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An asymptotic estimate of Aoki's function

Vito Lampret[1,✉]

[1] *Faculty of Civil and Geodetic Engineering, University of Ljubljana, Ljubljana 1000, Slovenia.*
vito.lampret@guest.arnes.si✉

ABSTRACT

The Aoki's function $A(x) := \left(1+\frac{1}{x}\right)^x + \left(1-\frac{1}{x}\right)^{-x}$ is sharply estimated for $x \gg 1$. For example, we have the zero approximation given as

$$2e\left(1+\frac{1}{4x^2-1}\right) < A(x) < 2e\left(1+\frac{3}{4x^2-1}\right), \quad x \geq \frac{29}{14}.$$

RESUMEN

Estimamos ajustadamente la función de Aoki $A(x) := \left(1+\frac{1}{x}\right)^x + \left(1-\frac{1}{x}\right)^{-x}$ para $x \gg 1$. Por ejemplo, tenemos la aproximación cero dada por

$$2e\left(1+\frac{1}{4x^2-1}\right) < A(x) < 2e\left(1+\frac{3}{4x^2-1}\right), \quad x \geq \frac{29}{14}.$$

Keywords and Phrases: Approximation, estimate, expansion, exponential function, inequality.

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1 Introduction

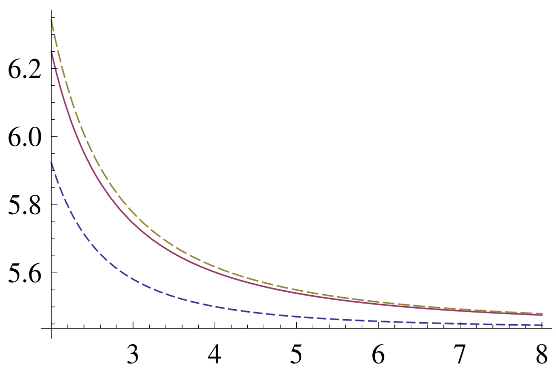
The Aoki's function $A(x)$,

$$A(x) := \left(1+\frac{1}{x}\right)^{x} + \left(1-\frac{1}{x}\right)^{-x}, \tag{1.1}$$

the sum of two strictly monotonic functions, increasing and decreasing respectively, has been estimated in [1, Theorem 1] as

$$\frac{e(2x^e-1)}{x^e-1} =: A_1(x) < A(x) < A_2(x) := \frac{e(2x^2-1)}{x^2-1} \quad (x>1). \tag{1.2}$$

Figure 1 (left), showing[1] the graphs of the functions $A_1(x)$, $A(x)$ and $A_2(x)$, discloses that the double inequality (1.2) is relatively rough. This fact has encouraged us to give more accurate approximations, which are illustrated in Figure 1 (right), where there are plotted the graphs of the functions $A_1^*(x)$, $A(x)$ and $A_2^*(x)$ from Example 3.5.



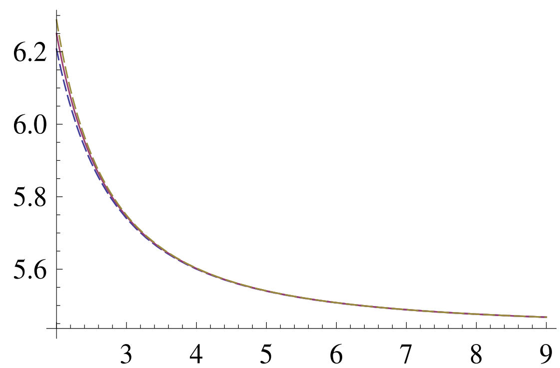


Figure 1: Left there are the graphs of the functions $A_1(x)$, $A(x)$ and $A_2(x)$. Right are illustrated the inequalities (3.1)–(3.2) in Example 3.5.

The main purpose of this article is to provide a sharp estimate of the function $A(x)$. The emphasis is on its brevity, a simple approach and its concrete sharpness (double inequalities), which is also important in some numerical treatments.

[1] All graphics in this paper are made using Mathematica [4].

2 Background – an expansion of the function $(1+y)^{1/y}$

According to [3, (20) and Theorem and Corollaries 1–2 on p. 105] there holds the following lemma.

Lemma 2.1. *For every real $y > -1$, we have the expansion*

$$(1+y)^{1/y} = \frac{2e}{y+2}\sum_{i=0}^{\infty} B_{2i}\cdot\left(\frac{y}{y+2}\right)^{2i}, \tag{2.1}$$

where the sequence B_{2n} is strictly monotonically decreasing, bounded as

$$B_2 = B_3 = \frac{5}{6} \quad and \quad \frac{7}{10} < \lim_{n\to\infty} B_n < B_n < \frac{8}{10}, \quad for\ n \geq 4, \tag{2.2}$$

and is given recursively as

$$B_0 = B_1 = 1, \quad B_{2m+1} = B_{2m} = \frac{1}{m}\sum_{j=1}^{m}\frac{4j+1}{4j+2}B_{2m-2j}, \quad for\ m \geq 1. \tag{2.3}$$

Lemma 2.1 implies the next lemma.

