CUBO, A Mathematical Journal
Vol. 28, no. 2, pp. 295–321, May 2026
DOI: 10.56754/0719-0646.2802.295

Applying the Riemann surfaces with extremal configurations of symmetries to the study of the real nerve of the moduli space of Riemann surfaces of odd genera

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ABSTRACT

In this paper we find the necessary and sufficient conditions for the geometrical dimension of the real nerve of the moduli space of Riemann surfaces of odd genus g to be maximal. Furthermore, we prove some properties of Riemann surfaces with extremal configuration of symmetries which lead to the conclusion that certain homology groups of the real nerve $\mathcal{N}_g$ of the moduli space of Riemann surfaces of odd genus g are nontrivial.

RESUMEN

En este artículo, encontramos las condiciones necesarias y suficientes para que la dimensión geométrica del nervio real del espacio de módulos de superficies de Riemann de género g impar sea máxima. Más aún, demostramos algunas propiedades de superficies de Riemann con configuraciones extremas de simetrías, que permiten concluir que ciertos grupos de homología del nervio real $\mathcal{N}_g$ del espacio de módulos de superficies de Riemann de género g impar son no triviales.

Keywords and Phrases: Riemann surface, symmetry of a Riemann surface, real form, automorphisms of Riemann surface, Fuchsian groups, Riemann uniformization theorem, separating symmetry

2020 AMS Mathematics Subject Classification: 30F99, 14H37, 20F.

Published: 18 May, 2026
Accepted: 11 March, 2026
Received: 26 September, 2025

1 Introduction

All Riemann surfaces in this paper are compact. A symmetry of a Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g \geq 2$, where Γ is a Fuchsian surface group, is just an antiholomorphic involution $\sigma \in G = \mathrm{Aut}^{\pm}(X)$. The set of points fixed by σ consists of no more than $g+1$ disjoint simple closed curves called *ovals*, see Harnack [15]. If the set $X \setminus \mathrm{Fix}(\sigma)$ is disconnected, then we call σ to be *separating* and we call it *non-separating* in the other case. In addition, we define the *topological type* of σ to be a symbol $\pm k$, where $k \geq 0$ denotes the number of ovals of σ, and the sign depends on the separability of σ: $+$ for separating, $-$ for a non-separating symmetry.

The study of Riemann surfaces with extremal configuration of symmetries has a long history and focuses on two threads:

(1) studying surfaces with the maximal possible number of nonconjugate symmetries;

(2) studying surfaces with the configuration of k nonconjugate symmetries admitting the maximal possible total number of ovals.

We shall call the surfaces with the first property *s-extremal* and the ones with the second property *o-extremal*. Now the maximal number of conjugacy classes of symmetries with fixed points was established in [2]:

Theorem 1.1. (Bujalance, Gromadzki, Izquierdo [2]) *Let X be a Riemann surface of genus $g \geq 2$ and let us write $g = 2^{r-1}u+1$ with u odd. Then the maximum number of nonconjugate symmetries with fixed points that X admits is 2^{r+1}. Furthermore, this bound is attained if and only if $u \geq 2^{r+1} - 3$.*

Now the case including the fixed-point free symmetries was studied in [1] and gives the same bound on the number of conjugacy classes of symmetries, while the only difference is that we have to require $u \geq r-2$. What is important, the group structure for s-extremal surfaces was established in [14], where it was shown that the group generated by the symmetries must be of the form $G = \mathrm{D}_{2^s} \times \mathbb{Z}_2^r$ for a Riemann surface of genus $g = 2^{r-1}u + 1$, u odd. This information shall be crucial in our investigations.

The topic of surfaces with the maximal total number of ovals was also studied extensively. The results concerning low values of k were obtained by Natanzon in [18–20], where he showed that the bound for such number is given by $2g + 2^{k-1}$ and it is attained for $g \equiv 1 \mod 2^{k-2}$ for $k = 2, 3, 4$. Later it was shown by Singerman in [22], that for arbitrary k there exist infinitely many values of g, for which there exists a Riemann surface of genus g, admitting k non-conjugate symmetries with $2g-2+2^{k-3}(9-k)$ ovals in total. In his work, Singerman also conjectured that this is in fact the best possible bound. This was shown in [9] to be false for $k > 9$ by Gromadzki, who proved that for $k \geq 9$, the maximal possible number of ovals is $2g - 2 + 2^{r-3}(9-k)$, where r denotes the smallest positive integer for which $k \leq 2^{r-1}$, and that this bound is attained for arbitrary $k \geq 9$ for infinitely many values of g. However, the Singerman's bound turned out to be true for $k = 5, 6, 7, 8$ in [11]. Similarly to the case of s-extremal surfaces, the structure of the automorphism group in the o-extremal case was established in [11] for $5 \leq k \leq 8$, in [14] for $k \geq 9$ and in [4] for lower values of k. The group structure is the same as for the s-extremal surfaces, with the group necessarily being abelian for all $k \neq 4, 5$. All these results are crucial for our studies.

The other aspect, that we need to underline is the topic of the real nerve $\mathcal{N}_g$ of the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g. The notion of real nerve in this setting was introduced in [12], where it was studied for even values of g, and then in [13] it was continued for odd values of g. However, by Theorem 1.1 the maximal number of possible symmetries is only 4 for even values of g. This makes the study for odd g much more complicated and in [13] a hypothesis was introduced, concerning the necessary and sufficient condition for the homological genus of $\mathcal{N}_g$ to be maximal. In this paper we prove this hypothesis to be wrong and we propose the new condition and prove it holds. As a byproduct of our studies we give some interesting properties of surfaces with extremal configurations of symmetries, the most important being the one that uses o-extremal surfaces to prove that certain homology groups of $\mathcal{N}_g$ are non-trivial for odd g.

2 Preliminaries

2.1 Non-euclidean crystallographic groups

We shall use the combinatorial approach, based on Riemann uniformization theorem, Fuchsian groups and non-euclidean crystallographic groups. Recall that a *NEC group* is just a discrete and cocompact subgroup of the group $\mathcal{G}$ of isometries of the hyperbolic plane $\mathcal{H}$, including those which reverse orientation, and if such a subgroup contains only orientation preserving isometries, then it is called a *Fuchsian* group. For every NEC group Λ we have the associated *signature*, which determines its algebraic structure. It has the form

$$(h; \pm; [m_1, \ldots, m_r]; \{C_1, \ldots, C_k, (-)^l\}). \tag{2.1}$$

The numbers $m_i \geq 2$ are called the *proper periods*, the non-empty brackets $C_i = (n_{i1}, \ldots, n_{is_i})$, $i = 1, \ldots, k$, or the empty ones $(-)$ are the *period cycles*, the numbers $n_{ij} \geq 2$ are the *link periods* and $h \geq 0$ is said to be the *orbit genus* of Λ. The orbit space $\mathcal{H}/\Lambda$ is a surface of topological genus h, having k boundary components, and being orientable or not according to the sign being $+$ or $-$.

The group with the signature (2.1) has the presentation given by the following generators and relations, where $s_i = 0$ if $i > k$:

$$\begin{cases} \text{generators:} \\ \text{(a)} \quad x_i,\ i = 1, \dots, r, \\ \text{(b)} \quad c_{ij},\ i = 1, \dots, k+l,\ j = 0, \dots, s_i, \\ \text{(c)} \quad e_i,\ i = 1, \dots, k+l, \\ \text{(d)} \quad a_i,\ b_i,\ i = 1, \dots, h \text{ if the sign is } +, \\ \qquad\ d_i,\ i = 1, \dots, h, \text{ if the sign is } -, \\ \text{relations:} \\ \text{(A)} \quad x_i^{m_i} = 1,\ i = 1, \dots, r, \\ \text{(B)} \quad c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1,\ i = 1, \dots, k+l,\ j = 1, \dots, s_i, \\ \text{(C)} \quad c_{is_i} = e_i^{-1} c_{i0} e_i,\ i = 1, \dots, k+l, \\ \text{(D)} \quad x_1 \cdots x_r e_1 \cdots e_{k+l} [a_1, b_1] \cdots [a_h, b_h] = 1, \text{ if the sign is } +, \\ \qquad\ x_1 \cdots x_r e_1 \cdots e_{k+l} d_1^2 \cdots d_h^2 = 1, \text{ if the sign is } -. \end{cases} \tag{2.2}$$

The generators $x_1, \dots, x_r$ are called *canonical elliptic generators*, $e_1, \dots, e_{k+l}$ are called the *canonical connecting generators* and c_{ij} are the *canonical reflections* of Λ.

