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Characterization of bc and strongly bc-polyharmonic functions

Abdelkader Abouricha[1]
Lamya Bouali[1]
Allal Ghanmi[1,✉]

[1] *Analysis, P.D.E & Spectral Geometry, Lab M.I.A.-S.I., CeReMAR, Department of Mathematics, P.O. Box 1014, Faculty of Sciences, Mohammed V University in Rabat, Morocco.*
abdelkader.abouricha@gmail.com
lamyabouali92@gmail.com
allal.ghanmi@fsr.um5.ac.ma✉

ABSTRACT

We provide new characterizations of the bicomplex harmonic and strongly bc-harmonic functions in terms of bc-holomorphic functions. An extension to the bc-polyharmonic setting is investigated. We also derive similar bicomplex analog for strongly bc-polyharmonic functions of finite bi-order.

RESUMEN

Entregamos nuevas caracterizaciones de las funciones bicomplejas armónicas y fuertemente bc-armónicas en términos de funciones bc-holomorfas. Se investiga una extensión al marco bc-poliarmónico. También derivamos análogos bicomplejos similares para funciones fuertemente bc-poliarmónicas de bi-orden finito.

Keywords and Phrases: bc-polyholomorphic functions, bc-harmonic functions, strongly bc-harmonic functions, bc-polyharmonicity, Almansi's theorem.

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1 Introduction

Polyharmonic functions with respect to the familiar Laplace operator are a natural extension of harmonic functions [7]. The latter have been extensively studied in the literature [7, 11, 28] and have played a crucial role in different areas of mathematics and physics, including the theory of holomorphic functions, the study of elliptic partial differential equations, minimal surfaces, digital processing and electrical engineering. Recall that a $2m$ times continuously differentiable complex-valued function f in the n-dimensional Euclidean space $\mathbb{R}^n$ is said to be polyharmonic of order m in a domain $\Omega \subset \mathbb{R}^n$, if it satisfies $\Delta^m f(x) = 0$ for $x \in \Omega$, where Δ^m is the m-th iterate of the Laplace operator

$$\Delta = \frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right), \quad x = (x_1 x_2, \dots, x_n).$$

For $m = 2$, they are the so-called biharmonic functions, intervening in elasticity theory. We should point out that polyharmonic functions have been studied by the end of the nineteenth century by the classical paper [4] by Almansi. His main result states that for every polyharmonic function f of order m on a star domain Ω, there exist some harmonic functions h_k, $k = 0, \dots, m$, on Ω such that

$$f(x) = |x|^{2m} h_m(x) + |x|^{2(m-1)} h_{m-1}(x) + \cdots + h_0(x).$$

This extends in fact the Gauss decomposition of a polynomial [3, 26]. The development of their theory is due to Nicolesco [30] and Aronszajn [6] works. Recently, they have been the subject of many investigations in a variety of mathematical and engineering fields, including numerical analysis, approximation of functions, wavelet analysis, the construction of multivariate splines and image processing. For a broader overview of these matters and its various applications see, *e.g.* [5, 8, 22, 26, 29] and the references therein.

On the other hand, the analysis within the bicomplex numbers generalizing complex numbers is currently a fully developed field of study. Its introduction goes back to Segre [39]. Next, they have been elaborated by the Italian school in the early twentieth century [14, 40]. Comprehensive studies were later carried out in [32, 34, 41]. In the last decades, they have been rediscovered, developed, and have attracted growing attention with some intriguing new advances with wide applications [2, 9, 12, 13, 18, 19, 21, 31, 37, 38, 42]. In fact, they have been used to discuss different aspects of the bicomplex neural networks [25, 43], and furthermore serve as an appropriate model for representing color image encoding in image processing [3, 17]. Bicomplex analysis was also investigated in the finite element method with a significant improvement when compared to the real and complex cases [33]. Moreover, they are an ideal context to extend the classical results concerning signal processing and time-frequency analysis using tools from frame theory [15].

One of the well-developed axes in bicomplex analysis is the theory of holomorphic functions of a bicomplex variable. In fact, it was widely studied in [32] (see also [14,36,40]) with a close connection with functional calculus, theory of function spaces and integral transforms [15, 19, 21]. Contrary to this theory, harmonic and potential theories are new areas of research that emerge within the framework of bicomplex numbers. For some of their fundamentals, one refers for instance to [1,16]. Notice that different bicomplex analogs of the classical mean value theorems (MVT) have been obtained in [1] for bc-harmonic and strongly bc-harmonic functions, as well as their analytical and geometrical converses, including the bicomplex analog of Hansen and Nadirashvili's result [23]. While a complete characterization of hyperbolic-valued bc-harmonic functions, in terms of the bicomplex holomorphic functions, has been provided in [16]. It is proved in particular that a real-valued bicomplex harmonic function is not necessarily the hyperbolic real part of a bicomplex holomorphic function, but of a bicomplex polyholomorphic one. A result that was next extended to the bicomplex polyharmonic functions.

In the present paper, we intend to pursue such investigation of extending to bicomplex context the fecund theory of harmonic and polyharmonic functions of complex variable. In fact, we are concerned with the bicomplex versions of some known results satisfied by the classical harmonic functions on the complex plane $\mathbb{C}$. Namely, we establish a concrete characterization of the strongly bc-harmonic functions (Theorem 3.1), as well as different bicomplex analogs of the additive decomposition theorem for bc-harmonic and strongly bc-harmonic functions. The initial motivation for the second task is a classical fact in complex analysis asserting that harmonic functions are exactly those that can be rewritten as $H + \overline{G}$ for certain holomorphic functions H and G, which usually is proved using the characterization of holomorphic functions in terms of the Wirtinger operators. The proof of "only if" can also be handled starting from the fact that a real-valued harmonic function is the real part of a holomorphic function, which fails when dealing with bc-harmonic functions as pointed out in [16]. Accordingly, it seems to be natural and interesting to know whether bc-harmonic (or bc-polyharmonic in general) functions can still have a similar additive decomposition. This paper contains then an answer to this question. To this end, one makes use of the expected characterization of an hyperbolic-valued bc-harmonic function F being the hyperbolic real part of a bc-holomorphic function if and only if F belongs to the kernels of some bicomplex first order differential operators. We also show that a bicomplex-valued function F on $\mathbb{BC}$ in $\ker(\partial_{\widetilde{Z}}) \cap \ker(\partial_{Z^\dagger})$ is bc-harmonic if and only if there exist certain bicomplex holomorphic functions H and G such that $F = H + G^*$, where $*$ denotes the complex conjugation in $\mathbb{BC}$ with respect to the bicomplex ij. More generally, we derive an additional decomposition without assuming the condition of belonging to $\ker(\partial_{\widetilde{Z}}) \cap \ker(\partial_{Z^\dagger})$, see Theorem 3.7. Similar characterization for bc-polyharmonic functions of finite order in terms of special subclass of bc-polyholomorphic functions is also obtained in Theorem 3.3. The main tool in its proof relies on [16, Proposition 3.8]. However, for a formal proof, see Remark 3.4, where one makes use of Proposition 4.4 in

