CUBO, A Mathematical Journal
Vol. 28, no. 1, pp. 79–97, January 2026
DOI: 10.56754/0719-0646.2801.079

Blow-up and global existence of solutions for a higher-order reaction diffusion equation with singular potential

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ABSTRACT

In this work, we consider the higher-order reaction-diffusion parabolic problem with time dependent coefficient. We prove the blow-up of solutions and obtain a lower and an upper bound for the blow-up time. Finally, we investigate the existence of a global weak solution to the problem.

RESUMEN

En este trabajo, consideramos un problema parabólico de reacción-difusión de alto orden con coeficiente dependiente del tiempo. Demostramos la explosión de soluciones y obtenemos cotas inferior y superior para el tiempo de explosión. Finalmente, investigamos la existencia de una solución débil global del problema.

Keywords and Phrases: Blow-up, higher-order, singular potential, global existence, reaction-diffusion.

2020 AMS Mathematics Subject Classification: 35B44, 35K25, 35K67.

Published: 22 January, 2026
Accepted: 27 October, 2025
Received: 16 October, 2024

1 Introduction

In this work, we investigate the following reaction-diffusion parabolic problem with singular potential:

$$\begin{cases} \dfrac{z_t}{|x|^{2m}} + \mathcal{A}z = \alpha(t)|z|^{r-1}z, & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial^i z(x,t)}{\partial \nu^i} = 0, \quad i = 0,1,\ldots,m-1, & (x,t) \in \partial\Omega \times (0,T), \\ z(x,0) = z_0(x) \in H_0^m(\Omega) \cap L^{r+1}(\Omega), \quad x \in \Omega, \end{cases} \tag{1.1}$$

here $n \geq 1$ and $\Omega \subset R^n$ is open and bounded with Lipschitz boundary, where $T > 0$, $r > 1$, $\mathcal{A} = (-\Delta)^m$, $m > 1$ is an integer constant and a unit outer normal ν, $x = (x_1, x_2, \ldots, x_n)$, $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. The coefficient $\alpha(t)$ is chosen such that

$$\alpha \in C^1[0,\infty), \ \alpha(0) > 0 \text{ and } \ \alpha'(t) \geq 0 \quad \text{for all } t \in [0,\infty). \tag{1.2}$$

Explosive phenomena commonly arise in solutions to reaction-diffusion partial differential equations of various types (see *e.g.* [4, 6, 15] and references therein). Understanding the conditions under which such phenomena occur is of practical interest. However, accurately computing the precise blow-up time is often challenging. Despite this challenge, it is still possible to estimate the blow-up time using various methods. Notable approaches for investigation include the first eigenvalue method proposed by Kaplan in 1963, the potential well method developed by Levine and Payne in 1970, the comparison method, and other techniques involving integration. A recent comprehensive overview of these methods can be found in the monograph by Hu [11], Al'shin *et al.* [2] and Pişkin [17]. Additionally, readers may refer to the survey articles by Galaktionov [8] and Levine [13] for insights into the blow-up properties of more general evolution problems. Specifically, sufficient conditions for blow-up estimates are discussed in works of Philippin [16] and Han [9] provided insights for the equation of the form:

$$z_t + \Delta^2 z = k(t) f(z).$$

In another study, Han [10] investigated the equation of the form

$$\frac{z_t}{|x|^2} - \Delta z = k(t)|z|^{p-1} z,$$

in which the author derived the lower and upper bounds on the blow-up time of weak solutions.

In [23], Thanh *et al.* considered the reaction-diffusion parabolic problem with time dependent coefficients

$$\frac{z_t}{|x|^4} + \Delta^2 z = k(t)|z|^{p-1} z.$$

They proved an upper and lower bound for blow-up time. Do *et al.* [5] investigated the existence of a global weak solution to the problem together with the decaying and blow-up properties using the potential well method.

Recently, Thanh *et al.* [24] proved the higher-order version $\Delta\left(|\Delta|^{m-2}\Delta\right)$ of the $p-$Laplacian and the function $k(t)$ non-Newtonian filtration equation and obtained the blow-up result with lower and upper bounds. The reader is directed to [19–21] for a detailed discussion of higher-order hyperbolic equations.

In our research, we employed various types of Dirichlet-Neumann boundary conditions in conjunction with a general nonlinear term. Additionally, we derived the primary outcomes of this paper using a methodology distinct from those discussed in prior works. While some of the literature has addressed blow-up solutions for higher-order parabolic equation, to the best of our knowledge, there is currently no article available that specifically explores the finite-time blow-up solutions for a higher-order parabolic equation with a variable coefficient term $\alpha(t)$. Consequently, we endeavored to investigate and present new and noteworthy findings in this regard. For a more in-depth exploration of blow-up phenomena in higher-order parabolic equation, readers are encouraged to consult the book by Galaktionov [7].

Motivated by above-mentioned papers, in this paper, we investigate to prove the upper and lower bounds for the blow-up time of solutions for problem (1.1), which was not previously studied, where we study higher-order parabolic equation with time dependent coefficient source terms $\alpha(t)|z|^{r-1}z$.

The rest of the work is as follows: In Section 2, we give some assumptions needed in this work. In Section 3, under suitable conditions, we obtain an upper bound for the blow-up time. In Section 4, we obtain a lower bound for the blow-up time. In Section 5, under suitable conditions, we investigate the existence of a global weak solution to the problem.

2 Preliminaries

In this part, we present certain lemmas and assumptions required for the formulation and proof of our results. Let $\|.\|$, $\|.\|_r$ and $\|.\|_{W^{m,r}(\Omega)}$ indicate the typical $L^2(\Omega)$, $L^r(\Omega)$ and $W^{m,r}(\Omega)$ norms (see [1, 18]).

