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A primitive associated with the Cantor–Bendixson derivative on Polish spaces

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ABSTRACT

Given a perfect Polish space X, a compact subset $K \subseteq X$ and a countable ordinal $\alpha < \omega_1$, we show that there exists a compact subset $\widehat{K} \subseteq X$ such that

$$\widehat{K}^{(\alpha)} = K,$$

where $\widehat{K}^{(\alpha)}$ denotes the α-th Cantor–Bendixson derivative of $\widehat{K}$. In other words, every compact subset of a perfect Polish space admits an *α-primitive* with respect to the Cantor–Bendixson derivative. This extends to perfect Polish spaces a result previously known for countable compact subsets of the real line. The proof proceeds in three steps: first, we construct primitives for singletons; then, for countable compact subsets; and finally, for arbitrary compact subsets, using separability of Polish spaces.

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RESUMEN

Dado un espacio polaco perfecto X, un subconjunto compacto $K \subseteq X$ y un ordinal numerable $\alpha < \omega_1$, mostramos que existe un subconjunto compacto $\widehat{K} \subseteq X$ tal que

$$\widehat{K}^{(\alpha)} = K,$$

donde $\widehat{K}^{(\alpha)}$ denota la α-ésima derivada de Cantor–Bendixson de $\widehat{K}$. En otras palabras, todo subconjunto compacto de un espacio polaco perfecto admite una *α-primitiva* con respecto a la derivada de Cantor–Bendixson. Esto extiende a espacios polacos perfectos un resultado conocido previamente para subconjuntos numerables compactos de la línea real. La demostración procede en tres pasos: primero construimos primitivas para singletons; luego, para subconjuntos numerables compactos; y finalmente, para subconjuntos compactos arbitrarios, usando la separabilidad de espacios polacos.

Keywords and Phrases: Cantor–Bendixson derivative, descriptive set theory, Polish spaces, primitive.

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1 Introduction

The Cantor–Bendixson derivative is a classical tool in topology and set theory. It was introduced by Cantor in [4] and later refined by Bendixson. It plays a significant role in several areas, including the study of Boolean frames [3], the structure of compact and σ-compact spaces [9], and applications to the semantics of finite logic programs [5].

In [3], Avilez García shows that the Cantor–Bendixson derivative can be used to measure how "Boolean" a frame is by producing for each element the largest Boolean interval containing it. This viewpoint leads to a tower of derivatives and to relationships between derivatives at different levels, which in turn characterize when a frame admits a Boolean reflection. In model theory, the Cantor–Bendixson rank is closely related to Morley rank in ω-stable theories. Furthermore, the Cantor–Bendixson derivative appears in the characterization of compact and σ-compact spaces and in the construction of co-inductive operators; see, for instance, [9]. In [2], the authors give a necessary and sufficient condition for an ordinal to be a Polish space and describe some properties of the Cantor–Bendixson derivative on compact and countable subsets of a Polish space.

These results illustrate the relevance and versatility of the Cantor–Bendixson derivative in different contexts and motivate further study of its structure and possible inverses.

In [1], Álvarez-Samaniego and Merino proved the existence of a "primitive" on the real line: for every countable compact set $K \subseteq \mathbb{R}$ and every countable ordinal $\alpha < \omega_1$, there exists a compact set $\widehat{K} \subseteq \mathbb{R}$ such that

$$\widehat{K}^{(\alpha)} = K.$$

The compact set $\widehat{K}$ is then called an α-primitive of K. The goal of the present paper is to generalize this result from $\mathbb{R}$ to perfect Polish spaces.

More precisely, we work in a perfect Polish space (X, d) and we prove that, given any compact subset $K \subseteq X$ and any countable ordinal $\alpha < \omega_1$, there exists a compact set $\widehat{K} \subseteq X$ such that $\widehat{K}^{(\alpha)} = K$. Our argument follows the same general strategy as in [1], but several technical modifications are needed to accommodate the more general setting.

The paper is organized as follows. In Section 2, we study the behavior of Cantor–Bendixson derivatives for certain countable unions of compact sets whose derivatives at a fixed level are singletons contained in a family of pairwise disjoint balls. In Section 3, we collect some basic properties of perfect Polish spaces and prove the existence of pairwise disjoint families of open balls around countable discrete subsets. In Section 4, we first construct an α-primitive for any singleton, and then use this to build primitives for countable compact sets and, finally, for arbitrary compact subsets of a perfect Polish space.

Throughout the paper, ω denotes the set of natural numbers, and we identify each $n \in \omega$ with the set $\{0, 1, \dots, n-1\}$. We write ω_1 for the first uncountable ordinal, and we only consider countable ordinals α (*i.e.*, $\alpha < \omega_1$). Let (X, τ) be a topological space, and let $A \subseteq X$. A point $x \in X$ is a *limit point* of A if every neighborhood V of x satisfies

$$(V \setminus \{x\}) \cap A \neq \emptyset.$$

The set of all limit points of A is denoted by A' and is called the *derivative* of A.

Using transfinite recursion, we define the Cantor–Bendixson derivative of a subset of a topological space.

