Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 28, no. 1, pp. 149–177, January 2026
DOI: 10.56754/0719-0646.2801.149

Inertial viscosity Mann-type subgradient extragradient algorithms for solving variational inequality and fixed point problems in real Hilbert spaces

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ABSTRACT

This paper presents two inertial viscosity Mann-type extrapolated algorithms for finding a common solution to the variational inequality problem involving a monotone and Lipschitz continuous operator and the fixed-point problem for a demicontractive mapping in real Hilbert spaces. The proposed algorithms feature an adaptive step size strategy, computed iteratively, which circumvents the need for prior knowledge of the operator's Lipschitz constant. Under appropriate assumptions, we establish two strong convergence theorems guaranteeing the robustness of the methods. Furthermore, we provide a comparative performance analysis of the proposed algorithms against some existing strongly convergent schemes, supported by numerical experiments with MATLAB-based graphical illustrations.

Published: 27 January, 2026
Accepted: 09 December, 2025
Received: 16 May, 2025

RESUMEN

Este artículo presenta dos algoritmos extrapolados de tipo Mann con viscosidad inercial para encontrar una solución común al problema de desigualdad variacional que involucra un operador continuo, monótono y Lipschitz y al problema de punto fijo para una aplicación semicontractiva en espacios de Hilbert reales. Los algoritmos propuestos presentan una estrategia de tamaño de paso adaptativo, calculado iterativamente, que evita la necesidad del conocimiento previo de la constante de Lipschitz del operador. Bajo hipótesis apropiadas, establecemos dos teoremas de convergencia fuertes que garantizan la robustez de los métodos. Más aún, entregamos un análisis comparativo del desempeño de los algoritmos propuestos contra algunos esquemas existentes fuertemente convergentes, sobre la base de experimentos numéricos con ilustraciones gráficas basadas en MATLAB.

Keywords and Phrases: Subgradient extragradient method, extragradient method, Mann-like method, inertial method, viscosity method.

2020 AMS Mathematics Subject Classification: 47H05, 47H09, 49J15, 47J20, 65K15.

1 Introduction

Consider a real Hilbert space $\mathcal{D}$ equipped with the inner product $\langle ., . \rangle$, and the corresponding norm $\|.\|$, and $\varnothing \neq E$ be a closed, convex subset of $\mathcal{D}$. This study is devoted to the pursuit of a common solution to problems involving variational inequalities and fixed point theory within the framework of real Hilbert spaces. The impetus for this investigation arises from the significant role these problems play in numerous mathematical models, where constraints are naturally formulated as variational inequalities and/or fixed point conditions. This situation occurs especially in practical problems, such as signal processing, composite minimization problems, optimal control problems, and image restoration. The relevance and applicability of this framework have been well-established in prior works [3, 17, 20, 23, 32]. Let us recall the involved problems.

The variational inequality problem associated with the operator $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ over the set E seeks to determine a point $v \in E$ such that the following condition is satisfied:

$$\langle \mathcal{F}v, s - v \rangle \geq 0, \quad \forall s \in E. \tag{VIP}$$

The solution set of the (VIP) is denoted by $VI(E, \mathcal{F})$. Variational inequality problems provide a useful and indispensable tool for investigating various interesting issues emerging in many areas, such as social, physics, engineering, economics, network analysis, medical imaging, inverse problems, transportation and much more; see, *e.g.*, [4, 12, 23]. Variational inequality theory has been proven to provide a simple, universal, and consistent structure to deal with possible problems. In the past few decades, researchers have shown tremendous interest in exploring different extensions of variational inequality problems. Recent advancements, as evidenced in works such as [1, 10, 24, 28, 29] underscore a growing emphasis on the development of efficient and practically implementable numerical algorithms for addressing variational inequalities. Under fairly general conditions, two prominent strategies have emerged for solving monotone variational inequalities: projection-type methods and regularization-based approaches. In this study, we concentrate on projection-type methods, with particular attention to the projection gradient method, arguably the most straightforward among them for solving (VIP) given as:

$$s_{n+1} = \mathcal{P}_E(s_n - \eta \mathcal{F} s_n),$$

where $\mathcal{P}_E$, denotes the metric projection onto the set E and $\eta > 0$ is an appropriately chosen step size.

It is worth emphasizing that the projected gradient method necessitates only a single projection onto the feasible set per iteration, making it computationally appealing. However, its convergence typically hinges on relatively strong assumptions, most notably, that the underlying operator is either strongly monotone or inverse strongly monotone. To relax these stringent conditions, Korpelevich [15] proposed the extragradient method, originally designed to solve saddle point problems in Euclidean spaces. The method introduces an additional intermediate step to enhance convergence properties under weaker assumptions. The iterative scheme of the extragradient method is given by:

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta\mathcal{F}s_n), \\ s_{n+1} = \mathcal{P}_E(s_n - \eta\mathcal{F}t_n), \end{cases} \tag{1.1}$$

where operator $\mathcal{F}$ is assumed to be monotone and $\mathcal{L}$-Lipschitz continuous, $\mathcal{P}_E$ represents the metric projection from $\mathcal{D}$ onto E, and $\eta \in (0, 1/\mathcal{L})$. It is established that the sequence $\{s_n\}$ produced by the process (1.1) converges to an element in $VI(E, \mathcal{F})$.

It is essential to recognize that solving the shortest distance problem is equivalent to computing the metric projection onto a closed convex set E. As previously noted, the extragradient method involves two projections onto E in each iteration. While effective, this requirement can pose significant computational challenges, particularly when E is a general closed and convex set with a complex structure. To mitigate this issue, Censor *et al.* [9] introduced the subgradient extragradient method as a refinement of the original extragradient algorithm. The key innovation in this approach lies in replacing the second projection onto E with a projection onto a carefully constructed half-space. This modification is advantageous because projecting onto a half-space is computationally explicit and significantly simpler. The modified algorithm is formulated as follows:

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta\mathcal{F}s_n), \\ T_n = \{s \in \mathcal{D} | \langle s_n - \eta\mathcal{F}s_n - t_n, s - t_n\rangle \leq 0\}, \\ s_{n+1} = \mathcal{P}_{T_n}(s_n - \eta\mathcal{F}t_n), \end{cases} \tag{1.2}$$

The sequence $\{s_n\}$ produced by (1.2) converges weakly to a solution of the variational inequality in this case where $VI(E, \mathcal{F}) \neq \varnothing$.

On the other hand, the fixed point problem plays a pivotal role in the theory and solution of variational inequalities. Let $\mathcal{S} : E \to E$ be a nonlinear mapping. A point $s \in \mathcal{D}$ is called a fixed point of the mapping $\mathcal{S}$ if it satisfies the condition $\mathcal{S}s = s$. The set of all fixed points of $\mathcal{S}$ is denoted as $Fix(\mathcal{S})$. The fixed point problem is formulated as follows:

$$\text{find } v \in E \text{ such that } \mathcal{S}v = v. \tag{FPP}$$

The principal objective of this paper is to determine a common solution to both the (VIP) and the (FPP). Specifically, the goal is to find a point v such that

$$v \in VI(E, \mathcal{F}) \cap Fix(\mathcal{S}). \tag{VIFPP}$$

A wide range of numerical algorithms have been developed to tackle the combined variational inequality and fixed point problem (VIFPP) in infinite-dimensional spaces as documented in [6, 7, 11, 36], and the references therein. Notably, Takahashi and Toyoda [26] proposed an iterative scheme for approximating a solution to the (VIFPP) which is described as follows:

$$s_{n+1} = (1-\zeta_n)s_n + \zeta_n \mathcal{S}\mathcal{P}_E(s_n - \eta_n \mathcal{F}s_n), \tag{1.3}$$

where $\mathcal{F} : E \to \mathcal{D}$ is μ-inverse strongly monotone, $\mathcal{S} : E \to E$ is nonexpansive, $\zeta_n \in (0,1)$ is a control sequence, $\eta_n > 0$ is a stepsize parameter, $\mathcal{P}_E$ denotes the metric projection onto the convex set E. They proved $\{s_n\}$ generated by (1.3) converges weakly to a solution of (VIFPP) under certain conditions. More recently, Censor *et al.* [8] established the following iterative scheme and proved its weak convergence to the solution of the (VIFPP),

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta \mathcal{F}s_n), \\ T_n = \{s \in \mathcal{D} | \langle s_n - \eta \mathcal{F}s_n - t_n, s - t_n\rangle \leq 0\}, \\ s_{n+1} = \zeta_n s_n + (1-\zeta_n)\mathcal{S}\mathcal{P}_{T_n}(s_n - \eta \mathcal{F}t_n). \end{cases} \tag{1.4}$$

In the context of infinite-dimensional Hilbert spaces, strong (norm) convergence is generally more desirable than weak convergence, particularly for practical applications. To ensure strong convergence when solving the combined (VIFPP), Kraikaew and Saejung [16] introduced the Halpern Subgradient Extragradient Method (HSEGM). This method integrates the Halpern iteration scheme with the subgradient extragradient framework, providing a robust approach for approximating common solutions to variational inequality and fixed point problems, which is described as:

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta \mathcal{F}s_n), \\ T_n = \{s \in \mathcal{D} | \langle s_n - \eta \mathcal{F}s_n - t_n, s - t_n\rangle \leq 0\}, \\ u_n = \zeta_n s_0 + (1-\zeta_n)\mathcal{P}_{T_n}(s_n - \eta \mathcal{F}t_n), \\ s_{n+1} = \tau_n s_n + (1-\tau_n)\mathcal{S}u_n, \end{cases} \tag{HSEGM}$$

They proved that the sequence $\{s_n\}$ generated by the (HSEGM) converges strongly to $\mathcal{P}_{VI\cap Fix(\mathcal{S})}(s_0)$, the metric projection of the initial point s_0 onto the set of common solutions of the variational inequality and fixed point problems.

Recently, Thong and Hieu [34] proposed the Modified Subgradient Extragradient Method (MSEGM) by integrating the subgradient extragradient technique with the Mann-type iteration scheme. The primary objective of this algorithm is to identify common solution elements belonging to both the solution set of the variational inequality problem (VIP) and the fixed point set of a demicontractive mapping. The algorithm is formally outlined as follows:

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta \mathcal{F} s_n), \\ T_n = \{s \in \mathcal{D} | \langle s_n - \eta \mathcal{F} s_n - t_n, s - t_n \rangle \leq 0\}, \\ u_n = \mathcal{P}_{T_n}(s_n - \eta \mathcal{F} t_n), \\ s_{n+1} = (1 - \zeta_n - \tau_n) u_n + \tau_n \mathcal{S} u_n, \end{cases} \tag{MSEGM}$$

They proved its strong convergence to an element $v \in VI(E, \mathcal{F}) \cap Fix(\mathcal{S})$, where $\|v\| = min\{\|u\| : u \in VI(E, \mathcal{F}) \cap Fix(\mathcal{S})\}$.