[16], giving a bicomplex analog of Almansi's theorem for the representation of bc-polyharmonic in terms of bc-harmonic functions. An explicit characterization of the so-called strongly bc-harmonic is also provided (Theorem 3.1). This result is then employed to give a precise description of the bc-harmonic functions arising as $H_0 + H_1^* + H_2^\dagger + \widetilde{H}_3$, for some bc-holomorphic functions H_ℓ, $\ell = 0,1,2,3$. See Theorem 3.5 for an exact statement. The motivation for considering strongly bc-harmonic functions lies in the fact that an explicit and complete description of some spectral aspects of the bc-harmonic functions needs in general an additional harmonicity condition with respect to the $*$-conjugation, see for example [1, 2]. This phenomena will be confirmed in the present investigation.

We anticipate that the findings will be helpful for ulterior uses and applications. In fact, we claim that they can be employed to give the explicit formula for special bicomplex Bergman and Bargmann spaces of bc-harmonic functions as well as the integral representation for their elements by Bargmann type transform. We also anticipate extending the obtained results to the bicomplex analog of the so-called (α,β)-harmonic functions (see *e.g.* [10, 20, 24] and the references therein), which are defined as those that are twice continuously differentiable functions u solutions of the homogeneous equation $L^\varepsilon_{\alpha,\beta} u = 0$ on the complex plane ($\varepsilon = 0$) or the hyperbolic unit disc ($\varepsilon = +1$), where

$$L^\varepsilon_{\alpha,\beta} := (1-\varepsilon|z|^2)\left\{(1-\varepsilon|z|^2)\Delta + \alpha z\partial_z + \beta\overline{z}\partial_{\overline{z}} - \alpha\beta\right\}.$$

Notice that for $\alpha = -\beta$, it has been initiated and implicitly investigated in [2], by considering a pair of bicomplex magnetic Laplacians on $\mathbb{BC}$ and the disc.

The paper is outlined as follows. In Section 2, we fix the notations, including those announced above and related to the bicomplex numbers. We also define the bicomplex Laplace type operator and different notions of bc-harmonicity that we will work with. Section 3 deals with the proof of Theorem 3.1, giving a complete description of strongly bc-harmonic functions, as well as the additive decomposition theorems characterizing the bc-harmonic (Theorems 3.2 and 3.7) and bc-polyharmonic (Theorem 3.3) functions. The last section deals with some concluding remarks to answer the question how can the obtained conclusions be properly adapted to product-type domains.

2 Preliminaries

In this section, we briefly review some basic and needed notions from bicomplex analysis, we fix notations, and we introduce the different notions of harmonicity in the bicomplex setting that we will consider in this paper.

2.1 Bicomplex numbers

Bicomplex numbers are defined by complexifying the complex numbers $z = x + iy \in \mathbb{C}$ $(x, y \in \mathbb{R})$. Their 4-dimensional real algebra is then defined as $\mathbb{BC} = \{Z = z_1 + jz_2; z_1, z_2 \in \mathbb{C}\}$, where j is an imaginary unit, $j^2 = -1$, independent of i and satisfying $ij = ji =: k$. This turns k into what is known as hyperbolic unit, leading to the particular subset $\mathbb{D}$ of hyperbolic numbers, which is constituted of the bi-reals $x + ky$. The computation rules in $\mathbb{BC}$ extend, in a natural way, those in $\mathbb{C}$, giving rise to similar algebraic properties, except for division. More precisely, the null cone coincides with $\mathcal{NC} = \{\lambda(1 \pm ij);\ \lambda \in \mathbb{C}, \lambda \neq 0\}$. The particular elements

$$e_+ = \frac{(1+ij)}{2} \quad \text{and} \quad e_- = \frac{(1-ij)}{2}$$

are idempotent and satisfy $e_+e_- = 0$. Moreover, they yield the idempotent decomposition $\alpha e_+ + \beta e_- = Z$ of every $Z = z_1 + jz_2 \in \mathbb{BC}$, with unique complex components

$$\alpha = z_1 - iz_2 =: Proj^+(z_1 + jz_2) \quad \text{and} \quad \beta = z_1 + iz_2 =: Proj^-(z_1 + jz_2).$$

Thus, the map $P = (Proj^+, Proj^-)$,

$$P(z_1 + jz_2) := (z_1 - iz_2, z_1 + iz_2) = (\alpha, \beta), \tag{2.1}$$

realizes the algebra isomorphism $\mathbb{BC} \simeq \mathbb{C} \oplus \mathbb{C}$. Given such decomposition, the set $\mathbb{D}$ reads equivalently as the set of all $xe_+ + ye_-$ with $x, y \in \mathbb{R}$, leading to the partial order $\preceq$ ($xe_+ + ye_- \preceq x'e_+ + y'e_-$ if $x \leq x'$ and $y \leq y'$ in $\mathbb{R}$). A particular exception in the theory of bicomplex numbers is the attribution of three complex conjugates $Z^\dagger = z_1 - jz_2 = \beta e_+ + \alpha e_-$, $\widetilde{Z} = \overline{z_1} + j\overline{z_2} = \overline{\beta}e_+ + \overline{\alpha}e_-$, $Z^* = \overline{z_1} - j\overline{z_2} = \overline{\alpha}e_+ + \overline{\beta}e_-$, to each bicomplex number $Z = z_1 + jz_2$. By means of the above projection operators, one defines

$$\Omega^\pm := Proj^\pm(\Omega) = \{z_1 \mp iz_2 \in \mathbb{C},\ z_1 + jz_2 \in \Omega\}, \tag{2.2}$$

for given $\Omega \subset \mathbb{BC}$. We will write $\Omega = \Omega^+e_+ + \Omega^-e_-$, whenever Ω is a generic product-type set in $\mathbb{BC}$, *i.e.* those for which there exists a one-to-one correspondence from Ω onto $\Omega^+e_+ + \Omega^-e_-$. By Theorem 8.6 in [32, p. 37], such product-type sets are exactly those subsets in $\mathbb{BC}$ such that $P(\Omega) = \Omega^+ \times \Omega^-$, where P is as in (2.1). It should be pointed out that the openness of the components $\Omega^\pm$ in $\mathbb{C}$ follows from the openness of Ω in $\mathbb{BC}$, which is seen as the four-dimensional Euclidean space (see Riley's notes [34] or [32, Theorem 8.7]). For further details on the different topological considerations related to $\mathbb{BC}$, one refers to [32, 34].

