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Hausdorff operators associated with the linear canonical Sturm-Liouville transform

Fethi Soltani[1,2,✉]
Maher Aloui[2]

[1] *Faculté des Sciences de Tunis, Laboratoire d'Analyse Mathématique et Applications LR11ES11, Université de Tunis El Manar, Tunis 2092, Tunisia.*
fethi.soltani@fst.utm.tn✉

[2] *Ecole Nationale d'Ingénieurs de Carthage, Université de Carthage, Tunis 2035, Tunisia.*
maher.aloui@fst.utm.tn

ABSTRACT

In the present paper, we introduce the canonical Sturm-Liouville operator $L^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b}\right)$, where A is a nonnegative function satisfying certain conditions. We prove the boundedness of the canonical Sturm-Liouville Hausdorff operators on the space $L^p(\mathbb{R}_+, A(x)\,\mathrm{d}x)$, $p \in [1, \infty)$. We investigate canonical Sturm-Liouville wavelet transform, and obtain some useful results. The relation between the canonical Sturm-Liouville wavelet transform and canonical Sturm-Liouville Hausdorff operator is also established. The properties of the adjoint canonical Sturm-Liouville Hausdorff operators are further discussed. The harmonic analysis associated with the operator L^M plays an important role in establishing the results of this paper.

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RESUMEN

En el presente aríCulo, introducimos el operador de Sturm-Liouville canónico $L^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b}\right)$, donde A es una función no-negativa que satisface ciertas condiciones. Demostramos el acotamiento de los operadores Hausdorff de Sturm-Liouville canónicos en el espacio $L^p(\mathbb{R}_+, A(x)\,\mathrm{d}x)$, $p \in [1, \infty)$. Investigamos la transformada de ondeletas de Sturm-Liouville canónica y obtenemos algunos resultados útiles. También se establece la relación entre la transformada de ondeletas de Sturm-Liouville canónica y el operador Hausdorff de Sturm-Liouville canónico. Se discuten las propiedades de los adjuntos a operadores Hausdorff de Sturm-Liouville canónicos. El análisis armónico asociado al operador L^M juega un rol importante para establecer los resultados de este artículo.

Keywords and Phrases: Canonical Sturm-Liouville transform, canonical Sturm-Liouville convolution, canonical Sturm-Liouville Hausdorff operators, canonical Sturm-Liouville wavelet transform.

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1 Introduction

The study of Hausdorff operators, which originated from some classical summation methods, has a long history in real and complex analysis. In the one-dimensional setting, Hausdorff operators on the real line were introduced in [10] and studied on the Hardy space in [18]. The natural generalization in several dimensions was introduced and studied in [3,5,16]. Particularly, Hausdorff operators are interesting operators in harmonic analysis [19]. It contains some important operators, such as Hardy operator, adjoint Hardy operator [6,15], and the Cesàro operator [14] in one dimension. The Hardy-Littlewood-Pólya operator and the Riemann-Liouville fractional integral operator can also be derived from the Hausdorff operator [1, 25]. The modern study of general Hausdorff operators on $L^1(\mathbb{R})$ and the real Hardy space $H^1(\mathbb{R})$ over the real line was pioneered by Liflyand and Móricz in [18]. Many research papers have addressed the boundedness of the Hausdorff operator on Hardy spaces. For instance, Liflyand and his collaborators in [16, 17] proved, by more effective ways, that the Hausdorff operator has the same behavior on the Hardy space $H^1(\mathbb{R})$ as that in the Lebesgue space $L^1(\mathbb{R})$. Recently, Daher and Saadi in [7, 8] investigated the Dunkl Hausdorff operator on the Lebesgue space $L^1_\alpha(\mathbb{R})$ and on the Hardy space $H^1_\alpha(\mathbb{R})$. Subsequently, Mondal and Poria [22] studied Hausdorff operators associated with the Opdam-Cherednik operator. Furthermore, Tyr [35] studied the boundedness of q-Hausdorff operators on q-Hardy spaces. Another fundamental tool in harmonic analysis is the canonical Sturm-Liouville Hausdorff operators, which is the main object of study in this paper.

Here, we denote by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitary matrix in $SL(2, \mathbb{R})$ such that $b > 0$. We define the canonical Sturm-Liouville operator L^M on $\mathbb{R}^*_+$ by

$$L^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b}\right),$$

where A is a nonnegative function satisfying certain conditions.

Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the operator L^M is reduced to the Sturm-Liouville operator L:

$$L := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x}.$$

The classical Sturm-Liouville operator L plays an important role in analysis [2, 39]. In particular, the two references [4,33] investigate standard constructions of harmonic analysis, such as translation operators, convolution product, and Fourier transform, in connection with the operator L.

Using the Sturm-Liouville harmonic analysis [4,33], for all $\lambda \in \mathbb{C}$, the system

$$\begin{cases} L^M u = -\dfrac{\lambda^2}{b^2} u, \\ u(0) = e^{\frac{id}{2b}\lambda^2}, \qquad u'(0) = 0, \end{cases}$$

admits a unique solution, denoted by φ_λ^M and given by

$$\varphi_\lambda^M(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)} \varphi_{\frac{\lambda}{b}}(x), \quad x \in \mathbb{R}_+,$$

where $\varphi_\lambda(x)$ is the Sturm-Liouville kernel [29,30].

In this paper, we introduce the canonical Sturm-Liouville transform $\mathscr{F}^M$:

$$\mathscr{F}^M(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda^M(x) f(x) A(x)\,\mathrm{d}x, \quad \lambda \in \mathbb{R}_+.$$

The canonical Sturm-Liouville transform $\mathscr{F}^M$ can be regarded as a generalization of the Sturm-Liouville transform $\mathscr{F}$ (see [20,27–32]):

$$\mathscr{F}(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda(x) f(x) A(x)\,\mathrm{d}x, \quad \lambda \in \mathbb{R}_+.$$

Let $\phi \in L^1(\mathbb{R}_+)$. We define the Hausdorff operator H_ϕ associated with the canonical Sturm-Liouville operator L^M for $f \in L^1(\mathbb{R}_+, A(x)\,\mathrm{d}x)$ by

$$H_\phi f(x) := \int_{\mathbb{R}_+} f_t(x)\phi(t)\,\mathrm{d}t,$$

where f_t is the dilation of f given by

$$f_t(x) := \frac{A\left(\frac{x}{t}\right)}{tA(x)} f\left(\frac{x}{t}\right), \quad x \in \mathbb{R}_+.$$

The main purpose of this paper is to extend some results of the classical Hausdorff operator given in [38] to the framework of canonical Sturm-Liouville theory, and to investigate the canonical Sturm-Liouville wavelet transform. We prove the boundedness of canonical Sturm-Liouville Hausdorff operator in space $L^p(\mathbb{R}_+, A(x)\,\mathrm{d}x)$, $p \in [1,\infty)$. The relation between the canonical Sturm-Liouville wavelet transform and the canonical Sturm-Liouville Hausdorff operator is also established. Next, we introduce the adjoint operator H_ϕ^* on $L^2(\mathbb{R}_+, A(x)\,\mathrm{d}x)$ by

$$H_\phi^* f(x) := \int_{\mathbb{R}_+} f(tx)\phi(t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+.$$

We present the properties of the adjoint operator H_ϕ^*, including its boundedness on $L^p(\mathbb{R}_+, A(x)\,\mathrm{d}x)$,

$p \in [1, \infty)$. We also establish a relation between the canonical Sturm-Liouville wavelet transform and the adjoint operator H_ϕ^*.

