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Series with harmonic numbers and the tail of $\zeta(2)$

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ABSTRACT

In this paper we solve an open problem related to the calculation of a quadratic series and we obtain that

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2 = 6\zeta(3) - \frac{19}{2}\zeta(4) + \frac{5}{2}\zeta(5) + 2\zeta(2)\zeta(3).$$

Also, we calculate the sum of the series involving the tail of $\zeta(2)$ and the square of the nth harmonic number:

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = 2\zeta(2)\zeta(3).$$

RESUMEN

En el presente artículo, resolvemos un problema abierto relacionado al cálculo de una serie cuadrática y obtenemos que

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2 = 6\zeta(3) - \frac{19}{2}\zeta(4) + \frac{5}{2}\zeta(5) + 2\zeta(2)\zeta(3).$$

También calculamos la serie que involucra la cola de $\zeta(2)$ y el cuadrado del $n$ésimo número armónico:

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = 2\zeta(2)\zeta(3).$$

Keywords and Phrases: Abel's summation formula, logarithmic integrals, polylogarithm integrals, quadratic zeta series, harmonic numbers, tail of $\zeta(2)$, Riemann zeta function values.

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1 Introduction

In this paper we solve the open problem **7.117** part (b) (see [11, p. 228]), which is about calculating the quadratic series involving the nth harmonic number $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ and the tail of $\zeta(2)$

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2.$$

In addition, we also calculate the series

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right),$$

which is *new* in the mathematical literature.

Further on we present some formulae and definitions which we shall need in our analysis.

Abel's summation formula ([1, p. 55], [5, Lemma A.1, p. 258], [11, p. 38]) states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of real numbers and $A_n = \sum_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}).$$

We will be using the *infinite version* of this formula

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}). \tag{1.1}$$

The Dilogarithm function $\mathrm{Li}_2(z)$ is defined, for $|z| \leq 1$, by ([13, p. 176])

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-t)}{t}\, \mathrm{d}t.$$

The special case $\mathrm{Li}_2(1) = \zeta(2)$ is worth mentioning.

Also, *the Trilogarithm function* $\mathrm{Li}_3(z)$ is defined by

$$\mathrm{Li}_3(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^3} = \int_0^z \frac{\mathrm{Li}_2(t)}{t}\, \mathrm{d}t, \quad |z| \leq 1.$$

The nth generalized harmonic number of order k is defined by $H_n^{(k)} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k}$, $n \geq 1$, $k \geq 1$.

A classical symmetric summation formula involving generalized harmonic numbers is given by

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} + \sum_{n=1}^{\infty} \frac{H_n^{(q)}}{n^p} = \zeta(p+q) + \zeta(p)\zeta(q), \quad p, q > 1. \tag{1.2}$$

A short proof of this formula is based on applying Abel's summation formula (1.1) to the series $\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}$, with $a_k = \frac{1}{k^q}$ and $b_k = H_k^{(p)}$. We have, since $A_n = H_n^{(q)}$ and $b_n - b_{n+1} = -\frac{1}{(n+1)^p}$, that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} &= \lim_{n\to\infty} H_n^{(q)} H_{n+1}^{(p)} - \sum_{n=1}^{\infty} \frac{H_n^{(q)}}{(n+1)^p} = \zeta(q)\zeta(p) - \sum_{n=1}^{\infty} \frac{H_{n+1}^{(q)} - \frac{1}{(n+1)^q}}{(n+1)^p} \\ &= \zeta(q)\zeta(p) - \sum_{n=1}^{\infty} \frac{H_n^{(q)}}{n^p} + \zeta(p+q),\end{aligned}$$

and the identity (1.2) is proved.

2 The main result

The main result of this paper is the following theorem.