A Fuchsian group can be regarded as a NEC group with the signature

$$(h; +; [m_1, \dots, m_r]; \{-\}), \tag{2.3}$$

which is usually shortened to $(h; m_1, \dots, m_r)$; a Fuchsian group without proper periods is called a *Fuchsian surface group.* An epimorphism $\theta : \Lambda \to G$, where Λ is a NEC group and G is a finite group, is said to be *smooth* if $\ker \theta$ is a surface group.

Any set of generators of a NEC group satisfying the above relations is called *a canonical set of generators*, and reflections c_{ij-1}, c_{ij} are said to be *consecutive.* Every NEC group has an associated fundamental region, whose hyperbolic area $\mu(\Lambda)$, for a NEC group Λ with signature (2.1), is given by

$$2\pi \left(\varepsilon h + k + l - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k+l} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right) \right) \tag{2.4}$$

where $\varepsilon = 2$ if the sign is $+$ and $\varepsilon = 1$ otherwise.

Now, if Γ is a finite index subgroup in a NEC group Λ, then it is a NEC group itself and the Hurwitz-Riemann formula applied to the covering $\mathcal{H}/\Gamma \to \mathcal{H}/\Lambda$ says:

$$[\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}.$$

Finally it is known that an abstract group with the presentation given by the generators and the relations in (2.2) can be realized as a NEC group Λ with the signature (2.1) if and only if (2.4) is positive.

2.2 Topological type of a symmetry

To retrieve the topological type of a symmetry, one needs to find its number of ovals and determine its separability type. The following two results are essential in solving this problem.

Theorem 2.1 (Gromadzki [8]). *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface whose group of automorphisms is $G = \Lambda/\Gamma$ for some NEC group Λ containing Γ as a normal subgroup and let $\theta : \Lambda \to G$ be the canonical epimorphism. Then the number of ovals of a symmetry σ of X, having fixed points, equals*

$$\sum [C(G, \theta(c_i)) \,:\, \theta(C(\Lambda, c_i))],$$

where C stands for the centralizer and the sum is taken over a set of representatives of all conjugacy classes of canonical reflections whose images under θ are conjugate to σ.

For a symmetry σ we shall denote by $\|\sigma\|$ the number of its ovals. In addition, we denote $\|X\|$ the number of ovals of all non-conjugate symmetries on the surface X.

The centralizers that appear in Theorem 2.1 have been described by Singerman in [21] and Gromadzki in [10] (see also [3] for more explicit explanation) and we shall use the following result, where $*$ stands for free product.

Theorem 2.2. *Let $c_0, c_1, \ldots, c_s, e$ be the system of canonical reflections corresponding to a period cycle $(n_1, \ldots, n_s)$ of a NEC group Λ with signature (2.1). If all n_i are even, then the centralizer $C(\Lambda, c_i)$ equals:*

$$\begin{aligned}
&\langle c_i\rangle \times \left(\langle (c_{i-1}c_i)^{n_i/2}\rangle * \langle (c_i c_{i+1})^{n_{i+1}/2}\rangle\right) = \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2) && \text{for } i \neq 0,\\
&\langle c_0\rangle \times \left(\langle (c_0 c_1)^{n_1/2}\rangle * \langle e^{-1}(c_{s-1}c_s)^{n_s/2} e\rangle\right) = \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2) && \text{for } i = 0,\\
&\langle c_0\rangle \times \langle e\rangle = \mathbb{Z}_2 \times \mathbb{Z} && \text{for } s = 0.
\end{aligned}$$

Now to determine the separability type, one can use the following technique described by Hoare and Singerman in [16]. Let us recall, that if Φ is a set of generators for Λ and $\tilde{\Lambda}$ is a subgroup of Λ, then a *right Schreier transversal S* is a set of words in Φ such that every initial segment of a word in S is also in S and the mapping $a \to \tilde{\Lambda}a$ is a $1-1$ correspondence from S to the cosets of $\tilde{\Lambda}$ in Λ. Now for each $a \in S$ and $\alpha \in \Phi$ there exists a unique $b \in S$ such that $\tilde{\Lambda}b = \tilde{\Lambda}a\alpha$. With these notations, the *Schreier generators* of $\tilde{\Lambda}$ are all the elements of the form $a\alpha b^{-1}$.

If Λ is a group with generators Φ and $\tilde{\Lambda}$ is its subgroup, then the *Schreier coset graph* $\mathcal{K}$ is the graph whose vertices are the cosets of $\tilde{\Lambda}$ in Λ and labelled directed edges at every vertex for each $a \in \Phi$ such that $a : \tilde{\Lambda}\alpha \to \tilde{\Lambda}\alpha a$. Now if a is a reflection and $\tilde{\Lambda}\alpha a = \tilde{\Lambda}a$, then the corresponding directed edge $a : \tilde{\Lambda}\alpha \to \tilde{\Lambda}\alpha a$ is called a *reflection loop*. Let us delete all the reflection loops from $\mathcal{K}$. In such a way we obtain the Schreier graph $\bar{\mathcal{K}}$. Observe, that each path in $\bar{\mathcal{K}}$ corresponds to a word in Φ, and so also to an element of Λ. We shall call a path *positive* if it corresponds to an orientation preserving element of Λ, otherwise we shall call a path *negative*.

We are ready to state the main result of [16]:

Theorem 2.3 (Hoare, Singerman [16])**.** *With the above notations, the following statements are equivalent:*

(i) $X/\tilde{\Lambda}$ *is orientable,*

(ii) the only orientation reversing Schreier generators are involutions (actually conjugations in Φ*),*

(iii) all circuits of $\bar{\mathcal{K}}$ *are positive,*

(iv) the cosets of $\tilde{\Lambda}$ *in* Λ *divide into two disjoint classes such that in the action* $a : \Lambda\gamma \mapsto \Lambda(\gamma a)$ *orientation preserving elements of* Φ *fix the classes and (apart from reflections fixing cosets) orientation reversing elements interchange the classes.*

Now let us assume that we have an epimorphism $\theta : \Lambda \to G$, where Λ is a NEC group and $G \cong \Lambda/\Gamma$ for $\Gamma = \ker\theta$ is the group of all, including orientation reversing, automorphisms of $X = \mathcal{H}/\Gamma$. Let us also assume that $\sigma \in G$ is a symmetry of X. Now let us consider the subgroup $\Gamma_\sigma = \theta^{-1}(\langle\sigma\rangle)$ of Λ. It is a NEC group as the index $[\Lambda : \Gamma_\sigma] < +\infty$. In addition it is a surface group, so its signature has the form

$$(h'; \pm; [-]; \{(-)^k\},$$

which also determines the topological type of the symmetry σ, *i.e.* the sign decides about the separability, as it decides about the orientability of the orbit space $\mathcal{H}/\Gamma_\sigma$. Moreover, the number of period cycles is equal to the number of connected components of the boundary of this space and hence it is equal to the number of ovals of the symmetry σ. We shall use the theorem above taking $\tilde{\Lambda} = \Gamma_\sigma$, which will allow us to determine the separability character of the symmetry σ.

2.3 The real nerve $\mathcal{N}_g$

We have introduced all the data necessary to deal with topological type of the symmetry, let us now focus on the real nerve $\mathcal{N}_g$ of $\mathcal{M}_g$. A smooth, irreducible, real, projective algebraic curve has three important topological invariants: the number of connected components, the algebraic genus being the ordinary genus of its complexification and its separability character in its complexification.

The complexification allows to map such curves of given genus g into the classical moduli space $\mathcal{M}_g$ of smooth, irreducible, complex projective algebraic curves of genus g. The image $\mathcal{M}_g^{\mathbb{R}}$, called the *real locus*, is covered by the subsets $\mathcal{M}_g^{\pm k}$ proceeding from the real algebraic curves with k connected components and given separability, as explained before. Now a subset $\mathcal{M}_g^{\pm k}$ overlaps a subset $\mathcal{M}_g^{\pm k'}$ if and only if there is a complex algebraic curve of genus g having two real forms of the types $\pm k$ and $\pm k'$. In this paper we study the nerve $\mathcal{N}_g$, corresponding to this covering, as in ([23, 3.1.6]), called the *real nerve* of complex algebraic curves of given genus g.