Note that if $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$, the operator L^M is reduced to the canonical Bessel operator L_α^M:

$$L_\alpha^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{2\alpha+1}{x} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + 2i(\alpha+1)\frac{a}{b}\right).$$

In this case $\varphi_\lambda^M(x) = \varphi_{\lambda,\alpha}^M(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)} j_\alpha(\frac{\lambda x}{b})$, where j_α is the normalized Bessel function of the first kind and order α. The canonical transform $\mathscr{F}^M$ is the canonical Fourier-Bessel transform $\mathscr{F}_\alpha^M$:

$$\mathscr{F}_\alpha^M(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_{\lambda,\alpha}^M(x) f(x) x^{2\alpha+1}\mathrm{d}x, \quad \lambda \in \mathbb{R}_+.$$

Recently, the canonical Fourier-Bessel transform $\mathscr{F}_\alpha^M$ is the goal of many applications in the harmonic analysis (see [9, 11, 12, 21, 26]).

This paper is organized as follows. In Section 2, we recall some results about the Sturm-Liouville transform $\mathscr{F}$, the Sturm-Liouville translation τ_y and the Sturm-Liouville convolution $*$. In Section 3, we introduce the canonical Sturm-Liouville operator L^M, and we investigate the properties of the canonical Sturm-Liouville transform $\mathscr{F}^M$, the canonical Sturm-Liouville translation τ_y^M and the canonical Sturm-Liouville convolution $*^M$ associated with this operator. In Section 4, we introduce the canonical Sturm-Liouville Hausdorff operators H_ϕ and we establish their properties. In the last section, we investigate the canonical Sturm-Liouville wavelet transform and derive its relation with the operators H_ϕ and H_ϕ^*.

2 Sturm-Liouville harmonic analysis

In this section we recall some results about the harmonic analysis associated with the Sturm-Liouville operator (Sturm-Liouville transform, Sturm-Liouville translation and Sturm-Liouville convolution).

We consider the second-order differential operator L defined on $\mathbb{R}_+^*$ by

$$L := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x},$$

where

$$A(x) = x^{2\alpha+1}B(x), \quad \alpha > -1/2,$$

for B a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0) = 1$. Moreover we assume that A satisfies the following conditions:

(i) A is increasing and $\lim_{x\to\infty} A(x) = \infty$.

(ii) $\dfrac{A'}{A}$ is decreasing and $\lim_{x\to\infty} \dfrac{A'(x)}{A(x)} = 0$.

(iii) There exists a constant $\delta > 0$ such that

$$\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + e^{-\delta x} D(x), \tag{2.1}$$

where D is an infinitely differentiable function on $\mathbb{R}_+^*$, bounded and with bounded derivatives on all intervals $[x_0, \infty)$, for $x_0 > 0$.

This operator was studied in [4, 33], and the following results have been established:

(I) For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} Lu = -\lambda^2 u, \\ u(0) = 1, \quad u'(0) = 0, \end{cases}$$

admits a unique solution, denoted by φ_λ, with the following properties:

- for $x \in \mathbb{R}_+$, the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic on $\mathbb{C}$.
- For $\lambda \in \mathbb{C}$, the function $x \mapsto \varphi_\lambda(x)$ is even and infinitely differentiable on $\mathbb{R}$.

(II) For nonzero $\lambda \in \mathbb{C}$, the equation

$$Lu = -\lambda^2 u,$$

has a solution Φ_λ satisfying

$$\Phi_\lambda(x) = \frac{e^{i\lambda x}}{\sqrt{A(x)}} V(x, \lambda),$$

with

$$\lim_{x\to\infty} V(x, \lambda) = 1.$$

Consequently there exists a function (spectral function) $\lambda \mapsto c(\lambda)$, such that

$$\varphi_\lambda(x) = c(\lambda)\Phi_\lambda(x) + c(-\lambda)\Phi_{-\lambda}(x), \quad x \in \mathbb{R}_+,$$

for nonzero $\lambda \in \mathbb{C}$.

Moreover there exist positive constants k_1, k_2, k, such that

$$k_1 |\lambda|^{2\alpha+1} \le |c(\lambda)|^{-2} \le k_2 |\lambda|^{2\alpha+1},$$

for all λ such that $\operatorname{Im}\lambda \le 0$ and $|\lambda| \ge k$.

(III) The Sturm–Liouville kernel $\varphi_\lambda(x)$ possesses the following integral representation of Mehler-type

$$\varphi_\lambda(x) = \int_0^x K(x,y)\cos(\lambda y)\,\mathrm{d}y, \quad x > 0, \tag{2.2}$$

where $K(x,.)$ is an even positive continuous function on $(-x,x)$ and supported in $[-x,x]$. Using the Mehler integral representation formula (2.2), we obtain

$$-1 \le \varphi_\lambda(x) \le 1, \quad \lambda, x \in \mathbb{R}_+. \tag{2.3}$$

We denote by

- μ the measure defined on $\mathbb{R}_+$ by

$$\mathrm{d}\mu(x) := A(x)\,\mathrm{d}x,$$

and by $L^p(\mu)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbb{R}_+$, such that

$$\|f\|_{L^p(\mu)} := \left[\int_{\mathbb{R}_+} |f(x)|^p\,\mathrm{d}\mu(x)\right]^{1/p} < \infty, \quad p \in [1,\infty),$$
$$\|f\|_{L^\infty(\mu)} := \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} |f(x)| < \infty.$$

- ν the measure defined on $\mathbb{R}_+$ by

$$\mathrm{d}\nu(\lambda) := \frac{\mathrm{d}\lambda}{2\pi|c(\lambda)|^2},$$

and by $L^p(\nu)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbb{R}_+$, such that $\|f\|_{L^p(\nu)} < \infty$.

The Sturm-Liouville transform is the Fourier transform associated with the operator L and is defined for $f \in L^1(\mu)$ by

$$\mathscr{F}(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda(x) f(x)\mathrm{d}\mu(x), \quad \lambda \in \mathbb{R}_+. \tag{2.4}$$

Some of the properties of the Sturm-Liouville transform $\mathscr{F}$ are collected bellow.

