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Normalized solutions for coupled Kirchhoff equations with critical and subcritical nonlinearities

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ABSTRACT

In this paper, we study Kirchhoff equations with constraint conditions

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u_1|^2\,dx\right)\Delta u_1=\lambda_1 u_1 \\ \qquad +\mu_1|u_1|^{p_1-2}u_1+\beta r_1|u_1|^{r_1-2}u_1|u_2|^{r_2} \quad \text{in } \mathbb{R}^3, \\ -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u_2|^2\,dx\right)\Delta u_2=\lambda_2 u_2 \\ \qquad +\mu_2|u_2|^{p_2-2}u_2+\beta r_2|u_1|^{r_1}|u_2|^{r_2-2}u_2 \quad \text{in } \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^3}|u_1|^2\,dx=c_1, \quad \int_{\mathbb{R}^3}|u_2|^2\,dx=c_2, \\ u_1\in H^1\left(\mathbb{R}^3\right), \quad u_2\in H^1\left(\mathbb{R}^3\right). \end{cases} \tag{P}$$

where a, b, β, μ_i, $c_i > 0$, $r_i > 1$, $2 < p_i < \frac{14}{3} < r := r_1 + r_2 \leq 2^*$ for $i = 1, 2$, and λ_1, $\lambda_2 \in \mathbb{R}$ appear as Lagrange multipliers. The existence of normalized solutions for p_1 and p_2 within a specific range of $(2, \frac{14}{3})$ has been considered both the Sobolev subcritical case ($r < 2^*$) and the critical case ($r = 2^*$) by the Minimax principle and variational methods. This paper provides a refinement and extension of the results for the normalized solutions to Kirchhoff equations.

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RESUMEN

En este artículo, estudiamos ecuaciones de Kirchhoff con condiciones de restricción

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u_1|^2\,dx\right)\Delta u_1=\lambda_1 u_1 \\ \qquad +\mu_1|u_1|^{p_1-2}u_1+\beta r_1|u_1|^{r_1-2}u_1|u_2|^{r_2} \quad \text{en } \mathbb{R}^3, \\ -\left(a+b\int_{\mathbb{R}^3}|\nabla u_2|^2\,dx\right)\Delta u_2=\lambda_2 u_2 \\ \qquad +\mu_2|u_2|^{p_2-2}u_2+\beta r_2|u_1|^{r_1}|u_2|^{r_2-2}u_2 \quad \text{en } \mathbb{R}^3, \\ \int_{\mathbb{R}^3}|u_1|^2\,dx=c_1, \quad \int_{\mathbb{R}^3}|u_2|^2\,dx=c_2, \\ u_1\in H^1\left(\mathbb{R}^3\right), \quad u_2\in H^1\left(\mathbb{R}^3\right). \end{cases} \tag{P}$$

donde a, b, β, μ_i, $c_i>0$, $r_i>1$, $2<p_i<\frac{14}{3}<r:=r_1+r_2\leq 2^*$ para $i=1,2$, y λ_1, $\lambda_2\in\mathbb{R}$ aparecen como multiplicadores de Lagrange. La existencia de soluciones normalizadas para p_1 y p_2 en un rango específico de $(2,\frac{14}{3})$ ha sido considerado tanto el caso Sobolev subcrítico ($r<2^*$) y el caso crítico ($r=2^*$) a través del principio Minimax y métodos variacionales. Este artículo entrega un refinamiento y una extensión de los resultados para soluciones normalizadas de ecuaciones de Kirchhoff.

Keywords and Phrases: Normalized solution, Kirchhoff equations, variational methods.

2020 AMS Mathematics Subject Classification: 35J60, 47J30, 35J20.

1 Introduction and main results

In this paper, we are concerned with the existence of normalized solutions to following Kirchhoff equations in $H^1\left(\mathbb{R}^3\right) \times H^1\left(\mathbb{R}^3\right)$,

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u_1|^2\,dx\right)\Delta u_1=\lambda_1 u_1+\mu_1|u_1|^{p_1-2}u_1+\beta r_1|u_1|^{r_1-2}u_1|u_2|^{r_2}, \\ -\left(a+b\int_{\mathbb{R}^3}|\nabla u_2|^2\,dx\right)\Delta u_2=\lambda_2 u_2+\mu_2|u_2|^{p_2-2}u_2+\beta r_2|u_1|^{r_1}|u_2|^{r_2-2}u_2, \end{cases} \tag{1.1}$$

under mass constraints,

$$\int_{\mathbb{R}^3}|u_1|^2\,dx=c_1,\quad \int_{\mathbb{R}^3}|u_2|^2\,dx=c_2, \tag{1.2}$$

where c_1, c_2 are prescribed positive constants.

The Kirchhoff-type problems, initially proposed by Kirchhoff in 1883 [18], extend the classical d'Alembert wave equations. Following the foundational work by Lions [22], Kirchhoff-type equations have attracted significant interest, leading to extensive exploration of their steady-state models. Early classical studies on Kirchhoff equations can be found in [1,12,13,19,23] and the references therein.

Currently, physicists are particularly interested in solutions that satisfy normalized conditions: $\int_{\mathbb{R}^3}|u_i|^2dx=c_i$, for $i=1,2$, due to their clear physical significance, particularly regarding mass. For example, from a physical perspective, the normalized condition can represent the number of particles in each component of Bose-Einstein condensates or the power supply in nonlinear optics. In this context, λ_i appears as an unknown quantity in the Kirchhoff equations (1.1). It is therefore natural to prescribe the value of the mass so that λ_i can be interpreted as Lagrange multipliers. From this perspective, problem (P) can be addressed by studying certain constrained variational problems, obtaining normalized solutions by identifying critical points of the energy functional $J: H^1\left(\mathbb{R}^3\right)\times H^1\left(\mathbb{R}^3\right)\to\mathbb{R}$ defined by

$$J(u_1,u_2)=\frac{a}{2}\sum_{i=1}^{2}\|\nabla u_i\|_2^2+\frac{b}{4}\sum_{i=1}^{2}\|\nabla u_i\|_2^4-\sum_{i=1}^{2}\frac{\mu_i}{p_i}\|u_i\|_{p_i}^{p_i}-\beta\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx,$$

constrained on $\mathcal{S}:=S(c_1)\times S(c_2)$, where $\|\cdot\|_p$ denotes the standard norm in $L^p\left(\mathbb{R}^3\right)$ for $p\in[1,+\infty)$ and $S(c):=\left\{u\in H^1\left(\mathbb{R}^3\right):\|u\|_2^2=c\right\}$ for any $c>0$.

When $b=0$, the Kirchhoff equations (1.1) reduce to a nonlinear Schrödinger equations. In this case, we note that the mass critical exponent $\frac{10}{3}$. If the problem (P) is purely mass subcritical, *i.e.*, $2<p_1,\,p_2,\,r<\frac{10}{3}$, Gou and Jeanjean [10] searched for a critical point of J as a global minimizer of J on $\mathcal{S}$. In the purely mass supercritical case, *i.e.*, $\frac{10}{3}<p,\,q,\,r<2^*$, Bartsch *et al.* [3] first considered the case of $p=q=r=4$. They obtained the existence of positive solutions to problem (P) provided $0<\beta<\beta_1(c_1,c_2)$ or $\beta>\beta_2(c_1,c_2)$. Bartsch and Jeanjean [2] extended these results of [3] to $\frac{10}{3} < p_1,\, p_2,\, r < 2^*$. Recently, Jeanjean *et al.* [17] focused on the coupled purely mass supercritical case and proved the existence of solutions for all c_1, c_2, and without restrictions on β. For the mixed cases such as $2 < p_1,\, p_2 < \frac{10}{3} < r < 2^*$ or $2 < r < \frac{10}{3} < p_1,\, p_2 < 2^*$, Gou and Jeanjean [11] explored the multiplicity of solutions to problem (P). Later, Bartsch and Jeanjean [2] used the mountain pass lemma and a compactness argument to show that problem (P) has a positive solution for suitable $c_1,\, c_2 > 0$ when $2 < p_1 < \frac{10}{3} < p_2$ and $r < 2^*$. In the Sobolev critical case, Li and Zou [21] investigated the condition that $2 < p_1,\, r < 2^*,\, p_2 \leq 2^*$. Bartsch *et al.* [4] also considered the Sobolev critical case with $2 < r < 2^* = p_1 = p_2$. When $\frac{10}{3} < p_1,\, p_2 < r = 2^*$, Liu and Fang [24] demonstrated that problem (P) has a mountain pass solution. Zhang and Han [34] obtained a positive ground state solution of problem (P) with $2 < p_1,\, p_2 < \frac{10}{3}$ and $r = 2^*$.

When $b > 0$, there are several results in the literature dealing with normalized solutions to problem (P). Ye [32, 33] considered this constrained problem for a single Kirchhoff equation

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u = \lambda u + \mu |u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^3} |u|^2 dx = c. \end{cases} \tag{1.3}$$

Ye proved that $p = \frac{14}{3}$ is a mass critical exponent for Kirchhoff equation. To be more precise, the functional corresponding to problem (1.3) is

$$I_\mu(u) := \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\mu}{p}\|u\|_p^p,$$

which is bounded from below on manifold $S(c)$ when $2 < p < \frac{14}{3}$. However, when $\frac{14}{3} < p < 6$, the functional is not bounded from below on $S(c)$. By Ekeland's variational principle and the strict monotonicity of a energy function, Cao *et al.* [5] considered the existence of positive solutions to problem (P) with the purely mass subcritical case $2 < p_1,\, p_2,\, r < \frac{14}{3}$. Recently, Yang [31] showed the existence of positive solutions to problem (P) in the purely mass supercritical case $\frac{14}{3} < p_1$, $p_2,\, r < 2^*$ and in the mixed case $2 < r < \frac{14}{3} < p_1,\, p_2 < 2^*$. Hu and Mao [15] further obtained the existence of two solution (local minimizer and Mountain-Pass type) for the mixed cases $2 < p_1$, $p_2 \leq \frac{10}{3}$ and $\frac{14}{3} < r < 2^*$. More results about the normalized solutions, we refer the readers to [8, 14, 29, 30].

To provide clarity in the discussion, we summarize some of the results on normalized solutions to problem (P) in Table 1.