A notable limitation of both the (HSEGM) and (MSEGM) algorithms is their reliance on prior knowledge of the Lipschitz constant of the mapping $\mathcal{F}$. However, in many practical situations, this information is either unavailable or difficult to estimate accurately. To address this issue, Thong and Hieu [35] proposed two extragradient-viscosity algorithms, designed to solve the combined (VIFPP) without requiring the Lipschitz constant. Their approach incorporates an adaptive step-size rule, allowing automatic updates at each iteration. The algorithms are formulated as follows:

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta_n \mathcal{F} s_n), \\ T_n = \{s \in \mathcal{D} | \langle s_n - \eta_n \mathcal{F} s_n - t_n, s - t_n \rangle \leq 0\}, \\ u_n = \mathcal{P}_{T_n}(s_n - \eta_n \mathcal{F} t_n), \\ s_{n+1} = \zeta_n \mathcal{J}(s_n) + (1 - \zeta_n)[(1 - \tau_n) u_n + \tau_n \mathcal{S} u_n], \end{cases} \tag{VSEGM}$$

and

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta_n \mathcal{F} s_n), \\ u_n = t_n - \eta_n (\mathcal{F} t_n - \mathcal{F} s_n), \\ s_{n+1} = \zeta_n \mathcal{J}(s_n) + (1 - \zeta_n)[(1 - \tau_n) u_n + \tau_n \mathcal{S} u_n], \end{cases} \tag{VTEGM}$$

where algorithms (VSEGM) and (VTEGM) update the step size $\{\eta_n\}$ by the following rule:

$$\eta_{n+1} = \begin{cases} \min \left\{ \dfrac{\nu \|s_n - t_n\|}{\|\mathcal{F} s_n - \mathcal{F} t_n\|}, \eta_n \right\}, & \text{if } \mathcal{F} s_n - \mathcal{F} t_n \neq 0 \\ \eta_n, & \text{otherwise,} \end{cases}$$

The sequences produced by (VTEGM) and (VTEGM) converges strongly under mild assumptions to $q \in Fix(\mathcal{S}) \cap VI(E, \mathcal{F})$, where $q = \mathcal{P}_{Fix(\mathcal{S}) \cap VI(E, \mathcal{F})}(\mathcal{J}(q))$.

In recent years, fast iterative algorithms have attracted considerable interest, especially those employing inertial techniques inspired by discrete analogues of second-order dissipative dynamical systems [2, 19]. These inertial methods accelerate convergence by incorporating momentum-like terms into the iterative process. Leveraging this framework, Tan *et al.* [33] proposed the following inertial algorithm for solving the combined variational inequality and fixed point problem (VIFPP)

$$\begin{cases} w_n = s_n + \mathcal{K}_n(s_n - s_{n-1}), \\ t_n = \mathcal{P}_E(w_n - \eta_n \mathcal{F} w_n), \\ T_n = \{s \in \mathcal{D} | \langle w_n - \eta_n \mathcal{F} w_n - t_n, s - t_n \rangle \leq 0\}, \\ u_n = \mathcal{P}_{T_n}(w_n - \eta_n \mathcal{F} t_n), \\ s_{n+1} = \zeta_n \mathcal{J}(s_n) + (1-\zeta_n)[(1-\tau_n)u_n + \tau_n \mathcal{S} u_n], \end{cases} \tag{IVSEGM}$$

where the step size $\{\mathcal{K}_n\}$ and $\{\eta_n\}$ are updated by the following rules:

$$\mathcal{K}_n = \begin{cases} \min\left\{\dfrac{\delta_n}{\|s_n - s_{n-1}\|}, \mathcal{K}\right\}, & \text{if } s_n \neq s_{n-1}, \\ \mathcal{K}, & \text{otherwise,} \end{cases}$$

and

$$\eta_{n+1} = \begin{cases} \min\left\{\dfrac{\nu\|s_n - t_n\|}{\|\mathcal{F}s_n - \mathcal{F}t_n\|}, \eta_n\right\}, & \text{if } \mathcal{F}s_n - \mathcal{F}t_n \neq 0, \\ \eta_n, & \text{otherwise.} \end{cases}$$

Recently, Mewomo *et al.* [18] integrated the inertial, viscosity, and Tseng's approaches and introduced two Generalized Viscosity Inertial Tseng Methods (GVITMs) for solving pseudomonotone variational inequalities with fixed point constraints, formulated as follows:

$$\begin{cases} w_n = s_n + \delta_n(s_n - s_{n-1}), \\ t_n = P_C(w_n - \gamma_n \mathcal{F} w_n), \\ z_n = t_n - \gamma_n(\mathcal{F}t_n - \mathcal{F}w_n), \\ u_n = \beta_{n,0} z_n + \sum_{i=1}^{m} \beta_{n,i} v_{n,i}, \quad v_{n,i} \in S_i z_n, \\ s_{n+1} = \alpha_n \gamma \mathcal{J}(w_n) + (I - \alpha_n G) u_n, \end{cases} \tag{GVITM$_\mathrm{I}$}$$

where δ_n and γ_n are updated by (1.5) and (1.6), respectively.

$$\delta_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|s_n - s_{n-1}\|}, \delta\right\}, & \text{if } s_n \neq s_{n-1}, \\ \delta, & \text{otherwise,} \end{cases} \tag{1.5}$$

$$\gamma_{n+1} = \begin{cases} \min\left\{\dfrac{\phi\|w_n - t_n\|}{\|\mathcal{F}w_n - \mathcal{F}t_n\|}, \gamma_n + \phi_n\right\}, & \text{if } \mathcal{F}w_n - \mathcal{F}t_n \neq 0, \\ \gamma_n + \phi_n, & \text{otherwise,} \end{cases} \tag{1.6}$$

and

$$\begin{cases} w_n = s_n + \delta_n(s_n - s_{n-1}), \\ t_n = P_C(w_n - \gamma_n \nabla\psi w_n), \\ z_n = t_n - \gamma_n(\nabla\psi t_n - \nabla\psi w_n), \\ u_n = \beta_{n,0} z_n + \sum_{i=1}^m \beta_{n,i} v_{n,i}, \quad v_{n,i} \in S_i z_n, \\ s_{n+1} = \alpha_n \gamma \mathcal{J}(w_n) + (I - \alpha_n G) u_n, \end{cases} \tag{GVITM$_{\mathrm{II}}$}$$

where δ_n and γ_n are updated by (1.5) and (1.7), respectively.

$$\gamma_{n+1} = \begin{cases} \min\left\{ \dfrac{\phi\|w_n - t_n\|}{\|\nabla\psi w_n - \nabla\psi t_n\|}, \gamma_n + \phi_n \right\}, & \text{if } \nabla\psi w_n - \nabla\psi y_n \neq 0, \\ \gamma_n + \phi_n, & \text{otherwise.} \end{cases} \tag{1.7}$$

where $\delta > 0$, $\gamma_1 > 0$, ϕ_n is a nonnegative sequence such that $\sum_{n=1}^\infty \phi_n < +\infty$, and $\phi \in (0,1)$. The authors established strong convergence results for the sequences generated by (GVITM$_{\mathrm{I}}$) and (GVITM$_{\mathrm{II}}$) without imposing the sequential weak continuity of the pseudomonotone operator and without requiring prior knowledge of the Lipschitz constants.

Recently, Kesornprom *et al.* [14] proposed a new variant of the proximal gradient algorithm incorporating double inertial extrapolation for solving constrained convex minimization problems in Hilbert spaces, formulated as follows:

$$\begin{aligned} z^n &= s^n + \theta_n(s^n - s^{n-1}) + \eta_n(s^{n-1} - s^{n-2}), \quad n \geq 1, \\ s^{n+1} &= P_E(prox_{\alpha_n g}(z^n - \alpha_n \nabla f(z^n))), \end{aligned}$$

where

$$\alpha_{n+1} = \begin{cases} \min\left\{ \dfrac{\delta\|z^n - prox_{\alpha_n g}(z^n - \alpha_n \nabla f(z^n))\|}{\|\nabla f(z^n) - \nabla f(prox_{\alpha_n g}(z^n - \alpha_n \nabla f(z^n)))\|}, \alpha_n \right\}, \\ \qquad\qquad \text{if } \nabla f(z^n) - \nabla f(prox_{\alpha_n g}(z^n - \alpha_n \nabla f(z^n))) \neq 0, \\ \alpha_n, \qquad\qquad \text{otherwise.} \end{cases}$$

where $\theta_n \geq 0$, $\eta_n \geq 0$, $\alpha_1 > 0$ and $\delta \in (0, \frac{1}{2})$. They established the weak convergence of the proposed method to a point in $\operatorname{argmin}(f+g) \cap E$. For an extensive discussion on fast iterative algorithms and their recent advancements, the reader may consult [21,25,31,38,39] and the references therein.

Motivated and inspired by existing studies in this area, the purpose of this paper is to develop two inertial extragradient algorithms that combine the Mann iteration, viscosity approximation, and subgradient extragradient methods with a new step size for discovering a common solution of a monotone and Lipschitz variational inequality problem and of the fixed point problem involving a demicontractive mapping in real Hilbert spaces. The suggested algorithms need to calculate the projection on the feasible set only once per iteration, which makes them faster. Strong convergence theorems of the algorithms are established without the prior information of the Lipschitz constant of the operator. Lastly, some computational tests appearing in finite and infinite dimensions are proposed to support the theoretical results.

The organizational structure of our paper is built up as follows. In Section 2, we recall some preliminary results and lemmas that need to be used in the next section. In Section 3, we propose the algorithms and analyse their convergence. Some numerical experiments to verify our theoretical results are presented in Section 4. At last, the paper ends with a brief summary in Section 5, the final section.

2 Preliminaries

Consider $\varnothing \neq E$ (closed, convex) subset of a real Hilbert space $\mathcal{D}$. The weak convergence and strong convergence of the sequence $\{s_n\}$ to s are denoted as $s_n \rightharpoonup s$ and $s_n \to s$, respectively. For any $s, t \in \mathcal{D}$ and $\zeta \in R$ the following statements hold:

(i) $\|s+t\|^2 = \|s\|^2 + 2\langle s, t\rangle + \|t\|^2$.

(ii) $\|s+t\|^2 \leq \|s\|^2 + 2\langle t, s+t\rangle$.

