Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 28, no. 2, pp. 323–347, May 2026
DOI: 10.56754/0719-0646.2802.323

Multivariate symmetrized, q-deformed and λ-parametrized hyperbolic tangent induced complex valued trigonometric and hyperbolic neural network enhanced approximation

George A. Anastassiou[1,✉]

[1] *Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A.*
ganastss@memphis.edu✉

ABSTRACT

Here we study the multivariate quantitative symmetrized approximation of complex valued continuous functions on a box by complex valued symmetrized and perturbed multivariate neural network operators. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the used function's high order partial derivatives. The kind of our approximations are trigonometric and hyperbolic. Our multivariate symmetrized operators are defined by using a multivariate density function generated by a q-deformed and λ-parametrized hyperbolic tangent function. These enhanced approximations are pointwise and of the uniform norm. The related complex valued feed-forward neural networks are with one hidden layer.

Published: 18 May, 2026
Accepted: 13 March, 2026
Received: 06 January, 2025

RESUMEN

Estudiamos la aproximación cuantitativa multivariada simetrizada de funciones continuas con valores complejos en una caja, a través de operadores de redes neuronales simetrizados y multivariados perturbados con valores complejos. Estas aproximaciones se derivan al establecer desigualdades de tipo Jackson, que involucran el módulo de continuidad de las derivadas parciales de alto orden de la función utilizada. Los tipos de nuestras aproximaciones son trigonométricas e hiperbólicas. Nuestros operadores multivariados simetrizados se definen usando una función de densidad multivariada generada por una función tangente hiperbólica q-deformada y λ-parametrizada. Estas aproximaciones mejoradas son puntuales y en la norma uniforme. Las redes neuronales prealimentadas con valores complejos relacionadas tienen una capa oculta.

Keywords and Phrases: q-deformed and λ-parametrized hyperbolic tangent, complex valued symmetrized multivariate neural network approximation, complex valued multivariate quasi-interpolation operator, modulus of continuity, trigonometric and hyperbolic enhanced approximation.

2020 AMS Mathematics Subject Classification: 41A17, 41A25, 41A99, 42A10.

1 Introduction

The author in [2] and [1], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining "bell-shaped" and "squashing" functions for these operators are assumed to be of compact support.

Again the author inspired by [10], continued his studies on neural network approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3], by treating both the univariate and multivariate cases.

The brain asymmetry has been observed in animals and humans in terms of structure, function and behaviour. This lateralization is thought to reflect evolutionary, hereditary, develop mental, experiential and pathological factors. Therefore it is natural to consider for our study deformed neural network activation functions and operators. So this article is a specific study under this philosophy of approaching reality as close as possible.

Consequently, the author here performs multivariate symmetrized q-deformed and λ-parametrized hyperbolic tangent function activated high order multivariate neural network approximations to continuous functions over boxes with complex values. All convergences are with rates expressed via the moduli of continuity of the involved functions high order partial derivatives, deriving by very tight multivariate Jackson type inequalities.

The basics of our higher order approximations here are some newly discovered by the author trigonometric and hyperbolic type multivariate Taylor's formulae.

Our boxes are not necessarily symmetric to the origin. The applied symmetrization technique and the newly introduced related multivariate operators cut in half the feed to neural networks, thus enhancing immensely their convergence speed to the unit operator.

A multilayer feed-forward neural network can be defined as follows (with $m \in \mathbb{N}$ hidden layers):

Let $x \in \mathbb{R}^s$; $s \in \mathbb{N}$, where $x = (x_1, \ldots, x_s)$; $\alpha_j, c_j \in \mathbb{R}^s$; $b_j \in \mathbb{R}$, with $0 \leq j \leq n$, $n \in \mathbb{N}$.

Here $\langle \alpha_j \cdot x \rangle$ is the inner product, thus $\sigma(\langle \alpha_j \cdot x \rangle + b_j) \in \mathbb{R}$; and $N_n(x) \in \mathbb{R}^s$, by $c_j \in \mathbb{R}^s$, as it is coming from $N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle \alpha_j \cdot x \rangle + b_j)$. We define:

$$N_n^{(2)}(x) = \sum_{j=0}^{n} c_j \sigma(\langle \alpha_j \cdot N_n(x) \rangle + b_j) = \sum_{j=0}^{n} c_j \sigma\left(\left\langle \alpha_j \cdot \left(\sum_{j=0}^{n} c_j \sigma(\langle \alpha_j \cdot x \rangle + b_j)\right)\right\rangle + b_j\right).$$

Furthermore, we can define

$$N_n^{(3)}(x) = \sum_{j=0}^{n} c_j \sigma\left(\left\langle \alpha_j \cdot N_n^{(2)}(x)\right\rangle + b_j\right).$$

And, in general we define:

$$N_n^{(m)}(x) = \sum_{j=0}^{n} c_j \sigma\left(\left\langle \alpha_j \cdot N_n^{(m-1)}(x)\right\rangle + b_j\right), \text{ for } m \in \mathbb{N}.$$

For more studies in neural networks read [11–17]

2 Basics

Initially we follow [8, pp. 455-460].

Our perturbed hyperbolic tangent activation function here to be used is

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - qe^{-\lambda x}}{e^{\lambda x} + qe^{-\lambda x}}, \quad \lambda, q > 0, \; x \in \mathbb{R}. \tag{1}$$

Above λ is the parameter and q is the deformation coefficient, typically it is $0 < \lambda, q \leq 1$.

For more read of [8, Chapter 18]: "q-Deformed and λ -Parametrized Hyperbolic Tangent based Banach space Valued Ordinary and Fractional Neural Network Approximation".

The chapters 17 and 18 of [8] motivate our current work.

The proposed "symmetrization method" aims to use half data feed to our multivariate neural networks.