Now, we consider some energy estimates: Let $n \geq 1$ and $\Omega \subset R^n$ be open bounded with Lipschitz boundary. For each $z \in H_0^m(\Omega) \cap L^{r+1}(\Omega)$ and $t \in [0,\infty)$ define the following functionals of the problem (1.1):

- Energy functional is as follows:

$$J(z,t) = \frac{1}{2}\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 - \frac{\alpha(t)}{r+1}\|z\|^{r+1},$$

- and Nehari functional is as follows:

$$I(z,t) = \left\|\mathcal{A}^{\frac{1}{2}} z\right\|^{2} - \alpha(t)\|z\|^{r+1}.$$

We strive to establish both upper and lower bounds for the blow-up time of a weak solution to equation (1.1), the precise definitions of which are provided in the following.

Definition 2.1. *A function z is termed a weak solution to equation* (1.1) *if* $z \in L^2\left(0,T; H_0^m(\Omega) \cap L^{r+1}(\Omega)\right)$ *and* $\frac{z_t}{|x|^{2m}} \in L^2\left(0,T; L^2(\Omega)\right)$ *where z satisfies the following equation:*

$$\left(\frac{z_t}{|x|^{2m}}, \varphi\right) + \left(\mathcal{A}^{\frac{1}{2}} z, \mathcal{A}^{\frac{1}{2}} \varphi\right) = \alpha(t)\left(|z|^{r-1} z, \varphi\right), \tag{2.1}$$

for all $\varphi \in H_0^m(\Omega) \cap L^{r+1}(\Omega)$ *and* $t \in [0,\infty)$.

When $\Omega \subset R^n$ is an open and bounded set with a Lipschitz boundary, the existence of a local weak solution can be established using standard Ordinary Differential Equation (ODE) theory, coupled with the Faedo-Galerkin approximation technique, as is well-known in the literature.

Definition 2.2. *Assume that $z(t)$ is a weak solution to* (1.1)*. If $z(t)$ exists for all t in the interval* $[0,T^*)$*, and the limit as to blow up at a finite time T^* if $z(t)$ exists for all $t \in [0,T^*)$ and*

$$\lim_{t\to T^*}\left\|\frac{z_t}{|x|^m}\right\|^2 = \infty. \tag{2.2}$$

Such a T^* is called the maximal existence time as well as the blow up time for $z(t)$. If (2.2) *does not happen for any finite time T^*, then $z(t)$ is called a global solution and the maximal existence time of $z(t)$ is ∞.*

We are able to define the stable and unstable sets as follows for each $t \geq 0$:

- Stable set:

$$\Sigma_1(t) = \left\{z \in H_0^m(\Omega) : J(z,t) < n_\infty \text{ and } I(z,t) > 0\right\}.$$

- Unstable set:

$$\Sigma_2(t) = \left\{z \in H_0^m(\Omega) : J(z,t) < n_\infty \text{ and } I(z,t) < 0\right\}.$$

$\Sigma_1(t)$ and $\Sigma_2(t)$ are crucial to our paper. Where

$$n_\infty = \lim_{t\to\infty} n(t).$$

Note that J, I, C_0, n, Σ_1 and Σ_2 are all time-dependent, as indicated by the presence of $\alpha(t)$ in

(1.1). The introduction of this time-dependent factor introduces additional technical complexity into our analysis.

Because of the presence of the inverse coefficient $1/|x|^{2m}$, it is important to highlight the distinction between the two cases when $0 \in \Omega$ and $0 \notin \Omega$. If $0 \in \Omega$ then $1/|x|^{2m}$ develops a singularity. This requires the application of Rellich's inequality, which is valid for $n \geq 2m+1$, in the proofs of our main results. However, if $0 \notin \Omega$ then there is no singularity and (1.1) can be considered as a slight extension of the model in [10]. In this case our results are valid for all $n \geq 1$. To deal with these two cases at the same time, we use the notation

$$n_\Omega = \begin{cases} 2m+1, & \text{if } 0 \in \Omega \\ 1, & \text{if } 0 \notin \Omega \end{cases} \quad \text{and} \quad 2^* = \begin{cases} \infty, & \text{if } n \leq 2m, \\ \frac{2n}{n-2m} = 2 + \frac{2m}{n-2m}, & \text{if } n \geq 2m+1. \end{cases}$$

Let us start with the following Rellich inequality Lemma.

Lemma 2.3. *Assume that $n \geq 2m+1$ and $\Omega \subset R^n$ be open bounded. Let $z \in H_0^m(\Omega)$. Then $\frac{z}{|x|^{2m}} \in L^2(\Omega)$ and*

$$\int_\Omega \frac{|z|^2}{|x|^{2m}} dx \leq \left(\frac{m^2}{n(m-1)(n-2m)} \right)^2 \int_\Omega \left| \mathcal{A}^{\frac{1}{2}} z \right|^2 dx = \mathcal{C} \int_\Omega \left| \mathcal{A}^{\frac{1}{2}} z \right|^2 dx.$$

Proof. Let $z \in H_0^m(\Omega)$ and $\check{z}$ be its zero extension to R^n. Then $\check{z} \in H^m(R^n)$ by [1, Lemma 3.27], and

$$\begin{aligned} \int_\Omega \frac{|z|^2}{|x|^{2m}} dx \leq \int_{R^n} \frac{|\check{z}|^2}{|x|^{2m}} dx &\leq \left(\frac{m^2}{n(m-1)(n-2m)} \right)^2 \int_{R^n} \left| \mathcal{A}^{\frac{1}{2}} \check{z} \right|^2 dx \\ &\leq \left(\frac{m^2}{n(m-1)(n-2m)} \right)^2 \int_{R^n} \left| \mathcal{A}^{\frac{1}{2}} z \right|^2 dx, \end{aligned} \tag{2.3}$$

here we used [3, Corollary 6.3.5], in the second step of the argument. This provides the justification for the claim. □

The next result below is the Gagliardo-Nirenberg inequality.