(iii) $\|\zeta s + (1-\zeta)t\|^2 = \zeta\|s\|^2 + (1-\zeta)\|t\|^2 - \zeta(1-\zeta)\|s-t\|^2$.

For any point $s \in \mathcal{D}$, there exists a distinct nearest point in the closed and convex subset E identified as $\mathcal{P}_E(s)$ satisfying $\mathcal{P}_E(s) = \operatorname{argmin}\{\|s-t\|, t \in E\}$. $\mathcal{P}_E$ is termed as the metric projection of $\mathcal{D}$ onto E. It is established that $\mathcal{P}_E$ is a nonexpansive mapping and it possesses the following fundamental properties:

(i) $\langle s - \mathcal{P}_E(s), t - \mathcal{P}_E(s)\rangle \leq 0$, $\forall t \in E$.

(ii) $\|\mathcal{P}_E(s) - \mathcal{P}_E(t)\|^2 \leq \langle \mathcal{P}_E(s) - \mathcal{P}_E(t), s-t\rangle$, $\forall t \in \mathcal{D}$.

Definition 2.1 ([27]). *A mapping $\mathcal{A} : \mathcal{D} \to \mathcal{D}$ is said to be:*

(i) $\mathcal{L}$-Lipschitz continuous with $\mathcal{L} > 0$ if

$$\|\mathcal{A}s - \mathcal{A}t\| \leq \mathcal{L}\|s-t\|, \quad \forall s, t \in \mathcal{D}.$$

(ii) ζ-strongly monotone if there exists $\zeta > 0$ such that

$$\langle \mathcal{A}s - \mathcal{A}t, s-t\rangle \geq \zeta\|s-t\|^2, \quad \forall s, t \in \mathcal{D}.$$

(iii) *ζ-inverse strongly monotone if there exists $\zeta > 0$ such that*

$$\langle \mathcal{A}s - \mathcal{A}t, s - t\rangle \geq \zeta \|\mathcal{A}s - \mathcal{A}t\|^2, \quad \forall s, t \in \mathcal{D}.$$

Remark 2.2 ([5]). *if $\mathcal{A} : \mathcal{D} \to \mathcal{D}$ be an injective operator so that $\mathcal{A}^{-1}$ is well defined, then:*

(a) If $\mathcal{A}$ is ζ-strongly monotone, then its inverse $\mathcal{A}^{-1}$ is ζ-inverse strongly monotone.

(b) If $\mathcal{A}$ is ζ-inverse strongly monotone, then its inverse $\mathcal{A}^{-1}$ is ζ-strongly monotone.

(iv) *monotone if*

$$\langle \mathcal{A}s - \mathcal{A}t, s - t\rangle \geq 0, \quad \forall s, t \in \mathcal{D}.$$

(v) *quasi-nonexpansive if*

$$\|\mathcal{A}s - u\| \leq \|s - t\|, \quad \forall u \in Fix(\mathcal{A}), \quad s \in \mathcal{D}.$$

(vi) *μ-strictly pseudocontractive with $0 \leq \mu < 1$ if*

$$\|\mathcal{A}s - \mathcal{A}t\|^2 \leq \|s - t\|^2 + \mu\|(I - \mathcal{A})s - (I - \mathcal{A})t\|^2, \quad \forall s, t \in \mathcal{D}.$$

(vii) *τ-demicontractive with $0 \leq \tau < 1$ if*

$$\|\mathcal{A}s - u\|^2 \leq \|s - u\|^2 + \tau\|(I - \mathcal{A})s\|^2, \quad \forall u \in Fix(\mathcal{A}), \quad s \in \mathcal{D}. \tag{2.1}$$

or equivalently

$$\langle \mathcal{A}s - s, s - u\rangle \leq \frac{\tau - 1}{2}\|s - \mathcal{A}s\|^2, \quad \forall u \in Fix(\mathcal{A}), \quad s \in \mathcal{D}. \tag{2.2}$$

Definition 2.3 ([37]). *If $\mathcal{A} : \mathcal{D} \to \mathcal{D}$ is a nonlinear operator with $Fix(\mathcal{A}) \neq \varnothing$. Then, $I - \mathcal{A}$ is said to be demiclosed at zero if for any $\{s_n\}$ in $\mathcal{D}$, the following implications holds:*

$$s_n \rightharpoonup s \text{ and } (I - \mathcal{A})s_n \to 0 \implies s \in Fix(\mathcal{A}).$$

Lemma 2.4 ([33]). *Consider $\mathcal{S} : \mathcal{D} \to \mathcal{D}$ as a τ-demicontractive operator with $Fix(\mathcal{S}) \neq \varnothing$. Let $\mathcal{S}_\mu = (1 - \mu)I + \mu\mathcal{S}$, where $\mu \in (0, 1 - \tau)$. Then:*

(i) $Fix(\mathcal{S}) = Fix(\mathcal{S}_\mu)$.

(ii) $\|\mathcal{S}_\mu s - u\|^2 \leq \|s - u\|^2 - \mu(1 - \tau - \mu)\|(I - \mathcal{S})s\|^2$, $\forall s \in \mathcal{D}$, $u \in Fix(\mathcal{S})$.

(iii) *$Fix(\mathcal{S})$ is a closed convex subset of $\mathcal{D}$.*

Lemma 2.5 ([16])**.** *Consider $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ as a monotone and $\mathcal{L}$-Lipschitz continuous mapping on E. Let $S = \mathcal{P}_E(I - \nu\mathcal{F})$, where $\nu > 0$. If s_n is a sequence in $\mathcal{D}$ such that $s_n \rightharpoonup q$ and $s_n - Ss_n \to 0$, then it follows that $q \in VI(E, \mathcal{F}) = Fix(S)$.*

Lemma 2.6 ([22])**.** *Consider a positive sequence $\{r_n\}$, a sequence of real numbers $\{b_n\}$ and a sequence $\{a_n\}$ in the interval $(0, 1)$ such that $\sum_{n=1}^{\infty} a_n = \infty$. Assuming*

$$r_{n+1} \leq a_n b_n + (1 - a_n) r_n, \quad \forall n \geq 1$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{r_{n_k}\}$ of $\{r_n\}$ satisfying $\liminf_{k\to\infty}(r_{n_k+1} - r_{n_k}) \geq 0$, then $\lim_{n\to\infty} r_n = 0$.

3 Main result

This section presents two inertial extragradient algorithms that are specifically designed to solve (VIFPP), and provides a convergence analysis of them. We first assume that the following conditions are met by the suggested algorithms.

(A1) $Fix(\mathcal{S}) \cap VI(E, \mathcal{F}) \neq \varnothing$.

(A2) $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is monotone and $\mathcal{L}$-Lipschitz continuous.

(A3) $\mathcal{S} : \mathcal{D} \to \mathcal{D}$ is μ-demicontractive such that $(I - \mathcal{S})$ is demiclosed at zero.

(A4) $\mathcal{J} : \mathcal{D} \to \mathcal{D}$ is $\mathcal{Q}$-contraction with constant $\mathcal{Q} \in [0, 1)$.

3.1 Algorithm-I

Algorithm 3.1 Algorithm-I

Initialization: Choose $\mathcal{K} > 0$, $\eta_1 > 0$, and $\nu \in (0, 1)$.
Select arbitrary s_0 and s_1 from $\mathcal{D}$.
Iterative step:
Step 1. Given the iterates s_{n-1} and s_n (for $n \geq 1$), set

$$w_n = s_n + \mathcal{K}_n(s_n - s_{n-1}),$$

where

$$\mathcal{K}_n = \begin{cases} \min\left\{\dfrac{\delta_n}{\|s_n - s_{n-1}\|}, \mathcal{K}\right\}, & \text{if } s_n \neq s_{n-1}; \\ \mathcal{K}, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute

$$t_n = \mathcal{P}_E(w_n - \eta_n \mathcal{F} w_n).$$

Algorithm 3.1 Algorithm-I

Step 3. Compute
$$u_n = \mathcal{P}_{T_n}(w_n - \eta_n \mathcal{F} t_n),$$
where the half-space T_n is defined by
$$T_n := \{s \in \mathcal{D} \mid \langle w_n - \eta_n \mathcal{F} w_n - t_n, s - t_n \rangle \leq 0\}.$$

Step 4. Compute
$$s_{n+1} = \zeta_n \mathcal{J}(s_n) + (1 - \zeta_n)\big[(1 - \gamma_n - \tau_n)u_n + \tau_n \mathcal{S} u_n\big]$$
and update
$$\eta_{n+1} = \begin{cases} \min\left\{ \dfrac{\nu\|w_n - t_n\|}{\|\mathcal{F}w_n - \mathcal{F}t_n\|}, \eta_n \right\}, & \text{if } \mathcal{F}w_n - \mathcal{F}t_n \neq 0; \\ \eta_n, & \text{otherwise.} \end{cases} \tag{3.2}$$

Set $n := n + 1$ and go to **Step 1**.

The subsequent lemmas prove to be valuable for analyzing the convergence of the algorithm.

Lemma 3.1 ([33]). *The sequence $\{\eta_n\}$ produced by (3.2) is a nonincreasing sequence and*
$$\lim_{n\to\infty} \eta_n = \eta \geq \min\left\{\eta_1, \frac{\nu}{\mathcal{L}}\right\}.$$

Lemma 3.2 ([30]). *Assume that condition* **(A2)** *holds. Let $\{u_n\}$ be a sequence produced by Algorithm 3.1, then*
$$\|u_n - v\|^2 \leq \|w_n - v\|^2 - \left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|t_n - w_n\|^2 - \left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|u_n - t_n\|^2 \tag{3.3}$$
for all $v \in VI(E, \mathcal{F})$.