We will employ the following density function

$$M_{q.\lambda}(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0, \quad \forall x \in \mathbb{R};\, q,\, \lambda > 0. \tag{2}$$

We have that

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \quad \forall x \in \mathbb{R};\, q,\, \lambda > 0, \tag{3}$$

and

$$M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x), \quad \forall x \in \mathbb{R};\, q,\, \lambda > 0. \tag{4}$$

Adding (3) and (4) we obtain

$$M_{q,\lambda}(-x) + M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x), \tag{5}$$

the key to this work. So that

$$\Phi(x) := \frac{M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x)}{2} \tag{6}$$

is an even function, symmetric with respect to the y-axis.

By [8, (18.18)], we have

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2} \quad \text{and} \quad M_{\frac{1}{q},\lambda}\left(-\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2}, \ \ \lambda > 0. \tag{7}$$

sharing the same maximum at symmetric points.

By [8, Theorem 18.1, p. 458], we have that

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) = 1, \ \forall x \in \mathbb{R}, \ \lambda, \ q > 0; \quad \text{and} \quad \sum_{i=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x-i) = 1, \ \forall x \in \mathbb{R}, \ \lambda, \ q > 0. \tag{8}$$

Consequently, we derive that

$$\sum_{i=-\infty}^{\infty} \Phi(x-i) = 1, \quad \forall x \in \mathbb{R}. \tag{9}$$

By [8, Theorem 18.2, p. 459], we have that

$$\int_{-\infty}^{\infty} M_{q,\lambda}(x)\, dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x)\, dx = 1, \tag{10}$$

so that

$$\int_{-\infty}^{\infty} \Phi(x)\, dx = 1, \tag{11}$$

therefore Φ is a density function.

By [8, Theorem 18.3, p. 459], we have:

Let $0 < \alpha < 1$ and $n \in \mathbb{N}$, with $n^{1-\alpha} > 2$; $q, \lambda > 0$. Then

$$\sum_{\substack{k=-\infty \\ |nx-k| \geq n^{1-\alpha}}}^{\infty} M_{q,\lambda}(nx-k) < 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} = Te^{-2\lambda n^{(1-\alpha)}}, \tag{12}$$

where $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Similarly, we get that

$$\sum_{k=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(nx-k) < Te^{-2\lambda n^{(1-\alpha)}}. \tag{13}$$

Consequently we obtain that

$$\sum_{\substack{k=-\infty \\ |nx-k|\geq n^{1-\alpha}}}^{\infty} \Phi\left(nx-k\right) < Te^{-2\lambda n^{(1-\alpha)}}, \tag{14}$$

where $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Here $\lceil\cdot\rceil$ denotes the ceiling of the number, and $\lfloor\cdot\rfloor$ its integral part.

We mention

Theorem ([8, Theorem 18.4, p. 459])**.** *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na\rceil \leq \lfloor nb\rfloor$. For $q, \lambda > 0$, we consider $\lambda_q > z_0 > 0$, such that $M_{q,\lambda}(z_0) = M_{q,\lambda}(0)$, and $\lambda_q > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right)} < \max\left\{\frac{1}{M_{q,\lambda}\left(\lambda_q\right)}, \frac{1}{M_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: \Delta\left(q\right). \tag{15}$$

Similarly, we consider $\lambda_{\frac{1}{q}} > z_1 > 0$, such that $M_{\frac{1}{q},\lambda}(z_1) = M_{\frac{1}{q},\lambda}(0)$, and $\lambda_{\frac{1}{q}} > 1$. Thus

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{\frac{1}{q},\lambda}\left(nx-k\right)} < \max\left\{\frac{1}{M_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}, \frac{1}{M_{q,\lambda}\left(\lambda_q\right)}\right\} = \Delta\left(q\right). \tag{16}$$

Hence

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right) > \frac{1}{\Delta\left(q\right)} \tag{17}$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{\frac{1}{q},\lambda}\left(nx-k\right) > \frac{1}{\Delta\left(q\right)}. \tag{18}$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(M_{q,\lambda}\left(nx-k\right) + M_{\frac{1}{q},\lambda}\left(nx-k\right)\right)}{2} > \frac{2}{2\Delta\left(q\right)} = \frac{1}{\Delta\left(q\right)}, \tag{19}$$

so that

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(M_{q,\lambda}\left(nx-k\right) + M_{\frac{1}{q},\lambda}\left(nx-k\right)\right)}{2}} < \Delta\left(q\right), \tag{20}$$

that is

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)}<\Delta\left(q\right). \tag{21}$$

We have proved

Theorem 2.1. *Let $x\in[a,b]\subset\mathbb{R}$ and $n\in\mathbb{N}$ so that $\lceil na\rceil\leq\lfloor nb\rfloor$. For $q,\lambda>0$, we consider $\lambda_q>z_0>0$, such that $M_{q,\lambda}\left(z_0\right)=M_{q,\lambda}\left(0\right)$, and $\lambda_q>1$. Also consider $\lambda_{\frac{1}{q}}>z_1>0$, such that $M_{\frac{1}{q},\lambda}\left(z_1\right)=M_{\frac{1}{q},\lambda}\left(0\right)$, and $\lambda_{\frac{1}{q}}>1$. Then*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)}<\Delta\left(q\right). \tag{22}$$

We make

Remark 2.2. *I) By [8, Remark 18.5, p. 460], we have*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}\left(nx_1-k\right)\neq1,\ \textit{for some } x_1\in[a,b], \tag{23}$$

and

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{\frac{1}{q},\lambda}\left(nx_2-k\right)\neq1,\ \textit{for some } x_2\in[a,b]. \tag{24}$$

Therefore it holds

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\left(M_{q,\lambda}\left(nx_1-k\right)+M_{\frac{1}{q},\lambda}\left(nx_2-k\right)\right)}{2}\neq1. \tag{25}$$

