be a bipartite graph containing a pendant vertex, say v, and H be the graph obtained from G by deleting v and its neighbor. Then, $\eta(G) = \eta(H)$.*

Lemma 2.12 ([9]). *Let $G = \bigcup_{i=1}^{t} G_i$, where $G_1, \ldots, G_t$ are connected components of G. Then, $\eta(G) = \sum_{i=1}^{t} \eta(G_i)$.*

We now mention our previous result.

Theorem 2.13 ([7]). *Let T be a tree other than a path. Let $\mathcal{B}_{\rm I}$ and $\mathcal{B}_{\rm II}$ be the sets of Type I and Type II branches in T, respectively. Let e_2 be the number of even tails in T. If T has an odd tail, then*

$$\dim(T) = \eta(T) - \eta(T - \mathcal{B}_{\rm I}) - |\mathcal{B}_{\rm II}| + e_2,$$

where $T - \mathcal{B}_{\rm I}$ is the graph obtained from T by deleting all Type I branches in T.

3 Main results

3.1 The metric dimension and nullity of bipartite or singular graphs

We first present an inequality involving $\dim(G)$ and $\eta(G)$ for any connected bipartite/singular graph having an odd tail. The proof of this theorem is similar to the proof of Theorem 2.13. However, for completeness, we present the proof.

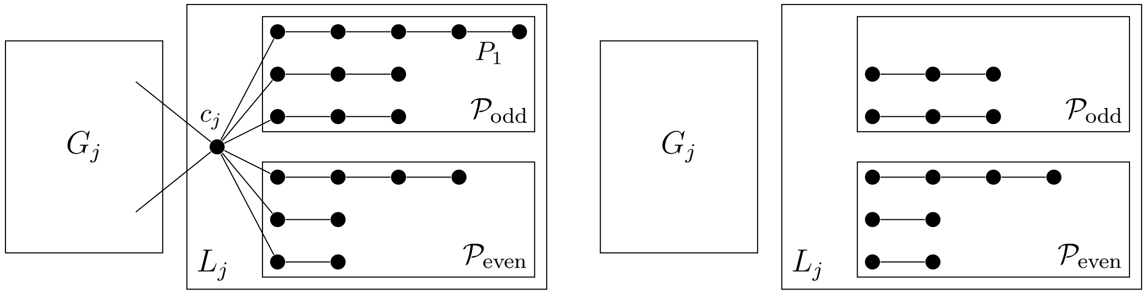
Theorem 3.1. *Let G be a connected bipartite or singular graph other than a path. Let $\mathcal{B}_{\rm I}$ and $\mathcal{B}_{\rm II}$ be the sets of Type I and Type II branches in G, respectively. Let e_2 be the number of even tails in G. If G has an odd tail, then*

$$\dim(G) \geq \eta(G) - \eta(G - \mathcal{B}_{\rm I}) - |\mathcal{B}_{\rm II}| + e_2$$

where $G - \mathcal{B}_{\rm I}$ is the graph obtained from G by deleting all Type I branches in G.

Proof. Let $B_1, \ldots, B_k$ be the branches in G. Since G has at least one odd tail, there exists a Type I branch in G. Suppose that $|\mathcal{B}_{\rm I}| = p \geq 1$. Without loss of generality, let $\mathcal{B}_{\rm I} = \{B_1, B_2, \ldots, B_p\}$ and $\mathcal{B}_{\rm II} = \{B_{p+1}, B_{p+2}, \ldots, B_k\}$. Observe that we may construct a sequence of graphs $G_0, G_1, \ldots, G_p$ where $G_0 := G$, $G_p = G - \mathcal{B}_{\rm I}$, and $G_j = G_{j-1} - B_j = G - \bigcup_{i=1}^{j} B_i$ for $j \in [1, p]$. So, the graph G_j is obtained from G by deleting the branches $B_1, B_2, \ldots, B_j$ of G.