Theorem 3.3. *Under the fulfillment of Conditions* **(A1)-(A4)**, *$\{\delta_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\delta_n}{\zeta_n} = 0$, where $\zeta_n \subset (0,1)$ satisfies $\sum_{n=1}^{\infty} \zeta_n = \infty$ and $\lim_{n\to\infty} \zeta_n = 0$. Furthermore, for some $a > 0$, $b > 0$, $\gamma_n \in (0,1)$, $\lim_{n\to\infty} \gamma_n = 0$, and $\sum_{n=0}^{\infty} \gamma_n = \infty$, let $\tau_n \in (a,b) \subset (0, (1-\mu)(1-\gamma_n))$, then the sequence $\{s_n\}$ produced by Algorithm 3.1 converges in norm to $v \in Fix(\mathcal{S}) \cap VI(E,\mathcal{F})$, where $v = \mathcal{P}_{Fix(\mathcal{S})\cap VI(E,\mathcal{F})}(\mathcal{J}(v))$.*

Proof. Since $VI(E,\mathcal{F})$ is a closed convex subset, and by Lemma 2.4, $Fix(\mathcal{S})$ is also a closed convex subset. Therefore, the mapping $\mathcal{P}_{Fix(\mathcal{S})\cap VI(E,\mathcal{F})}(\mathcal{J}) : \mathcal{D} \to \mathcal{D}$ forms a contraction. By applying the Banach contraction principle, there exists a unique point $v \in \mathcal{D}$ such that $v = \mathcal{P}_{Fix(\mathcal{S})\cap VI(E,\mathcal{F})}(\mathcal{J})$. Specifically, $v \in Fix(\mathcal{S}) \cap VI(E,\mathcal{F})$, and
$$\langle \mathcal{J}(v) - v, u - v\rangle \leq 0, \quad \forall u \in Fix(\mathcal{S}) \cap VI(E,\mathcal{F}).$$

The proof is split up into four sections.

Claim 1. $\{s_n\}$ is a bounded sequence. Put, $t_n = (1-\gamma_n-\tau_n)u_n + \tau_n \mathcal{S}u_n$, we have

$$\begin{aligned}\|t_n - v\| &= \|(1-\gamma_n-\tau_n)u_n + \tau_n \mathcal{S}u_n - v\| \\ &= \|(1-\gamma_n-\tau_n)(u_n - v) + \tau_n(\mathcal{S}u_n - v) - \gamma_n v\| \\ &= \|(1-\gamma_n-\tau_n)(u_n - v) + \tau_n(\mathcal{S}u_n - v)\| + \|\gamma_n v\|. \end{aligned} \tag{3.4}$$

Additionally, it can be deduced from (2.1), (2.2), and Lemma 3.2 that

$$\begin{aligned}\|(1-\gamma_n-\tau_n)(u_n - v) + \tau_n(\mathcal{S}u_n - v)\|^2 &= (1-\gamma_n-\tau_n)^2\|(u_n - v)\|^2 \\ &+ 2(1-\gamma_n-\tau_n)\tau_n\langle \mathcal{S}u_n - v, u_n - v\rangle + \tau_n^2\|\mathcal{S}u_n - v\|^2 \\ &\le (1-\gamma_n-\tau_n)^2\|(u_n - v)\|^2 \\ &+ 2(1-\gamma_n-\tau_n)\tau_n\left[\|u_n - v\|^2 - \frac{1-\mu}{2}\|u_n - \mathcal{S}u_n\|^2\right] \\ &+ \tau_n^2\left[\|u_n - v\|^2 + \mu\|u_n - \mathcal{S}u_n\|^2\right] \\ &= (1-\gamma_n)^2\|u_n - v\|^2 + \tau_n\left(\tau_n - (1-\gamma_n)(1-\mu)\right)\|u_n - \mathcal{S}u_n\|^2 \\ &\le (1-\gamma_n)^2\|u_n - v\|^2 \le (1-\gamma_n)^2\|w_n - v\|^2\end{aligned}$$

signifying that

$$\|(1-\gamma_n-\tau_n)(u_n - v) + \tau_n(\mathcal{S}u_n - v)\| \le (1-\gamma_n)\|w_n - v\|. \tag{3.5}$$

By the definition of w_n, we obtain

$$\|w_n - v\| = \|s_n + \mathcal{K}_n(s_n - s_{n-1}) - v\| \le \|s_n - v\| + \zeta_n\frac{\mathcal{K}_n}{\zeta_n}\|s_n - s_{n-1}\|.$$

From (3.1), it can be deduced that

$$\lim_{n\to\infty}\frac{\mathcal{K}_n}{\zeta_n}\|s_n - s_{n-1}\| = 0.$$

This result holds true, since $\mathcal{K}_n\|s_n - s_{n-1}\| \le \delta_n$ for all $n \ge 1$. Moreover, considering the limit $\lim_{n\to\infty}\frac{\delta_n}{\zeta_n} = 0$, it follows that

$$\lim_{n\to\infty}\|s_n - s_{n-1}\| \le \lim_{n\to\infty}\frac{\delta_n}{\zeta_n} = 0.$$

Therefore, there exists a constant $\mathcal{M}_* > 0$ such that

$$\frac{\mathcal{K}_n}{\zeta_n}\|s_n - s_{n-1}\| \le \mathcal{M}_*, \quad \forall n \ge 1. \tag{3.6}$$

Thus utilizing above, we get

$$\|w_n - v\| \le \|s_n - v\| + \zeta_n \mathcal{M}_*. \tag{3.7}$$

which in turn implies

$$\|(1-\gamma_n-\tau_n)(u_n - v) + \tau_n(\mathcal{S}u_n - v)\| \le \|s_n - v\| + \zeta_n \mathcal{M}_*.$$

Referring to (3.4), we obtain

$$\|t_n - v\| \le \|s_n - v\| + \zeta_n \left[\mathcal{M}_* + \frac{\gamma_n}{\zeta_n}\|v\|\right] \le \|s_n - v\| + \zeta_n \mathcal{M}, \tag{3.8}$$

where $\left[\mathcal{M}_* + \frac{\gamma_n}{\zeta_n}\|v\|\right] \le \mathcal{M}$ for some $\mathcal{M} > 0$. Now,

$$\begin{aligned}
\|s_{n+1} - v\| &\le \|\zeta_n \mathcal{J}(s_n) + (1-\zeta_n)\eta_n - v\| \\
&\le \zeta_n\|\mathcal{J}(s_n) - \mathcal{J}(v)\| + \zeta_n\|\mathcal{J}(v) - v\| + (1-\zeta_n)\|t_n - v\| \\
&\le \zeta_n \mathcal{Q}\|s_n - v\| + \zeta_n\|\mathcal{J}(v) - v\| + (1-\zeta_n)[\|s_n - v\| + \zeta_n\mathcal{M}] \\
&= (1-\zeta_n(1-\mathcal{Q}))\|s_n - v\| + \zeta_n(\|\mathcal{J}(v) - v\| + \mathcal{M}) \\
&\le \max\left\{\|s_n - v\|, \frac{\|\mathcal{J}(v) - v\| + \mathcal{M}}{1-\mathcal{Q}}\right\} \le \cdots \le \max\left\{\|s_0 - v\|, \frac{\|\mathcal{J}(v) - v\| + \mathcal{M}}{1-\mathcal{Q}}\right\}.
\end{aligned}$$

This implies that the sequence $\{s_n\}$ is bounded. Consequently, the sequences $\{w_n\}$, $\mathcal{J}(s_n)$, $\{t_n\}$, and $\{u_n\}$ are also bounded.

Claim 2.

$$\begin{aligned}
&(1-\zeta_n)\left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|t_n - w_n\|^2 + (1-\zeta_n)\left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|u_n - t_n\|^2 \\
&\quad + (1-\zeta_n)\tau_n[(1-\mu) - \tau_n]\|u_n - \mathcal{S}u_n\| \\
&\quad \le \|s_n - v\|^2 - \|s_{n+1} - v\|^2 + \zeta_n\|\mathcal{J}(s_n) - v\|^2 + \zeta_n\mathcal{M}_{**} + (1-\zeta_n)\gamma_n\mathcal{M}_{***}.
\end{aligned}$$

Since from (3.7),

$$\|w_n - v\|^2 \le (\|s_n - v\| + \zeta_n\mathcal{M}_*)^2 = \|s_n - v\|^2 + \zeta_n\left(2\mathcal{M}_*\|s_n - v\| + \zeta_n\mathcal{M}_*^2\right) \le \|s_n - v\|^2 + \zeta_n\mathcal{M}_{**}, \tag{3.9}$$

for some $\mathcal{M}_{**} > 0$.

$$\|s_{n+1} - v\|^2 = \|\zeta_n(\mathcal{J}(s_n) - v) + (1-\zeta_n)(t_n - v)\|^2 \le \zeta_n\|\mathcal{J}(s_n) - v\|^2 + (1-\zeta_n)\|t_n - v\|^2. \tag{3.10}$$

Now,

$$\begin{aligned}\|t_n - v\|^2 &= \|(1-\gamma_n-\tau_n)u_n + \tau_n\mathcal{S}u_n - v\|^2 = \|(u_n - v) + \tau_n(\mathcal{S}u_n - u_n) - \gamma_n u_n\|^2\\ &\leq \|(u_n - v) + \tau_n(\mathcal{S}u_n - u_n)\|^2 - 2\gamma_n\langle u_n, \eta_n - v\rangle\\ &= \|u_n - v\|^2 + \tau_n^2\|\mathcal{S}u_n - u_n\|^2 + 2\tau_n\langle \mathcal{S}u_n - u_n, u_n - v\rangle + 2\gamma_n\langle u_n, v - \eta_n\rangle.\end{aligned}$$

It follows from Lemma(2.4),

$$\begin{aligned}\|t_n - v\|^2 &\leq \|u_n - v\|^2 + \tau_n^2\|\mathcal{S}u_n - u_n\|^2 - \tau_n(1-\mu)\|u_n - \mathcal{S}u_n\|^2 + 2\gamma_n\langle u_n, v - \eta_n\rangle\\ &\leq \|w_n - v\|^2 + \tau_n[\tau_n - (1-\mu)]\|u_n - \mathcal{S}u_n\|^2 + \gamma_n\mathcal{M}_{***}. \end{aligned} \tag{3.11}$$

for some $\mathcal{M}_{***} > 0$, from (3.10)

$$\begin{aligned}\|s_{n+1} - v\|^2 &\leq \zeta_n\|\mathcal{J}(s_n) - v\|^2\\ &\quad + (1-\zeta_n)\left[\|w_n - v\|^2 + \tau_n[\tau_n - (1-\mu)]\|u_n - \mathcal{S}u_n\|^2 + \gamma_n\mathcal{M}_{***}\right]\\ &\leq \zeta_n\|\mathcal{J}(s_n) - v\|^2 + \|s_n - v\|^2 + \zeta_n\mathcal{M}_{**}\\ &\quad - (1-\zeta_n)\left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|t_n - w_n\|^2 - (1-\zeta_n)\left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|u_n - t_n\|^2\\ &\quad - (1-\zeta_n)\tau_n[(1-\mu) - \tau_n]\|u_n - \mathcal{S}u_n\| + (1-\zeta_n)\gamma_n\mathcal{M}_{***}.\end{aligned}$$

By a straightforward manipulation, we attain the desired result.