Theorem 2.1 ([2, 4, 33, 39]). *(i)* ***Plancherel theorem.*** *The Sturm–Liouville transform $\mathscr{F}$ extends uniquely to an isometric isomorphism of $L^2(\mu)$ onto $L^2(\nu)$. In particular,*

$$\|f\|_{L^2(\mu)} = \|\mathscr{F}(f)\|_{L^2(\nu)}.$$

(ii) ***Inversion theorem.*** *Let $f \in L^1(\mu)$, such that $\mathscr{F}(f) \in L^1(\nu)$. Then*

$$f(x) = \int_{\mathbb{R}_+} \varphi_\lambda(x)\mathscr{F}(f)(\lambda)\,\mathrm{d}\nu(\lambda), \quad a.e.\ x \in \mathbb{R}_+.$$

The Sturm-Liouville kernel φ_λ satisfies the product formula [4, 33]

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_{\mathbb{R}_+} \varphi_\lambda(z)w(x,y,z)\,\mathrm{d}\mu(z) \quad \text{for} \quad x, y \in \mathbb{R}_+; \tag{2.5}$$

where $w(x,y,.)$ is a measurable positive function on $\mathbb{R}_+$, with support in $[|x-y|, x+y]$, satisfying

$$\int_{\mathbb{R}_+} w(x,y,z)\,\mathrm{d}\mu(z) = 1,$$

$$w(x,y,z) = w(y,x,z) \quad \text{for} \quad z \in \mathbb{R}_+, \tag{2.6}$$

$$w(x,y,z) = w(x,z,y) \quad \text{for} \quad z > 0. \tag{2.7}$$

We now define the generalized translation operator induced by (2.5). For $f \in L^1(\mu)$, the linear operator

$$\tau_y f(x) := \int_{\mathbb{R}_+} f(z)w(x,y,z)\,\mathrm{d}\mu(z), \quad x, y \in \mathbb{R}_+, \tag{2.8}$$

will be called Sturm-Liouville translation [4, 33].

As a first remark, we note that the relation (2.6) means that

$$\tau_y f(x) = \tau_x f(y), \quad x, y \in \mathbb{R}_+.$$

Theorem 2.2 ([23, 29, 30]). *(i) For all $y \geq 0$ and $f \in L^p(\mu)$, $p \in [1, \infty]$, we have*

$$\|\tau_y f\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)}.$$

(ii) For $f \in L^2(\mu)$ and $y \in \mathbb{R}_+$, we have

$$\mathscr{F}(\tau_y f)(\lambda) = \varphi_\lambda(y)\mathscr{F}(f)(\lambda), \quad \lambda \in \mathbb{R}_+.$$

Let $f, g \in L^2(\mu)$. The Sturm-Liouville convolution $f * g$ of f and g is defined by

$$f * g(x) := \int_{\mathbb{R}_+} \tau_x f(y)g(y)\,\mathrm{d}\mu(y), \quad x \in \mathbb{R}_+. \tag{2.9}$$

The convolution $*$ is commutative, associative and satisfies the Young inequality (see [23]). Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then for $f \in L^p(\mu)$ and $g \in L^q(\mu)$ we have

$$\|f * g\|_{L^r(\mu)} \leq \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Theorem 2.3 ([23, 34]). *(i) For $f, g \in L^2(\mu)$, the function $f * g$ belongs to $L^\infty(\mu)$, and*

$$f * g(x) = \int_{\mathbb{R}_+} \varphi_\lambda(x)\, \mathscr{F}(f)(\lambda) \mathscr{F}(g)(\lambda)\, \mathrm{d}\nu(\lambda), \quad x \in \mathbb{R}_+.$$

(ii) Let $f, g \in L^2(\mu)$. Then

$$\int_{\mathbb{R}_+} |f * g(x)|^2 \, \mathrm{d}\mu(x) = \int_{\mathbb{R}_+} |\mathscr{F}^M(f)(\lambda)|^2 \, |\mathscr{F}^M(g)(\lambda)|^2 \, \mathrm{d}\nu(\lambda),$$

where both sides are finite or infinite.

Example 2.4 ([13, 24]). *Note that if $A(x) = x^{2\alpha+1}$, with $\alpha > -1/2$, the operator L is reduced to the Bessel operator L_α:*

$$L_\alpha := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\alpha + 1}{x} \frac{\mathrm{d}}{\mathrm{d}x}.$$

In this case $\varphi_\lambda(x) = j_\alpha(\lambda x)$, where j_α is the normalized Bessel function of the first kind and order α. We denote by μ_α the measure defined by $\mathrm{d}\mu_\alpha(x) := x^{2\alpha+1}\mathrm{d}x$.

The Fourier-Bessel transform $\mathscr{F}_\alpha$ is defined for $f \in L^1(\mu_\alpha)$ by

$$\mathscr{F}_\alpha(f)(\lambda) := \int_{\mathbb{R}_+} j_\alpha(\lambda x) f(x) \, \mathrm{d}\mu_\alpha(x), \quad \lambda \in \mathbb{R}_+.$$

The Fourier-Bessel translation operators are defined for $f \in L^1(\mu_\alpha)$ by

$$\tau_y^\alpha f(x) := \int_{\mathbb{R}_+} f(z) w_\alpha(x, y, z) \, \mathrm{d}\mu_\alpha(z), \quad x, y \in \mathbb{R}_+,$$

being $w_\alpha(x, y, .)$ the kernel given by

$$w_\alpha(x, y, z) = a_\alpha \frac{[(x+y)^2 - z^2]^{\alpha - \frac{1}{2}} [z^2 - (x-y)^2]^{\alpha - \frac{1}{2}}}{2^{2\alpha - 1} (xyz)^{2\alpha}} \, \chi_{(|x-y|, x+y)}(z), \tag{2.10}$$

where $a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})}$ and $\chi_{(|x-y|, x+y)}$ is the characteristic function of the interval $(|x-y|, x+y)$. Let $f, g \in L^2(\mu_\alpha)$. The Fourier-Bessel convolution $f *_\alpha g$ of f and g is defined by

$$f *_\alpha g(x) := \int_{\mathbb{R}_+} \tau_x^\alpha f(y) g(y) \, \mathrm{d}\mu_\alpha(y), \quad x \in \mathbb{R}_+.$$

3 Canonical Sturm-Liouville operator

Throughout this paper, we denote by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitary matrix in $SL(2,\mathbb{R})$ such that $b > 0$. We define the canonical Sturm-Liouville operator L^M on $\mathbb{R}_+^*$ by

$$L^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b}\right),$$

where A is the nonnegative function given in Section 2.

Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the operator L^M is reduced to the Sturm-Liouville operator L:

$$L := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x}.$$

For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} L^M u = -\dfrac{\lambda^2}{b^2}u, \\ u(0) = e^{\frac{id}{2b}\lambda^2}, \quad u'(0) = 0, \end{cases}$$

admits a unique solution, denoted by φ_λ^M and given by

$$\varphi_\lambda^M(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)}\varphi_{\frac{\lambda}{b}}(x), \quad x \in \mathbb{R}_+.$$

For $f \in L^1(\mu)$, we define the canonical Sturm-Liouville transform $\mathscr{F}^M(f)$ by

$$\mathscr{F}^M(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda^M(x) f(x)\,\mathrm{d}\mu(x), \quad \lambda \in \mathbb{R}_+.$$

This transform can be written as

$$\mathscr{F}^M(f)(\lambda) = e^{\frac{id}{2b}\lambda^2}\mathscr{F}\left(e^{\frac{ia}{2b}x^2}f\right)\left(\frac{\lambda}{b}\right), \quad f \in L^1(\mu), \tag{3.1}$$

where $\mathscr{F}$ is the Sturm-Liouville transform given by (2.4).