Definition 1.1 (Cantor–Bendixson derivative)**.** *Let A be a subset of a topological space X, and let $\alpha < \omega_1$ be an ordinal. The α-th Cantor–Bendixson derivative of A, denoted by $A^{(\alpha)}$, is defined by*

- $A^{(0)} = A$;
- $A^{(\alpha+1)} = \left(A^{(\alpha)}\right)'$ *for every ordinal* α;
- $A^{(\alpha)} = \bigcap\limits_{\gamma<\alpha} A^{(\gamma)}$ *for every nonzero limit ordinal* α.

Definition 1.2. *A topological space (X, τ) is* metrizable *if there exists a metric d on X such that the topology induced by d coincides with τ. This metric d is called a* compatible metric *for (X, τ).*

Definition 1.3. *A topological space X is called a* Polish space *if it is separable and completely metrizable, that is, if there exists a compatible metric d on X for which (X, d) is complete.*

A subset P of a topological space is called *perfect* if $P' = P$, that is, if every point of P is a limit point of P.

2 The derivative of a countable union

In this section, we study the behavior of Cantor–Bendixson derivatives for certain countable unions of compact sets. We will work in a fixed topological space (X, τ) and consider the collection of countable compact subsets of X.

Definition 2.1. *Let (X, τ) be a topological space. We set*

$$\mathcal{K}_X = \{K \subseteq X : K \textit{ is countable and compact}\}.$$

We begin with the following elementary fact; see, for example, [8].

Proposition 2.2. *Let (X, τ) be a topological space, let $\alpha < \omega_1$ and let $\mathcal{A}$ be an arbitrary family of subsets of X. Then*

$$\bigcup_{A \in \mathcal{A}} A^{(\alpha)} \subseteq \left(\bigcup_{A \in \mathcal{A}} A \right)^{(\alpha)}.$$

In a Hausdorff space, the Cantor–Bendixson derivatives of a closed set form a decreasing family of closed subsets of X (see, for instance, [8]). For countable compact sets, we obtain the following.

Proposition 2.3. *Let (X, τ) be a Hausdorff topological space and let $K \in \mathcal{K}_X$. Then, for every ordinal $\alpha < \omega_1$, the set $K^{(\alpha)}$ belongs to $\mathcal{K}_X$, and the family $\left(K^{(\alpha)}\right)_{\alpha<\omega_1}$ is decreasing: if $\alpha \geq \beta$ then $K^{(\alpha)} \subseteq K^{(\beta)}$.*

Indeed, K is closed and compact, so each $K^{(\alpha)}$ is closed, and the standard transfinite recursion defining the derivatives yields the monotonicity (see [7] for details). Moreover, $K^{(\alpha)} \subseteq K$ for all α, so each $K^{(\alpha)}$ is countable and compact, hence $K^{(\alpha)} \in \mathcal{K}_X$.

Our next goal is to compute the derivative of a special type of countable union. We will work in a metrizable space and consider a convergent sequence $(x_n)_{n\in\omega}$ together with a family of countable compact sets whose derivatives at a fixed level α are the points x_n, and that are contained in pairwise disjoint balls. Under these hypotheses, we can describe all derivatives up to level α of the union of the K_n together with the limit point of the sequence.

The following proposition generalizes a construction that appears in the proof of [1, Theorem 2.1].

Proposition 2.4. *Let (X, d) be a metrizable space, let $\alpha < \omega_1$, and let $(x_n)_{n\in\omega} \in X^\omega$ be a sequence satisfying:*

(i) *$(x_n)_{n\in\omega}$ converges to a point $x \in X$;*

(ii) *there exists a family $(K_n)_{n\in\omega}$ in $\mathcal{K}_X$ such that $K_n^{(\alpha)} = \{x_n\}$ and*

$$K_n \subseteq B(x_n, r_n)$$

for all $n \in \omega$, where $(B(x_n, r_n))_{n\in\omega}$ is a family of pairwise disjoint open balls.

Set

$$K = \bigcup_{n\in\omega} K_n \cup \{x\}.$$

Then, for every $\beta \leq \alpha$, we have

$$K^{(\beta)} = \bigcup_{n\in\omega} K_n^{(\beta)} \cup \{x\}.$$

Proof. Let $\beta \leq \alpha$. Using Proposition 2.2 and the monotonicity of the derivative with respect to the inclusion (see [8]), we obtain

$$\bigcup_{n\in\omega} K_n^{(\beta)} \subseteq \left(\bigcup_{n\in\omega} K_n\right)^{(\beta)} \subseteq \left(\bigcup_{n\in\omega} K_n \cup \{x\}\right)^{(\beta)} = K^{(\beta)}.$$

On the other hand, by Proposition 2.3 and the monotonicity of the derivative with respect to the inclusion, we have

$$\{x_n : n \in \omega\} = \bigcup_{n\in\omega} K_n^{(\alpha)} \subseteq \bigcup_{n\in\omega} K_n^{(\beta)} \subseteq K^{(\beta)}.$$

Since $(x_n)_{n\in\omega}$ converges to x and $x_n \in K^{(\beta)}$ for all n, it follows that x is a limit point of $K^{(\beta)}$, so

$$x \in K^{(\beta+1)} \subseteq K^{(\beta)}.$$

Thus, for every $\beta \leq \alpha$,

$$\bigcup_{n\in\omega} K_n^{(\beta)} \cup \{x\} \subseteq K^{(\beta)}.$$

It remains to prove the reverse inclusion. We again proceed by transfinite induction on β.