The above covering of the real locus $\mathcal{M}_g^{\mathbb{R}}$ gives rise to the associated nerve $\mathcal{N}_g$, which we call *the real nerve*, being the simplicial complex whose vertices are the topological types $\pm k$. The sequence of distinct types $\left(\pm k_0, \ldots, \pm k_n\right)$ is an n-simplex in $\mathcal{N}_g$ if and only if there exists a Riemann surface X of genus g having $n+1$ symmetries of the types $\pm k_0, \ldots, \pm k_n$.

First of all, $\mathcal{N}_g$ has $[(3g+4)/2]$ vertices, by the mentioned above results of Harnack and Weichold (*c.f.* [6]). By the results of Buser, Seppälä and Silhol [5], $\mathcal{N}_g$ is connected and furthermore it was shown by Costa and Izquierdo in [7], that given g and a type $\pm k$ there exists a Riemann surface X of genus g, having two symmetries σ, τ of the types $\pm k$ and -1 respectively, which means that -1 is a *spine* for $\mathcal{N}_g$ for arbitrary g. In [12, 13] the properties of the nerve were studied, resulting in some answers concerning its geometrical and homological dimension. In this paper we complete the answer concerning geometrical dimension of $\mathcal{N}_g$ for odd genera and we use o-extremal Riemann surfaces to retrieve new information concerning homology groups of $\mathcal{N}_g$.

3 The combinatorial problem

In the last part of the paper [13] the authors have asked a following question:

Problem 1. *Let us consider a number of points situated on a circle, coloured by $k \geq 3$ colours in such a way that no two consecutive points have the same colour. Moreover, we put weights on our points in such a way that the weight is 2 if a point has neighbours with the same colour and the weight is 1 otherwise. Next, for every colour we define its weight as the sum of all the weights of points coloured with it. What is the smallest possible number of points $\varphi(k)$, for which there exists such a colouring and all the colours have distinct weights?*

Figure 1 depicts a toy example of a correctly coloured circle, where the number of colours equals 4. Since it uses 10 points, we can conclude that $\varphi(4) \leq 10$.

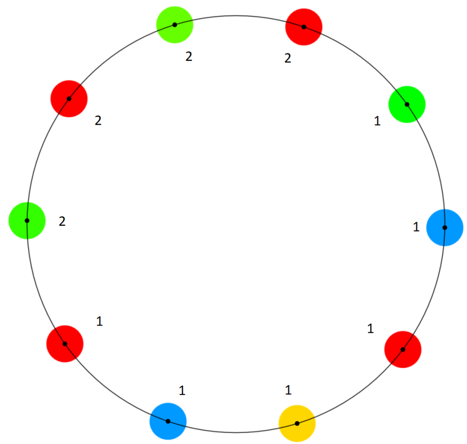

Figure 1: Correctly coloured circle with weights of each point. Weights of colours are as follows: yellow - 1, blue - 2, green - 5, red - 6.

Here, the most important is the case $k = 2^{r+1} - 1$ and it was conjectured that $\varphi(k) = 2^r(2^r + 1) - 1$. However, as we prove below, the conjecture is not true. The problem is closely related to the later part of this article, where we study properties of the real nerve of the moduli space of Riemann surfaces. To state the general solution of the above problem, we need a combinatorial lemma and preceding it a technical definition. The lemma will prove useful in lowerbounding $\varphi(k)$.

Definition 3.1. *Let $k \geqslant 4$ be the number of colours. Consider $s \geqslant k$ points on a circle indexed clockwise from 0 to $s-1$ and coloured with k colours in such a way, that no two neighbour points have the same colour. We say that the colour $\boldsymbol{a}$ satisfies the property $\star$ if there exists an index $j \in \{0, \ldots, s-1\}$, such that the point of index j is coloured with the colour $\boldsymbol{a}$ and the point of index $j+2 \mod s$ is not coloured with it.*

One may think of the property $\star$ in the following, equivalent way - the colour $\boldsymbol{a}$ possesses the property $\star$ if more than a half of points have colour different than $\boldsymbol{a}$.

Lemma 3.2. (see also [17]) *Suppose that $k \geqslant 4$ colours has been used to colour $s \geqslant k$ points on a circle in such a way, that no two neighbour points have the same colour. Suppose also that each colour a_i, $i \in \{1, \ldots, k\}$ satisfies the property $\star$. Then at least k points have neighbours of different colours.*

Proof. Without loss of generality, we can assume that consecutive points are coloured in the following way

$$a_1\ a_2\ a_p\ a_q\ \ldots\ a_r. \tag{3.1}$$

The last point is a neighbour of the first one, since the points are located on a circle and indexed in a clockwise order. Suppose that $i_p \in \{2 \ldots s-1\}$ is the smallest index of a point coloured with the colour a_p, $p \in \{3, \ldots, k\}$. Since it is the first encounter of the colour a_p, then the point of index $i_p - 1$ has neighbours of different colours. In total it gives us $k-2$ points with neighbours of different colours.

Consider now a finite sequence

$$\underbrace{\{a_1, a_2\}}_{1}, \underbrace{\{a_2, a_p\}}_{2}, \underbrace{\{a_p, a_q\}}_{3}, \ldots, \underbrace{\{a_r, a_1\}}_{s}, \underbrace{\{a_1, a_2\}}_{s+1}. \tag{3.2}$$

of unordered pairs associated with the colouring given in equation (3.1). Each pair contains a colour of a point and colour of its consequent. More precisely, i-th pair contains colours of points of indices $i-1 \mod s$ and $i \mod s$. There are $s+1$ such pairs, since the first pair is repeated at the last position.

$$\ldots, \underbrace{\{a_p, a_q\}}_{i}, \underbrace{\{a_q, a_r\}}_{i+1}, \ldots \quad //i, i+1 \text{ denote indices of unordered pairs,}$$

$$\ldots \underbrace{a_p}_{i-1}\ \underbrace{a_q}_{i}\ \underbrace{a_r}_{i+1} \ldots \quad //i-1, i, i+1 \text{ denote indices of points.}$$

Observe that:

(1) Two consecutive pairs have one common element and can differ on the second one,

(2) If $(i+1)$-th pair differs from i-th one, where $i \in \{1, \ldots, s\}$, then the neighbours of the i-th point have different colours,

(3) From the property $\star$ there exists a pair that does not contain the colour a_1 and there exists a pair that does not contain the colour a_2.

Let $i_1 \leq s$ be the smallest index of a pair which does not contain the colour a_1 and $j_1 \leq s+1$ be the smallest, but greater than i_1, index of a pair which contain the colour a_1. Above observation (point 3) and the fact that the last pair is $\{a_1, a_2\}$ guarantees that such indices always exist.

$$\ldots, \underbrace{\{a_1, a_p\}}_{i_1-1}, \underbrace{\{a_p, a_q\}}_{i_1}, \ldots, \underbrace{\{a_u, a_r\}}_{j_1-1}, \underbrace{\{a_r, a_1\}}_{j_1}, \ldots \quad (p, q, u, r \neq 1)$$

$$\ldots \underbrace{a_1}_{i_1-2}\ \underbrace{a_p}_{i_1-1}\ \underbrace{a_q}_{i_1} \ldots \underbrace{a_u}_{j_1-2}\ \underbrace{a_r}_{j_1-1}\ \underbrace{a_1}_{j_1} \ldots \quad (p, q, u, r \neq 1)$$

Now, $j_1 \mod s$ is an index of a point with neighbours of different colours. Clearly it was not taken into account in first $k-2$ such points, since we considered colours different than a_1 and a_2. So we pointed out one more such point. We can repeat above arguments for colour a_2, In total it gives us $k-2+1+1=k$ points with neighbours of different colour. □

In what follows, the following, easy to check, equality will be useful.

$$\sum_{n=1}^{k}\left\lceil \frac{n}{2}\right\rceil = \begin{cases} \dfrac{k^2+2k}{4} & \text{if } 2|k, \\ \\ \dfrac{k^2+2k+1}{4} & \text{if } 2\nmid k. \end{cases} \tag{3.3}$$

Theorem 3.3. *For a number of colours $k \geqslant 4$, the minimal number of points required to satisfy conditions from Problem 1 is equal to*

$$\varphi(k) = \begin{cases} \dfrac{k^2+3k+2}{4} & \textit{if } k \equiv 2 \mod 4 \textit{ or } k \equiv 3 \mod 4, \\ \\ \dfrac{k^2+3k}{4} & \textit{if } k \equiv 0 \mod 4 \textit{ or } k \equiv 1 \mod 4. \end{cases} \tag{3.4}$$

Proof. In the first part of the proof we will construct the required colouring to upper-bound the number $\varphi(k)$. Let us start with a special case, where $k=4$. We use labels a_1, a_2, a_3, a_4 to colour the consecutive 7 points on circle in the following way

$$a_4\ a_2\ a_4\ a_3\ a_4\ a_3\ a_1.$$

Observe, that the weight of colour a_i is i for $i = 1, 2, 3, 4$. Total number of points is 7 which is equal to the postulated value $\dfrac{4^2+3\cdot 4}{4} = 7$.