For an arbitrary $j \in [1, p]$, consider the graph G_{j-1} and Type I branch B_j with stem vertex c_j. Suppose that B_j has $e^{(j)}$ tails, $e_1^{(j)}$ odd tails, and $e_2^{(j)}$ even tails, hence $e^{(j)} = e_1^{(j)} + e_2^{(j)}$ and $e_2 = \sum_{i=1}^{k} e_2^{(i)}$. Let $\mathcal{P}_{\rm odd}$ be the set of all odd tails of B_j, and let $\mathcal{P}_{\rm even}$ be the set of all even tails of B_j. Pick an arbitrary odd tail, say P_1, and then delete P_1 and c_j from G_{j-1}. Since P_1 is an odd



(a) G_{j-1} (b) $G_{j-1} - P_1 - c_j$

Figure 2: The grouping of the vertices in G_{j-1} and $G_{j-1} - P_1 - c_j$

tail, we have $\eta(G_{j-1}) = \eta(G_{j-1} - P_1 - c_j)$ by Lemma 2.11. Observe that the graph $G_{j-1} - P_1 - c_j$ has several connected components (see Figure 2): G_j, odd tails of B_j except P_1, and even tails of B_j. By Lemma 2.11, we have

$$\eta(P) = \begin{cases} 1, & \text{if } P \in \mathcal{P}_{\text{odd}}, \\ 0, & \text{if } P \in \mathcal{P}_{\text{even}}, \end{cases}$$

since successively deleting a pendant vertex and its neighbor of a path yields a single vertex if it has an odd order, and an empty graph if it has an even order.

Consequently, by Lemma 2.12, we have

$$\eta(G_{j-1}) = \eta(G_{j-1} - P_1 - c_j) = \eta(G_j) + \sum_{P \in \mathcal{P}_{\text{odd}}} \eta(P) + \sum_{P \in \mathcal{P}_{\text{even}}} \eta(P) = \eta(G_j) + \left(e_1^{(j)} - 1\right).$$

Therefore, we have the relation $\eta(G_j) = \eta(G_{j-1}) - \left(e_1^{(j)} - 1\right)$ for $j \in [1, p]$. By applying this relation successively, we obtain

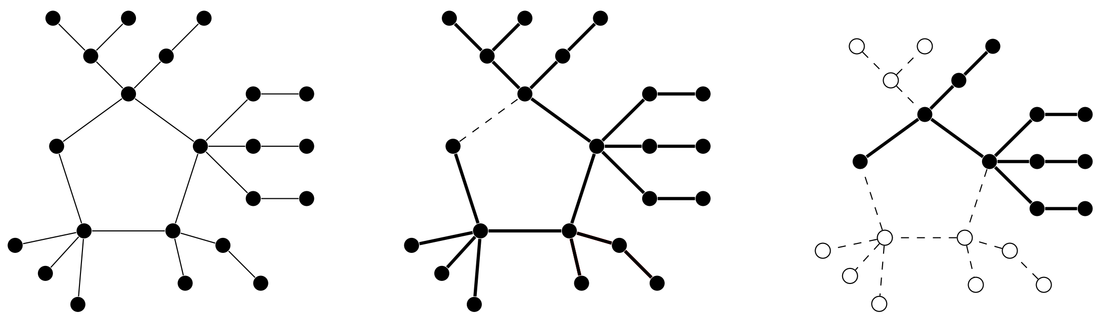
$$\eta(G - \mathcal{B}_{\text{I}}) = \eta(G_p) = \eta(G_0) - \sum_{i=1}^{p} \left(e_1^{(i)} - 1\right) = \eta(G) - \sum_{i=1}^{p} \left(e_1^{(i)} - 1\right).$$