2.2 Bicomplex holomorphy

Recall that a bicomplex-valued function

$$F(Z) = F_1(z_1, z_2) + jF_2(z_1, z_2),$$

on a given open set $\Omega \subset \mathbb{BC}$, is said in [32] to be bicomplex holomorphic (bc-holomorphic for short) in Ω, if for every $Z_0 \in \Omega$, the bicomplex limit

$$\lim_{\substack{H \to 0 \\ H \notin \mathcal{NC}}} \frac{F(Z_0 + H) - F(Z_0)}{H}$$

is finite. Another interesting characterization of the bc-holomorphicity is the Ringleb decomposition theorem [35] (see also [32, Theorem 15.5]), asserting that a bicomplex-valued function f is bc-holomorphic if and only if it is of the form

$$f(Z) = f(\alpha e_+ + \beta e_-) = \phi^+(\alpha)e_+ + \phi^-(\beta)e_-, \tag{2.3}$$

where $\phi^\pm : \mathbb{C} \longrightarrow \mathbb{C}$ are holomorphic $\mathbb{C}$-valued functions on $\mathbb{C}$. For a product-type domain this remains equivalent to F_1, F_2 be holomorphic in the complex variables $(z_1, z_2) \in \Omega^+ \times \Omega^-$ and satisfying in addition the complex Cauchy-Riemann equations [36, Theorem 1]

$$\frac{\partial F_1}{\partial z_1} = \frac{\partial F_2}{\partial z_2} \quad \text{and} \quad \frac{\partial F_2}{\partial z_1} = -\frac{\partial F_1}{\partial z_2}.$$

Analogously to the classical complex derivatives $\partial_z = \partial/\partial z$ and its complex conjugate $\partial_{\overline{z}} = \partial/\partial \overline{z}$, there are the first order differential operators with respect to the different bicomplex conjugates

$$\partial_Z = \frac{\partial}{\partial Z} := \frac{1}{2}\left(\frac{\partial}{\partial z_1} - j\frac{\partial}{\partial z_2}\right), \quad \partial_{Z^*} = \frac{\partial}{\partial Z^*} := \frac{1}{2}\left(\frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}}\right),$$
$$\partial_{Z^\dagger} = \frac{\partial}{\partial Z^\dagger} := \frac{1}{2}\left(\frac{\partial}{\partial z_1} + j\frac{\partial}{\partial z_2}\right), \quad \partial_{\widetilde{Z}} = \frac{\partial}{\partial \widetilde{Z}} := \frac{1}{2}\left(\frac{\partial}{\partial \overline{z_1}} - j\frac{\partial}{\partial \overline{z_2}}\right),$$

which can be used to provide a special realization of the so-called bicomplex holomorphic functions as solutions of a system of linear differential equations with constant coefficients. Namely, a real differentiable bicomplex-valued function F on an open set in $\mathbb{BC}$ is bc-holomorphic if and only if it is solution of (see [13, Theorem 2.7] or also [27, p. 159])

$$\frac{\partial F}{\partial Z^*} = \frac{\partial F}{\partial Z^\dagger} = \frac{\partial F}{\partial \widetilde{Z}} = 0. \tag{2.4}$$

The system provided in (2.4) is a central tool in the theory of bc-holomorphic functions, and can be used to extend the bc-holomorphy to polyanalytic setting, so that the discussed bc-holomorphic functions appear as the $(1,1,1)$-bc holomorphic functions in the definition below.

Definition 2.1 ([21]). *A bicomplex–valued function F having continuous partial derivatives on an open set $\Omega \subset \mathbb{BC}$, up to order $2\max(m,n,k)$, and satisfying the system*

$$\partial^m_{Z^*} F = \partial^n_{\widetilde{Z}} F = \partial^k_{Z^\dagger} F = 0 \tag{2.5}$$

is said to be (m,n,k)-bc-polyholomorphic on Ω.

An explicit characterization of these functions has been obtained in [16, Proposition 3.8].

Proposition 2.2. *The bicomplex-valued (m,n,k)-bc-polyholomorphic functions on $\mathbb{BC}$ are exactly those that can be expanded as*

$$F(Z) = \sum_{\ell_1=0}^{m-1} \sum_{\ell_2=0}^{n-1} \sum_{\ell_3=0}^{k-1} Z^{*\ell_1} \widetilde{Z}^{\ell_2} Z^{\dagger\ell_3} H_{\ell_1,\ell_2,\ell_3}(Z) \tag{2.6}$$

for given bc-holomorphic functions H_{ℓ_1,ℓ_2,ℓ_3}.

This result leads to an immediate extension of the Ringleb result (2.3) to these class of functions, which reads simply for the $(m,1,1)$ case as

$$F(Z = \alpha e_+ + \beta e_-) = \left(\sum_{k=0}^{m-1} \overline{\alpha}^k \phi_k(\alpha)\right) e_+ + \left(\sum_{k=0}^{m-1} \overline{\beta}^k \psi_k(\beta)\right) e_-,$$

for certain bc-holomorphic functions ϕ_k and ψ_k.

2.3 Bicomplex harmonicity

The existence of the different types of conjugates in the set of bicomplex numbers leads to different natural analogs of the classical Laplace operator

$$\Delta_z = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{\partial^2}{\partial z \partial \overline{z}}, \quad z = x + iy, \tag{2.7}$$

see [16] for details. The so-called bc-Laplacian Δ_{bc} as well as its †-conjugate $\Delta^\dagger_{bc}$ given by

$$\Delta_{bc} := \frac{\partial^2}{\partial Z \partial Z^*} \quad \text{and} \quad \Delta^\dagger_{bc} := \frac{\partial^2}{\partial Z^\dagger \partial \widetilde{Z}}.$$

are examples of such Laplacians. Their action on a given sufficiently real differential bicomplex-valued function is well-defined and to be understood in the sense of Remark 2.5 in [16]. Thus, for a twice continuously differentiable function $F = F^+ e_+ + F^- e_-$, we have the idempotent decomposition $\Delta_{bc} = \Delta_\alpha e_+ + \Delta_\beta e_-$ and $\Delta^\dagger_{bc} = \Delta_\beta e_+ + \Delta_\alpha e_-$. By considering the complex-valued component functions $h^\pm(\alpha,\beta) := F^\pm(Z)$ with $Z = \alpha e_+ + \beta e_-$, this action reads

$$[\Delta_{bc} F](Z) = ([\Delta_\alpha h^+](\alpha,\beta))e_+ + ([\Delta_\beta h^-](\alpha,\beta))e_-. \tag{2.8}$$