We denote by ν_b, $b > 0$, the measure defined on $\mathbb{R}_+$ by

$$\mathrm{d}\nu_b(\lambda) := \frac{\mathrm{d}\lambda}{2\pi b|c(\frac{\lambda}{b})|^2},$$

and by $L^p(\nu_b)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbb{R}_+$, such that $\|f\|_{L^p(\nu_b)} < \infty$.

Theorem 3.1. *(i) Let $f \in L^1(\mu)$, such that $\mathscr{F}^M(f) \in L^1(\nu_b)$. Then*

$$f(x) = \int_{\mathbb{R}_+} \varphi_\lambda^N(x)\mathscr{F}^M(f)(\lambda)\,\mathrm{d}\nu_b(\lambda), \quad a.e. \quad x \in \mathbb{R}_+,$$

where N is the matrix given by $N = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$.

(ii) For $f \in L^2(\mu)$ we have

$$\|\mathscr{F}^M(f)\|_{L^2(\nu_b)} = \|f\|_{L^2(\mu)}.$$

Proof. (i) follows from Theorem 2.1 (ii) and relation (3.1). (ii) follows from Theorem 2.1 (i) and relation (3.1). □

For $f \in L^1(\mu)$, we define the canonical Sturm-Liouville translation operators by

$$\tau_y^N f(x) := e^{-\frac{ia}{2b}(x^2+y^2)} \int_{\mathbb{R}_+} f(z)e^{\frac{ia}{2b}z^2} w(x,y,z)\,\mathrm{d}\mu(z), \quad x,y \in \mathbb{R}_+. \tag{3.2}$$

It is easy to prove the following results.

Theorem 3.2. *The operators τ_y^N, $y \in \mathbb{R}_+$, satisfy:*

(i) $\tau_y^N f(x) = \tau_x^N f(y)$, $x, y \in \mathbb{R}_+$.

(ii) $\tau_y^N f(x) = e^{-\frac{ia}{2b}(x^2+y^2)}\tau_y\left(f(z)e^{\frac{ia}{2b}z^2}\right)(x)$, where τ_y is the Sturm-Liouville translation given by (2.8).

(iii) $\tau_y^M \varphi_\lambda^M(x) = e^{-\frac{id}{2b}\lambda^2}\varphi_\lambda^M(x)\varphi_\lambda^M(y)$.

Theorem 3.3. *(i) For all $y \in \mathbb{R}_+$ and $f \in L^p(\mu)$, $p \in [1,\infty]$, we have*

$$\|\tau_y^N f\|_{L^p(\mu)} \le \|f\|_{L^p(\mu)}.$$

(ii) For $f \in L^2(\mu)$ and $y \in \mathbb{R}_+$, we have

$$\mathscr{F}^M(\tau_y^N f)(\lambda) = e^{\frac{id}{2b}\lambda^2}\varphi_\lambda^N(y)\mathscr{F}^M(f)(\lambda), \quad \lambda \in \mathbb{R}_+,$$

where $N = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$.

Proof. (i) follows from Theorem 2.2 (i) and Theorem 3.2 (ii).

(ii) Let $f \in L^1(\mu) \cap L^2(\mu)$. Then

$$\begin{aligned}\mathscr{F}^M(\tau_y^N f)(\lambda) &= \int_{\mathbb{R}_+} \tau_y^N f(x)\varphi_\lambda^M(x)\,\mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}_+} \left[e^{-\frac{ia}{2b}(x^2+y^2)} \int_{\mathbb{R}_+} f(z)e^{\frac{ia}{2b}z^2} w(x,y,z)\,\mathrm{d}\mu(z)\right] \varphi_\lambda^M(x)\,\mathrm{d}\mu(x).\end{aligned}$$

By using Fubini's theorem, (2.6) and (2.7) we obtain

$$\mathscr{F}^M(\tau_y^N f)(\lambda) = e^{-\frac{ia}{2b}y^2} \int_{\mathbb{R}_+} f(z)e^{\frac{ia}{2b}z^2} \left[\int_{\mathbb{R}_+} \varphi_\lambda^M(x)e^{-\frac{ia}{2b}x^2} w(z,y,x)\,\mathrm{d}\mu(x)\right] \mathrm{d}\mu(z).$$

And by Theorem 3.2 (iii) we deduce that

$$\mathscr{F}^M(\tau_y^N f)(\lambda) = e^{\frac{id}{2b}\lambda^2} \varphi_\lambda^N(y)\mathscr{F}^M(f)(\lambda), \quad \lambda \in \mathbb{R}_+. \tag{3.3}$$

Since $L^1(\mu) \cap L^2(\mu)$ is dense in $L^2(\mu)$, the formula (3.3) remains valid for $f \in L^2(\mu)$. □

Let $f, g \in L^2(\mu)$. The canonical Sturm-Liouville convolution $f *^N g$ of f and g is defined by

$$f *^N g(x) := \int_{\mathbb{R}_+} \tau_x^N f(y) \left[e^{\frac{ia}{b}y^2} g(y)\right] \mathrm{d}\mu(y), \quad x \in \mathbb{R}_+. \tag{3.4}$$

Then we can write

$$f *^N g(x) = e^{-\frac{ia}{2b}x^2} \left(e^{\frac{ia}{2b}z^2} f\right) * \left(e^{\frac{ia}{2b}z^2} g\right)(x), \quad x \in \mathbb{R}_+, \tag{3.5}$$

where $*$ is the Sturm-Liouville convolution given by (2.9).

The canonical Sturm-Liouville convolution $*^N$ is commutative, associative and satisfies the Young inequality. Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then for $f \in L^p(\mu)$ and $g \in L^q(\mu)$ we have

$$\|f *^N g\|_{L^r(\mu)} \leq \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Theorem 3.4. *(i) For $f, g \in L^2(\mu)$, the function $f *^N g$ belongs to $L^\infty(\mu)$, and*

$$f *^N g(x) = \int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \varphi_\lambda^N(x)\mathscr{F}^M(f)(\lambda)\mathscr{F}^M(g)(\lambda)\,\mathrm{d}\nu_b(\lambda), \quad x \in \mathbb{R}_+.$$

(ii) Let $f, g \in L^2(\mu)$. Then

$$\int_{\mathbb{R}_+} |f *^N g(x)|^2\,\mathrm{d}\mu(x) = \int_{\mathbb{R}_+} |\mathscr{F}^M(f)(\lambda)|^2 |\mathscr{F}^M(g)(\lambda)|^2\,\mathrm{d}\nu_b(\lambda),$$

where both sides are finite or infinite.

Proof. (i) follows from (3.5), Theorem 2.3 (i) and (3.1). (ii) follows from (3.5), Theorem 2.3 (ii) and (3.1). □

Example 3.5 ([9,11,12,21,26])**.** *Note that if* $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$, *the operator* L^M *is reduced to the canonical Bessel operator* L_α^M*:*

$$L_\alpha^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{2\alpha+1}{x} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + 2i(\alpha+1)\frac{a}{b}\right).$$

In this case $\varphi_\lambda^M(x) = \varphi_{\lambda,\alpha}^M(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)} j_\alpha(\frac{\lambda x}{b})$.