Finally, since $\dim(G) \geq \sum_{i=1}^{k} (e^{(i)} - 1)$ by Lemma 2.5, we have

$$\begin{aligned} \eta(G - \mathcal{B}_{\text{I}}) &= \eta(G) - \sum_{i=1}^{k} \left(e_1^{(i)} - 1\right) + \sum_{i=p+1}^{k} \left(e_1^{(i)} - 1\right) \\ &= \eta(G) - \sum_{i=1}^{k} \left(e^{(i)} - 1 - e_2^{(i)}\right) + \sum_{i=p+1}^{k} (0 - 1) \\ &= \eta(G) - \sum_{i=1}^{k} \left(e^{(i)} - 1\right) + \sum_{i=1}^{k} e_2^{(i)} - (k - p) \\ &\geq \eta(G) - \dim(G) + e_2 - |\mathcal{B}_{\text{II}}|. \end{aligned}$$

□

Example 3.2. *Let G be the graph shown in Figure 3a. The graph $G - \mathcal{B}_{\mathrm{I}}$ is the bold subgraph shown in Figure 3c. With some calculations, we obtain $\eta(G) = 4$ (so G is singular), $\eta(G - \mathcal{B}_{\mathrm{I}}) = 1$, $|\mathcal{B}_{\mathrm{II}}| = 2$, and $e_2 = 5$. Thus, by Theorem 3.1, we obtain* $\dim(G) \geq \eta(G) - \eta(G - \mathcal{B}_{\mathrm{I}}) - |\mathcal{B}_{\mathrm{II}}| + e_2 = 4 - 1 - 2 + 5 = 6$.



(a) G (b) T (c) $G - \mathcal{B}_{\mathrm{I}}$

Figure 3: The graph G, spanning tree T of G, and $G - \mathcal{B}_{\mathrm{I}}$

3.2 The metric dimension and nullity of the rooted product of some graphs

Next, we discuss some relationships between the metric dimension and nullity of the rooted product of some graphs. For certain conditions, this product will establish an infinite class of graphs whose metric dimension and nullity are equal. For that, we need a useful class of graph called *branch graph* which is simply a subdivision of $K_{1,n}$ for some positive integer n. The number of subdivision processes in each "leg" of $K_{1,n}$ is arbitrary. The following proposition gives the metric dimension of $G(\mathcal{H})$ for *any* set of branch graphs $\mathcal{H}$ (see Figure 4). Observe that this proposition generalizes Theorems 2.6 and 2.7.

Proposition 3.3. *Let $\mathcal{H} = \{B_1, B_2, \ldots, B_n\}$ be a set of $n \geq 1$ branch graphs. For every $i \in [1, n]$, the graph B_i has $e_i \geq 2$ tails, and the center of B_i is chosen as the root of B_i. For every connected graph G of order n,* $\dim(G(\mathcal{H})) = \sum_{i=1}^{n}(e_i - 1) = p(G(\mathcal{H})) - n$.

Proof. Let G be a connected graph of order n. First, we show that $\dim(G(\mathcal{H})) \geq \sum_{i=1}^{n}(e_i - 1)$. Let $V(G) = \{v_1, \ldots, v_n\}$. The graph $G(\mathcal{H})$ is obtained by identifying v_i with the center of B_i. Consequently, the pendant vertices of all B_i's become the pendant vertices in $G(\mathcal{H})$, so $p(G(\mathcal{H})) = \sum_{i=1}^{n} e_i$. Moreover, all vertices in G become the exterior major vertices in $G(\mathcal{H})$, so $\mathrm{ex}(G(\mathcal{H})) = n$. Thus, by Lemma 2.5, we have

$$\dim(G(\mathcal{H})) \geq p(G(\mathcal{H})) - \mathrm{ex}(G(\mathcal{H})) = \sum_{i=1}^{n} e_i - n = \sum_{i=1}^{n}(e_i - 1).$$

Next, we show that $\dim(G(\mathcal{H})) \leq \sum_{i=1}^{n}(e_i - 1)$. For every $v_i \in V(G) \subset V(G(\mathcal{H}))$, let $T_i := \{v_i^1, v_i^2, \dots, v_i^{e_i}\}$ be the set of all terminal vertices of v_i, where v_i^j is the terminal vertex of v_i in the jth tail, $j \in [1, e_i]$. Let $S = \bigcup_{i=1}^{n}(T_i \setminus \{v_i^{e_i}\})$. We will show that S is a resolving set of $G(\mathcal{H})$. Let $x, y \in V(G(\mathcal{H}))$ be two distinct vertices. There are some cases for x and y.

