

Rings in which every ideal disjoint with S is S -almost prime

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ABSTRACT

Let R be a commutative ring with identity and S a multiplicative subset of R . In this study, we introduce the concept of rings in which every ideal disjoint with S is S -almost prime. We investigate the possible transfer of the above ring property in the quotient rings, localizations, direct products, trivial ring extensions, and amalgamation algebra.

RESUMEN

Sea R un anillo conmutativo con identidad y S un conjunto multiplicativo de R . En este estudio, introducimos el concepto de anillos en los cuales todo ideal disjunto con S es S -casi primo. Investigamos la posible transferencia de la propiedad de anillos anterior en anillos cociente, localizaciones, productos directos, extensiones triviales de anillos y el álgebra de amalgamación.

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1 Introduction

Throughout this paper, all rings are assumed to be commutative with unity, all modules are unital. By $I(R)$ (and $I(R)^*$), we denote the set of (proper) ideals of a ring R . It is well-known that $Q \in I(R)^*$ is called prime if whenever $x, y \in R$ with $xy \in Q$ then either $x \in Q$ or $y \in Q$. Equivalently, if whenever $IJ \subseteq Q$ for some ideals I, J of R , then either $I \subseteq Q$ or $J \subseteq Q$. Since the notion of prime ideals plays an important role in commutative ring theory, several generalizations of the concept of prime ideals have been studied in the literature, for example: almost prime, strongly prime, weakly prime, S -prime and S -almost prime ideals. Recall from [8] that $Q \in I(R)^*$ is said to be weakly prime if for $x, y \in R$ with $0 \neq xy \in Q$, then either $x \in Q$ or $y \in Q$. Clearly every prime ideal is weakly prime, but the converse is not true, obviously $\{0\}$ is always weakly prime, but not prime provided that R is a ring which is not an integral domain. For non-trivial examples, refer to [8]. In 2005, Bhatwadekar and Sharma [11] said $Q \in I(R)^*$ where R is an integral domain to be almost prime if for $a, b \in R$ with $ab \in Q - Q^2$, then either $a \in Q$ or $b \in Q$. It is evident that this definition can be applied to any commutative ring R . Consequently, any weakly prime ideal is almost prime. Furthermore, an ideal $Q \in I(R)^*$ is almost prime if and only if Q/Q^2 is a weakly prime in the quotient ring R/Q^2 .

Consider a multiplicative set (briefly, m.s) S of a ring R that satisfies $0 \notin S$, $1 \in S$, and $xy \in S$ for all $x, y \in S$. In recent times, the notion of S -extensions of certain ideal structures has assumed considerable significance within the domain of commutative algebra, thus attracting the interest of numerous authors. The concept of (resp. weakly) S -prime ideals has been introduced and thoroughly investigated in (resp. [5]) [16]. An ideal I of a ring R disjoint with an m.s S is said to be (resp. weakly) S -prime ideal if there exists $s \in S$ such that for all $a, b \in R$ if $ab \in I$ (resp. $0 \neq ab \in I$), then $sa \in I$ or $sb \in I$. (See also, [1, 20–22].)

The primary focus of this study is the definition and study of almost prime ideals in [4]. Let P be an ideal of R disjoint with an m.s S . Then P is an S -almost prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in P - P^2$, then $sa \in P$ or $sb \in P$.

This paper presents a new type of ring of which every ideal disjoint with S is S -almost prime (called S -AP ring) as a generalization of ring in which every proper ideal is almost prime (called AP ring). If $S \subseteq U(R)$, then the concepts S -AP ring and AP-ring coincide. Nevertheless, these two types of rings are significantly dissimilar in general. Subsequently, an example is presented in order to demonstrate that [4, Theorems 2.20 and 2.21] do not hold in general. The subsequent investigation will address the potential transfer of the ring property that every ideal disjoint with S is S -almost prime in localization, direct product, homomorphic image, trivial ring extensions, and the amalgamation rings along an ideal. This property is employed to construct new and intriguing examples. An example is provided in order to demonstrate that the statement [4, Theorem 2.9] is not generally valid. In order to conclude the relationship between an S -AP ring

and its homomorphic images, we discuss the extensions and contractions of an S -almost prime ideals under a homomorphic image. Afterwards, we present the corrected and generalized version of [4, Theorem 2.9] (see Proposition 3.1 and Corollary 3.2, and then Theorem 3.3). We give characterizations for $(S \times E)$ -AP rings regarding the trivial ring extension of a ring R by an R -module E denoted by $R \times E$ (see Theorem 4.1). Moreover, we investigate the relationship between S -AP rings A and the ideals of the ring $A \bowtie^f J$ the amalgamation of A and B along J with respect to f (see Theorem 4.7 and Corollaries 4.8, 4.9).