Lemma 2.2. *The equation* (2.1) *holds for any real y such that $|y| < 1$.*

Remark 2.3. *Instead of Lemma 2.1, we could also use the results of the paper [2], which provides the expansion $(1+x)^{1/x} = e\sum_{j=0}^{\infty}(-1)^j b_j x^j$ ($b_j \in \mathbb{R}^+$, $-1 < x \leq 1$). However, in this expansion, the convergence of the series is slower than the convergence of the series in the expansion $(1+x)^{1/x} = e\cdot\sum_{j=0}^{\infty}(-1)^j B_j\cdot\left(\frac{x}{x+2}\right)^j$ ($B_j \in \mathbb{R}^+$, $-1 < x \neq 0$), given in the paper [3].*

3 Expansion of the Aoki's function

Using $y = \pm\frac{1}{x}$ in Lemma 2.2, we get the following theorem.

Theorem 3.1. *The expansion*

$$A(x) = 2e\,x\sum_{i=0}^{\infty} B_{2i}\cdot\left(\frac{1}{(2x+1)^{2i+1}} + \frac{1}{(2x-1)^{2i+1}}\right)$$

holds true for any $x > 1$.

Proof. For $x > 1$, we have $\left|\pm\frac{1}{x}\right| < 1$. Consequently, using Lemma 2.2, the equation (2.1) holds for $y = \frac{1}{x}$ and also for $y = -\frac{1}{x}$. Therefore we obtain

$$\left(1+\frac{1}{x}\right)^x = \frac{2ex}{1+2x}\sum_{i=0}^{\infty} B_{2i}\cdot\left(\frac{1}{1+2x}\right)^{2i}$$

and

$$\left(1-\frac{1}{x}\right)^{-x}=\frac{2ex}{2x-1}\sum_{i=0}^{\infty}B_{2i}\cdot\left(\frac{1}{2x-1}\right)^{2i}. \qquad \square$$

Corollary 3.2. *For any integer $m \geq 0$ and every real $x > 1$, we have*

$$A(x)=A_m^*(x)+\delta_m(x),$$

where
$$A_m^*(x) := 2e\,x\sum_{i=0}^{m}B_{2i}\cdot\left(\frac{1}{(2x+1)^{2i+1}}+\frac{1}{(2x-1)^{2i+1}}\right)$$

and
$$0<\delta_m(x)<\delta_m^*(x) := \frac{e\,B_{2m+2}}{(x-1)(2x-1)^{2m+1}}<\frac{e}{(x-1)(2x-1)^{2m+1}}\,.$$

Proof. Referring to Theorem 3.1 and (2.2) in Lemma 2.1, we have

$$\begin{aligned}0<\delta_m(x) &= 2e\,x\cdot\sum_{i=m+1}^{\infty}B_{2i}\cdot\frac{2}{(2x-1)^{2i+1}}<4e\,x\cdot B_{2m+2}\cdot(2x-1)^{-(2m+3)}\sum_{i=0}^{\infty}(2x-1)^{-2i}\\ &= 4e\,x\cdot B_{2m+2}\cdot(2x-1)^{-(2m+3)}\cdot\frac{1}{1-(2x-1)^{-2}}=\frac{e\,B_{2m+2}}{(x-1)(2x-1)^{2m+1}}\,.\end{aligned}$$

Hence, referring to the estimates (2.2), we prove Corollary 3.2. $\square$

Remark 3.3. *In Corollary 3.2, m is a parameter that affect the error term $\delta_m(x)$.*

Example 3.4 (zero approximation)**.** *Setting $m = 0$ in Corollary 3.2 and using (2.2), we estimate*

$$\begin{aligned}2e\left(1+\frac{1}{4x^2-1}\right)<A(x)&<2e\left(1+\frac{1}{4x^2-1}\right)+\frac{5e}{6(x-1)(2x-1)},\quad x>1\\ &\leq 2e\left(1+\frac{3}{4x^2-1}\right),\quad x\geq\frac{29}{14}.\end{aligned}$$

Example 3.5. *Putting $m = 1$ in Corollary 3.2 and considering the equality $B_4 = \frac{287}{360}$, given by* (2.3)*, we obtain the following inequalities*

$$A(x)>2e\left(1+\frac{1}{4x^2-1}+\frac{10x^2(4x^2+3)}{3(4x^2-1)^3}\right) \tag{3.1}$$

$$A(x)<2e\left(1+\frac{1}{4x^2-1}+\frac{10x^2(4x^2+3)}{3(4x^2-1)^3}\right)+\frac{287e}{360(x-1)(2x-1)^3}\,. \tag{3.2}$$