The canonical Fourier-Bessel transform $\mathscr{F}_\alpha^M$ *is defined for* $f \in L^1(\mu_\alpha)$ *by*

$$\mathscr{F}_\alpha^M(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_{\lambda,\alpha}^M(x) f(x)\,\mathrm{d}\mu_\alpha(x), \quad \lambda \in \mathbb{R}_+.$$

Recently, the canonical Fourier-Bessel transform $\mathscr{F}_\alpha^M$ *is the goal of many applications in the harmonic analysis.*

The canonical Fourier-Bessel translation operators are defined for $f \in L^1(\mu_\alpha)$ *by*

$$\tau_y^{\alpha,N} f(x) := e^{-\frac{ia}{2b}(x^2+y^2)} \int_{\mathbb{R}_+} f(z) e^{\frac{ia}{2b}z^2} w_\alpha(x,y,z)\,\mathrm{d}\mu_\alpha(z), \quad x, y \in \mathbb{R}_+,$$

being $w_\alpha(x,y,.)$ *the kernel given by* (2.10).

Let $f, g \in L^2(\mu_\alpha)$. *The canonical Fourier-Bessel convolution* $f *_\alpha^N g$ *of* f *and* g *is defined by*

$$f *_\alpha^N g(x) := \int_{\mathbb{R}_+} \tau_x^{\alpha,N} f(y) \left[e^{\frac{ia}{b}y^2} g(y)\right] \mathrm{d}\mu_\alpha(y), \quad x \in \mathbb{R}_+.$$

4 Canonical Sturm-Liouville Hausdorff operator

In this section we define and study the Hausdorff operator associated with the canonical Sturm-Liouville operator L^M.

Let $f \in L^p(\mu)$, $p \in [1, \infty)$ and $t > 0$. We define the dilation function f_t by

$$f_t(x) := \frac{A\left(\frac{x}{t}\right)}{tA(x)} f\left(\frac{x}{t}\right), \tag{4.1}$$

and satisfies

$$\|f_t\|_{L^p(\mu)} \leq \left(\frac{k(t)}{t}\right)^{1-\frac{1}{p}} \|f\|_{L^p(\mu)}, \tag{4.2}$$

where

$$k(t) = \sup_{x \in \mathbb{R}_+} \left(\frac{A(x)}{A(tx)}\right).$$

From (2.1), there exist two constants $C_1, C_2 > 0$, such that

$$C_1 x^{2\alpha+1} \leq A(x) \leq C_2 x^{2\alpha+1}, \quad x \in \mathbb{R}_+^*.$$

Therefore,

$$\frac{1}{Ct^{2\alpha+1}} \leq k(t) \leq \frac{C}{t^{2\alpha+1}}, \quad t > 0,$$

where $C = \frac{C_2}{C_1}$.

Let $\phi \in L^1(\mathbb{R}_+)$. We define the Hausdorff operator H_ϕ associated with the canonical Sturm-Liouville operator L^M for $f \in L^1(\mu)$ by

$$H_\phi f(x) := \int_{\mathbb{R}_+} f_t(x)\phi(t)\,\mathrm{d}t. \tag{4.3}$$

If we choose $\phi(t) = \beta(1-t)^{\beta-1}\chi_{(0,1)}(t)$, $\beta > 0$, we obtain the canonical Sturm-Liouville Cesàro operator of order β denoted by $\mathcal{C}_\beta$ and given by

$$\mathcal{C}_\beta f(x) := \beta \int_0^1 f_t(x)(1-t)^{\beta-1}\,\mathrm{d}t.$$

A brief history of the study of Cesàro operator can be found in [14].

If we choose $\phi(t) = \frac{1}{t}\chi_{(1,\infty)}(t)$, we obtain the canonical Sturm-Liouville Hardy operator denoted by $\mathcal{H}$ and given by

$$\mathcal{H}f(x) := \int_1^\infty f_t(x)\,\frac{\mathrm{d}t}{t}.$$

It is well known that Hardy operators are important operators in harmonic analysis, for instance, see [6, 15].

If we choose $\phi(t) = \frac{1}{\max(1,t)}$, we obtain the canonical Sturm-Liouville Hardy-Littlewood-Pólya operator denoted by $\mathcal{P}$ and given by

$$\mathcal{P}f(x) := \int_0^1 f_t(x)\mathrm{d}t + \int_1^\infty f_t(x)\,\frac{\mathrm{d}t}{t}.$$

The study of Hardy-Littlewood-Pólya operators can be found in [1].

If we choose $\phi(t) = \frac{1}{\Gamma(\eta)}\frac{(1-\frac{1}{t})^{\eta-1}}{t}\chi_{(1,\infty)}(t)$, $\eta > 0$ we obtain the canonical Sturm-Liouville Riemann-Liouville fractional integral operator denoted by $\mathcal{I}$ and given by

$$\mathcal{I}f(x) := \frac{1}{\Gamma(\eta)}\int_1^\infty f_t(x)\left(1-\frac{1}{t}\right)^{\eta-1}\frac{\mathrm{d}t}{t}.$$

The study of Riemann-Liouville fractional integral operators can be found in [25].

Theorem 4.1. *Let $\phi \in L^1(\mathbb{R}_+)$. Then for $f \in L^1(\mu)$, we have*

$$\mathscr{F}^M(H_\phi f)(\lambda) = \int_{\mathbb{R}_+} \mathscr{F}^M(f_t)(\lambda)\phi(t)\,\mathrm{d}t, \quad \lambda \in \mathbb{R}_+.$$

Proof. Let $\phi \in L^1(\mathbb{R}_+)$, and let $f \in L^1(\mu)$. Then by (4.3) we have

$$\mathscr{F}^M(H_\phi f)(\lambda) = \int_{\mathbb{R}_+} H_\phi f(x)\varphi_\lambda^M(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}_+}\left[\int_{\mathbb{R}_+} f_t(x)\phi(t)\,\mathrm{d}t\right]\varphi_\lambda^M(x)\,\mathrm{d}\mu(x).$$

Since

$$\int_{\mathbb{R}_+^2} |f_t(x)||\phi(t)||\varphi_\lambda^M(x)|\,\mathrm{d}t\,\mathrm{d}\mu(x) \leq \|\phi\|_{L^1(\mathbb{R}_+)}\|f\|_{L^1(\mu)} < \infty,$$

by Fubini's theorem we obtain

$$\mathscr{F}^M(H_\phi f)(\lambda) = \int_{\mathbb{R}_+}\left[\int_{\mathbb{R}_+} f_t(x)\varphi_\lambda^M(x)\,\mathrm{d}\mu(x)\right]\phi(t)\,\mathrm{d}t = \int_{\mathbb{R}_+} \mathscr{F}^M(f_t)(\lambda)\phi(t)\,\mathrm{d}t.$$