(1) Let $x, y \in V(B_i)$, $i \in [1, n]$, that is, x and y are in the same branch.

  (a) If x and y are in the same tail, say the jth tail, $j \in [1, e_i]$, then $d(x, v_i^1) \neq d(y, v_i^1)$.

  (b) Suppose that x and y are in different tails, say j_1th and j_2th tails, respectively. Observe that at least one of $v_i^{j_1}$ and $v_i^{j_2}$ must be in S; say $v_i^{j_1} \in S$ without loss of generality. Consequently, $d(y, v_i^{j_1}) = d(y, v_i) + d(v_i, x) + d(x, v_i^{j_1}) > d(x, v_i^{j_1})$ since $d(y, v_i) > 0$.

  (c) Suppose that $x = v_i$ and y is in the jth tail. If $j \in [1, e_i - 1]$, then $d(y, v_i^j) < d(x, v_i^j)$. If $j = e_i$, then $d(y, v_i^1) = d(y, x) + d(x, v_i^1) > d(x, v_i^1)$ since $d(y, x) > 0$.

(2) Let $x \in V(B_s)$ and $y \in V(B_t)$, $s \neq t \in [1, n]$, that is, x and y are in different branches. Consequently, $d(v_s, v_t) > 0$.

  (a) If x is in the jth tail, $j \in [1, e_s - 1]$, then wherever y may be in B_t, we have $d(y, v_s^j) = d(y, v_t) + d(v_t, v_s) + d(v_s, x) + d(x, v_s^j) > d(x, v_s^j)$. Similar argument also applies if y is in the jth tail, $j \in [1, e_t - 1]$, that is, $d(x, v_t^j) > d(y, v_t^j)$ wherever x may be in B_s.

  (b) If $x = v_s$ and $y = v_t$, then $d(y, v_s^1) = d(y, v_t) + d(v_t, v_s) + d(v_s, v_s^1) > d(v_s, v_s^1) = d(x, v_s^1)$.

  (c) For the last case, suppose that x and y are in the e_sth and e_tth tails, respectively. If $d(x, v_s^1) \neq d(y, v_s^1)$, then we are done. Now, let us assume that $d(x, v_s^1) = d(y, v_s^1)$. Observe that since $d(v_t, v_s) > 0$, we have

$$
\begin{aligned}
d(x, v_t^1) &= d(x, v_s) + d(v_s, v_t) + d(v_t, v_t^1) \\
&= (d(x, v_s) + d(v_s, v_s^1)) + d(v_s, v_t) + d(v_t, v_t^1) - d(v_s, v_s^1) \\
&= d(x, v_s^1) + d(v_s, v_t) + d(v_t, v_t^1) - d(v_s, v_s^1) \\
&= d(y, v_s^1) + d(v_s, v_t) + d(v_t, v_t^1) - d(v_s, v_s^1) \\
&= (d(y, v_t) + d(v_t, v_s) + d(v_s, v_s^1)) + d(v_s, v_t) + d(v_t, v_t^1) - d(v_s, v_s^1) \\
&= (d(y, v_t) + d(v_t, v_t^1)) + 2d(v_t, v_s) \\
&= d(y, v_t^1) + 2d(v_t, v_s) \\
&> d(y, v_t^1).
\end{aligned}
$$

Thus, for every case of x and y, there is an element of S resolving them. Consequently, S is a resolving set of $G(\mathcal{H})$, and since $|S| = \sum_{i=1}^{n}(e_i - 1)$, we have $\dim(G(\mathcal{H})) \leq \sum_{i=1}^{n}(e_i - 1)$. Therefore, $\dim(G(\mathcal{H})) = \sum_{i=1}^{n}(e_i - 1)$. □