Theorem 2.1. *The following identities hold:*

(a) A linear series with the tail of $\zeta(2)$

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = 2\zeta(2)\zeta(3).$$

(b) A quadratic series with the tail of $\zeta(2)$

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2 = 6\zeta(3) - \frac{19}{2}\zeta(4) + \frac{5}{2}\zeta(5) + 2\zeta(2)\zeta(3).$$

The convergence of series in Theorem 2.1 is based on the behavior of the sequence $(r_n)_{n\geq 1}$, defined by

$$r_n := \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}, \quad n \geq 1.$$

Using Cesàro-Stolz lemma, the 0/0 case ([5, Theorem B.2, p. 265]), [11, p. 11], one can prove that

$$\lim_{n\to\infty} n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = 1. \tag{2.1}$$

This implies that, for large values of n, we have $\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\sim\frac{1}{n}$. Thus,

$$\frac{H_n^2}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)\sim\frac{H_n^2}{n^2}$$

and

$$H_n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2\sim\frac{H_n^2}{n^2},$$

and since $\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}=\frac{17}{4}\zeta(4)$ (see [11, pp. 245–249]), we have that both series in Theorem 2.1 converge.

3 Intermediate results

In this section we prove some lemmas which we shall need in obtaining our main result, *i.e.* Theorem 2.1.

Lemma 3.1. *An Euler sum and a logarithmic integral*

The following identities hold:

(a) $\sum_{n=1}^{\infty}\frac{H_n}{n^4}=3\zeta(5)-\zeta(2)\zeta(3)$;

(b) $\int_0^1\frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x=8\zeta(5)-4\zeta(2)\zeta(3)$.

Proof. We prove Lemma 3.1 by calculating the logarithmic integral $\int_0^1\frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x$ by two different ways.

First, we use the generating function for the nth harmonic number $\sum_{n=1}^{\infty}H_nx^n=-\frac{\ln(1-x)}{1-x}$, for $x\in(-1,1)$ ([11, Problem, 3.63, part (a), p. 93]), combined with the formula $\int_0^1 x^k\ln^3 x\,\mathrm{d}x=-\frac{6}{(k+1)^4}$, $k\geq 0$. We integrate by parts, with $f(x)=\ln^2(1-x)$, $f'(x)=-\frac{2\ln(1-x)}{1-x}$, $g'(x)=\frac{\ln^2 x}{x}$, $g(x)=\frac{1}{3}\ln^3 x$, and we have that

$$\begin{aligned}\int_0^1\frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x&=\frac{1}{3}\ln^2(1-x)\ln^3 x\Big|_0^1+\frac{2}{3}\int_0^1\frac{\ln(1-x)}{1-x}\ln^3 x\,\mathrm{d}x\\&=\frac{2}{3}\int_0^1\frac{\ln(1-x)}{1-x}\ln^3 x\,\mathrm{d}x=-\frac{2}{3}\int_0^1\left(\sum_{n=1}^{\infty}H_nx^n\right)\ln^3 x\,\mathrm{d}x\\&=-\frac{2}{3}\sum_{n=1}^{\infty}H_n\int_0^1 x^n\ln^3 x\,\mathrm{d}x=4\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^4}\end{aligned}$$

$$= 4\sum_{n=1}^{\infty}\frac{H_{n+1}-\dfrac{1}{n+1}}{(n+1)^4} = 4\sum_{n=1}^{\infty}\frac{H_n}{n^4} - 4\zeta(5). \tag{3.1}$$

Second, we calculate the same integral by using the Taylor series expansion of the logarithmic function $-\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$, $-1\le x<1$. It follows that

$$\int_0^1 \frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x = \int_0^1 \frac{\ln^2 x}{x}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{n+m}}{nm}\right)\mathrm{d}x = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\int_0^1 x^{n+m-1}\ln^2 x\,\mathrm{d}x$$
$$= 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm(n+m)^3},$$

since $\int_0^1 x^{k-1}\ln^2 x\,\mathrm{d}x = \frac{2}{k^3}$, $\forall k\in\mathbb{N}=\{1,2,\ldots\}$. A calculation shows that

$$\frac{1}{nm(n+m)^3} = \frac{1}{n^3}\cdot\frac{1}{m(n+m)} - \frac{1}{n^3}\cdot\frac{1}{(n+m)^2} - \frac{1}{n^2}\cdot\frac{1}{(n+m)^3}, \qquad n,m\ge 1.$$

