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Modified convergences to the Euler-Mascheroni constant

Cristinel Mortici[1,2,3] ✉

[1] *Academy of Romanian Scientists, Strada IIfov nr. 3, 050044 Bucureşti, Romania.*
cristinel.mortici@valahia.ro ✉

[2] *Department of Science and Advanced Technologies, Faculty of Sciences and Arts, Valahia University of Târgovişte, Aleea Sinaia 13, 130004 Târgovişte, Romania.*

[2] *National University of Science and Technology Politehnica Bucureşti, Splaiul Independenţei 313, 060042 Bucureşti, Romania.*

ABSTRACT

We introduce in this paper some new sequences that converge to the Euler-Mascheroni constant. These sequences have a higher convergence rate than the classical one. Further properties are given.

RESUMEN

En este artículo introducimos nuevas sucesiones que convergen a la constante de Euler-Mascheroni. Estas sucesiones tienen una tasa de convergencia mayor que la clásica. También entregamos propiedades adicionales de las mismas.

Keywords and Phrases: Euler-Mascheroni constant, Euler's gamma function, digamma function, approximations, speed of convergence.

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1 Introduction and motivation

The Euler-Mascheroni constant, represented by the symbol gamma γ, is a key mathematical constant that appears in numerous areas of number theory and analysis. Introduced by the Swiss mathematician Leonhard Euler in 1734, this constant is defined as the limit of the difference between the harmonic series and the natural logarithm. Mathematically, it is defined as the limit of the sequence:

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} + \ln \frac{1}{n}.$$

The approximate value of γ is $0.57721\ldots$, although its precise nature –whether it is rational or irrational– remains unresolved in the field of mathematics.

Throughout history, the Euler-Mascheroni constant has been extensively studied and computed. Euler initially determined its value to six decimal places, and later mathematicians, including the Italian mathematician Lorenzo Mascheroni, have worked to refine this calculation.

Despite its long-standing history, many aspects of γ continue to captivate mathematicians, making it a subject of ongoing research and investigation.

In particular, many researchers are focused on developing new, rapidly converging sequences to approximate γ.

This interest stems from the hypothesis that the unresolved question of whether γ is rational or irrational may be attributed to the slow convergence rate of the classical sequence $(\gamma_n)_{n\geq 1}$.

Recent studies have introduced various sequences with faster convergence rates (but a sacrifice of simplicity), aiming to shed light on the true nature of this enigmatic number. The methods used range from modifying some terms from the harmonic series to changing the argument of the logarithm to polynomial or rational functions. See, *e.g.*, [2–5].

This paper aims to introduce some new faster convergences to γ, keeping a simple form.

2 The results

Along with the classical sequence $(\gamma_n)_{n\geq 1}$ (that converges to γ decreasingly), the following sequence

$$\gamma_n' = \sum_{k=1}^{n} \frac{1}{k} + \ln \frac{1}{n+1}$$

converges increasingly to γ.

Both sequences $(\gamma_n)_{n\geq 1}$ and $(\gamma_n')_{n\geq 1}$ converge to γ like n^{-1}, since

$$\lim_{n\to\infty} n\,(\gamma_n - \gamma) = \frac{1}{2} \quad \text{and} \quad \lim_{n\to\infty} n\,(\gamma_n' - \gamma) = -\frac{1}{2}.$$

We introduce in this paper new sequences by modifying the argument of the logarithm to $\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)$, then to $\frac{1}{n^2}+\frac{1}{(n+1)^2}$.

For the sake of simplicity, we propose the sequence

$$\mu_n=\sum_{k=1}^{n}\frac{1}{k}+\ln\left(\frac{1}{n}+\frac{1}{n+1}\right)$$

that converges (to $\gamma+\ln 2$) at a higher rate of convergence, as we can see from the following:

Theorem 2.1. *a) The sequence $(\mu_n)_{n\geq 1}$ converges decreasingly to $\gamma+\ln 2$, at a rate of convergence n^{-2}. More precisely,*

$$\lim_{n\to\infty} n^2\left(\mu_n-(\gamma+\ln 2)\right)=\frac{7}{24}.$$

b) The following inequalities hold true, for every integer $n\geq 1$:

$$\frac{7}{24(n+1)(n+2)}\leq \mu_n-(\gamma+\ln 2)\leq \frac{7}{24n(n+1)}.$$

Keeping in mind that the number $\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)$, which appears in the expression of the sequence $(\mu_n)_{n\geq 1}$, is the arithmetic mean of $\frac{1}{n}$ and $\frac{1}{n+1}$, we introduce the following sequence involving the quadratic mean of $\frac{1}{n}$ and $\frac{1}{n+1}$:

$$\eta_n=\sum_{k=1}^{n}\frac{1}{k}+\frac{1}{2}\ln\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}\right).$$

