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Generalized translation and convolution operators in the realm of linear canonical deformed Hankel transform with applications

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ABSTRACT

Among the class of generalized Fourier transformations, the linear canonical transform is of pivotal importance mainly due to its higher degrees of freedom in lieu of the conventional Fourier and fractional Fourier transforms. This article is a continuation of our recent work *"Linear canonical deformed Hankel transform and the associated uncertainty principles, J. Pseudo-Differ. Oper. Appl.(2023), 14:29"*. Building upon this, we formulate the generalized translation and convolution operators associated with this newly proposed transformation. Besides, the obtained results are invoked to examine and obtain an analytical solution of the generalized heat equation. Finally, we study the heat semigroup pertaining to the generalized heat equation.

RESUMEN

Entre la clase de transformaciones de Fourier generalizadas, la transformada lineal canónica es de importancia central, mayormente debido a sus grados de libertad más altos en lugar de las transformadas convencionales de Fourier y de Fourier fraccionaria. Este artículo es una continuación de nuestro trabajo reciente *"Linear canonical deformed Hankel transform and the associated uncertainty principles, J. Pseudo-Differ. Oper. Appl.(2023), 14:29"*. Construyendo a partir de esto, formulamos los operadores de traslación y convolución generalizados asociados a esta nueva transformación propuesta. Además, los resultados obtenidos se utilizan para examinar y obtener una solución analítica de la ecuación de calor generalizada. Finalmente, estudiamos el semigrupo de calor pertinente a la ecuación de calor generalizada.

Keywords and Phrases: Deformed Hankel transform, linear canonical deformed Hankel transform, generalized translation, generalized convolution, heat semigroup, heat equation.

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1 Introduction

The Fourier transform is regarded as one of the remarkable discoveries in mathematical sciences as it profoundly influenced diverse branches of science and engineering. In the realm of harmonic analysis, the Fourier transform plays a pivotal role in analyzing signals wherein the characteristics are statistically invariant over time [6]. In the higher-dimensional scenario, there are several ways to arrive at the definition of the Fourier transform. The most basic formulation in $\mathbb{R}^d$ is given by the integral transform

$$\mathcal{F}(f)(\lambda) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)\, e^{-i\langle \lambda, x\rangle}\, dx. \tag{1.1}$$

Alternatively, one can rewrite the transform as

$$\mathcal{F}(f)(\lambda) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)\, \mathcal{K}(\lambda, x)\, dx, \tag{1.2}$$

where $\mathcal{K}(\lambda, x)$ is the unique solution to the system of partial differential equations

$$\begin{cases} \partial_{x_j}\mathcal{K}(\lambda, x) = -i\lambda_j \mathcal{K}(\lambda, x), & j = 1, \ldots, d, \\ \mathcal{K}(\lambda, 0) = 1, & \lambda \in \mathbb{R}^d \end{cases}$$

Yet another mathematical description of the higher-dimensional Fourier transform was proposed by Howe [44] via the Laplace operator $\triangle$ on $\mathbb{R}^d$ as follows:

$$\mathcal{F} = \exp\left(\frac{i\pi d}{4}\right) \exp\left(\frac{i\pi}{4}\left(\triangle - \|x\|^2\right)\right). \tag{1.3}$$

It is pertinent to mention that each of the above alternative representations has its specific use cases, and a detailed description regarding different ramifications of the Fourier transform can be found in [10]. Many generalizations of the Fourier transform can be attributed to a deeper understanding of the fundamental operators in Harmonic analysis. In the d-dimensional Euclidean space, the three elementary operators are the Laplace operator $\triangle$, norm $\|\cdot\|$, and the Euler operator $\mathbb{E}$, respectively defined as follows:

$$\triangle := \sum_{j=1}^{d} \partial_{x_j}^2, \quad \|x\|^2 := \sum_{j=1}^{d} x_j^2, \quad \mathbb{E} := \sum_{j=1}^{d} x_j \partial_{x_j},$$

As observed in [44], the operators

$$E = \frac{\|x\|^2}{2}, \quad F = -\frac{\triangle}{2}, \quad \text{and} \quad H = E + \frac{d}{2}$$

are invariant under $O(d)$ and generate the Lie algebra $\mathfrak{sl}_2$:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

Recently, there has been a lot of interest in other differential or difference operator realizations of $\mathfrak{sl}_2$ or other Lie (super) algebras. The focus is in particular on the generalized Fourier transforms that subsequently arise from these operator theoretic notions including the Dunkl transform [13], various discrete Fourier transforms in $\mathbb{R}^d$ [23], Fourier transforms in Clifford algebras [11] and many more. However, the hard problem in this context is to find explicit closed formulas for the integral kernel of the associated Fourier transforms. For further useful details regarding the generalized Fourier transforms and their implications, we refer the interested reader to [10].

Very recently, Ben Said *et al.* [3] have given a foundation for the deformation theory of the classical case, by constructing a generalization $\mathcal{F}_{k,a}$ of the Fourier transform, and the holomorphic semigroup $\mathcal{I}_{k,a}$ with infinitesimal generator

$$\mathcal{L}_{k,a,d} := \|x\|^{2-a} \triangle_k - \|x\|^a, \quad a > 0, \tag{1.4}$$

acting on a concrete Hilbert space deforming $L^2(\mathbb{R}^d)$, where $\triangle_k$ is the Dunkl Laplace operator. The authors have analyzed $\mathcal{F}_{k,a}$ and $\mathcal{I}_{k,a}(z)$ in the context of integral operators as well as representation theory. The deformation parameters consist of a real parameter a coming from the interpolation of the minimal unitary representations of two different reductive groups by keeping smaller symmetries, and a parameter k coming from Dunkl's theory of differential-difference operators associated with a finite Coxeter group (see [3]). In case $a = \frac{2}{n}$, $n \in \mathbb{N}$ and $d = 1$, we call the generalized Fourier transform $\mathcal{F}_{k,\frac{2}{n}}$, the deformed *Hankel transform* and will be denoted by $\mathcal{F}_{k,n}$.

As of now, the deformed Hankel transform $\mathcal{F}_{k,n}$ has witnessed an ample amount of research in the realm of harmonic analysis, which includes the study of kernel of the deformed Hankel transform [9], the generalized translation operator [2, 5, 30], the generalized maximal function [2], the Flett potentials [4], the deformed wavelet packets [19], uncertainty principles [25], the (k, n)-generalized wavelet multipliers [26], the (k, n)-generalized wavelet transform [27,29], the localization operators [34], the (k, n)-generalized Gabor transform [28], the (k, n)-generalized Stockwell transform [30], the (k, n)-generalized Wigner transform [32] and many more.

This paper is a continuation of the recent work carried out in the article *Linear canonical deformed Hankel transform and the associated uncertainty principles* [33]. Nonetheless, in [33], we have introduced and studied the linear canonical transform in the deformed Hankel frame (*i.e.* special case $a = \frac{2}{n}$, $n \in \mathbb{N}$ and $d = 1$). Recall that the classical linear canonical transform (LCT) was independently introduced by Collins [8] in paraxial optics, and Moshinsky, and Quesne [35] in quantum mechanics, to study the conservation of information and uncertainty under linear maps of phase space. The LCT is an integral transformation associated with a general homogeneous lossless linear mapping in phase space endowed with a total of three free parameters. The involved parameters constitute a 2×2 uni-modular matrix mapping the position x and the wave number y

into

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $ad - bc = 1$. The transformation maps any convex body into another convex body while preserving the area of the body. Such transformations constitute the homogeneous special group $SL(2,\mathbb{R})$. The linear canonical transform of any signal f with respect to a real matrix $M = (a,b;c,d) \in SL(2,\mathbb{R})$ with $b \neq 0$ is defined by

$$\mathcal{F}^M\big[f(x)\big](y) = \frac{1}{\sqrt{ib}} \int_{\mathbb{R}} f(x)\, \mathcal{K}^M(x,y)\, dy, \tag{1.5}$$

where

$$\mathcal{K}^M(x,y) = \exp\left\{ \frac{i}{2} \left(\frac{dx^2 + ay^2 - ixy}{b} \right) \right\}. \tag{1.6}$$

It is important to emphasize that the LCT provides a unified treatment of many generalized Fourier transforms in the sense that it is an embodiment of several well-known integral transforms including the Fourier transform [6,42], the fractional Fourier transform [1], the Fresnel transform [24], scaling operations and so on [7,21]. Due to the extra degrees of freedom and simple geometrical manifestation, the LCT is more flexible than other transforms and is as such suitable as well as a powerful tool for investigating deep problems in optics, quantum physics and signal processing [7,21]. Indeed, over a couple of decades, the application areas for LCT have been growing at an exponential rate and is as such befitting for investigating deep problems in signal analysis, filter design, phase retrieval problems, pattern recognition, radar analysis, holographic three-dimensional television, quantum physics, and many more. Apart from applications, the theoretical framework of LCT has likewise been extensively studied and investigated which has led to the formulation of convolution theorems [40], sampling theorems [22], Poisson summation formulae [45] and uncertainty principles [41]. For more about LCT and their applications, we allude to [7,21,37–39].

The main goal of this article is twofold. First, by employing the fundamental tools associated with the linear canonical deformed Hankel transform (LCDHT) [33], we introduce and investigate a generalized translation operator corresponding to the LCDHT. This operator is then utilized to define a convolution product, and several of its essential properties are examined. Subsequently, we establish the main theorems pertaining to the harmonic analysis in the framework of the LCDHT. Recognizing that the LCDHT represents a recent addition to the class of integral transforms, offering several additional degrees of freedom, we are further motivated to apply it to the heat equation. Therefore, the second objective of this paper is to study the generalized heat equation and the corresponding heat semigroup within the LCDHT setting. Thus, we can conclude that the principal contribution of this work lies in developing the harmonic analysis and exploring the generalized heat equation associated with a family of integral transforms such as the Dunkl, Bessel, and linear canonical Bessel (LCB) transforms [12,15–17]. Besides, our analysis extends to other integral transforms that have not yet been studied in this context, including the Dunkl fractional transform, the Dunkl Fresnel transform, and the LCD transform.

The remainder of this paper is organized as follows. Section 2 recalls the main results of the harmonic analysis associated with the deformed Hankel transform and the linear canonical deformed Hankel transform (LCDHT). Section 3 introduces and investigates the generalized translation operator corresponding to the LCDHT, along with an examination of its fundamental properties, including symmetry, commutativity, and continuity on certain functional spaces. Section 4 is devoted to the development and analysis of the generalized convolution product. In Section 5, we consider the generalized heat equation and the associated heat semigroup operator within the LCDHT framework. Finally, Section 6 presents the concluding remarks, summarizing the principal findings and outlining possible directions for future research.

2 Deformed Hankel transforms, translation and convolutions

In this section, we shall present the prerequisites concerning the deformed Hankel transform which shall be frequently used in formulating the main results. More precisely, we shall briefly review the conventional translation operators, deformed Hankel transform and the corresponding generalized translation and convolutions. For a detailed perspective, we refer to the articles [3, 5, 30] and the references therein.

2.1 Deformed Hankel transform

Let $L^p_{k,n}(\mathbb{R}), 1 \leq p \leq \infty$, be the space of measurable functions on $\mathbb{R}$ such that

$$\|f\|_{L^p_{k,n}(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p d\gamma_{k,n}(x)\right)^{1/p} < \infty, \quad \text{if} \quad 1 \leq p < \infty,$$
$$\|f\|_{L^\infty_{k,n}(\mathbb{R})} = \operatorname{ess}\sup_{x\in\mathbb{R}} |f(x)| < \infty,$$

where

$$d\gamma_{k,n}(x) := M_{k,n}|x|^{\frac{(2k-2)n+2}{n}} dx, \quad M_{k,n} = \frac{n^{\frac{n(2k-1)}{2}}}{2^{\frac{n(2k-1)+2}{2}}\Gamma\left(\frac{n(2k-1)+2}{2}\right)}, \quad k \geq \frac{n-1}{n}, \quad n \in \mathbb{N}.$$

For $p = 2$, the space is equipped with the scalar product:

$$\langle f, g\rangle_{L^2_{k,n}(\mathbb{R})} := \int_{\mathbb{R}} f(x)\overline{g(x)}\, d\gamma_{k,n}(x).$$

To facilitate our narrative, we set some notations as under:

- $C_b(\mathbb{R})$ the space of bounded continuous functions on $\mathbb{R}$.
- $C_{b,e}(\mathbb{R})$ the space of even bounded continuous functions on $\mathbb{R}$.
- $C_0(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$ and vanishing at infinity. We provide $C_0(\mathbb{R})$ with the topology of uniform convergence.
- $C_c(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$ and with compact support.
- $C^p(\mathbb{R})$ the space of functions of class C^p on $\mathbb{R}$.
- $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.
- $\mathfrak{S}_{k,n}(\mathbb{R})$ the space of all functions $f \in C^\infty(\mathbb{R}^*)$ such that

$$\sup_{x\in\mathbb{R}^*} |(|x|^{\frac{2}{n}})^j (|x|^{2-\frac{2}{n}} \triangle_k)^s (x^m f^{(m)}(x))| < \infty, \quad \text{for all } j, s, m \in \mathbb{N}_0.$$

- $SL(2,\mathbb{R})$ the group of 2×2 real matrices with determinant one.

We are now in a position to recall the notion of Dunkl operator. In this direction, we have the following definition:

For any $f \in C^1(\mathbb{R})$, the Dunkl operator T_k on $\mathbb{R}$ is defined by

$$T_k f(x) := f'(x) + 2k\, \frac{f(x) - f(-x)}{x}, \tag{2.1}$$

where as the corresponding Dunkl-Laplace operator $\triangle_k$, for any $f \in C^2(\mathbb{R})$, is given by

$$\triangle_k f(x) := T_k^2 f(x) = f''(x) + 2k \left(\frac{f'(x)}{x} - \frac{f(x) - f(-x)}{2x^2} \right). \tag{2.2}$$

Consider the operator

$$\Delta_{k,n} := \big|x\big|^{2-\frac{2}{n}} \triangle_k - \big|x\big|^{\frac{2}{n}}. \tag{2.3}$$

In the following, we recall some spectral properties of the differential-difference operator $\Delta_{k,n}$.

- $\Delta_{k,n}$ is an essentially self-adjoint operator on $L^2_{k,n}(\mathbb{R})$.
- There is no continuous spectrum of $\Delta_{k,n}$.
- The discrete spectrum of $-\Delta_{k,n}$ is $\big\{ \frac{4m}{n} + 2k + \frac{2}{n} \pm 1 : m \in \mathbb{N} \big\}$.

Definition 2.1. *For any* $f \in L^1_{k,n}(\mathbb{R})$ *and* $k \geq \dfrac{n-1}{n}$, $n \in \mathbb{N}$, *the deformed Hankel transform is denoted by* $\mathcal{F}_{k,n}(f)$ *and is given as*

$$\mathcal{F}_{k,n}(f)(\xi) = \int_{\mathbb{R}} f(x)\, B_{k,n}(\lambda, x)\, d\gamma_{k,n}(x), \quad \textit{for all } \lambda \in \mathbb{R}, \tag{2.4}$$

where $B_{k,n}(\lambda, x)$ *is the deformed Hankel kernel given by*

$$B_{k,n}(\lambda, x) = \jmath_{nk-\frac{n}{2}}\left(n|\lambda x|^{\frac{1}{n}}\right) + (-i)^n \left(\frac{n}{2}\right)^n \frac{\Gamma\left(nk - \frac{n}{2} + 1\right)}{\Gamma\left(nk + \frac{n}{2} + 1\right)} \lambda x \jmath_{nk+\frac{n}{2}}\left(n|\lambda x|^{\frac{1}{n}}\right). \tag{2.5}$$

Observe that

$$\jmath_\alpha(u) := \Gamma(\alpha+1)\left(\frac{u}{2}\right)^{-\alpha} J_\alpha(u) = \Gamma(\alpha+1) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\alpha+m+1)} \left(\frac{u}{2}\right)^{2m} \tag{2.6}$$

denotes the normalized Bessel function of index α.