Now, let $k \geq 5$. The general idea is that we will define blocks of consecutive coloured points which we will use to assign the colour to each point. To keep the notation simple, to this end we will use the following notation

$$(a_ia_j)^n = \underbrace{a_ia_ja_ia_j\cdots a_ia_j}_{n}.$$

We consider the following four cases:

$k \equiv 3 \mod 4$**:** We define the block B_i as follows

$$B_i = (a_{4i+2}\ a_{4i})^{2i}\ a_{4i+2}\ a_{4i+3}\ a_{4i+2}\ (a_{4i+3}\ a_{4i+1})^{2i+1},$$

We use these blocks to compose the following sequence

$$a_3\ a_1\ a_2\ a_3\ a_2\ B_1 \dots B_{\frac{k-3}{4}}.$$

Observe that in each block B_i, weights of colours $a_{4i}, a_{4i+1}, a_{4i+2}, a_{4i+3}$ are $4i, 4i+1, 4i+2, 4i+3$ respectively. The construction has been prepared in such a way that the weight of each colour a_j is equal to j for each $j \in \{1, \dots, n\}$. So weights of each colour are different. It remains to calculate total number of points used in above construction. Each block B_i has exactly $(8i+5)$ points, so the total number of points is

$$5 + \sum_{i=1}^{\frac{k-3}{4}} (8i+5) = 5 + 5 \cdot \frac{k-3}{4} + 8 \cdot \frac{1+\frac{k-3}{4}}{2} \frac{k-3}{4} \frac{5k+5}{4} + \frac{(k-3)(k+1)}{4} = \frac{k^2+3k+2}{4},$$

which equals to the postulated value.

$k \equiv 0 \mod 4$**:** We define blocks C_i as follows

$$C_i = (a_{4i}\ a_{4i-2})^{2i-1}\ a_{4i}\ a_{4i-1}\ a_{4i}\ (a_{4i-1}\ a_{4i-3})^{2i-1}.$$

We use these block to compose the following sequence

$$C_1 \dots C_{\frac{k}{4}}.$$

Observe that in each block C_i, weights of colors $a_{4i-3}, a_{4i-2}, a_{4i-1}, a_{4i}$ are $4i-3, 4i-2, 4i-1, 4i$ respectively. But i goes from 1 to $k/4$, so the weight of color a_j is equal to j for each $j \in \{1, \dots, k\}$. Each block C_i has exactly $2 \cdot (2i-1) + 3 + 2 \cdot (2i-1) = 8i - 1$ points. In total it gives us

$$\sum_{i=1}^{\frac{k}{4}} (8i-1) = 8 \cdot \frac{1+\frac{k}{4}}{2} \frac{k}{4} - \frac{k}{4} = \frac{k(k+4)}{4} - \frac{k}{4} = \frac{k^2+3k}{4},$$

points, which equals to the postulated value.

$k \equiv 1 \mod 4$**:** We define blocks D_i as follows

$$D_i = (a_{4i}\ a_{4i-2})^{2i-1}\ a_{4i}\ a_{4i+1}\ a_{4i}\ (a_{4i+1}\ a_{4i-1})^{2i}.$$

We use these blocks to compose the following sequence

$$a_1\ D_1 \dots D_{\frac{k-1}{4}}.$$

Observe that in each block D_i, weights of colours $a_{4i-2}, a_{4i-1}, a_{4i}, a_{4i+1}$ are $4i-2, 4i-1, 4i, 4i+1$ respectively. But i goes from 1 to $(k-1)/4$, so the weight of colour a_j is equal to j for each $j \in \{2, \ldots, k\}$. The weight of colour a_1 is equal to 1, since it appears on only one point which has neighbours of different colours. Each block D_i consists of $2 \cdot (2i-1) + 3 + 2 \cdot 2i = 8i + 1$ points. In total it gives us

$$1 + \sum_{i=1}^{\frac{k-1}{4}} (8i+1) = 1 + 8 \cdot \frac{1 + \frac{k-1}{4}}{2} \frac{k-1}{4} + \frac{k-1}{4} = \frac{4 + (k+3)(k-1)}{4} + \frac{k-1}{4} = \frac{k^2 + 3k}{4},$$

points, which equals to the postulated value.

$k \equiv 2 \mod 4$: At first, we have to consider a special case, when $k = 6$. The following sequence

$$a_6 \; a_1 \; (a_5 \; a_6)^3 \; (a_3 \; a_4)^2 \; a_2 \; a_4$$

is correct colouring of points on a circle. It is straightforward to check that for each $j \in \{1, 2, 3, 4, 5, 6\}$, the weight of the colour a_j is equal to j. This sequence consists of

$$14 = \frac{6^2 + 3 \cdot 6 + 2}{4}$$

points, which equals to the postulated value.

Now let $k \geq 10$. We define blocks E_i as follows

$$E_i = (a_{4i+2} \; a_{4i})^{2i} \; a_{4i+2} \; a_{4i+1} \; a_{4i+2} \; (a_{4i+1} \; a_{4i-1})^{2i}.$$

We use these blocks to compose the following sequence

$$a_2 \; a_1 \; E_1 \; a_2 \; E_2 \ldots E_{\frac{k-2}{4}}.$$

Observe that in each block E_i, weights of colours $a_{4i-1}, a_{4i}, a_{4i+1}, a_{4i+2}$ are $4i-1, 4i, 4i+1, 4i+2$ respectively. But i goes from 1 to $(k-2)/4$, so the weight of colour a_j is equal to j for each $j \in \{3, \ldots, k\}$. The weights of colours a_1, a_2 are 1 and 2 respectively, since a_1 appears only once and a_2 twice, each time between neighbours of different colours. Each block E_i consists of $2 \cdot 2i + 3 + 2 \cdot 2i = 8i + 3$ points. In total it gives us

$$3 + \sum_{i=1}^{\frac{k-2}{4}} (8i+3) = \frac{3k - 6 + 12}{4} + 8 \cdot \frac{1 + \frac{k-2}{4}}{2} \frac{k-2}{4} = \frac{3k+6}{4} + \frac{(k+2)(k-2)}{4} = \frac{k^2 + 3k + 2}{4},$$

points, which equals to the postulated value.

Above constructions give us an upper bound on $\varphi(k)$. Now we will prove a lower bound which will match it.

Notice that the description of the Problem 1 requires that colours have distinct weights. We start by showing that the choice of weights $\{1, 2, \ldots, k\}$ is optimal. We use the word "optimal" in the sense that if one has the correct colouring of s_1 points on a circle and colours have weights $\{w_1, \ldots, w_k\}$, then there exists a correct colouring of s_2 points on a circle in which weights have colours $\{1, 2, \ldots, k\}$ and $s_2 \leq s_1$. For the sake of clarity, we will assume that the set of weights $\{w_1, \ldots, w_k\}$ is sorted, *i.e.* if $i < j$ then $w_i < w_j$.

Observe that each occurrence of a colour on some point on a circle can add 1 or 2 to the total weight of a colour. So the minimal number of points required to achieve the weight w_i is $\lceil w_i/2 \rceil$ ($w_i = 2 + \ldots + 2$ if $2|w_i$ or $w_i = 2 + \ldots + 2 + 1$ otherwise). To realize all the weights from the set $\{w_1, \ldots, w_k\}$, we will need at least

$$\sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil \tag{3.5}$$

points on a circle. Consider now the special case, where every second point has a colour a_ℓ. Suppose its weight is equal to w_ℓ. The number of points of colour a_ℓ is the same as the number of all other points, so there are at least

$$2 \cdot \left(\sum_{i=1}^{\ell-1} \left\lceil \frac{w_i}{2} \right\rceil + \sum_{i=\ell+1}^{k} \left\lceil \frac{w_i}{2} \right\rceil \right)$$

points on a circle. This sum is minimized for $\ell = k$ and the following weights $\{1, \ldots, k\}$. We can lower-bound this sum

$$2 \cdot \sum_{i=1}^{k-1} \left\lceil \frac{w_i}{2} \right\rceil \geqslant 2 \cdot \sum_{n=1}^{k-1} \left\lceil \frac{n}{2} \right\rceil \geqslant 2 \cdot \frac{(k-1)^2 + 2(k-1)}{4} = 2 \cdot \frac{k^2 - 1}{4} = \frac{k^2 - 1}{2}, \tag{3.6}$$

which for $k \geq 4$ is greater than our upper bound on $\varphi(k)$. It implies that having half of points coloured in the same colour is not an optimal strategy. We can discard it and assume that assumptions of Lemma 3.2 are satisfied.