Motivated by the aforementioned works, we study normalized solutions to problem (P) in three distinct cases: (H_1): $\frac{10}{3} < p_1,\, p_2 < \frac{14}{3} < r < 2^*$; (H_2): $2 < p_1 < \frac{14}{3} < p_2,\, r < 2^*$ and (H_3): $2 < p_1,\, p_2 < \frac{10}{3},\, r = 2^*$. To address compactness issues, we work within the radial space $\mathcal{S}_r := S_r(c_1) \times S_r(c_2)$, where $S_r(c) := \left\{u \in H_r^1\left(\mathbb{R}^3\right) : \|u\|_2^2 = c\right\}$, and $H_r^1\left(\mathbb{R}^3\right)$ denotes the space of

Table 1

b	p_1, p_2, r	Types of solutions	References
$b = 0$	$2 < p_1, p_2, r < \frac{10}{3}$	a global minimizer	[2, 10]
$b = 0$	$\frac{10}{3} < p_1, p_2, r < 6$	Mountain Pass solution	[2, 3]
$b = 0$	$2 < p_1 < \frac{10}{3} < p_2, r < 6$	Mountain Pass solution	[2]
$b = 0$	$2 < r < \frac{10}{3} < p_1, p_2 < 6$	Mountain Pass solution, a local minimizer	[11]
$b = 0$	$r = 6$ or $p_1, p_2 = 6$	Mountain Pass solution, ground state solution	[4, 21, 24, 34]
$b > 0$	$2 < p_1, p_2, r < \frac{14}{3}$	a global minimizer	[5]
$b > 0$	$\frac{14}{3} < p_1, p_2, r < 6$; $2 < r < \frac{14}{3} < p_1, p_2 < 6$	Mountain Pass solution, a local minimizer	[31]
$b > 0$	$2 < p_1, p_2 \leq \frac{10}{3}, \frac{14}{3} < r < 6$	Mountain Pass solution, a local minimizer	[15]
$b > 0$	$\frac{10}{3} < p_1, p_2 < \frac{14}{3}, \frac{14}{3} < r < 6$	open problem	
$b > 0$	$2 < p_1 < \frac{14}{3} < p_2, r < 6$	open problem	
$b > 0$	$2 < p_1, p_2 < \frac{14}{3}, r = 6$	open problem	

radial functions on $\mathbb{R}^3$. By the principle of symmetric criticality, the critical points of J constrained on $\mathcal{S}_r$ are also critical points of J constrained on $\mathcal{S}$.

It is known that critical points of $J|_{\mathcal{S}_r}$ stay in

$$\mathcal{P} := \{(u_1, u_2) \in \mathcal{S}_r : P(u_1, u_2) = 0\},$$

as a consequence of Pohozaev identity, where

$$P(u_1, u_2) := a\sum_{i=1}^{2} \|\nabla u_i\|_2^2 + b\sum_{i=1}^{2} \|\nabla u_i\|_2^4 - \sum_{i=1}^{2} \mu_i \gamma_{p_i} \|u_i\|_{p_i}^{p_i} - \beta r \gamma_r \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx.$$

Moreover, we define for $u \in S(c)$ the map

$$(s \star u)(x) := e^{\frac{3s}{2}} u(e^s x), \quad s \in \mathbb{R},$$

which preserves the L^2 norm and plays a special role in the study of structures of $J(u_1, u_2)$ and $P(u_1, u_2)$ on the constraint $\mathcal{S}_r$. We introduce the fiber mapping for $J(u_1, u_2)$,

$$\begin{aligned} \Phi_{u_1,u_2}(s) :&= J(s \star u_1, s \star u_2) \\ &= \frac{ae^{2s}}{2} \sum_{i=1}^{2} \|\nabla u_i\|_2^2 + \frac{be^{4s}}{4} \sum_{i=1}^{2} \|\nabla u_i\|_2^4 - \sum_{i=1}^{2} \frac{\mu_i e^{p_i \gamma_{p_i} s}}{p_i} \|u_i\|_{p_i}^{p_i} - \beta e^{r\gamma_r s} \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx, \end{aligned} \tag{1.4}$$

for any $(u_1, u_2) \in \mathcal{S}_r$. It is easy to verify that $(s \star u, s \star v) \in \mathcal{P}$ if and only if s is a critical point of $\Phi_{u_1,u_2}(s)$. In particular, $(u, v) \in \mathcal{P}$ if only if $s = 0$ is a critical point of $\Phi_{u_1,u_2}(s)$.

We will require some preliminary results regarding problem (1.3). Let $m(c, \mu)$ denote the ground state level, defined as

$$m(c, \mu) := \inf\left\{ I_\mu(u) : u \in S(c) \text{ such that } \left(I_\mu|_{S(c)}\right)'(u) = 0 \right\},$$

and introduce the Pohozaev set for the single Kirchhoff equation:

$$V(c) := \left\{u \in S(c) : 0 = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \mu\gamma_p\|u\|_p^p\right\}.$$

Now, we state the first result about the mass sub-critical case as follows.

Theorem 1.1. *Assume the following assumptions* (H_1) *holds,*

$$(H_1): \ \frac{10}{3} < p_1, \ p_2 < \frac{14}{3} < r < 2^*.$$

There exists $\beta_0 := \beta_0(c_1, c_2) > 0$*, such that for* $0 < \beta \leq \beta_0$ *and* $c_1, c_2 < c^*$*, problem* (P) *has a positive normalized solution.*

Inspired by [2], Bartsch and Jeanjean constructed a minimax level and proved the existence of a positive normalized solution for Schrödinger equations with $2 < p_1 < \frac{10}{3} < p_2$, $r < 2^*$. Our second result deals with the case

$$(H_2): \ 2 < p_1 < \frac{14}{3} < p_2, \ r < 2^*; \ 2 < r_2 < \frac{10}{3}.$$

which we call it mix mass sup-critical case.

Theorem 1.2. *Assume that* (H_2) *holds. For*

(p_1) $2 < p_1 \leq \frac{10}{3}$ *and* $c_1 > 0$*, or* $\frac{10}{3} < p_1 < \frac{14}{3}$ *and* $c_1 > c_*$*, where* c_* *is positive constant only depend on* a, b, μ_1*,*

if $m(c_1, \mu_1) + m(c_2, \mu_2) < 0$*, problem* (P) *has a positive normalized solution.*

As a corollary of Theorem 1.2, we obtain the following results.

Corollary 1.3. *Assume that* (H_2) *holds.*

- (i) *For any* $c_2 > 0$*, there exists* $\bar{c}_1$*, such that for* $c_1 \geq \bar{c}_1$*, problem* (P) *has a positive normalized solution.*
- (ii) *For any* $c_1 > c_*$*, there exists* $\bar{c}_2$*, such that for* $c_2 \geq \bar{c}_2$*, problem* (P) *has a positive normalized solution.*

Last, we consider the mass sub-critical and Sobolev critical case,

$$(H_3): \ 2 < p_1, \ p_2 \leq \frac{10}{3}, \ r = 2^*.$$

Theorem 1.4. *Assume that* (H_3) *holds. There exist* $\beta_* := \beta_*(c_1, c_2)$ *and* μ_**, such that for* $0 < \beta < \beta_*$ *and* μ_1*,* $\mu_2 < \mu_*$*, problem* (P) *has a ground state solution.*

Remark 1.5. (i) *Theorem 1.1 serves as a complement to the work of Hu and Mao [15], specifically addressing the case of problem* (P) *with* $2 < p_1, p_2 \leq \frac{10}{3}$ *and* $\frac{14}{3} < r < 2^*$*. Compared with a single equation, the main difficulty for systems is how to exclude the semi-trivial solutions. In [15], the authors heavily rely on* $p < \frac{10}{3}$ *since that* $m(c, \mu) < 0$ *to excluding semi-trivial solutions. However, we partially extend to the case that* $\frac{10}{3} < p_1, p_2 < \frac{14}{3}$ *with the mass constrained suitable small to overcome this difficulty.*

(ii) *Theorems 1.2 and 1.4 complement the results of Zhang and Han [34] and Bartsch and Jeanjean [2], which extended the study from Schrödinger equations to Kirchhoff equations.*

(iii) *Compared Kirchhoff equations with single Kirchhoff equation, the existence and types of solutions to problem* (P) *are similar to the result of single equation,*

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u = \lambda u + \mu|u|^{q-2}u + |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^3} |u|^2 dx = c, \end{cases} \tag{1.5}$$

where a*,* b*,* c *are positive constants and* $2 < q < p \leq 2^*$*. Feng et al. in [7] have proven that under condition* $2 < q < \frac{10}{3} < p = 2^*$*, problem* (1.5) *has a second solution. It is an interesting question whether problem* (P) *also has a second solution under condition* (H_3)*?*

The rest of this paper is organized as follows. In Section 2, we present some preliminary results. Sections 3-5 are devoted to the proofs of Theorems 1.1-1.4.

Notation: In this paper, we denote $H := H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ and $H_r := H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3)$. $\to$ and $\rightharpoonup$ denote the strong and weak convergence in the related function space, respectively. $H^{-1}(\mathbb{R}^3)$ is the dual space of $H^1(\mathbb{R}^3)$. C, $C(\cdot), \ldots$ denote positive constants. $o_n(1)$ represents a real sequence with $o_n(1) \to 0$ as $n \to +\infty$. $D^{1,2}(\mathbb{R}^3)$ denotes the closure of the function space $C_c^\infty(\mathbb{R}^3)$ with the norm $\|u\|_{D^{1,2}(\mathbb{R}^3)} = \|\nabla u\|_2$. The best Sobolev constant S is given by $S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}$.

2 Preliminary results

Before we proceed further, let us first revisit the Gagliardo-Nirenberg inequality in [27, 28]. For $2 \leq p \leq 2^*$, there exists a constant $C_p > 0$ such that for any $u \in H^1(\mathbb{R}^3)$,

$$\|u\|_p \leq C_p \|\nabla u\|_2^{\gamma_p} \|u\|_2^{1-\gamma_p},$$

where $\gamma_p = \frac{3(p-2)}{2p}$. For $2 \le r_1 + r_2 \le 2^*$, there exists $q > 1$ such that

$$\max\left\{\frac{2}{r_1}, \frac{2^*}{2^* - r_2}\right\} \le q \le \min\left\{\frac{2^*}{r_1}, \frac{2}{(2-r_2)^+}\right\}. \tag{2.1}$$

Set $q' := \frac{q}{q-1}$, $2 \le r_1 q$, $r_2 q' \le 2^*$, by the Hölder inequality, we have

$$\int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2}\, dx \le \|u_1\|_{r_1 q}^{r_1} \|u_2\|_{r_2 q'}^{r_2} < \infty,$$

which implies that the functional J is well defined. For $\frac{14}{3} < r = r_1 + r_2 < 2^*$, by the Hölder inequality and the Gagliardo-Nirenberg inequality, we know

$$\begin{aligned}\int_{\mathbb{R}^3} |u_1|^{r_1}|u_2|^{r_2}\, dx &\le \|u_1\|_r^{r_1}\|u_2\|_r^{r_2} \le C\|\nabla u_1\|_2^{r_1\gamma_r}\|\nabla u_2\|_2^{r_2\gamma_r} \\ &\le C\left(\sum_{i=1}^2 \|\nabla u_i\|_2^2\right)^{\frac{r_1\gamma_r}{2}}\left(\sum_{i=1}^2 \|\nabla u_i\|_2^2\right)^{\frac{r_2\gamma_r}{2}} \le C\left(\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2\right)^{\frac{r\gamma_r}{2}}. \end{aligned} \tag{2.2}$$

Specifically, for $r = 2^*$, $r\gamma_r = 2^*$, then $C = S^{-\frac{2^*}{2}}$. Next, we need a splitting lemma similar to Brézis-Lieb Lemma as follows.

Lemma 2.1 ([11, Lemma 2.4], [6, Lemma 2.3])**.** *Assume that r_1, $r_2 > 1$, $2 < r_1 + r_2 \le 2^*$. If*

$$(u_1^n, u_2^n) \rightharpoonup (u_1, u_2) \text{ in } H,$$

then up to a subsequence

$$\int_{\mathbb{R}^3} |u_1^n|^{r_1} |u_2^n|^{r_2}\, dx = \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2}\, dx + \int_{\mathbb{R}^3} |u_1^n - u_1|^{r_1} |u_2^n - u_2|^{r_2}\, dx + o_n(1).$$

Moreover, a description of the PPS sequence is also needed as follows.