Hence

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\left(M_{q,\lambda}\left(nx_1-k\right)+M_{\frac{1}{q},\lambda}\left(nx_1-k\right)\right)}{2}\neq1, \tag{26}$$

even if

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{\frac{1}{q},\lambda}\left(nx_1-k\right)=1, \tag{27}$$

because then

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{M_{q,\lambda}\left(nx_1-k\right)}{2}+\frac{1}{2}\neq1, \tag{28}$$

equivalently

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{M_{q,\lambda}(nx_1-k)}{2}\neq\frac{1}{2}, \tag{29}$$

true by

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx_1-k)\neq 1. \tag{30}$$

II) Let $[a,b]\subset\mathbb{R}$. *For large* n *we always have* $\lceil na\rceil\leq\lfloor nb\rfloor$. *Also* $a\leq\frac{k}{n}\leq b$, *iff* $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$. *So in general it holds*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\leq 1. \tag{31}$$

Next, we move on to the multivariate case, see [8, Chapter 17, pp. 419-452], as a model of action.

We make

Remark 2.3. *We introduce*

$$Z(x_1,\ldots,x_N):=Z(x):=\prod_{i=1}^{N}\Phi(x_i)=\frac{1}{2^N}\prod_{i=1}^{N}\left(M_{q,\lambda}+M_{\frac{1}{q},\lambda}\right)(x_i), \tag{32}$$

$x=(x_1,\ldots,x_N)\in\mathbb{R}^N$; $\lambda,q>0$, $N\in\mathbb{N}$.

Properties:

(i)

$$Z(x)>0,\quad\forall\ x\in\mathbb{R}^N, \tag{33}$$

(ii)

$$\sum_{k=-\infty}^{\infty}Z(x-k):=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}Z(x_1-k_1,\ldots,x_N-k_N)=1, \tag{34}$$

$k:=(k_1,\ldots,k_n)\in\mathbb{Z}^N$, $\forall\ x\in\mathbb{R}^N$, *hence*

(iii)

$$\sum_{k=-\infty}^{\infty}Z(nx-k)=1, \tag{35}$$

$\forall\ x\in\mathbb{R}^N$, $n\in\mathbb{N}$,

(iv)

$$\int_{\mathbb{R}^N}Z(x)\ dx=1, \tag{36}$$

that is Z *is a multivariate density function.*

Here denote:

$\|x\|_\infty := \max\{|x_1|, \ldots, |x_N|\}$, $x \in \mathbb{R}^N$, $\infty := (\infty, \ldots, \infty)$, $-\infty := (-\infty, \ldots, -\infty)$, $\lceil na \rceil := (\lceil na_1 \rceil, \ldots, \lceil na_N \rceil)$, $\lfloor nb \rfloor := (\lfloor nb_1 \rfloor, \ldots, \lfloor nb_N \rfloor)$, $a := (a_1, \ldots, a_N)$, $b := (b_1, \ldots, b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \Phi(nx_i - k_i) \right) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi(nx_i - k_i) \right). \tag{37}$$

(v) *We derive that*

$$\sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k) < Te^{-2\lambda n^{(1-\beta)}}, \quad \text{where } 0 < \beta < 1, \tag{38}$$

with $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^{N} [a_i, b_i]$.

(vi) *It holds*

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} < (\Delta(q))^N, \tag{39}$$

$\forall\, x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $n \in \mathbb{N}$.

It is clear that

(vii)

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx-k) < Te^{-2\lambda n^{(1-\beta)}}, \tag{40}$$

where $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.

Furthermore, it holds

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \neq 1, \tag{41}$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

Here $\left(X, \|\cdot\|_\gamma \right)$ *is a Banach space.*

Let $f \in C\left(\prod_{i=1}^{N} [a_i, b_i], X \right)$, $x = (x_1, \ldots, x_N) \in \prod_{i=1}^{N} [a_i, b_i]$, $n \in \mathbb{N} : \lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \ldots, N$.

We introduce and define the following multivariate linear normalized symmetrized neural network operator, let $x := (x_1, \ldots, x_N) \in \prod_{i=1}^{N} [a_i, b_i]$*:*

$$\theta_n^s(f, x_1, \ldots, x_N) := \theta_n^s(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} =$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \left(M_{q,\lambda}(nx_i - k_i) + M_{\frac{1}{q},\lambda}(nx_i - k_i)\right)\right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \left(M_{q,\lambda}(nx_i - k_i) + M_{\frac{1}{q},\lambda}(nx_i - k_i)\right)\right)}. \tag{42}$$

For large enough $n \in \mathbb{N}$*, we always obtain* $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \ldots, N$*. Also* $a_i \leq \frac{k_i}{n} \leq b_i$*, iff* $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, \ldots, N$*.*

When $g \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$ *we define the companion operator*

$$\widetilde{\theta}_n^s(g, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}. \tag{43}$$

Clearly, $\widetilde{\theta}_n^s$ *is a positive linear operator. We have that*

$$\widetilde{\theta}_n^s(1, x) = 1, \quad \forall\, x \in \prod_{i=1}^{N} [a_i, b_i].$$

Notice that $\theta_n^s(f) \in C\left(\prod_{i=1}^{N} [a_i, b_i], X\right)$ *and* $\widetilde{\theta}_n^s(g) \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

Furthermore, it holds

$$\|\theta_n^s(f, x)\|_\gamma \leq \widetilde{\theta}_n^s\left(\|f\|_\gamma, x\right), \quad \forall \in \prod_{i=1}^{N} [a_i, b_i] \tag{44}$$

and

$$\theta_n^s(cg, x) = c\widetilde{\theta}_n^s(g, x), \quad \forall\, x \in \prod_{i=1}^{N} [a_i, b_i], \tag{45}$$

and

$$\theta_n^s(c) = c, \quad \text{any } c \in X.$$

We call

$$^*\theta_n^s(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k). \tag{46}$$