To achieve minimal number of points, each odd weight has to be composed from only 2s and one 1 and each even weight has to be composed from only 2s. Lemma 3.2 implies that there had to be at least k points with weight 1. Suppose that there are $s \leq k$ odd numbers in the set $\{w_1, \ldots, w_k\}$. Then Lemma 3.2 forces us to add additional $k - s$ points of weight 1. Consider two cases

$2|k - s$: we swap $(k - s)/2$ points of weight 2 into $k - s$ points of weight 1. Then minimal number of points required to realize the weights from the set $\{w_1, \ldots, w_k\}$ is

$$\sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil - \frac{k-s}{2} + k - s = \sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil + \frac{k-s}{2}.$$

Such configuration consists of k points of weight 1 (which is required by Lemma 3.2) and uses minimal number of points required to realize weights from the set $\{w_1, \dots, w_k\}$.

$2 \nmid (k-s)$: we swap $(k-s+1)/2$ points of weight 2 into $k-s+1$ points of weight 1. Then minimal number of points required to realize the weights from the set $\{w_1, \dots, w_k\}$ is

$$\sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil - \frac{k-s+1}{2} + k - s + 1 = \sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil + \frac{k-s+1}{2}.$$

These expressions lower-bound minimal number of points required to realize weights $\{w_1, \dots, w_k\}$. They consist of two factors: main sum and the correction forced by Lemma 3.2. Observe that

- If the weights $\{w_1, \dots, w_k\}$ are not subsequent integer numbers, *i.e.* there exists i such that $w_i + 1 < w_{i+1}$, then there exists a set of weights $\{\tilde{w}_1, \dots, \tilde{w}_k\}$ for which minimal number of points is not greater than for $\{w_1, \dots, w_k\}$. Indeed, if we define $\{\tilde{w}_1, \dots, \tilde{w}_k\}$ as follows

$$\begin{cases} \tilde{w}_j = w_j & \text{if } j \neq i+1 \\ \tilde{w}_{i+1} = w_i + 1 & \end{cases},$$

  then three cases can occur:

  - $\tilde{w}_{i+1}$ and w_{i+1} have the same parity, *i.e.* both are even or both are odd and $w_{i+1} - \tilde{w}_{i+1} \geq 2$. Then the penalty term does not change and the following inequality holds

$$\sum_{i=1}^{k} \left\lceil \frac{\tilde{w}_i}{2} \right\rceil + 2 \leqslant \sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil.$$

    The penalty term will not increase, so the new set of weights gives us lower number of points.

  - $\tilde{w}_{i+1}$ is odd and w_{i+1} is even. Surely

$$\sum_{i=1}^{k} \left\lceil \frac{\tilde{w}_i}{2} \right\rceil \leqslant \sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil.$$

  - $\tilde{w}_{i+1}$ is even and w_{i+1} is odd. Then $\lceil w_{i+1}/2 \rceil - \lceil \tilde{w}_{i+1}/2 \rceil \geq 1$, so

$$\sum_{i=1}^{k} \left\lceil \frac{\tilde{w}_i}{2} \right\rceil + 1 \leqslant \sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil.$$

    The penalty term can increase but at most by 1, so the number of points for weights $\{\tilde{w}_1, \dots, \tilde{w}_k\}$ will not increase.

- If $w_1 > 1$ then we can define the set $\{\tilde{w}_1, \ldots, \tilde{w}_k\}$ as follows

$$\begin{cases} \hat{w}_j = w_j & \text{if } j \neq 1 \\ \hat{w}_1 = 1 \end{cases}.$$

  The same arguments as above can show that the minimal number of points required to realize $\{\tilde{w}_1, \ldots, \tilde{w}_k\}$ is not greater then for $\{w_1, \ldots, w_k\}$.

It follows from above observation that the minimal number of points required to realize subsequent weights starting from 1 is not greater than the minimal number of points required to realize every other set $\{w_1, \ldots, w_k\}$ of different weights.

We showed that $\{1, 2, \ldots, k\}$ is an optimal choice of weights. Now we can lower-bound the number of points required to realize weights from this set. To achieve this, we consider the following four cases

- $k \equiv 0 \mod 4$:

$$\sum_{n=1}^{k} \left\lceil \frac{n}{2} \right\rceil + \frac{k - \frac{k}{2}}{2} = \frac{k^2 + 2k}{4} + \frac{k}{4} = \frac{k^2 + 3k}{4},$$

- $k \equiv 1 \mod 4$:

$$\sum_{n=1}^{k} \left\lceil \frac{n}{2} \right\rceil + \frac{k - \frac{k+1}{2}}{2} = \frac{k^2 + 2k + 1}{4} + \frac{k-1}{4} = \frac{k^2 + 3k}{4},$$

- $k \equiv 2 \mod 4$:

$$\sum_{n=1}^{k} \left\lceil \frac{n}{2} \right\rceil + \frac{k - \frac{k}{2} + 1}{2} = \frac{k^2 + 2k}{4} + \frac{k+2}{4} = \frac{k^2 + 3k + 2}{4},$$

- $k \equiv 3 \mod 4$:

$$\sum_{n=1}^{k} \left\lceil \frac{n}{2} \right\rceil + \frac{k - \frac{k+1}{2} + 1}{2} = \frac{k^2 + 2k + 1}{4} + \frac{k+1}{4} = \frac{k^2 + 3k + 2}{4}.$$

As one can see, the lower bound matches the upper bound. This finishes the proof. □

Authors of [13] were interested in the special case, where $k = 2^{r+1} - 1$. Our solution gives the following answer to the problem $\varphi(2^{r+1} - 1) = 2^r \left(2^r + \frac{1}{2}\right)$.

4 Topological types of the symmetries on Riemann surfaces

Let us start with observations concerning possible topological types which are admissible on Riemann surfaces of our interest, that is Riemann surfaces which admit the maximal possible number of nonconjugate symmetries. By the results of [2] it is known that for $g = 2^{r-1}u + 1$, u odd, the maximal possible number of conjugacy classes of symmetries of a Riemann surface of genus g is 2^{r+1} and by the results of [14] it is known that the automorphism group generated by these symmetries is isomorphic to $\mathrm{D}_{2^s} \times \mathbb{Z}_2^r$. We shall call the surfaces realizing the maximal value *s-extremal*. In the sequel we shall assume that g is odd, which means that $r \geq 2$.