Lemma 2.2 ([15, Lemma 2.5, 2.6])**.** *Assume that $2 < p_1, p_2 < 2^*$, $2 < r < 2^*$. If $\{(u_1^n, u_2^n)\}$ is a bounded Palais-Smale sequence for J on $\mathcal{S}_r$, there exist $(u_1, u_2) \in H_r$ and a sequence $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R}^2$, such that up to a subsequence*

(i) $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ *in* H_r, $(u_1^n, u_2^n) \to (u_1, u_2)$ *in* $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ *for* $p \in (2, 2^*)$.

(ii) $(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$ *in* $\mathbb{R}^2$.

(iii) $J'(u_1^n, u_2^n) - \lambda_1^n (u_1^n, 0) - \lambda_2^n (0, u_2^n) \to 0$ *in* $H_r^{-1}(\mathbb{R}^3) \times H_r^{-1}(\mathbb{R}^3)$.

(iv) (u_1, u_2) *is a solution of equations* (1.1) *for* λ_1, $\lambda_2 \le 0$ *if* $P(u_1^n, u_2^n) \to 0$. *In addition,* $u_1^n \to u_1$ *in* $H_r^1(\mathbb{R}^3)$ *if* $\lambda_1 < 0$. *Similarly,* $u_2^n \to u_2$ *in* $H_r^1(\mathbb{R}^3)$ *if* $\lambda_2 < 0$.

Lemma 2.3 ([16])**.** *Let $p \in (1,3]$. If $u \in L^p\left(\mathbb{R}^3\right) \cap C^2\left(\mathbb{R}^3\right)$ is non-negative and satisfies $-\Delta u \geq 0$ in $\mathbb{R}^3$, then $u = 0$.*

Lemma 2.4. *Let $p_i \in (2, 2^*)$, $i = 1, 2$. If $(u_1, u_2) \in H_r$ is a solution of Kirchhoff equations* (1.1) *with $u_1 \geq 0$, $u_1 \neq 0$, and $u_2 \geq 0$, then $\lambda_1 < 0$. Similarly, if $u_1 \geq 0$, $u_2 \geq 0$, and $u_2 \neq 0$, then $\lambda_2 < 0$.*

Proof. Similar proofs can be referenced in [5, Lemma 2.4]. □

The following existing results concerning the single Kirchhoff equation is rather significant to the main proof of Theorems.

Proposition 2.5. *Assume that $p \in (2, 2^*)$ and $\mu > 0$. Then*

(*i*) *[5, Lemma 2.2], [26, Theorem 1.1, 1.4]: Assume that $2 < p < \frac{10}{3}$, the problem* (1.3) *has a unique positive ground state solution for any $c > 0$. If $p = \frac{10}{3}$, there exists c' such that the problem* (1.3) *has a unique positive ground state solution for $c > c'$. Moreover, $m(c, \mu) < 0$, $m(c, \mu) \to -\infty$ as $c \to \infty$.*

(*ii*) *[5, Lemma 2.2], [26, Theorem 1.1], [25, Theorem 1.1]: Assume that $p \in (\frac{10}{3}, \frac{14}{3})$, there exists $0 < c^* < c_*$, such that the problem* (1.3) *admits exactly two positive normalized solutions w_1, w_2 if $c > c^*$ and no solution if $c < c^*$. If $c \geq c_*$, one of the above positive solutions is the unique normalized ground state solution. Without loss of generality, let w_1 be the normalized ground state and w_2 be the high-energy, then there holds that $I_\mu(w_1) = m(c, \mu) \leq 0 < I_\mu(w_2)$, and $m(c, \mu) \to -\infty$ as $c \to \infty$.*

(*iii*) *[33], [31, Lemma 3.1]: If $p \in (\frac{14}{3}, 2^*)$ and problem* (1.3) *admits a unique solution u_c for any $c > 0$, $m(c, \mu) = I_\mu(u_c) = \max_{s \in \mathbb{R}} \Phi_{u_c}(s) = \min_{u \in V(c)} I_\mu(u) > 0$, where*

$$\Phi_u(s) := I_\mu(s \star u) = \frac{ae^{2s}}{2}\|\nabla u\|_2^2 + \frac{be^{4s}}{4}\|\nabla u\|_2^4 - \frac{\mu e^{p\gamma_p s}}{p}\|u\|_p^p.$$

Moreover, $m(c, \mu)$ is strictly decreasing with respect to c.

3 The proof of Theorem 1.1

We shall investigate the mountain pass geometry of $J(u_1, u_2)$ on $\mathcal{S}_r$.

Lemma 3.1. *Assume that (H_1) holds.*

(*i*) *There exist $\rho_0 = \rho_0(c_1, c_2)$ and $\beta_0 = \beta_0(c_1, c_2) > 0$, such that for $0 < \beta \leq \beta_0$,*

$$\inf_{A(2\rho_0) \setminus A(\rho_0)} J(u_1, u_2) > 0,$$

where $A(\rho_0) := \left\{(u_1,u_2) \in \mathcal{S}_r : \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2 < \rho_0\right\}$ *for* $\rho_0 > 0$.

(*ii*) *There exists* $(u_1,u_2) \in \mathcal{S}_r \backslash A(2\rho_0)$, *such that* $J(u_1,u_2) < 0$.

Proof. (*i*) Let $\rho := \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2$. By (2.2) and the Gagliardo-Nirenberg inequality, for $(u_1,u_2) \in \mathcal{S}_r$, we have:

$$\begin{aligned} J(u_1,u_2) &= \frac{a}{2}\rho + \frac{b}{4}\sum_{i=1}^{2}\|\nabla u_i\|_2^4 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}\|u_i\|_{p_i}^{p_i} - \beta\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}\,dx \\ &\geq \frac{b}{8}\rho^2 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}C_i\|\nabla u_i\|_2^{p_i\gamma_{p_i}} - \beta C_3\rho^{\frac{r\gamma_r}{2}} \\ &\geq \frac{b}{8}\rho^2 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}C_i\rho^{\frac{p_i\gamma_{p_i}}{2}} - \beta C_3\rho^{\frac{r\gamma_r}{2}}, \end{aligned}$$

where $C_i := C(c_1,c_2)$ for $(i = 1,2,3)$. If (H_1) holds, then $2 < p_i\gamma_{p_i} < 4$ and $4 < r\gamma_r < 2^*$. Let $\rho_0 > 0$ be large enough such that

$$\sum_{i=1}^{2}\frac{\mu_i}{p_i}C_i(\rho_0)^{\frac{p_i\gamma_{p_i}-4}{2}} \leq \frac{b}{32}, \tag{3.1}$$

and then choose $\beta_0 > 0$ small enough such that

$$\beta_0 C_3(2\rho_0)^{\frac{r\gamma_r-4}{2}} \leq \frac{b}{32}.$$

Hence, for any $0 < \beta \leq \beta_0$ and $(u_1,u_2) \in A(2\rho_0)\backslash A(\rho_0)$, *i.e.*, $\rho_0 \leq \rho < 2\rho_0$, we have

$$\begin{aligned} J(u_1,u_2) &\geq \frac{b}{8}\rho^2 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}C_i\rho^{\frac{p_i\gamma_{p_i}}{2}} - \beta C_3\rho^{\frac{r\gamma_r}{2}} = \rho^2\left(\frac{b}{8} - \sum_{i=1}^{2}C_i\rho^{\frac{p_i\gamma_{p_i}-4}{2}} - \beta C_3\rho^{\frac{r\gamma_r-4}{2}}\right) \\ &\geq b\rho_0^2\left(\frac{1}{8} - \frac{1}{32} - \frac{1}{32}\right) = \frac{b}{16}\rho_0^2. \end{aligned}$$

(*ii*) Let $u^t(x) := t^{\frac{3}{2}}u(tx)$. Then,

$$\|u^t\|_2^2 = \|u\|_2^2, \quad \|\nabla u^t\|_2^2 = t^2\|\nabla u\|_2^2, \quad \|u^t\|_p^p = t^{p\gamma_p}\|u\|_p^p, \quad \text{for all } p \in (2,2^*).$$

Fix $(u_1,u_2) \in \mathcal{S}_r$, $(u_1^t,u_2^t) \in \mathcal{S}_r\backslash A(2\rho_0)$ when t is sufficiently large. Since

$$J(u_1^t,u_2^t) = \frac{a}{2}t^2\sum_{i=1}^{2}\|\nabla u_i\|_2^2 + \frac{b}{4}t^4\sum_{i=1}^{2}\|\nabla u_i\|_2^4 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}t^{p_i\gamma_{p_i}}\|u_i\|_{p_i}^{p_i} - \beta t^{r\gamma_r}\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx,$$

it is straightforward to check that $\psi_{(u_1,u_2)}(t) := J(u_1^t,u_2^t) < 0$ for t large enough. □

Thanks to Lemma 3.1, we introduce a minimax structure of the mountain pass type. Specifically, there exists,

$$\gamma\left(c_1, c_2\right) := \inf_{g\in\Gamma} \max_{t\in[0,1]} J(g(t)),$$

where $\Gamma := \left\{g \in C\left([0,1], \mathcal{S}_r\right) : g(0) \in \partial A(\rho_0),\ g(1) \notin \overline{A\left(2\rho_0\right)},\ J(g(1)) < 0\right\}$. This framework allows us to search for a critical point of the mountain pass type at the level $\gamma\left(c_1, c_2\right)$. It is clear that $\gamma\left(c_1, c_2\right) \geq \inf_{u\in\partial A(\rho_0)} J\left(u_1, u_2\right) > 0$.