Example 2.2. *The function* α_t, $t > 0$, *defined on* $\mathbb{R}$ *by*

$$\alpha_t(x) = \frac{1}{(2t)^{\frac{2nk+2-n}{2}}} e^{-\frac{n|x|^{\frac{2}{n}}}{4t}},$$

satisfies

$$\mathcal{F}_{k,n}(\alpha_t)(\xi) = e^{-nt|\xi|^{\frac{2}{n}}}, \quad \forall\, \xi \in \mathbb{R}.$$

Here, we list some important properties of the deformed Hankel kernel and transform:

(i) $B_{k,n}(z,t) = B_{k,n}(t,z)$, $\quad B_{k,n}(z,0) = 1$, $\overline{B_{k,n}(z,t)} = B_{k,n}((-1)^n z, t)$,
$B_{k,n}(\lambda z, t) = B_{k,n}(z, \lambda t)$, $\forall z, t, \lambda \in \mathbb{R}$.

(ii) $B_{k,n}(.,.)$ solves the following differential-difference equations on $\mathbb{R} \times \mathbb{R}$

$$\begin{cases} |\lambda|^{2-\frac{2}{n}} \triangle_k^\lambda B_{k,n}(\lambda, x) = -|x|^{\frac{2}{n}} B_{k,n}(\lambda, x), \\ |x|^{2-\frac{2}{n}} \triangle_k^x B_{k,n}(\lambda, x) = -|\lambda|^{\frac{2}{n}} B_{k,n}(\lambda, x). \end{cases}$$

where the superscript in $\triangle_k^x$ denotes the relevant variable.

(iii) For $k \geq 1/2$, $B_{k,n}(.,.)$ satisfies the following inequality

$$\left|B_{k,n}(x,y)\right| \leq 1, \quad \forall\, x, y \in \mathbb{R} \tag{2.7}$$

(iv) $B_{k,n}(.,.)$ is bounded if and only if

$$k \geq \frac{n-1}{2n}. \tag{2.8}$$

(v) Under the bounded condition (2.8), there always exists a finite positive constant C depending on n and k such that

$$\left|B_{k,n}(x,y)\right| \leq C, \quad \forall\, x,y \in \mathbb{R}. \tag{2.9}$$

(vi) ([31]). For $x,y \in \mathbb{R}$ and $\delta \in \mathbb{C}$ with $\mathrm{Re}\,\delta > 0$, we have

$$\int_{\mathbb{R}} e^{-\delta|\xi|^{2/n}}\, B_{k,n}(x,\xi)\, B_{k,n}(y,\xi)\, d\gamma_{k,n}(\xi) = \frac{e^{-(n^2/4\delta)(|x|^{2/n}+|y|^{2/n})}}{\left(\frac{2\delta}{n}\right)^{\frac{(2k-1)n+2}{2}}}\, B_{k,n}\left(\frac{x}{(\frac{2\delta}{n})^n}, (-i)^n y\right). \tag{2.10}$$

(vii) Under the bounded condition (2.8), the deformed Hankel transform $\mathcal{F}_{k,n}$ is bounded on $L^1_{k,n}(\mathbb{R})$. In particular, if $k \geq 1/2$,

$$\left\|\mathcal{F}_{k,n}(f)\right\|_{L^\infty_{k,n}(\mathbb{R})} \leq \left\|f\right\|_{L^1_{k,n}(\mathbb{R})}. \tag{2.11}$$

(viii) The deformed Hankel transform $\mathcal{F}_{k,n}$ provides a natural generalization of the conventional Hankel transform. For instance, if we set

$$B^{even}_{k,n}(x,y) = \frac{1}{2}\left(B_{k,n}(x,y) + B_{k,n}(x,-y)\right) = j_{nk-\frac{n}{2}}\left(n|xy|^{\frac{1}{n}}\right). \tag{2.12}$$

Then, $\mathcal{F}_{k,n}$ of an even function f on $\mathbb{R}$ specializes to a Hankel type transform on $\mathbb{R}_+$. In fact, when $f(x) = F(|x|)$ is an even function on $\mathbb{R}$ and belongs to $L^1_{k,n}(\mathbb{R})$, then

$$\mathcal{F}_{k,n}(f)(\xi) = \frac{(\frac{n}{2})^{(\frac{2nk-n}{2})}}{\Gamma\left(\frac{2nk+2-n}{2}\right)} \int_0^\infty F(r) j_{\frac{2nk-n}{2}}\left(n(r|\xi|)^{\frac{1}{n}}\right) r^{\frac{(2k-2)n+2}{n}}\, dr, \quad \forall\, \xi \in \mathbb{R}. \tag{2.13}$$

(ix) The deformed Hankel transform $f \mapsto \mathcal{F}_{k,n}(f)$ is an isometric isomorphism on $L^2_{k,n}(\mathbb{R})$ and satisfies [3]

$$\int_{\mathbb{R}} \left|\mathcal{F}_{k,n}(f)(\lambda)\right|^2 d\gamma_{k,n}(\lambda) = \int_{\mathbb{R}} \left|f(x)\right|^2 d\gamma_{k,n}(x). \tag{2.14}$$

(x) For all $f,g \in L^2_{k,n}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \mathcal{F}_{k,n}(f)(\lambda)\overline{\mathcal{F}_{k,n}(g)(\lambda)}\, d\gamma_{k,n}(\lambda) = \int_{\mathbb{R}} f(x)\overline{g(x)}\, d\gamma_{k,n}(x). \tag{2.15}$$

(xi) The deformed Hankel transform $\mathcal{F}_{k,n}$ is an involutive unitary operator on $L^1_{k,n}(\mathbb{R})$, that is;

$$\mathcal{F}^{-1}_{k,n}(f)(x) = \mathcal{F}_{k,n}(f)((-1)^n x), \quad x \in \mathbb{R}. \tag{2.16}$$

(xii) For any $f \in L^p_{k,n}(\mathbb{R}), 1 \leq p \leq 2$, the deformed Hankel transform $\mathcal{F}_{k,n}(f)$ belongs to $L^{p'}_{k,n}(\mathbb{R})$ and satisfies the following inequality:

$$\|\mathcal{F}_{k,n}(f)\|_{L^{p'}_{k,n}(\mathbb{R})} \leq \|f\|_{L^p_{k,n}(\mathbb{R})}, \tag{2.17}$$

where p' denotes the conjugate exponent of p.

(xiii) $\mathcal{F}_{k,n}(\mathcal{S}(\mathbb{R})) \subset C^\infty(\mathbb{R})$ if and only if $n = 1$.

(xiv) $\mathcal{F}_{k,n}(\mathcal{S}(\mathbb{R})) = \mathcal{S}(\mathbb{R})$ if and only if $n = 1$.

(xv) For any $f \in \mathcal{S}(\mathbb{R})$, we have

$$\mathcal{F}_{k,n}(f)(y) = F_1\left(|y|^{\frac{1}{n}}\right) + yF_2\left(|y|^{\frac{1}{n}}\right), \tag{2.18}$$

where the even functions $F_1, F_2 \in \mathcal{S}(\mathbb{R})$.

(xvi) The space $\mathfrak{S}_{k,n}(\mathbb{R})$ satisfies the following properties: (see [14]).

- $\mathcal{F}_{k,n}(\mathfrak{S}_{k,n}(\mathbb{R})) = \mathfrak{S}_{k,n}(\mathbb{R})$.
- The embedding $\mathfrak{S}_{k,n}(\mathbb{R}) \hookrightarrow L^p_{k,n}(\mathbb{R})$, $1 \leq p < \infty$, is continuous.
- $\mathfrak{S}_{k,n}(\mathbb{R})$ is a dense subset of $L^p_{k,n}(\mathbb{R})$, $1 \leq p < \infty$.

(xvii) The unitary operator $\mathcal{F}_{k,n}$ satisfies the following intertwining relations on a dense subspace of $L^2_{k,n}(\mathbb{R})$:

$$\mathcal{F}_{k,n} \circ |x|^{\frac{2}{n}} = -|x|^{2-\frac{2}{n}}\Delta_k \circ \mathcal{F}_{k,n}, \quad \mathcal{F}_{k,n} \circ |x|^{2-\frac{2}{n}}\Delta_k = -|x|^{\frac{2}{n}} \circ \mathcal{F}_{k,n}. \tag{2.19}$$

2.2 Generalized translation and convolution operators

Definition 2.3 ([27]). *The generalized translation operator $f \mapsto \tau^{k,n}_x f$ on $L^2_{k,n}(\mathbb{R})$ is defined by*

$$\mathcal{F}_{k,n}(\tau^{k,n}_x f) = \overline{B_{k,n}(.,x)}\mathcal{F}_{k,n}(f). \tag{2.20}$$

It is fruitful to have a class of functions in which (2.20) holds pointwise. One such class is the generalized Wigner space $\mathcal{W}_{k,n}(\mathbb{R})$ given by

$$\mathcal{W}_{k,n}(\mathbb{R}) := \left\{f \in L^1_{k,n}(\mathbb{R}) : \mathcal{F}_{k,n}(f) \in L^1_{k,n}(\mathbb{R})\right\}.$$

Following, we give several properties of the generalized translation operator [27].

(i) For any $f \in L^2_{k,n}(\mathbb{R})$, we have

$$\left\|\tau_x^{k,n} f\right\|_{L^2_{k,n}(\mathbb{R})} \leq \left\|f\right\|_{L^2_{k,n}(\mathbb{R})}, \quad \forall\, x \in \mathbb{R}. \tag{2.21}$$

(ii) For any $f \in \mathcal{W}_{k,n}(\mathbb{R})$, we have

$$\tau_x^{k,n} f(y) = \int_{\mathbb{R}} B_{k,n}((-1)^n x, \xi) B_{k,n}((-1)^n y, \xi) \mathcal{F}_{k,n}(f)(\xi)\, d\gamma_{k,n}(\xi), \quad \forall x, y \in \mathbb{R}. \tag{2.22}$$

(iii) For any $f \in \mathcal{W}_{k,n}(\mathbb{R})$, we have

$$\tau_x^{k,n} f(y) = \tau_y^{k,n}(f)(x), \quad \forall x, y \in \mathbb{R}. \tag{2.23}$$

(iv) For all f in $\mathcal{W}_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \tau_x^{k,n} f(y) g(y)\, d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y) \tau^{k,n}_{(-1)^n x} g(y)\, d\gamma_{k,n}(y), \quad \forall x \in \mathbb{R}. \tag{2.24}$$

(v) ([31]). For every $\delta > 0$, the (k,n)-generalized translation of the generalized Gaussian function is given by

$$\tau_x^{k,n}\left(e^{-\frac{n^2 |s|^{\frac{2}{n}}}{4\delta}}\right)(y) = e^{-n^2 \frac{|x|^{\frac{2}{n}} + |y|^{\frac{2}{n}}}{4\delta}} B_{k,n}\left(\frac{x}{(\frac{2\delta}{n})^n}, (i)^n y\right). \tag{2.25}$$

Recently, an explicit formula for the generalized translation operator $\tau_x^{k,n}$ has been reported in [5]:

Theorem 2.4. *For any $f \in C_b(\mathbb{R})$ and $k \geq \dfrac{n-1}{n}$, the generalized translation operator $\tau_x^{k,n}$ is given by*

$$\tau_x^{k,n} f(y) = \int_{\mathbb{R}} f(z)\, d\zeta^{k,n}_{x,y}(z), \tag{2.26}$$

where

$$d\zeta^{k,n}_{x,y}(z) = \begin{cases} \mathcal{K}_{k,n}(x,y,z) d\gamma_{k,n}(z), & \text{if } xy \neq 0, \\ d\delta_x(z), & \text{if } y = 0, \\ d\delta_y(z), & \text{if } x = 0, \end{cases} \tag{2.27}$$

$$\mathcal{K}_{k,n}(x,y,z) = K_{\mathrm{B}}^{nk-\frac{n}{2}}(|x|^{\frac{1}{n}}, |y|^{\frac{1}{n}}, |z|^{\frac{1}{n}})\, \nabla_{k,n}(x,y,z), \tag{2.28}$$

having support on the set $\left\{z \in \mathbb{R} : \left|\, |x|^{\frac{1}{n}} - |y|^{\frac{1}{n}} \right| < |z|^{\frac{1}{n}} < |x|^{\frac{1}{n}} + |y|^{\frac{1}{n}}\right\}$,

$$\nabla_{k,n}(x,y,z) = \frac{M_{k,n}}{2n}\left\{1 + (-1)^n \frac{n!\,\mathrm{sgn}(xy)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}\left(\Delta(|x|^{\frac{2}{n}}, |y|^{\frac{2}{n}}, |z|^{\frac{2}{n}})\right)\right.$$
$$\left. + \frac{n!\,\mathrm{sgn}(xz)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}\left(\Delta(|z|^{\frac{2}{n}}, |x|^{\frac{2}{n}}, |y|^{\frac{2}{n}})\right) + \frac{n!\,\mathrm{sgn}(yz)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}\left(\Delta(|z|^{\frac{2}{n}}, |y|^{\frac{2}{n}}, |x|^{\frac{2}{n}})\right)\right\}, \tag{2.29}$$

$$\Delta(u,v,w) = \frac{1}{2\sqrt{uv}}(u+v-w), \quad u,v,w \in \mathbb{R}_+^*, \tag{2.30}$$

$C_n^{nk-\frac{n}{2}}$ *the Gegenbauer polynomials and* $K_{\mathrm{B}}^{nk-\frac{n}{2}}$ *is the positive kernel given by*

$$K_{\mathrm{B}}^{nk-\frac{n}{2}}(u,v,w) =$$
$$\begin{cases} \frac{\Gamma(nk-\frac{n}{2}+1)}{2^{2nk-n-1}\Gamma(nk-\frac{n-1}{2})\Gamma(\frac{1}{2})} \frac{\left\{\left[(u+v)^2-w^2\right]\left[w^2-(u-v)^2\right]\right\}^{nk-\frac{n+1}{2}}}{(uvw)^{2nk-n}} & \text{if } |u-v| < w < u+v, \\ 0 & \text{elsewhere.} \end{cases}$$

Remark 2.5. *(i) For all $x, y, \lambda \in \mathbb{R}$, we have the following product formula:*

$$\tau_x^{k,n} B_{k,n}(\lambda, y) = B_{k,n}(\lambda, x) B_{k,n}(\lambda, y). \tag{2.31}$$

(ii) For all $x, y \in \mathbb{R}^*$, we have

$$\int_{\mathbb{R}} \mathcal{K}_{k,n}(x,y,z)\, d\gamma_{k,n}(z) = 1. \tag{2.32}$$

(iii) For all $x, y, z \in \mathbb{R}^*$, we have

$$\mathcal{K}_{k,n}(x,y,z) = \mathcal{K}_{k,n}(y,x,z). \tag{2.33}$$

(iv) For all $x, y, z \in \mathbb{R}^*$, we have

$$\mathcal{K}_{k,n}(x,y,z) = \mathcal{K}_{k,n}((-1)^n x, z, y). \tag{2.34}$$

(v) For all $x, y, z \in \mathbb{R}^*$, we have

$$\mathcal{K}_{k,n}(x,(-1)^n y,z) = \mathcal{K}_{k,n}(x,(-1)^n z, y). \tag{2.35}$$

(vi) For any $x, y \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} |\mathcal{K}_{k,n}(x,y,z)|\, d\gamma_{k,n}(z) \le 4. \tag{2.36}$$

On what follows we will recall the "trigonometric" form of the generalized translation operator proved in [30].