The sequence $(\eta_n)_{n\geq 1}$ converges $\left(\text{to } \gamma+\frac{1}{2}\ln 2\right)$ with a rate of convergence n^{-2}, as we can see from the following:

Theorem 2.2. *a) The sequence $(\eta_n)_{n\geq 1}$ converges decreasingly to $\gamma+\frac{1}{2}\ln 2$, at a rate of convergence n^{-2}. More precisely,*

$$\lim_{n\to\infty} n^2\left(\eta_n-\left(\gamma+\frac{1}{2}\ln 2\right)\right)=\frac{5}{12}.$$

b) The following inequalities hold, for every integer $n\geq 1$:

$$\frac{5}{12(n+1)(n+2)}\leq \eta_n-\left(\gamma+\frac{1}{2}\ln 2\right)\leq \frac{5}{12n(n+1)}.$$

3 The proofs

A main tool for computing the speed of convergence is the following lemma, first stated in [6].

Lemma 3.1. *If* $(x_n)_{n\geq 1}$ *is convergent to zero and*

$$\lim_{n\to\infty} n^k (x_n - x_{n+1}) = l \in (-\infty, \infty),$$

for some $k > 1$ *and* $l \neq 0$, *then*

$$\lim_{n\to\infty} n^{k-1} x_n = \frac{l}{k-1}.$$

This lemma is useful especially when the sequence $(x_n)_{n\geq 1}$ is defined as a sum and consequently, the difference $x_n - x_{n+1}$ becomes of a simpler form.

Proof of Theorem 1. *a*) We have $\mu_n - \mu_{n+1} = f(n)$, where

$$f(x) = -\frac{1}{x+1} + \ln\left(\frac{1}{x} + \frac{1}{x+1}\right) - \ln\left(\frac{1}{x+1} + \frac{1}{x+2}\right).$$

This function f is decreasing on $(0, \infty)$, since

$$f'(x) = -\frac{14x + 7x^2 + 6}{x(2x+3)(2x+1)(x+2)(x+1)^2} < 0.$$

As $\lim_{x\to\infty} f(x) = 0$, it follows that $f > 0$ on $(0, \infty)$ and consequently, the sequence $(\mu_n)_{n\geq 1}$ is decreasing.

By standard calculations (or faster, using the Maple software) we get:

$$\lim_{n\to\infty} n^3 (\mu_n - \mu_{n+1}) = \frac{7}{12}.$$

According to Lemma 3.1, we obtain:

$$\lim_{n\to\infty} n^2 (\mu_n - (\gamma + \ln 2)) = \frac{7}{24}.$$

b) First we prove the following inequalities, for every integer $n \geq 1$:

$$\frac{7}{12n(n+1)(n+2)} - \frac{7}{4n(n+1)(n+2)(n+3)} < \mu_n - \mu_{n+1} < \frac{7}{12n(n+1)(n+2)}, \quad (3.1)$$

namely $u(x) < 0$ and $v(x) > 0$, for all $x \in (0, \infty)$, where

$$u(x) = f(x) - \frac{7}{12x(x+1)(x+2)}$$

and

$$v(x) = f(x) - \left(\frac{7}{12x(x+1)(x+2)} - \frac{7}{4x(x+1)(x+2)(x+3)} \right).$$

The function u is increasing, while the function v is decreasing, as

$$u'(x) = \frac{94x + 47x^2 + 42}{12x^2(2x+3)(2x+1)(x+2)^2(x+1)^2} > 0, \quad x > 0,$$

and

$$v'(x) = -\frac{4305x + 4748x^2 + 2137x^3 + 336x^4 + 1296}{12x(2x+3)(2x+1)(x+3)^2(x+2)^2(x+1)^2} < 0, \quad x > 0.$$

But $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = 0$, thus $u(x) < 0$ and $v(x) > 0$, for all $x \in (0, \infty)$, as we have announced before. The inequality (3.1) is true.