It follows that

$$\int_0^1 \frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x = 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{1}{n^3}\cdot\frac{1}{m(n+m)} - \frac{1}{n^3}\cdot\frac{1}{(n+m)^2} - \frac{1}{n^2}\cdot\frac{1}{(n+m)^3}\right]$$
$$= 2\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{1}{m(n+m)} - 2\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{1}{(n+m)^2} - 2\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{m=1}^{\infty}\frac{1}{(n+m)^3}$$
$$= 2\sum_{n=1}^{\infty}\frac{H_n}{n^4} - 2\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right) - 2\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)$$
$$= 2\sum_{n=1}^{\infty}\frac{H_n}{n^4} - 4\zeta(2)\zeta(3) + 2\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n^3}+\frac{H_n^{(3)}}{n^2}\right) = 2\sum_{n=1}^{\infty}\frac{H_n}{n^4} - 2\zeta(2)\zeta(3) + 2\zeta(5). \tag{3.2}$$

We used in the previous calculations the formula $\sum_{m=1}^{\infty}\frac{1}{m(n+m)} = \frac{H_n}{n}$ and identity (1.2) with $p=2$ and $q=3$. Combining (3.1) and (3.2), we have that the desired results hold and parts (a) and (b) of Lemma 3.1 are proved. ☐

Remark 3.2. *We mention that the linear Euler sum* $\sum_{n=1}^{\infty}\frac{H_n}{n^4} = 3\zeta(5)-\zeta(2)\zeta(3)$ *is a special case of a classical series formula due to Euler, which states that* $2\sum_{k=1}^{\infty}\frac{H_k}{k^n} = (n+2)\zeta(n+1) - \sum_{k=1}^{n-2}\zeta(n-k)\zeta(k+1)$, $n\in\mathbb{N}\setminus\{1\}$. *For reference materials related to this formula the reader is referred to [5, p. 208]. Lemma 3.1 gives another proof of the identity* $\sum_{n=1}^{\infty}\frac{H_n}{n^4} = 3\zeta(5)-\zeta(2)\zeta(3)$, *which we believe is new in the mathematical literature, based on calculating a quadratic logarithmic integral in two different ways.*

Lemma 3.3. *Logarithm and polylogarithm integrals*

The following formulae are valid:

(a) $\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}=-\int_0^1\frac{x^n}{1-x}\ln x\,\mathrm{d}x$, $\forall n\ge 1$;

(b) $\int_0^1 x^{n-1}\mathrm{Li}_2(x)\,\mathrm{d}x=\frac{\zeta(2)}{n}-\frac{H_n}{n^2}$, $\forall n\ge 1$;

(c) $\int_0^1\frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x=2\zeta(2)\zeta(3)-3\zeta(5)$;

(d) $\int_0^1\frac{\mathrm{Li}_2(x)\ln^2 x}{1-x}\,\mathrm{d}x=6\zeta(2)\zeta(3)-11\zeta(5)$;

(e) $\int_0^1\frac{\mathrm{Li}_2(x)\ln^2 x}{x}\,\mathrm{d}x=2\zeta(5)$;

(f) $\int_0^1\frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\,\mathrm{d}x=-2\zeta(2)\zeta(3)+\frac{9}{2}\zeta(5)$.

Proof. *(a)* Using the formula $\int_0^1 x^{n-1}\ln x\,\mathrm{d}x=-\frac{1}{n^2}$, $\forall n\ge 1$, we have that

$$\begin{aligned}\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}&=\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots\\&=-\int_0^1 x^n\ln x\,\mathrm{d}x-\int_0^1 x^{n+1}\ln x\,\mathrm{d}x-\cdots\\&=-\int_0^1\left(x^n+x^{n+1}+\cdots\right)\ln x\,\mathrm{d}x=-\int_0^1\frac{x^n}{1-x}\ln x\,\mathrm{d}x.\end{aligned}$$

(b) We have

$$\begin{aligned}\int_0^1 x^{n-1}\mathrm{Li}_2(x)\,\mathrm{d}x&=\int_0^1 x^{n-1}\left(\sum_{m=1}^{\infty}\frac{x^m}{m^2}\right)\mathrm{d}x=\sum_{m=1}^{\infty}\frac{1}{m^2}\int_0^1 x^{n+m-1}\,\mathrm{d}x\\&=\sum_{m=1}^{\infty}\frac{1}{m^2(n+m)}=\frac{1}{n}\sum_{m=1}^{\infty}\left(\frac{1}{m^2}-\frac{1}{m(n+m)}\right)=\frac{\zeta(2)}{n}-\frac{H_n}{n^2},\end{aligned}$$

since $\sum_{m=1}^{\infty}\frac{1}{m(n+m)}=\frac{H_n}{n}$.