Being indeed, since both ∂_Z and ∂_{Z^*} are seen as $\mathbb{BC}$-linear operators and $e_+ \cdot e_- = 0$, we have

$$\frac{\partial F}{\partial Z^*}(Z) = \left(\frac{\partial}{\partial \overline{\alpha}} e_+ + \frac{\partial}{\partial \overline{\beta}} e_-\right)\left(h^+(\alpha,\beta)e_+ + h^-(\alpha,\beta)e_-\right) = \frac{\partial h^+}{\partial \overline{\alpha}}(\alpha,\beta)e_+ + \frac{\partial h^-}{\partial \overline{\beta}}(\alpha,\beta)e_-,$$

and moreover

$$\begin{aligned}[\Delta_{bc}F](Z) &= \left[\frac{\partial}{\partial Z}\left(\frac{\partial F}{\partial Z^*}\right)\right](Z) = \left(\frac{\partial}{\partial \alpha} e_+ + \frac{\partial}{\partial \beta} e_-\right)\left(\frac{\partial h^+}{\partial \overline{\alpha}}(\alpha,\beta)e_+ + \frac{\partial h^-}{\partial \overline{\beta}}(\alpha,\beta)e_-\right)\\ &= \frac{\partial^2 h^+}{\partial \alpha \partial \overline{\alpha}}(\alpha,\beta)e_+ + \frac{\partial^2 h^-}{\partial \beta \partial \overline{\beta}}(\alpha,\beta)e_-.\end{aligned}$$

Accordingly, one suggests the following definition.

Definition 2.3 ([16]). *Let F be a bicomplex-valued function on an open set $\Omega \subset \mathbb{BC}$.*

(i) F is said to be bicomplex harmonic (bc-harmonic) if it is twice continuously real differentiable and satisfies the bc-Laplace equation $\Delta_{bc} = 0$ on Ω. We denote their set by $\mathcal{BH}arm(\Omega)$.

(ii) F is said to be bc-polyharmonic of order m if it is continuously real differentiable up to order $2m$ and satisfies the m-th bc-Laplace equation $\Delta_{bc}^m = 0$ on Ω.

It should be noticed here that the bc-polyharmonic functions are closely connected to a special class of bc-polyholomorphic functions as expected in [16]. Their representations in terms of bc-harmonic functions were obtained in [16, Proposition 4.4], which itself is a bicomplex extension of Almansi's result [4] for the classical polyharmonic complex-valued functions. For its exact statement, we let $|Z|_{bc}^{2k} := Z^k Z^{*k}$ for every $Z \in \mathbb{BC}$ and $k = 0, 1, 2, \dots$

Proposition 2.4. *For every bc-polyharmonic function F on $\mathbb{BC}$ of order m, there are certain bc-harmonic functions H_k, $k = 0, 1, \dots, m-1$, such that*

$$F(Z) = \sum_{k=0}^{m-1} |Z|_{bc}^{2k} H_k(Z). \tag{2.9}$$

Remark 2.5. *The component functions H_k in Proposition 2.4 are bc-harmonic and they implicitly depend on $Z^\dagger$ and $\widetilde{Z}$. More precisely, identity (2.9) reads equivalently*

$$F(Z) = \sum_{k=0}^{+\infty}\sum_{k=0}^{m-1}\left(Z^{n+k}Z^{*k}A_{n,k}(\widetilde{Z}, Z^\dagger) + Z^k Z^{*n+k} B_{n,k}(\widetilde{Z}, Z^\dagger)\right), \tag{2.10}$$

for given bicomplex-valued functions $A_{n,k}$ and $B_{n,k}$ belonging to $\ker(\partial_Z) \cap \ker(\partial_{Z^*})$.

Definition 2.6. *Let F be a bicomplex-valued function on an open set $\Omega \subset \mathbb{BC}$.*

(i) It is said to be strongly bicomplex harmonic if F and $F^\dagger$ are both bc-harmonic.

(ii) It is said to be strongly bc-polyharmonic of bi-order (m,n), if it has continuous partial derivatives up to order $2\max(m,n)$ and verifies $\Delta_{bc}^m F = 0$ and $\Delta_{bc}^n F^\dagger = 0$ on Ω.

We conclude this section by providing explicit examples for the different classes of bicomplex holomorphic, polyholomorphic, harmonic and polyharmonic functions, in the $i, j, ij = k$ representation as well as in the idempotent representation, which can easily constructed making use of the obtained characterizations. Thus, the functions

$$(Z^m + Z^n) + k(Z^m - Z^n) = 2\alpha^m e_+ + 2\beta^n e_-$$

are the elementary bc-holomorphic functions on $\mathbb{BC}$, while

$$(Z^m Z^* + Z^n Z^\dagger) + k(Z^m Z^* - Z^n Z^\dagger) = 2\alpha^m \overline{\alpha} e_+ + 2\alpha\beta^n e_-$$

is an example of a $(2,2,1)$-polyholomorphic function. The following

$$h_0(Z) = ZZ^\dagger + Z\widetilde{Z} + Z^*\widetilde{Z} + Z^*\widetilde{Z} = 2\Re\left(\alpha(\beta + \overline{\beta})\right)$$

is a fundamental example of bc-harmonic function which can not be the real part of any bc-harmonic function. An example of polyharmonic function is given by the the biharmonic function

$$Z^* Z^\dagger h_0(Z) + h_0(Z) = 2\left\{(\overline{\alpha}\beta + 1)\, e_+ + \left(\alpha\overline{\beta} + 1\right) e_-\right\} \Re\left(\alpha(\beta + \overline{\beta})\right).$$

3 Main results

3.1 Characterization of strongly bc-harmonic functions

The following result provides an explicit characterization of the strongly bc-harmonic functions.

Theorem 3.1. *Let F be a bicomplex-valued function on $\mathbb{BC}$. Then, the function F is strongly bc-harmonic if and only if there are some sequences $(a_{m,n})_{m,n}$, $(b_{m,n})_{m,n}$, $(c_{m,n})_{m,n}$ and $(d_{m,n})_{m,n}$ of bicomplex numbers such that F has a power series expansion of the form*

$$F(Z) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\left(a_{m,n}Z^m Z^{\dagger n} + b_{m,n}Z^m \widetilde{Z}^n + c_{m,n}Z^{*m}Z^{\dagger n} + d_{m,n}Z^{*m}\widetilde{Z}^n\right), \tag{3.1}$$

converging absolutely and uniformly on any compact set of $\mathbb{BC}$.