Lemma 3.2. *Assume that (H_1) holds. There exists a Palais-Smale sequence $\{(u_1^n, u_2^n)\}$ for $J|_{\mathcal{S}_r}$ at the level $\gamma\left(c_1, c_2\right)$, which satisfies $\{u_1^n\}^- \to 0$, $\{u_2^n\}^- \to 0$, and $P\left(u_1^n, u_2^n\right) \to 0$.*

Proof. The proof of the theorem is standard, and we omit the detailed steps here. For a comprehensive explanation, refer to [15, Lemma 3.1], [2, Lemma 5.5], and [9, Theorem 4.1]. □

Lemma 3.3. *Assume that (H_1) holds. There exists a pair of positive solution (u_1, u_2) to equations (1.1) for some (λ_1, λ_2), and $J\left(u_1, u_2\right) = \gamma\left(c_1, c_2\right) > 0$.*

Proof. By Lemma 3.2, there exists a Palais-Smale sequence $\{(u_1^n, u_2^n)\}$ for $J|_{\mathcal{S}_r}$ at the level $\gamma\left(c_1, c_2\right)$. We first prove that $\{(u_1^n, u_2^n)\}$ is bounded in H_r. Since $P\left(u_1^n, u_2^n\right) \to 0$, we have

$$a\sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + b\sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 = \sum_{i=1}^{2} \mu_i\gamma_{p_i}\|u_i^n\|_{p_i}^{p_i} + \beta r\gamma_r \int_{\mathbb{R}^3} |u_1^n|^{r_1}\,|u_2^n|^{r_2}\,dx + o_n(1). \tag{3.2}$$

Thus,

$$\begin{aligned}
\gamma(c_1, c_2) + o_n(1) &= \frac{a}{2}\sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + \frac{b}{4}\sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 - \sum_{i=1}^{2} \frac{\mu_i}{p_i}\|u_i^n\|_{p_i}^{p_i} - \beta\int_{\mathbb{R}^3} |u_1^n|^{r_1}\,|u_2^n|^{r_2}\,dx \\
&= a\left(\frac{1}{2} - \frac{1}{r\gamma_r}\right)\sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + b\left(\frac{1}{4} - \frac{1}{r\gamma_r}\right)\sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 \\
&\quad - \sum_{i=1}^{2} \mu_i\gamma_{p_i}\left(\frac{1}{p_i\gamma_{p_i}} - \frac{1}{r\gamma_r}\right)\|u_i^n\|_{p_i}^{p_i} \\
&\geq a\left(\frac{1}{2} - \frac{1}{r\gamma_r}\right)\rho + \frac{b}{2}\left(\frac{1}{4} - \frac{1}{r\gamma_r}\right)\rho^2 - \sum_{i=1}^{2} C_i\mu_i\gamma_{p_i}\left(\frac{1}{p_i\gamma_{p_i}} - \frac{1}{r\gamma_r}\right)\rho^{\frac{p_i\gamma_{p_i}}{2}},
\end{aligned}$$

where $\rho = \|\nabla u_1^n\|_2^2 + \|\nabla u_2^n\|_2^2$, $4 < r\gamma_r < 2^*$, $2 < p_i\gamma_{p_i} < 4$. Hence, $\{(u_1^n, u_2^n)\}$ is bounded in H_r. Then, for p, $q \in (2, 2^*)$, we may assume that

$$(u_1^n, u_2^n) \rightharpoonup (u_1, u_2) \text{ in } H_r, \quad (u_1^n, u_2^n) \to (u_1, u_2) \text{ in } L^p(\mathbb{R}^3) \times L^q(\mathbb{R}^3). \tag{3.3}$$

By Lemmas 2.2, 3.2, there exists a sequence $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R}^2$, such that $(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$, $\lambda_1, \lambda_2 \leq 0$. Consequently, (u_1, u_2) is a solution to equations (1.1) and satisfies $P\left(u_1, u_2\right) = 0$.

Since $(u_1^n)^- \to 0$, $(u_2^n)^- \to 0$, it follows that $u_1,\, u_2 \geq 0$.

Now, we prove $J(u_1, u_2) = \gamma(c_1, c_2)$. By (3.3) and Lemma 2.1, the right hand side of (3.2) converges to

$$\sum_{i=1}^{2} \mu_i \gamma_{p_i} \|u_i\|_{p_i}^{p_i} + \beta r \gamma_r \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx.$$

Combining this with $P(u_1, u_2) = 0$, we have

$$\lim_{n \to +\infty} a \sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + b \sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 = a \sum_{i=1}^{2} \|\nabla u_i\|_2^2 + b \sum_{i=1}^{2} \|\nabla u_i\|_2^4.$$

Therefore, $J(u_1^n, u_2^n) \to J(u_1, u_2)$, and hence, $J(u_1, u_2) = \gamma(c_1, c_2)$. □

Proof of Theorem 1.1. As known from Lemma 3.3, it is sufficient to prove that $(u_1, u_2) \in \mathcal{S}_r$. Using the fact that (u_1, u_2) is a solution to equations (1.1), we deduce that

$$\lambda_1 \|u_1\|_2^2 + \lambda_2 \|u_2\|_2^2 = a \sum_{i=1}^{2} \|\nabla u_i\|_2^2 + b \sum_{i=1}^{2} \|\nabla u_i\|_2^4 - \sum_{i=1}^{2} \mu_i \|u_i\|_{p_i}^{p_i} - \beta r \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} \, dx.$$

Combining Pohozaev identity and the fact that $\gamma_{p_i}, \gamma_r < 1$, we get

$$\lambda_1 \|u_1\|_2^2 + \lambda_2 \|u_2\|_2^2 = \sum_{i=1}^{2} \mu_i \left(\gamma_{p_i} - 1\right) \|u_i\|_{p_i}^{p_i} + \beta r \left(\gamma_r - 1\right) \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} \, dx < 0.$$

Hence, at least one of λ_1 and λ_2 is negative. Without loss of generality, we may assume $\lambda_1 < 0$. By Lemma 2.2, we have $u_1^n \to u_1$ in $H_r^1(\mathbb{R}^3)$, and then $u_1 \in S_r(c_1)$. For the sake of contradiction, suppose that $\lambda_2 \geq 0$, then

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u_2|^2 \, dx\right) \Delta u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2 - 2} u_2 + \beta r_2 |u_1|^{r_1} |u_2|^{r_2 - 2} u_2 \geq 0.$$

It follows from Lemma 2.3 that $u_2 = 0$. Thus, $J(u_1, u_2) = J(u_1, 0)$, and $u_1 \in S_r(c_1)$ satisfies the equation

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right) \Delta u = \lambda_1 u + \mu_1 |u|^{p_1 - 2} u. \tag{3.4}$$

However, this equation contradicts Proposition 2.5 (ii) that equation (3.4) admits no solution if $c < c^*$. Therefore, $\lambda_2 < 0$, and then, $u_2 \in S_r(c_2)$. Finally, by the maximum principle, we deduce that $u_1, u_2 > 0$ in $\mathbb{R}^3$. □

4 The proof of Theorem 1.2

Inspired by [2], let p_1 and p_2 be in different ranges *i.e.*, (H_2). For any $K > 0$, set

$$T_K := \left\{u_2 \in S(c_2) : \|\nabla u_2\|_2^2 \leq K\right\} \quad \text{and} \quad B_K := \left\{u_2 \in S(c_2) : \|\nabla u_2\|_2^2 = 2K\right\}.$$

Rewriting that $J_{u_1}(u_2) := J(u_1, u_2)$ for $u_1 \in S(c_1)$ and

$$J_{u_1}(u_2) = J_{u_1}(0) + \frac{a}{2}\|\nabla u_2\|_2^2 + \frac{b}{4}\|\nabla u_2\|_2^4 - \frac{\mu_2}{p_2}\|u_2\|_{p_2}^{p_2} - \beta\int_{\mathbb{R}^3} |u_1|^{r_1}|u_2|^{r_2}\,dx.$$

Lemma 4.1. *Assume that (H_2) holds. There exists a continuous function K from $S(c_1)$ to $\mathbb{R}$, $u_1 \mapsto K(u_1)$, such that*

$$\sup_{T_{K(u_1)}} J_{u_1}(u_2) < \inf_{B_{K(u_1)}} J_{u_1}(u_2), \quad \textit{for all } u_1 \in S(c_1).$$

The function K is bounded, and it is bounded away from 0 on bounded subsets of $S(c_1)$.

Proof. Fixing $u_1 \in S(c_1)$, for $u_2 \in T_K$, we have that,

$$J_{u_1}(u_2) \leq J_{u_1}(0) + \frac{a}{2}\|\nabla u_2\|_2^2 + \frac{b}{4}\|\nabla u_2\|_2^4 \leq J_{u_1}(0) + \frac{aK(u_1)}{2} + \frac{bK(u_1)^2}{4}.$$

For $u_2 \in B_K$, $\gamma' := \frac{3(r_2 q' - 2)}{2q'}$, where q' is defined in (2.1). Using the Gagliardo-Nirenberg inequality and (2.2), we obtain,

$$\begin{aligned}J_{u_1}(u_2) &\geq J_{u_1}(0) + aK(u_1) + bK(u_1)^2 - \frac{\mu_2}{p_2}C\|\nabla u_2\|_2^{p_2\gamma_{p_2}}\|u_2\|_2^{p_2(1-\gamma_{p_2})} - C\beta\|u_1\|_{r_1 q}^{r_1}\|u_2\|_{r_2 q'}^{r_2}\\ &\geq J_{u_1}(0) + aK(u_1) + bK(u_1)^2 - C_1 K(u_1)^{\frac{p_2\gamma_{p_2}}{2}} - C_2\|u_1\|_{r_1 q}^{r_1} K(u_1)^{\frac{\gamma'}{2}}.\end{aligned}$$

Observe that $C_1 K(u_1)^{\frac{p_2\gamma_{p_2}}{2}} \leq \frac{a}{8}K(u_1)$ if $K(u_1) > 0$ is sufficiently small for $\frac{p_2\gamma_{p_2}}{2} > 1$. Similarly, $C_2\|u_1\|_{r_1 q}^{r_1} K(u_1)^{\frac{\gamma'}{2}} \leq \frac{a}{8}K(u_1)$ if $K(u_1) > 0$ is sufficiently small for $\frac{\gamma'}{2} > 1$, provided that $q < \frac{6}{10-3r_2}$. We can choose q satisfying this inequality and (2.1) because

$$\frac{6}{10 - 3r_2} > \max\left\{\frac{2}{r_1}, \frac{2^*}{2^* - r_2}\right\},$$

which is a consequence of $r_1 + r_2 > \frac{14}{3}$ and $2 < r_2 < \frac{10}{3}$. More precisely, let $K : S(a_1) \to \mathbb{R}^+$ satisfy

$$K(u_1) \leq \min\left\{\left(\frac{a}{8C_1}\right)^{\frac{2}{p_2\gamma_{p_2}-2}}, \left(\frac{a}{8C_2\|u_1\|_{r_1 q}^{r_1}}\right)^{\frac{2}{\gamma'-2}}\right\}. \tag{4.1}$$

For $u_2 \in B_{K(u_1)}$, we have

$$\begin{aligned} J_{u_1}(u_2) &\geq J_{u_1}(0) + aK(u_1) + bK^2(u_1) - \frac{a}{8}K(u_1) - \frac{a}{8}K(u_1) \\ &> J_{u_1}(0) + \frac{a}{2}K(u_1) + \frac{b}{4}K^2(u_1) \geq \sup_{T_{K(u_1)}} J_{u_1}(u_2). \end{aligned} \tag{4.2}$$

Clearly, we define a continuous function $K : S(c_1) \to \mathbb{R}^+$ that satisfies (4.1) and is bounded away from 0 on bounded subsets of $S(c_1)$. In fact, the right-hand side of (4.1) can serve as a definition. By (4.1), K is also bounded from above. □

Now, we denote

$$T(u_1) := T_{K(u_1)}, \quad B(u_1) := B_{K(u_1)},$$

and

$$B := \{(u_1, u_2) : u_1 \in S(c_1), u_2 \in B(u_1)\}.$$

It follows from the assumption (p_1) in Theorem 1.2 and Proposition 2.5 that there exists a ground state solution $\underline{u} \in S(c_1)$ for problem (1.3) satisfying

$$J(\underline{u}, 0) = m(c_1, \mu_1) = I_{\mu_1}(\underline{u}) = \min_{u \in S(c_1)} J(u, 0) < 0.$$

Lemma 4.2. *Assume that (H_2) holds. There exist $\bar{v} \in T(\underline{u})$ and $\bar{w} \in S(c_2) \backslash T_{2K(\underline{u})}$ such that*

$$\max\{J(\underline{u}, \bar{v}), J(\underline{u}, \bar{w})\} < \inf_{(u_1, u_2) \in B} J(u_1, u_2).$$

Proof. Since $J(\underline{u}, u_2) \to J(\underline{u}, 0)$ as $\|\nabla u_2\|_2 \to 0$, to obtain $\bar{v} \in T(\underline{u})$, we claim that $J(\underline{u}, 0) < \inf_B J$. The functional $J(\cdot, 0) : S(c_1) \to \mathbb{R}$ is coercive because $2 < p_1 < \frac{14}{3}$. Choose $R > 0$ such that $J(u_1, 0) \geq J(\underline{u}, 0) + 1$ if $\|\nabla u_1\|_2 \geq R$. It follows from (4.2) and $(u_1, u_2) \in B$ with $\|\nabla u_1\|_2 \geq R$ that

$$J(u_1, u_2) \geq J(u_1, 0) + \frac{3}{4}K(u_1) > J(\underline{u}, 0) + 1. \tag{4.3}$$

For $(u_1, u_2) \in B$ with $\|\nabla u_1\|_2 \leq R$, there holds,

$$J(u_1, u_2) \geq J(u_1, 0) + \frac{3}{4}K(u_1) \geq J(\underline{u}, 0) + \frac{3}{4}\varepsilon, \tag{4.4}$$

where $\varepsilon := \inf_{\|\nabla u_1\|_2 \leq R} K(u_1) > 0$ from Lemma 4.1. By (4.3) and (4.4), the claim holds.