Theorem 2.6. *(i) For $f \in C_b(\mathbb{R})$ write $f = f_{\mathrm{e}} + f_{\mathrm{o}}$ as a sum of even and odd functions. Then*

$$\begin{aligned}\tau_x^{k,n} f(y) = \frac{M_{k,n}}{2n} \Bigg[\int_0^\pi f_{\mathrm{e}}\big(\langle\langle x,y\rangle\rangle_{\phi,n}\big) \left\{ 1 + (-1)^n \frac{n!\,\mathrm{sgn}(xy)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}(\cos\phi) \right\} (\sin\phi)^{2nk-n} d\phi \\ + \int_0^\pi f_{\mathrm{o}}\left(\langle\langle x,y\rangle\rangle_{\phi,n}\right) \left\{ \frac{n!\,\mathrm{sgn}(x)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}} \left(\frac{|x|^{\frac{1}{n}} - |y|^{\frac{1}{n}}\cos\phi}{\langle\langle x,y\rangle\rangle_{\phi,n}^{\frac{1}{n}}} \right) \right. \\ \left. + \frac{n!\,\mathrm{sgn}(y)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}} \left(\frac{|y|^{\frac{1}{n}} - |x|^{\frac{1}{n}}\cos\phi}{\langle\langle x,y\rangle\rangle_{\phi,n}^{\frac{1}{n}}} \right) \right\} (\sin\phi)^{2nk-n} d\phi \Bigg], \end{aligned} \tag{2.37}$$

where

$$\langle\langle x,y\rangle\rangle_{\phi,n} := \left(|x|^{\frac{2}{n}} + |y|^{\frac{2}{n}} - 2|xy|^{\frac{1}{n}}\cos\phi \right)^{\frac{n}{2}}. \tag{2.38}$$

(ii) For every $f \in C_{b,\mathrm{e}}(\mathbb{R})$, we have

$$\tau_x^{k,n} f(y) = \frac{M_{k,n}}{2n} \int_0^\pi f\big(\langle\langle x,y\rangle\rangle_{\phi,n}\big) \left\{ 1 + (-1)^n \frac{n!\,\mathrm{sgn}(xy)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}(\cos\phi) \right\} (\sin\phi)^{2nk-n} d\phi. \tag{2.39}$$

(iii) For every $\lambda > 0$, we have

$$\tau_x^{k,n}\left(e^{-\lambda|.|^{\frac{2}{n}}}\right)(y) = \frac{M_{k,n}}{2n} e^{-\lambda\left(|x|^{\frac{2}{n}} + |y|^{\frac{2}{n}}\right)} V_{k,n}(\lambda; x, y), \tag{2.40}$$

where

$$V_{k,n}(\lambda; x, y) := \int_0^\pi e^{2\lambda|xy|^{\frac{1}{n}}\cos\phi} \left\{ 1 + (-1)^n \frac{n!\,\mathrm{sgn}(xy)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}(\cos\phi) \right\} (\sin\phi)^{2nk-n} d\phi.$$

(iv) ([30]). Using (2.40), properties of the Gegenbauer polynomials and by simple calculations, we obtain

$$\left| \tau_x^{k,n}\left(e^{-\lambda|.|^{\frac{2}{n}}}\right)(y) \right| \leq \frac{M_{k,n}}{2n} e^{-\lambda\left(|x|^{\frac{1}{n}} - |y|^{\frac{1}{n}}\right)^2}. \tag{2.41}$$

Theorem 2.7 ([5]). *Let $\tau_x^{k,n}$ be the generalized translation operation as defined in* (2.19). *Then,*

(i) For any $f \in L^1_{\mathrm{loc}}(d\gamma_{k,n})$ and $k \geq \dfrac{n-1}{n}$, we have

$$\tau_x^{k,n} f(y) = \tau_y^{k,n} f(x), \quad \tau_0^{k,n} f = f.$$

(ii) For any $f \in L^p_{k,n}(\mathbb{R}), 1 \leq p \leq \infty$, we have

$$\big\|\tau_x^{k,n} f\big\|_{L^p_{k,n}(\mathbb{R})} \leq 4\big\|f\big\|_{L^p_{k,n}(\mathbb{R})}. \tag{2.42}$$

(iii) For every $f \in L^1_{k,n}(\mathbb{R})$, we have

$$\mathcal{F}_{k,n}(\tau_x^{k,n} f)(\lambda) = B_{k,n}((-1)^n\lambda, x)\mathcal{F}_{k,n} f(\lambda), \quad \lambda \in \mathbb{R}.$$

(iv) For any $f \in L^p_{k,n}(\mathbb{R}), 1 \leq p \leq 2$, we have

$$\mathcal{F}_{k,n}(\tau_x^{k,n} f)(\lambda) = B_k((-1)^n\lambda, x)\mathcal{F}_k(f)(\lambda), \quad a.e.\ \lambda \in \mathbb{R}. \tag{2.43}$$

(v) For all $f \in C_b(\mathbb{R})$ or belongs in $L^p_{k,n}(\mathbb{R}), 1 \leq p \leq \infty$, we have

$$\tau_x^{k,n}\tau_y^{k,n}(f) = \tau_y^{k,n}\tau_x^{k,n}(f). \tag{2.44}$$

Proposition 2.8. *If $f \in C_0(\mathbb{R})$, then we have*

$$\lim_{|x|\to\infty} \tau_x^{k,n}(f)(y) = 0.$$

Proof. For $f \in C_0(\mathbb{R})$, $y \in \mathbb{R}$ and $\phi \in [0,\pi]$, we have

$$\lim_{|x|\to\infty} f_{\mathrm{e}}\big(\langle\!\langle x, y\rangle\!\rangle_{\phi,n}\big) = \lim_{|x|\to\infty} f_{\mathrm{o}}\big(\langle\!\langle x, y\rangle\!\rangle_{\phi,n}\big) = 0.$$

Using Theorem 2.6 *(i)*, the properties of the Gegenbauer polynomials, an application of dominated convergence theorem give the desired result. ☐

Theorem 2.9 ([30]). *Let $L^p_{k,n,e}(\mathbb{R})$ be the space of even functions in $L^p_{k,n}(\mathbb{R})$. Then,*

(i) For every bounded and non-negative function $f \in L^1_{k,n,e}(\mathbb{R})$, we have $\tau^{k,n}_x f \geq 0$, $\tau^{k,n}_x f \in L^1_{k,n}(\mathbb{R})$, $\forall\, x \in \mathbb{R}$, and

$$\int_{\mathbb{R}} \tau^{k,n}_x f(y)\, d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y)\, d\gamma_{k,n}(y). \tag{2.45}$$

(ii) For any $f \in L^p_{k,n,e}(\mathbb{R})$, we have

$$\big\|\tau^{k,n}_x f\big\|_{L^p_{k,n,e}(\mathbb{R})} \leq \big\|f\big\|_{L^p_{k,n,e}(\mathbb{R})}. \tag{2.46}$$

(iii) For every $f \in L^1_{k,n}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \tau^{k,n}_x f(y)\, d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y)\, d\gamma_{k,n}(y). \tag{2.47}$$

(iv) If f_1 and f_2 are two suitable functions, we have

$$\int_{\mathbb{R}} \tau^{k,n}_y f_1((-1)^n t) f_2(t) d\gamma_{k,n}(t) = \int_{\mathbb{R}} \tau^{k,n}_y f_2((-1)^n t) f_1(t) d\gamma_{k,n}(t), \quad y \in \mathbb{R}. \tag{2.48}$$

Definition 2.10. *The generalized convolution product of two suitable functions $f, g \in L^2_{k,n}(\mathbb{R})$ is defined by*

$$f *_{k,n} g(x) = \int_{\mathbb{R}} \tau^{k,n}_x f((-1)^n y)\, g(y)\, d\gamma_{k,n}(y). \tag{2.49}$$

It is pertinent to mention that the convolution product (2.49) is both commutative and associative. We culminate this subsection by giving the following important results.

Proposition 2.11 ([5]). *Let $f *_{k,n} g(x)$ be the generalized convolution as defined in (2.49). Then,*

(i) For any $f \in L^2_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R})$, we have

$$f *_{k,n} g(x) = \int_{\mathbb{R}} \tau^{k,n}_x f((-1)^n y) g(y)\, d\gamma_{k,n}(y). \tag{2.50}$$

(ii) For every $f \in L^p_{k,n}(\mathbb{R})$ and $g \in L^q_{k,n}(\mathbb{R})$ with $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$, the convolution product $f *_{k,n} g$ belongs to $L^r_{k,n}(\mathbb{R})$ and satisfies the inequality:

$$\big\|f *_{k,n} g\big\|_{L^r_{k,n}(\mathbb{R})} \leq 4\|f\|_{L^p_{k,n}(\mathbb{R})}\|g\|_{L^q_{k,n}(\mathbb{R})}. \tag{2.51}$$

(iii) For every $f \in L^2_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R})$, we have

$$\mathcal{F}_{k,n}(f *_{k,n} g) = \mathcal{F}_{k,n}(f)\mathcal{F}_{k,n}(g). \tag{2.52}$$

(iv) *For $f,g \in L^2_{k,n}(\mathbb{R})$, the convolution $f *_{k,n} g \in L^2_{k,n}(\mathbb{R})$ if and only if $\mathcal{F}_{k,n}(f)\mathcal{F}_{k,n}(g) \in L^2_{k,n}(\mathbb{R})$ and satisfies [27]*

$$\mathcal{F}_{k,n}(f *_{k,n} g) = \mathcal{F}_{k,n}(f)\,\mathcal{F}_{k,n}(g). \tag{2.53}$$

(v) *For every $f,g \in L^2_{k,n}(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} \left|f *_{k,n} g(x)\right|^2 d\gamma_{k,n}(x) = \int_{\mathbb{R}} \left|\mathcal{F}_{k,n}(f)(\xi)\right|^2 |\mathcal{F}_{k,n}(g)(\xi)|^2 d\gamma_{k,n}(\xi). \tag{2.54}$$

2.3 Deformed Hankel transform in linear canonical domain

In this section, we recall some results proved in [33].

Definition 2.12. *The deformed linear canonical Hankel transform of any function $f \in L^1_{k,n}(\mathbb{R})$, with respect to the uni-modular matrix $M = (a,b;c,d) \in SL(2,\mathbb{R})$ is defined by*

$$\mathcal{F}^M_{k,n}(f)(x) = \frac{1}{(ib)^{\frac{(2k-1)n+2}{2n}}} \int_{\mathbb{R}} K^M_{k,n}(x,y) f(y)\, d\gamma_{k,n}(y), \tag{2.55}$$

where

$$K^M_{k,n}(x,y) = e^{\frac{i}{2}\left(\frac{d}{b}x^2 + \frac{a}{b}y^2\right)} B_{k,n}\left(\frac{x}{b}, y\right). \tag{2.56}$$

Definition 2.12 allows us to make the followings comments:

(i) For $M = (1,b,0,1)$, the deformed linear canonical Hankel transform (2.55) coincides with the Fresnel transform associated with the deformed Hankel transform:

$$\mathcal{W}^b_{k,n} f(x) = \begin{cases} \dfrac{1}{(ib)^{\frac{(2k-1)n+2}{2n}}} \displaystyle\int_{\mathbb{R}} E^b_{k,n}(x,y) f(y)\, d\gamma_{k,n}(y), & b \neq 0, \\ f(x), & b = 0, \end{cases}$$

where $E^b_{k,n}(x,y) = e^{\frac{i}{2b}(x^2+y^2)} B_{k,n}\left(\frac{x}{b}, y\right)$.

(ii) For $M = (\cosh(b), \sinh(b); \sinh(b), \cosh(b))$, $b \in \mathbb{R}$, the deformed linear canonical Hankel transform (2.55) boils down to the following integral transform

$$\mathcal{V}^b_{k,n} f(x) = \begin{cases} \dfrac{1}{(i\sinh(b))^{\frac{(2k-1)n+2}{2n}}} \displaystyle\int_{\mathbb{R}} R^b_{k,n}(x,y) f(y)\, d\gamma_{k,n}(y), & b \neq 0, \\ f(x), & b = 0, \end{cases}$$

where $R^b_{k,n}(x,y) = e^{\frac{i}{2}\coth(b)(x^2+y^2)} B_{k,n}\left(\frac{x}{\sinh(b)}, y\right)$.

(iii) For $M = (\cos\alpha, \sin\alpha; -\sin\alpha, \cos\alpha)$, $\alpha \in \mathbb{R}$, the deformed linear canonical Hankel transform (2.55) coincides with the fractional deformed Hankel transform $\mathcal{F}^{\alpha}_{k,n}$:

$$\mathcal{F}^{\alpha}_{k,n} f(x) = \begin{cases} \dfrac{e^{i\left(\frac{(2k-1)n+2}{2n}\right)(\alpha - 2n\pi) - \hat{\alpha}\pi/2)}}{|\sin(\alpha)|^{\frac{(2k-1)n+2}{2n}}} \displaystyle\int_{\mathbb{R}} \mathcal{K}^{\alpha}_{k,n}(x,y) f(y)\, d\gamma_{k,n}(y), & (2j-1)\pi < \alpha < (2j+1)\pi, \\ f(x), & \alpha = 2j\pi, \\ f(-x), & \alpha = (2j+1)\pi, \end{cases}$$

where $\hat{\alpha} = \mathrm{sgn}(\sin(\alpha))$, $\mathcal{K}^{\alpha}_{k,n}(x,y) = e^{-\frac{i}{2}\cot(\alpha)(x^2+y^2)} B_{k,n}\left(\frac{x}{\sin(\alpha)}, y\right)$.

Definition 2.13. *For any uni-modular matrix $M \in SL(2,\mathbb{R})$, the differential-difference operator $\triangle^M_{k,n}$ is defined by*

$$\triangle^M_{k,n} := |x|^{2(1-\frac{1}{n})} \left\{ \frac{d^2}{dx^2} + \left(\frac{2k}{x} - 2i\frac{d}{b}x \right) \frac{d}{dx} - \left(\frac{d^2}{b^2} x^2 + (2k+1)i\frac{d}{b} + \frac{k}{x^2}(1-s) \right) \right\}, \tag{2.57}$$

where $s(u(x)) := u(-x)$.

Definition 2.13 allows us to make the following comments:

(i) For $M = (0,1;-1,0)$, $\triangle^M_{k,n}$ boils down to the deformed Laplace operator $\triangle_{k,n}$ whereas $\mathcal{F}^M_{k,n}$ coincides with the deformed Hankel transform $\mathcal{F}_{k,n}$ (except for a constant unimodular factor $(e^{i\frac{\pi}{2}})^{\frac{(2k-1)n+2}{2n}}$).