- For $\beta = 0$, the result is immediate. Assume $\beta < \alpha$ and

  $$K^{(\beta)} \subseteq \bigcup_{n\in\omega} K_n^{(\beta)} \cup \{x\}. \tag{2.1}$$

  Let $z \in K^{(\beta+1)} = \left(K^{(\beta)}\right)'$. Then $z \in K^{(\beta)}$, and by (2.1) either $z = x$ or there exists $M \in \omega$ such that

  $$z \in K_M^{(\beta)} \subseteq B(x_M, r_M).$$

  If $z = x$, then $z \in \bigcup_{n\in\omega} K_n^{(\beta+1)} \cup \{x\}$ and we are done. Suppose $z \neq x$. We necessarily have $z \in K_M^{(\beta+1)}$; otherwise, z would be an isolated point of $K_M^{(\beta)}$, so there exists $\varepsilon_1 > 0$ such that

  $$B(z, \varepsilon_1) \cap K_M^{(\beta)} = \{z\}. \tag{2.2}$$

  Let

  $$\varepsilon = \min\{\varepsilon_1,\ r_M - d(z, x_M),\ d(x, z)\}.$$

  Then $B(z, \varepsilon) \subseteq B(x_M, r_M)$, and since the balls $B(x_n, r_n)$ are pairwise disjoint and $K_n \subseteq B(x_n, r_n)$ for all n, we have

  $$B(z, \varepsilon) \cap \left(\bigcup_{m\neq M} K_m^{(\beta)}\right) = \emptyset. \tag{2.3}$$

Combining (2.2) and (2.3), we obtain

$$\{z\} = B(z,\varepsilon) \cap \left(K_M^{(\beta)} \cup \bigcup_{m \neq M} K_m^{(\beta)} \cup \{x\} \right) = B(z,\varepsilon) \cap \left(\bigcup_{n\in\omega} K_n^{(\beta)} \cup \{x\} \right) = B(z,\varepsilon) \cap K^{(\beta)},$$

which contradicts $z \in \left(K^{(\beta)}\right)' = K^{(\beta+1)}$. Hence $z \in K_M^{(\beta+1)}$, and therefore

$$K^{(\beta+1)} \subseteq \bigcup_{n\in\omega} K_n^{(\beta+1)} \cup \{x\}.$$

- Let $\gamma \leq \alpha$ be a nonzero limit ordinal, and suppose that

$$K^{(\delta)} \subseteq \bigcup_{n\in\omega} K_n^{(\delta)} \cup \{x\},$$

for all $\delta < \gamma$. Let $z \in K^{(\gamma)}$; using the induction hypothesis, we have that

$$K^{(\gamma)} = \bigcap_{\delta<\gamma} K^{(\delta)} \subseteq \bigcap_{\delta<\gamma} \left(\bigcup_{n\in\omega} K_n^{(\delta)} \cup \{x\} \right).$$

Then, for every $\delta < \gamma$, either $z = x$ or there exists $m \in \omega$ such that $z \in K_m^{(\delta)}$.

If $z = x$, we are done. Otherwise, $z \neq x$ and for $\delta = 0$ there exists $M \in \omega$ with

$$z \in K_M^{(0)} = K_M \subseteq B(x_M, r_M).$$

We claim that $z \in K_M^{(\delta)}$ for every $\delta < \gamma$. Suppose not, and let $\delta_0 < \gamma$ be minimal such that $z \notin K_M^{(\delta_0)}$. Thus, since $z \in K^{(\lambda)}$, $z \neq x$ and $z \notin K_M^{(\delta_0)}$, for δ_0, there exists $m_0 \in \omega$ with $m_0 \neq M$ such that

$$z \in K_{m_0}^{(\delta_0)} \subseteq K_{m_0} \subseteq B(x_{m_0}, r_{m_0}).$$

Thus $B(x_M, r_M) \cap B(x_{m_0}, r_{m_0}) \neq \emptyset$, contradicting the fact that the balls $B(x_n, r_n)$ are pairwise disjoint. Hence

$$z \in \bigcap_{\delta<\gamma} K_M^{(\delta)} = K_M^{(\gamma)}.$$

We conclude that

$$K^{(\gamma)} \subseteq \bigcup_{n\in\omega} K_n^{(\gamma)} \cup \{x\}.$$

Combining the three steps, we obtain

$$K^{(\beta)} = \bigcup_{n\in\omega} K_n^{(\beta)} \cup \{x\},$$

for every $\beta \leq \alpha$, as required. □

The previous result suggests replacing the singleton $\{x\}$ by a more general countable compact set. This leads to the following variant.

Proposition 2.5. *Let (X, d) be a metrizable space, let $\alpha < \omega_1$ and let $K \subseteq X$ be a closed set satisfying:*

(i) $K \setminus K' = \{x_n : n \in \omega\}$ is infinite and the points x_n are pairwise distinct;

(ii) there exists a family $(K_n)_{n\in\omega}$ in $\mathcal{K}_X$ such that $K_n^{(\alpha)} = \{x_n\}$ and

$$K_n \subseteq B(x_n, r_n)$$

for all $n \in \omega$, where $(B(x_n, r_n))_{n\in\omega}$ is a family of pairwise disjoint open balls.