To find $\bar{w} \in S(c_2) \backslash T_{2K(\underline{u})}$ as required, consider any $u \in S(c_2)$. Clearly, $t \star u \in S(c_2)$ for every $t > 0$, and $\|\nabla(t \star u)\|_2 \to \infty$ as $t \to \infty$. Since $p_2 > \frac{14}{3}$, fixing an arbitrary $u \in S(c_2)$, we see that $J(\underline{u}, (t \star u)) \to -\infty$ as $t \to \infty$. □

As a result of Lemma 4.2, the set

$$\Gamma_1 := \Big\{g' \in \mathcal{C}([0,1], \mathcal{S}_r) : g'(0) = (v_1, v_2)\,,\, g'(1) = (w_1, w_2)\,,\, v_2 \in T\,(v_1)\,,\, w_2 \notin T_{2K(w_1)},$$
$$\max\left\{J\,(v_1, v_2)\,, J\,(w_1, w_2)\right\} < \inf_B J\Big\},$$

is nonempty.

Lemma 4.3. $\bar{\gamma}\,(c_1, c_2) := \inf_{g' \in \Gamma_1} \max_{t \in [0,1]} J(g'(t)) \geq \inf_B J$.

Proof. It is sufficient to show that for each $g'(t) := (g_1'(t), g_2'(t)) \in \Gamma_1$, there exists $t \in [0,1]$ such that $g'(t) \in B$. Consider the map $\alpha : [0,1] \to \mathbb{R}$ defined by $t \to \|\nabla g_2'(t)\|_2^2 - 2K\,(g_1'(t))$. This map satisfies

$$\alpha(0) = \|\nabla v_2\|_2^2 - 2K\,(v_1) \leq K\,(v_1) - 2K\,(v_1) < 0,$$

and $\alpha(1) = \|\nabla w_2\|_2^2 - 2K\,(w_1) > 0$. Thus, there exists $t \in [0,1]$ such that $\alpha(t) = 0$, which implies that $g'(t) \in B$. □

Lemma 4.4. *Assume that the conditions of Theorem 1.2 hold. Then, we have*

$$\bar{\gamma}\,(c_1, c_2) \leq m(c_1, \mu_1) + m(c_2, \mu_2).$$

Proof. By Proposition 2.5 (iii), there exists $\bar{u} \in V\,(c_2)$ such that

$$\min_{u \in V(c_2)} I_{\mu_2}(u) = \max_{t \in \mathbb{R}} I_{\mu_2}(t \star \bar{u}) = m(c_2, \mu_2) = I_{\mu_2}(0 \star \bar{u}) = I_{\mu_2}(\bar{u}) = J(0, \bar{u}). \tag{4.5}$$

Next, we consider the path $h : [0,1] \to \mathcal{S}_r$ defined by $h(t) = (\underline{u}, h_s(t))$, where

$$h_s(t)(x) = \mathrm{e}^{\frac{s(2t-1)3}{2}}\,\bar{u}\left(\mathrm{e}^{s(2t-1)}x\right).$$

Here, $s > 0$ is chosen sufficiently large so that

$$h_s(0)(\cdot) = \mathrm{e}^{\frac{-3s}{2}}\,\bar{u}\left(\mathrm{e}^{-s}\cdot\right) \in T(\underline{u}), \quad h_s(1)(\cdot) = \mathrm{e}^{\frac{3s}{2}}\,\bar{u}\left(\mathrm{e}^{s}\cdot\right) \notin T_{2K(\underline{u})},$$

and

$$\max\left\{J\left(\underline{u}, h_s(0)\right), J\left(\underline{u}, h_s(1)\right)\right\} < \inf_B J.$$

Therefore, h belongs to Γ_1. Utilizing (4.5) and $\beta \geq 0$, we get

$$\max_{t \in [0,1]} J(h(t)) \leq J(\underline{u}, 0) + \max_{t \in [0,1]} J\left(0, h_s(t)\right) = m(c_1, \mu_1) + m(c_2, \mu_2).$$

This completes the proof. □

Lemma 4.5. *Assume that* (H_2) *holds. There exists a Palais-Smale sequence* $\{(u_1^n, u_2^n)\} \subset \mathcal{S}_r$ *for* J *at the level* $\bar{\gamma}(c_1, c_2)$ *that satisfies* $\{u_1^n\}^- \to 0$, $\{u_2^n\}^- \to 0$ *in* H_r, *and the additional property that* $P(u_1^n, u_2^n) \to 0$. *Moreover, the sequence* $\{(u_1^n, u_2^n)\}$ *is bounded.*

Proof. The existence of the sequence $\{(u_1^n, u_2^n)\}$ can be referenced in Lemma 3.2. Here, we only provide the proof of boundedness. Given that $P(u_1^n, u_2^n) = 0$, for any $\varepsilon > 0$, we have:

$$\begin{aligned} J(u_1^n, u_2^n) &= \frac{a}{2}\sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + \frac{b}{4}\sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 - \sum_{i=1}^{2} \frac{\mu_i}{p_i}\|u_i^n\|_{p_i}^{p_i} - \beta \int_{\mathbb{R}^3} |u_1^n|^{r_1}|u_2^n|^{r_2}\,dx \\ &= \frac{(1+\epsilon)a}{4}\sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + \frac{\epsilon b}{4}\sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 + \delta_1(\epsilon)\|u_1^n\|_{p_1}^{p_1} + \delta_2(\epsilon)\|u_2^n\|_{p_2}^{p_2} \\ &+ \beta\delta_3(\epsilon)\int_{\mathbb{R}^3} |u_1^n|^{r_1}|u_2^n|^{r_2}\,dx + \frac{(1-\epsilon)}{4}P(u_1^n, u_2^n), \end{aligned}$$

where

$$\delta_1(\epsilon) = \frac{(1-\epsilon)\mu_1\gamma_{p_1}}{4} - \frac{\mu_1}{p_1}, \quad \delta_2(\epsilon) = \frac{(1-\epsilon)\mu_2\gamma_{p_2}}{4} - \frac{\mu_2}{p_2}, \quad \delta_3(\epsilon) = \frac{(1-\epsilon)r\gamma_r}{4} - 1.$$

Note that the coefficients satisfy $\delta_1(\epsilon) < 0$ and $\delta_2(\epsilon), \delta_3(\epsilon) > 0$ for sufficiently small $\varepsilon > 0$. Although $\delta_1(\epsilon) < 0$, the term $\|u_1^n\|_{p_1}^{p_1}$ is controlled by $\sum_{i=1}^{2} \|\nabla u_i^n\|_2^4$ because $p_1 < \frac{14}{3}$. Hence, we conclude that J is coercive. Consequently, the sequence $\{(u_1^n, u_2^n)\} \subset \mathcal{S}_r$ is bounded. □

Proof of Theorem 1.2. By Lemmas 2.2 and 4.5, we can assume that $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ in H_r, where $u_1 \geq 0$ and $u_2 \geq 0$. As shown in Lemma 3.3, we have $J(u_1, u_2) = \bar{\gamma}(c_1, c_2)$. To establish strong convergence, it suffices to show, according to Lemmas 2.4 and 2.2 (iv), that $u_1 \neq 0$ and $u_2 \neq 0$.

We first claim that: if $\bar{\gamma}(c_1, c_2) < 0$, *then* $u_1 \neq 0$ *and* $u_2 \neq 0$.

For contradiction, that at least one of u_1 or u_2 is zero. Then, by Lemma 2.1,

$$(u_1^n, u_2^n) \to (u_1, u_2) \text{ in } L^p(\mathbb{R}^3) \times L^q(\mathbb{R}^3) \text{ for } p, q \in (2, 2^*) \text{ and } \beta \int_{\mathbb{R}^3} |u_1^n|^{r_1}|u_2^n|^{r_2}\,dx \to 0.$$

For the sequence $\{(u_1^n, u_2^n)\}$ satisfying $P(u_1^n, u_2^n) \to 0$, we deduce that

$$a\sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + b\sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 - \sum_{i=1}^{2} \mu_i\gamma_{p_i}\|u_i^n\|_{p_i}^{p_i} = o_n(1).$$

By the weak lower semi-continuity, we have

$$\begin{aligned} J\left(u_1^n, u_2^n\right) &= \frac{a}{2}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^2 + \frac{b}{4}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^4 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}\|u_i^n\|_{p_i}^{p_i} - \beta\int_{\mathbb{R}^3}|u_1^n|^{r_1}|u_2^n|^{r_2}dx \\ &= \frac{a}{4}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^2 - \sum_{i=1}^{2}\mu_i\gamma_{p_i}\left(\frac{1}{p_i\gamma_{p_i}} - \frac{1}{4}\right)\|u_i^n\|_{p_i}^{p_i} + o_n(1) \\ &\geq \frac{a}{4}\sum_{i=1}^{2}\|\nabla u_i\|_2^2 - C_1\|u_1\|_{p_1}^{p_1} + C_2\|u_2\|_{p_2}^{p_2}, \end{aligned} \tag{4.6}$$

where $C_1 > 0$ and $C_2 > 0$. We now distinguish three cases.

Case 1. $(u_1 = u_2 = 0)$**:** From (4.6), we obtain $J\left(u_1^n, u_2^n\right) \geq 0$. Since we have assumed that $\gamma\left(c_1, c_2\right) < 0$, this case cannot occur.

Case 2. $(u_1 = 0$ **and** $u_2 \neq 0)$**:** By Lemmas 2.2, 2.4, we have $\lambda_2 < 0$, and hence $u_2^n \to u_2 \in S_r\left(c_2\right)$. From (4.6), we get

$$0 > \bar{\gamma}(c_1, c_2) = J\left(u_1^n, u_2^n\right) \geq \frac{a}{4}\|\nabla u_2\|_2^2 + C_2\|u_2\|_{p_2}^{p_2} > 0, \quad \text{as} \quad n \to \infty. \tag{4.7}$$

This results in a contradiction.