(ii) $\triangle^M_{k,n}$ is related to the deformed Laplace operator $\triangle_{k,n}$ via

$$e^{-\frac{i}{2}\frac{d}{b}x^2} \circ \triangle^M_{k,n} \circ e^{\frac{i}{2}\frac{d}{b}x^2} = \triangle_{k,n} + |x|^{\frac{2}{n}}. \tag{2.58}$$

(iii) For any $f, g \in \mathcal{S}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \triangle^M_{k,n} f(x) \overline{g(x)}\, d\gamma_{k,n}(x) = \int_{\mathbb{R}} f(x) \overline{\triangle^M_{k,n} g(x)}\, d\gamma_{k,n}(x). \tag{2.59}$$

(iv) For each $y \in \mathbb{R}$, the kernel $K^M_{k,n}(.,y)$ of the linear canonical deformed Hankel transform $\mathcal{F}^M_{k,n}$ satisfy the following:

$$\begin{cases} \triangle^M_{k,n} K^M_{k,n}(.,y) = -|\frac{y}{b}|^{\frac{2}{n}} K^M_{k,n}(.,y), \\ K^M_{k,n}(0,y) = e^{\frac{i}{2}\frac{a}{b}y^2}. \end{cases} \tag{2.60}$$

(v) For each $x, y \in \mathbb{R}$, we have

$$|K^M_{k,n}(x,y)| \le 1. \tag{2.61}$$

Theorem 2.14. *Let $M = (a, b; c, d) \in SL(2, \mathbb{R})$. Then,*

(i) For any $f \in L^1_{k,n}(\mathbb{R})$, $\mathcal{F}^M_{k,n}(f)$ belongs to $C_0(\mathbb{R})$ and satisfies the following inequality:

$$\left\|\mathcal{F}^M_{k,n}(f)\right\|_{L^\infty_{k,n}(\mathbb{R})} \leq |b|^{-\frac{(2k-1)n+2}{2n}} \left\|f\right\|_{L^1_{k,n}(\mathbb{R})}. \tag{2.62}$$

(ii) For every $f \in L^1_{k,n}(\mathbb{R})$ with $\mathcal{F}^M_{k,n}(f) \in L^1_{k,n}(\mathbb{R})$, we have

$$\left(\mathcal{F}^M_{k,n} \circ \mathcal{F}^{M^{-1}}_{k,n}\right)(f) = \left(\mathcal{F}^{M^{-1}}_{k,n} \circ \mathcal{F}^M_{k,n}\right)(f) = s_{n+1}(f) \quad a.e., \tag{2.63}$$

where $s_j(f)(x) := f((-1)^j x)$, $\forall\, x \in \mathbb{R}$, $j \in \mathbb{N}$.

(iii) $\mathcal{F}^M_{k,n}$ is a topological isomorphism from $L^2_{k,n}(\mathbb{R})$ into itself.

(iv) $\mathcal{F}^M_{k,n}$ is a topological isomorphism from $\mathfrak{S}_{k,n}(\mathbb{R})$ into itself.

(v) For any $f, g \in L^1_{k,n}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \mathcal{F}^M_{k,n}(f)(x)\overline{g(x)}\, d\gamma_{k,n}(x) = \int_{\mathbb{R}} f(x)\overline{\mathcal{F}^{M^{-1}}_{k,n}(g)(x)}\, d\gamma_{k,n}(x).$$

(vi) If $f \in L^1_{k,n}(\mathbb{R}) \cap L^2_{k,n}(\mathbb{R})$, then $\mathcal{F}^M_{k,n}(f) \in L^2_{k,n}(\mathbb{R})$ and

$$\left\|\mathcal{F}^M_{k,n}(f)\right\|_{L^2_{k,n}(\mathbb{R})} = \left\|f\right\|_{L^2_{k,n}(\mathbb{R})}. \tag{2.64}$$

(vii) For any $f, g \in L^2_{k,n}(\mathbb{R})$, we have

$$\left\langle \mathcal{F}^M_{k,n}(f), g\right\rangle_{L^2_{k,n}(\mathbb{R})} = \left\langle f, \mathcal{F}^{M^{-1}}_{k,n} g\right\rangle_{L^2_{k,n}(\mathbb{R})}. \tag{2.65}$$

(viii) (Operational formulas). Let $M \in SL(2, \mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$. Then we have

$$\mathcal{F}^M_{k,n}\left[|y|^{\frac{2}{n}} f(y)\right] = -|b|^{\frac{2}{n}} \triangle^M_{k,n}\left[\mathcal{F}^M_{k,n}(f)\right], \tag{2.66}$$

and

$$|x|^{\frac{2}{n}} \mathcal{F}^M_{k,n}(f) = -|b|^{\frac{2}{n}} \mathcal{F}^M_{k,n}\left[\triangle^{M^{-1}}_{k,n}(f)\right]. \tag{2.67}$$

Definition 2.15. *The deformed linear canonical Hankel transform of any function $f \in L^p_{k,n}(\mathbb{R})$, $1 \leq p \leq 2$ with respect to the uni-modular matrix $M = (a, b; c, d) \in SL(2, \mathbb{R})$ is defined by*

$$\mathcal{F}^M_{k,n}(f) = e^{-i\left(\frac{(2k-1)n+2}{2n}\right)\frac{\pi}{2}\,\mathrm{sgn}(b)}\Big(\mathbf{L}_{\frac{d}{b}} \circ \Delta_b \circ \mathcal{F}_{k,n} \circ \mathbf{L}_{\frac{a}{b}}\Big)(f), \tag{2.68}$$

where $\mathcal{F}_{k,n} : L^p_{k,n}(\mathbb{R}) \to L^{p'}_{k,n}(\mathbb{R})$ is the deformed Hankel transformation on $L^p_{k,n}(\mathbb{R})$, $\mathbf{L}_{d/b}$ and Δ_b are the chirp multiplication and dilation operators, defined respectively, by

$$\mathbf{L}_s f(x) = e^{\frac{is}{2}x^2} f(x),\ s \in \mathbb{R} \quad \text{and} \quad \Delta_s f(x) = \frac{1}{|s|^{\frac{(2k-1)n+2}{2n}}} f\left(\frac{x}{s}\right),\ s \in \mathbb{R}^*. \tag{2.69}$$

Theorem 2.16 (Young's inequality)**.** *For any uni-modular matrix $M = (a, b; c, d) \in SL(2, \mathbb{R})$ and $1 \leq p \leq 2$, $\mathcal{F}^M_{k,n}$ satisfies the following inequality:*

$$\left\|\mathcal{F}^M_{k,n}(f)\right\|_{L^{p'}_{k,n}(\mathbb{R})} \leq |b|^{\left(\frac{(2k-1)n+2}{2n}\right)\left(\frac{2}{p'}-1\right)} \left\|f\right\|_{L^p_{k,n}(\mathbb{R})}. \tag{2.70}$$

3 Generalized translations associated with LCDHT

Definition 3.1. *Let $M = (a, b; c, d) \in SL(2, \mathbb{R})$, $b \neq 0$, a given uni-modular matrix. For suitable function f, we define the generalized translation operator associated with the operator $\triangle^M_{k,n}$ by*

$$T^{M,k,n}_x f(y) = e^{\frac{i}{2}\frac{d}{b}(x^2+y^2)} \tau^{k,n}_x \left[e^{-\frac{i}{2}\frac{d}{b}s^2} f(s)\right](y), \tag{3.1}$$

where $\tau^{k,n}_x$ is the (k, n)-generalized translation operator associated with $\Delta_{k,n}$.

We will rely on this definition for each function on the following spaces:

- $L^p_{k,n}(\mathbb{R}), 1 \leq p \leq \infty$.
- $C_b(\mathbb{R})$.

Some important properties of the generalized translation operator $T^{M,k,n}_x$ are assembled in the following theorem.

Theorem 3.2. *Let $M = (a, b; c, d) \in SL(2, \mathbb{R})$, $b \neq 0$, then the generalized translation operator $T^{M,k,n}_x$ as defined in (3.1) satisfies the following properties:*

(i) Linearity: $T^{M,k,n}_x [\alpha f + \beta g](y) = \alpha T^{M,k,n}_x f(y) + \beta T^{M,k,n}_x g(y)$, $\alpha, \beta \in \mathbb{R}$.

(ii) Symmetry: $T^{M,k,n}_0 = Id$, $T^{M,k,n}_x f(y) = T^{M,k,n}_y f(x)$, $\forall\, x, y \in \mathbb{R}$.

(iii) Product Formula: For every $x, , y, z \in \mathbb{R}$, *we have*

$$T_x^{M,k,n}\left[K_{k,n}^M(.,y)\right](z) = e^{-\frac{i}{2}\frac{a}{b}y^2}\, K_{k,n}^M(x,y)K_{k,n}^M(z,y). \tag{3.2}$$

(iv) Commutative: We have

$$T_x^{M,k,n}\circ T_y^{M,k,n} = T_y^{M,k,n}\circ T_x^{M,k,n} \quad \textit{and} \quad \triangle_{k,n}^M \circ T_x^{M,k,n} = T_x^{M,k,n}\circ \triangle_{k,n}^M. \tag{3.3}$$

(v) Let $f \in \mathfrak{S}_{k,n}(\mathbb{R})$. *The function* $u(x,y) = T_x^{M,k,n}f(y)$ *is a solution of the problem*

$$\begin{cases} \Delta_{x,k,n}^M\ u(x,y) = \Delta_{y,k,n}^M\ u(x,y) \\ u(x,0) = f(x). \end{cases} \tag{3.4}$$

(vi) For all $x, y \in \mathbb{R}$, *we have*

$$T_x^{M,k,n}f(y) = \int_{\mathbb{R}} e^{-i\frac{d}{b}z^2} f(z)\ \mathcal{W}_{k,n}^M(x,y,z)\, d\gamma_{k,n}(z), \tag{3.5}$$

where

$$\mathcal{W}_{k,n}^M(x,y,z) = e^{\frac{i}{2}\frac{d}{b}(x^2+y^2+z^2)}\, \mathcal{K}_{k,n}(x,y,z). \tag{3.6}$$

(vii) The generalized translation operator $T_x^{M,k,n}$ *is continuous from* $C_b(\mathbb{R})$ *into itself. Moreover, the operator is also continuous from* $L_{k,n}^p(\mathbb{R})$, $1 \leq p \leq \infty$, *into itself and satisfies the following inequality:*

$$\left\|T_x^{M,k,n}f\right\|_{L_{k,n}^p(\mathbb{R})} \leq 4\, \|f\|_{L_{k,n}^p(\mathbb{R})}\,. \tag{3.7}$$

(viii) For any $f \in L_{k,n}^1(\mathbb{R})$ *and* $g \in C_b(\mathbb{R})$, *we have*

$$\int_{\mathbb{R}} \left[T_x^{M,k,n}f((-1)^n y)\right]\ \left[e^{-i\frac{d}{b}y^2} g(y)\right] d\gamma_{k,n}(y) = \int_{\mathbb{R}} \left[e^{-i\frac{d}{b}y^2} f(y)\right] \left[T_x^{M,k,n}g((-1)^n y)\right] d\gamma_{k,n}(y). \tag{3.8}$$

(ix) For any $f \in L_{k,n}^1(\mathbb{R})$, *we have*

$$\mathcal{F}_{k,n}^M\left[T_x^{M^{-1},k,n}f\right](\lambda) = e^{\frac{i}{2}\frac{d}{b}\lambda^2}\ \overline{K_{k,n}^M(\lambda,x)}\, \mathcal{F}_{k,n}^M(f)(\lambda), \quad \lambda \in \mathbb{R}. \tag{3.9}$$

(x) For every $f \in L_{k,n}^p(\mathbb{R})$, $1 < p \leq 2$, *we have*

$$\mathcal{F}_{k,n}^M\left[T_x^{M^{-1},k,n}f\right](\lambda) = e^{\frac{i}{2}\frac{d}{b}\lambda^2}\ \overline{K_{k,n}^M(\lambda,x)}\, \mathcal{F}_{k,n}^M(f)(\lambda), \quad a.e. \tag{3.10}$$

(xi) If $f \in C_0(\mathbb{R})$, then we have

$$\lim_{|x|\to\infty} T_x^{M^{-1},k,n} f(y) = 0, \quad y \in \mathbb{R}. \tag{3.11}$$

Proof. Using (3.1), we establish the proof of (i) and (ii).

(iii) Invoking Definition 3.1 and (2.31), we observe that

$$\begin{aligned} T_x^{M,k,n}\left[K_{k,n}^M(\cdot,y)\right](z) &= e^{\frac{i}{2}\frac{d}{b}(x^2+z^2)}\tau_x^{k,n}\left[s\mapsto e^{\frac{i}{2}\frac{a}{b}y^2}B_{k,n}\left(\frac{s}{b},y\right)\right](z)\\ &= e^{\frac{i}{2}\frac{d}{b}(x^2+z^2)}e^{\frac{i}{2}\frac{a}{b}y^2}\tau_x^{k,n}\left[s\mapsto B_{k,n}\left(\frac{s}{b},y\right)\right](z)\\ &= e^{\frac{i}{2}\frac{d}{b}(x^2+z^2)}e^{\frac{i}{2}\frac{a}{b}y^2}B_{k,n}\left(\frac{x}{b},y\right)B_{k,n}\left(\frac{z}{b},y\right)\\ &= e^{-\frac{i}{2}\frac{a}{b}y^2}K_{k,n}^M(x,y)K_{k,n}^M(z,y). \end{aligned}$$

(iv) For any $f \in L^p_{k,n}(\mathbb{R})$, $1 \le p \le \infty$ (or $f \in C_b(\mathbb{R})$), (3.1) and Theorem 2.7 imply that

$$\begin{aligned} \left[T_x^{M,k,n}\circ T_y^{M,k,n}\right]f(z) &= e^{\frac{i}{2}\frac{d}{b}(x^2+y^2+z^2)}\left[\tau_x^{k,n}\circ\tau_y^{k,n}\right]\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](z)\\ &= e^{\frac{i}{2}\frac{d}{b}(x^2+y^2+z^2)}\left[\tau_y^{k,n}\circ\tau_x^{k,n}\right]\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](z)\\ &= \left[T_y^{M,k,n}\circ T_x^{M,k,n}\right]f(z). \end{aligned}$$

Moreover, for any $f \in \mathfrak{S}_{k,n}(\mathbb{R})$, identities (2.58) and (2.19) imply that

$$\begin{aligned} \left[\triangle_{k,n}^M\circ T_x^{M,k,n}\right]f(y) &= e^{\frac{i}{2}\frac{d}{b}(x^2+y^2)}\left[|x|^{2-\frac{2}{n}}\triangle_k\circ\tau_x^{k,n}\right]\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](y)\\ &= e^{\frac{i}{2}\frac{d}{b}(x^2+y^2)}\left[\tau_x^{k,n}\circ|x|^{2-\frac{2}{n}}\triangle_k\right]\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](y)\\ &= \left[T_x^{M,k,n}\circ\triangle_{k,n}^M\right]f(y). \end{aligned}$$

(v) Since system (3.4) is equivalent to

$$\begin{cases} |x|^{2-\frac{2}{n}}\triangle_{k,x}\tilde{u}(x,y) = |y|^{2-\frac{2}{n}}\triangle_{k,y}\tilde{u}(x,y),\\ \tilde{u}(x,0) = e^{-\frac{i}{2}\frac{d}{b}x^2}f(x), \end{cases}$$

where $\tilde{u}(x,y) = e^{-\frac{i}{2}\frac{d}{b}(x^2+y^2)}u(x,y)$. Therefore, by invoking the transmutation property

$$e^{-\frac{i}{2}\frac{d}{b}x^2}\circ\triangle_{k,n}^M\circ e^{\frac{i}{2}\frac{d}{b}x^2} = |x|^{2-\frac{2}{n}}\triangle_k,$$

together with the identity (2.19) and $\tau_x^{k,n}\triangle_k = \triangle_k\tau_x^{k,n}$, we obtain that the function

$$\tilde{u}(x,y) = \tau_x^{k,n}\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](y)$$

is a solution of the previous system. Consequently, we get

$$u(x,y) = e^{\frac{i}{2}\frac{d}{b}(x^2+y^2)}\tau_x^{k,n}\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](y) = T_x^{M,k,n}(f)(y)$$

is a solution of (3.4).

(vi) This is a direct consequence of (3.1) and (2.26).