Proof. The "if" follows by direct computation. However, the strongly bc-harmonicity of F in (3.1) in the sense of Definition 2.6 can be handled by observing that the uniformly convergent series in (3.1) can be rewritten as $F = H + G^*$, with some functions H and G that can expanded as

$$\sum_{m=0}^{+\infty} Z^m \left((\psi(Z))^\dagger + \widetilde{\varphi(Z)} \right)$$

for given bc-holomorphic functions ψ and φ, and next employing using the useful facts $\partial_{Z^*}(\phi^\dagger) = \left(\partial_{\widetilde{Z}}(\phi)\right)^\dagger$, $\partial_{Z^*}(\widetilde{\phi}) = \widetilde{\partial_{Z^\dagger}(\phi)}$, $\partial_{\widetilde{Z}}(\phi^\dagger) = (\partial_{Z^*}(\phi))^\dagger$, and $\partial_{Z^\dagger}(\widetilde{\phi}) = \widetilde{\partial_{Z^*}(G)}$ as well as $\partial_Z(G^*) = (\partial_{Z^*}(G))^*$ and $\partial_{Z^\dagger}(G^*) = \left(\partial_{\widetilde{Z}}(G)\right)^*$.

For the proof of the "only if", let $F(\alpha e_+ + \beta e_-) = F^+(\alpha,\beta)e_+ + F^-(\alpha,\beta)e_-$ be a strongly bc-harmonic function, with $F^+, F^- : \mathbb{BC} \longrightarrow \mathbb{C}$. Thus, from $\Delta_{bc}F = 0$ and $\Delta_{bc}F^\dagger = 0$, and in particular $\Delta_\alpha F^+(\cdot,\beta) = 0$ and $\Delta_\alpha F^-(\cdot,\beta) = 0$, for every fixed complex number β, one observes that both the partial components $\alpha \longmapsto F^+(\alpha,\beta)$ and $\alpha \longmapsto F^-(\alpha,\beta)$ are complex-valued harmonic functions in the complex plane, for every fixed $\beta \in \mathbb{C}$. Therefore, there exist some complex-valued holomorphic functions $H^{+,\beta}$, $H^{-,\beta}$, $G^{+,\beta}$ and $G^{-,\beta}$ on $\mathbb{C}$ with power series expansions centered at the origin such that

$$F^+(\alpha,\beta) = H^{+,\beta}(\alpha) + \overline{G^{+,\beta}(\alpha)} = \sum_{m=0}^{+\infty} a_m^+(\beta)\alpha^m + b_m^+(\beta)\overline{\alpha}^m \tag{3.2}$$

and

$$F^-(\alpha,\beta) = H^{-,\beta}(\alpha) + \overline{G^{-,\beta}(\alpha)} = \sum_{m=0}^{+\infty} a_m^-(\beta)\alpha^m + b_m^-(\beta)\overline{\alpha}^m, \tag{3.3}$$

for all $\alpha \in \mathbb{C}$. However, since the partial functions $\beta \longmapsto F^\pm(\alpha,\beta)$ being harmonic, the involved coefficients

$$a_m^\pm(\beta) = \frac{1}{m!}\frac{\partial^m F^\pm}{\partial \alpha^m}(0,\beta), \quad m = 0,1,2,\dots,$$

and

$$b_m^\pm(\beta) = \frac{1}{m!}\frac{\partial^m F^\pm}{\partial \overline{\alpha}^m}(0,\beta), \quad m = 0,1,2,\dots,$$

which are independent of α and $\overline{\alpha}$ and seen as functions in the β-variable, become $\mathcal{C}^\infty$ and moreover harmonic on the complex plane. Thus, we write

$$a_m^\pm = H_{1,m}^\pm + \overline{H_{2,m}^\pm} \quad \text{and} \quad b_m^\pm(\beta) = G_{1,m}^\pm + \overline{G_{2,m}^\pm},$$

for certain holomorphic functions $H_0^\pm$, $G_0^\pm$, $H_{1,m}^\pm$, $H_{2,m}^\pm$, $G_{1,m}^\pm$ and $G_{2,m}^\pm$ on $\mathbb{C}$. Returning back to (3.2)-(3.3) and using the expansion series of the involved holomorphic functions, we get

$$F^{\pm}(\alpha,\beta) = H_0^{\pm}(\beta) + \overline{G_0^{\pm}(\beta)} + \sum_{m=1}^{+\infty} \left(H_{1,m}^{\pm}(\beta) + \overline{H_{2,m}^{\pm}(\beta)} \right) \alpha^m + \left(G_{1,m}^{\pm}(\beta) + \overline{G_{2,m}^{\pm}(\beta)} \right) \overline{\alpha}^m$$
$$= \sum_{m,n=0}^{+\infty} \left(a_{1,m,n}^{\pm}\beta^n + a_{2,m,n}^{\pm}\overline{\beta}^n \right) \alpha^m + \left(b_{1,m,n}^{\pm}\beta^n + b_{2,m,n}^{\pm}\overline{\beta}^n \right) \overline{\alpha}^m,$$

which gives rise to (3.1). □

3.2 Additive decomposition theorems

We begin with the following.

Theorem 3.2. *A bicomplex-valued function F is of the form $F = H+G^*$, for some bc-holomorphic functions H and G, if and only if it is bc-harmonic on $\mathbb{BC}$ such that $\partial_{\widetilde{Z}} F = \partial_{Z^\dagger} F = 0$.*

Proof. For given $F = H+G^*$ such that H and G are bc-holomorphic, the function F is bc-harmonic for the smooth function F satisfies

$$\frac{\partial^2 F}{\partial Z \partial Z^*} = \frac{\partial}{\partial Z}\left(\frac{\partial H}{\partial Z^*}\right) + \frac{\partial}{\partial Z^*}\left(\left(\frac{\partial G}{\partial Z^*}\right)\right)^* = 0.$$

Moreover, using the facts $\partial_{\widetilde{Z}}(G^*) = (\partial_{Z^\dagger}(G))^*$ and $\partial_{Z^\dagger}(G^*) = \left(\partial_{\widetilde{Z}}(G)\right)^*$, and keeping in mind (2.8) it becomes clear that

$$\partial_{\widetilde{Z}} F = \partial_{\widetilde{Z}}(H) + \partial_{\widetilde{Z}}(G^*) = \partial_{\widetilde{Z}}(H) + (\partial_{Z^\dagger}(G))^* = 0$$

and

$$\partial_{Z^\dagger} F = \partial_{Z^\dagger}(H) + \partial_{Z^\dagger}(G^*) = \partial_{Z^\dagger}(H) + \left(\partial_{\widetilde{Z}}(G)\right)^* = 0$$

hold.