Corollary 3.6. *For an integer $m \geq 0$ and a real $x > 1$, the relative error*

$$\rho_m(x) := \frac{A(x)-A_m^*(x)}{A(x)}$$

of the approximation $A(x) \approx A_m^*(x)$ satisfies the double inequality

$$0<\rho_m(x)<\rho_m^*(x) := \frac{B_{2m+2}}{2(x-1)(2x-1)^{2m+1}}<\frac{1}{2(x-1)(2x-1)^{2m+1}}\,.$$

Proof. According to Example 3.4, we have $A(x) > 2e$. Therefore, using Corollary 3.2, we get

$$\rho_m(x) = \frac{\big(A_m^*(x) + \delta_m(x)\big) - A_m^*(x)}{A(x)} < \frac{\delta_m^*(x)}{2e} = \frac{B_{2m+2}}{2(x-1)(2x-1)^{2m+1}}. \qquad \square$$

Example 3.7. *Thanks to Lemma 2.1 and Corollary 3.6, we have*

$$\rho_0^*(x) = \frac{5}{12(x-1)(2x-1)} \quad \textit{and} \quad \rho_1^*(x) = \frac{287}{720(x-1)(2x-1)^3}, \quad x > 1.$$

Figure 2 shows the graphs of the errors $\rho_1(x)$ and $\rho_1^*(x)$ on the left and the graphs of the quotient $\rho_1^*(x)/\rho_1(x)$ on the right respectively.

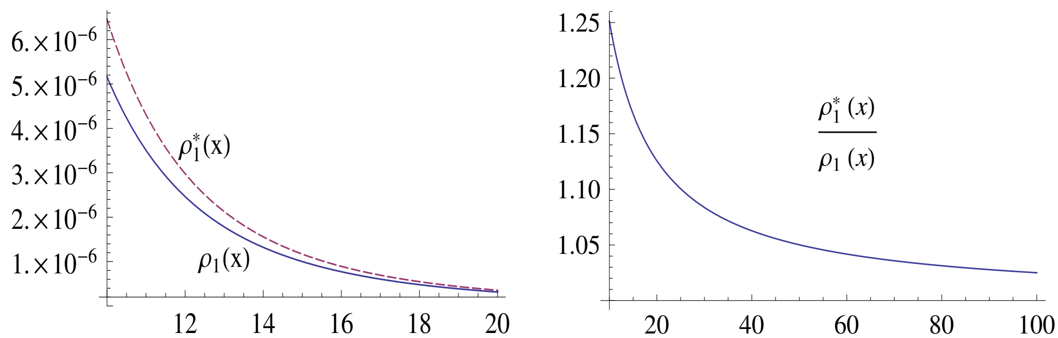


Figure 2: On the left are the graphs of the errors $\rho_1(x)$ and $\rho_1^*(x)$; on the right is the graph of the quotient $\rho_1^*(x)/\rho_1(x)$.

Remark 3.8. *A reviewer of this article suggested that the author rewrite the article following reviewer's suggestions, which, in his opinion, also include a better and much simpler approach to the problem at hand. The result of reviewer's intervention is his expansion*

$$A(x) = \sum_{n=0}^{\infty} \frac{a_{-2n}}{x^{2n}} = \sum_{n=0}^{m} \frac{a_{-2n}}{x^{2n}} + E_m(x),$$

where $a_{-2n} := \dfrac{2e}{(2n)!} D_{2n}$ *with* D_n *defined recursively as*

$$D_0 := 1, \quad D_m := \sum_{j=0}^{m-1} (-1)^{m-j} \frac{(m-1)!}{j!} \cdot \frac{m-j}{m+1-j} D_j, \quad m \geq 1$$

and estimated as

$$\big|D_m\big| < \frac{m!}{2}, \quad \big|E_m(x)\big| < \frac{2e|D_{2m+2}|}{(2m+2)! \cdot (x^{2m+2} - x^{2m+1})} < \frac{e}{(x-1)x^{2m+1}},$$

for $m \geq 1$. *However, the sequence* $(D_n)_{n\geq 0}$ *is not simple. Additionally, the crucial fact is that the series* $\sum_{n=0}^{\infty} \frac{a_{-2n}}{x^{2n}}$ *converges more slowly than the series* $\sum_{i=0}^{\infty} B_{2i} \cdot \left(\frac{1}{(2x+1)^{2i+1}} + \frac{1}{(2x-1)^{2i+1}}\right)$, *see Corollary 3.2.*

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