Definition 2.4 ([6, p. 274])**.** *Let M be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$, and $\left(X, \|\cdot\|_\gamma\right)$ be a Banach space. Let $f \in C(M, X)$. We define the first modulus of continuity of f as*

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in M \\ \|x - y\|_p \leq \delta}} \|f(x) - f(y)\|_\gamma, \quad 0 < \delta \leq diam(M). \tag{47}$$

If $\delta > diam(M)$, then

$$\omega_1(f, \delta) = \omega_1(f, diam(M)). \tag{48}$$

Notice $\omega_1(f, \delta)$ is increasing in $\delta > 0$. For $f \in C_B(M, X)$ (continuous and bounded functions) $\omega_1(f, \delta)$ is defined similarly.

Lemma 2.5 ([6, p. 274])**.** *We have $\omega_1(f, \delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C(M, X)$, where M is a convex compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$.*

In this study we work only for the case of $p = \infty$.

Clearly we have also: $f \in C_U\left(\mathbb{R}^N, X\right)$ (uniformly continuous functions), iff $\omega_1(f, \delta) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (47). The space $C_B\left(\mathbb{R}^N, X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

Let now $f \in C^m\left(\prod_{i=1}^N [a_i, b_i]\right)$, $m, N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, \ldots, N$, and $|\alpha| := \sum_{i=1}^N \alpha_i = l$, where $l = 0, 1, \ldots, m$. We write also $f_\alpha := \dfrac{\partial^n f}{\partial x^n}$ and we say it is of order l.

We denote

$$\omega_{1,m}^{\max}(f_\alpha, h) := \max_{\alpha : |\alpha| = m} \omega_1(f_\alpha, h). \tag{49}$$

Call also

$$\|f_\alpha\|_{\infty, m}^{\max} := \max_{|\alpha| = m} \{\|f_\alpha\|_\infty\}, \tag{50}$$

where $\|\cdot\|_\infty$ is the supremum norm.

From now on we use $X = (\mathbb{C}, |\cdot|)$ the complex numbers, which is a Banach space.

(II) Multivariate New Taylor formulae

We will use

Theorem 2.6 ([7]). *Let $f \in C^2([c,d], \mathbb{C})$, where $a, x \in [c,d]$. Then*

$$f(x) - f(a) = f'(a)\sin(x-a) + 2f''(a)\sin^2\left(\frac{x-a}{2}\right) + \int_a^x \left[\Big(f''(t) + f(t)\Big) - \left(f''(a) + f(a)\right)\right]\sin(x-t)\,dt. \quad (51)$$

We make

Remark 2.7. *Let now Q be an open convex subset of $\mathbb{R}^k$, $k \geq 2$; $z = (z_1, \ldots, z_k)$, $x_0 := (x_{01}, \ldots, x_{0k}) \in Q$. We consider $f \in C^2(Q, \mathbb{C})$ each second order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$ and $|\alpha| := \sum_{i=1}^k \alpha_i = 2$. We consider $g_z(t) := f(x_0 + t(z - x_0))$, $0 \leq t \leq 1$. Clearly $x_0 + t(z - x_0) \in Q$. Then*

$$g_z(0) = f(x_0), \quad g_z(1) = f(z),$$

$$g_z'(t) = \sum_{i=1}^k (z_i - x_{0i}) \frac{\partial f}{\partial x_i}(x_{01} + t(z_1 - x_{01}), \ldots, x_{0k} + t(z_k - x_{0k})), \quad (52)$$

$$g_z'(0) = \sum_{i=1}^k (z_i - x_{0i}) \frac{\partial f}{\partial x_i}(x_{01}, \ldots, x_{0k}),$$

and

$$g_z''(t) = \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^2 f\right](x_{01} + t(z_1 - x_{01}), \ldots, x_{0k} + t(z_k - x_{0k})), \quad (53)$$

$$g_z''(0) = \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^2 f\right](x_{01}, \ldots, x_{0k}).$$

Notice above the second order partials commute.

Clearly $g_z \in C^2([0,1], \mathbb{C})$, and by Theorem 2.6 we obtain

$$f(z_1, \ldots, z_k) - f(x_{01}, \ldots, x_{0k}) = g_z(1) - g_z(0) =$$

$$g_z'(0)\sin(1) + 2g_z''(0)\sin 2\left(\frac{1}{2}\right) + \int_0^1 \left[\Big(g_z''(t)\, g_z(t)\Big) - \Big(g_z''(0) + g_z(0)\Big)\right]\sin(1-t)\,dt. \quad (54)$$

We also mention

Theorem 2.8 ([7])**.** *Let $f \in C^2([c,d], \mathbb{C})$, where $a, x \in [c,d]$. Then*

$$f(x) - f(a) = f'(a)\sinh(x-a) + 2f''(a)\sinh^2\left(\frac{x-a}{2}\right) + \int_a^x \Big[\big(f''(t) - f(t)\big) - \big(f''(a) - f(a)\big)\Big]\sinh(x-t)\,dt. \quad (55)$$

We make

Remark 2.9. *Consequently, we get that*

$$f(z_1,\ldots,z_k) - f(x_{01},\ldots,x_{0k}) = g_z(1) - g_z(0) = g_z'(0)\sinh(1) + 2g_z''(0)\sinh^2\left(\frac{1}{2}\right) + \int_0^1 \Big[\big(g_z''(t) - g_z(t)\big) - \big(g_z''(0) - g_z(0)\big)\Big]\sinh(1-t)\,dt. \quad (56)$$