The theorem is proved. □

Theorem 4.2. *Let ϕ be a measurable function on $\mathbb{R}_+$ such that*

$$C_{\phi,p} := \int_{\mathbb{R}_+}\left(\frac{k(t)}{t}\right)^{1-\frac{1}{p}} |\phi(t)|\,\mathrm{d}t < \infty. \tag{4.4}$$

Then the Hausdorff operator H_ϕ is bounded on $L^p(\mu)$, $p \in [1,\infty)$ with

$$\|H_\phi f\|_{L^p(\mu)} \leq C_{\phi,p}\|f\|_{L^p(\mu)}.$$

Proof. By using Minkowski's inequality for integrals, we have

$$\begin{aligned}\|H_\phi f\|_{L^p(\mu)} &= \left[\int_{\mathbb{R}_+}\left|\int_{\mathbb{R}_+} f_t(x)\phi(t)\,\mathrm{d}t\right|^p \mathrm{d}\mu(x)\right]^{1/p} \leq \left[\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+} |f_t(x)||\phi(t)|\,\mathrm{d}t\right)^p \mathrm{d}\mu(x)\right]^{1/p} \\ &\leq \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+} |f_t(x)|^p|\phi(t)|^p\,\mathrm{d}\mu(x)\right)^{1/p}\mathrm{d}t = \int_{\mathbb{R}_+} \|f_t\|_{L^p(\mu)}|\phi(t)|\,\mathrm{d}t.\end{aligned}$$

Then by (4.2) we obtain

$$\|H_\phi f\|_{L^p(\mu)} \leq C_{\phi,p}\|f\|_{L^p(\mu)}.$$

Going back to the definition of

$$\left[\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+} |f_t(x)||\phi(t)|\,\mathrm{d}t\right)^p \mathrm{d}\mu(x)\right]^{1/p},$$

we deduce that the integral

$$H_\phi f(x) = \int_{\mathbb{R}_+} f_t(x)\phi(t)\,\mathrm{d}t,$$

is absolutely convergent for almost all $x \in \mathbb{R}_+$, and defines a function $H_\phi f \in L^p(\mathbb{R}_+)$. □

Let $f, g \in L^2(\mu)$, and let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition

$$C_{\phi,2} := \int_{\mathbb{R}_+} \left(\frac{k(t)}{t}\right)^{\frac{1}{2}} |\phi(t)|\,\mathrm{d}t < \infty. \tag{4.5}$$

We define the adjoint operator H_ϕ^* by the relation

$$\int_{\mathbb{R}_+} H_\phi^* f(x) g(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}_+} f(x) H_\phi g(x)\,\mathrm{d}\mu(x).$$

Theorem 4.3. *Let $f \in L^2(\mu)$, and let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition* (4.5). *Then*

$$H_\phi^* f(x) = \int_{\mathbb{R}_+} f(tx)\phi(t)\,\mathrm{d}t. \tag{4.6}$$

Proof. Let $f, g \in L^2(\mu)$, and let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition (4.5). From (4.3) and Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{R}_+} f(x) H_\phi g(x)\,\mathrm{d}\mu(x) &= \int_{\mathbb{R}_+} f(x) \left[\int_{\mathbb{R}_+} g_t(x)\phi(t)\,\mathrm{d}t\right] \mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} f(x) g_t(x)\,\mathrm{d}\mu(x)\right] \phi(t)\,\mathrm{d}t = \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} f(tx) g(x)\,\mathrm{d}\mu(x)\right] \phi(t)\,\mathrm{d}t.\end{aligned}$$

Using (4.2), this calculation is justified by the fact that

$$\int_{\mathbb{R}_+^2} |f(x)||g_t(x)|\mathrm{d}\mu(x)|\phi(t)|\,\mathrm{d}t \le C_{\phi,2} \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)} < \infty.$$

Then according to Fubini's theorem we obtain

$$\int_{\mathbb{R}_+} f(x) H_\phi g(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} f(tx)\phi(t)\,\mathrm{d}t\right] g(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}_+} H_\phi^* f(x) g(x)\,\mathrm{d}\mu(x),$$

where

$$H_\phi^* f(x) = \int_{\mathbb{R}_+} f(tx)\phi(t)\,\mathrm{d}t.$$

This calculation is justified by the fact that

$$\int_{\mathbb{R}_+^2} |f(tx)||g(x)|\,\mathrm{d}\mu(x)|\phi(t)|\,\mathrm{d}t \le C_{\phi,2} \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)} < \infty.$$

This completes the proof of the theorem. □

Remark 4.4. *From Theorem 4.2, the operator* H_ϕ^* *is bounded on* $L^p(\mu)$*,* $p \in [1, \infty)$*, with*

$$\|H_\phi^* f\|_{L^p(\mu)} \leq C_{\phi, \frac{p}{p-1}} \|f\|_{L^p(\mu)},$$

where $C_{\phi,p}$ *is the constant given by* (4.4).

As in the same of Theorem 4.1, we obtain the following result.

Theorem 4.5. *Let* ϕ *be a measurable function on* $\mathbb{R}_+$ *satisfying the condition*

$$C_{\phi,\infty} := \int_{\mathbb{R}_+} \frac{k(t)}{t} |\phi(t)| \,\mathrm{d}t < \infty. \tag{4.7}$$

Then for $f \in L^1(\mu)$*, we have*

$$\mathscr{F}^M(H_\phi^* f)(\lambda) = \int_{\mathbb{R}_+} \mathscr{F}^M(f_t^*)(\lambda)\phi(t)\,\mathrm{d}t, \quad \lambda \in \mathbb{R}_+,$$

where $f_t^*(x) = f(tx)$*.*

Proof. Let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition (4.7), and let $f \in L^1(\mu)$. Then by (4.6) we have

$$\mathscr{F}^M(H_\phi^* f)(\lambda) = \int_{\mathbb{R}_+} H_\phi^* f(x) \varphi_\lambda^M(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} f(tx)\phi(t)\,\mathrm{d}t\right] \varphi_\lambda^M(x)\,\mathrm{d}\mu(x).$$

Since

$$\int_{\mathbb{R}_+^2} |f(tx)||\phi(t)||\varphi_\lambda^M(x)|\,\mathrm{d}t\,\mathrm{d}\mu(x) \leq C_{\phi,\infty}\|f\|_{L^1(\mu)} < \infty,$$

by Fubini's theorem we obtain

$$\mathscr{F}^M(H_\phi^* f)(\lambda) = \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} f(tx)\varphi_\lambda^M(x)\,\mathrm{d}\mu(x)\right] \phi(t)\,\mathrm{d}t = \int_{\mathbb{R}_+} \mathscr{F}^M(f_t^*)(\lambda)\phi(t)\,\mathrm{d}t.$$

The theorem is proved. □

Example 4.6. *Note that if* $A(x) = x^{2\alpha+1}$*,* $\alpha > -1/2$*, we have*

$$f_t(x) = \frac{1}{t^{2\alpha+2}} f\left(\frac{x}{t}\right), \quad k(t) = \frac{1}{t^{2\alpha+1}}, \quad C_{\phi,p} = \int_{\mathbb{R}_+} \frac{|\phi(t)|}{t^{(2\alpha+2)(1-\frac{1}{p})}}\,\mathrm{d}t.$$