Lemma 2.4. *Let $n \geq 2m+1$ and Ω be open and bounded subset of R^n, $1 < r < 1 + \frac{4m}{n-2m}$. Then there exists $C_0 = C_0(\Omega, n, r) > 0$ so that*

$$\|z\|_{L^{r+1}(\Omega)}^{r+1} \leq C_0 \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{\beta(r+1)} \|z\|^{(1-\beta)(r+1)}, \quad \forall z \in H_0^m(\Omega),$$

where

$$\beta = \frac{n(r-1)}{4(r+1)} \in (0,1). \tag{2.4}$$

Proof. Let $z \in H_0^m(\Omega)$. It follows from Gagliardo-Nirenberg inequality that

$$\|z\|_{L^{r+1}(\Omega)}^{r+1} \leq C(\Omega, n, r) \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{\beta(r+1)} \|z\|^{(1-\beta)(r+1)},$$

where used

$$\left\| \nabla^2 z \right\| \leq C(\Omega, n) \left\| \mathcal{A}^{\frac{1}{2}} z \right\|,$$

by [22, Chapter 3, Proposition 3]. □

Lemma 2.5. *Assume that* $n \geq 1$ *and* $\Omega \subset R^n$ *be open and bounded with Lipschitz boundary. Suppose* α *is defined by* (1.2). *Let* z *be a weak solution to equation* (1.1) *with* $T > 0$. *Then the following identities hold:*

(H1)

$$J(z(h), h) + \int_0^h \left(\left\| \frac{z_t(s)}{|x|^m} \right\|^2 - \frac{\alpha'(s)}{r+1} \|z\|_{L^{r+1}(\Omega)}^{r+1} \right) ds = J(z_0, 0),$$

and

(H2)

$$\frac{d}{dt}\left(\frac{1}{2} \left\| \frac{z(h)}{|x|^m} \right\|^2 \right) = \left(\frac{z(h)}{|x|^{2m}}, z_t(h) \right) = -I(z(h), h),$$

for a.e. $h \in [0, T)$.

Proof. For **(H1)**, first assume that $z_t \in L^2\left(0, T; H_0^m(\Omega) \cap L^{r+1}(\Omega)\right)$. Then, utilizing z_t as a test function in (2.1) we have

$$\left\| \frac{z_t}{|x|^m} \right\|^2 + \left(\mathcal{A}^{\frac{1}{2}} z, \mathcal{A}^{\frac{1}{2}} z_t \right) = \alpha(t) \left(|z|^{r-1} z, z_t \right).$$

Moreover, direct calculations provide

$$\frac{d}{dt} J(z(t), t) = \left(\mathcal{A}^{\frac{1}{2}} z, \mathcal{A}^{\frac{1}{2}} z_t \right) - \alpha(t) \left(|z|^{r-1} z, z_t \right) - \frac{\alpha'(t)}{r+1} \|z\|_{L^{r+1}(\Omega)}^{r+1},$$

as a function of t in the interval $[0, T)$. Combining these two identities together results in

$$\frac{d}{dt} J(z(t), t) = -\left\| \frac{z_t}{|x|^m} \right\|^2 - \frac{\alpha'(t)}{r+1} \|z\|_{L^{r+1}(\Omega)}^{r+1}, \tag{2.5}$$

as a function of t in the interval $[0, T)$.

Now **(H1)** follows by integrating both sides of (2.5) with respect to t over $(0, h)$, where $h \in (0, T)$.

To conclude, with an approximation argument we examine that (2.5) holds without the assumption that $z_t \in L^2\left(0, T; H_0^m(\Omega) \cap L^{r+1}(\Omega)\right)$.

The proof of **(H2)** is the same way and is omitted. □

The result we give below is obtained directly from Lemma 2.4 and the Friedrichs inequality (*cf.* [14, Theorem 1.10]).

Lemma 2.6. *Let $n \geq 1$, $z \in H_0^m(\Omega)$ and $2 < r+1 < 2^*$. Then there exists a constant $S_r = S_r(n,r) > 0$ so that*

$$\|z\|_{L^{r+1}(\Omega)} \leq S_r \|\Delta z\|.$$

In addition, we note that the constant S_r may be made explicit and sharp when $n \geq 2m+1$.

Our next result is known as the concavity argument, which is widely used in the literature and is used for the sufficient condition of blow-up.

Lemma 2.7 ([12,13])**.** *Suppose that a positive, twice-differentiable on $(0,\infty)$ function $\psi(t)$ satisfies the inequality*

$$\psi''(t)\psi(t) - (1+\theta)(\psi'(t))^2 \geq 0,$$

where $\theta > 0$. If $\psi(0) > 0$ and $\psi'(0) > 0$ for all $t \in (0,\infty)$. Then there exists $T > 0$ such that

$$\lim_{t\to T^-} \psi(t) = \infty, \quad \text{and} \quad T \leq \frac{\psi(0)}{\theta\psi'(0)}.$$

3 Upper bound for blow-up time

In this part, we are going to obtain the upper bounds for the finite time blow-up results. For simplicity, we shall write

$$\mathcal{L}(t) = \frac{1}{2}\left\|\frac{z(t)}{|x|^m}\right\|^2,$$

for each $t \in [0,T)$.

We start with the proof of Theorem 3.1. This is related to the upper limit on the explosion time of the weak solution when the initial energy functional is negative (1.1) .

Theorem 3.1. *Assume that $n \geq 2m+1$ and $\Omega \subset R^n$ be open and bounded with Lipschitz boundary. Let $r > 1$ and α be given by (1.2). Such that $0 \neq z_0 \in H_0^m(\Omega) \cap L^{r+1}(\Omega)$ and*

$$J(z_0,0) = \frac{1}{2}\left\|\mathcal{A}^{\frac{1}{2}} z_0\right\|^2 - \frac{\alpha(0)}{r+1}\|z_0\|_{L^{r+1}(\Omega)}^{r+1} < 0.$$

Suppose that $z(t)$ is a weak solution to (1.1) with $T > 0$. Then z blows up at a finite time T^* which satisfies

$$T^* \leq \frac{\left\|\frac{z_0}{|x|^m}\right\|^2}{(1-r^2)J(z_0,0)}.$$

Proof. Here we set $T^* < \infty$, where $T^* \geq 0$ is the maximum existence time of z, and then we aim to provide an upper bound for T^*.