(vii) The continuous property of $T_x^{M,k,n}$ follows directly from the fact that

$$T_x^{M,k,n}f = \left[\mathbf{L}_{\frac{d}{b},x}\circ\mathbf{L}_{\frac{d}{b},y}\circ\tau_x^{k,n}\circ\mathbf{L}_{-\frac{d}{b}}\right]f$$

where $\mathbf{L}_{\frac{d}{b}}$, $\mathbf{L}_{-\frac{d}{b}}, \tau_x^{k,n}$ are continuous from $C_b(\mathbb{R})$ into itself and $L^p_{k,n}(\mathbb{R})$ into itself, respectively. Moreover, for any $f\in L^p_{k,n}(\mathbb{R})$, the operator $T_x^{M,k,n}f$ belongs to $L^p_{k,n}(\mathbb{R})$ and satisfies

$$\left\|T_x^{M,k,n}f\right\|_{L^p_{k,n}(\mathbb{R})} = \left\|\tau_x^{k,n}\left[\mathbf{L}_{-\frac{d}{b}}f\right]\right\|_{L^p_{k,n}(\mathbb{R})} \le 4\left\|\mathbf{L}_{-\frac{d}{b}}f\right\|_{L^p_{k,n}(\mathbb{R})} = 4\left\|f\right\|_{L^p_{k,n}(\mathbb{R})}.$$

(viii) For any $f\in L^1_{k,n}(\mathbb{R})$ and $g\in C_b(\mathbb{R})$, (3.1) and (2.49) yield

$$\begin{aligned}\int_{\mathbb{R}}&\left[T_x^{M,k,n}f((-1)^ny)\right]\left[e^{-i\frac{d}{b}y^2}g(y)\right]d\gamma_{k,n}(y)\\
&= e^{\frac{i}{2}\frac{d}{b}x^2}\int_{\mathbb{R}}\tau_x^{k,n}\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right]((-1)^ny)\left[e^{-\frac{i}{2}\frac{d}{b}y^2}g(y)\right]d\gamma_{k,n}(y)\\
&= e^{\frac{i}{2}\frac{d}{b}x^2}\int_{\mathbb{R}}\left[e^{-\frac{i}{2}\frac{d}{b}y^2}f(y)\right]\tau_x^{k,n}\left[e^{-\frac{i}{2}\frac{d}{b}s^2}g(s)\right]((-1)^ny)\,d\gamma_{k,n}(y)\\
&= \int_{\mathbb{R}}\left[e^{-i\frac{d}{b}y^2}f(y)\right]\left[T_x^{M,k,n}g((-1)^ny)\right]d\gamma_{k,n}(y).\end{aligned}$$

(ix) For any $f\in L^1_{k,n}(\mathbb{R})$, (2.55), (2.56), (3.1) and Theorem 2.7 imply that

$$\begin{aligned}\left[(ib)^{\frac{(2k-1)n+2}{2n}}\right]&\mathcal{F}^M_{k,n}\left[T_x^{M^{-1},k,n}f\right](\lambda)\\
&= e^{\frac{i}{2}(\frac{d}{b}\lambda^2-\frac{a}{b}x^2)}\int_{\mathbb{R}}\tau_x^{k,n}\left[e^{\frac{i}{2}\frac{a}{b}s^2}f(s)\right](y)B_{k,n}\left(\frac{\lambda}{b},y\right)d\gamma_{k,n}(y)\\
&= e^{\frac{i}{2}(\frac{d}{b}\lambda^2-\frac{a}{b}x^2)}\int_{\mathbb{R}}e^{\frac{i}{2}\frac{a}{b}y^2}f(y)\tau_x^{k,n}\left[s\mapsto B_{k,n}\left(\frac{\lambda}{b},s\right)\right]((-1)^ny)\,d\gamma_{k,n}(y)\\
&= e^{\frac{i}{2}(\frac{d}{b}\lambda^2-\frac{a}{b}x^2)}\overline{B_{k,n}\left(\frac{\lambda}{b},x\right)}\int_{\mathbb{R}}e^{\frac{i}{2}\frac{a}{b}y^2}f(y)B_{k,n}\left(\frac{\lambda}{b},y\right)d\gamma_{k,n}(y)\\
&= \left[(ib)^{\frac{(2k-1)n+2}{2n}}\right]e^{\frac{i}{2}\frac{d}{b}\lambda^2}\,\overline{K^M_{k,n}(\lambda,x)}\,\mathcal{F}^M_{k,n}(f)(\lambda).\end{aligned}$$

(x) For any $f\in L^1_{k,n}(\mathbb{R})\cap L^p_{k,n}(\mathbb{R})$, the result follows directly by virtue of property (ix) while as Young inequality (2.70) and relation (3.7) show that the mappings $f\mapsto\mathcal{F}^M_{k,n}\left[T_x^{M^{-1},k,n}f\right]$ and $f\mapsto\mathcal{F}^M_{k,n}(f)$ are continuous from $L^p_{k,n}(\mathbb{R})$ into $L^{p'}_{k,n}(\mathbb{R})$. As such, the result follows immediately by the density of $L^1_{k,n}(\mathbb{R}) \cap L^p_{k,n}(\mathbb{R})$ in $L^p_{k,n}(\mathbb{R})$.

(xi) Using the relation (3.1) and Proposition 2.8, we derive the result. □

Corollary 3.3. *For any $f \in \mathcal{S}(\mathbb{R})$, we have*

$$T_x^{M^{-1},k,n} f(y) = \frac{1}{(-ib)^{\frac{(2k-1)n+2}{2n}}} e^{-\frac{i}{2}\frac{a}{b}y^2} \int_{\mathbb{R}} B_{k,n}\left((-1)^n \frac{\lambda}{b}, y\right) \overline{K^M_{k,n}(\lambda, x)}\, \mathcal{F}^M_{k,n}(f)(\lambda)\, d\gamma_{k,n}(\lambda). \tag{3.12}$$

Proof. For any $f \in \mathcal{S}(\mathbb{R})$, inequality (3.7) implies that $y \mapsto [T_x^{M^{-1},k,n} f](y)$ is continuous function of $L^1_{k,n}(\mathbb{R})$. Therefore, as a consequence of (3.9) and the inversion formula of the deformed linear canonical Hankel transform, the result follows immediately. □

We conclude this section with the following important result.

Theorem 3.4. *Let $T_y^{M,k,n}$ be the generalized translation operator associated with the uni-modular matrix $M = (a, b; c, d)$, $b \neq 0$. Then,*

(i) For all $f \in C_0(\mathbb{R})$, we have

$$\lim_{y \to 0} \left\| T_y^{M,k,n} f - f \right\|_\infty = 0. \tag{3.13}$$

(ii) For any $f \in L^p_{k,n}(\mathbb{R})$, $1 \leq p < \infty$, we have

$$\lim_{y \to 0} \left\| T_y^{M,k,n} f - f \right\|_{L^p_{k,n}(\mathbb{R})} = 0. \tag{3.14}$$

Proof. (i) **First step:** We shall prove the result for any $f \in C_c(\mathbb{R})$. Using the fact that

$$\frac{M_{k,n}}{2n} \int_0^\pi (\sin\phi)^{2nk-n} d\phi = 1 \quad \text{and} \quad \int_0^\pi C_n^{nk-\frac{n}{2}}(\cos\phi)(\sin\phi)^{2nk-n} d\phi = 0,$$

the generalized translation operator $T_y^{M,k,n}$ we can be expressed

$$T_y^{M,k,n} f(x) - f(x) = a_y(x) + b_y(x) + c_y(x) + d_y(x), \tag{3.15}$$

where

$$a_y(x) = \frac{M_{k,n}}{2n} f_e(x) \int_0^\pi \left[e^{i\frac{d}{2b}(x^2+y^2-\langle\langle x,y\rangle\rangle^2_{\phi,n})} - 1\right] \left\{1 + (-1)^n \frac{n!\,\mathrm{sgn}(xy)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}(\cos\phi)\right\} (\sin\phi)^{2nk-n} d\phi,$$

$$b_y(x) = \frac{M_{k,n}}{2n} \int_0^\pi e^{i\frac{d}{2b}\left(x^2+y^2-\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}-2|xy|^{\frac{1}{n}}\cos\phi\right)^n\right)} \left[f_e\left(\langle\langle x,y\rangle\rangle_{\phi,n}\right) - f_e(x)\right] (\sin\phi)^{2nk-n} d\phi$$

$$c_y(x) = \frac{M_{k,n}}{2n} f_o(x) \int_0^\pi \left[e^{i\frac{d}{2b}(x^2+y^2-\langle\langle x,y\rangle\rangle^2_{\phi,n})} - 1\right] R_{k,n}(x,y,\phi)(\sin\phi)^{2nk-n} d\phi$$

$$d_y(x) = \frac{M_{k,n}}{2n} \int_0^\pi e^{i\frac{d}{2b}\left(x^2+y^2-\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}-2|xy|^{\frac{1}{n}}\cos\phi\right)^n\right)} \left[f_o\left(\langle\langle x,y\rangle\rangle_{\phi,n}\right) - f_o(x)\right] R_{k,n}(x,y,\phi)(\sin\phi)^{2nk-n} d\phi,$$

and

$$R_{k,n}(x,y,\phi)=\frac{n!\,\mathrm{sgn}(x)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\left(\frac{|x|^{\frac{1}{n}}-|y|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}\right)+\frac{n!\,\mathrm{sgn}(y)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\left(\frac{|y|^{\frac{1}{n}}-|x|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}\right).$$

Invoking the properties of the Gegenbauer polynomials, we observe that there exists a positive constant $\mathfrak{C}(k,n)$ such that

$$\left\|a_y\right\|_\infty\leq\mathfrak{C}(k,n)\|f\|_\infty\int_0^\pi\left|e^{i\frac{d}{b}(x^2+y^2-\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^2)}-1\right|(\sin\phi)^{2nk-n}d\phi.$$

Therefore, we have

$$\lim_{y\to0}e^{i\frac{d}{b}(x^2+y^2-\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^2)}-1=0,\quad |e^{i\frac{d}{b}(x^2+y^2-\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^2)}-1|\leq2,$$

and

$$\int_0^\pi(\sin\phi)^{2nk-n}d\phi=\frac{2n}{M_{k,n}}<\infty.$$

Then, an application of dominated convergence theorem implies that

$$\lim_{y\to0}\int_0^\pi\left|e^{i\frac{d}{b}(x^2+y^2-\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^2)}-1\right|(\sin\phi)^{2nk-n}d\phi=0.$$

So, we derive that

$$\lim_{y\to0}\left\|a_y\right\|_\infty=0.$$

As $\lim_{y\to0}f_e\left(\langle\!\langle x,y\rangle\!\rangle_{\phi,n}\right)=f_e(x)$, we derive from the uniform continuity of f, that for given $\epsilon>0$, there exists $\delta>0$ such that $|y|<\delta$ and

$$\left|b_y(x)\right|\leq\frac{M_{k,n}}{2n}\int_0^\pi\left|f_e\left(\langle\!\langle x,y\rangle\!\rangle_{\phi,n}\right)-f_e(x)\right|(\sin\phi)^{2nk-n}d\phi\leq\epsilon.$$

Hence

$$\lim_{y\to0}\|b_y\|_\infty=0.$$

Similarly, one can prove that

$$\lim_{y\to0}\|c_y\|_\infty=\lim_{y\to0}\|d_y\|_\infty=0.$$

Thus, we conclude that for any $f\in C_c(\mathbb{R})$, we have

$$\lim_{y\to0}\left\|T_y^{M,k,n}f-f\right\|_\infty=0.$$

Second step: Assume that $f \in C_0(\mathbb{R})$. Using the fact that $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, there exists a function $g \in C_c(\mathbb{R})$ such that $\|f-g\|_\infty \leq \frac{\epsilon}{10}$ so that

$$\left\|T_y^{M,k,n} f - f\right\|_\infty \leq \left\|T_y^{M,k,n}(f-g)\right\|_\infty + \left\|T_y^{M,k,n} g - g\right\|_\infty + \|f-g\|_\infty$$
$$\leq 5\,\|f-g\|_\infty + \left\|T_y^{M,k,n} g - g\right\|_\infty \leq \frac{\epsilon}{2} + \left\|T_y^{M,k,n} g - g\right\|_\infty.$$

From the first step, for sufficiently small values of y, the quantity $\left\|T_y^{M,k,n} g - g\right\|_\infty$ can be made less than $\epsilon/2$. As such, we shall get the desired result.

(ii) Let $f \in C_c(\mathbb{R})$ such that $\operatorname{supp} f \subset [-R, R]$ and $y \in [-1,1]$. Involving Theorem 3.2 of [4], we derive that the functions $T_y^{M,k,n} f$ are also supported in a common compact set $[-(R^{\frac{1}{n}} + |y|^{\frac{1}{n}})^n, (R^{\frac{1}{n}} + |y|^{\frac{1}{n}})^n] \subset [-2^n(R+1), 2^n(R+1)]$. Consequently, we have

$$\left\|T_y^{M,k,n} f - f\right\|^p_{L^p_{k,n}(\mathbb{R})} \leq \left(\int_{-2^n(R+1)}^{2^n(R+1)} d\gamma_{k,n}(x)\right) \left\|T_y^{M,k,n} f - f\right\|_\infty \to 0, \quad \text{as } y \to 0.$$

Therefore, the general case follows immediately by the density of $C_c(\mathbb{R})$ in $L^p_{k,n}(\mathbb{R})$. This completes the proof of the theorem. □

4 Generalized convolutions product associated with LCDHT

Definition 4.1. *For a given uni-modular matrix $M = (a,b;c,d) \in SL(2,\mathbb{R})$, $b \neq 0$, the generalized convolution product, associated with $\mathcal{F}^M_{k,n}$, for two suitable functions f and g is defined by*

$$f \underset{M,k,n}{\odot} g(x) = \int_{\mathbb{R}} \left[T_x^{M,k,n} f\right]((-1)^n y) \ \left[e^{-i\frac{d}{b}y^2} g(y)\right] d\gamma_{k,n}(y). \tag{4.1}$$

Some elementary properties of convolution (4.1) are summarized below:

(i) An application of Fubini's theorem together with (2.35), (3.5) and (3.6), we have

$$f \underset{M,k,n}{\odot} g = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\frac{d}{b}z^2} f(z) \mathcal{W}^M_{k,n}(x,(-1)^n y, z)\, d\gamma_{k,n}(z)\right] \left[e^{-i\frac{d}{b}y^2} g(y)\right] d\gamma_{k,n}(y)$$
$$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\frac{d}{b}y^2} g(y) \mathcal{W}^M_{k,n}(x,(-1)^n y, z)\, d\gamma_{k,n}(y)\right] \left[e^{-i\frac{d}{b}z^2} f(z)\right] d\gamma_{k,n}(z) = g \underset{M,k,n}{\odot} f.$$

(ii) Using Fubini's theorem, we have

$$T_x^{M,k,n}\left(f \underset{M,k,n}{\odot} g\right)(y) = \int_{\mathbb{R}} e^{-i\frac{d}{b}z^2} \left(f \underset{M,k,n}{\odot} g\right)(z) \mathcal{W}^M_{k,n}(x,y,z)\, d\gamma_{k,n}(z)$$
$$= \int_{\mathbb{R}} e^{-i\frac{d}{b}z^2} \left[\int_{\mathbb{R}} \left[T_z^{M,k,n} f((-1)^n s)\right] \left[e^{-i\frac{d}{b}s^2} g(s)\right] d\gamma_{k,n}(s)\right] \mathcal{W}^M_{k,n}(x,y,z)\, d\gamma_{k,n}(z)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\frac{d}{b}z^2} \left[T^{M,k,n}_{(-1)^n s} f(z) \right] \mathcal{W}^M_{k,n}(x,y,z)\, d\gamma_{k,n}(z) \right] \left[e^{-i\frac{d}{b}s^2} g(s) \right] d\gamma_{k,n}(s) \\
&= \int_{\mathbb{R}} T^{M,k,n}_x \left[T^{M,k,n}_{(-1)^n s} f \right](y) \left[e^{-i\frac{d}{b}s^2} g(s) \right] d\gamma_{k,n}(s) \\
&= \int_{\mathbb{R}} T^{M,k,n}_y \left[T^{M,k,n}_x f \right]((-1)^n s) \left[e^{-i\frac{d}{b}s^2} g(s) \right] d\gamma_{k,n}(s) \\
&= \left(\left[T^{M,k,n}_x f \right] \underset{M,k,n}{*} g \right)(y).
\end{aligned}
$$

The following proposition contain the basic facts about convolutions of $L^p_{k,n}(\mathbb{R})$, $1 \leq p \leq \infty$.