We make

Remark 2.10. *Let $f \in C^2\left(\prod_{i=1}^N [a_i,b_i], \mathbb{C}\right)$, $N \in \mathbb{N}$.*
Clearly, the mixed partials commute.
Here $\frac{k}{n} := \left(\frac{k_1}{n},\ldots,\frac{k_N}{n}\right)$, and $x := (x_1,\ldots,x_N)$, with $\frac{k}{n}, x \in \left(\prod_{i=1}^N [a_i,b_i]\right)$, then (by (54)*, where $g_{\frac{k}{n}}(t) := f\left(x + t\left(\frac{k}{n} - x\right)\right)$, $0 \le t \le 1$) we have*

$$\begin{aligned} f\left(\frac{k}{n}\right) - f(x) = &\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial f}{\partial x_i}(x)\right)\sin(1) + 2\left\{\left[\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x)\right\}\sin^2\left(\frac{1}{2}\right) \\ &+ \int_0^1 \left\{\left\{\left[\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right]\left(x + t\left(\frac{k}{n} - x\right)\right) + f\left(x + t\left(\frac{k}{n} - x\right)\right)\right\}\right. \\ &\left. - \left\{\left[\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x) + f(x)\right\}\right\}\sin(1-t)\,dt. \end{aligned} \quad (57)$$

Denote the remainder

$$\begin{aligned} R := \int_0^1 &\left\{\left\{\left[\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right]\left(x + t\left(\frac{k}{n} - x\right)\right) + f\left(x + t\left(\frac{k}{n} - x\right)\right)\right\}\right. \quad (58) \\ &\left. - \left\{\left[\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x) + f(x)\right\}\sin(1-t)\,dt \right. \end{aligned}$$

$$= \int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N,\, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{k_i}{n} - x_i \right)^{\alpha_i} \right) \left[f_\alpha \left(x + t \left(\frac{k}{n} - x \right) \right) - f_\alpha (x) \right] \right.$$

$$\left. + \left(f \left(x + t \left(\frac{k}{n} - x \right) \right) - f(x) \right) \right\} \sin(1-t)\, dt.$$

Therefore it holds

$$|R| \leq \int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N,\, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) \left| f_\alpha \left(x + t \left(\frac{k}{n} - x \right) \right) - f_\alpha(x) \right| \right. \tag{59}$$

$$\left. + \left| f \left(x + t \left(\frac{k}{n} - x \right) \right) - f(x) \right| \right\} |\sin(1-t)|\, dt$$

$$\leq \int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) \omega_1 \left(f_\alpha, t \left\| \frac{k}{n} - x \right\|_\infty \right) \right.$$

$$\left. + \omega_1 \left(f, t \left\| \frac{k}{n} - x \right\|_\infty \right) \right\} |\sin(1-t)|\, dt \leq (*)$$

Notice here that $(0 < \beta < 1)$

$$\left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \iff \left| \frac{k_i}{n} - x_i \right| \leq \frac{1}{n^\beta}, \quad i = 1, \ldots, N. \tag{60}$$

We further see that:

$$(*) \leq \left\{ \omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) \left(\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N,\, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \frac{1}{n^{\beta\alpha_i}} \right) \right) + \omega_1\left(f, \frac{1}{n^\beta}\right) \right\} \tag{61}$$

$$\int_0^1 |\sin(1-t)|\, dt \left[\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) \left(\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N,\, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \right) \frac{1}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right) \right]$$

$$(1-\cos(1)) = (1-\cos(1)) \left\{ \frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right) \right\}.$$

We have proved that

$$|R| \leq (1-\cos(1)) \left\{ \frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right) \right\}, \tag{62}$$

given that $\left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}$.

We notice also that

$$|R| \leq \int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) 2 \|f_\alpha\|_\infty + 2\|f\|_\infty \right\} |\sin(1-t)|\, dt \tag{63}$$

$$\leq \left\{ \left(\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \right) 2\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} + 2\|f\|_\infty \right\} \left(\int_0^1 |\sin(1-t)|\, dt \right)$$

$$= \left(2\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + 2\|f\|_\infty \right)(1-\cos(1)),$$

where $a := (a_1, \ldots, a_N)$, $b = (b_1, \ldots, b_N)$.

We have proved that

$$|R| \leq \left(2 \|b-a\|_{\infty}^{2} \|f_{\alpha}\|_{\infty,2}^{\max} N^{2} + 2 \|f\|_{\infty}\right) (1-\cos(1)) =: \rho. \tag{64}$$

3 Main results

We start with symmetrized and perturbed trigonometric approximation by using the smoothness of f.

Theorem 3.1. *Let* $f \in C^{2}\left(\prod_{i=1}^{N} [a_i, b_i], \mathbb{C}\right)$, $0 < \beta < 1$; $n, N \in \mathbb{N}$, $n^{1-\beta} > 2$; $x, x_0 \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $a := (a_1, \ldots, a_N)$, $b := (b_1, \ldots, b_N)$. *Then:*

(i)

$$\Bigg| \theta_n^{s}(f,x) - f(x) - \left(\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} \theta_n^{s}((\cdot - x_i), x)\right) \sin(1) - \tag{65}$$

$$4 \left\{ \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N, |\alpha| := \sum_{i=1}^{N} \alpha_i = 2}} f_{\alpha}(x) \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right) \theta_n^{s}\left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i}, x\right) \right\} \sin 2\left(\frac{1}{2}\right) \Bigg|$$

$$\leq (\Delta(q))^{N} \left\{ \left[(1-\cos(1)) \left\{ \frac{\omega_{1,2}^{\max}\left(f_{\alpha}, \frac{1}{n^{\beta}}\right) N^{2}}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^{\beta}}\right) \right\} \right] + \right.$$

$$\left. \left[2 \|b-a\|_{\infty}^{2} \|f_{\alpha}\|_{\infty,2}^{\max} N^{2} + 2 \|f\|_{\infty} \right] (1-\cos(1)) T e^{-2\lambda n^{(1-\beta)}} \right\},$$