Theorem 4.1. *Let X be an s-extremal Riemann surface of odd genus g such that at most one of the symmetries in question is fixed-point free. Then all the symmetries in question are non-separating.*

Proof. Let X be an s-extremal Riemann surface of genus $g = 2^{r-1}u + 1$, where u is odd. This means that X admits the maximal number of 2^{r+1} nonconjugate symmetries and the automorphism group G of X is isomorphic to $\mathrm{D}_{2^s} \times \mathbb{Z}_2^r = \langle a, b\rangle \times \langle \sigma_1, \ldots, \sigma_r\rangle$. We know that G may be viewed as a quotient group Λ/Γ for a NEC group Λ with a general signature (2.1) and Γ being a kernel of an epimorphism $\theta : \Lambda \to G$. Clearly we can assume that the representatives of the 2^{r+1} conjugacy classes of all the symmetries of X are of the forms

$$ax, \quad bx, \quad \sigma, \quad (ab)^{2^{s-1}}\sigma,$$

where x is a word of even length in $\sigma_1, \ldots, \sigma_r$ and σ is a word of odd length in $\sigma_1, \ldots, \sigma_r$. The first two ones are non-central and the remaining two are central. Now it is clear that for all but at most one of the symmetries above there is a canonical reflection which contributes ovals to it. We can also assume, without loss of generality, that there are canonical reflections

$$c_a, \quad c_b, \quad c_{\sigma_i},$$

for $i = 1, \ldots, r$, which are mapped to

$$a, \quad b, \quad \sigma_i,$$

by the canonical epimorphism θ. Now it is clear that any of the symmetries can be written as a word of odd length in a, b, σ_i. This also means that any of the non-generating symmetries can be seen as the image of a certain product of canonical reflections c_a, c_b, c_{σ_i}. This clearly makes any of the non-generating symmetries non-separating. Now actually the same applies also to any of the generating symmetries a, b, σ_i as each of them can be written as a product of three other symmetries with fixed points. Say, if $\sigma = a\sigma_i\sigma_j$ is an image of c_σ, then a is the image of $c_\sigma c_{\sigma_i} c_{\sigma_j}$. This means all the symmetries are nonseparating. □

Now we shall consider the total maximal number of ovals of the symmetries on s-extremal Riemann surfaces with the automorphism group as before. The key elements here are the centralizers mentioned in Theorem 2.2. Let us first consider a central symmetry σ. Obviously, its centralizer in G has order $|G| = 2^{r+s+1}$. Now, we shall observe what is a possible contribution of any reflection c_σ such that $\theta(c_\sigma) = \sigma$. If the reflection corresponds to an empty period cycle, then the possible numbers of ovals contributed are $|G|/2$ or $|G|/4$, depending on whether $\theta(e) = 1$ or not for the corresponding connecting generator of the empty cycle. Now in the case of a nonempty period cycle, the possible values are bounded from above by $|G|/4$ if there are equal neighbors, meaning that the images of the neighboring reflections under θ are the same, and from below by $|G|/2^{s+2}$ if the neighbors generate a dihedral group of order 2^{s+1}. Therefore, we see that any such occurrence has to contribute at least 2^{r-1} ovals. Surprisingly, the case is exactly the same for noncentral symmetries. Even though their centralizers in G are of order 2^{r+2}, the possible numbers of ovals are 2^{r+1} or 2^r for an empty period cycle and at least $2^{r+2}/8 = 2^{r-1}$ for a nonempty period cycle.

Theorem 4.2. *The maximal total number of ovals for $2^{r+1} - 1$ symmetries of an s-extremal Riemann surface with G as above is equal to*

$$2^r u + (7 - 2^{r+1})2^{r+s-2} + 2^r.$$

Proof. By the general result of [9] we know that the maximal total number of ovals of k non-conjugate symmetries, on a Riemann surface X of genus g, which generate the group G, is

$$2g - 2 + (9 - k) \cdot \frac{|G|}{8}.$$

In our case $g = 2^{r-1}u + 1$ and we shall assume that our surface is s-extremal, so $G = \mathrm{D}_{2^s} \times \mathbb{Z}_2^r$. As we are going to look for the maximal total number of ovals, by the proof of the main result in [9], we may assume that the signature of a NEC group Λ in the quotient $G = \Lambda/\Gamma$ contains just one, nonempty, period cycle

$$(h; \pm; [m_1, \dots, m_v]; \{(n_1, \dots, n_s)\}).$$

Furthermore, by the Hurwitz-Riemann formula

$$\varepsilon h - 1 + \sum_{i=1}^{v} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{2^{r-1}u}{2^{r+s}} = \frac{u}{2^{s+1}},$$

and since u is odd, this means that there is an odd number of link periods equal 2^s in the signature of Λ. Going even further into the details,

$$\frac{u}{2^{s+1}} \geq -1 + \frac{1}{2} - \frac{1}{2^{s+1}} + \frac{s-1}{4}$$

and so

$$s \leq \frac{u}{2^{s-1}} + 3 + \frac{4}{2^{s+1}}. \tag{4.1}$$

As there is an odd number of link periods 2^s in the signature, it means that at least k reflections in the cycle have neighbors labeled with different labels, meaning they have distinct images under θ.

Indeed, otherwise, by Lemma 3.2, half of the reflections of the cycle would have the same image under θ, which contradicts the assumptions. Therefore, summing up, at least $k = 2^{r+1} - 1$ reflections have neighbors with distinct labels. Therefore, the total number of ovals holds inequality

$$\|X\| \leq k \cdot \frac{|G|}{8} + (s-k) \cdot \frac{|G|}{4} = s \cdot \frac{|G|}{4} + \frac{|G|}{8} - 2^{r+1} \cdot \frac{|G|}{8}$$

and using the inequality (4.1) on s we obtain

$$\|X\| \leq \left(\frac{u}{2^{s-1}} + 3 + \frac{4}{2^{s+1}}\right) \cdot 2^{r+s-1} + 2^{r+s-2} - 2^{r+1} \cdot 2^{r+s-2} = 2^r u + 2^r + 2^{r+s-2} \cdot (7 - 2^{r+1}).$$

□

As a corollary, we obtain the following interesting result concerning relations between s-extremal and o-extremal Riemann surfaces.

Corollary 4.3. *An s-extremal Riemann surface is never o-extremal.*

Theorem 4.4. *If a Riemann surface of genus $g = 2^{r-1}u + 1$, odd u, has 2^{r+1} nonconjugate symmetries such that each of the symmetries has a distinct topological type, then $u \geq 2^{2r} + 2^{r-1} - 3$.*

Proof. First of all, we shall assume that one of the symmetries has 0 ovals. This assumption is natural: our aim is to minimize u, which is strictly connected to the signature of Λ in $G = \Lambda/\Gamma$. To make sure that all the symmetries have distinct numbers of ovals, we are allowed to assume that at most one of them is fixed-point free. However, if we allowed all the symmetries to have fixed points, then our number k in the previous theorem would be equal to 2^{r+1} as we have to incorporate one more colour in our setting, being the symmetry as the image of a canonical reflection. This makes the total admissible number of ovals even smaller and the conditions of the theorem are fulfilled automatically. Hence, let us consider the case where one of the symmetries is fixed-point free. Now, as by Theorem 4.1, all the symmetries are nonseparating and the minimal contribution of a reflection is 2^{r-1}, the minimal number of ovals that they have together is

$$1 \cdot 2^{r-1} + 2 \cdot 2^{r-1} + \cdots + (2^{r+1} - 1) \cdot 2^{r-1},$$

which is equal to $2^{2r-1}(2^{r+1}-1)$. Now it has to be that

$$2^{2r-1}(2^{r+1}-1) \leq 2^r u + (7-2^{r+1})2^{r+s-2} + 2^r.$$

This in turn means that

$$u \geq 2^{2r} + 2^{r-1}(2^s-1) - 1 - 7 \cdot 2^{s-2}.$$

For $r \geq 3$, $s \geq 2$ we get that

$$u \geq 2^{2r} + 2^{r-1} - 3.$$

Now if $r=2, s \geq 3$ we need to show that $u \geq 15$ and we have

$$u \geq 16 - 2 - 1 + 2 = 15.$$

If $r = s = 2$ we get

$$u \geq 16 - 2 - 1 + 1 = 14$$

and is $r=2, s=1$ we get

$$u \geq 16 - 2 - 2 + \frac{1}{2} = 13\frac{1}{2}.$$

However, u is an odd integer, which means that $u \geq 15$ in both cases. □

Theorem 4.5. *The bound on u is sharp. Moreover, for any odd $u \geq u_{min} = 2^r(2^r+1/2)-3$ there exists a Riemann surface of genus $g = 2^{r-1}u+1$ having 2^{r+1} symmetries of distinct topological types.*

Proof. In this proof, we will use the construction from the solution of the combinatorial Problem 1. It is necessary to show, how it can be applied to define a Riemann surface. Let $X = \mathcal{H}/\Gamma$ be a Riemann surface of genus g with the following automorphism group

$$\mathrm{Aut}(X) = \Lambda/\Gamma = \mathbb{Z}_2^{r+2} = \langle\sigma_1\rangle \oplus \cdots \oplus \langle\sigma_{r+2}\rangle,$$

where Λ is a NEC group with signature

$$s(\Lambda) = (0; +; [-], \{(\underbrace{2, \ldots, 2}_{\alpha})\}).$$

Theorem 2.1 gives us the method for calculating number of ovals for symmetries. Suppose that the epimorphism $\theta : \Lambda \longrightarrow \Lambda/\Gamma$ is defined on generators c_{i-1}, c_i, c_{i+1} as follows

$$\theta(c_{i-1}) = \sigma_a\ , \quad \theta(c_i) = \sigma_b\ , \quad \theta(c_{i+1}) = \sigma_c.$$