Therefore,

- *the canonical Bessel-Hausdorff operator is given by*

$$H_\phi f(x) = \int_{\mathbb{R}_+} f\left(\frac{x}{t}\right) \frac{\phi(t)}{t^{2\alpha+2}}\,\mathrm{d}t.$$

- *The canonical Bessel-Cesàro operator of order β is given by*

$$\mathcal{C}_\beta f(x) = \beta \int_0^1 f\left(\frac{x}{t}\right) \frac{(1-t)^{\beta-1}}{t^{2\alpha+2}}\,\mathrm{d}t.$$

- *The canonical Bessel-Hardy operator is given by*

$$\mathcal{H} f(x) = \int_1^\infty f\left(\frac{x}{t}\right) \frac{\mathrm{d}t}{t^{2\alpha+3}}.$$

- *The canonical Bessel-Hardy-Littlewood-Pólya operator is given by*

$$\mathcal{P} f(x) = \int_0^1 f\left(\frac{x}{t}\right) \frac{\mathrm{d}t}{t^{2\alpha+2}} + \int_1^\infty f\left(\frac{x}{t}\right) \frac{\mathrm{d}t}{t^{2\alpha+3}}.$$

- *The canonical Bessel-Riemann-Liouville fractional integral operator is given by*

$$\mathcal{I} f(x) = \frac{1}{\Gamma(\eta)} \int_1^\infty f\left(\frac{x}{t}\right) \left(1-\frac{1}{t}\right)^{\eta-1} \frac{\mathrm{d}t}{t^{2\alpha+3}}.$$

5 Canonical Sturm-Liouville wavelet transform

In this section, we first recall some fundamental results on the canonical Sturm-Liouville wavelet transform. The classical Sturm-Liouville wavelet transform has been studied extensively in [23, 34] where detailed definitions, illustrative examples, and comprehensive discussions of its properties can be found. In the following we establish a relation between the canonical Sturm-Liouville wavelet transform and the canonical Sturm-Liouville Hausdorff operator.

As in the same of [23, 34] and by using Theorem 3.1 (ii), we prove following lemma.

Theorem 5.1. *Let $g \in L^2(\mu)$, and $t > 0$. Then there exists a function $g_r^\sharp$ in $L^2(\mu)$, such that*

$$\mathscr{F}^M(g_r^\sharp)(\lambda) = \mathscr{F}^M(g)(r\lambda), \quad \lambda \in \mathbb{R}_+, \tag{5.1}$$

and satisfies

$$\|g_r^\sharp\|_{L^2(\mu)} \leq \frac{\ell_b(r)}{\sqrt{r}} \|g\|_{L^2(\mu)}, \tag{5.2}$$

where

$$\ell_b(r) = \sup_{\lambda>0} \frac{|c(\frac{\lambda}{b})|}{|c(\frac{\lambda}{rb})|}.$$

We say that a function $g \in L^2(\mu)$ is a canonical Sturm-Liouville wavelet, if it satisfies the admissibility condition

$$0 < \omega_g := \int_{\mathbb{R}_+} |\mathscr{F}^M(g)(\lambda)|^2 \frac{\mathrm{d}\lambda}{\lambda} < \infty. \tag{5.3}$$

Example 5.2. *The function g given by*

$$g(x) := \int_{\mathbb{R}_+} \lambda^2 e^{-\lambda^2} \varphi_\lambda^N(x) \,\mathrm{d}\nu_b(\lambda), \quad x \in \mathbb{R}_+,$$

is a canonical Sturm-Liouville wavelet and $\omega_g = \frac{1}{8}$. Note that if $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$, we have

$$g(x) := -\frac{e^{-\frac{ia}{2b}x^2}}{2^\alpha \Gamma(\alpha+1)} \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{e^{-\frac{x^2}{2(ibd+2tb^2)}}}{(ibd+2tb^2)^{\alpha+1}}\right]_{t=0}, \quad x \in \mathbb{R}_+,$$

For a function $g \in L^2(\mu)$ and for $(r,s) \in \mathbb{R}_+^* \times \mathbb{R}_+$ we denote by $g_{r,s}$ the function defined on $\mathbb{R}_+$ by

$$g^\sharp_{r,s}(y) := \tau_s^N g^\sharp_r(y),$$

where τ_s^N are the generalized translation operators given by (3.2).

From Theorem 3.3 (i) and (5.2), the function $g^\sharp_{r,s}$ satisfies

$$\|g^\sharp_{r,s}\|_{L^2(\mu)} \leq \frac{\ell_b(r)}{\sqrt{r}} \|g\|_{L^2(\mu)}. \tag{5.4}$$

Let $g \in L^2(\mu)$ be a canonical Sturm-Liouville wavelet. We define for regular functions on $\mathbb{R}_+$, the canonical Sturm-Liouville wavelet transform by

$$\Phi_g^N(f)(r,s) := \int_{\mathbb{R}_+} e^{\frac{ia}{b}y^2} f(y) g^\sharp_{r,s}(y) \,\mathrm{d}\mu(y), \tag{5.5}$$

which can also be written in the form

$$\Phi_g^N(f)(r,s) = f *^N g^\sharp_r(s), \tag{5.6}$$

where $*^N$ is the generalized convolution product given by (3.4).

From (5.4) and (5.5) with Hölder's inequality, we have

$$\|\Phi_g^N(f)(r,.)\|_{L^\infty(\mu)} \leq \frac{\ell_b(r)}{\sqrt{r}} \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)}.$$

From (5.6), Theorem 3.4 (i) and (5.1), we have

$$\Phi_g^N(f)(r,s) = \int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \varphi_\lambda^N(s) \mathscr{F}^M(f)(\lambda) \mathscr{F}^M(g)(r\lambda) \,\mathrm{d}\nu_b(\lambda). \tag{5.7}$$

We denote by γ the measure defined on $\mathbb{R}_+^2$ by

$$\mathrm{d}\gamma(r,s) := \mathrm{d}\mu(s)\,\frac{\mathrm{d}r}{r},$$

and by $L^2(\gamma)$ the space of measurable functions f on $\mathbb{R}_+^2$, such that

$$\|f\|_{L^2(\gamma)} := \left[\int_{\mathbb{R}_+^2} |f(r,s)|^2\,\mathrm{d}\mu(s)\,\frac{\mathrm{d}r}{r}\right]^{1/2} < \infty.$$

Theorem 5.3. *Let $g \in L^2(\mu)$ be a canonical Sturm-Liouville wavelet.*

(i) Plancherel formula for Φ_g^N. For $f \in L^2(\mu)$ we have

$$\|f\|_{L^2(\mu)}^2 = \frac{1}{\omega_g}\|\Phi_g^N(f)\|_{L^2(\gamma)}^2.$$

(ii) Parseval formula for Φ_g^N. For $f, h \in L^2(\mu)$ we have

$$\langle f, h\rangle_{L^2(\mu)} = \frac{1}{\omega_g}\langle \Phi_g^N(f), \Phi_g^N(h)\rangle_{L^2(\gamma)}.$$