(ii) assume that $\frac{\partial f(x_0)}{\partial x_i} = 0$, $i = 1, \ldots, N$, *and* $f_{\alpha}(x_0) = 0$, $\alpha : |\alpha| = 2$, *we have that*

$$|\theta_n^{s}(f,x) - f(x)| \leq (\Delta(q))^{N} \left\{ \left[(1-\cos(1)) \left\{ \frac{\omega_{1,2}^{\max}\left(f_{\alpha}, \frac{1}{n^{\beta}}\right) N^{2}}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^{\beta}}\right) \right\} \right] + \right.$$

$$\left.\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)Te^{-2\lambda n^{(1-\beta)}}\right\}, \tag{66}$$

and

(iii)

$$\left|\theta_{n}^{s}\left(f,x\right)-f\left(x\right)\right|\leq\left(\Delta\left(q\right)\right)^{N}\left\{\left\{\left\{\sum_{i=1}^{N}\left|\frac{\partial f\left(x\right)}{\partial x_{i}}\right|\left\{\frac{1}{n^{\beta}}+\left(b_{i}-a_{i}\right)Te^{-2\lambda n^{(1-\beta)}}\right\}\right\}\sin\left(1\right)+\right.\right.\right.$$

$$\left.4\left\{\sum_{\alpha:|\alpha|=2}\left|f_{\alpha}\left(x\right)\right|\left(\frac{1}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}\left(b_{i}-a_{i}\right)^{\alpha_{i}}\right)Te^{-2\lambda n^{(1-\beta)}}\right]\right\}\sin^{2}\left(\frac{1}{2}\right)\right\}+$$

$$\left\{\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]\right. \tag{67}$$

$$\left.\left.+\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)Te^{-2\lambda n^{(1-\beta)}}\right\}\right\},$$

and

(iv)

$$\left\|\theta_{n}^{s}\left(f\right)-f\right\|_{\infty}\leq\left(\Delta\left(q\right)\right)^{N}\left\{\left\{\left\{\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{\infty}\left\{\frac{1}{n^{\beta}}+\left(b_{i}-a_{i}\right)Te^{-2\lambda n^{(1-\beta)}}\right\}\right\}\sin\left(1\right)+\right.\right.\right.$$

$$\left.4\left\{\sum_{\alpha:|\alpha|=2}\left\|f_{\alpha}\right\|_{\infty}\left(\frac{1}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}\left(b_{i}-a_{i}\right)^{\alpha_{i}}\right)Te^{-2\lambda n^{(1-\beta)}}\right]\right\}\sin^{2}\left(\frac{1}{2}\right)\right\} \tag{68}$$

$$+\left\{\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]\right.$$

$$+\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)Te^{-2\lambda n^{\left(1-\beta\right)}}\Bigg\}\Bigg\}=:\xi_{n}\left(f\right).$$

We observe that $\theta_n^s \to I$ (unit operator), as $n \to \infty$, pointwise and uniformly.

Proof. Here R is as in (58). We see that

$$U_{n}:=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx-k\right)R=\sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}Z\left(nx-k\right)R+\sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}Z\left(nx-k\right)R. \quad (69)$$

Therefore

$$\left|U_{n}\right|\leq\left(\sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}Z\left(nx-k\right)\right) \quad (70)$$

$$\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\rho Te^{-2\lambda n^{\left(1-\beta\right)}}$$

$$\leq\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\rho Te^{-2\lambda n^{\left(1-\beta\right)}}.$$

We have established that

$$\left|U_{n}\right|\leq\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]$$

$$+\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)Te^{-2\lambda n^{\left(1-\beta\right)}}. \quad (71)$$

By (57) we observe that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\right) \tag{72}$$

$$= \left(\sum_{i=1}^{N}\left(\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\left(\frac{k_i}{n}-x_i\right)\right)\frac{\partial f}{\partial x_i}(x)\right)\right)\sin(1)$$

$$2\left\{\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\,\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,\,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}} f_\alpha(x)\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\left(\prod_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)\right)\right\}\sin^2\left(\frac{1}{2}\right)+U_n.$$

The last says

$$^*\theta_n^s(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\right) - \left(\sum_{i=1}^{N}\frac{\partial f(x)}{\partial x_i}\ {}^*\theta_n^s((\cdot - x_i),x)\right)\sin(1) -$$

$$2\left\{\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\,\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,\,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}} f_\alpha(x)\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\ {}^*\theta_n^s\left(\prod_{i=1}^{N}(\cdot - x_i)^{\alpha_i},x\right)\right\}\sin^2\left(\frac{1}{2}\right) = U_n. \tag{73}$$

We notice that

$$\left|^*\theta_n^s((\cdot - x_i),x)\right| \leq^* \theta_n^s(|\cdot - x_i|,x) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left|\frac{k_i}{n}-x_i\right| Z(nx-k) \tag{74}$$

$$= \sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\left|\frac{k_i}{n}-x_i\right| Z(nx-k) + \sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\left|\frac{k_i}{n}-x_i\right| Z(nx-k)$$

$$\leq \frac{1}{n^\beta} + (b_i - a_i)\sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\lfloor nb\rfloor} Z(nx-k) \leq \frac{1}{n^\beta} + (b_i-a_i)\,Te^{-2\lambda n^{(1-\beta)}}.$$

We have proved that

$$\left|^*\theta_n^s((\cdot - x_i),x)\right| \leq \frac{1}{n^\beta} + (b_i-a_i)\,Te^{-2\lambda n^{(1-\beta)}}, \tag{75}$$

$i = 1,\ldots,N$.