(c) We have, based on part (b), that

$$\begin{aligned}\int_0^1\frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x&=\int_0^1\mathrm{Li}_2(x)\left(\sum_{n=1}^{\infty}\frac{x^{n-1}}{n^2}\right)\mathrm{d}x=\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{n-1}\mathrm{Li}_2(x)\,\mathrm{d}x\\&=\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{\zeta(2)}{n}-\frac{H_n}{n^2}\right)=\zeta(2)\zeta(3)-\sum_{n=1}^{\infty}\frac{H_n}{n^4},\end{aligned}$$

and the result follows based on part (a) of Lemma 3.1.

(d) We calculate the integral by parts, with $f(x) = \mathrm{Li}_2(x)\ln^2 x$, $f'(x) = -\frac{\ln(1-x)}{x}\ln^2 x + \frac{2\mathrm{Li}_2(x)\ln x}{x}$, $g'(x) = \frac{1}{1-x}$, $g(x) = -\ln(1-x)$, and we have that

$$\begin{aligned}\int_0^1 \frac{\mathrm{Li}_2(x)\ln^2 x}{1-x}\,\mathrm{d}x &= -\ln(1-x)\mathrm{Li}_2(x)\ln^2 x\Big|_0^1 \\ &\quad + \int_0^1 \ln(1-x)\left(\frac{2\mathrm{Li}_2(x)\ln x}{x} - \frac{\ln(1-x)}{x}\ln^2 x\right)\mathrm{d}x \\ &= 2\int_0^1 \frac{\mathrm{Li}_2(x)\ln x\ln(1-x)}{x}\,\mathrm{d}x - \int_0^1 \frac{\ln^2(1-x)\ln^2 x}{x}\,\mathrm{d}x \\ &= \left[-\ln x\,\mathrm{Li}_2^2(x)\Big|_0^1 + \int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x\right] - \int_0^1 \frac{\ln^2(1-x)\ln^2 x}{x}\,\mathrm{d}x \\ &= \int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x - \int_0^1 \frac{\ln^2(1-x)\ln^2 x}{x}\,\mathrm{d}x \\ &= 6\zeta(2)\zeta(3) - 11\zeta(5),\end{aligned}$$

where the last equality follows based on part (c) of Lemma 3.3 and part (b) of Lemma 3.1.

(e) We have

$$\int_0^1 \frac{\mathrm{Li}_2(x)\ln^2 x}{x}\,\mathrm{d}x = \int_0^1 \frac{\ln^2 x}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}\right)\mathrm{d}x = \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{n-1}\ln^2 x\,\mathrm{d}x = \sum_{n=1}^{\infty}\frac{2}{n^5} = 2\zeta(5).$$

(f) We need the following Landen formula for the Dilogarithm function Li_2 ([13, entry 10, p. 177])

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \zeta(2) - \ln x\ln(1-x), \quad x \in (0,1). \tag{3.3}$$

We have, based on (3.3), that

$$\begin{aligned}\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\,\mathrm{d}x &= \int_0^1 \frac{\mathrm{Li}_2(x)}{x}\left[\zeta(2) - \ln x\ln(1-x) - \mathrm{Li}_2(x)\right]\mathrm{d}x \\ &= \zeta(2)\int_0^1 \frac{\mathrm{Li}_2(x)}{x}\,\mathrm{d}x - \int_0^1 \frac{\mathrm{Li}_2(x)}{x}\ln x\ln(1-x)\,\mathrm{d}x - \int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x \\ &= \zeta(2)\mathrm{Li}_3(x)\Big|_0^1 - \int_0^1 \frac{\mathrm{Li}_2(x)}{x}\ln x\ln(1-x)\,\mathrm{d}x - \int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x \\ &= \zeta(2)\zeta(3) - \left[-\frac{\ln x}{2}\mathrm{Li}_2^2(x)\,\Bigg|_0^1 + \frac{1}{2}\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x\right] - \int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x \\ &= \zeta(2)\zeta(3) - \frac{3}{2}\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x \\ &= -2\zeta(2)\zeta(3) + \frac{9}{2}\zeta(5),\end{aligned}$$

where the last equality follows from part (c) of Lemma 3.3. □

Lemma 3.4. *The generating function of the sequence* $\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)_{n\geq 1}$.