Proof. (i) Using Fubini's theorem, Theorem 3.4 (ii), and the relation (5.6), we obtain

$$\begin{aligned}\frac{1}{\omega_g}\|\Phi_g^N(f)\|_{L^2(\gamma)}^2 &= \frac{1}{\omega_g}\int_{\mathbb{R}_+^2} |f *^N g_r^\sharp(s)|^2\,\mathrm{d}\mu(s)\,\frac{\mathrm{d}r}{r}\\ &= \frac{1}{\omega_g}\int_{\mathbb{R}_+^2} |\mathscr{F}^M(f)(\lambda)|^2 |\mathscr{F}^M(g_r^\sharp)(\lambda)|^2\,\mathrm{d}\nu_b(\lambda)\,\frac{\mathrm{d}r}{r}\\ &= \int_{\mathbb{R}_+} |\mathscr{F}^M(f)(\lambda)|^2 \left(\frac{1}{\omega_g}\int_{\mathbb{R}_+} |\mathscr{F}^M(g)(r\lambda)|^2\,\frac{\mathrm{d}r}{r}\right)\mathrm{d}\nu_b(\lambda).\end{aligned}$$

By relation (5.3) we have

$$\frac{1}{\omega_g}\int_{\mathbb{R}_+} |\mathscr{F}^M(g)(r\lambda)|^2\,\frac{\mathrm{d}r}{r} = 1.$$

Then we deduce the desired result from Theorem 3.1 (ii).

(ii) The result is easily deduced from (i). □

We obtain a relation between the canonical Sturm-Liouville wavelet transform and the canonical Sturm-Liouville Hausdorff operator.

Theorem 5.4. *Let $g \in L^2(\mu)$ be a canonical Sturm-Liouville wavelet, and let $\phi \in L^1(\mathbb{R}_+)$ satisfying the condition* (4.5)*. Then for $f \in L^1(\mu) \cap L^2(\mu)$ we have*

$$\Phi_g^N(H_\phi f)(r,s) = \int_{\mathbb{R}_+} \Phi_g^N(f_t)(r,s)\phi(t)\,\mathrm{d}t,$$

where f_t is the dilation of f given by (4.1)*.*

Proof. Let $g \in L^2(\mu)$ be a canonical Sturm-Liouville wavelet, and let $f \in L^1(\mu) \cap L^2(\mu)$. From Theorem 4.2 we have $H_\phi f \in L^2(\mu)$. Then by (5.7) and Theorem 4.1, we get

$$\begin{aligned}
\Phi_g^N(H_\phi f)(r,s) &= \int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \mathscr{F}^M(H_\phi f)(\lambda)\mathscr{F}^M(g)(r\lambda)\varphi_\lambda^N(s)\,\mathrm{d}\nu_b(\lambda) \\
&= \int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2}\left[\int_{\mathbb{R}_+} \mathscr{F}^M(f_t)(\lambda)\phi(t)\,\mathrm{d}t\right]\mathscr{F}^M(g)(r\lambda)\varphi_\lambda^N(s)\,\mathrm{d}\nu_b(\lambda) \\
&= \int_{\mathbb{R}_+}\left[\int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2}\mathscr{F}^M(f_t)(\lambda)\mathscr{F}^M(g)(r\lambda)\varphi_\lambda^N(s)\,\mathrm{d}\nu_b(\lambda)\right]\phi(t)\,\mathrm{d}t \\
&= \int_{\mathbb{R}_+} \Phi_g^N(f_t)(r,s)\phi(t)\,\mathrm{d}t.
\end{aligned}$$

Using (4.2), this calculation is justified by the fact that

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |\mathscr{F}^M(f_t)(\lambda)||\mathscr{F}^M(g_r^\sharp)(\lambda)|\,\mathrm{d}\nu_b(\lambda)|\phi(t)|\,\mathrm{d}t \le C_{\phi,2}\|f\|_{L^2(\mu)}\|g_r^\sharp\|_{L^2(\mu)} < \infty.$$

This ends the proof of the theorem. □

As in the same of Theorem 5.4, we obtain the following result.

Theorem 5.5. *Let $g \in L^2(\mu)$ be a canonical Sturm-Liouville wavelet, and Let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the conditions* (4.5) *and* (4.7)*. Then for $f \in L^1(\mu) \cap L^2(\mu)$ we have*

$$\Phi_g^N(H_\phi^* f)(r,s) = \int_{\mathbb{R}_+} \Phi_g^N(f_t^*)(r,s)\phi(t)\,\mathrm{d}t,$$

where $f_t^*(x) = f(tx)$.

Proof. Let $g \in L^2(\mu)$ be a canonical Sturm-Liouville wavelet, and let $f \in L^1(\mu) \cap L^2(\mu)$. From Remark 4.4 we have $H_\phi^* f \in L^2(\mu)$. Then by (5.7) and Theorem 4.5, we get

$$\begin{aligned}
\Phi_g^N(H_\phi^* f)(r,s) &= \int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \mathscr{F}^M(H_\phi^* f)(\lambda)\mathscr{F}^M(g)(r\lambda)\varphi_\lambda^N(s)\,\mathrm{d}\nu_b(\lambda) \\
&= \int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2}\left[\int_{\mathbb{R}_+} \mathscr{F}^M(f_t^*)(\lambda)\phi(t)\,\mathrm{d}t\right]\mathscr{F}^M(g)(r\lambda)\varphi_\lambda^N(s)\,\mathrm{d}\nu_b(\lambda)
\end{aligned}$$

$$= \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \mathscr{F}^M(f_t^*)(\lambda) \mathscr{F}^M(g)(r\lambda) \varphi_\lambda^N(s) \, \mathrm{d}\nu_b(\lambda) \right] \phi(t) \, \mathrm{d}t$$
$$= \int_{\mathbb{R}_+} \Phi_g^N(f_t^*)(r,s)\phi(t) \, \mathrm{d}t.$$

This calculation is justified by the fact that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |\mathscr{F}^M(f_t^*)(\lambda)||\mathscr{F}^M(g_r^\sharp)(\lambda)| \, \mathrm{d}\nu_b(\lambda)|\phi(t)| \, \mathrm{d}t \leq C_{\phi,2}\|f\|_{L^2(\mu)}\|g_r^\sharp\|_{L^2(\mu)} < \infty.$$

This ends the proof of the theorem. □

Conclusion

In this work we have succeeded in generalizing the results of Móricz for the classical Hausdorff operator [38], Upadhyay *et al.* for the Hankel Hausdorff operator [36, 37] and Daher *et al.* for the Dunkl Hausdorff operator [7, 8] to the setting of canonical Sturm-Liouville theory. In this paper, we have studied the canonical Sturm-Liouville Hausdorff operator on the Lebesgue space $L^p(\mu)$, $p \in [1, \infty)$. Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we obtain the results of the classical Sturm-Liouville case.

Conflicts of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Data availability statement

There are no data used in this manuscript.

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