Next we see that

$$\left| {}^{*}\theta_n^s \left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i}, x \right) \right| \leq {}^{*}\theta_n^s \left(\prod_{i=1}^{N} |\cdot - x_i|^{\alpha_i}, x \right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) Z(nx-k) \tag{76}$$

$$= \sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) Z(nx-k) + \sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) Z(nx-k)$$

$$\leq \frac{1}{n^{2\beta}} + \left(\prod_{i=1}^{N} (b_i - a_i)^{\alpha_i} \right) T e^{-2\lambda n^{(1-\beta)}}.$$

We have proved that

$$\left| {}^{*}\theta_n^s \left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i}, x \right) \right| \leq \frac{1}{n^{2\beta}} + \left(\prod_{i=1}^{N} (b_i - a_i)^{\alpha_i} \right) T e^{-2\lambda n^{(1-\beta)}}. \tag{77}$$

At last we observe that

$$\Bigg| \theta_n^s (f, x) - f(x) - \left(\sum_{=1}^{N} \frac{\partial f(x)}{\partial x_i} \theta_n^s ((\cdot - x_i), x) \right) \sin(1) - \tag{78}$$

$$4 \left\{ \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N, |\alpha| := \sum_{i=1}^{N} \alpha_i = 2}} f_\alpha(x) \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \theta_n^s \left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i}, x \right) \right\} \sin^2\left(\frac{1}{2}\right) \Bigg|$$

$$\leq (\Delta(q))^N |U_n| = (\Delta(q))^N \Bigg| {}^{*}\theta_n^s (f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \right) -$$

$$\left(\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} \, {}^{*}\theta_n^s ((\cdot - x_i), x) \right) \sin(1) -$$

$$4 \left\{ \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N, |\alpha| := \sum_{i=1}^{N} \alpha_i = 2}} f_\alpha(x) \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) {}^{*}\theta_n^s \left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i}, x \right) \right\} \sin^2\left(\frac{1}{2}\right) \Bigg|.$$

Putting all of the above together we prove the theorem. □

We continue with the hyperbolic symmetrized and perturbed approximation.

Theorem 3.2. *Let* $f \in C^2\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$, $0<\beta<1$; $n,N\in\mathbb{N}$, $n^{1-\beta}>2$; $x,x_0\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $a:=(a_1,\dots,a_N)$, $b:=(b_1,\dots,b_N)$. *Then*

(i)

$$\Bigg|\theta_n^s(f,x)-f(x)-\left(\sum_{=1}^{N}\frac{\partial f(x)}{\partial x_i}\theta_n^s((\cdot-x_i),x)\right)\sinh(1)-$$

$$4\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\,\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,\,|\alpha|:=\sum limits_{i=1}^{N}\alpha_i=2}} f_\alpha(x)\left(dfrac1\prod_{i=1}^{N}\alpha_i!\right)\theta_n^s\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right\}\sinh^2\left(\frac{1}{2}\right)\Bigg|$$

$$\le(\Delta(q))^N\cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]+\right. \tag{79}$$

$$\left.\left[\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+\|f\|_\infty\right]Te^{-2\lambda n^{(1-\beta)}}\right\},$$

(ii) assume that $\frac{\partial f(x_0)}{\partial x_i}=0$, $i=1,\dots,N$, *and* $f_\alpha(x_0)=0$, $\alpha:|\alpha|=2$, *we have that*

$$|\theta_n^s(f,x)-f(x)|\le(\Delta(q))^N(\cosh(1))\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]+\right.$$

$$\left.\left[\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+\|f\|_\infty\right]Te^{-2\lambda n^{(1-\beta)}}\right\}, \tag{80}$$

(iii)

$$|\theta_n^s(f,x)-f(x)|\le(\Delta(q))^N\left\{\left\{\left\{\sum_{i=1}^{N}\left|\frac{\partial f(x)}{\partial x_i}\right|\left\{\frac{1}{n^\beta}+(b_i-a_i)Te^{-2\lambda n^{(1-\beta)}}\right\}\right\}\sinh(1)+\right.\right.$$

$$4\left\{\sum_{\alpha:|\alpha|=2}|f_{\alpha}(x)|\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)Te^{-2\lambda n^{(1-\beta)}}\right]\right\}\sinh^2\left(\frac{1}{2}\right)\Bigg\}$$

$$+\cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\right. \tag{81}$$

$$\left.\left[\|b-a\|_{\infty}^2\|f_{\alpha}\|_{\infty,2}^{\max}N^2+\|f\|_{\infty}\right]Te^{-2\lambda n^{(1-\beta)}}\right\}\Bigg\},$$

and

(iv)

$$\|\theta_n^s(f)-f\|_{\infty}\leq(\Delta(q))^N\left\{\left\{\left\{\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}\left\{\frac{1}{n^{\beta}}+(b_i-a_i)Te^{-2\lambda n^{(1-\beta)}}\right\}\right\}\sinh(1)+\right.\right.$$

$$4\left\{\sum_{\alpha:|\alpha|=2}\|f_{\alpha}\|_{\infty}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)Te^{-2\lambda n^{(1-\beta)}}\right]\right\}\sinh^2\left(\frac{1}{2}\right)\Bigg\}$$

$$+\cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\right.$$

$$\left.\left[\|b-a\|_{\infty}^2\|f_{\alpha}\|_{\infty,2}^{\max}N^2+\|f\|_{\infty}\right]Te^{-2\lambda n^{(1-\beta)}}\right\}=:\psi_n(f). \tag{82}$$

We observe that $\theta_n^s\to I$ (unit operator), as $n\to\infty$, pointwise and uniformly.