For the proof of the converse, we proceed into two steps.

Step 1: Assume that $F : \mathbb{BC} \longrightarrow \mathbb{D}$ is a hyperbolic-valued bc-harmonic function belonging to $\ker(\partial_{\widetilde{Z}}) \cap \ker(\partial_{Z^\dagger})$. Next, observe that by means of [16, Theorem 1.1] there exists a bc-holomorphic function T such that $F = \Re e_{hyp}(T) := (T+T^*)/2$, which infers $F = H + G^*$ with $H = G = T/2$.

Step 2: For the general case when F does not take values in $\mathbb{D}$, we rewrite it as $F = F_1 + iF_2$, with

$$F_1 = \frac{F+F^*}{2} \quad \text{and} \quad F_2 = \frac{F-F^*}{2i}.$$

Both F_1 and F_2 are hyperbolic-valued functions on $\mathbb{BC}$. From this, it becomes clear that F is a bc-harmonic if and only if F_1 and F_2 are bc-harmonic. Moreover, we necessarily have

$$2\partial_{\widetilde{Z}}F_1 = -2i\partial_{\widetilde{Z}}F_1 = \partial_{\widetilde{Z}}F^* = (\partial_{Z^\dagger}F)^* = 0,$$

and

$$2\partial_{\widetilde{Z}}F_1 = -2i\partial_{Z^\dagger}F_1 = \partial_{Z^\dagger}F^* = \left(\partial_{\widetilde{Z}}F\right)^* = 0.$$

This implies that the functions F_1 and F_2 belong to $\ker(\partial_{\widetilde{Z}}) \cap \ker(\partial_{Z^\dagger})$. However, from the first step, we easily conclude that $F_1 = H_1 + G_1^*$ and $F_2 = H_2 + G_2^*$, for some bc-holomorphic functions H_ℓ and G_ℓ, $\ell = 1, 2$. Now, since $i^* = -i$, it follows

$$F = (H_1 + G_1^*) + i\,(H_2 + G_2^*) = H + G^*,$$

with $H = H_1 + iH_2$ and $G = G_1 - iG_2$. □

The following result extends the previous one to the bc-polyharmonic functions of arbitrary finite order. The argument in the presented proof is completely different from the one provided for Theorem 3.2.

Theorem 3.3. *Let F be a bicomplex-valued bc-polyharmonic function of order m on $\mathbb{BC}$. Then, there exist certain $(m, 1, 1)$-bc-polyholomorphic functions H and G such that $F = H + G^*$ if and only if $\partial_{\widetilde{Z}}F = \partial_{Z^\dagger}F = 0$.*

Proof. In the sense of Definition 2.1, the function $H + G^*$ is clearly bc-polyharmonic, whenever H and G are bc-polyholomorphic of order $(m, 1, 1)$ and $(n, 1, 1)$, respectively. Indeed, by setting $\ell = \max(m, n)$, we have

$$\Delta_{bc}^{\ell}(H + G^*) = \frac{\partial^\ell}{\partial Z^\ell}\left(\frac{\partial^\ell H}{\partial Z^{*\ell}}\right) + \frac{\partial^\ell}{\partial Z^{*\ell}}\left(\frac{\partial^\ell G}{\partial Z^{*\ell}}\right)^* = 0.$$

To prove the converse, let F be a bc-polyharmonic function of order m. Then, $\partial_{Z^*}^m(\partial_Z^m F) = \Delta_{bc}^m F = 0$. But, under the assumption $\partial_{\widetilde{Z}}F = \partial_{Z^\dagger}F = 0$, the function $\partial_Z^m F$ becomes $(m, 1, 1)$-bc-polyholomorphic. Accordingly, it can be expanded as

$$\partial_Z^m F = \sum_{\ell=0}^{m-1} {Z^*}^{\ell}\psi_\ell,$$

by means of Proposition 2.2 (with $n = k = 1$). The involved functions ψ_ℓ, $\ell = 0, 1, \dots, m-1$, are bc-holomorphic and can always be rewritten as $\psi_\ell = \partial_Z^m \varphi_\ell$ for certain bc-holomorphic functions φ_ℓ. Thus, by considering the function

$$G = \sum_{\ell=0}^{m-1} {Z^*}^{\ell}\varphi_\ell,$$

we get $\partial^m_{Z^*}(F^* - G^*) = 0$. But, using again the assumption $\partial_{\widetilde{Z}} F = \partial_{Z^\dagger} F = 0$, it becomes clear that $F^* - G^* = H$ is a $(m,1,1)$-bc-polyholomorphic function. □

Remark 3.4. *The proof of Theorem 3.3 can be handled using Almansi's theorem for bc-polyharmonic functions (see Proposition 2.4 or Remark 2.5) and by viewing Z and $Z^\dagger$ as independent variables. In fact, for F being a bc-polyharmonic function of order m, there exist some bc-harmonic functions F_k, $k = 0, 1, \dots, m-1$, such that*

$$F(Z) = F_0 + |Z|_{bc} F_1 + \cdots + |Z|_{bc}^{2(m-1)} F_{m-1}.$$

Accordingly, the assumption $\partial_{\widetilde{Z}} F = \partial_{Z^\dagger} F = 0$ becomes equivalent to $\partial_{\widetilde{Z}} F_k = \partial_{Z^\dagger} F_k = 0$ for every $k = 0, 1, \dots, m-1$. Therefore, making appeal to the discussion provided in the proof of Theorem 3.2 for each F_k, there exist some bc-holomorphic functions H_k and G_k such that $F_k = H_k + G_k^*$. Hence, one derives $F = H + G^*$, where

$$H = \sum_{k=0}^{m-1} |Z|_{bc}^{2k} H_k \quad \textit{and} \quad G = \sum_{k=0}^{m-1} |Z|_{bc}^{2k} G_k.$$

Given such result (Theorem 3.3), the next one provides a sufficient condition to decompose a given strongly bc-harmonic function F as $F = H_0 + H_1^* + H_2^\dagger + \widetilde{H}_3$ for certain bc-holomorphic function H. Notice, that the converse is clear since the different bicomplex conjugates H^*, $H^\dagger$, $\widetilde{H}$ of a bc-holomorphic function H are obviously bc-harmonic, and moreover they are strongly bc-harmonic, which shows that the functions $H_0 + H_1^* + H_2^\dagger + \widetilde{H}_3$, arising as the sum of the different conjugates of bc-holomorphic functions for some bicomplex holomorphic functions H_ℓ, $\ell = 0, 1, 2, 3$, are strongly bc-harmonic.