Set

$$\widehat{K} = \bigcup_{n\in\omega} K_n \cup K.$$

Then, for every $\beta \leq \alpha$, we have

$$\widehat{K}^{(\beta)} = \bigcup_{n\in\omega} K_n^{(\beta)} \cup K.$$

Proof. We first prove that $K \subseteq \widehat{K}^{(\beta)}$ for every $\beta \leq \alpha$, using transfinite induction.

- For $\beta = 0$, the result is immediate.
- Assume that $K \subseteq \widehat{K}^{(\beta)}$ for some $\beta < \alpha$. Then $K' \subseteq \widehat{K}^{(\beta+1)}$, and, using (ii),

  $$K \setminus K' = \bigcup_{n\in\omega} \{x_n\} = \bigcup_{n\in\omega} K_n^{(\alpha)} \subseteq \bigcup_{n\in\omega} K_n^{(\beta+1)} \subseteq \widehat{K}^{(\beta+1)}.$$

  Thus

  $$K = K' \cup (K \setminus K') \subseteq \widehat{K}^{(\beta+1)}.$$

- Let $\gamma \leq \alpha$ be a nonzero limit ordinal, and suppose

  $$K \subseteq \widehat{K}^{(\delta)},$$

  for all $\delta < \gamma$. Then

  $$K \subseteq \bigcap_{\delta<\gamma} \widehat{K}^{(\delta)} = \widehat{K}^{(\gamma)}.$$

Hence $K \subseteq \widehat{K}^{(\beta)}$ for every $\beta \leq \alpha$. On the other hand, for each $\beta \leq \alpha$, Proposition 2.2 yields

$$\bigcup_{n\in\omega} K_n^{(\beta)} \subseteq \left(\bigcup_{n\in\omega} K_n\right)^{(\beta)} \subseteq \left(\bigcup_{n\in\omega} K_n \cup K\right)^{(\beta)} = \widehat{K}^{(\beta)}.$$

Combining these inclusions, we obtain

$$\bigcup_{n\in\omega} K_n^{(\beta)} \cup K \subseteq \widehat{K}^{(\beta)}, \quad \text{for all } \beta \leq \alpha.$$

To prove the opposite inclusion, we again use transfinite induction on β.

- For $\beta = 0$, again, the result is immediate.
- Assume that $\beta < \alpha$ and
$$\widehat{K}^{(\beta)} \subseteq \bigcup_{n\in\omega} K_n^{(\beta)} \cup K. \tag{2.4}$$
Let $x \in \widehat{K}^{(\beta+1)} = \left(\widehat{K}^{(\beta)}\right)'$. Then $x \in \widehat{K}^{(\beta)}$, so by (2.4) either $x \in K$ or there exists $M \in \omega$ with
$$x \in K_M^{(\beta)} \subseteq B(x_M, r_M).$$
If $x \in K$, there is nothing to prove. Suppose $x \notin K$. If $x \notin K_M^{(\beta+1)} = \left(K_M^{(\beta)}\right)'$, then x is an isolated point of $K_M^{(\beta)}$, and there exists $\varepsilon_1 > 0$ such that
$$B(x, \varepsilon_1) \cap K_M^{(\beta)} = \{x\}. \tag{2.5}$$
Since K is closed and $x \notin K$, we have
$$d(x, K) := \inf\{d(x, z) : z \in K\} > 0.$$
Let
$$\varepsilon = \min\{\varepsilon_1,\ r_M - d(x, x_M),\ d(x, K)\}.$$
Then $B(x, \varepsilon) \subseteq B(x_M, r_M)$ and
$$B(x, \varepsilon) \cap \Big(\bigcup_{m\neq M} K_m^{(\beta)}\Big) = \emptyset. \tag{2.6}$$

Moreover, by the definition of ε, $B(x,\varepsilon)\cap K=\emptyset$. Combining these facts with (2.5) gives

$$\{x\} = B(x,\varepsilon)\cap\left(K_M^{(\beta)}\cup\bigcup_{m\neq M}K_m^{(\beta)}\cup K\right) = B(x,\varepsilon)\cap\left(\bigcup_{n\in\omega}K_n^{(\beta)}\cup K\right) = B(x,\varepsilon)\cap\widehat{K}^{(\beta)},$$

which contradicts $x\in\left(\widehat{K}^{(\beta)}\right)'=\widehat{K}^{(\beta+1)}$. Hence $x\in K_M^{(\beta+1)}$, and therefore

$$\widehat{K}^{(\beta+1)}\subseteq\bigcup_{n\in\omega}K_n^{(\beta+1)}\cup K.$$

- Let $\gamma\leq\alpha$ be a nonzero limit ordinal, and suppose

$$\widehat{K}^{(\delta)}\subseteq\bigcup_{n\in\omega}K_n^{(\delta)}\cup K, \tag{2.7}$$

for all $\delta<\gamma$. Let $z\in\widehat{K}^{(\gamma)}=\bigcap_{\delta<\gamma}\widehat{K}^{(\delta)}$. By (2.7), for each $\delta<\gamma$, either $z\in K$ or there exists $m\in\omega$ such that $z\in K_m^{(\delta)}$.