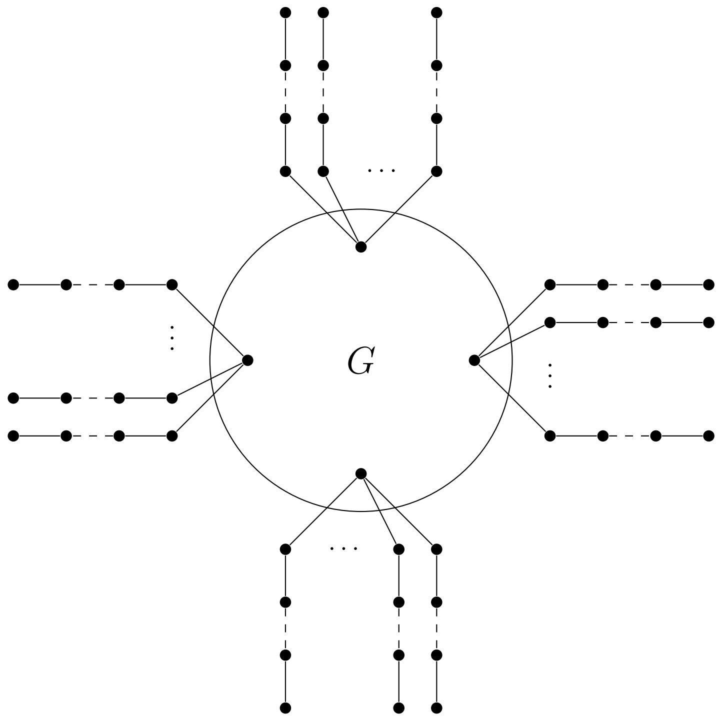


Figure 4: The graph $G(\mathcal{H})$

Theorem 3.4. *Let $\mathcal{H} = \{B_1, B_2, \dots, B_n\}$ be a set of $n \geq 1$ branch graphs whose tails are all odd tails. For every $i \in [1, n]$, the graph B_i has $e_i \geq 2$ tails, and the center of B_i is chosen as the root of B_i. For every connected bipartite graph G of order n,*

$$\dim(G(\mathcal{H})) = \eta(G(\mathcal{H})) = \sum_{i=1}^{n}(e_i - 1).$$

Proof. From Proposition 3.3, $\dim(G(\mathcal{H})) = \sum_{i=1}^{n}(e_i - 1)$. We only need to show that $\eta(G(\mathcal{H})) = \sum_{i=1}^{n}(e_i - 1)$. Observe that G is bipartite implies $G(\mathcal{H})$ is also bipartite. Consider an arbitrary branch B_i in $G(\mathcal{H})$. By applying Lemma 2.11 consecutively, we may delete one tail from B_i together with the vertex v_i without changing the nullity, that is, the nullity of the resulting graph is the same as of $G(\mathcal{H})$. Moreover, this deletion leaves only $e_i - 1$ tails of B_i. From Lemma 2.11 again, these $e_i - 1$ tails leave $e_i - 1$ isolated vertices (since every tail in B_i is an odd tail) without changing the nullity. Thus, the deletion process on the branch B_i leaves the graph $G(\mathcal{H}) - B_i$ and $e_i - 1$ isolated vertices with the same nullity as $G(\mathcal{H})$. By applying the same process to the other branches, we get a graph consisting of $\sum_{i=1}^{n}(e_i - 1)$ isolated vertices whose nullity equals the nullity of $G(\mathcal{H})$. Thus, $\eta(G(\mathcal{H})) = \sum_{i=1}^{n}(e_i - 1)$. Therefore, $\dim(G(\mathcal{H})) = \eta(G(\mathcal{H}))$. □

The following corollary is a consequence of Theorem 3.4 by observing that corona product of graphs and caterpillar graphs are special cases of rooted product of graphs.