Proof. As it is similar to the proof of [9, Theorem 5.8, pp. 175-179], it is omitted. □

We give

Remark 3.3. *By (42) we get that $\|\theta_n^s(f)\|_{\infty}\leq\|f\|_{\infty}<\infty$, and $\theta_n^s(f)\in C\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$, given that $f\in C\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$.*

Clearly then

$$\left\|\theta_n^{s2}(f)\right\|_\infty = \left\|\theta_n^s\left(\theta_n^s(f)\right)\right\|_\infty \leq \left\|\theta_n^s(f)\right\|_\infty \leq \|f\|_\infty\,, \tag{83}$$

etc.

Therefore we get

$$\left\|\left(\theta_n^s\right)^k(f)\right\|_\infty \leq \|f\|_\infty\,, \quad \forall\ k \in \mathbb{N}, \tag{84}$$

the contraction property.

Also we see that

$$\left\|\left(\theta_n^s\right)^k(f)\right\|_\infty \leq \left\|\left(\theta_n^s\right)^{k-1}(f)\right\|_\infty \leq \cdots \leq \left\|\theta_n^s(f)\right\|_\infty \leq \|f\|_\infty\,. \tag{85}$$

Also $\theta_n^s(1) = 1$, $\left(\theta_n^s\right)^k(1) = 1$, $\forall\ k \in \mathbb{N}$.

Following [5, 18.14, pp. 401-402], similarly we obtain that

$$\left\|\left(\theta_n^s\right)^r f - f\right\|_\infty \leq r\left\|\theta_n^s(f) - f\right\|_\infty\,, \quad r \in \mathbb{N}. \tag{86}$$

We give

Theorem 3.4. *All as in Theorems 3.1, 3.2. Then*

(i)

$$\left\|\left(\theta_n^s\right)^r f - f\right\|_\infty \leq r\xi_n(f)\,, \tag{87}$$

where $\xi_n(f)$ *as in* (68).

(ii)

$$\left\|\left(\theta_n^s\right)^r f - f\right\|_\infty \leq r\psi_n(f)\,, \tag{88}$$

where $\psi_n(f)$ *as in* (82).

So that the speed of convergence to the unit operator of $\left(\theta_n^s\right)^r$ *is not worse than of* θ_n^s*, see also [4].*

References

[1] G. Anastassiou, *Quantitative approximations.* Boca Raton, FL: Chapman & Hall/CRC, 2001.

[2] G. A. Anastassiou, "Rate of convergence of some neural network operators to the unit-univariate case," *J. Math. Anal. Appl.*, vol. 212, no. 1, pp. 237–262, 1997, doi: 10.1006/jmaa.1997.5494.

[3] G. A. Anastassiou, *Intelligent systems. Approximation by artificial neural networks*, ser. Intell. Syst. Ref. Libr. Berlin: Springer, 2011, vol. 19, doi: 10.1007/978-3-642-21431-8.

[4] G. A. Anastassiou, "Approximation by neural networks iterates," in *Advances in Applied Mathematics and Approximation Theory: Contributions from AMAT 2012.* Springer, 2013, pp. 1–20.

[5] G. A. Anastassiou, *Intelligent systems II. Complete approximation by neural network operators*, ser. Stud. Comput. Intell. Cham: Springer, 2016, vol. 608, doi: 10.1007/978-3-319-20505-2.

[6] G. A. Anastassiou, *Intelligent computations: abstract fractional calculus, inequalities, approximations*, ser. Stud. Comput. Intell. Cham: Springer, 2018, vol. 734, doi: 10.1007/978-3-319-66936-6.

[7] G. A. Anastassiou, "Opial and Ostrowski type inequalities based on trigonometric and hyperbolic type Taylor formulae," *Malaya J. Mat*, vol. 11, pp. 1–26, 2023, doi: 10.26637/mjm11S/001.

[8] G. A. Anastassiou, *Parametrized, deformed and general neural networks*, ser. Stud. Comput. Intell. Cham: Springer, 2023, vol. 1116, doi: 10.1007/978-3-031-43021-3.

[9] G. A. Anastassiou, *Trigonometric and hyperbolic generated approximation theory*, ser. Ser. Concr. Appl. Math. Singapore: World Scientific, 2025, vol. 24, doi: 10.1142/13857.

[10] Z. Chen and F. Cao, "The approximation operators with sigmoidal functions," *Computers & Mathematics with Applications*, vol. 58, no. 4, pp. 758–765, 2009, doi: 10.1016/j.camwa.2009.05.001.

[11] D. Costarelli and M. Piconi, "Asymptotic analysis of neural network operators employing the Hardy-Littlewood maximal inequality," *Mediterr. J. Math.*, vol. 21, no. 7, 2024, Art. ID 199, doi: 10.1007/s00009-024-02752-8.

[12] D. Costarelli and R. Spigler, "Approximation results for neural network operators activated by sigmoidal functions," *Neural Netw.*, vol. 44, pp. 101–106, 2013, doi: 10.1016/j.neunet.2013.03.015.

[13] D. Costarelli and R. Spigler, "Multivariate neural network operators with sigmoidal activation functions," *Neural Netw.*, vol. 48, pp. 72–77, 2013, doi: 10.1016/j.neunet.2013.07.009.

[14] S. Haykin, *Neural networks. A comprehensive foundation.* New York, NY: Macmillan College Publ. Co., 1994.

[15] W. S. McCulloch and W. Pitts, "A logical calculus of the ideas immanent in nervous activity," *Bull. Math. Biophys.*, vol. 5, pp. 115–133, 1943, doi: 10.1007/BF02478259.

[16] T. Mitchell, *Machine learning.* New York, NY: WCB-McGraw-Hill, 1997.

[17] D. Yu and F. Cao, "Construction and approximation rate for feedforward neural network operators with sigmoidal functions," *Journal of Computational and Applied Mathematics*, vol. 453, 2025, Art. ID 116150, doi: 10.1016/j.cam.2024.116150.