Now we plan to sum the inequalities (3.1) from n to $n+k-1$, where k is any positive number:

$$\frac{7}{12}\sum_{i=n}^{n+k-1} \frac{1}{i(i+1)(i+2)} - \frac{7}{4}\sum_{i=n}^{n+k-1} \frac{1}{i(i+1)(i+2)(i+3)} < \mu_n - \mu_{n+k} < \frac{7}{12}\sum_{i=n}^{n+k-1} \frac{1}{i(i+1)(i+2)}. \tag{3.2}$$

These are telescopic sums, as

$$\frac{1}{i(i+1)(i+2)} = \frac{1}{2}\left(\frac{1}{i(i+1)} - \frac{1}{(i+1)(i+2)} \right) \tag{3.3}$$

and

$$\frac{1}{i(i+1)(i+2)(i+3)} = \frac{1}{3}\left(\frac{1}{i(i+1)(i+2)} - \frac{1}{(i+1)(i+2)(i+3)} \right). \tag{3.4}$$

The inequality (3.2) becomes:

$$\frac{7}{24}\left(\frac{1}{n(n+1)} - \frac{1}{(n+k)(n+k+1)} \right) - \frac{7}{12}\left(\frac{1}{n(n+1)(n+2)} - \frac{1}{(n+k)(n+k+1)(n+k+2)} \right) < \mu_n - \mu_{n+k} < \frac{7}{24}\left(\frac{1}{n(n+1)} - \frac{1}{(n+k)(n+k+1)} \right).$$

By taking the limit as $k \to \infty$, we obtain:

$$\frac{7}{24}\frac{1}{n(n+1)} - \frac{7}{12}\frac{1}{n(n+1)(n+2)} \le \mu_n - (\gamma + \ln 2) \le \frac{7}{24}\frac{1}{n(n+1)},$$

which is the conclusion. □

Proof of Theorem 2. *a*) We have $\eta_n - \eta_{n+1} = g(n)$, where

$$g(x) = -\frac{1}{x+1} + \frac{1}{2}\ln\left(\frac{1}{x^2} + \frac{1}{(x+1)^2}\right) - \frac{1}{2}\ln\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2}\right)$$

This function g is decreasing on $(0,\infty)$, since

$$g'(x) = -\frac{38x + 59x^2 + 40x^3 + 10x^4 + 10}{x(x+2)(2x+2x^2+1)(6x+2x^2+5)(x+1)^2} < 0.$$

As $\lim_{x\to\infty} g(x) = 0$, it follows that $g > 0$ on $(0,\infty)$ and consequently, the sequence $(\eta_n)_{n\geq 1}$ is decreasing.

By standard calculations (or faster, using the Maple software) we get:

$$\lim_{n\to\infty} n^3 (\eta_n - \eta_{n+1}) = \frac{5}{6}.$$

According to the Lemma 3.1, we obtain:

$$\lim_{n\to\infty} n^2 \left(\eta_n - \left(\gamma + \frac{1}{2}\ln 2\right)\right) = \frac{5}{12}.$$

b) First we prove the following inequalities, for every integer $n \geq 1$:

$$\frac{5}{6n(n+1)(n+2)} - \frac{5}{2n(n+1)(n+2)(n+3)} < \eta_n - \eta_{n+1} < \frac{5}{6n(n+1)(n+2)}, \tag{3.5}$$

namely $s(x) < 0$ and $t(x) > 0$, for all $x \in (0,\infty)$, where

$$s(x) = g(x) - \frac{5}{6x(x+1)(x+2)}$$

and

$$t(x) = g(x) - \left(\frac{5}{6x(x+1)(x+2)} - \frac{5}{2x(x+1)(x+2)(x+3)}\right).$$

The function s is increasing, while the function t is decreasing, as

$$s'(x) = \frac{190x + 279x^2 + 184x^3 + 46x^4 + 50}{6x^2(2x+2x^2+1)(6x+2x^2+5)(x+2)^2(x+1)^2} > 0$$

and

$$t'(x) = -\frac{5089x + 10\,460x^2 + 11\,283x^3 + 6620x^4 + 1994x^5 + 240x^6 + 1080}{6x(2x+2x^2+1)(6x+2x^2+5)(x+3)^2(x+2)^2(x+1)^2} < 0.$$

But $\lim_{x\to\infty} s(x) = \lim_{x\to\infty} t(x) = 0$, thus $s(x) < 0$ and $t(x) > 0$, for all $x \in (0,\infty)$, as we have announced before. The inequality (3.5) is true.