Corollary 3.5. *The condition* $\dim(G) = \eta(G)$ *holds if* G *is one of the following graphs:*

(1) $H \odot \overline{K_p}$ *for every connected bipartite graph* H *and positive integer* $p \geq 2$*, or*

(2) $CP(n_1, n_2, \ldots, n_k)$ *for every positive integers* k *and* $n_i \geq 2$*,* $i \in [1, k]$.

In contrast to Theorem 3.4, if all branch graphs in $\mathcal{H}$ have only even tails, then the metric dimension of $G(\mathcal{H})$ is strictly greater than its nullity as we show in the following theorem.

Theorem 3.6. *Let* $\mathcal{H}$ *be a set of* $n \geq 2$ *branch graphs with at least 2 tails whose tails are all even tails, and for every* $B \in \mathcal{H}$*, the center of* B *is chosen as the root of* B*. For every connected bipartite graph* G *of order* n*,* $\dim(G(\mathcal{H})) > \eta(G(\mathcal{H}))$.

Proof. Let $\mathcal{H} = \{B_1, \ldots, B_n\}$, where every $B_i \in \mathcal{H}$ has $e_i \geq 2$ tails. Assume to the contrary that there exists a connected bipartite graph G of order n satisfying $\dim(G(\mathcal{H})) \leq \eta(G(\mathcal{H}))$. Since G is connected and has an order $n \geq 2$, we have $G \neq \overline{K_n}$, so $\eta(G) \leq n - 1$ from Lemma 2.10. From Proposition 3.3, we have $\dim(G(\mathcal{H})) = \sum_{i=1}^{n} e_i - n$, and by applying Lemma 2.11 on $G(\mathcal{H})$ consecutively, we obtain $\eta(G(\mathcal{H})) = \eta(G)$. Therefore,

$$n = 2n - n \leq \sum_{i=1}^{n} e_i - n = \dim(G(\mathcal{H})) \leq \eta(G(\mathcal{H})) = \eta(G) \leq n - 1,$$

a contradiction. □

3.3 The metric dimension and distance matrix of graphs

Finally, we discuss a relationship between the metric dimension of a graph and its distance matrix. For that, we need the following notations. For a connected graph G and $\emptyset \neq S \subseteq V(G)$, the distance matrix $\mathbf{D}$ of G can be partitioned into

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}[S] & \mathbf{D}[V \backslash S] \end{bmatrix}$$

where $\mathbf{D}[S] \in \mathbb{R}^{|G| \times |S|}$ and $\mathbf{D}[V \backslash S] \in \mathbb{R}^{|G| \times |V \backslash S|}$ are the submatrices obtained from $\mathbf{D}$ by taking all the columns corresponding to the elements of S and $V \backslash S$, respectively. Observe that the vth row of $\mathbf{D}[S]$ is $r(v|S)^\top$. Observation 3.7 is a direct consequence of this definition. Recall that a resolving set S of G is called *minimal* if S does not contain a smaller resolving set of G. A basis is a minimal resolving set, but the converse is not necessarily true.

Observation 3.7. *Let* G *be a connected graph with distance matrix* $\mathbf{D}$ *and* $\emptyset \neq S \subseteq V(G)$.

(1) S *is a resolving set of* G *if and only if* $\mathbf{D}[S]$ *has no two identical rows.*

(2) *S is a minimal resolving set of G if and only if (1) $\mathbf{D}[S]$ has no two identical rows, and (2) for every $s \in S$, $\mathbf{D}[S\backslash\{s\}]$ has two identical rows.*

Theorem 3.8. *Let G be a connected graph other than a path with distance matrix $\mathbf{D}$. If S is a minimal resolving set of G, then $\operatorname{rank}(\mathbf{D}[S]) = |S|$. Consequently, $\dim(G) \le \operatorname{rank}(\mathbf{D})$.*