Case 3. $(u_1 \neq 0$ **and** $u_2 = 0)$**:** Since $u_2 = 0$ and $J(u_1, u_2) = \bar{\gamma}\left(c_1, c_2\right)$, we have

$$\bar{\gamma}(c_1, c_2) = J\left(u_1, u_2\right) = J\left(u_1, 0\right) = I_{\mu_1}\left(u_1\right).$$

We know u_1 satisfies

$$-\left(a + b\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)\Delta u = \lambda_1 u + \mu_1|u|^{p_1-2}u.$$

For $2 < p_1 \leq \frac{10}{3}$, u_1 is a positive ground state solution by Propsition 2.5 (i). Then $m(c_1, \mu_1) = I_{\mu_1}(u_1)$. From Lemmas 4.1, 4.3 and the definitions of B, Γ_1, we know that

$$\bar{\gamma}\left(c_1, c_2\right) \geq \inf_B J > J(u_1, 0) = I_{\mu_1}(u_1) = m(c_1, \mu_1), \tag{4.8}$$

which contradicts $\bar{\gamma}\left(c_1, c_2\right) = m(c_1, \mu_1)$. When $\frac{10}{3} < p_1 < \frac{14}{3}$, u_1 can be characterized as either a high energy solution or a ground state solution. If u_1 is ground state solution, we can get a contradiction similar to (4.8). If u_1 is high energy solution, we have a contradiction as $0 < I_\mu(u_1) = \bar{\gamma}(c_1, c_2) < 0$. Thus, the claim holds.

In view of Lemmas 2.2, 4.4 and 4.5, to establish the theorem, it is enough to prove that $m(c_1, \mu_1) + m(c_2, \mu_2) < 0$. Note also that $u_1 > 0$ and $u_2 > 0$ follow directly from the strong maximum principle. □

Proof of Corollary 1.3. The Corollary is a straightforward consequence of Theorem 1.2 and Proposition 2.5. □

5 The proof of Theorem 1.4

In this section, we first consider the case that (H_3). Recalling Proposition 2.5 (i), for $2 < p_1$, $p_2 \leq \frac{10}{3}$, there exist u^1 and u^2 such that

$$m(c_1, \mu_1) = I_{\mu_1}(u^1) \quad \text{and} \quad m(c_2, \mu_2) = I_{\mu_2}(u^2).$$

Lemma 5.1. *Assume that* (H_3) *holds. There exist* $\beta_1 := \beta_1(c_1, c_2)$ *and* $\rho_* := \rho_*(c_1, c_2) > \|\nabla u^1\|_2^2 + \|\nabla u^2\|_2^2$ *such that*

$$J(u_1, u_2) > 0 \text{ on } A\left(2\rho_*\right) \setminus A\left(\rho_*\right) \quad \text{for } 0 < \beta < \beta_1,$$

where $A(\rho_*) = \left\{(u_1, u_2) \in \mathcal{S}_r : \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2 < \rho_*\right\}$ *for* $\rho_* > 0$.

Proof. Recalling the proof of Lemma 3.1, we can take a sufficiently large ρ_* such that

$$\rho_* > \|\nabla u^1\|_2^2 + \|\nabla u^2\|_2^2,$$

and

$$\sum_{i=1}^{2} \frac{\mu_i}{p_i} C_i \left(\rho_*\right)^{\frac{p_i \gamma_{p_i} - 4}{2}} \leq \frac{b}{32}. \tag{5.1}$$

Next, we choose $\beta_1 > 0$ to be sufficiently small, such that

$$\beta_1 C_3 \left(2\rho_*\right)^{\frac{2^* - 4}{2}} \leq \frac{b}{32}. \tag{5.2}$$

The lemma follows directly from (5.1) and (5.2). □

Now we can set

$$\gamma'(c_1, c_2) := \inf_{A(2\rho_*)} J(u_1, u_2).$$

The following lemma plays a crucial role in overcoming compactness.

Lemma 5.2. *Assume that* (H_3) *holds. Then, for any* $0 < \beta < \beta_1$, *the following statements are true:*

(i) $\gamma'(c_1, c_2) < m(c_1, \mu_1) + m(c_2, \mu_2) < 0$.

(ii) $\gamma'(c_1, c_2) \leq \gamma'(c_1', c_2')$, *for all* $0 < c_1' \leq c_1$, $0 < c_2' \leq c_2$.

Proof. (*i*) From Lemma 5.1, we know that $\left(u^1, u^2\right) \in A(\rho_*)$. Furthermore, using Proposition 2.5 (*i*) and the fact that $\beta > 0$, we deduce that

$$\gamma'(c_1, c_2) \leq J(u^1, u^2) = I_{\mu_1}(u^1) + I_{\mu_2}(u^2) - \beta \int_{\mathbb{R}^3} \left|u^1\right|^{r_1} \left|u^2\right|^{r_2} dx < m(c_1, \mu_1) + m(c_2, \mu_2) < 0.$$

(*ii*) To prove this, we need to show that for any $\varepsilon > 0$, $\gamma'(c_1, c_2) \leq \gamma'(c_1', c_2') + \varepsilon$, for all $0 < c_1' \leq c_1$ and $0 < c_2' \leq c_2$. Let $\varphi(x) \in C_c^\infty\left(\mathbb{R}^N\right)$ be a cut-off function such that

$$0 \leq \phi(x) \leq 1 \quad \text{and} \quad \phi(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

By the definition of $\gamma'(c_1', c_2')$ and Lemma (5.1), there exists $(u_1, u_2) \in A\left(\rho_*\right)$ such that

$$J(u_1, u_2) \leq \gamma'(c_1', c_2') + \frac{\varepsilon}{2}. \tag{5.3}$$

For any $\delta > 0$, we define $(u_{\delta_1}(x), u_{\delta_2}(x)) := (u_1\phi(\delta x), u_2\phi(\delta x))$. Since $(u_{\delta_1}, u_{\delta_2}) \to (u_1, u_2)$ in H_r as $\delta \to 0^+$, there exists a sufficiently small δ such that

$$J\left(u_{\delta_1}, u_{\delta_2}\right) \leq J(u_1, u_2) + \frac{\varepsilon}{4} \quad \text{and} \quad \|\nabla u_{\delta_1}\|_2^2 + \|\nabla u_{\delta_2}\|_2^2 \leq \frac{3}{2}\rho_*. \tag{5.4}$$

Let $\varphi(x) \in C_c^\infty\left(\mathbb{R}^3\right)$ such that $\text{supp}(\varphi) \subset \left\{x \in \mathbb{R}^3 : \frac{4}{\delta} \leq |x| \leq 1 + \frac{4}{\delta}\right\}$ and set

$$(\tilde{u}_1, \tilde{u}_2) = \left(\frac{\sqrt{c_1 - \|u_{\delta_1}\|_2^2}}{\|\varphi\|_2}\varphi, \frac{\sqrt{c_2 - \|u_{\delta_2}\|_2^2}}{\|\varphi\|_2}\varphi\right).$$

Noting that, for any $s \leq 0$,

$$\text{supp}\left(u_{\delta_1}\right) \cap \text{supp}(s \star \tilde{u}_1) = \emptyset \quad \text{and} \quad \text{supp}\left(u_{\delta_2}\right) \cap \text{supp}(s \star \tilde{u}_2) = \emptyset.$$

As $s \to -\infty$, we have

$$J(s \star \tilde{u}_1, s \star \tilde{u}_2) \to 0 \quad \text{and} \quad \|\nabla s \star \tilde{u}_1\|_2^2 + \|\nabla s \star \tilde{u}_2\|_2^2 \leq \frac{\varepsilon}{12\rho_* b}. \tag{5.5}$$

It follows that

$$\left(u_{\delta_1} + s \star \tilde{u}_1, u_{\delta_2} + s \star \tilde{u}_2\right) \in A\left(2\rho_*\right),$$

and by (5.3)-(5.5), for $s<0$ large enough, we have

$$\begin{aligned}\gamma'(c_1,c_2) &\leq J\left(u_{\delta_1}+s\star\tilde{u}_1, u_{\delta_2}+s\star\tilde{u}_2\right)\\ &= J\left(u_{\delta_1},u_{\delta_2}\right)+J(s\star\tilde{u}_1,s\star\tilde{u}_2)+\frac{b}{2}\sum_{i=1}^{2}\|\nabla u_{\delta_i}\|_2^2\,\|\nabla s\star\tilde{u}_i\|_2^2\\ &\leq J(u_1,u_2)+\frac{\varepsilon}{4}+\frac{\varepsilon}{8}+\frac{\varepsilon}{8}\leq\gamma'(c_1',c_2')+\varepsilon.\end{aligned}$$

The proof is completed. □

Lemma 5.3. *Assume that (H_3) holds. For any $0<\beta<\beta_1$, there exists*

$$\mu_*:=\mu_*(a,b,c_1,c_2,p_1,p_2,\beta,\rho)$$

such that for $\mu_1,\mu_2<\mu_*$ and $(u_1,u_2)\in\mathcal{S}_r$, the function $\Phi_{u_1,u_2}(s)$, defined in (1.4) has two critical points $t_{u_1,u_2}<\tau_{u_1,u_2}$ and two zeros $c_{u_1,u_2}<d_{u_1,u_2}$ with $t_{u_1,u_2}<c_{u_1,u_2}<\tau_{u_1,u_2}<d_{u_1,u_2}$. Moreover, for $s\in\mathbb{R}$,

(i) If $(s\star u_1,s\star u_2)\in\mathcal{P}$, then either $s=t_{u_1,u_2}$ or $s=\tau_{u_1,u_2}$.

(ii) $\|\nabla s\star u_1\|_2^2+\|\nabla s\star u_2\|_2^2\leq\rho_*$ for every $s\leq c_{u_1,u_2}$ and

$$J(t_{u_1,u_2}\star u_1,t_{u_1,u_2}\star u_2)=\min\left\{J(s\star u_1,s\star u_2):\|\nabla s\star u_1\|_2^2+\|\nabla s\star u_2\|_2^2\leq\rho_*\right\}<0.$$

(iii) We have $J\left(\tau_{u_1,u_2}\star u_1,\tau_{u_1,u_2}\star u_2\right)=\max\{J(s\star u_1,s\star u_2):s\in\mathbb{R}\}$.