Claim 3.

$$\|s_{n+1} - v\|^2 = (1-(1-\mathcal{Q})\zeta_n)\|s_n - v\|^2 + \zeta_n(1-\mathcal{Q})\left[\frac{(1-\zeta_n)\zeta_n\mathcal{N} + 2\langle\mathcal{J}(v) - v, s_{n+1} - v\rangle}{1-\mathcal{Q}}\right].$$

Since by (3.8),

$$\|t_n - v\|^2 \leq [\|s_n - v\| + \zeta_n\mathcal{M}]^2 = \|s_n - v\|^2 + \zeta_n^2\mathcal{M}^2 + 2\zeta_n\langle\mathcal{M}, s_n - v\rangle \leq \|s_n - v\|^2 + \zeta_n^2\mathcal{N},$$

where $\mathcal{M}^2 + \frac{2}{\zeta_n}\langle\mathcal{M}, s_n - v\rangle \leq \mathcal{N}$ for some $\mathcal{N} > 0$.

$$\begin{aligned}\|s_{n+1} - v\|^2 &= \|\zeta_n\mathcal{J}(s_n) + (1-\zeta_n)t_n - v\|^2\\ &= \|\zeta_n(\mathcal{J}(s_n) - \mathcal{J}(v)) + (1-\zeta_n)(t_n - v) + \zeta_n(\mathcal{J}(v) - v)\|^2\\ &\leq \|\zeta_n(\mathcal{J}(s_n) - \mathcal{J}(v)) + (1-\zeta_n)(t_n - v)\|^2 + 2\zeta_n\langle\mathcal{J}(v) - v, s_{n+1} - v\rangle\\ &\leq \zeta_n\mathcal{Q}\|s_n - v\|^2 + (1-\zeta_n)\|t_n - v\|^2 + 2\zeta_n\langle\mathcal{J}(v) - v, s_{n+1} - v\rangle\\ &\leq \zeta_n\mathcal{Q}\|s_n - v\|^2 + (1-\zeta_n)\left[\|s_n - v\|^2 + \zeta_n^2\mathcal{N}\right] + 2\zeta_n\langle\mathcal{J}(v) - v, s_{n+1} - v\rangle\\ &= (1-(1-\mathcal{Q})\zeta_n)\|s_n - v\|^2 + \zeta_n(1-\mathcal{Q})\left[\frac{(1-\zeta_n)\zeta_n\mathcal{N} + 2\langle\mathcal{J}(v) - v, s_{n+1} - v\rangle}{1-\mathcal{Q}}\right].\end{aligned}$$

Claim 4. The sequence $\|s_n - v\|^2$ converges to zero. In fact, using Lemma 2.6, it is sufficient to show that for each subsequence $\|s_{n_k} - v\|$ of $\|s_n - v\|$ satisfying $\limsup_{k\to\infty}\langle \mathcal{J}(v)-v, s_{n+1}-v\rangle \leq 0$ with

$$\liminf_{k\to\infty}(\|s_{n_k+1} - v\| - \|s_{n_k} - v\|) \geq 0. \tag{3.12}$$

We assume that $\|s_{n_k} - v\|$ is a subsequence of $\|s_n - v\|$, such that (3.12) holds, for the purposes of this analysis. Next,

$$\begin{aligned}\liminf_{k\to\infty}&\left(\|s_{n_k+1} - v\|^2 - \|s_{n_k} - v\|^2\right)\\ &= \liminf_{k\to\infty}\left[(\|s_{n_k+1} - v\| - \|s_{n_k} - v\|)(\|s_{n_k+1} - v\| + \|s_{n_k} - v\|)\right] \geq 0.\end{aligned}$$

Based on Claim 2, we have,

$$\begin{aligned}\limsup_{k\to\infty}&(1-\zeta_{n_k})\left(1-\nu\frac{\eta_{n_k}}{\eta_{n_k+1}}\right)\|t_{n_k} - w_{n_k}\|^2 + (1-\zeta_{n_k})\left(1-\nu\frac{\eta_{n_k}}{\eta_{n_k+1}}\right)\|u_{n_k} - t_{n_k}\|^2\\ &+ (1-\zeta_{n_k})\tau_{n_k}[(1-\mu) - \tau_{n_k}]\|u_{n_k} - \mathcal{S}u_{n_k}\|\\ &\leq \limsup_{k\to\infty}\left[\|s_{n_k} - v\|^2 - \|s_{n_k+1} - v\|^2 + \zeta_{n_k}\|\mathcal{J}(s_{n_k}) - v\|^2\right.\\ &\left.+ \zeta_{n_k}\mathcal{M}_{**} + (1-\zeta_{n_k})\gamma_{n_k}\mathcal{M}_{***}\right]\\ &= -\liminf_{k\to\infty}\left[\|s_{n_k+1} - v\|^2 - \|s_{n_k} - v\|^2\right]\end{aligned}$$

signifying that

$$\lim_{k\to\infty}\|w_{n_k} - t_{n_k}\| = 0, \lim_{k\to\infty}\|u_{n_k} - \mathcal{S}u_{n_k}\| = 0, \|u_{n_k} - t_{n_k}\| = 0. \tag{3.13}$$

Therefore, we can infer that $\lim_{k\to\infty}\|u_{n_k} - w_{n_k}\| = 0$. Referring to the definition of w_n, we have

$$\|s_{n_k} - w_{n_k}\| = \mathcal{K}_{n_k}\|s_{n_k} - s_{n_{k-1}}\| = \zeta_{n_k}\frac{\mathcal{K}_{n_k}}{\zeta_{n_k}}\|s_{n_k} - s_{n_{k-1}}\| \to 0 \text{ as } k \to \infty. \tag{3.14}$$

This in conjunction with $\lim_{k\to\infty}\|u_{n_k} - w_{n_k}\| = 0$, implies that

$$\lim_{k\to\infty}\|u_{n_k} - s_{n_k}\| = 0. \tag{3.15}$$

Considering $t_{n_k} = (1-\gamma_{n_k} - \tau_{n_k})u_{n_k} + \tau_{n_k}\mathcal{S}u_{n_k}$, it is evident that

$$\|t_{n_k} - u_{n_k}\| \leq \tau_n\|(\mathcal{S}u_n - u_{n_k})\| + \gamma_n\|u_{n_k}\|.$$

Hence, we obtain

$$\|t_{n_k} - u_{n_k}\| = 0. \tag{3.16}$$

By using (3.15) and (3.16), we can deduce that

$$\begin{aligned}\|s_{n_{k+1}} - s_{n_k}\| &\leq \|\zeta_{n_k}\mathcal{J}(s_{n_k}) + (1-\zeta_{n_k})t_{n_k} - s_{n_k}\| \\ &\leq \zeta_{n_k}\|\mathcal{J}(s_{n_k}) - s_{n_k}\| + (1-\zeta_{n_k})\|t_{n_k} - s_{n_k}\| \\ &\leq \zeta_{n_k}\|\mathcal{J}(s_{n_k}) - s_{n_k}\| + \|t_{n_k} - u_{n_k}\| + \|u_{n_k} - s_{n_k}\| \to 0 \text{ as } k \to \infty. \quad (3.17)\end{aligned}$$

Given that the sequence $\{s_{n_k}\}$ is bounded, it can be inferred that there exists a subsequence $\{s_{n_{k_j}}\}$ of $\{s_{n_k}\}$ such that $s_{n_{k_j}} \rightharpoonup u$. This further implies that

$$\limsup_{k\to\infty}\langle\mathcal{J}(v) - v, s_{n_k} - v\rangle = \lim_{j\to\infty}\langle\mathcal{J}(v) - v, s_{n_{k_j}} - v\rangle = \langle\mathcal{J}(v) - v, u - v\rangle. \quad (3.18)$$

From (3.14), it follows that $w_{n_k} \rightharpoonup u$. Combining (3.13), $\lim_{n\to\infty}\eta_n = \eta$ and Lemma 2.5, one can conclude that $u \in VI(E, \mathcal{F})$. Utilizing (3.15), we have $u_{n_k} \rightharpoonup u$. By the demiclosedness of $(I - \mathcal{S})$, we obtain $u \in Fix(\mathcal{S})$. Consequently, $u \in Fix(\mathcal{S}) \cap VI(E, \mathcal{F})$. Combining (3.18), the definition of v and $u \in Fix(\mathcal{S}) \cap VI(E, \mathcal{F})$, we obtain

$$\limsup_{k\to\infty}\langle\mathcal{J}(v) - v, s_{n_k} - v\rangle = \langle\mathcal{J}(v) - v, u - v\rangle \leq 0, \quad (3.19)$$

which in conjunction with (3.19) and (3.18), implies that

$$\begin{aligned}\limsup_{k\to\infty}\langle\mathcal{J}(v) - v, s_{n_k+1} - v\rangle &\leq \limsup_{k\to\infty}\langle\mathcal{J}(v) - v, s_{n_k+1} - s_{n_k}\rangle + \limsup_{k\to\infty}\langle\mathcal{J}(v) - v, s_{n_k} - v\rangle \\ &= \langle\mathcal{J}(v) - v, u - v\rangle \leq 0 \quad (3.20)\end{aligned}$$

Therefore (3.20) and Claim 3 in the light of Lemma 2.6 indicates that $s_n \to v$ as $n \to \infty$. Thus, completes the proof. □

Specifically, we may design a new algorithm for (VIP) if $\mathcal{S} = I$ (identity operator) in Algorithm 3.1. To be more exact, we have the corollary that follows:

Corollary 3.4. *If $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is Lipschitz continuous, monotone and $\mathcal{J} : \mathcal{D} \to \mathcal{D}$ is a $\mathcal{Q}$-contraction with $\mathcal{Q} \in [0, 1)$. If the sequences γ_n, ζ_n, and τ_n be same as in Theorem 3.3 and if $VI(E, \mathcal{F}) \neq \varnothing$, let $s_0, s_1 \in \mathcal{D}$ and let the sequence $\{s_n\}$ be generated by*

$$\begin{cases} w_n = s_n + \mathcal{K}_n(s_n - s_{n-1}), \\ t_n = \mathcal{P}_E(w_n - \eta_n\mathcal{F}w_n), \\ u_n = \mathcal{P}_{T_n}(w_n - \eta_n\mathcal{F}t_n), \ \textit{where the half-space } T_n \textit{ is defined by} \\ T_n := \{s \in \mathcal{D} | \langle w_n - \eta_n\mathcal{F}w_n - t_n, s - t_n\rangle \leq 0\}, \\ s_{n+1} = \zeta_n\mathcal{J}(s_n) + (1-\zeta_n)((1-\gamma_n)u_n), \end{cases} \quad (3.21)$$

where $\mathcal{K}_n$ and η_n are defined by (3.1) *and* (3.2)*, respectively. Then the iterative sequence* $\{s_n\}$ *generated by* (3.21) *converges to* $v \in VI(E, \mathcal{F})$ *in norm, where* $v = \mathcal{P}_{VI(E,\mathcal{F})}(\mathcal{J}(v))$.