Set for this purpose

$$\mathcal{K}(t) = -J(z(t), t),$$

for every $t \in [0, T^*)$. According to the hypothesis $\mathcal{L}(0) > 0$ and $\mathcal{K}(0) > 0$.

We can also write from Lemma 2.5:

$$\mathcal{K}'(t) = -\frac{d}{dt} J(z(t), t) = \left\| \frac{z_t}{|x|^m} \right\|^2 + \frac{\alpha'(s)}{r+1} \|z\|_{L^{r+1}(\Omega)}^{r+1} \geq 0, \tag{3.1}$$

for each $t \in [0, T^*)$, so $\mathcal{K}$ increases over $[0, T^*)$. Consequently, $\mathcal{K}(t) \geq \mathcal{K}(0) > 0$ for all $t \in [0, T^*)$.

Assume that $t \in [0, T^*)$. Same way,

$$\mathcal{L}'(t) = \left(\frac{z}{|x|^{2m}}, z_t \right) = -I(z(t), t) = \frac{r-1}{2} \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 - (r+1) J(z(t), t) \geq (r+1) \mathcal{K}(t). \tag{3.2}$$

Thus,

$$\mathcal{L}(t) \mathcal{K}'(t) \geq \frac{1}{2} \left\| \frac{z}{|x|^m} \right\|^2 \left\| \frac{z_t}{|x|^m} \right\|^2 \geq \frac{1}{2} \left(\frac{z}{|x|^{2m}}, z_t \right)^2 = \frac{1}{2} (\mathcal{L}'(t))^2 \geq \frac{r+1}{2} \mathcal{L}'(t) \mathcal{K}(t). \tag{3.3}$$

From (3.1), (3.2) and (3.3), we get

$$\left(\mathcal{K}(t) \mathcal{L}^{-(r+1)/2}(t) \right)' = \mathcal{L}^{-(r+3)/2}(t) \left(\mathcal{K}'(t) \mathcal{L}(t) - \frac{r+1}{2} \mathcal{K}(t) \mathcal{L}'(t) \right) \geq 0.$$

This means that $\mathcal{K}\mathcal{L}^{-(r+1)/2}$ strictly increases over $[0, T^*)$, which follows:

$$\begin{aligned} 0 < \xi_0 = \mathcal{K}(0) \mathcal{L}^{-(r+1)/2}(0) &< \mathcal{K}(t) \mathcal{L}^{-(r+1)/2}(t) \\ &\leq \frac{1}{r+1} \mathcal{L}'(t) \mathcal{L}^{-(r+1)/2}(t) = \frac{2}{1-r^2} \left(\mathcal{L}^{(1-r)/2}(t) \right)', \end{aligned}$$

here we used (3.2).

Integrating this last representation with respect to t over $(0, \tau)$, where $\tau \in (0, T^*)$, we obtain:

$$\xi_0 \tau \leq \frac{2}{1-r^2} \left(\mathcal{L}^{(1-r)/2}(\tau) - \mathcal{L}^{(1-r)/2}(0) \right).$$

Since this inequality only holds for a finite period of time, we deduce $T^* < \infty$. Moreover,

$$0 \leq \mathcal{L}^{(1-r)/2}(\tau) \leq \mathcal{L}^{(1-r)/2}(0) - \frac{(r^2-1)\xi_0}{2} \tau,$$

for all $\tau \in [0, T^*)$. This reveals that

$$T^* \leq \frac{2}{(r^2-1)\,\xi_0}\mathcal{L}^{(1-r)/2}(0) = \frac{2\mathcal{L}(0)}{(1-r^2)\,J(z_0,0)}.$$

The proof is complete. □

Next we state and prove Theorem 3.2. Here it provides an upper bound on the explosion time for a weak solution to (1.1) when the initial energy functional is positive.

Theorem 3.2. *Suppose that* $n \geq 2m+1$ *and* $\Omega \subset R^n$ *be open and bounded with Lipschitz boundary. Let* $r > 1$ *and* α *be given by* (1.2). *Assume that* $0 \neq z_0 \in H_0^m(\Omega) \cap L^{r+1}(\Omega)$ *and*

$$0 \leq C_1 J(z_0, 0) < \frac{1}{2}\left\|\frac{z_0}{|x|^m}\right\|^2 = \mathcal{L}(0),$$

where

$$C_1 = \frac{(r+1)\,\mathcal{C}}{r-1} \quad \text{and} \quad \mathcal{C} = \left(\frac{m^2}{n(m-1)(n-2m)}\right)^2.$$

Suppose that $z(t)$ *be a weak solution to* (1.1) *with* $T > 0$. *Then* z *blows up at a finite time* T^* *which satisfies*

$$T^* \leq \frac{4rC_1\mathcal{L}(0)}{(r-1)^2(r+1)(\mathcal{L}(0) - C_1 J(z_0,0))}.$$

Proof. Here we set $T^* < \infty$, where $T^* \geq 0$ is the maximum existence time of z, and then we aim to provide an upper bound for T^*.

From (3.2)

$$\begin{aligned}\mathcal{L}'(t) &\geq \frac{r-1}{2}\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 - (r+1)\,J(z(t),t) \geq \frac{r-1}{2\mathcal{C}}\left\|\frac{z(t)}{|x|^m}\right\|^2 - (r+1)\,J(z(t),t)\\ &= \frac{r-1}{\mathcal{C}}\left[\mathcal{L}(t) - C_1 J(z(t),t)\right] = \frac{r-1}{\mathcal{C}}\mathcal{M}(t),\end{aligned}$$

for each $t \in (0, T^*)$, where in the second step we used Lemma 2.3.