Proof. (i) Since $p_i\gamma_{p_1}<2$ for $i=1$, 2, and $r=2^*$, it is evident that $\Phi_{u_1,u_2}(-\infty)=0^-$ and $\Phi_{u_1,u_2}(+\infty)=-\infty$. By Lemma 5.1, we know that $\Phi_{u_1,u_2}(s)$ has at least two critical points $t_{u_1,u_2}<\tau_{u_1,u_2}$, where t_{u_1,u_2} is a local minimum point of $\Phi_{u_1,u_2}(s)$ at negative level and τ_{u_1,u_2} is a global maximum point at positive level. On the other hand, it is standard to prove that $\Phi_{u_1,u_2}(s)$ has at most two critical points as in [20, Lemma 4.5]. The (ii) and (iii) follow from Lemma 5.1 and (i). □

Proof of Theorem 1.4. Consider a minimizing sequence $\{(u_1^n,u_2^n)\}\subset\mathcal{S}_r$ for $J|_{A(2\rho_*)}$. By Lemma 5.3, we have $\|\nabla t_{u_1^n,u_2^n}\star u_1^n\|_2^2+\|\nabla t_{u_1^n,u_2^n}\star u_2^n\|_2^2\leq\rho_*$, and the sequence $\left\{t_{u_1^n,u_2^n}\star u_1^n,t_{u_n,u_2^n}\star u_2^n\right\}$ remains a minimizing sequence for $J|_{A(2\rho_*)}$. According to [9, Theorem 4.1], there exists a new minimizing sequence, still denoted by $\{(u_1^n,u_2^n)\}\subset A\left(2\rho_*\right)$, such that

$$J\left(u_1^n,u_2^n\right)\to\gamma'(c_1,c_2),\quad P\left(u_1^n,u_2^n\right)\to 0,\quad J'|_{\mathcal{S}_r}\left(u_1^n,u_2^n\right)\to 0,\qquad\text{as } n\to\infty. \tag{5.6}$$

Given that $J'|_{\mathcal{S}_r}(u_1^n, u_2^n) \to 0$, there exist sequences $\{\lambda_1^n\} \subset \mathbb{R}$ and $\{\lambda_2^n\} \subset \mathbb{R}$ such that

$$\begin{aligned} a\sum_{i=1}^{2}\int_{\mathbb{R}^3}\nabla u_i^n\nabla\varphi_i dx + b\sum_{i=1}^{2}\|\nabla u_i^n\|_2^2\int_{\mathbb{R}^3}\nabla u_i^n\nabla\varphi_i dx - \sum_{i=1}^{2}\mu_i\int_{\mathbb{R}^3}|u_i^n|^{p_i-2}u_i^n\varphi_i dx \\ - \beta r_1\int_{\mathbb{R}^3}|u_1^n|^{r_1-2}|u_2^n|^{r_2}u_1^n\varphi_1 dx - \beta r_2\int_{\mathbb{R}^3}|u_1^n|^{r_1}|u_2^n|^{r_2-2}u_2^n\varphi_2 dx \\ = \int_{\mathbb{R}^3}(\lambda_1^n u_1^n\varphi_1 + \lambda_2^n u_2^n\varphi_2)dx + o_n(1), \end{aligned} \tag{5.7}$$

for any $(\varphi_1, \varphi_2) \in H_r$. Taking $(u_1^n, 0)$ and $(0, u_2^n)$ as test functions, we have

$$\begin{cases} \lambda_1^n c_1 + o_n(1) = a\|\nabla u_1^n\|_2^2 + b\|\nabla u_1^n\|_2^4 - \mu_1\|u_1^n\|_{p_1}^{p_1}, \\ \lambda_2^n c_2 + o_n(1) = a\|\nabla u_2^n\|_2^2 + b\|\nabla u_2^n\|_2^4 - \mu_2\|u_2^n\|_{p_2}^{p_2}. \end{cases}$$

Since the sequence $\{u_1^n, u_2^n\} \subset A(2\rho_*)$ is bounded, we suppose that $\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_i^n|^2 dx = A_i \geq 0$. Without loss of generality, let us assume that, up to a subsequence, $(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2) \in \mathbb{R}^2$, $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2) \in H_r$ and $(u_1^n, u_2^n) \to (u_1, u_2)$ in $L^p(\mathbb{R}^3) \times L^q(\mathbb{R}^3)$ for any p, $q \in (2, 2^*)$. Then, we know that,

$$\begin{cases} -(a + bA_1)\Delta u_1 = \lambda_1 u_1 + \mu_1|u_1|^{p_1-2}u_1 + \beta r_1|u_1|^{r_1-2}u_1|u_2|^{r_2}, \\ -(a + bA_2)\Delta u_2 = \lambda_2 u_2 + \mu_2|u_2|^{p_2-2}u_2 + \beta r_2|u_1|^{r_1}|u_2|^{r_2-2}u_2. \end{cases} \tag{5.8}$$

From (5.8), we have

$$0 = P_A(u_1, u_2) := a\sum_{i=1}^{2}\|\nabla u_i\|_2^2 + b\sum_{i=1}^{2}A_i\|\nabla u_i\|_2^2 - \sum_{i=1}^{2}\mu_i\gamma_{p_i}\|u_i\|_{p_i}^{p_i} - \beta 2^*\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx.$$

Let $(\bar{u}_1^n, \bar{u}_2^n) := (u_1^n - u_1, u_2^n - u_2)$. Then $\bar{u}_1^n \to 0$ in $L^{p_1}(\mathbb{R}^3)$, $\bar{u}_2^n \to 0$ in $L^{p_2}(\mathbb{R}^3)$ and we have

$$\begin{aligned} P(u_1^n, u_2^n) &= P_A(u_1, u_2) + a\sum_{i=1}^{2}\|\nabla\bar{u}_i^n\|_2^2 + b\sum_{i=1}^{2}A_i\|\nabla\bar{u}_i^n\|_2^2 - \beta 2^*\int_{\mathbb{R}^3}|\bar{u}_1^n|^{r_1}|\bar{u}_2^n|^{r_2}dx \\ &= a\sum_{i=1}^{2}\|\nabla\bar{u}_i^n\|_2^2 + b\sum_{i=1}^{2}A_i\|\nabla\bar{u}_i^n\|_2^2 - \beta 2^*\int_{\mathbb{R}^3}|\bar{u}_1^n|^{r_1}|\bar{u}_2^n|^{r_2}dx + o_n(1). \end{aligned} \tag{5.9}$$

From (2.2), (5.9) and Lemma 2.1, we obtain

$$\begin{aligned} a\sum_{i=1}^{2}\|\nabla\bar{u}_i^n\|_2^2 &\leq a\sum_{i=1}^{2}\|\nabla\bar{u}_i^n\|_2^2 + b\sum_{i=1}^{2}A_i\|\nabla\bar{u}_i^n\|_2^2 = \beta 2^*\int_{\mathbb{R}^3}|\bar{u}_1^n|^{r_1}|\bar{u}_2^n|^{r_2}dx + o_n(1) \\ &\leq \beta 2^* S^{-\frac{2^*}{2}}\left(\sum_{i=1}^{2}\|\nabla\bar{u}_i^n\|_2^2\right)^{\frac{2^*}{2}} + o_n(1). \end{aligned} \tag{5.10}$$

Up to a subsequence, we assume that $\sum_{i=1}^2 \|\nabla \bar{u}_i^n\|_2^2 \to l \geq 0$. According to (5.10), we have $l = 0$ or $l \geq \left(\frac{a}{\beta 2^*}\right)^{\frac{1}{2}} S^{\frac{3}{2}}$. If $l \geq \left(\frac{a}{\beta 2^*}\right)^{\frac{1}{2}} S^{\frac{3}{2}}$, then from (5.6), (5.10), and Lemma 2.1, we conclude

$$
\begin{aligned}
\gamma'(c_1, c_2) &= \lim_{n\to\infty} J\left(u_1^n, u_2^n\right) = J(u_1, u_2) + \lim_{n\to\infty} J\left(\bar{u}_1^n, \bar{u}_2^n\right) + \frac{b}{2}\sum_{i=1}^2 \|\nabla u_i^n\|_2^2 \|\nabla u_i\|_2^2 \\
&\geq J(u_1, u_2) + \lim_{n\to\infty} J\left(\bar{u}_1^n, \bar{u}_2^n\right) \\
&\geq \gamma'\left(\|u_1\|_2^2, \|u_2\|_2^2\right) + \lim_{n\to\infty}\left(\frac{a}{2}\sum_{i=1}^2 \|\nabla \bar{u}_i^n\|_2^2 + \frac{b}{4}\sum_{i=1}^2 \|\nabla \bar{u}_i^n\|_2^4 - \beta\int_{\mathbb{R}^3} |\bar{u}_1^n|^{r_1} |\bar{u}_2^n|^{r_2}\, dx\right) \\
&\geq \gamma'\left(\|u_1\|_2^2, \|u_2\|_2^2\right) + \lim_{n\to\infty}\left(\frac{a}{2}\sum_{i=1}^2 \|\nabla \bar{u}_i^n\|_2^2 - \beta\int_{\mathbb{R}^3} |\bar{u}_1^n|^{r_1} |\bar{u}_2^n|^{r_2}\, dx\right) \\
&\geq \gamma'\left(\|u_1\|_2^2, \|u_2\|_2^2\right) + \lim_{n\to\infty}\left[\frac{a}{2}\sum_{i=1}^2 \|\nabla \bar{u}_i^n\|_2^2\left(1 - \frac{2}{a}\beta S^{-\frac{2^*}{2}}\left(\sum_{i=1}^2 \|\nabla \bar{u}_i^n\|_2^2\right)^2\right)\right].
\end{aligned}
$$

By $\sum_{i=1}^2 \|\nabla \bar{u}_i^n\|_2^2 \leq \rho_*$, there exists $\beta_* < \beta_1$ such that $\left(\frac{a}{2\beta_*} S^{\frac{2^*}{2}}\right)^{\frac{1}{2}} \geq \rho_*$. Then

$$
\left(1 - \frac{2}{a}\beta S^{-\frac{2^*}{2}}\left(\sum_{i=1}^2 \|\nabla \bar{u}_i\|_2^2\right)^2\right) \geq 0,
$$

when $\beta < \beta_*$, which contradicts with (ii) of Lemma 5.2. Thus, $\sum_{i=1}^2 \|\nabla \bar{u}_i^n\|_2^2 \to 0$, as $n \to \infty$. Then $(u_1^n, u_2^n) \to (u_1, u_2)$ in $D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ and (u_1, u_2) is a solution to equations (1.1).

Finally, we will prove that $(u_1^n, u_2^n) \to (u_1, u_2)$ in H_r. Taking (u_1^n, u_2^n) as the test function in (5.7), we obtain

$$
\langle J'\left(u_1^n, u_2^n\right), \left(u_1^n, u_2^n\right)\rangle = \lambda_1^n c_1 + \lambda_2^n c_2 + o_n(1).
$$

Given that $P\left(u_1^n, u_2^n\right) \to 0$, $(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$, we have

$$
\lambda_1 c_1 + \lambda_2 c_2 = \lambda_1^n c_1 + \lambda_2^n c_2 + o_n(1) = \sum_{i=1}^2 \mu_i\left(\gamma_{p_i} - 1\right)\|u_i^n\|_{p_i}^{p_i} < 0.
$$

Since $\lambda_1 c_1 + \lambda_2 c_2 < 0$, at least one of λ_1 and λ_2 is negative. Next, we consider three possible conditions.

Case 1. ($\lambda_1 < 0$ and $\lambda_2 < 0$): Using the fact that

$$
\langle J'\left(u_1^n, u_2^n\right) - \lambda_1^n\left(u_1^n, 0\right), \left(u_1^n, 0\right)\rangle \to \langle J'\left(u_1, u_2\right) - \lambda_1\left(u_1, 0\right), \left(u_1, 0\right)\rangle = 0,
$$

we have

$$
\begin{cases}
\lambda_1\|u_1^n\|_2^2 + o_n(1) = a\|\nabla u_1^n\|_2^2 + b\|\nabla u_1^n\|_2^4 - \mu_1\|u_1^n\|_{p_1}^{p_1}, \\
\lambda_1\|u_1\|_2^2 = a\|\nabla u_1\|_2^2 + b\|\nabla u_1\|_2^4 - \mu_1\|u_1\|_{p_1}^{p_1}.
\end{cases}
$$

Since Lemma 2.1, $\lambda_1^n \to \lambda_1 < 0$, $\|u_1^n\|_{p_1}^{p_1} \to \|u_1\|_{p_1}^{p_1}$, and $u_1^n \to u_1$ in $D^{1,2}(\mathbb{R}^3)$, we get $\|u_1^n\|_2^2 \to \|u_1\|_2^2$, leading to strong convergence. The case where $\lambda_2 < 0$ is treated similarly.