3.2 Algorithm-II

Algorithm 3.2 Algorithm-II

Initialization: Choose $\mathcal{K} > 0, \eta_1 > 0, \nu \in (0, 1)$. Let $s_0, s_1 \in \mathcal{D}$ be arbitrary.
Iterative step: Calculate s_{n+1} as follows:
Step 1. Given the iterates s_{n-1} and $s_n (n \geq 1)$. Set $w_n = s_n + \mathcal{K}_n(s_n - s_{n-1})$, where $\mathcal{K}_n$ is defined by (3.1).
Step 2. Compute $t_n = \mathcal{P}_E(w_n - \eta_n \mathcal{F} w_n)$.
Step 3. Compute $u_n = \mathcal{P}_{T_n}(w_n - \eta_n \mathcal{F} t_n)$, where the half-space T_n is defined by

$$T_n := \{s \in \mathcal{D} \mid \langle w_n - \eta_n \mathcal{F} w_n - t_n, s - t_n \rangle \leq 0\}.$$

Step 4. Compute $s_{n+1} = \zeta_n \mathcal{J}(s_n) + (1 - \zeta_n)[(1 - \tau_n)(\gamma_n u_n) + \tau_n \mathcal{S} u_n]$, and update η_{n+1} by (3.2).
Set $n := n + 1$ and go to **Step 1.**

Theorem 3.5. *Let conditions* **(A1)-(A4)** *holds and* $\{\delta_n\}$ *be a positive sequence with* $\lim_{n\to\infty} \frac{\delta_n}{\zeta_n} = 0$*, where* $\zeta_n \subset (0, 1)$ *satisfies* $\sum_{n=1}^{\infty} \zeta_n = \infty$ *and* $\lim_{n\to\infty} \zeta_n = 0$*. Furthermore, for some* $a > 0$*,* $\gamma_n \in (0, 1)$*,* $\lim_{n\to\infty} \gamma_n = 1$*, and* $\sum_{n=0}^{\infty}(1 - \gamma_n) = \infty$*, let* $\tau_n \in \left(a, \frac{(1-\mu)\gamma_n}{2+\mu+\gamma_n}\right) \subset (a, 1 - \mu)$*, then the sequence* $\{s_n\}$ *produced by Algorithm 3.2 converges in norm to* $v \in Fix(\mathcal{S}) \cap VI(E, \mathcal{F})$*, where* $v = \mathcal{P}_{Fix(\mathcal{S})\cap VI(E,\mathcal{F})}\mathcal{J}(v)$.

Proof. **Claim 1.** The sequence s_n is bounded. Define $t_n = (1 - \tau_n)(\gamma_n u_n) + \tau_n \mathcal{S} u_n$.

$$\begin{aligned} \|t_n - v\| &= \|(1 - \tau_n)(\gamma_n u_n) + \tau_n \mathcal{S} u_n - v\| \\ &\leq \|(1 - \tau_n)\gamma_n(u_n - v) + \tau_n(\mathcal{S} u_n - v)\| + (1 - \tau_n)(1 - \gamma_n)\|v\|. \end{aligned} \tag{3.22}$$

On the other hand,

$$\begin{aligned} \|(1 - \tau_n)\gamma_n(u_n - v) + \tau_n(\mathcal{S} u_n - v)\|^2 &= ((1 - \tau_n)\gamma_n)^2\|u_n - v\|^2 + \tau_n^2\|\mathcal{S} u_n - v\|^2 \\ &+ 2(1 - \tau_n)\gamma_n\tau_n\langle \mathcal{S} u_n - v, u_n - v\rangle \\ &\leq ((1 - \tau_n)\gamma_n + \tau_n)^2 \|u_n - v\|^2 \\ &+ \tau_n (\mu\tau_n - (1 - \mu)(1 - \tau_n)\gamma_n) \|\mathcal{S} u_n - u_n\|^2 \\ &\leq ((1 - \tau_n)\gamma_n + \tau_n)^2 \|u_n - v\|^2. \end{aligned} \tag{3.23}$$

we obtained the above inequality because $\tau_n < \dfrac{(1 - \mu)\gamma_n}{2 + \mu + \gamma_n}$.

Thus it is implied from (3.23) that

$$\begin{aligned}\|(1-\tau_n)\gamma_n(u_n-v)+\tau_n(\mathcal{S}u_n-v)\| &\leq ((1-\tau_n)\gamma_n+\tau_n)\|u_n-v\| \\ &\leq (1-(1-\tau_n)(1-\gamma_n))\|u_n-v\| \\ &\leq (1-(1-\tau_n)(1-\gamma_n))\|w_n-v\| \\ &\leq (1-(1-\tau_n)(1-\gamma_n))\left[\|s_n-v\|+\zeta_n\mathcal{M}_*\right]. \end{aligned} \tag{3.24}$$

From (3.22), we have

$$\begin{aligned}\|t_n-v\| &\leq (1-(1-\tau_n)(1-\gamma_n))\left[\|s_n-v\|+\zeta_n\mathcal{M}_*\right]+(1-\tau_n)(1-\gamma_n)\|v\| \\ &\leq (1-(1-\tau_n)(1-\gamma_n))\|s_n-v\|+\zeta_n\mathcal{M}_*+(1-\tau_n)(1-\gamma_n)\|v\| \\ &= (1-(1-\tau_n)(1-\gamma_n))\|s_n-v\| \\ &\quad +(1-\tau_n)(1-\gamma_n)\left[\frac{\zeta_n\mathcal{M}_*}{(1-\tau_n)(1-\gamma_n)}+\|v\|\right] \\ &\leq \max\left\{\|s_n-v\|,\frac{\zeta_n\mathcal{M}_*}{(1-\tau_n)(1-\gamma_n)}+\|v\|\right\}:=M^*\end{aligned}$$

for some $M^*>0$, hence

$$\begin{aligned}\|s_{n+1}-v\| &= \|\zeta_n\mathcal{J}(s_n)+(1-\zeta_n)t_n-v\| \\ &\leq \zeta_n\|\mathcal{J}(s_n)-\mathcal{J}(v)\|+\zeta_n\|\mathcal{J}(v)-v\|+(1-\zeta_n)\|t_n-v\| \\ &\leq \zeta_n\mathcal{Q}\|s_n-v\|+\zeta_n\|\mathcal{J}(v)-v\|+(1-\zeta_n)M^* \\ &= \zeta_n\mathcal{Q}\|s_n-v\|+(1-\zeta_n)\left[M^*+\frac{\zeta_n}{1-\zeta_n}\|\mathcal{J}(v)-v\|\right] \\ &\leq \max\left\{M^*+\frac{\zeta_n}{1-\zeta_n}\|\mathcal{J}(v)-v\|,\mathcal{Q}\|s_n-v\|\right\} \\ &\leq \cdots \leq \max\left\{M^*,\mathcal{Q}\|s_0-v\|\right\}.\end{aligned}$$

Which ensures the boundedness of $\{s_n\}$, so the sequences $\{w_n\}$, $\{\mathcal{J}(s_n)\}$, $\{t_n\}$, and $\{u_n\}$ are also bounded.

Claim 2.

$$\begin{aligned}&(1-\zeta_n)\left(1-\nu\frac{\eta_n}{\eta_{n+1}}\right)\|t_n-w_n\|+(1-\zeta_n)\left(1-\nu\frac{\eta_n}{\eta_{n+1}}\right)\|u_n-t_n\| \\ &+(1-\zeta_n)\tau_n(1-\mu-\tau_n)\|\mathcal{S}u_n-u_n\|^2 \\ &\leq \|s_n-v\|^2-\|s_{n+1}-v\|^2+\zeta_n\|\mathcal{J}(s_n)-v\|^2+(1-\gamma_n)M^{**}+\zeta_nM^{***}.\end{aligned}$$

$$\begin{aligned}\|t_n-v\|^2 &= \|(1-\tau_n)(\gamma_nu_n)+\tau_n\mathcal{S}u_n-v\|^2 \\ &= \|(u_n-v)+\tau_n(\mathcal{S}u_n-u_n)-(1-\tau_n)(1-\gamma_n)u_n\|^2 \\ &\leq \|(u_n-v)+\tau_n(\mathcal{S}u_n-u_n)\|^2-2(1-\tau_n)(1-\gamma_n)\langle u_n,\eta_n-v\rangle\end{aligned}$$

$$= \|u_n - v\|^2 + \tau_n^2 \|\mathcal{S}u_n - u_n\|^2 + 2\tau_n \langle \mathcal{S}u_n - u_n, u_n - v\rangle -$$
$$2(1-\tau_n)(1-\gamma_n)\langle u_n, \eta_n - v\rangle$$
$$\leq \|u_n - v\|^2 + \tau_n^2 \|\mathcal{S}u_n - u_n\|^2 - \tau_n(1-\mu)\|\mathcal{S}u_n - u_n\|^2$$
$$- 2(1-\tau_n)(1-\gamma_n)\langle u_n, \eta_n - v\rangle$$
$$= \|u_n - v\|^2 - \tau_n(1-\mu-\tau_n)\|\mathcal{S}u_n - u_n\|^2$$
$$- 2(1-\tau_n)(1-\gamma_n)\langle u_n, \eta_n - v\rangle$$
$$\leq \|u_n - v\|^2 - \tau_n(1-\mu-\tau_n)\|\mathcal{S}u_n - u_n\|^2 + (1-\gamma_n)M^{**}$$

for some $M^{**} > 0$. Now,

$$\begin{aligned}
\|s_{n+1} - v\|^2 &= \|\zeta_n(\mathcal{J}(s_n) - v) + (1-\zeta_n)(t_n - v)\|^2 \\
&\leq \zeta_n \|\mathcal{J}(s_n) - v\|^2 + (1-\zeta_n)\|t_n - v\|^2 \\
&\leq \zeta_n \|\mathcal{J}(s_n) - v\|^2 + \|s_n - v\|^2 + \zeta_n M^{***} \\
&- (1-\zeta_n)\left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|t_n - w_n\| - (1-\zeta_n)\left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|u_n - t_n\| \\
&- (1-\zeta_n)\tau_n(1-\mu-\tau_n)\|\mathcal{S}u_n - u_n\|^2 + (1-\gamma_n)M^{**}.
\end{aligned}$$

Hence, by simple deformation, we obtain the desired result.