Observe from the inequality above:

$$\mathcal{M}'(t) = \mathcal{L}'(t) - C_1\frac{d}{dt}J(z(t),t) \geq \mathcal{L}'(t) \geq \frac{r-1}{\mathcal{C}}\mathcal{M}(t),$$

for each $t \in (0, T^*)$, here we used (3.1) in the second step.

Moreover,

$$\mathcal{M}(0) = \mathcal{L}(0) - C_1 J(z_0, 0) > 0,$$

by assumption. Consequently, an application of Gronwall's inequality gives

$$\mathcal{M}(t) \geq \mathcal{M}(0) \exp\left(\frac{r-1}{\mathcal{C}} t\right) > 0.$$

This means that $\mathcal{L}'(t) > 0$ for every $t \in (0, T^*)$. That is, $\mathcal{L}$ increases strictly over $[0, T^*)$ and hence

$$\mathcal{L}(t) > \mathcal{L}(0), \tag{3.4}$$

for every $t \in [0, T^*)$.

And by C_1 and $\mathcal{C}$ given in the statement of this theorem. Fix $\tau \in [0, T^*)$ and

$$\beta \in \left(0, \frac{r+1}{rC_1}\right) \mathcal{M}(0) \quad \text{and} \quad \sigma \in \left(\frac{\mathcal{L}(0)}{(r-1)\beta}, \infty\right). \tag{3.5}$$

The choices of β and σ are justified below with (3.8) and (3.9) respectively. Define non-negative functional

$$\Psi(h) = \int_0^h \mathcal{L}(s)\, ds + (\tau - h)\mathcal{L}(0) + \beta(h+\sigma)^2,$$

where $h \in [0, \tau]$. Then

$$\Psi'(h) = \mathcal{L}(h) - \mathcal{L}(0) + 2\beta(h+\sigma) = 2\int_0^h \left(\frac{z(s)}{|x|^m}, z_t(s)\right) ds + 2\beta(h+\sigma),$$

and

$$\begin{aligned}
\Psi''(h) &= 2\left(\frac{z(h)}{|x|^m}, z_t(h)\right) + 2\beta = -2I(z(h), h) + 2\beta \\
&\geq -2(r+1)J(z(h), h) + (r-1)\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 + 2\beta \\
&\geq -2(r+1)\left[J(z_0, 0) - \int_0^h \left(\left\|\frac{z_t(s)}{|x|^m}\right\|^2 + \frac{\alpha'(s)}{r+1}\|z\|_{L^{r+1}(\Omega)}^{r+1}\right) ds\right] + (r-1)\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 + 2\beta \\
&\geq -2(r+1)\left[J(z_0, 0) - \int_0^h \left(\left\|\frac{z_t(s)}{|x|^m}\right\|^2 + \frac{\alpha'(s)}{r+1}\|z\|_{L^{r+1}(\Omega)}^{r+1}\right) ds\right] + \frac{2(r-1)}{\mathcal{C}}\mathcal{L}(h) + 2\beta,
\end{aligned} \tag{3.6}$$

for each $h \in [0, \tau)$, where we used Lemmas 2.5 and 2.3 in the third and fourth lines, respectively.

In what follows it is convenient to denote

$$\begin{aligned}
\theta(h) = &\left(2\int_0^h \mathcal{L}(s)\, ds + \beta(h+\sigma)^2\right)\left(\int_0^h \left\|\frac{z_t(s)}{|x|^m}\right\|_{L^2(\Omega)}^2 ds + \beta\right) \\
&- \left(\int_0^h \left(\frac{z(s)}{|x|^m}, z_t(s)\right) ds + \beta(h+\sigma)\right)^2 \geq 0,
\end{aligned} \tag{3.7}$$

for every $h \in [0,\tau]$, where in the last step of (3.7) we used the Cauchy-Schwarz inequality.

From Lemma 2.7, (3.6) and (3.4), we obtain

$$\begin{aligned}
\Psi(h)\Psi''(h) - \frac{r+1}{2}\left(\Psi'(h)\right)^2 &= \Psi(h)\Psi''(h) - 2(r+1)\left[\int_0^h \left(\frac{z(s)}{|x|^m}, z_t(s)\right) ds + \beta(h+\sigma)\right]^2 \\
&= \Psi(h)\Psi''(h) + 2(r+1)\left[\theta(h) - (\Psi(h) - (\tau-h)\mathcal{L}(0))\left(\int_0^h \left\|\frac{z_t(s)}{|x|^m}\right\|^2 ds + \beta\right)\right] \\
&\geq \Psi(h)\Psi''(h) - 2(r+1)\Psi(h)\left(\int_0^h \left\|\frac{z_t(s)}{|x|^m}\right\|^2 ds + \beta\right) \\
&\geq \Psi(h)\left[\Psi''(h) - 2(r+1)\left(\int_0^h \left\|\frac{z_t(s)}{|x|^m}\right\|^2 ds + \beta\right)\right] \\
&\geq \Psi(h)\left[-2(r+1)J(z_0,0) + \frac{2(r-1)}{\mathcal{C}}\mathcal{L}(h) - 2r\beta\right] \\
&\geq \Psi(h)\left[-2(r+1)J(z_0,0) + \frac{2(r-1)}{\mathcal{C}}\mathcal{L}(0) - 2r\beta\right] \\
&= 2(r+1)\Psi(h)\left[-J(z_0,0) + \frac{1}{C_1}\mathcal{L}(0) - \frac{r\beta}{r+1}\right] \geq 0,
\end{aligned} \tag{3.8}$$

for all $h \in [0,\tau]$.