If $z\in K$, we are done. Otherwise, $z\notin K$ and there exists $M\in\omega$ such that $z\in K_M^{(0)}=K_M\subseteq B(x_M,r_M)$. We claim that $z\in K_M^{(\delta)}$ for all $\delta<\gamma$. If not, let $\delta_0<\gamma$ be minimal with $z\notin K_M^{(\delta_0)}$. Then, for δ_0, there exists $m_0\in\omega$ with $m_0\neq M$ such that

$$z\in K_{m_0}^{(\delta_0)}\subseteq K_{m_0}\subseteq B(x_{m_0},r_{m_0}),$$

so $B(x_M,r_M)\cap B(x_{m_0},r_{m_0})\neq\emptyset$, contradicting that the balls are pairwise disjoint. Hence

$$z\in\bigcap_{\delta<\gamma}K_M^{(\delta)}=K_M^{(\gamma)}.$$

Thus

$$\widehat{K}^{(\gamma)}\subseteq\bigcup_{n\in\omega}K_n^{(\gamma)}\cup K.$$

This completes the induction and the proof. □

3 Some properties of perfect Polish spaces

In this section, we collect several standard facts about perfect Polish spaces that will be needed later. We also construct, for each countable discrete subset, a family of pairwise disjoint open balls, as required in Propositions 2.4 and 2.5.

The next lemma expresses the fact that in a perfect metric space, every point is the limit of a sequence of distinct points.

Proposition 3.1. *Let (X, d) be a Polish space and let $A \subseteq X$ be perfect. Then, for every $x \in A$, there exists a sequence $(x_n)_{n\in\omega} \in (A \setminus \{x\})^\omega$ such that $(x_n)_{n\in\omega}$ converges to x.*

The next statements are well-known, and their proofs are straightforward, so we omit them. Proposition 3.2 follows from the fact that every non-empty open subset of a perfect Polish space is uncountable. Proposition 3.3 is a standard consequence of basic properties of compact countable sets. Finally, Lemma 3.4 is the classical construction of pairwise disjoint balls around a countable discrete set in a metric space.

Proposition 3.2. *Let (X, d) be a Polish space and let $K \in \mathcal{K}_X$. Then K is not perfect.*

Proposition 3.3. *Let (X, d) be a Polish space and let $K \in \mathcal{K}_X$ be infinite. Then K has infinitely many isolated points.*

We now construct disjoint balls around a countable discrete subset. This is a standard fact; we provide a version convenient for our purposes.

Lemma 3.4. *Let (X, d) be a metric space and let $A \subseteq X$ be a countable, infinite, discrete set. Write $A = \{x_n : n \in \omega\}$ with $x_n \neq x_m$ for $n \neq m$. Then there exists a sequence of positive real numbers $(r_n)_{n\in\omega}$ such that*

$$\{B(x_n, r_n)\}_{n\in\omega}$$

is a family of pairwise disjoint open balls. Moreover, we may additionally require $r_n \leq \dfrac{1}{n+1}$ for all $n \in \omega$.

4 Existence of primitives

In this section, we prove the main results of the paper. We begin by constructing, for each point x lying in an appropriate Cantor–Bendixson derivative of a Polish space and each ordinal $\alpha < \omega_1$, an α-primitive contained in an arbitrarily small ball around x. We then use this construction to obtain primitives for countable compact sets, and finally for all compact subsets of a perfect Polish space.

Although the main theorem could be stated and proved in a single step, we intentionally develop the argument in three stages: first for singletons, then for countable compact sets, and finally for arbitrary compact sets. This stepwise organization reflects the way the construction naturally extends from isolated points to countable compact sets and then to general compact subsets.

4.1 Primitives for singletons

We begin with the key step: primitives for singletons.

Proposition 4.1. *Let (X, d) be a Polish space, let $\alpha < \omega_1$, let $x \in X^{(\alpha)}$ and let $r > 0$. Then there exists $K \in \mathcal{K}_X$ such that*

$$K \subseteq B(x, r) \quad \textit{and} \quad K^{(\alpha)} = \{x\}.$$

Proof. We proceed by transfinite induction on $\alpha < \omega_1$.

- If $\alpha = 0$, it suffices to take $K = \{x\}$. Then $K \subseteq B(x, r)$ and $K^{(0)} = K = \{x\}$.
- Assume that for some $\alpha < \omega_1$ the following statement holds: for every $\widetilde{x} \in X^{(\alpha)}$ and every $\widetilde{r} > 0$, there exists $\widetilde{K} \in \mathcal{K}_X$ such that

  $$\widetilde{K} \subseteq B(\widetilde{x}, \widetilde{r}) \quad \text{and} \quad \widetilde{K}^{(\alpha)} = \{\widetilde{x}\}.$$

  We prove the statement for $\alpha + 1$.

  Let $x \in X^{(\alpha+1)}$ and $r > 0$. Since $x \in X^{(\alpha+1)}$, by Proposition 3.1 there exists a sequence $(x_n)_{n\in\omega} \in \left(X^{(\alpha)} \setminus \{x\}\right)^{\omega}$ converging to x. Using Lemma 3.4, we may choose a family of pairwise disjoint balls $\{B(x_n, r_n)\}_{n\in\omega}$ with $r_n \leq 1/(n + 1)$ for all n.