Claim 3.

$$\begin{aligned}
\|s_{n+1} - v\|^2 &= (1 - (1-\mathcal{Q})\zeta_n)\|s_n - v\|^2 \\
&\quad + \zeta_n(1-\mathcal{Q})\left[\frac{(1-\zeta_n)\zeta_n \mathcal{M}_* + 2\langle \mathcal{J}(v) - v, s_{n+1} - v\rangle}{1-\mathcal{Q}}\right].
\end{aligned}$$

By using the identical reasons as in Claim 3 of Theorem 3.3, the required result can be produced.

Claim 4. Sequence $\{\|s_n - v\|^2\}$ converges to zero. We do not include the proof here because it is comparable to Claim 4 of Theorem 3.3. □

The following Corollary will be obtained if we put $\mathcal{S} = I$ in Algorithm 3.2.

Corollary 3.6. *Consider $\mathcal{F}, \mathcal{J}$ as in Corollary 3.4 and let ζ_n, γ_n, τ_n be same as in Theorem 3.5. Then the sequence $\{s_n\}$ with $s_0, s_1 \in \mathcal{D}$ generated by* (3.25)

$$\begin{cases}
w_n = s_n + \mathcal{K}_n(s_n - s_{n-1}), \\
t_n = \mathcal{P}_E(w_n - \eta_n \mathcal{F} w_n), \\
u_n = \mathcal{P}_{T_n}(w_n - \eta_n \mathcal{F} t_n), \text{where the half-space } T_n \text{ is defined by} \\
T_n := \{s \in \mathcal{D} | \langle w_n - \eta_n \mathcal{F} w_n - t_n, s - t_n\rangle \leq 0\}, \\
s_{n+1} = \zeta_n \mathcal{J}(s_n) + (1-\zeta_n)(\gamma_n u_n + \tau_n(1-\gamma_n)u_n),
\end{cases} \tag{3.25}$$

converges to $v \in VI(E, \mathcal{F})$ *in norm, where* $v = \mathcal{P}_{VI(E,\mathcal{F})}(\mathcal{J}(v))$*. where* $\mathcal{K}_n$ *and* η_n *are defined by* (3.1) *and* (3.2)*, respectively.*

4 Numerical example

In this section, we provide a numerical example to illustrate the behavior of the proposed algorithms and compare them with some existing strongly convergent algorithms. The parameters are set as follows: $\zeta_n = \frac{1}{n+1}$, $\tau_n = \frac{n}{2n+1}$, $\gamma_n = \frac{n}{30n+1}$, $\eta_1 = 1$, $\nu = 0.5$, $\mathcal{J}(s) = 0.5s$, $\mathcal{K} = 0.3$, $\delta_n = \frac{100}{(n+1)^2}$. The solution s^* is known, so we use $D_n = \|s_n - s^*\|$ to measure the n-th iteration error and convergence of D_n to 0 indicates that $\{s_n\}$ converges to the problem's solution.

Example 4.1. *We take the nonlinear operator* $\mathcal{F} : R^2 \to R^2$ *defined by* $\mathcal{F}(s,t) = (s+t+\sin s, -s+t+\sin s)$*, feasible set* $E = [-1,1] \times [-1,1]$*. Clearly* $\mathcal{F}$ *is monotone and Lipschitz continuous with constant* $\mathcal{L} = 3$ *and let the matrix* $F = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$*. We consider the mapping* $\mathcal{S} : R^2 \to R^2$ *by* $\mathcal{S}u = \|F\|^{-1}Fu$*, where* $u = (s,t)^T$*. It is obvious to see that* $\mathcal{S}$ *is* 0*-demicontractive and thus* $\tau = 0$*. The solution of the problem is* $s^* = (0,0)^T$*. The initial values* $s_0 = s_1$ *are randomly generated by* $k * rand(2,1)$ *in MATLAB. The numerical results of all the algorithms with different initial values are described in Figures (Figure 1, Figure 2, Figure 3, Figure 4).*

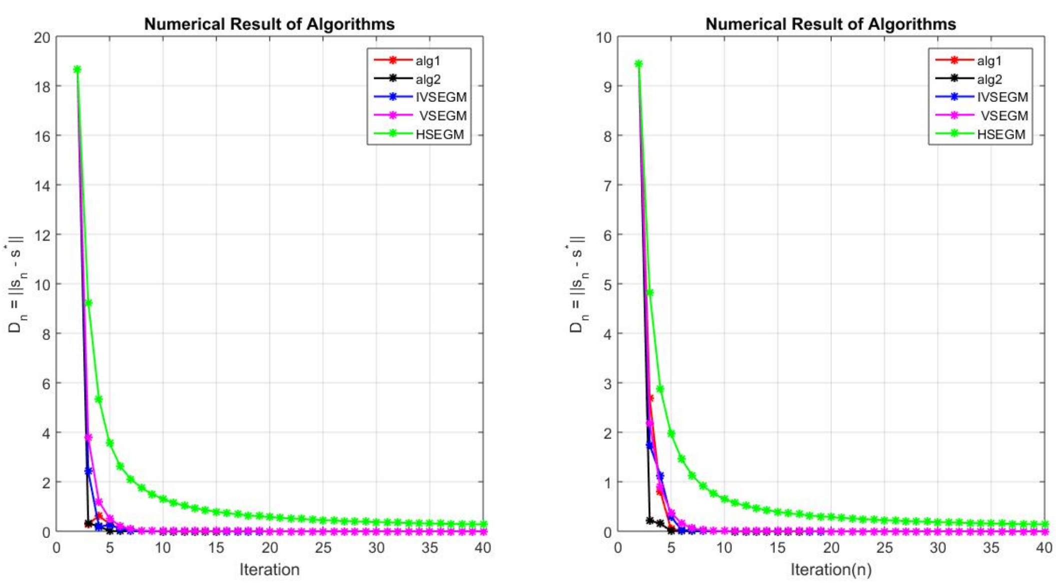

Figure 1: The convergence graphs of $\{D_n = \|s_n - s^*\|\}$ vs iteration ($n = 40$).

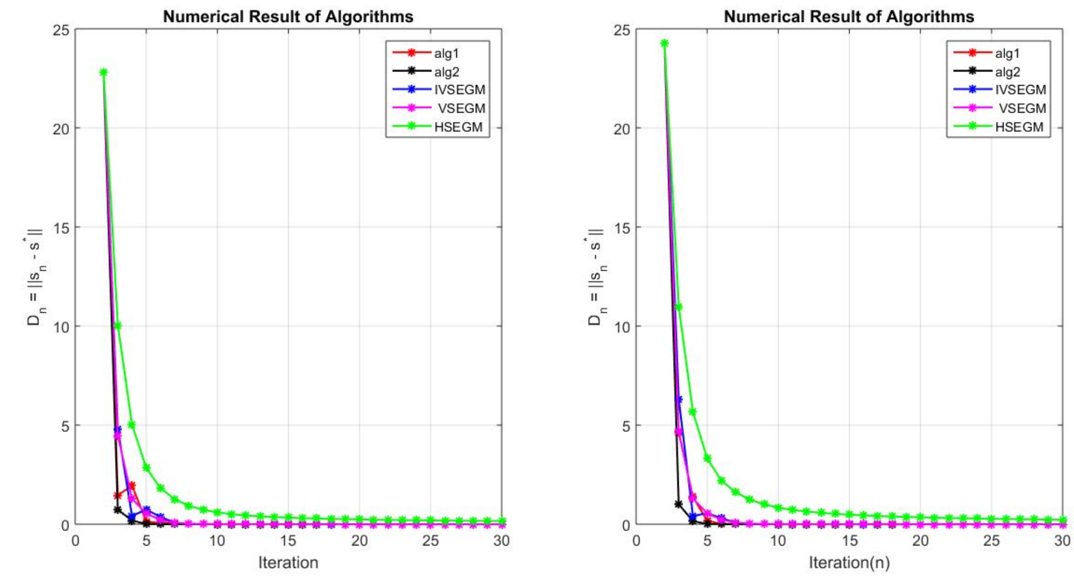

Figure 2: The convergence graphs of $\{D_n = \|s_n - s^*\|\}$ vs iteration $(n = 30)$.

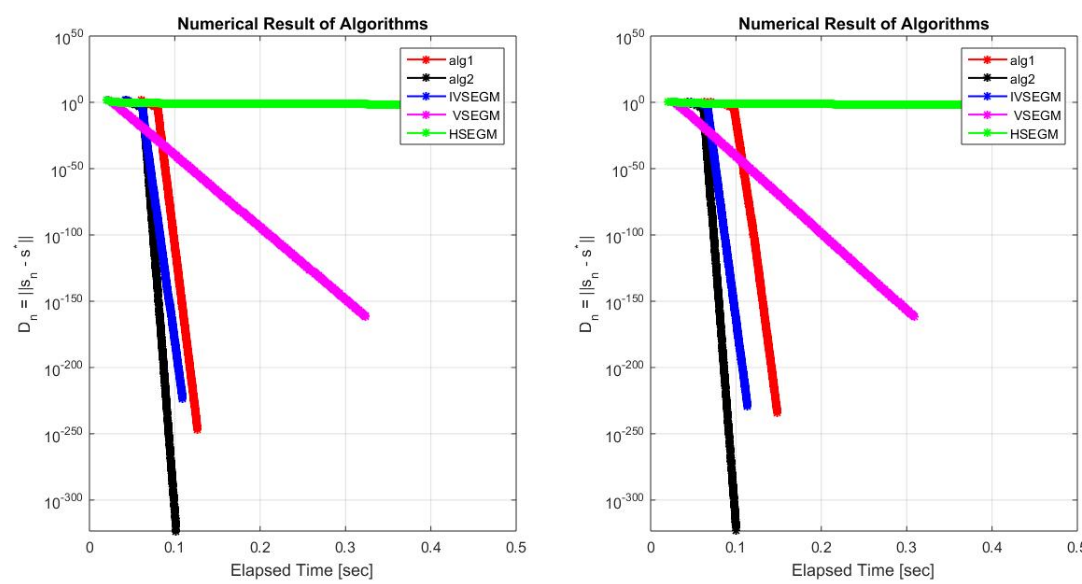

Figure 3: The Elapsed time graph of the sequence $\{D_n = \|s_n - s^*\|\}$ with initial values $s_0 = s_1 = 30rand(2,1)$ and $n = 300$