Since $\mathbb{Z}_2^{r+2}$ is abelian, then $|C(\mathbb{Z}_2^{r+2}, \sigma_b)| = |\mathbb{Z}_2^{r+2}| = 2^{r+2}$. Theorem 2.2 describes the structure of $C(\Lambda, c_i)$ and since each period of period cycles equals 2, then

$$\theta(C(\Lambda, c_i)) = \langle\theta(c_{i-1})\rangle \oplus \langle\theta(c_i)\rangle \oplus \langle\theta(c_{i+1})\rangle,$$

$$\theta(C(\Lambda, c_i)) = \begin{cases} \mathbb{Z}_2^3 & \text{if } \theta(c_{i-1}) \neq \theta(c_{i+1}), \\ \mathbb{Z}_2^2 & \text{if } \theta(c_{i-1}) = \theta(c_{i+1}). \end{cases}$$

Hence, $\theta(c_i)$ contributes to σ_b either $\dfrac{2^{r+2}}{8}$ or $\dfrac{2^{r+2}}{4}$ ovals depending on whether $\theta(c_{i-1}) \neq \theta(c_{i+1})$ or $\theta(c_{i-1}) = \theta(c_{i+1})$ respectively.

Now, the correspondence between constructing epimorphism θ and combinatorial problem is straightforward. Colours correspond to symmetries and points on circle correspond to the generators $c_0, \ldots, c_s$ of a NEC group Λ. Assigning colours to the points corresponds to the definition of epimorphism θ. Moreover, weights of points correspond to the number of ovals contributed to symmetries by each image $\theta(c_i)$. To conclude, the solution of combinatorial problem, allows us to define epimorphism θ in such a way that each symmetry has distinct number of ovals.

Let $g = 2^{r-1}u + 1$, where u is odd. We will construct a Riemann surface of genus g which has exactly 2^{r+1} symmetries of distinct number of ovals.

Let

$$\alpha = 2^r\left(2^r + \frac{1}{2}\right) + 1, \tag{4.2}$$

and Λ be NEC group with the following signature

$$s(\Lambda) = (0; +; [-], \{(\underbrace{2, \ldots, 2}_{\alpha})\}).$$

Let $X = \mathcal{H}/\Gamma$ be a Riemann surface of genus g with the following automorphism group

$$\mathrm{Aut}(X) = \Lambda/\Gamma = \mathbb{Z}_2^{r+2} = \langle\sigma_1\rangle \oplus \cdots \oplus \langle\sigma_{r+2}\rangle.$$

Observe that based on Hurwitz-Riemann formula, the necessary condition for a Riemann surface X to be realised as $\mathcal{H}/\Gamma$ is that $u = 2^r\left(2^r + \frac{1}{2}\right) - 3$. Indeed,

$$2g - 2 = \left|\mathbb{Z}_2^{r+2}\right|\left(-2 + 1 + \frac{1}{2}\sum_{i=1}^{\alpha}\frac{1}{2}\right) \iff 2\left(2^{r-1}u + 1\right) - 2 = 2^{r+2}\left(-1 + \frac{\alpha}{4}\right)$$
$$\iff 2^r u = 2^r(\alpha - 4) \iff u = 2^r\left(2^r + \frac{1}{2}\right) - 3.$$

Now, we will define an epimorphism $\theta : \Lambda \longrightarrow \Lambda/\Gamma$. On a generator e_1 of a group Λ we define $\theta(e_1) = 1$. Next, the construction from Theorem 3.4 gives us a recipe how to define the epimorphism

θ on generators c_i. Notice, that one symmetry can have no fixed points, so we do not include it in the construction from Theorem 3.4. For this construction we will need

$$\varphi\left(2^{r+1}-1\right) = \frac{(2^{r+1}-1)^2 + 3\cdot(2^{r+1}-1)+2}{4} = 2^r\left(2^r+\frac{1}{2}\right)$$

periods from period cycles. But there are $2^r\left(2^r+\frac{1}{2}\right)+1$ periods from period cycles in the signature $s(\Lambda)$. It is not a problem at all, since we can modify the construction from Theorem 3.4 by adding one point of colour a_{4k+3} in the following way

$$a_3\ a_1\ a_2\ a_3\ a_2\ a_{4k+3}\ B_1\cdots B_{\frac{k-4}{4}},$$

where in our case $k = 2^{r+1}$. Weight of colour a_{4k+3} will increase by 1 and weights of all other colours will not change, so the construction will remain valid. With small modification of a solution for combinatorial problem, we have constructed the epimorphism θ whose kernel $\ker(\theta) = \Gamma$ is a surface Fuchsian group of genus g. We have constructed then a Riemann surface $\mathcal{H}/\Gamma$ of genus $g = 2^{r-1}u+1$, where $u = 2^r\left(2^r+\frac{1}{2}\right)-3$, which has 2^{r+1} symmetries of distinct topological types.

Now, let $u > u_{min}$ and Λ be a NEC group with the following signature

$$s(\Lambda) = (0;+;[\underbrace{(2,\ldots,2)}_{\beta}],\{\underbrace{(2,\ldots,2)}_{\alpha}\}),$$

where

$$\beta = \frac{1}{2}\left(u - 2^r\left(2^r+\frac{1}{2}\right)+3\right),$$

and α is defined as in (4.2). We define the epimorphism θ on generators c_i the same way as above. Moreover for each generator x_i we put $\theta(x_i) = \sigma_1\sigma_2$. It only remains to define θ on the generator e_1. We do it as follows: if β is even, we put $\theta(e_1) = 1$ and $\theta(e_1) = \sigma_1\sigma_2$ otherwise.

Observe that the process of counting ovals is the same as earlier. The only one change may occur, when $\theta(e_1) = \sigma_1\sigma_2$. However, since our group is abelian it does not introduce any change in the ovals count. Such definition of the epimorphism θ is correct, since relations from NEC group presentation are satisfied and the kernel of θ, $\ker(\theta) = \Gamma$ is a surface Fuchsian group. It only remains to check, if Γ (so as the constructed Riemann surface $\mathcal{H}/\Gamma$) has a genus $g = 2^{r-1}u+1$. We do this by checking Hurwitz-Riemann formula

$$2g-2=\left|\mathbb{Z}_2^{r+2}\right|\left(-2+1+\sum_{i=1}^{\beta}\frac{1}{2}+\frac{1}{2}\sum_{i=1}^{\alpha}\frac{1}{2}\right)\iff 2g-2=2^{r+2}\left(-1+\frac{\beta}{2}+\frac{\alpha}{4}\right)$$

$$\iff g=2^{r-1}\left(2\beta+\alpha-4\right)+1\iff g=2^{r-1}\left(u-2^r\left(2^r+\frac{1}{2}\right)+3+2^r\left(2^r+\frac{1}{2}\right)+1-4\right)-1$$

$$\iff g=2^{r-1}u+1.$$

For each $u \geq u_{min}$ we have constructed a Riemann surface of genus $g = 2^{r-1}u + 1$ having 2^{r+1} symmetries of distinct number of ovals, which implies distinct topological types. □

As a Corollary, we obtain the main result of this paper, which also gives an answer to the question posed in [13]

Corollary 4.6. *The geometrical dimension of the real nerve $\mathcal{N}_g$ for odd values of g is maximal if and only if $u \geq u_{min}$.*

Now we shall present another application of o-extremal surfaces. This time we shall show extremal configurations of symmetries which prove the nontriviality of certain homology groups. Before we proceed to the main result, let us make an easy observation about o-extremal surfaces. By the results of [9], we know that on a Riemann surface of genus g, k non-conjugate symmetries with fixed points have at most

$$2g-2+(9-k)\cdot\frac{|G|}{8} \tag{4.3}$$

ovals. As g is odd, we are going to play with at least 8 symmetries on an o-extremal Riemann surface, it is known by [14] and [12] that they commute. Therefore the structure of the group is known, being $\mathbb{Z}_2^{a+2}$, where a is the smallest possible integer such that $k \leq 2^{a+1}$ for $k \geq 9$, or $\mathbb{Z}_2^8$ for $k = 8$. What can be easily observed by (4.3) is, that with the growth of k, the maximal possible number of ovals does not grow, and even more, it drops down for any $k \geq 9$ being the power of 2. Therefore the idea for the next results is as follows. We are going to prove that there exists a nontrivial cycle in certain $(k-1)$-th homology group. To show this, we shall construct $(k-1)$-dimensional facets of what-could-be a simplex of dimension k, but we do it in such a way, that one of the facets comes from an o-extremal Riemann surface. This means, that the sum of absolute values of all vertices is maximal. Therefore the said hypothetical k-dimensional simplex cannot exist, as long as we are sure that the total maximal number of ovals for k symmetries is strictly smaller than the one for $k-1$ symmetries. This is guaranteed for the values of k being of the form $k = 2^{a+1}$ for some integer a. Now we can proceed with the main result.