  Since $x_n \to x$, there exists $N \in \omega$ such that, for all $n > N$,

  $$d(x_n, x) < \frac{r}{2} \quad \text{and} \quad \frac{1}{n + 1} < \frac{r}{2}.$$

  If $n > N$ and $z \in B(x_n, r_n)$, then

  $$d(z, x) \leq d(z, x_n) + d(x_n, x) < r_n + \frac{r}{2} \leq \frac{1}{n + 1} + \frac{r}{2} < r,$$

  so $B(x_n, r_n) \subseteq B(x, r)$ for all $n > N$.

  For each $m \in \omega$, apply the induction hypothesis to $x_m \in X^{(\alpha)}$ and $r_m > 0$ to obtain $K_m \in \mathcal{K}_X$ such that

  $$K_m \subseteq B(x_m, r_m) \quad \text{and} \quad K_m^{(\alpha)} = \{x_m\}.$$

  Define

  $$K = \bigcup_{m>N+1} K_m \cup \{x\}.$$

We now verify the required properties.

- $K \subseteq B(x,r)$. For $m > N+1$, $K_m \subseteq B(x_m, r_m) \subseteq B(x,r)$, and clearly $\{x\} \subseteq B(x,r)$, so $K \subseteq B(x,r)$.
- *K is countable and compact.* The set K is a countable union of countable sets, so it is countable. To see that K is compact, it suffices (in a metric space) to show that K is sequentially compact.

  Let $(z_k)_{k\in\omega} \in K^\omega$ be a sequence in K and set $S = \{z_k : k \in \omega\}$. We distinguish three cases.

  (a) If $S \cap K$ is infinite, then there is a subsequence of $(z_k)_{k\in\omega}$ contained in K, and since K is compact, this subsequence has a further convergent subsequence.

  (b) If there exists $m \in \omega$ such that $S \cap K_m$ is infinite, then a subsequence of $(z_k)_{k\in\omega}$ lies in K_m, and since K_m is compact, it has a convergent subsequence.

  (c) Suppose that $S \cap K$ is finite and $S \cap K_n$ is finite for every $n \in \omega$. Then the set

  $$I = \{n \in \omega : S \cap K_n \neq \emptyset\}$$

  is infinite. For $N, M \in \omega$, set

  $$S_M = \{z_k : k > M\} \quad \text{and} \quad I(N,M) = \{n \in \omega : S_M \cap K_n \neq \emptyset,\ n > N\}.$$

  A simple counting argument shows that $I(N,M)$ is infinite for every $N, M \in \omega$ (otherwise I would be finite). Using this, we recursively define strictly increasing functions $\sigma, \psi \colon \omega \to \omega$ as follows:

  $$\sigma(0) = \min I, \qquad \psi(0) = \min\{k \in \omega : z_k \in K_{\sigma(0)}\},$$

  and, for $m \in \omega$,

  $$\sigma(m+1) = \min I(\sigma(m), \psi(m)),$$

  $$\psi(m+1) = \min\{k \in \omega : k > \psi(m) \text{ and } z_k \in K_{\sigma(m+1)}\}.$$

  Then ψ is strictly increasing and $z_{\psi(m)} \in K_{\sigma(m)} \subseteq B(x_{\sigma(m)}, r_{\sigma(m)})$ for all $m \in \omega$. Hence

  $$d\big(z_{\psi(m)}, x_{\sigma(m)}\big) < r_{\sigma(m)} \leq \frac{1}{\sigma(m)+1}.$$

  Since $\big(x_{\sigma(m)}\big)_{n\in\omega}$ is a sequence in the compact set K (recall that each $x_n \in X^{(\alpha)} \subseteq X$ and later we will use them in a compact set), it has a convergent subsequence $\big(x_{\sigma(\varphi(m))}\big)_{n\in\omega}$. The corresponding subsequence $\big(z_{\psi(\varphi(m))}\big)_{n\in\omega}$ is then Cauchy and convergent in X. Its limit belongs to K by closedness. Thus $(z_k)_{k\in\omega}$ has a convergent subsequence.

  In all cases, K is sequentially compact; it is compact.

  - $K^{(\alpha+1)} = \{x\}$. Since $K_m^{(\alpha)} = \{x_m\}$ for each m and the balls $B(x_m, r_m)$ are pairwise disjoint, Proposition 2.4 applied to the sequence $(x_m)_{n\in\omega}$ and the family $\{K_m\}_{m>N+1}$ yields

$$K^{(\alpha)} = \left(\bigcup_{m>N+1} K_m \cup \{x\} \right)^{(\alpha)} = \bigcup_{m>N+1} K_m^{(\alpha)} \cup \{x\} = \{x_m : m > N+1\} \cup \{x\}.$$

    Since $x_m \to x$, the only limit point of $K^{(\alpha)}$ is x, so

$$K^{(\alpha+1)} = \left(K^{(\alpha)}\right)' = \{x\}.$$

- Let $\lambda < \omega_1$ be a nonzero limit ordinal, and assume the claim holds for all $\rho < \lambda$. Let $x \in X^{(\lambda)}$ and $r > 0$. Choose a strictly increasing sequence of ordinals $(\rho_m)_{m\in\omega}$ with $\rho_m < \lambda$ for all m and $\sup\limits_{m\in\omega} \rho_m = \lambda$.