Then observe this

$$\Psi(0) = \tau\mathcal{L}(0) + \beta\sigma^2 > 0, \quad \text{and} \quad \Psi'(0) = 2\beta\sigma > 0.$$

Consequently, from Lemma 2.7:

$$\tau \leq \frac{2\Psi(0)}{(r-1)\Psi'(0)} = \frac{2\left(\tau\mathcal{L}(0) + \beta\sigma^2\right)}{2(r-1)\beta\sigma} = \frac{\mathcal{L}(0)}{(r-1)\beta\sigma}\tau + \frac{\sigma}{r-1}.$$

This is as a result

$$\tau\left(1 - \frac{\mathcal{L}(0)}{(r-1)\beta\sigma}\right) \leq \frac{\sigma}{r-1},$$

or equivalently, we can write

$$\tau \leq \frac{\sigma}{r-1}\left(1 - \frac{\mathcal{L}(0)}{(r-1)\beta\sigma}\right)^{-1} = \frac{\beta\sigma^2}{(r-1)\beta\sigma - \mathcal{L}(0)}. \tag{3.9}$$

Reducing the expression mentioned in (3.5) across the range of σ results in

$$\tau \leq \frac{4\mathcal{L}(0)}{(r-1)^2\beta}. \tag{3.10}$$

Next, we aim to minimize the expression referenced by (3.10) within the specified range of β as outlined in (3.5). This leads to the following inequality:

$$\tau \leq \frac{4rC_1\mathcal{L}(0)}{(r-1)^2(r+1)\mathcal{M}(0)}. \tag{3.11}$$

Finally, the inequality stated in reference (3.11) remains valid for all $\tau \in (0, T^*)$. From this, we can conclude that

$$T^* \leq \frac{4rC_1\mathcal{L}(0)}{(r-1)^2(r+1)\mathcal{M}(0)},$$

as needed. □

4 Lower bound for blow-up time

In this section we consider with the lower bound for the finite time blow-up results. This is the content of Theorem 4.1. For simplicity, we shall write

$$\mathcal{L}(t) = \frac{1}{2}\left\|\frac{z(t)}{|x|^m}\right\|^2,$$

for each $t \in [0, T)$.

We start with the proof of Theorem 4.1. This is related to the lower limit on the explosion time of the weak solution when the initial energy functional is negative (1.1) .

Theorem 4.1. *Assume that $n \geq 2m+1$ and $\Omega \subset R^n$ be open bounded with Lipschitz boundary. Let α is given by (1.2) which enjoys a further property that*

$$\alpha_\infty = \lim_{t\to\infty} \alpha(t) < \infty.$$

Suppose that $1 < r < 1+\frac{4m}{n}$. Let $z(t)$ be a weak solution to (1.1) with $T > 0$ and $0 \neq z_0 \in H_0^m(\Omega)$. Assume that $z(t)$ blows up at T^*. Then

$$T^* \geq \frac{\mathcal{L}^{1-\gamma}(0)}{C^*(\gamma-1)},$$

where

$$\beta = \frac{n(r-1)}{4(r+1)} \in (0,1), \quad \gamma = \frac{(1-\beta)(r+1)}{2-\beta(r+1)} > 1,$$

and

$$C^* = \frac{2-\beta(r+1)}{2}\left(\frac{2}{\alpha_\infty C_0 \beta(r+1)}\right)^{-\beta(r+1)/(2-\beta(r+1))}\left(\sup_{x\in\Omega}|x|\right)^{4\gamma},$$

with $C_0 = C_0(\Omega, n, r) > 0$.

Proof. By assumption $1 < r < 1 + \frac{4m}{n}$ this leads to

$$0 < \beta(r+1) = \frac{(r-1)n}{4} < m.$$

This allows us to apply Young's inequality below.

Based on the constants defined in the expression of this theorem and utilizing the Lemma 2.4 and Young's inequality. We get

$$\begin{aligned}
\mathcal{L}'(h) &= \left(\frac{z(h)}{|x|^m}, z_t(h)\right) = -I(z(h), h) = \alpha(h)\|z\|_{L^{r+1}(\Omega)}^{r+1} - \left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 \\
&\le C_0\alpha_\infty \left\|\mathcal{A}^{\frac{1}{2}}z\right\|^{\beta(r+1)} \|z\|^{(1-\beta)(r+1)} - \left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 \\
&\le \left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 + \frac{2-\beta(r+1)}{2}\left(\frac{2}{\alpha_\infty \mathcal{C}_0\beta(r+1)}\right)^{-\beta(r+1)/(2-\beta(r+1))} \|z\|^{2\gamma} - \left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 \\
&= \frac{2-\beta(r+1)}{2}\left(\frac{2}{\alpha_\infty \mathcal{C}_0\beta(r+1)}\right)^{-\beta(r+1)/(2-\beta(r+1))} \|z\|^{2\gamma} \\
&\le \frac{2-\beta(r+1)}{2}\left(\frac{2}{\alpha_\infty \mathcal{C}_0\beta(r+1)}\right)^{-\beta(r+1)/(2-\beta(r+1))} \left(\sup_{x\in\Omega}|z|\right)^{4\gamma} \mathcal{L}(t)^\gamma \\
&= C^*\mathcal{L}(t)^\gamma,
\end{aligned}$$

for all $h \in (0, T^*)$. Equivalency one has

$$\frac{\mathcal{L}'(t)}{\mathcal{L}(t)^\gamma} \le C^*,$$

where do we get it

$$\frac{1}{1-\gamma}\left(\mathcal{L}^{1-\gamma}(t) - \mathcal{L}^{1-\gamma}(0)\right) \le C^* t.$$