The following equality holds

$$\sum_{n=1}^{\infty}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)x^n=\frac{\zeta(2)x-\operatorname{Li}_2(x)}{1-x},\quad x\in[-1,1).$$

Proof. We apply formula (1.1), with $a_n=x^n$ and $b_n=\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}$, and we have, since $b_n-b_{n+1}=\frac{1}{(n+1)^2}$ and $A_n=x+x^2+\cdots+x^n=\frac{x-x^{n+1}}{1-x}$, that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)x^n&=\lim_{n\to\infty}\frac{x-x^{n+1}}{1-x}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}\right)\\&\quad+\sum_{n=1}^{\infty}\frac{x-x^{n+1}}{1-x}\cdot\frac{1}{(n+1)^2}=\frac{1}{1-x}\sum_{n=1}^{\infty}\frac{x-x^{n+1}}{(n+1)^2}\\&=\frac{1}{1-x}\sum_{n=1}^{\infty}\frac{x-x^n}{n^2}=\frac{\zeta(2)x-\operatorname{Li}_2(x)}{1-x},\end{aligned}$$

and Lemma 3.4 is proved. ☐

Lemma 3.5. *The following formulae hold:*

Linear and nonlinear Euler sums

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^3}=\frac{\pi^4}{72}=\frac{5}{4}\zeta(4);$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}=\frac{7}{2}\zeta(5)-\zeta(2)\zeta(3).$

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(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)=\zeta(3);$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)=\frac{7}{4}\zeta(4);$

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^2}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)=\zeta(2)\zeta(3)-\zeta(5).$

Proof. *(a)* The proof of this series, due to the famous German mathematician Christian Goldbach, can be found in [11, pp. 239–240].

(b) This nonlinear harmonic series is recorded in [4, p. 24], [10, p. 209], and it also appears as problem **4.23** in [14], with a detailed solution on pages 394–395.

(c) This is problem **3.20** in [5, p. 142], with a solution in [5, p. 178].

(d) This is problem **3.62** in [5, p. 149], whose proof can be found in the same book on pages 211–213. An alternative solution is given in [8]. Another method for proving this equality, which is based on an application of Abel's summation formula, can be found in [6]. We give below a new solution which uses the special generating function given in Lemma 3.4, combined with the formula $\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x = -\frac{H_n}{n}$, $n \in \mathbb{N}$. For a history of this formula and reference materials related to it the reader is referred to [5, p. 206]. We have:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right) &= -\sum_{n=1}^{\infty}\int_0^1 x^{n-1}\ln(1-x)\,\mathrm{d}x\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)\\
&= -\int_0^1 \frac{\ln(1-x)}{x}\sum_{n=1}^{\infty}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)x^n\,\mathrm{d}x\\
&\overset{\text{Lemma 3.4}}{=} -\int_0^1 \frac{x\zeta(2)-\operatorname{Li}_2(x)}{1-x}\cdot\frac{\ln(1-x)}{x}\,\mathrm{d}x\\
&= \int_0^1 \operatorname{Li}_2(x)\frac{\ln(1-x)}{x}\,\mathrm{d}x - \int_0^1 (\zeta(2)-\operatorname{Li}_2(x))\frac{\ln(1-x)}{1-x}\,\mathrm{d}x\\
&= -\frac{\operatorname{Li}_2^2(x)}{2}\bigg|_0^1 - \int_0^1 (\zeta(2)-\operatorname{Li}_2(x))\frac{\ln(1-x)}{1-x}\,\mathrm{d}x\\
&= -\frac{\pi^4}{72} - \left[-\frac{\ln^2(1-x)}{2}(\zeta(2)-\operatorname{Li}_2(x))\bigg|_0^1 + \frac{1}{2}\int_0^1 \frac{\ln^3(1-x)}{x}\,\mathrm{d}x\right]\\
&= -\frac{\pi^4}{72} - \frac{1}{2}\int_0^1 \frac{\ln^3(1-x)}{x}\,\mathrm{d}x = -\frac{\pi^4}{72} - \frac{1}{2}\int_0^1 \frac{\ln^3 x}{1-x}\,\mathrm{d}x\\
&= -\frac{5}{4}\zeta(4) - \frac{1}{2}\int_0^1 \ln^3 x\left(\sum_{n=0}^{\infty} x^n\right)\mathrm{d}x = -\frac{5}{4}\zeta(4) - \frac{1}{2}\sum_{n=0}^{\infty}\int_0^1 x^n\ln^3 x\,\mathrm{d}x\\
&= -\frac{5}{4}\zeta(4) + 3\sum_{n=0}^{\infty}\frac{1}{(n+1)^4} = -\frac{5}{4}\zeta(4) + 3\zeta(4) = \frac{7}{4}\zeta(4).
\end{aligned}$$