  Since $x \in X^{(\lambda)}$, for each $m \in \omega$ we may choose a point

$$x_m \in X^{(\rho_m)} \setminus \{x\}$$

  such that $x_m \to x$ and $x_m \in B(x, r)$. Applying Lemma 3.4 and shrinking the radii if necessary, we obtain a family of pairwise disjoint open balls $\{B(x_m, r_m)\}_{m\in\omega}$ such that, for some $N \in \omega$, we have $B(x_m, r_m) \subseteq B(x, r)$ whenever $m > N$.

  For each m we apply the induction hypothesis (at level ρ_m) to x_m and $r_m > 0$, obtaining $K_m \in K_X$ such that

$$K_m \subseteq B(x_m, r_m) \quad \text{and} \quad K_m^{(\rho_m)} = \{x_m\}.$$

  Set

$$K = \bigcup_{m>N+1} K_m \cup \{x\}.$$

  As in the successor step, K is countable, contained in $B(x, r)$, and compact.

  For each m, the set $K_m^{(\lambda)}$ is empty, since $\rho_m + 1 < \lambda$ and the derivatives form a decreasing family. Applying Proposition 2.4 with the sequence $(x_m)_{n\in\omega}$ and the family $\{K_m\}_{m>N+1}$, we obtain

$$K^{(\lambda)} = \left(\bigcup_{m>N+1} K_m \cup \{x\} \right)^{(\lambda)} = \bigcup_{m>N+1} K_m^{(\lambda)} \cup \{x\} = \{x\}.$$

This completes the induction and the proof. □

4.2 Primitives for countable compact sets

We now use the previous proposition to construct primitives for countable compact subsets of a perfect Polish space.

Theorem 4.2. *Let* (X, d) *be a perfect Polish space, let* $K \in \mathcal{K}_X$*, and let* $\alpha < \omega_1$*. Then there exists* $\widehat{K} \in \mathcal{K}_X$ *such that*

$$\widehat{K}^{(\alpha)} = K.$$

Proof. If $\alpha = 0$, we may simply take $\widehat{K} = K$, so we assume $\alpha > 0$.

By Proposition 3.2, K is not perfect, so $K' \neq K$. We consider two cases.

Case 1: K **is infinite.** By Proposition 3.3, the set K has infinitely many isolated points, so $K \setminus K'$ is infinite. Enumerate

$$K \setminus K' = \{x_n : n \in \omega\}$$

with pairwise distinct x_n.

Each point $x \in K \setminus K'$ is isolated in K, so there exists a neighborhood V of x such that $V \cap K = \{x\}$. In particular,

$$\{x\} = V \cap K \subseteq V \cap (K \setminus K') \subseteq V \cap K = \{x\},$$

so x is also isolated in $K \setminus K'$, and the set $K \setminus K'$ is discrete.

Applying Lemma 3.4 to the discrete set $\{x_n : n \in \omega\}$, we obtain a family of pairwise disjoint open balls $\{B(x_n, r_n)\}_{n \in \omega}$ with $r_n \leq 1/(n+1)$ for all n.

Since X is perfect, then $x_n \in X^{(\alpha)}$ for every $n \in \omega$. For each n, applying Proposition 4.1 to $x_n \in X^{(\alpha)}$ and $r_n > 0$, we obtain $K_n \in \mathcal{K}_X$ such that

$$K_n \subseteq B(x_n, r_n) \quad \text{and} \quad K_n^{(\alpha)} = \{x_n\}.$$

Since the balls $B(x_n, r_n)$ are pairwise disjoint, so are the sets K_n.

Define

$$\widehat{K} = \bigcup_{n \in \omega} K_n \cup K.$$

The set $\widehat{K}$ is countable as a countable union of countable sets. A compactness argument analogous to the one used in the proof of Proposition 4.1 (successor step) shows that $\widehat{K}$ is compact. Hence $\widehat{K} \in \mathcal{K}_X$.

Finally, applying Proposition 2.5 with the family $\{K_n\}$ and the set K, and using that $K_n^{(\alpha)} = \{x_n\}$ for all n, we obtain

$$\widehat{K}^{(\alpha)} = \left(\bigcup_{n\in\omega} K_n \cup K\right)^{(\alpha)} = \bigcup_{n\in\omega} K_n^{(\alpha)} \cup K = \bigcup_{n\in\omega} \{x_n\} \cup K = K.$$

Case 2: K is finite. In this case, $K \setminus K'$ is also finite. We may write

$$K \setminus K' = \{x_n : n \in M\},$$

for some finite subset $M \subseteq \omega$. Since X is perfect, we have $x_n \in X^{(\alpha)}$ for all $n \in M$.