Lastly, since $\gamma > 1$ and $\lim_{t\to T^*}\mathcal{L}(t) = \infty$, allowing $t \to T^*$ in the above inequality, we have

$$T^* \ge \frac{\mathcal{L}^{1-\gamma}(0)}{C^*(\gamma-1)},$$

as required. □

5 Global existence

In this Section, we establish the existence of a global weak solution to the equation referenced as (1.1), which corresponds to Theorem 5.2. While the proof follows the conventional arguments of Faedo-Galerkin approximation, the presence of the fourth-order operator in (1.1) requires a thorough justification, particularly when the initial datum z_0 belongs to the stable set Σ_1. For the sake of simplicity in notation, we utilize the dot notation in this part

$$z_k' = (z_k)_t = \frac{\partial}{\partial t} z_k.$$

Hereafter

$$a \wedge b = \min\{a, b\} \quad \text{and} \quad a \vee b = \max\{a, b\}.$$

Remember we set

$$n_\Omega = \begin{cases} 2m+1, & \text{if } 0 \in \Omega \\ 1, & \text{if } 0 \notin \Omega \end{cases} \quad \text{and} \quad 2^* = \begin{cases} \infty, & \text{if } n \leq 2m, \\ \frac{2n}{n-2m}, & \text{if } n \geq 2m+1, \end{cases}$$

with

$$\Sigma_1(t) = \{z \in H_0^m(\Omega) : J(z,t) < n_\infty \ \text{ and } \ I(z,t) > 0\},$$

and

$$\Sigma_2(t) = \{z \in H_0^m(\Omega) : J(z,t) < n_\infty \ \text{ and } \ I(z,t) > 0\},$$

for every $t \geq 0$.

We begin with a problem of approximation.

Lemma 5.1 ([5])**.** *Assume that $n \geq n_\Omega$ and $2 < r+1 < 2^*$. Suppose that $k \in N$, $T > 0$ and $z_{k0} \in C_c^\infty(\Omega)$. Then the problem*

$$\begin{cases} \rho_k(x) z_k' + \mathcal{A} z_k = \beta_k(z_k), & (x,t) \in \Omega \times (0,T], \\ \dfrac{\partial^i z_k(x,t)}{\partial \nu^i} = 0, \ i = 0,1,\ldots,m-1, & (x,t) \in \Omega \times (0,T], \\ z_k(x,0) = z_{k0}, & x \in \Omega, \end{cases} \tag{5.1}$$

accepts a global solution $z_k \in C([0,T]; H_0^m(\Omega))$ so that $z_k' \in L^2(0,T; H_0^m(\Omega))$, where

$$\rho_k(x) = |x|^{-2m} \wedge n \quad \text{and} \quad \beta_k(z_k) = \alpha(t)\left[(-k) \vee \left(|z_k|^{r-1} z_k\right) \wedge k\right].$$

Finally, we present the existence of a global weak solution to (1.1) when the initial datum z_0 belongs to the stable set Σ_1.

Theorem 5.2. *Suppose that $n \geq n_\Omega$ and $\Omega \subset R^n$ be open bounded with Lipschitz boundary. Assume that $2 < r+1 < 2^*$. Let $z_0 \in \sum_1(0)$. Suppose $\alpha \in C^1[0,\infty)$ satisfies $\alpha(0) > 0$ and $\alpha'(t) \geq 0$ for all $t \in [0,\infty)$. Morever suppose that $\lim_{t\to\infty} \alpha(t) = 1$. Then there exists a global weak solution to* (1.1).

Proof. Since $z_0 \in \Sigma_1(0)$, there exists a constant $\epsilon_0 > 0$ so that

$$J(u_0, 0) + \epsilon_0 < n_\infty.$$

From Lemma 5.1 for every $k \in N$ there exists a weak solution $z_k \in C([0,T]; H_0^m(\Omega))$ with $z_k' \in L^2(0,T; H_0^m(\Omega))$ to the problem (5.1), here $z_{k0} \in C_c^\infty(\Omega)$ is so that

$$\lim_{k\to\infty} z_{k0} = z_0 \quad \text{in } H_0^m(\Omega).$$

By choosing a sufficiently large $k \in N$, we can also assume that

$$J(z_{k0}, 0) \le J(z_0, 0) + \epsilon_0 < n_\infty. \tag{5.2}$$

Using z_k' as a test function in (5.1), we get

$$\int_0^t \int_\Omega \rho_k^2 z_k'(s)^2\,dx\,ds + \int_0^t \int_\Omega \mathcal{A} z_k(s)\, z_k'(s)\,dx\,ds$$
$$= \int_0^t \int_\Omega \beta_k(z_k)\, z_k'(s)\,dx\,ds \le \int_0^t \int_\Omega |z_k(s)|^{r-1} z_k(s)\, z_k'(s)\,dx\,ds.$$

When you realize this

$$\int_\Omega \mathcal{A} z_k z_k' dx = \frac{d}{dt}\left(\frac{1}{2}\int_\Omega \left\|\mathcal{A}^{1/2} z_k\right\|^2 dx\right),$$

and

$$\int_\Omega |z_k|^{r-1} z_k z_k' dx = \frac{d}{dt}\left(\frac{1}{r+1}\int_\Omega \|z_k\|_{L^{r+1}(\Omega)}^{r+1} dx\right).$$

We can rewrite the above inequality as follows:

$$\int_0^t \int_\Omega \rho_k z_k'(s)^2\,dx\,ds + J(z_k(t), t) \le J(z_{k0}, 0) < n_\infty, \tag{5.3}$$

here we used (5.2) in the last step. This implies $z_k(t) \in \Sigma_1$ for every $t \in [0,T]$. Indeed, let us express the opposite statement by way of contradiction. Let t^* denote the minimal time at which $z_k(t^*) \notin \Sigma_1$. Utilizing the fact that $z_k \in C([0,T]; H_0^m(\Omega))$ we deduce that $z_k(t^*) \in \partial\Sigma_1$. In other words, either $J(z_k(t^*), t^*) = n_\infty$ or $I(z_k(t^*), t^*) = 0$. The former is impossible due to (5.3).