Proposition 4.2 (Young's Inequality). *Let $1 \leq p, q, r \leq \infty$ with $p^{-1}+q^{-1} = r^{-1}+1$. If $f \in L^p_{k,n}(\mathbb{R})$ and $g \in L^q_{k,n}(\mathbb{R})$, then $f \underset{M,k,n}{\odot} g \in L^r_{k,n}(\mathbb{R})$ and satisfies the following inequality:*

$$
\left\| f \underset{M,k,n}{\odot} g \right\|_{L^r_{k,n}(\mathbb{R})} \leq 4 \left\| f \right\|_{L^p_{k,n}(\mathbb{R})} \left\| g \right\|_{L^q_{k,n}(\mathbb{R})}. \tag{4.2}
$$

Proof. Using Hölder's inequality, we obtain

$$
\begin{aligned}
&\left| T^{M,k,n}_x f((-1)^n y) e^{-i\frac{d}{b}y^2} g(y) \right| \\
&\qquad = \left(\left| T^{M,k,n}_x f((-1)^n y) \right|^p |g(y)|^q \right)^{1/r} \left(\left| T^{M,k,n}_x f((-1)^n y) \right|^p \right)^{1/p-1/r} \left(|g(y)|^q \right)^{1/q-1/r}.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\int_{\mathbb{R}} \left| T^{M,k,n}_x f((-1)^n y) e^{-i\frac{d}{b}y^2} g(y) \right| d\gamma_{k,n}(y) &\leq \left(\int_{\mathbb{R}} \left| T^{M,k,n}_x f((-1)^n y) \right|^p |g(y)|^q \, d\gamma_{k,n}(y) \right)^{1/r} \\
&\quad \left(\int_{\mathbb{R}} \left| T^{M,k,n}_x f((-1)^n y) \right|^p d\gamma_{k,n}(y) \right)^{\frac{r-p}{rp}} \left(\int_{\mathbb{R}} |g(y)|^q \, d\gamma_{k,n}(y) \right)^{\frac{r-q}{rq}},
\end{aligned}
$$

which leads us to

$$
\begin{aligned}
\left| \left(f \underset{M,k,n}{\odot} g \right)(x) \right|^r &\leq \left(\int_{\mathbb{R}} \left| T^{M,k,n}_x f((-1)^n y) \right|^p d\gamma_{k,n}(y) \right)^{\frac{r-p}{p}} \|g\|^{r-q}_{L^q_{k,n}(\mathbb{R})} \\
&\quad \int_{\mathbb{R}} \left| T^{M,k,n}_x f((-1)^n y) \right|^p |g(y)|^q \, d\gamma_{k,n}(y).
\end{aligned}
$$

By invoking (3.7), we observe that

$$
\left| \big(f \underset{M,k,n}{\odot} g \big)(x) \right|^r \leq 4^{r-p} \|f\|^{r-p}_{L^p_{k,n}(\mathbb{R})} \|g\|^{r-q}_{L^q_{k,n}(\mathbb{R})} \int_{\mathbb{R}} \left| T^{M,k,n}_x f((-1)^n y) \right|^p |g(y)|^q \, d\gamma_{k,n}(y).
$$

After multiply both sides by $d\gamma_{k,n}(x)$ and integrating over $\mathbb{R}$, we get

$$\begin{aligned}\left\| f \underset{M,k,n}{\odot} g \right\|^r_{L^r_{k,n}(\mathbb{R})} &\leq 4^{r-p} \|f\|^{r-p}_{L^p_{k,n}(\mathbb{R})} \|g\|^{r-q}_{L^q_{k,n}(\mathbb{R})} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left|T^{M,k,n}_x f((-1)^n y)\right|^p |g(y)|^q \, d\gamma_{k,n}(y)\right] d\gamma_{k,n}(x)\\ &= 4^{r-p} \|f\|^{r-p}_{L^p_{k,n}(\mathbb{R})} \|g\|^{r-q}_{L^q_{k,n}(\mathbb{R})} \int_{\mathbb{R}} |g(y)|^q \left[\int_{\mathbb{R}} \left|T^{M,k,n}_{(-1)^n y} f(x)\right|^p d\gamma_{k,n}(x)\right] d\gamma_{k,n}(y)\\ &\leq 4^r \|f\|^r_{L^p_{k,n}(\mathbb{R})} \|g\|^r_{L^q_{k,n}(\mathbb{R})}.\end{aligned}$$

Or equivalently,

$$\left\| f \underset{M,k,n}{\odot} g \right\|_{L^r_{k,n}(\mathbb{R})} \leq 4 \|f\|_{L^p_{k,n}(\mathbb{R})} \|g\|_{L^q_{k,n}(\mathbb{R})}.$$ □

Theorem 4.3. *Let* $\underset{M,k,n}{\odot}$ *be the generalized convolution as defined by (4.1) associated with unimodular matrix* $M = (a, b; c, d) \in SL(2, \mathbb{R})$, $b \neq 0$. *Then,*

(i) For any $f, g \in L^1_{k,n}(\mathbb{R})$, *we have*

$$\mathcal{F}^M_{k,n}\left(f \underset{M^{-1},k,n}{\odot} g\right)(x) = \left((ib)^{\frac{(2k-1)n+2}{2n}}\right) e^{-\frac{i}{2}\frac{d}{b}x^2} \mathcal{F}^M_{k,n}(f)(x)\mathcal{F}^M_{k,n}(g)(x), \quad \text{for all } x \in \mathbb{R}. \tag{4.3}$$

(ii) For any $f \in L^1_{k,n}(\mathbb{R})$ *and* $g \in L^p_{k,n}(\mathbb{R})$, $1 < p \leq 2$, *we have*

$$\mathcal{F}^M_{k,n}\left(f \underset{M^{-1},k,n}{\odot} g\right)(x) = \left((ib)^{\frac{(2k-1)n+2}{2n}}\right) e^{-\frac{i}{2}\frac{d}{b}x^2} \mathcal{F}^M_{k,n}(f)(x)\mathcal{F}^M_{k,n}(g)(x), \quad a.e.\ x \in \mathbb{R}. \tag{4.4}$$

(iii) For $f, g, h \in L^1_{k,n}(\mathbb{R})$, *we have*

$$\left(f \underset{M,k,n}{\odot} g\right) \underset{M,k,n}{\odot} h = f \underset{M,k,n}{\odot} \left(g \underset{M,k,n}{\odot} h\right). \tag{4.5}$$

Proof. (i) Using the definition of $\mathcal{F}^M_{k,n}$ along with (3.9), it follows that

$$\begin{aligned}&\mathcal{F}^M_{k,n}\left(f \underset{M^{-1},k,n}{\odot} g\right)(x)\\ &= \frac{1}{(ib)^{\frac{(2k-1)n+2}{2n}}} \int_{\mathbb{R}} K^M_{k,n}(x,y) \left[\int_{\mathbb{R}} T^{M^{-1},k,n}_y f((-1)^n z) \left[e^{i\frac{a}{b}z^2} g(z)\right] d\gamma_{k,n}(z)\right] d\gamma_{k,n}(y)\\ &= \frac{1}{(ib)^{\frac{(2k-1)n+2}{2n}}} \int_{\mathbb{R}} \left[e^{i\frac{a}{b}z^2} g(z)\right] \left[\int_{\mathbb{R}} K^M_{k,n}(x,y) T^{M^{-1},k,n}_{(-1)^n z} f(y) \, d\gamma_{k,n}(y)\right] d\gamma_{k,n}(z)\\ &= \int_{\mathbb{R}} \left[e^{i\frac{a}{b}z^2} g(z)\right] \left[\mathcal{F}^M_{k,n}\left(T^{M^{-1},k,n}_{(-1)^n z} f\right)(x)\right] d\gamma_{k,n}(z)\\ &= \left((ib)^{\frac{(2k-1)n+2}{2n}}\right) e^{-\frac{i}{2}\frac{d}{b}x^2} \mathcal{F}^M_{k,n}(f)(x) \, \mathcal{F}^M_{k,n}(g)(x).\end{aligned}$$

It is pertinent to mention that Fubini theorem has been used in the second line as

$$\int_{\mathbb{R}^2} \left| K_{k,n}^M(x,y) T_y^{M^{-1},k,n} f((-1)^n z) e^{i\frac{a}{b}z^2} g(z) \right| d\gamma_{k,n}(y)\, d\gamma_{k,n}(z)$$
$$\leq C \int_{\mathbb{R}^2} \left| T_y^{M^{-1},k,n} f((-1)^n z) \right| |g(z)|\, d\gamma_{k,n}(y)\, d\gamma_{k,n}(z) \leq 4C \|f\|_{L^1_{k,n}(\mathbb{R})} \|g\|_{L^1_{k,n}(\mathbb{R})} < \infty.$$

(ii) The result is true for $g \in L^1_{k,n}(\mathbb{R}) \cap L^p_{k,n}(\mathbb{R})$ by virtue of (i). On the other hand, the Young's inequality (2.70) for the deformed linear canonical Hankel transform and Proposition 4.2 show that the mappings $g \mapsto \mathcal{F}_{k,n}^M \left(f \underset{M^{-1},k,n}{*} g \right)$ and $g \mapsto \mathcal{F}_{k,n}^M(f)\ \mathcal{F}_{k,n}^M(g)$ are continuous from $L^p_{k,n}(\mathbb{R})$ into $L^{p'}_{k,n}(\mathbb{R})$. Finally, the result follows directly from density of $L^1_{k,n}(\mathbb{R}) \cap L^p_{k,n}(\mathbb{R})$ in $L^p_{k,n}(\mathbb{R})$.

(iii) The result follows immediately by an application of result (i). □

5 Generalized heat equation and the associated operators

In this section, we shall illustrate our proposed theory developed in previous sections to the following generalized heat equation associated with the operator $\triangle_{k,n}^{M^{-1}}$:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \sigma \triangle_{k,n}^{M^{-1}} u(t,x), \ (t,x) \in (0,\infty) \times \mathbb{R} \\ u(0,x) = f(x), \end{cases} \tag{5.1}$$

where f is defined on the Banach space $\mathfrak{B}$ which could be either $L^p_{k,n}(\mathbb{R})$, $1 \leq p \leq \infty$, $(C_b(\mathbb{R}), \|.\|_\infty)$ or $(C_0(\mathbb{R}), \|.\|_\infty)$, $\sigma > 0$ is the coefficient of heat conductivity and the initial data $u(0,x) = f(x)$ means that $u(t,x) \to f(x)$ as $t \to 0$ in the norm of $\mathfrak{B}$.

5.1 Generalized heat kernel associated with $\sigma \triangle_{k,n}^{M^{-1}}$

Given a uni-modular matrix $M = (a,b;c,d) \in SL(2,\mathbb{R})$, $b \neq 0$ and $\sigma, t > 0$, we define

$$\mathcal{P}_t^{M^{-1}}(y) := \frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} \exp\left\{ -\frac{iay^2}{2b} - \frac{ny^2}{2\sigma t} \right\}, \quad y \in \mathbb{R}. \tag{5.2}$$

Using the relations (2.55), (2.56), (5.2) and Example 2.2, we obtain

$$\mathcal{F}_{k,n}^M \left(\mathcal{P}_t^{M^{-1}} \right)(x) = \exp\left\{ \frac{idx^2}{2b} - t\sigma \left(\frac{x}{|b|} \right)^2 \right\}, \quad \forall\, t > 0, \quad x \in \mathbb{R}. \tag{5.3}$$

Definition 5.1. *Given a uni-modular matrix $M = (a,b;c,d) \in SL(2,\mathbb{R})$, $b \neq 0$, the generalized heat kernel associated with $\triangle_{k,n}^{M^{-1}}$ is denoted as $G_t^{M^{-1}}$ and defined by*

$$G_t^{M^{-1}}(x,y) = T_x^{M^{-1},k,n}\left[\mathcal{P}_t^{M^{-1}}\right](y), \quad x,y \in \mathbb{R}, \quad t > 0. \tag{5.4}$$

We collect some basic properties of the generalized heat kernel $G_t^{M^{-1}}$ in the following proposition.

Proposition 5.2. *The generalized heat kernel $G_t^{M^{-1}}$ as defined in (5.4) satisfies the following properties:*

(i) For $t > 0$, we have

$$G_t^{M^{-1}}(x,y) = \frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}}\exp\left\{\frac{-ia(x^2+y^2)}{2b} - \frac{n(|x|^{\frac{2}{n}} + |y|^{\frac{2}{n}})}{2\sigma t}\right\} B_{k,n}\left(\frac{x}{(\sigma t)^n}, (-i)^n y\right). \tag{5.5}$$

(ii) For $t > 0$, there exists a positive constant $\mathcal{C}(k,n)$ such that

$$\left|G_t^{M^{-1}}(x,y)\right| \leq \mathcal{C}(k,n)\frac{e^{-n\frac{\left(|x|^{\frac{1}{n}} - |y|^{\frac{1}{n}}\right)^2}{2\sigma t}}}{(\sigma t)^{\frac{(2k-1)n+2}{2}}}. \tag{5.6}$$

(iii) For $t > 0$, we have

$$\int_{\mathbb{R}} e^{\frac{i}{2}\frac{a}{b}(x^2+y^2)} G_t^{M^{-1}}(x,y)\, d\gamma_{k,n}(y) = 1. \tag{5.7}$$

(iv) For $s, t > 0$, we have

$$G_{t+s}^{M^{-1}}(x,y) = \int_{\mathbb{R}} G_t^{M^{-1}}(x,z)\, G_s^{M^{-1}}(y,z)\; e^{i\frac{a}{b}z^2} d\gamma_{k,n}(z). \tag{5.8}$$

(v) For fixed $t > 0$ and $y \in \mathbb{R}$, we have

$$\mathcal{F}_{k,n}^M\left(G_t^{M^{-1}}(.,y)\right)(\xi) = e^{i\frac{d}{b}\xi^2}\overline{K_{k,n}^M(\xi,y)}\exp\left\{-t\sigma\left|\frac{\xi}{b}\right|^{\frac{2}{n}}\right\}. \tag{5.9}$$

(vi) For a fixed $y \in \mathbb{R}$, $u(t,x) = G_t^{M^{-1}}(x,y)$ is the solution of the generalized heat equation (5.1).

Proof. (i) Using the Definition 3.1, we observe that

$$G_t^{M^{-1}}(x,y) = \frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} e^{-\frac{i}{2}\frac{a}{b}(x^2+y^2)} \tau_x^{k,n}\left[e^{-\frac{n|s|^{\frac{2}{n}}}{2\sigma t}}\right](y). \tag{5.10}$$

Therefore, by simple application of (2.25), we derive the desired assertion.

(ii) The assertion follows directly from the relation (5.10) and inequality (2.41).

(iii) An application of (5.10) leads us to

$$\int_{\mathbb{R}} e^{\frac{i}{2}\frac{a}{b}(x^2+y^2)} G_t^{M^{-1}}(x,y)\, d\gamma_{k,n}(y) = \frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} \int_{\mathbb{R}} \tau_x^{k,n}\left(e^{-\frac{n|s|^{\frac{2}{n}}}{2\sigma t}}\right)(y)\, d\gamma_{k,n}(y).$$

Thus, we obtain the desired result by applying (2.47) and simple calculations.