Fix $r > 0$. For each $n \in M$, apply Proposition 4.1 to $x_n \in X^{(\alpha)}$ and $r > 0$ to obtain $K_n \in \mathcal{K}_X$ such that

$$K_n \subseteq B(x_n, r) \quad \text{and} \quad K_n^{(\alpha)} = \{x_n\}.$$

Define

$$\widehat{K} = \bigcup_{n\in M} K_n \cup K.$$

Since M is finite and each K_n and K is compact, $\widehat{K}$ is compact. It is also countable; hence $\widehat{K} \in \mathcal{K}_X$.

We now show that $\widehat{K}^{(\alpha)} = K$. First, observe that $K \subseteq \widehat{K}^{(\beta)}$ for every $\beta \leq \alpha$, by a transfinite induction entirely analogous to the one used in Proposition 2.5. In particular, $K \subseteq \widehat{K}^{(\alpha)}$.

On the other hand, since $\bigcup_{n\in M} K_n^{(\alpha)} = \{x_n : n \in M\}$, we have

$$\widehat{K}^{(\alpha)} = \left(\bigcup_{n\in M} K_n \cup K\right)^{(\alpha)} = \bigcup_{n\in M} K_n^{(\alpha)} \cup K^{(\alpha)} \subseteq \bigcup_{n\in M} \{x_n\} \cup K = K.$$

Combining these inclusions gives $\widehat{K}^{(\alpha)} = K$.

In both cases, we have constructed $\widehat{K} \in \mathcal{K}_X$ such that $\widehat{K}^{(\alpha)} = K$, as required. □

4.3 Primitives for arbitrary compact sets

We finally remove the countability assumption on K.

Lemma 4.3. *Let (X, d) be a Polish space, and let $A \subseteq X$ be such that every point of A is isolated in A. Then A is countable.*

Proof. See, for instance, [6, p. 126]. □

We can now state and prove the main theorem.

Theorem 4.4. *Let (X,d) be a perfect Polish space, let $K \subseteq X$ be compact, and let $\alpha < \omega_1$. Then there exists a compact subset $\widehat{K} \subseteq X$ such that*

$$\widehat{K}^{(\alpha)} = K.$$

Proof. If K is perfect, then $K^{(\alpha)} = K$ for all $\alpha < \omega_1$. Thus, taking $\widehat{K} = K$ gives the result. Hence, we may assume that K is not perfect.

If $\alpha = 0$, we can again take $\widehat{K} = K$, so we assume $\alpha > 0$. Since K is compact and not perfect, the set $K \setminus K'$ consists precisely of the isolated points of K and is therefore countable by Lemma 4.3. We write

$$K \setminus K' = \{x_n : n \in M\},$$

for some (finite or infinite) $M \subseteq \omega$.

We distinguish two cases.

Case 1: M **is infinite.** As in the proof of Theorem 4.2, we may choose the points x_n to be pairwise distinct and apply Lemma 3.4 to obtain a family of pairwise disjoint open balls $\{B(x_n, r_n)\}_{n\in\omega}$ with $r_n \leq 1/(n+1)$ for all n.

Since X is perfect, then $x_n \in X^{(\alpha)}$ for all $n \in \omega$. For each n, applying Proposition 4.1 to x_n and r_n yields $K_n \in \mathcal{K}_X$ such that

$$K_n \subseteq B(x_n, r_n) \quad \text{and} \quad K_n^{(\alpha)} = \{x_n\}.$$

The sets K_n are pairwise disjoint.

Define

$$\widehat{K} = \bigcup_{n\in\omega} K_n \cup K.$$

An argument analogous to the one in the proof of Theorem 4.2 shows that $\widehat{K}$ is compact. Moreover, by the same reasoning as in Case 1 of Theorem 4.2, we obtain

$$\widehat{K}^{(\alpha)} = K.$$

Case 2: M **is finite.** In this case, $K \setminus K'$ is finite, and we may write

$$K \setminus K' = \{x_n : n \in M\},$$

for some finite $M \subseteq \omega$. Since X is perfect, then $x_n \in X^{(\alpha)}$ for all $n \in M$.

Fix $r > 0$. For each $n \in M$, applying Proposition 4.1 with x_n and r, we obtain $K_n \in \mathcal{K}_X$ such that

$$K_n \subseteq B(x_n, r) \quad \text{and} \quad K_n^{(\alpha)} = \{x_n\}.$$

Define

$$\widehat{K} = \bigcup_{n \in M} K_n \cup K.$$

Since M is finite and each K_n and K is compact, the set $\widehat{K}$ is compact. Repeating the argument from Case 2 of Theorem 4.2, we conclude that $\widehat{K}^{(\alpha)} = K$.

In both cases, we obtain a compact subset $\widehat{K} \subseteq X$ such that $\widehat{K}^{(\alpha)} = K$, as desired. □

Although Theorem 4.4 is stated for compact sets, the argument actually extends to arbitrary closed subsets of a perfect Polish space.

Theorem 4.5 (Extension to closed sets)**.** *Let (X, d) be a perfect Polish space, let $F \subseteq X$ be a closed set, and let $\alpha < \omega_1$. Then there exists a closed set $\widehat{F} \subseteq X$ such that*

$$\widehat{F}(\alpha) = F.$$

Indeed, the key ingredient, Proposition 2.5, only requires K to be closed rather than compact. We restrict ourselves to the compact case in order to simplify the management of details in the construction; nevertheless, the same method yields the corresponding result for closed sets without any essential changes.

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