Theorem 3.5. *A bicomplex-valued strongly bc-harmonic function F in $\mathbb{BC}$ is of the form $F = H_0 + H_1^* + H_2^\dagger + \widetilde{H}_3$, for some bc-holomorphic functions H_ℓ, $\ell = 0, 1, 2, 3$, if*

$$\frac{\partial^{m+n+j+k} F}{\partial Z^m Z^{*n} Z^{\dagger j} \widetilde{Z}^k}(0) = 0, \tag{3.4}$$

holds, for every non-negative integers m, n, j and k such that $mn = jk = 0$.

Proof. The key observation is contained in the characterization provided by Theorem 3.1. In fact, the involved bicomplex constants in (3.1) are given by

$$a_{m,n} = \frac{1}{m!\,n!} \frac{\partial^{m+n} F}{\partial Z^m Z^{\dagger n}}(0), \quad b_{m,n} = \frac{1}{m!\,n!} \frac{\partial^{m+n} F}{\partial Z^m \widetilde{Z}^n}(0),$$
$$c_{m,n} = \frac{1}{m!\,n!} \frac{\partial^{m+n} F}{\partial Z^{*m} Z^{\dagger n}}(0), \quad d_{m,n} = \frac{1}{m!\,n!} \frac{\partial^{m+n} F}{\partial Z^{*m} \widetilde{Z}^n}(0).$$

Accordingly, under the assumption (3.4), which reads equivalently as

$$\frac{\partial^{m+j}F}{\partial Z^m Z^{\dagger j}}(0) = \frac{\partial^{m+j}F}{\partial Z^{*m} Z^{\dagger j}}(0) = \frac{\partial^{k+n}F}{\partial Z^k \widetilde{Z}^n}(0) = \frac{\partial^{k+n}F}{\partial Z^{*k} \widetilde{Z}^n}(0) = 0, \tag{3.5}$$

we get $a_{m,n} = d_{m,n} = 0$, for every $n \geq 1$, and $b_{m,n} = c_{m,n} = 0$, for any $m \geq 1$. Thus, the expansion series of F reduces further to $F = H_0 + H_1^* + H_2^\dagger + \widetilde{H}_3$, where H_0, H_1, H_2 and H_3 are the bc-holomorphic functions given by

$$H_0 := \sum_{m=0}^{+\infty} a_m Z^m, \quad H_1 := \sum_{m=0}^{+\infty} d_m^{*m} Z^m, \quad H_2 := \sum_{n=0}^{+\infty} c_n^\dagger Z^n, \quad \text{and} \quad H_3 =: \sum_{n=0}^{+\infty} \widetilde{b_n} Z^n,$$

where we have set $a_k := a_{k,0}$, $d_k := d_{k,0} c_k := c_{0,k}$ and $b_k := b_{0,k}$. □

Remark 3.6. *Theorem 3.5 can be reproved by considering an equivalent sufficient condition, leading to $a_{m,n} = d_{m,n} = 0$ for every $m \geq 1$ and $b_{m,n} = c_{m,n} = 0$ for any $n \geq 1$.*

Below, we give an additional additive decomposition theorem, which is specific for the bc-harmonic functions.

Theorem 3.7. *We have $\mathcal{BH}arm(\mathbb{BC}) = \left(\ker(\partial_{Z^*}) + \ker\left(\partial_{\widetilde{Z}}\right)\right) \cap \mathcal{C}^\infty(\mathbb{BC})$. More precisely, H is a bc-harmonic function if and only if it can be expanded as*

$$H(Z) = \sum_{k=0}^{+\infty} Z^k A_k(Z^\dagger, \widetilde{Z}) + Z^{\dagger k} B_k(Z, Z^*), \tag{3.6}$$

for some $A_k \in \ker(\partial_Z) \cap \ker(\partial_{Z^*})$ and $B_k \in \ker(\partial_{Z^\dagger}) \cap \ker\left(\partial_{\widetilde{Z}}\right)$.

Proof. Let H be a bc-harmonic function and write $H(Z) = \widehat{H}^+(\alpha,\beta)e_+ + \widehat{H}^-(\alpha,\beta)e_-$. Hence, the functions $\widehat{H}^+(\cdot,b) : \mathbb{C} \longrightarrow \mathbb{C}$ and $\widehat{H}^-(a,\cdot) : \mathbb{C} \longrightarrow \mathbb{C}$ are harmonic on $\mathbb{C}$. Thus, for every fixed $a,b \in \mathbb{C}$, the involved functions can be decomposed as $\widehat{H}^+(\alpha,b) = h_b^{+,1}(\alpha) + h_b^{+,2}(\overline{\alpha})$ and $\widehat{H}^-(a,\beta) = h_a^{-,1}(\beta) + h_a^{-,2}(\overline{\beta})$ for some holmorphic functions $h_b^{+,1}, h_b^{+,2} : \mathbb{C} \longrightarrow \mathbb{C}$ and $h_a^{-,1}, h_a^{-,2} : \mathbb{C} \longrightarrow \mathbb{C}$, thanks to the additive decomposition theorem for classical harmonic functions. Therefore, by setting

$$H^{(1)}(Z|a,b) := h_b^{+,1}(\alpha)e_+ + h_a^{-,1}(\beta)e_-$$

and

$$H^{(2)}(Z|a,b) := h_b^{+,2}(\overline{\alpha})e_+ + h_a^{-,2}(\overline{\beta})e_-,$$

we have $\partial_{Z^*}(H^{(1)}(\cdot|a,b)) = \partial_{\widetilde{Z}}(H^{(2)}(\cdot|a,b)))$ and

$$H(Z) = H^{(1)}(Z|\alpha,\beta) + H^{(2)}(Z|\alpha,\beta), \quad Z = \alpha e_+ + \beta e_-. \tag{3.7}$$

The functions $H^{(1)}$ and $H^{(2)}$ belong clearly to $\ker(\partial_{Z^*})$ and $\ker\left(\partial_{\widetilde{Z}}\right)$, respectively. The inverse inclusion is immediate. □

In Proposition 3.10 below, it is proved that the involved $H^{(1)}$, $H^{(2)}$, A_k and B_k in (3.7) and (3.6) are connected to each other by some additive separate bc-holomorphic function, which extends the notion of separate holomorphy to the bicomplex setting. Let F be a given bicomplex-valued function on $\mathbb{BC}$, identified to $\widehat{F}(\alpha,\beta) := F(\alpha e_+ + \beta e_-)$ on $\mathbb{C}^2$. Define the partial functions $F_\alpha : \mathbb{C} \longrightarrow \mathbb{BC}$ and $F^\beta : \mathbb{C} \longrightarrow \mathbb{BC}$ given by

$$F_\alpha(\beta) = F^\beta(\alpha) =: \widehat{F}(\alpha,\beta).$$

Definition 3.8. *A bicomplex-valued function F on $\mathbb{BC}$ is said to be separately holomorphic if F_α and F^β are both holomorphic in $\mathbb{C}$.*

Accordingly, we have the following characterization.