As a result, it is necessary to satisfy $I(z_k(t^*), t^*) = 0$ or equivalently,

$$\left\|\mathcal{A}^{\frac{1}{2}} z_k(t^*)\right\|^2 = \alpha(t^*)\, \|z_k(t^*)\|_{L^{r+1}(\Omega)}^{r+1},$$

which implies

$$J(z_k(t^*), t^*) = \frac{r-1}{2(r+1)} \left\|\mathcal{A}^{\frac{1}{2}} z_k(t^*)\right\|^2 \ge \frac{r-1}{2(r+1)} S_r^{-2} \|z_k(t^*)\|_{L^{r+1}(\Omega)}^2$$
$$= \frac{r-1}{2(r+1)} S_r^{-2} \left(\frac{\alpha(t^*)^{-1/2} \left\|\mathcal{A}^{\frac{1}{2}} z_k(t^*)\right\|}{\|z\|_{L^{r+1}(\Omega)}}\right)^{\frac{2}{r+1}\left(\frac{1}{2}-\frac{1}{r+1}\right)^{-1}}$$

$$\geq \frac{r-1}{2(r+1)} \alpha (t^*)^{2/(1-r)} S_r^{-2(r+1)/(r-1)} = n(t^*) \geq n_\infty.$$

This statement contradicts the information provided in inequality (5.3). Therefore, $z_k(t)$ belongs to the set Σ_1 for each t in the interval $[0,T]$, as asserted.

For $t \in [0,T]$, if $z_k(t) \in \Sigma_1$, it implies

$$\left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 = \alpha(t) \|z_k(t)\|_{L^{r+1}(\Omega)}^{r+1}.$$

By utilizing equation (5.3) we can derive the following inequality:

$$\int_0^t \int_\Omega \rho_k z_k'(s)^2 \, dx \, ds + \left(\frac{1}{2} - \frac{\alpha(t)}{r+1} \right) \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 < J(z_{k0}, 0) < n_\infty. \tag{5.4}$$

There is one in particular

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{r+1} \right) \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 &= \left(\frac{1}{2} - \frac{\alpha_\infty}{r+1} \right) \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 \\ &< \left(\frac{1}{2} - \frac{\alpha(t)}{r+1} \right) \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 < J(z_{k0}, 0), \end{aligned} \tag{5.5}$$

here $\alpha_\infty = \lim_{t\to\infty} \alpha(t) = 1$ by hypothesis. Utilizing the Lemma 2.6, (5.5) and (5.2), we get

$$\begin{aligned} \int_\Omega |z_k(t)|^{r+1} dx &< S_r^{r+1} \left(\left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 \right)^{(r+1)/2} = S_r^{r+1} \left(\left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 \right)^{(r+1)/2-1} \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 \\ &< S_r^{r+1} \left[\left(\frac{1}{2} - \frac{1}{r+1} \right)^{-1} J(z_{k0}, 0) \right]^{(r+1)/2-1} \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 \\ &< S_r^{r+1} \left[\left(\frac{1}{2} - \frac{1}{r+1} \right)^{-1} (J(z_0, 0) + \epsilon_0) \right]^{(r+1)/2-1} \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 \\ &= \delta \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2. \end{aligned} \tag{5.6}$$

Note that

$$0 < \delta < S_r^{r+1} \left[\left(\frac{1}{2} - \frac{1}{r+1} \right)^{-1} d_\infty \right]^{(r+1)/2-1} = \left[\left(\frac{1}{2} - \frac{1}{r+1} \right)^{-1} \frac{r-1}{2(r+1)} \right]^{(r-1)/2} = 1.$$

Next, we employ z_k as a test function in (5.1) to obtain

$$\begin{aligned} \frac{1}{2} \int_\Omega \rho_k z_k^2 \, dx + \int_0^t \int_\Omega \left| \mathcal{A}^{\frac{1}{2}} z_k(s) \right|^2 dx \, ds &\leq \int_0^t \int_\Omega |z_k(s)|^{r+1} dx \, ds + \frac{1}{2} \int_\Omega \rho_k z_{k0}^2 \, dx \\ &< \delta \int_0^t \int_\Omega \left| \mathcal{A}^{\frac{1}{2}} z_k(s) \right|^2 dx \, ds + \frac{1}{2} \int_\Omega \rho_k z_{k0}^2 \, dx, \end{aligned}$$

where we utilized reference (5.6) in the second step.

It can be deduced that

$$\frac{1}{2}\int_{\Omega}\rho_k z_k^2\,dx+(1-\delta)\int_0^t\int_{\Omega}\left|\mathcal{A}^{\frac{1}{2}}z_k\left(s\right)\right|^2dx\,ds<\frac{1}{2}\int_{\Omega}\rho_k z_{k0}^2\,dx<C, \tag{5.7}$$

here $C>0$ is independent of k and T. As a result, the sequence $\{z_k\}_{k\in N}$ is uniformly bounded in $L^2\left(0,T;H_0^m\left(\Omega\right)\right)$.

By (5.4) and (5.7), the following properties are satisfied:

$$\begin{cases} z_k \to z & \text{a.e. in } (0,T)\times\Omega, \\ \rho_k^{1/2} z_k \overset{\omega}{\to} \frac{z_t}{|x|^m} & \text{in } L^2\left(0,T;L^2\left(\Omega\right)\right), \\ \mathcal{A}^{\frac{1}{2}}z_k \overset{\omega}{\to} \mathcal{A}^{\frac{1}{2}}z & \text{in } L^2\left(0,T;L^2\left(\Omega\right)\right), \\ z_k \overset{\omega}{\to} z & \text{in } L^2\left(0,T;L^{r+1}\left(\Omega\right)\right), \\ z_k \overset{\omega}{\to} z & \text{in } L^2\left(0,T;L^{r+1}\left(\Omega\right)\right), \end{cases}$$

for all $T>0$. The theorem now follows by taking limits as $k\to\infty$ in (5.1). Since $T>0$ is arbitrary, the solution is global. □

Conflict of interest and data availability

The authors state that there is no conflict of interest and data availability not applicable to this article.

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