Now we plan to sum the inequalities (3.5) from n to $n+k-1$, where k is any positive number:

$$\frac{5}{6}\sum_{i=n}^{n+k-1}\frac{1}{i(i+1)(i+2)}-\frac{5}{2}\sum_{i=n}^{n+k-1}\frac{1}{i(i+1)(i+2)(i+3)} < \eta_n-\eta_{n+k} < \frac{5}{6}\sum_{i=n}^{n+k-1}\frac{1}{i(i+1)(i+2)}. \quad (3.6)$$

These are telescopic sums, as we can see from (3.3)-(3.4). The inequality (3.6) becomes:

$$\frac{5}{12}\left(\frac{1}{n(n+1)}-\frac{1}{(n+k)(n+k+1)}\right) - \frac{5}{6}\left(\frac{1}{n(n+1)(n+2)}-\frac{1}{(n+k)(n+k+1)(n+k+2)}\right) < \eta_n-\eta_{n+k} < \frac{5}{12}\left(\frac{1}{n(n+1)}-\frac{1}{(n+k)(n+k+1)}\right).$$

By taking the limit as $k\to\infty$, we obtain:

$$\frac{5}{12(n+1)(n+2)} \le \eta_n-\left(\gamma+\frac{1}{2}\ln 2\right) \le \frac{5}{12n(n+1)}.$$

□

4 Further remarks

We believe that the ideas in this paper could be of interest to other researchers to obtain new generalizations, or results.

To be more precisely, recall that the harmonic sum is closely related to the digamma function ψ, *i.e.* the logaritmic derivative of the Euler-gamma function:

$$\psi(x)=\frac{d}{dx}(\ln\Gamma(x)).$$

Here,

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt, \quad x>0.$$

We have $\psi(1)=-\gamma$ and for every integer $n\ge 2$,

$$\psi(n)=-\gamma+\sum_{k=1}^{n-1}\frac{1}{k}.$$

Furthermore,

$$\psi(x+1)=\psi(x)+\frac{1}{x}.$$

For proofs and other properties, please see [1, p. 258].

Under these circumstances, the sequences we deal with in the above sections admit continuous forms on $(1,\infty)$, as follows:

$$\gamma(x)=\gamma+\psi(x)+\ln\frac{1}{x-1} \tag{4.1}$$

$$\mu(x)=\gamma+\psi(x)+\ln\left(\frac{1}{x-1}+\frac{1}{x}\right) \tag{4.2}$$

$$\eta(x)=\gamma+\psi(x)+\frac{1}{2}\ln\left(\frac{1}{(x-1)^2}+\frac{1}{x^2}\right), \tag{4.3}$$

for $x>1$. We have: $\gamma_n=\gamma(n+1)$, $\mu_n=\mu(n+1)$, $\eta_n=\eta(n+1)$, for all integers $n\geq 1$.

Bounds for the functions γ, μ, η given in (4.1)-(4.3) and consequently for the sequences $(\gamma_n)_{n\geq 1}$, $(\mu_n)_{n\geq 1}$, $(\eta_n)_{n\geq 1}$ can be obtained by using the asymptotic series of the digamma function:

$$\psi(x)\sim\ln x-\frac{1}{2x}-\sum_{i=1}^{\infty}\frac{B_{2i}}{2ix^{2i}}=\ln x-\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\frac{1}{252x^5}+\ldots,\ \text{as } x\to\infty. \tag{4.4}$$

Here, B_j are the Bernoulli numbers given by the generating function:

$$\frac{t}{e^t-1}=\sum_{j=0}^{\infty}B_j\frac{t^j}{j!}.$$

We have $B_1=-1/2$ and $B_{2j+1}=0$, for all positive integers j, while the first few Bernoulli numbers are $B_2=1/6$, $B_4=-1/30$, $B_6=1/42\ldots$ For detalis, see, *e.g.*, [1, p. 804].

The above announced bounds can be obtained by truncation of the (4.4) series. More precisely, under and upper approximations are given by alternatively truncate the (4.4) series:

$$\ln x-\frac{1}{2x}-\sum_{i=1}^{2m-1}\frac{B_{2i}}{2ix^{2i}}<\psi(x)<\ln x-\frac{1}{2x}-\sum_{i=1}^{2n}\frac{B_{2i}}{2ix^{2i}}.$$

In this way, along bounds, other monotonicity, even complete monotonicity properties can be discovered.

Declarations

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Availability of data and material

Not applicable.

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