Proposition 3.9. *Let F be a bicomplex-valued function on $\mathbb{BC}$. Then, the following assertions are equivalent.*

(i) F is separate holomorphic on Ω.

(ii) F satisfies

$$\frac{\partial F}{\partial Z^*} = \frac{\partial F}{\partial \widetilde{Z}} = 0. \tag{3.8}$$

(iii) F has the expansion

$$F(Z) = \sum_{m,n=0}^{+\infty} C_{m,n} Z^m Z^{\dagger n}, \quad C_{m,n} \in \mathbb{BC}. \tag{3.9}$$

Proof. The separate holomorphy of F reads $\partial \widehat{F}/\partial\overline{\alpha} = \partial \widehat{F}/\partial\overline{\beta} = 0$, and therefore

$$\frac{\partial \widehat{F}^+}{\partial\overline{\alpha}} = \frac{\partial \widehat{F}^-}{\partial\overline{\alpha}} = \frac{\partial \widehat{F}^+}{\partial\overline{\beta}} = \frac{\partial \widehat{F}^-}{\partial\overline{\beta}} = 0. \tag{3.10}$$

This is in fact also equivalent to

$$\frac{\partial\left(\widehat{F}^+ e_+ + \widehat{F}^- e_-\right)}{\partial Z^*} = \frac{\partial\left(\widehat{F}^- e_+ + \widehat{F}^+ e_-\right)}{\partial Z^*} = \left(\frac{\partial\left(\widehat{F}^+ e_+ + \widehat{F}^- e_-\right)}{\partial \widetilde{Z}}\right)^\dagger$$

be identically zero on $\mathbb{BC}$, which infers (3.8). Next, by means of (3.10), the functions in (ii) are those for which we have

$$\widehat{F}^\pm(\alpha,\beta) = \sum_{m,n=0}^{+\infty} a^\pm_{m,n} \alpha^m \beta^n, \quad a^\pm_{m,n} \in \mathbb{C},$$

and therefore

$$F(Z)=\sum_{m,n=0}^{+\infty} C_{m,n} Z^m Z^{\dagger n}, \quad \text{with} \quad C_{m,n}=a^+_{m,n}e_+ + a^-_{n,m}e_-.$$

The converse (iii) implies (ii) is clearly immediate. □

Proposition 3.10. *Keep the notations of $H^{(1)}$, $H^{(2)}$, A_k and B_k as above. Then, for any $H\in \mathcal{BH}arm(\mathbb{BC})$, there exists a separate bc-holomorphic function G such that*

$$H^{(1)}(Z|\alpha,\beta)=\sum_{k=0}^{+\infty} Z^k A_k(Z^\dagger,\widetilde{Z})+G(Z) \quad \text{and} \quad H^{(2)}(Z|\alpha,\beta)=\sum_{k=0}^{+\infty} Z^{\dagger k} B_k(Z,Z^*)-G(Z).$$

Proof. For every $Z=\alpha e_+ + \beta e_-$, set

$$G^{(1)}(Z):=H^{(1)}(Z|\alpha,\beta)-\sum_{k=0}^{+\infty} Z^k A_k(Z^\dagger,\widetilde{Z}) \quad \text{and} \quad G^{(2)}(Z):=H^{(2)}(Z|\alpha,\beta)-\sum_{k=0}^{+\infty} Z^{\dagger k} B_k(Z,Z^*).$$

Then, from (3.7) and (3.6), we conclude that $G^{(1)}=-G^{(2)}$. However, since $\partial_Z(A_k)=\partial_{Z^*}(A_k)=0$, $\partial_{Z^\dagger}(B_k)=\partial_{\widetilde{Z}}(B_k)=0$ and $\partial_{Z^*}(H^{(1)}(\cdot|a,b))=\partial_{\widetilde{Z}}(H^{(2)}(\cdot|a,b))=0$, we get $\partial_{Z^*}G=\partial_{Z^*}G^{(1)}=0$ and $\partial_{\widetilde{Z}}G=\partial_{\widetilde{Z}}G^{(1)}=0$. This completes the proof by setting $G:=G^{(1)}=-G^{(2)}\in \ker(\partial_{Z^*})\cap \ker(\partial_{\widetilde{Z}})$. □

4 Concluding remarks

The conclusions of Theorems 3.2, 3.3, and 3.7 remain valid for arbitrary generic product-type domains in $\mathbb{BC}$ without additional assumptions, while Theorems 3.1 and 3.5 remain correct on special product-type domains in $\mathbb{BC}$. In fact, the statements of Theorems 3.5 and 3.7 are both valid on a given $D(0,r_1,r_2)$, where

$$D(Z_0,r_1,r_2):=\{Z\in\mathbb{BC};\ ZZ^*\preceq r_1e_+ + r_2e_-\},$$

for given nonnegative reals r_1 and r_2. Assertion of Theorem 3.5 also holds for arbitrary $D(Z_0,r_1,r_2)$ by imposing

$$\frac{\partial^{m+n+j+k}F}{\partial Z^m Z^{*n} Z^{\dagger j}\widetilde{Z}^k}(Z_0)=0, \tag{4.1}$$

for every positive integers m,n,j and k, as a sufficient condition for a given strongly bc-harmonic bicomplex-valued function F on $D(Z_0,r_1,r_2)$ to be of the form $F=H_0+H_1^*+H_2^\dagger+\widetilde{H}_3$, for some bc-holomorphic functions H_ℓ, $\ell=0,1,2,3$, on $D(Z_0,r_1,r_2)$. Analogously to Theorem 3.1, one asserts that a given bicomplex-valued function F is strongly bc-harmonic on a product domain Ω

if and only if for any $Z_0 \in \Omega$ and any $r_1, r_2 > 0$ such that $\overline{D(Z_0, r_1, r_2)} \subset \Omega$, F can be expanded as

$$F(Z) = \sum_{m,n=0}^{+\infty} a_{m,n}(Z - Z_0)^m (Z - Z_0)^{\dagger n} + b_{m,n}(Z - Z_0)^m \widetilde{(Z - Z_0)}^n + c_{m,n}(Z - Z_0)^{*m}(Z - Z_0)^{\dagger n} + d_{m,n}(Z - Z_0)^{*m}\widetilde{(Z - Z_0)}^n$$

on $D(Z_0, r_1, r_2)$.

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Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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