Theorem 4.7. *Let $g = 2^{r-1}u + 1$ for some odd u and $r \geq 2$. Let $a \leq r-1$ be a positive integer. Then, for $u \geq 2^{2r} + 2^r - 3$, the homology groups $H_{2^{a+1}-1}(\mathcal{N}_g) \neq 0$.*

Proof. Let us consider a NEC group Λ with signature

$$(0; +; [2, \overset{m}{\ldots}, 2]; \{(2, \overset{s}{\ldots}, 2)\}), \tag{4.4}$$

where $m = u \cdot 2^{r-1-a} + 2 - (2^{a+1} + 2)(2^{a+1} - 1)$ and $s = (2^{a+2} + 4)(2^{a+1} - 1)$. Our goal will be to define $\theta : \Lambda \to G = \mathbb{Z}_2^{a+2} = \langle \sigma_i \,|\, \sigma_i^2, i = 1, \ldots, a+2 \rangle$ in such a way, that $X = \mathcal{H}/\ker\theta$ is an o-extremal Riemann surface, having 2^{a+1} non-conjugate symmetries with distinct topological types. Next, we shall define $2^{a+1} - 1$ Riemann surfaces X_i, $i = 1, \ldots, 2^{a+1} - 1$ with the same genus and automorphism group as X in such a way that each of them has 2^{a+1} non-conjugate symmetries with distinct topological types, but the types for X_i are the same as the ones for X, with the only exception of σ_i which shall have $2^{a+1} = |G|/2$ ovals in the new construction. First, we construct X by defining θ on the canonical generators: all the elliptic generators x_i, $i = 1, \ldots, m$, are mapped to $\sigma_1\sigma_2$ and the connecting generator e_1 is mapped to 1 if m is even and $\sigma_1\sigma_2$ if m is odd, to satisfy the long relation. Now, reflections corresponding to the unique nonempty period cycle are mapped in the following way. We divide the cycle into $2^{a+1} - 1$ segments S_i of increasing length - the length of S_i is $(i+1) \cdot 4$ and its reflections are mapped consecutively to σ_1 and σ_{i+1}:

$$\sigma_1,\ \sigma_{i+1},\ \sigma_1,\ \sigma_{i+1}, \ldots, \sigma_1,\ \sigma_{i+1},\ \sigma_1,\ \sigma_{i+1}.$$

Obviously, each of the reflections contributing to σ_{i+1} yields $|G|/4$ ovals, as its neighbors have the same images under θ. It is almost the same for reflections corresponding to σ_1, except for the first one, which has neighbors with distinct images and so it contributes $|G|/8$ ovals to σ_1. Therefore, the total contribution to σ_1 from the segment S_i is $(i+1) \cdot |G|/2 - |G|/8$ and the contribution to σ_{i+1} is $(i+1)|G|/2$. As the symmetries, except σ_1, do not appear outside their respective segments, all of them have distinct numbers of ovals. It is also clearly visible that the total number of ovals is maximal possible and so we have constructed an o-extremal Riemann surface of genus $g = 2^{r-1}u + 1$. Now by the method that could be called surgery of the signature (4.4) of Λ, we shall construct the surfaces X_{i+1} mentioned above. Let first $i + 1 \geq 2$. We shall replace the signature (4.4) above by the signature

$$(0; +; [2, \overset{m}{\ldots}, 2]; \{(2, \overset{s_i}{\ldots}, 2), (-)^i)\}), \tag{4.5}$$

where $s_i = s - 4i$. Let us call the corresponding NEC group Λ_{i+1} an now we modify θ to $\theta_{i+1} : \Lambda_{i+1} \to G$ - all the segments remain unchanged except from the segment S_i which is replaced by a segment of length 4 whose reflections are mapped to

$$\sigma_1,\ \sigma_{i+1},\ \sigma_1,\ \sigma_{i+1}.$$

This exchange can be seen as a surgery on S_i: we cut the middle of the cycle out and leave only the first two and the last two elements of the segment. Now the symmetry σ_{i+1} has $|G|/2$ ovals. Clearly σ_1 lost $i \cdot |G|/2$ ovals. These ovals are added again in the empty cycles of Λ_i, as all the canonical reflections are mapped by θ_{i+1} to σ_1 and all the corresponding connecting generators are mapped to 1, except for e_1 for odd m, where $\theta(e_1) = \sigma_1\sigma_2$. In such a way, we have obtained the surface X_{i+1}, of genus $g = 2^{r-1}u + 1$ which has 2^{a+1} symmetries of distinct topological types and all the types are the same except the one corresponding to σ_{i+1}.

Let us now construct X_1. We have to replace the number of ovals of σ_1 by $|G|/2$. This clearly requires a different approach than the one presented above for $i + 1 \geq 2$. The idea is as follows: for all $i \geq 3$, we replace the segment S_i by $i + 1$ empty period cycles. All of these period cycles have corresponding reflections being mapped by θ_1 to σ_{i+1} and the connecting generators to 1. Observe that in this construction, each period cycle contributes $|G|/2$ ovals to σ_{i+1} and so in total it has $(i + 1)|G|/2$ ovals again. Now let us consider S_1 and S_2: together they have 20 reflections contributing $|G|$ ovals to σ_2 and $3|G|/2$ ovals to σ_3. We have to define θ_1 in such a way that these numbers of ovals stay the same for σ_2, σ_3 and σ_1 gets $|G|/2$ ovals. We introduce a new period cycle of length 8 which maps its reflections consecutively to σ_2, σ_3 giving each of them $|G|$ ovals. Then we add two more empty period cycle, one of them contributing to σ_1 and the other to σ_3. Summing up, we consider a NEC group

$$(1; -; [2, \overset{m}{\ldots}, 2]; \{(2, 2, 2, 2, 2, 2, 2, 2), (-)^l, (-), (-)\}),$$

where m is as before and $l = (2^a + 2)(2^{a+1} - 3)$. Now d_1 is mapped to σ_1, all the elliptic generators x_i, $i = 1, \ldots, m$ are mapped to $\sigma_1\sigma_2$, all the connecting generators e_i, $i \geq 2$ are mapped to 1, the connecting generator e_1 is mapped to 1 if m is even and $\sigma_1\sigma_2$ if m is odd. Now the reflections of the unique nonempty period cycle are mapped consecutively to σ_2, σ_3, and we divide the empty period cycles into segments in the following way: there are $2^{a+1} - 3$ segments and segment E_i consists of $i+3$ empty period cycles where the canonical reflections involved in the segment E_i are mapped to σ_{i+3}. The reflections of the last two empty period cycles are mapped to σ_3 and σ_1. In such a way we obtain the surface X_1, of the same genus as X, as announced above.

Now, when it comes to homology groups of the real nerve $\mathcal{N}_g$, in such a way we have constructed a cycle which is not a boundary in dimension $2^{a+1}-1$. Indeed, if it were a boundary, then there would have to exist a simplex with more than 2^{a+1} vertices where the sum of absolute values of marks of the vertices, coming from the corresponding symmetry types, is at least the same as the one for simplex defining X. This is clearly not possible, as X is o-extremal and any o-extremal surface with more than 2^{a+1} symmetries has total number of ovals which is strictly smaller than the maximal value for 2^{a+1} symmetries. Therefore indeed the homology group is nontrivial in this dimension. □

Remark 4.8. *The same method cannot be used in any other dimension. In fact, in such a case our symmetries do not exhaust all the orientation reversing involutions in the group G and we can always add a 0-vertex to our simplex and obtain a simplex of a larger dimension which admits X as a face, without changing the maximal possible number of ovals. This means, that in all the other dimensions, our constructed cycle is a boundary of a $(k+1)$-dimensional simplex, where the new vertex corresponds to a fixed-point free symmetry.*

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