Case 2. ($\lambda_1 < 0$ and $\lambda_2 \geq 0$): Using the method of Case 1, it can be concluded that $u_1^n \to u_1$ in $H_r^1\left(\mathbb{R}^3\right)$ and $u_1 \in S_r\left(c_1\right)$. Assume, by contradiction, that $\lambda_2 \geq 0$, then

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u_2|^2\,dx\right)\Delta u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2-2} u_2 + \beta r_2 |u_1|^{r_1} |u_2|^{r_2-2} u_2 \geq 0.$$

By Lemma 2.3, we deduce that $u_2 = 0$. Thus, $J\left(u_1, u_2\right) = J\left(u_1, 0\right)$, $u_1^n \to u_1$, and $u_1 \in S_r\left(c_1\right)$ satisfies the equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)\Delta u = \lambda_1 u + \mu_1 |u|^{p_1-2} u.$$

Therefore, $I_{\mu_1}(u_1) \geq m(c_1, \mu_1)$. On the other hand, by Hölder inequality,

$$0 \leq \int_{\mathbb{R}^3} |u_1^n|^{r_1} |u_2^n|^{r_2}\,dx \leq \|u_1^n\|_{2^*}^{r_1} \|u_2^n\|_{2^*}^{r_2}.$$

Using the fact $u_2^n \to 0$ in $D^{1,2}\left(\mathbb{R}^3\right)$, we have

$$\gamma'(c_1, c_2) = \lim_{n\to\infty} J\left(u_1^n, u_2^n\right) = I_{\mu_1}(u_1) + \lim_{n\to\infty} I_{\mu_2}\left(u_2^n\right) - \beta \lim_{n\to\infty} \int_{\mathbb{R}^3} |u_1^n|^{r_1} |u_2^n|^{r_2}\,dx \geq m(c_1, \mu_1),$$

which contradicts Lemma 5.2 (i)

Case 3. ($\lambda_2 < 0$ and $\lambda_1 \geq 0$): By similar arguments as in Case 2, we obtain a contradiction $\gamma'(c_1, c_2) \geq m(c_2, \mu_2)$. Therefore, we conclude that $(u_1^n, u_2^n) \to (u_1, u_2)$ in H_r.

By Lemma 5.3 and $\gamma'(c_1, c_2) < 0$, we have

$$\gamma'(c_1, c_2) = J(u_1, u_2) = \inf_{\mathcal{P}} J = \inf_{A(\rho_*)} J < 0.$$

This implies that (u_1, u_2) is a ground state solution. The proof of Theorem 1.4 is completed. □

References

[1] C. O. Alves, F. J. S. A. Corrêa, and T. F. Ma, "Positive solutions for a quasilinear elliptic equation of Kirchhoff type," *Comput. Math. Appl.*, vol. 49, no. 1, pp. 85–93, 2005, doi: 10.1016/j.camwa.2005.01.008.

[2] T. Bartsch and L. Jeanjean, "Normalized solutions for nonlinear Schrödinger systems," *Proc. Roy. Soc. Edinburgh Sect. A*, vol. 148, no. 2, pp. 225–242, 2018, doi: 10.1017/S0308210517000087.

[3] T. Bartsch, L. Jeanjean, and N. Soave, "Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$," *J. Math. Pures Appl. (9)*, vol. 106, no. 4, pp. 583–614, 2016, doi: 10.1016/j.matpur.2016.03.004.

[4] T. Bartsch, H. Li, and W. Zou, "Existence and asymptotic behavior of normalized ground states for Sobolev critical Schrödinger systems," *Calc. Var. Partial Differential Equations*, vol. 62, no. 1, 2023, Art. ID 9, doi: 10.1007/s00526-022-02355-9.

[5] X. Cao, J. Xu, and J. Wang, "The existence of solutions with prescribed L^2-norm for Kirchhoff type system," *J. Math. Phys.*, vol. 58, no. 4, 2017, Art. ID 041502, doi: 10.1063/1.4982037.

[6] Z. J. Chen and W. M. Zou, "Existence and symmetry of positive ground states for a doubly critical Schrödinger system," *Transactions of the American Mathematical Society*, vol. 367, pp. 3599–3646, 2015.

[7] X. J. Feng, H. D. Liu, and Z. T. Zhang, "Normalized solutions for Kirchhoff type equations with combined nonlinearities: The Sobolev critical case," *Discrete and Continuous Dynamical Systems*, vol. 43, pp. 2935–2972, 2023.

[8] Y. Gao, L. S. Liu, N. Wei, and Y. H. Wu, "Multiple nontrivial solutions for a Kirchhoff-type transmission problem in $\mathbb{R}^3$ with concave–convex nonlinearities," *Nonlinear Analysis: Real World Applications*, vol. 85, 2025, Art. ID 104377.

[9] N. Ghoussoub, *Duality and Perturbation Methods in Critical Point Theory.* Cambridge University Press, 1993.

[10] T. Gou and L. Jeanjean, "Existence and orbital stability of standing waves for nonlinear Schrödinger systems," *Nonlinear Anal.*, vol. 144, pp. 10–22, 2016, doi: 10.1016/j.na.2016.05.016.

[11] T. Gou and L. Jeanjean, "Multiple positive normalized solutions for nonlinear Schrödinger systems," *Nonlinearity*, vol. 31, no. 5, pp. 2319–2345, 2018, doi: 10.1088/1361-6544/aab0bf.

[12] X. He and W. Zou, "Infinitely many positive solutions for Kirchhoff-type problems," *Nonlinear Anal.*, vol. 70, no. 3, pp. 1407–1414, 2009, doi: 10.1016/j.na.2008.02.021.

[13] X. He and W. Zou, "Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$," *J. Differential Equations*, vol. 252, no. 2, pp. 1813–1834, 2012, doi: 10.1016/j.jde.2011.08.035.

[14] D. Hu, D. D. Qin, and X. Tang, "Concentration of semiclassical ground states for an N-Laplacian Kirchhoff-type problem with critical exponential growth," *Manuscripta Mathematica*, vol. 176, 2025, Art. ID 12.

[15] J. Hu and A. Mao, "Normalized solutions to nonlocal Schrödinger systems with L^2-subcritical and supercritical nonlinearities," *J. Fixed Point Theory Appl.*, vol. 25, no. 3, 2023, Art. ID 77, doi: 10.1007/s11784-023-01077-5.

[16] N. Ikoma, "Compactness of minimizing sequences in nonlinear Schrödinger systems under multiconstraint conditions," *Advanced Nonlinear Studies*, vol. 14, pp. 115–136, 2014.

[17] L. Jeanjean, J. Zhang, and X. Zhong, "Normalized ground states for a coupled Schrödinger system: mass super-critical case," *NoDEA Nonlinear Differential Equations Appl.*, vol. 31, no. 5, 2024, Art. ID 85, doi: 10.1007/s00030-024-00972-1.

[18] G. Kirchhoff, *Mechanik.* Teubner, 1883.

[19] C.-Y. Lei, J.-F. Liao, and C.-L. Tang, "Multiple positive solutions for Kirchhoff type of problems with singularity and critical exponents," *J. Math. Anal. Appl.*, vol. 421, no. 1, pp. 521–538, 2015, doi: 10.1016/j.jmaa.2014.07.031.

[20] G. B. Li, X. Luo, and T. Yang, "Normalized solutions to a class of Kirchhoff equations with Sobolev critical exponent," *Annales Fennici Mathematici*, vol. 47, pp. 895–925, 2022.

[21] H. Li and W. Zou, "Normalized ground states for semilinear elliptic systems with critical and subcritical nonlinearities," *J. Fixed Point Theory Appl.*, vol. 23, no. 3, 2021, Art. ID 43, doi: 10.1007/s11784-021-00878-w.

[22] J.-L. Lions, "On some questions in boundary value problems of mathematical physics," in *Contemporary developments in continuum mechanics and partial differential equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977)*, ser. North-Holland Math. Stud. North-Holland, Amsterdam-New York, 1978, vol. 30, pp. 284–346.

[23] J. Liu, J.-F. Liao, and C.-L. Tang, "Positive solutions for Kirchhoff-type equations with critical exponent in $\mathbb{R}^N$," *J. Math. Anal. Appl.*, vol. 429, no. 2, pp. 1153–1172, 2015, doi: 10.1016/j.jmaa.2015.04.066.

[24] M.-Q. Liu and X.-D. Fang, "Normalized solutions for the Schrödinger systems with mass supercritical and double Sobolev critical growth," *Z. Angew. Math. Phys.*, vol. 73, no. 3, 2022, Art. ID 108, doi: 10.1007/s00033-022-01757-1.

[25] S. Mo and S. W. Ma, "Normalized solutions to Kirchhoff equation with nonnegative potential," 2023, arXiv:2301.07926.

[26] S. Qi and W. Zou, "Exact number of positive solutions for the Kirchhoff equation," *SIAM J. Math. Anal.*, vol. 54, no. 5, pp. 5424–5446, 2022, doi: 10.1137/21M1445879.

[27] C. A. Stuart, "Bifurcation from the continuous spectrum in the L^2 theory of elliptic equations on $\mathbb{R}^n$," in *Recent Methods in Nonlinear Analysis and Applications*, 1981, pp. 231–300.

[28] M. I. Weinstein, "Nonlinear Schrödinger equations and sharp interpolation estimates," *Communications in Mathematical Physics*, vol. 87, pp. 567–576, 1982.

[29] L. Xu, F. Li, and Q. Xie, "Existence and multiplicity of normalized solutions with positive energy for the Kirchhoff equation," *Qual. Theory Dyn. Syst.*, vol. 23, no. 3, 2024, Art. ID 135, doi: 10.1007/s12346-024-01001-3.

[30] L. Xu, C. Song, and Q. Xie, "Multiplicity of normalized solutions for Schrödinger equation with mixed nonlinearity," *Taiwanese J. Math.*, vol. 28, no. 3, pp. 589–609, 2024, doi: 10.11650/tjm/240202.

[31] Z. Yang, "Normalized ground state solutions for Kirchhoff type systems," *J. Math. Phys.*, vol. 62, no. 3, 2021, Art. ID 031504, doi: 10.1063/5.0028551.

[32] H. Ye, "The existence of normalized solutions for L^2-critical constrained problems related to Kirchhoff equations," *Z. Angew. Math. Phys.*, vol. 66, no. 4, pp. 1483–1497, 2015, doi: 10.1007/s00033-014-0474-x.

[33] H. Ye, "The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations," *Math. Methods Appl. Sci.*, vol. 38, no. 13, pp. 2663–2679, 2015, doi: 10.1002/mma.3247.

[34] P. Zhang and Z. Han, "Normalized ground states for Schrödinger system with a coupled critical nonlinearity," *Appl. Math. Lett.*, vol. 150, 2024, Art. ID 108947, doi: 10.1016/j.aml.2023.108947.