(iv) Using the identity (5.5), we obtain

$$\int_{\mathbb{R}} G_t^{M^{-1}}(x,z)\, G_s^{M^{-1}}(y,z)\; e^{i\frac{a}{b}z^2} d\gamma_{k,n}(z) = \frac{1}{((\sigma)^2 ts)^{\frac{(2k-1)n+2}{2}}} e^{\frac{-i}{2}\frac{a}{b}(x^2+y^2)-\left[n\frac{|x|^{\frac{2}{n}}}{2\sigma t}+n\frac{|y|^{\frac{2}{n}}}{2\sigma s}\right]}$$

$$\int_{\mathbb{R}} e^{-n\left[\frac{|z|^{\frac{2}{n}}}{2\sigma t}+\frac{|z|^{\frac{2}{n}}}{2\sigma s}\right]} B_{k,n}\left(\frac{x}{(\sigma t)^n}, (-i)^n z\right) B_{k,n}\left(\frac{y}{(\sigma s)^n}, (-i)^n z\right) d\gamma_{k,n}(z).$$

From the relation (2.10), we deduce that

$$\int_{\mathbb{R}} e^{-n\left[\frac{|z|^{\frac{2}{n}}}{2\sigma t}+\frac{|z|^{\frac{2}{n}}}{2\sigma s}\right]} B_{k,n}\left(\frac{x}{(\sigma t)^n}, (i)^n z\right) B_{k,n}\left(\frac{y}{(\sigma s)^n}, (i)^n z\right) d\gamma_{k,n}(z)$$

$$= \left(\frac{\sigma ts}{t+s}\right)^{\frac{(2k-1)n+2}{2}} e^{n\left[\frac{s|x|^{\frac{2}{n}}}{2\sigma t(t+s)}+\frac{t|y|^{\frac{2}{n}}}{2\sigma s(t+s)}\right]} B_{k,n}\left(\frac{x}{(\sigma(s+t))^n}, (i)^n z\right),$$

which leads to the given desired result.

(v) Involving the relations (5.4), (3.9) and (5.3), we get the desired result.

(vi) For fixed $y \in \mathbb{R}$ and $t > 0$, we put $v(x,t) := G_t^{M^{-1}}(x,y)$. Using (5.4) and Corollary 3.3, we deduce that

$$G_t^{M^{-1}}(x,y) = \frac{1}{(-ib)^{\frac{(2k-1)n+2}{2n}}} \int_{\mathbb{R}} e^{i\frac{d}{b}\lambda^2} B_{k,n}\left((-1)^n\frac{\lambda}{b}, y\right) \overline{K_{k,n}^M(\lambda,x)} \exp\left\{-t\sigma\left|\frac{\lambda}{b}\right|^{\frac{2}{n}}\right\} d\gamma_{k,n}(\lambda). \tag{5.11}$$

By taking differentiations under integral, the identities (2.66), (2.60) and by standard analysis, we see that

$$\left[\frac{\partial}{\partial t} - \sigma\triangle_{k,n}^{M^{-1}}\right] G_t^{M^{-1}}(x,y) = 0.$$

This completes the proof of the Proposition 5.2. □

Theorem 5.3. *Assume that $M = (a,b;c,d) \in SL(2,\mathbb{R})$ such that $b \neq 0$. Let $\mathfrak{B}$ be one of the Banach spaces $L^p_{k,n}(\mathbb{R})$ $(1 \leq p \leq \infty)$, $(C_b(\mathbb{R}), \|.\|_\infty)$ or $(C_0(\mathbb{R}), \|.\|_\infty)$. Then:*

(i) For each $f \in X$, the function $u(t,x) = \left(\mathcal{P}_t^{M^{-1}} \underset{M^{-1},k,n}{\odot} f\right)(x)$ satisfies the generalized heat equation

$$\frac{\partial u(t,x)}{\partial t} = \sigma \triangle_{k,n}^{M^{-1}} u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \tag{5.12}$$

and

$$\|u(t,\cdot)\|_{L^r_{k,n}(\mathbb{R})} \leq \frac{4\left(2\Gamma\left(\frac{(2k-1)n+2}{nq}\right) M_{k,n}\right)^{1/q}}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} \|f\|_{L^p_{k,n}(\mathbb{R})}, \tag{5.13}$$

where $p,q,r \in [1,\infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.

(ii) Let $f(x) = e^{-\frac{i}{2}\frac{a}{b}x^2} p\left(|x|^{\frac{2}{n}}\right)$ with $p(s) = \sum_{j=0}^{m} c_j s^j$.

We define the function u as $u(t,x) = \left(\mathcal{P}_t^{M^{-1}} \underset{M^{-1},k,n}{\odot} f\right)(x)$. We have

$$u(t,x) = e^{-\frac{i}{2}\frac{a}{b}x^2} \sum_{j=0}^{n} j!\, c_j \left(\frac{2\sigma t}{n}\right)^j L_j^{\left(\frac{(2k-1)n}{2}\right)}\left(-\frac{n|x|^{\frac{2}{n}}}{2\sigma t}\right), \tag{5.14}$$

where $L_j^{\left(\frac{(2k-1)n}{2}\right)}$ denote the Laguerre functions of degree j [43]. Moreover,

$$\frac{\partial u(t,x)}{\partial t} = \sigma \triangle_{k,n}^{M^{-1}} u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \quad \text{with } u(0,x) = f(x).$$

Proof. (i) In view of (5.11) and Fubini's theorem, the function $u(t,x)$ can be expressed as

$$u(t,x) = \frac{1}{(-ib)^{\frac{(2k-1)n+2}{2n}}} \int_{\mathbb{R}} e^{i\frac{d}{b}\lambda^2} \overline{K_{k,n}^M(\lambda,x)} \exp\left\{-t\sigma\left|\frac{\lambda}{b}\right|^{\frac{2}{n}}\right\} \mathcal{F}_{k,n}(f)(\lambda)\, d\gamma_{k,n}(\lambda). \tag{5.15}$$

Moreover, as above take again differentiation under the integral in (5.15) and (2.66), we derive the result.

Furthermore, the Young's inequality (4.2) implies that

$$\|u(t,.)\|_{L^r_{k,n}(\mathbb{R})} = \left\|\mathcal{P}_t^{M^{-1}} \underset{M^{-1},k,n}{\odot} f\right\|_{L^r_{k,n}(\mathbb{R})} \leq 4 \left\|\mathcal{P}_t^{M^{-1}}\right\|_{L^q_{k,n}(\mathbb{R})} \|f\|_{L^p_{k,n}(\mathbb{R})}. \tag{5.16}$$

Using (5.16) and the fact that

$$\left\|\mathcal{P}_t^{M^{-1}}\right\|_{L^q_{k,n}(\mathbb{R})} = \frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}}\left(\int_{\mathbb{R}} e^{-\frac{nq}{2\sigma t}|y|^{\frac{2}{n}}}\, d\gamma_{k,n}(y)\right)^{1/q} = \frac{\left(2\Gamma\left(\frac{(2k-1)n+2}{nq}\right)M_{k,n}\right)^{1/q}}{(\sigma t)^{\frac{(2k-1)n+2}{2}}},$$

we obtain the desired inequality (5.12).

(ii) Firstly, it is easy to see that

$$u(t,x) = \int_{\mathbb{R}} G_t^{M^{-1}}(x,(-1)^n y)\, e^{\frac{i}{2}\frac{a}{b}y^2} p\left(|y|^{\frac{2}{n}}\right)\, d\gamma_{k,n}(y). \tag{5.17}$$

Now, if we write $p\left(|y|^{\frac{2}{n}}\right) = \sum_{j=1}^{m} c_j |y|^{\frac{2j}{n}}$, then using (5.5) and by the change of variables $u = \frac{y}{(\sigma t)^{\frac{n}{2}}}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} G_t^{M^{-1}}(x,(-1)^n y)\, e^{\frac{i}{2}\frac{a}{b}y^2} p\left(|y|^{\frac{2}{n}}\right)\, d\gamma_{k,n}(y) \\
&\quad= \frac{M_{k,n} e^{-\frac{i}{2}\frac{a}{b}x^2} e^{-\frac{n|x|^{\frac{2}{n}}}{2\sigma t}}}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} \sum_{j=1}^{m} c_j \int_{\mathbb{R}} e^{n\frac{-|y|^{\frac{2}{n}}}{2\sigma t}} B_{k,n}\left(\frac{x}{(\sigma t)^n}, (i)^n y\right) |y|^{\frac{(2k-2)n+2+2j}{n}}\, dy \\
&\quad= \sum_{j=1}^{m} c_j \left(\frac{2\sigma t}{n}\right)^j e^{-\frac{i}{2}\frac{a}{b}x^2} e^{-n\frac{|x|^{\frac{2}{n}}}{2\sigma t}}\; I_t(x),
\end{aligned} \tag{5.18}$$

where

$$I_t(x) = \frac{2}{\Gamma\left(\frac{(2k-1)n+2}{2}\right)} \int_{\mathbb{R}} e^{-u^2} j_{\frac{(2k-1)n}{2}}\left(\frac{2i|x|^{\frac{1}{n}}u}{\sqrt{\frac{2\sigma t}{n}}}\right) u^{(2k-1)n+1+2j}\, du.$$

Using the identity (6.631(10) in [20]), we get

$$\int_0^{\infty} e^{-u^2} j_\alpha(uz) u^{2j+2\alpha+1} du = \frac{\Gamma(\alpha+1)}{2} j!\, e^{-\frac{z^2}{4}} L_j^{\alpha}\left(\frac{z^2}{4}\right), \quad z \geq 0.$$

Further, by simple calculations, we see that

$$I_t(x) = j!\, e^{-\frac{i}{2}\frac{a}{b}x^2} e^{n\frac{|x|^{\frac{2}{n}}}{2\sigma t}} L_j^{\left(\frac{(2k-1)n}{2}\right)}\left(-\frac{n|x|^{2/n}}{2\sigma t}\right), \quad x \in \mathbb{R}. \tag{5.19}$$

Substituting (5.19) in (5.18), we get the desired identity:

$$u(t,x) = e^{-\frac{i}{2}\frac{a}{b}x^2} \sum_{j=0}^{n} j!\, c_j \left(\frac{2\sigma t}{n}\right)^j L_j^{\left(\frac{(2k-1)n}{2}\right)}\left(-\frac{n|x|^{\frac{2}{n}}}{2\sigma t}\right).$$

Finally, using (i) we observe that the function u solves (5.12). Moreover using the identity, (*cf.* [43]),

$$\left(\frac{2\sigma t}{n}\right)^j L_j^{\frac{(2k-1)n}{2}}\left(-\frac{n|x|^{2/n}}{2\sigma t}\right)\Big|_{t=0} = \frac{|x|^{\frac{2j}{n}}}{j!}$$

we derive that $u(0,x) = f(x)$. This completes the proof of the Theorem 5.3. □

5.2 Heat semi-groups associated with $\sigma\triangle_{k,n}^{M^{-1}}$

We begin this subsection by recalling the necessary tools on semigroups.

Definition 5.4 ([36]). *Let X be a Banach space. A one-parameter family $S = \{S(t);\ t \geq 0\}$ of bounded linear operators on X is called a strongly continuous semigroup if it satisfies:*

(i) $S(0) = Id_X$.

(ii) $S(t+s) = S(t)S(s)$ *for all* $t, s \geq 0$.

(iii) The mapping $t \mapsto S(t)u$ is continuous on $[0,\infty)$ for all $u \in X$. A strongly continuous semigroup is called a contraction semigroup, if $\|S(t)\| \leq 1$ for all $t \geq 0$.

Let $S = (S(t))_{t\geq 0}$ be a strongly continuous semigroup. The generator $\mathfrak{O}$ of S is defined by the formula

$$\mathfrak{O}u = \lim_{t\to 0}\frac{S(t)u-u}{t} = \frac{d}{dt}S(t)u\Big|_{t=0},$$

the domain $\mathcal{D}(\mathfrak{O})$ of $\mathfrak{O}$ being the set of all $u \in X$ for which the limit defined above exists.

In this subsection, we shall denote $\mathfrak{B}$ as one of the Banach spaces $L^p_{k,n}(\mathbb{R})$ $(1 \leq p < \infty)$ or $(C_0(\mathbb{R}), \|.\|_\infty)$.

Definition 5.5. *Let $M = (a,b;c,d) \in SL(2,\mathbb{R})$ be a uni-modular matrix such that $b \neq 0$. Then, for each $t \geq 0$ and $f \in X$, we define a family of operators*

$$S_{k,n}^{M^{-1}}(t)f = \begin{cases} \frac{1}{4}\left[\mathcal{P}_t^{M^{-1}} \underset{M^{-1},k,n}{\odot} f\right] & \text{if} \quad t > 0, \\ f & \text{if} \quad t = 0. \end{cases} \tag{5.20}$$

The family of operators (5.20) is often called the heat semigroup associated with $\sigma\triangle_{k,n}^{M^{-1}}$.

Theorem 5.6. *The family of operators* $\left\{S_{k,n}^{M^{-1}}(t) : t \geq 0\right\}$ *is strongly continuous contraction on* $\mathfrak{B}$.

Proof. We shall divide the proof of the theorem into two steps.

First step: (i) Assume that $\mathfrak{B} = C_0(\mathbb{R})$. Then, the result is trivial when $t = 0$. For any $f \in C_0(\mathbb{R})$ and $t > 0$, (3.8) and (5.2), implies that

$$\begin{aligned}\left(S_{k,n}^{M^{-1}}(t)f\right)(x) &= \frac{1}{4}\int_{\mathbb{R}}\left[T_x^{M^{-1},k,n}\mathcal{P}_t^{M^{-1}}\right]((-1)^n y)\left[e^{i\frac{a}{b}y^2}f(y)\right]d\gamma_{k,n}(y)\\ &= \frac{1}{4}\int_{\mathbb{R}} e^{i\frac{a}{b}y^2}\mathcal{P}_t^{M^{-1}}(y)\left[T_x^{M^{-1},k,n}f\right]((-1)^n y)\,d\gamma_{k,n}(y)\\ &= \frac{1}{4}\frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}}\int_{\mathbb{R}} e^{\frac{i}{2}\frac{a}{b}y^2}e^{\frac{-n|y|^{\frac{2}{n}}}{2\sigma t}}\left[T_x^{M^{-1},k,n}f\right]((-1)^n y)\,d\gamma_{k,n}(y)\\ &= \frac{1}{4}\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}\int_{\mathbb{R}} e^{i\frac{a}{2b}\left(\frac{2\sigma t}{n}\right)^n}e^{-|v|^{\frac{2}{n}}}\left[T_x^{M^{-1},k,n}f\right]\left((-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v\right)d\gamma_{k,n}(v).\end{aligned} \tag{5.21}$$

Clearly the mapping $(x, v) \longmapsto \left[T_x^{M^{-1},k,n}f\right]\left((-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v\right)$ is continuous on $\mathbb{R}^2$. Moreover, using (3.11) and (3.7), we have

$$\lim_{|x|\to\infty}\left[T_x^{M^{-1},k,n}f\right]\left((-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v\right) = 0$$

and

$$\left|e^{i\frac{a}{2b}\left(\frac{2\sigma t}{n}\right)^n}e^{-|v|^{\frac{2}{n}}}\left[T_x^{M^{-1},k,n}f\right]\left((-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v\right)\right| \leq 4\|f\|_\infty e^{-|v|^{\frac{2}{n}}} \in L^1_{k,n}(\mathbb{R}).$$

Therefore, it follows by the dominated convergence theorem that $S_{k,n}^{M^{-1}}(t)f \in C_0(\mathbb{R})$ and satisfies the inequality:

$$\left\|S_{k,n}^{M^{-1}}(t)f\right\|_\infty \leq \left\{\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}\int_{\mathbb{R}} e^{-|y|^{\frac{2}{n}}}d\gamma_{k,n}(y)\right\}\|f\|_\infty = \|f\|_\infty.$$

By taking supremum over all $f \in C_0(\mathbb{R})$ and noting that $\|f\|_\infty \leq 1$, we obtain $\left\|S_{k,n}^{M^{-1}}(t)\right\| \leq 1$.