Another proof of part (d) is based on part (a) of Lemma 3.3, *i.e.* the identity $\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} = -\int_0^1 \frac{x^n}{1-x}\ln x\,\mathrm{d}x$, $n \geq 1$, combined with the generating function of the sequence $\left(\frac{H_n}{n}\right)_{n\geq 1}$, *i.e.* $\sum_{n=1}^{\infty}\frac{H_n}{n}x^n = \operatorname{Li}_2(x) + \frac{1}{2}\ln^2(1-x)$, $x \in [-1, 1)$ (see [10, entry (25), p. 216]). We leave the details to the interested reader.

(e) This part of the lemma follows from Euler's series $\sum_{n=1}^{\infty}\frac{H_n}{n^2} = 2\zeta(3)$ (see [11, p. 238]) and the identity $\sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^2} = \zeta(2)\zeta(3) + \zeta(5)$ (see [10, Theorem 6, p. 210], [14, Problem 4.25, p. 293]). □

4 Proof of the main result

In this section we prove Theorem 2.1.

Proof. *(a)* We need the generating function of the sequence $\left(\frac{H_n^2}{n}\right)_{n\geq 1}$

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n}x^n=\mathrm{Li}_3(x)-\ln(1-x)\mathrm{Li}_2(x)-\frac{1}{3}\ln^3(1-x),\quad x\in[-1,1).$$

The previous formula is recorded in [15, entry (4.42), p. 401] and it can also be found in an equivalent form in [10, entry (38), p. 222]. Then, we have, based on part (a) of Lemma 3.3, that

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)=-\sum_{n=1}^{\infty}\frac{H_n^2}{n}\int_0^1\frac{x^n}{1-x}\ln x\,\mathrm{d}x$$
$$=-\int_0^1\frac{\ln x}{1-x}\left(\sum_{n=1}^{\infty}\frac{H_n^2}{n}x^n\right)\mathrm{d}x=\int_0^1\frac{\ln x}{1-x}\left(\frac{1}{3}\ln^3(1-x)+\ln(1-x)\mathrm{Li}_2(x)-\mathrm{Li}_3(x)\right)\mathrm{d}x.$$

Let I be the previous integral. We calculate I by parts, with $f(x)=\frac{1}{3}\ln^3(1-x)+\ln(1-x)\mathrm{Li}_2(x)-\mathrm{Li}_3(x)$, $f'(x)=-\frac{\ln^2(1-x)}{1-x}-\frac{\mathrm{Li}_2(x)}{1-x}-\frac{\ln^2(1-x)}{x}-\frac{\mathrm{Li}_2(x)}{x}$, $g'(x)=\frac{\ln x}{1-x}$, $g(x)=\mathrm{Li}_2(1-x)$, and we have that