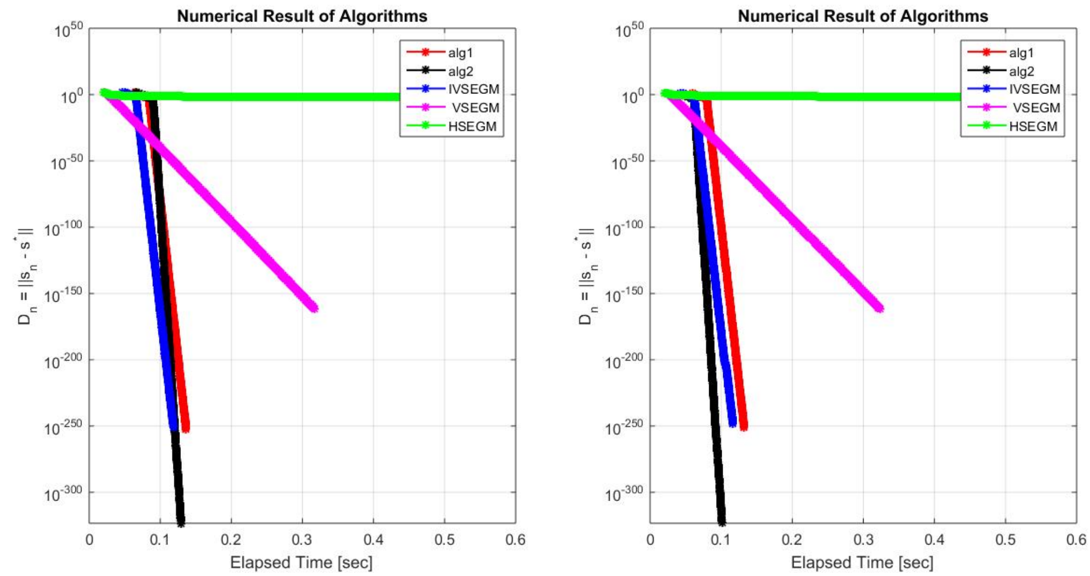

Figure 4: The Elapsed time graph of the sequence $\{D_n = \|s_n - s^*\|\}$ with initial values $s_0 = s_1 = 40rand(2,1)$ and $n = 250$.

Example 4.2. *Consider the linear operator* $\mathcal{F} : R^m \to R^m$ *(*$m = 50, 100, 150, 200$*) in the form* $\mathcal{F}(s) = Ms + q$*, where* $q \in R^m$ *and* $M = NN^T + Q + D$*,* N *is a* $m \times m$ *matrix,* Q *is a* $m \times m$ *skew-symmetric matrix, and* D *is a* $m \times m$ *diagonal matrix with its diagonal entries being nonnegative (hence* M *is positive symmetric definite). The feasible set* E *is given by* $E = \{s \in R^m : -2 \leq s_i \leq 5, i = 1, \ldots, m\}$*. It is clear that* $\mathcal{F}$ *is monotone and Lipschitz continuous with constant* $L = \|M\|$*. In this experiment, all entries of* N, D *are generated randomly in* $[0, 2]$*,* Q *is generated randomly in* $[-2, 2]$ *and* $q = 0$*. Let* $\mathcal{S} : \mathcal{D} \to \mathcal{D}$ *be given by* $\mathcal{S}s = 0.5s$*. It is easy to see that the solution of the problem in this case is* $s^* = \{0\}$*. The initial values* $s_0 = s_1$ *are randomly generated by* $k*rand(2,1)$ *in MATLAB. Figure 5 shows the numerical behavior of all the algorithms in different dimensions (*$m = 50$*,* $m = 100$*,* $m = 150$*,* $m = 200$*).*

Example 4.3. *Finally, we consider our problem in the infinite-dimensional Hilbert space* $\mathcal{D} = L^2([0,1])$ *with inner product* $\langle s, y\rangle = \int_0^1 s(t)y(t)dt$ *and norm* $\|s\| = \left(\int_0^1 |s(t)|^2 dt\right)^{\frac{1}{2}}$*,* $\forall s, y \in \mathcal{D}$*. Let the feasible set be the unit ball* $E = \{s \in \mathcal{D} : \|s\| \leq 1\}$*. Define an operator* $\mathcal{F} : E \to \mathcal{D}$ *by*

$$(\mathcal{F}s)(t) = \int_0^1 (s(t) - G(t,u)g(s(u)))du + h(t)), \quad t \in [0,1], \quad s \in E,$$

where,

$$G(t,u) = \frac{2tue^{t+u}}{e\sqrt{e^2-1}}, \quad g(s) = \cos(s), \quad h(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

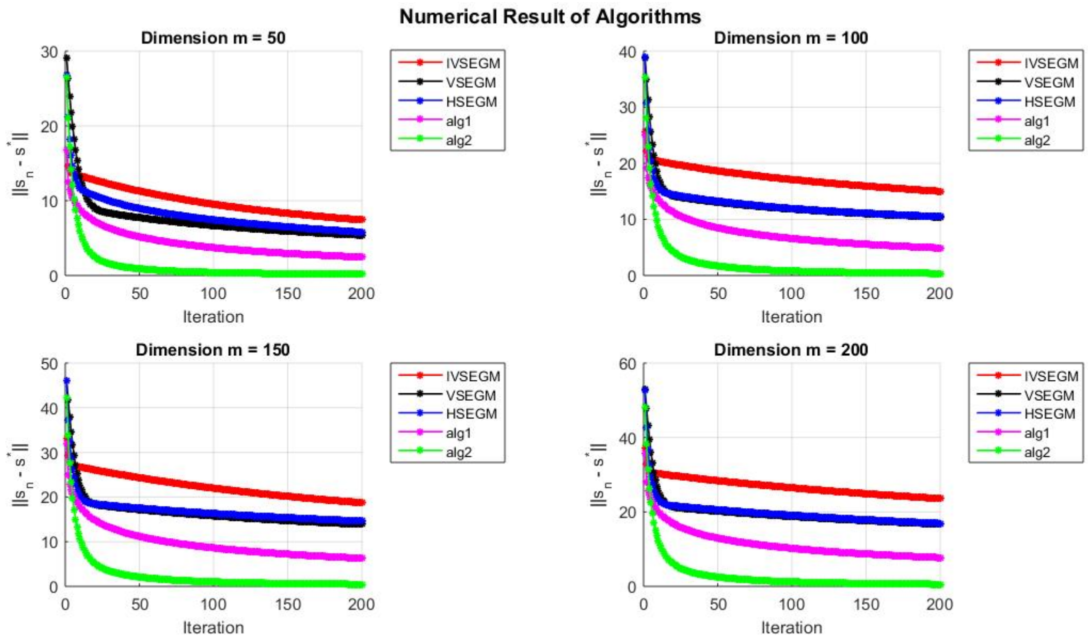

Figure 5: The convergence graphs of $\{D_n = \|s_n - s^*\|\}$ vs iteration(n = 200).

It is known that $\mathcal{F}$ is monotone and L-Lipschitz continuous with $L = 2$ ([13]). The projection on E is inherently explicit, that is,

$$P_E(s) = \begin{cases} \dfrac{s}{\|s\|}, & \text{if } \|s\| > 1; \\ s, & \text{if } \|s\| \leq 1. \end{cases}$$

The mapping $\mathcal{S} : L^2([0,1]) \to L^2([0,1])$ is of the form

$$(\mathcal{S}s)(t) = \int_0^1 ts(u)\,du, \quad t \in [0,1].$$

A straightforward computation implies that $\mathcal{S}$ is 0-demicontractive. The solution of the problem is $s^*(t) = 0$. The maximum number of iterations 50 is used as a common stopping criterion for all algorithms. Figure 6 shows the behaviors of $D_n = \|s_n(t) - s^*(t)\|$ generated by all the algorithms with four starting points.

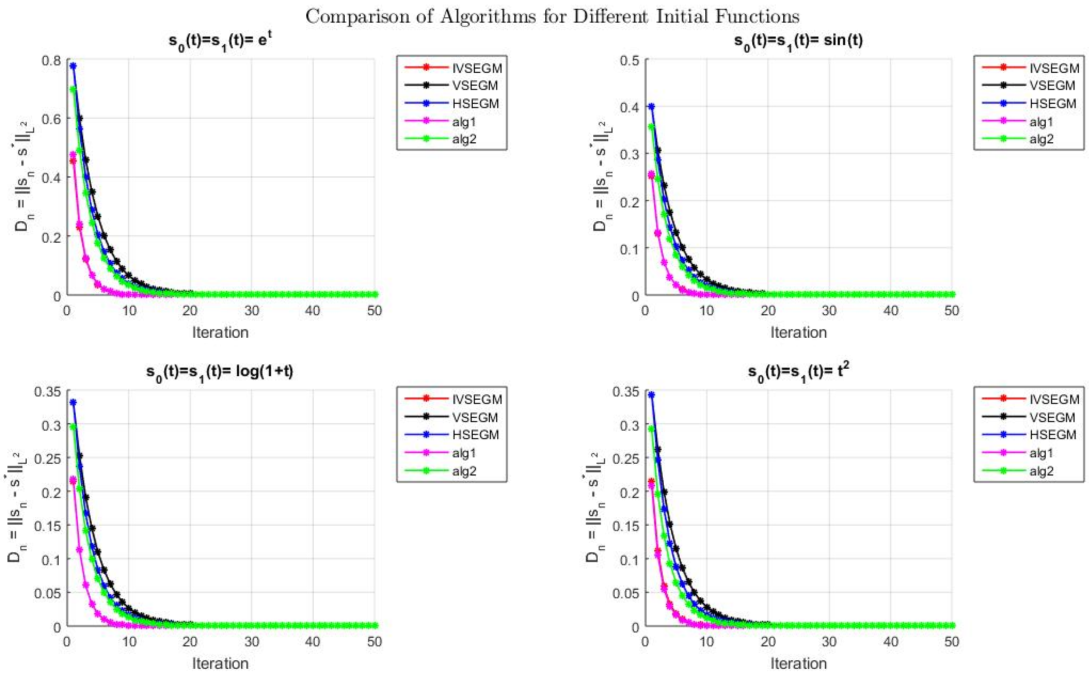

Figure 6: The convergence graphs of $\{D_n = \|s_n - s^*\|\}$ vs iteration $(n = 50)$.

5 Conclusion

In this study, we investigated two self-adaptive iterative schemes for seeking a common solution to the variational inequality problem involving a monotone and Lipschitz continuous mapping and the fixed point problem with a demicontractive mapping. We proposed two new inertial extragradient methods with a new step size to compute the approximate solutions of problems in a real Hilbert space. The strong convergence of the suggested methods is established under standard and suitable conditions. Finally, some computational tests are given to explain our convergent results. The algorithms obtained in this paper improved and summarized some of the recent results in the literature.

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