Proof. Let S be a minimal resolving set of G with $|S| = k$. Since G is not a path, we have $k \ge 2$ from Theorem 2.1. Let $i \in [1,k]$ be arbitrary. According to Observation 3.7, there are two rows $\mathbf{d}_u = (d_{u1}, \ldots, d_{uk})^\top$ and $\mathbf{d}_v = (d_{v1}, \ldots, d_{vk})^\top$ ($u \ne v$) of $\mathbf{D}[S]$ such that $d_{us} = d_{vs}$ for every $s \in [1,k]\backslash\{i\}$, but $d_{ui} > d_{vi}$, without loss of generality. Define $c_i := d_{ui} - d_{vi} > 0$. Observe that $\frac{1}{c_i}(\mathbf{d}_u - \mathbf{d}_v) = \mathbf{e}_i$ where $\mathbf{e}_i := (0, \ldots, 0, 1, 0, \ldots, 0)^\top$ with entry 1 is in the ith column. This means that $\mathbf{e}_i$ is in the row space of $\mathbf{D}[S]$. Since $i \in [1,k]$ is arbitrary, the linearly independent set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$ is contained in the row space of $\mathbf{D}[S]$, hence $\operatorname{rank}(\mathbf{D}[S]) \ge |S|$. By the property of rank, we obtain $\operatorname{rank}(\mathbf{D}[S]) \le \min\{|G|, |S|\} = |S|$. Therefore, $\operatorname{rank}(\mathbf{D}[S]) = |S|$. Consequently, $\dim(G) \le |S| = \operatorname{rank}(\mathbf{D}[S]) \le \operatorname{rank}(\mathbf{D})$. □

The contrapositive of Theorem 3.8 and the fact that $\operatorname{rank}(\mathbf{D}[S]) \le |S|$ produce the following corollary.

Corollary 3.9. *Let G be a connected graph other than a path with distance matrix $\mathbf{D}$. If S is a resolving set of G and $\operatorname{rank}(\mathbf{D}[S]) < |S|$, then S contains a smaller resolving set of G.*

4 Conclusion and open problems

In this paper, we gave a lower bound of the metric dimension $\dim(G)$ of any connected bipartite/singular graph G in terms of its nullity $\eta(G)$. Then, we gave infinite examples of graphs having equal metric dimension and nullity using the rooted product of graphs. We found that $\dim(G(\mathcal{H})) = \eta(G(\mathcal{H}))$ if $\mathcal{H}$ is the set of branch graphs having only odd tails and having at least two tails. It is still an open problem to characterize or list other graphs having equal metric dimension and nullity.

Problem 4.1. *Give other examples of graphs G with $\dim(G) = \eta(G)$.*

Another interesting problem is to investigate $\dim(G(\mathcal{H}))$ when $\mathcal{H}$ is the set of complete graphs of order at least 3. As a preliminary observation, it is known that for every integer $n \ge 2$, $\dim(K_n) = n - 1$. On the other hand, we also have $m_{\mathbf{A}(K_n)}(-1) = n - 1$, thus $\dim(K_n) = m_{\mathbf{A}(K_n)}(-1)$. We conjectured that there is a relationship between the metric dimension of a graph with the multiplicity of eigenvalue -1 through the existence of a clique.

Problem 4.2. *Investigate the relationships between the metric dimension of a graph having cliques and the multiplicity of -1 in their spectrum. In particular, if $F = G(\mathcal{H})$ where G is any connected* *bipartite graph and $\mathcal{H}$ is the set of complete graphs of order at least 3, then compare* $\dim(F)$ *and* $m_{\mathbf{A}(F)}(-1)$.

Lastly, we gave a relationship between the metric dimension of a graph and its distance matrix. We showed that if S is a minimal resolving set of G having distance matrix $\mathbf{D}$, then $\mathbf{D}[S]$ is full-rank. Since the metric dimension of a graph is closely related to the graph distance, there may be more relationships between the metric dimension and the distance matrix of a graph.

Problem 4.3. *Find other relationships between the metric dimension of a graph and its distance matrix.*

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