$$\begin{aligned}I&=\mathrm{Li}_2(1-x)\left[\frac{1}{3}\ln^3(1-x)+\ln(1-x)\mathrm{Li}_2(x)-\mathrm{Li}_3(x)\right]\Bigg|_0^1\\&\quad+\int_0^1\mathrm{Li}_2(1-x)\left[\frac{\ln^2(1-x)}{1-x}+\frac{\mathrm{Li}_2(x)}{1-x}+\frac{\ln^2(1-x)}{x}+\frac{\mathrm{Li}_2(x)}{x}\right]\mathrm{d}x\\&=\int_0^1\frac{\mathrm{Li}_2(1-x)\ln^2(1-x)}{1-x}\mathrm{d}x+\int_0^1\frac{\mathrm{Li}_2(1-x)\mathrm{Li}_2(x)}{1-x}\mathrm{d}x\\&\quad+\int_0^1\frac{\mathrm{Li}_2(1-x)\ln^2(1-x)}{x}\mathrm{d}x+\int_0^1\frac{\mathrm{Li}_2(1-x)\mathrm{Li}_2(x)}{x}\mathrm{d}x\\&=\int_0^1\frac{\mathrm{Li}_2(x)\ln^2 x}{x}\mathrm{d}x+2\int_0^1\frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\mathrm{d}x+\int_0^1\frac{\mathrm{Li}_2(x)\ln^2 x}{1-x}\mathrm{d}x=2\zeta(2)\zeta(3),\end{aligned}$$

where the last equality follows based on parts (d), (e) and (f) of Lemma 3.3.

(b) Let $r_n=\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}$. We calculate the quadratic series $\sum_{n=1}^{\infty}H_n^2r_n^2$ by Abel's summation formula (1.1), with $a_k=H_k^2$ and $b_k=r_k^2$. A calculation shows that $\forall n\geq 1$:

$$A_n=\sum_{k=1}^{n}H_k^2=(n+1)H_n^2-(2n+1)H_n+2n=(n+1)H_{n+1}^2-[2(n+1)+1]H_{n+1}+2(n+1),$$

and we observe that

$$b_n - b_{n+1} = r_n^2 - r_{n+1}^2 = \frac{1}{(n+1)^2}(r_n + r_{n+1}).$$

We have

$$\begin{aligned}
\sum_{n=1}^{\infty} H_n^2 r_n^2 &= \lim_{n\to\infty} A_n r_{n+1}^2 + \sum_{n=1}^{\infty} A_n (r_n^2 - r_{n+1}^2) = \sum_{n=1}^{\infty} A_n (r_n^2 - r_{n+1}^2) \\
&= \sum_{n=1}^{\infty} \left[(n+1)H_{n+1}^2 - [2(n+1)+1]H_{n+1} + 2(n+1)\right] \frac{r_n + r_{n+1}}{(n+1)^2} \\
&= \sum_{n=1}^{\infty} \left[\frac{H_{n+1}^2}{n+1} - \left(\frac{2}{n+1} + \frac{1}{(n+1)^2}\right) H_{n+1} + \frac{2}{n+1}\right] (r_n + r_{n+1}) \\
&= \sum_{n=1}^{\infty} \left[\frac{H_{n+1}^2}{n+1} - \left(\frac{2}{n+1} + \frac{1}{(n+1)^2}\right) H_{n+1} + \frac{2}{n+1}\right] \left(2r_{n+1} + \frac{1}{(n+1)^2}\right) \\
&= \sum_{n=1}^{\infty} \left[\frac{H_n^2}{n} - \left(\frac{2}{n} + \frac{1}{n^2}\right) H_n + \frac{2}{n}\right] \left(2r_n + \frac{1}{n^2}\right) \\
&= 2\sum_{n=1}^{\infty} \frac{H_n^2}{n} r_n + \sum_{n=1}^{\infty} \frac{H_n^2}{n^3} - 4\sum_{n=1}^{\infty} \frac{H_n}{n} r_n - 2\sum_{n=1}^{\infty} \frac{H_n}{n^2} r_n \\
&\quad - 2\sum_{n=1}^{\infty} \frac{H_n}{n^3} - \sum_{n=1}^{\infty} \frac{H_n}{n^4} + 4\sum_{n=1}^{\infty} \frac{r_n}{n} + 2\sum_{n=1}^{\infty} \frac{1}{n^3} \\
&= 6\zeta(3) - \frac{19}{2}\zeta(4) + \frac{5}{2}\zeta(5) + 2\zeta(2)\zeta(3),
\end{aligned}$$