(ii) For all $t, s > 0$ and $f \in C_0(\mathbb{R})$, from (5.8) we have

$$\begin{aligned}
&S_{k,n}^{M^{-1}}(s)\left(S_{k,n}^{M^{-1}}(t)f\right)(x)\\
&\quad= \frac{1}{4}\int_{\mathbb{R}} G_s^{M^{-1}}(x,z)e^{i\frac{a}{b}z^2}\left(\int_{\mathbb{R}} G_t^{M^{-1}}(y,z)e^{i\frac{a}{b}y^2}f(y)\,d\gamma_{k,n}(y)\right)d\gamma_{k,n}(z)\\
&\quad= \frac{1}{4}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} G_s^{M^{-1}}(x,z)G_t^{M^{-1}}(y,z)\;e^{i\frac{a}{b}z^2}d\gamma_{k,n}(z)\right)e^{i\frac{a}{b}y^2}f(y)\,d\gamma_{k,n}(y)\\
&\quad= \frac{1}{4}\int_{\mathbb{R}} G_{s+t}^{M^{-1}}(x,y)e^{i\frac{a}{b}y^2}f(y)\,d\gamma_{k,n}(y)\\
&\quad= S_{k,n}^{M^{-1}}(s+t)f(x).
\end{aligned}$$

(iii) Using the fact

$$\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}\int_{\mathbb{R}} e^{-|y|^{\frac{2}{n}}}d\gamma_{k,n}(y) = 1,$$

identity (5.21) gives the freedom to write

$$\left(S_{k,n}^{M^{-1}}(t)f\right)(x) - f(x) = a_t(x) + b_t(x) \tag{5.22}$$

where

$$a_t(x) = \frac{\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}}{4}\int_{\mathbb{R}} e^{-|v|^{\frac{2}{n}}}\left(e^{i\frac{a}{2b}\left(\frac{2\sigma t}{n}\right)^n|v|^2}-1\right)f(x)\,d\gamma_{k,n}(v), \tag{5.23}$$

$$b_t(x) = \frac{\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}}{4}\int_{\mathbb{R}} e^{i\frac{a}{2b}\left(\frac{2\sigma t}{n}\right)^n|v|^2}e^{-|v|^{\frac{2}{n}}}\left(\left[T_x^{M^{-1},k,n}f\right]\left((-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v\right)-f(x)\right)d\gamma_{k,n}(v). \tag{5.24}$$

Using the fact that

$$\left\|T^{M^{-1},k,n}_{(-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v}f - f\right\|_\infty \le 5\|f\|_\infty \quad\text{and}\quad \lim_{t\to 0}\left\|T^{M^{-1},k,n}_{(-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v}f - f\right\|_\infty = 0,$$

together as above with an application of the dominated convergence theorem, we get the desired result as

$$\|a_t\|_\infty \le \frac{\left(\dfrac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}}{4}\left[\int_{\mathbb{R}} e^{-|v|^{\frac{2}{n}}}\left|e^{i\frac{\alpha}{2b}\left(\frac{2\sigma t}{n}\right)^n|v|^2}-1\right|d\gamma_{k,n}(v)\right]\|f\|_\infty \longrightarrow 0, \quad \text{as } t\to 0,$$

$$\|b_t\|_\infty \le \frac{\left(\dfrac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}}{4}\int_{\mathbb{R}} e^{-|v|^{\frac{2}{n}}}\left\|T^{M^{-1},k,n}_{(-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v}f - f\right\|_\infty d\gamma_{k,n}(v) \longrightarrow 0, \quad \text{as } t\to 0.$$

Second step: (i) Assume that $X = L^p_{k,n}(\mathbb{R})$, $1 \le p < \infty$. For any $f \in L^p_{k,n}(\mathbb{R})$, Young's inequality (4.2) yields

$$\left\| S^{M^{-1}}_{k,n}(t) f \right\|_{L^p_{k,n}(\mathbb{R})} = \frac{1}{4} \left\| \mathcal{P}^{M^{-1}}_t \underset{M^{-1},k,n}{\odot} f \right\|_{L^r_{k,n}(\mathbb{R})} \le \left\| \mathcal{P}^{M^{-1}}_t \right\|_{L^1_{k,n}(\mathbb{R})} \|f\|_{L^p_{k,n}(\mathbb{R})}.$$

Since

$$\left\| \mathcal{P}^{M^{-1}}_t \right\|_{L^1_{k,n}(\mathbb{R})} = \frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} \int_{\mathbb{R}} e^{-\frac{n|y|^{\frac{2}{n}}}{2\sigma t}} d\gamma_{k,n}(y) = 1.$$

Thus, we obtain

$$\left\| S^{M^{-1}}_{k,n}(t) f \right\|_{L^p_{k,n}(\mathbb{R})} \le \|f\|_{L^p_{k,n}(\mathbb{R})}.$$

By taking supremum over all $f \in L^p_{k,n}(\mathbb{R})$ and noting that $\|f\|_{L^p_{k,n}(\mathbb{R})} \le 1$, we obtain for each $t \ge 0$, $S^{M^{-1}}_{k,n}(t)$ is a bounded linear operator on $L^p_{k,n}(\mathbb{R})$ and $\|S^{M^{-1}}_{k,n}(t)\| \le 1$.

(ii) Since $\mathcal{S}(\mathbb{R}) \subset C_0(\mathbb{R})$, we derive that

$$S^{M^{-1}}_{k,n}(s+t) = S^{M^{-1}}_{k,n}(s)\, S^{M^{-1}}_{k,n}(t) \quad \text{on } \mathcal{S}(\mathbb{R}).$$

On the other hand, $S^{M^{-1}}_{k,n}(s)$, $S^{M^{-1}}_{k,n}(t)$ and $S^{M^{-1}}_{k,n}(s+t)$ are continuous from $L^p_{k,n}(\mathbb{R})$ into itself. Therefore, the result follows immediately by the density of $\mathcal{S}(\mathbb{R})$ in $L^p_{k,n}(\mathbb{R})$.

(iii) Firstly, we show that if $f \in C_c(\mathbb{R})$, then

$$\lim_{t \to 0} \left\| S^{M^{-1}}_{k,n}(t) f - f \right\|_{L^p_{k,n}(\mathbb{R})} = 0. \tag{5.25}$$

By virtue of the relation (5.22), it follows that

$$\left\| S^{M^{-1}}_{k,n}(t) f - f \right\|_{L^p_{k,n}(\mathbb{R})} \le \|a_t\|_{L^p_{k,n}(\mathbb{R})} + \|b_t\|_{L^p_{k,n}(\mathbb{R})},$$

with

$$\|a_t\|_{L^p_{k,n}(\mathbb{R})} \le \frac{\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}}{4} \left[\int_{\mathbb{R}} e^{-|v|^{\frac{2}{n}}} \left| e^{i\frac{a}{2b}\left(\frac{2\sigma t}{n}\right)^n |v|^2} - 1 \right| d\gamma_{k,n}(v) \right] \|f\|_{L^p_{k,n}(\mathbb{R})} \longrightarrow 0, \quad \text{as } t \to 0,$$

whereas the Minkowski's inequality yields that

$$\|b_t\|_{L^p_{k,n}(\mathbb{R})} \le \frac{\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}}{4} \int_{\mathbb{R}} e^{-|v|^{\frac{2}{n}}} \left\| T^{M^{-1},k,n}_{(-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}} v} f - f \right\|_{L^p_{k,n}(\mathbb{R})} d\gamma_{k,n}(v) \longrightarrow 0, \quad \text{as } t \to 0.$$

Implementation of the dominated convergence theorem implies that

$$\left\| T^{M^{-1},k,n}_{(-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v} f - f \right\|_{L^p_{k,n}(\mathbb{R})} \leq 5\left\| f \right\|_{L^p_{k,n}(\mathbb{R})} \quad \text{(by (3.7))},$$

$$\lim_{t\to 0}\left\| T^{M^{-1},k,n}_{(-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v} f - f \right\|_{L^p_{k,n}(\mathbb{R})} = 0, \quad \text{(see Theorem 3.4)},$$

and $v \mapsto e^{-|v|^{\frac{2}{n}}} \in L^1_{k,n}(\mathbb{R})$.

Since $C_c(\mathbb{R})$ is dense in $L^p_{k,n}(\mathbb{R})$, therefore, for any $f \in L^p_{k,n}(\mathbb{R})$, there exists $g \in C_c(\mathbb{R})$ such that

$$\left\| f - g \right\|_{L^p_{k,n}(\mathbb{R})} \leq \frac{\epsilon}{3},$$

and

$$\begin{aligned} \left\| S^{M^{-1}}_{k,n}(t)f - f \right\|_{L^p_{k,n}(\mathbb{R})} &\leq \left\| S^{M^{-1}}_{k,n}(t)(f-g) \right\|_{L^p_{k,n}(\mathbb{R})} + \left\| S^{M^{-1}}_{k,n}(t)g - g \right\|_{L^p_{k,n}(\mathbb{R})} \\ &+ \left\| f - g \right\|_{L^p_{k,n}(\mathbb{R})} \\ &\leq 2\left\| f - g \right\|_{L^p_{k,n}(\mathbb{R})} + \left\| S^{M^{-1}}_{k,n}(t)g - g \right\|_{L^p_{k,n}(\mathbb{R})} \\ &\leq \frac{2\epsilon}{3} + \left\| S^{M^{-1}}_{k,n}(t)g - g \right\|_{L^p_{k,n}(\mathbb{R})}. \end{aligned}$$

Further the relation (5.25) implies that, for sufficiently small values of t, we have

$$\left\| S^{M^{-1}}_{k,n}(t)g - g \right\|_{L^p_{k,n}(\mathbb{R})} \leq \frac{\epsilon}{3}.$$

Subsequently, we obtain

$$\lim_{t\to 0}\left\| S^{M^{-1}}_{k,n}(t)f - f \right\|_{L^p_{k,n}(\mathbb{R})} = 0.$$

This completes the proof of Theorem 5.6. □

We close this section by the following statement for the semigroup $\left(S^{M^{-1}}_{k,n}(t), t \geq 0\right)$ acting on the Banach spaces $\mathfrak{B} = L^p_{k,n}(\mathbb{R})$ $(1 \leq p < \infty)$ or $(C_0(\mathbb{R}), \|.\|_\infty)$.

Proposition 5.7. *The operator $\triangle^{M^{-1}}_{k,n}$ is closable and its closure generates the semigroup $\left(S^{M^{-1}}_{k,n}(t), t \geq 0\right)$ acting on the Banach spaces $\mathfrak{B}$.*

Proof. Let $f \in \mathfrak{S}_{k,n}(\mathbb{R})$. Involving the relations (5.20) and (5.15), we observe that

$$\mathcal{F}^M_{k,n}\left(\frac{S^{M^{-1}}_{k,n}(t) - Id}{t} f\right)(\lambda) = \frac{e^{-t\sigma\left|\frac{\lambda}{b}\right|^{\frac{2}{n}}} - 1}{t}\mathcal{F}^M_{k,n}(f)(\lambda).$$

Thus, we derive that

$$\lim_{t\to 0}\mathcal{F}^M_{k,n}\left(\frac{S^{M^{-1}}_{k,n}(t)-Id}{t}f\right)(\lambda)=-\sigma\left|\frac{\lambda}{b}\right|^{\frac{2}{n}}\mathcal{F}^M_{k,n}(f)(\lambda)=\mathcal{F}^M_{k,n}\left(\sigma\triangle^{M^{-1}}_{k,n}f\right)(\lambda).$$

Using the injectivity of $\mathcal{F}^M_{k,n}$ on $\mathfrak{S}_{k,n}(\mathbb{R})$, we infer that the generator of the semigroup $\left(S^{M^{-1}}_{k,n}(t), t\geq 0\right)$, denoted by $\mathfrak{O}_{k,n}$, satisfies

$$\mathfrak{O}_{k,n}f=\lim_{t\to 0}\frac{S^{M^{-1}}_{k,n}(t)-Id}{t}f=\sigma\triangle^{M^{-1}}_{k,n}f.$$

As $\mathfrak{S}_{k,n}(\mathbb{R})$ is invariant under $\mathcal{F}_{k,n}$, we derive that $\mathfrak{S}_{k,n}(\mathbb{R})$ is invariant under $\left(S^{M^{-1}}_{k,n}(t), t\geq 0\right)$ which is a strongly continuous semigroup of contractions on $\mathfrak{B}$. So, we observe that $\mathfrak{S}_{k,n}(\mathbb{R})$ is subset of $\mathfrak{O}_{k,n}$. Moreover since $\mathfrak{S}_{k,n}(\mathbb{R})$ is dense in $\mathfrak{B}$, Then by [36, Corollary 1.2.2], it follows that $\mathfrak{S}_{k,n}(\mathbb{R})$ is a core for the generator $\mathfrak{O}_{k,n}$ and the desired result is proved. □

6 Potential applications and simulation perspectives

The theoretical framework developed in this article admits several potential applications in diverse areas of harmonic analysis, signal processing, and mathematical physics. Owing to the additional degrees of freedom offered by the parameters of the linear canonical deformed Hankel transform (LCDHT), the corresponding generalized translation and convolution operators introduced here extend the analytical and practical scope of existing transform methods.

6.1 Uncertainty principles

The LCDHT provides a natural platform for establishing new variants of classical uncertainty relations, including the Heisenberg, Donoho–Stark, and Hardy-type inequalities. By incorporating linear canonical and deformed Hankel parameters, the LCDHT allows sharper localization bounds in both the time and transform domains. Such results are expected to find applications in quantum mechanics, optical tomography, and time–frequency localization theory, where precise phase–space characterizations are essential.

6.2 Signal reconstruction

The generalized translation and convolution structures developed in this work constitute the foundation for signal reconstruction and sampling theorems in the LCDHT domain. These results facilitate the recovery of signals that are bandlimited with respect to the LCDHT rather than the classical Fourier transform, offering significant advantages in nonuniform sampling, filter design, and inverse problems. Potential applications include optical field recovery, radar and sonar imaging, seismic data interpretation, and medical image reconstruction, where signals often exhibit Hankel-type or radial symmetries.

6.3 Simulation and error analysis perspectives

Although the present work is primarily theoretical, the proposed framework can be extended toward numerical validation and simulation studies. A theoretical error analysis may focus on the stability and convergence of the generalized translation and convolution operators under discretization or kernel truncation. Synthetic test signals, such as Gaussian–Bessel or chirp-type functions, may be used to verify reconstruction accuracy and energy preservation. Quantitative measures like mean square error (MSE) and signal-to-noise ratio (SNR) would help assess computational fidelity. Such experiments would not only corroborate the analytical findings but also demonstrate the robustness and applicability of the LCDHT in signal reconstruction and time–frequency localization problems.

7 Conclusion and future work

In this paper, we have investigated the generalized translation and convolution operators within the framework of the linear canonical deformed Hankel transform (LCDHT). Although the results presented here are primarily theoretical, they have been effectively applied to the analysis of the generalized heat equation and the associated heat semigroup. It is pertinent to mention that the proposed transform not only unifies several existing integral transforms such as the classical and fractional Fourier transforms, as well as the linear canonical transform in the Dunkl and Hankel settings but also leads to the formulation of new integral transforms, including the fractional (k, n)-generalized Fourier transform and the generalized Fresnel transform. Furthermore, building upon the harmonic analysis developed in the earlier sections, we have explored the Gabor, wavelet, Wigner, and wavelet multiplier transforms in the context of the LCDHT framework [18]. For future research, we plan to extend this work by investigating additional applications in time-frequency analysis and by developing the reproducing kernel theory associated with the LCDHT. These directions are expected to further enrich the theoretical foundations and broaden the applicability of this new class of integral transforms.

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Data Availability

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Conflict of Interest

The authors declare that they have no conflicts of interest.

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