where the last equality follows based on part (a) of Theorem 2.1, part (a) of Lemma 3.1 and Lemma 3.5. We used in the previous calculations that $\lim_{n\to\infty} A_n r_{n+1}^2 = 0$. This can be proved, based on (2.1), as follows

$$\begin{aligned}
\lim_{n\to\infty} A_n r_{n+1}^2 &= \lim_{n\to\infty} \frac{A_n}{(n+1)^2} \cdot \lim_{n\to\infty} ((n+1)r_{n+1})^2 = \lim_{n\to\infty} \frac{A_n}{(n+1)^2} \\
&= \lim_{n\to\infty} \left(\frac{H_n^2}{n+1} - \frac{2n+1}{n+1} \cdot \frac{H_n}{n+1} + \frac{2n}{(n+1)^2}\right) = 0.
\end{aligned}$$

□

5 Concluding remarks

We illustrate the identities given in Theorem 2.1 by some numerical examples given in the next tables.

Let $u_n = \frac{H_n^2}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)$, $n \in \mathbb{N}$. In Theorem 2.1, part (a), we have proved that $\lim_{n\to\infty} \sum_{k=1}^{n} u_k = 2\zeta(2)\zeta(3)$. For the right-hand side we have

$$2\zeta(2)\zeta(3) = 3.954608700\ldots$$

and for the left-hand side we have the numerical results given in Table 1[1].

Table 1: Theorem 2.1, part (a)

n	$\sum_{k=1}^{n} u_k$
100000	$3.952885141\ldots$
200000	$3.953653756\ldots$
300000	$3.953934247\ldots$
400000	$3.954082231\ldots$
500000	$3.954174493\ldots$
600000	$3.954237863\ldots$
700000	$3.954284244\ldots$
800000	$3.954319753\ldots$
900000	$3.954347867\ldots$
1000000	$3.954370713\ldots$

Let $v_n = H_n^2\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2$, $n \in \mathbb{N}$. In Theorem 2.1, part (b), we have proved that $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} v_k = 6\zeta(3) - \frac{19}{2}\zeta(4) + \frac{5}{2}\zeta(5) + 2\zeta(2)\zeta(3)$. For the right-hand side we have

$$6\zeta(3) - \frac{19}{2}\zeta(4) + \frac{5}{2}\zeta(5) + 2\zeta(2)\zeta(3) = 3.477198776\ldots$$

and for the left-hand side we have the numerical results given in Table 2

Table 2: Theorem 2.1, part (b)

n	$\sum_{k=1}^{n} v_k$
100000	$3.475475488\ldots$
200000	$3.476244100\ldots$
300000	$3.476524591\ldots$
400000	$3.476672575\ldots$
500000	$3.476764837\ldots$
600000	$3.476828207\ldots$
700000	$3.476874588\ldots$
800000	$3.476910097\ldots$
900000	$3.476938211\ldots$
1000000	$3.476961057\ldots$

---

[1] For the numerical calculations given in this section we have used Maple 13.

The numerical data from the two tables above illustrate that the sequences $\left(\sum_{k=1}^{n} u_k\right)_{n\in\mathbb{N}}$ and $\left(\sum_{k=1}^{n} v_k\right)_{n\in\mathbb{N}}$ are slowly convergent to their limits.

The calculation of series, linear and nonlinear, involving combination of harmonic numbers and tails of Riemann zeta function values is a relatively new topic in the theory of series. This direction of research has been extensively studied in recent years by Furdui [5, Chapter 3], Furdui and Vălean [7], Hoffman [9], Sîntămărian and Furdui [11, Chapters 2, 3, 4, 7], Somu, Haw, Nguyen and Khanh Tran [12] and Vălean [14, Chapter 4], [15, Chapter 4].

In general, the computation of such series is connected to the evaluation of linear or nonlinear Euler series, *i.e.*, series of the form $\sum_{n=1}^{\infty} \frac{H_n^k}{n^m}$, where $k \geq 1$, $m \geq 2$ are positive integers; see [2–4, 13] and the references therein for more information about the evaluation of such series.

For the numerical calculations given in this section we have used Maple 13.

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