2 Main results

To avoid repetition, throughout the article unless otherwise stated, let R be a ring and S a multiplicative set (briefly, m.s) of R . As it is frequently used in this sequel, we should recall from [4] that an ideal $P \in I(R)$ disjoint with S is said to be an S -almost prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in P - P^2$, then $sa \in P$ or $sb \in P$. In this case, P is said to be an S -almost prime ideal associated with s . Now the following is our key definition.

Definition 2.1. *A ring R with an m.s S is called S -AP ring if every ideal disjoint with S is S -almost prime.*

It is interesting to note that the authors in [4] have been studying this class of rings in [4, Theorems 2.20 and 2.21]. However, we show by the following example that those theorems do not hold in general.

Example 2.2. *Let $R = D \times K$ where D is a domain and K is a field, and let the multiplicative subset $S = D^* \times \{1\}$ of R where $D^* = D - \{0\}$. Then the ideals of R which are disjoint with S are just $0 \times K$ and $I \times 0$ where I is an ideal of D . Note that $0 \times K$ is prime and so S -almost prime. Clearly, 0×0 is almost prime and so S -almost prime. Finally, if I is a nonzero ideal of D , then for $a \in I - \{0\}$, we conclude that $(I \times 0 : (a, 1)) = D \times 0$ is a prime ideal of R , and hence, $I \times 0$ is S -prime by [16, Proposition 1]. Thus, $I \times 0$ is also an S -almost prime ideal of R . Thus, R is an S -AP-ring.*

Remark 2.3. (1) *Let S be a multiplicative set of a ring R . If R is an AP ring, then R is an S -AP ring.*

(2) *If $S \subseteq U(R)$, then R is an S -AP ring if and only if R is an AP ring.*

(3) *Let $S_1 \subseteq S_2$ be multiplicative subsets of R , and P an ideal of R disjoint with S_2 . Clearly, if P is an S_1 -almost prime ideal of R , then P is S_2 -almost prime. Nevertheless, it has been demonstrated that the converse is not true in general. Indeed, let $R = \mathbb{Z}[X]$, $S = \{2^n : n \in \mathbb{N}\}$ and $T = \{1\} \subseteq U(R)$. Consider the ideal $P = 4X\mathbb{Z}[X]$ of R . From [16, Example 1],*

P is an S -prime ideal of R , and hence P is an S -almost prime ideal of R . Note that P is not a T -almost prime ideal of R as $4X \in P - P^2$ but neither $1 \cdot 4 \in P$ nor $1 \cdot X \in P$.

The converse of Remark 2.3(1) is not true in general since not every almost prime ideal is an S -almost prime ideal by [4, Example 2.3].

Let R_1, \dots, R_n be rings and S_1, \dots, S_n be m.ss of rings R_1, \dots, R_n , respectively. Then $S = \prod_{i=1}^n S_i$ is an m.s of the ring $R = \prod_{i=1}^n R_i$. We study the property of the stability of S -AP rings under direct products.

Theorem 2.4. *Let S_1, \dots, S_n be m.ss of rings R_1, \dots, R_n , respectively. Put $R = \prod_{i=1}^n R_i$ and $S = \prod_{i=1}^n S_i$. If R is an S -AP ring, then R_i is an S_i -AP ring for each $1 \leq i \leq n$.*

In order to demonstrate the validity of the aforementioned theorem, it is necessary to verify the following lemma.

Lemma 2.5. *Let R_1 and R_2 be commutative rings, S_1 and S_2 be m.ss of R_1 and R_2 , respectively, $R = R_1 \times R_2$ and $S = S_1 \times S_2$. If I and J be proper ideals of R_1 and R_2 , respectively, it can thus be concluded that the following statements are true.*

- (1) I is an S_1 -almost prime ideal of R_1 if and only if $I \times R_2$ is an S -almost prime ideal of R .
- (2) J is an S_2 -almost prime ideal of R_2 if and only if $R_1 \times J$ is an S -almost prime ideal of R .

Proof. (1) Assume that I is an S_1 -almost prime ideal of R_1 . Let $(a, b), (a', b') \in R$ such that $(a, b)(a', b') = (aa', bb') \in I \times R_2 - (I \times R_2)^2$. Then $aa' \in I - I^2$ which implies that there exists $s_1 \in S_1$ such that $s_1a \in I$ or $s_1a' \in I$. Put $s = (s_1, 1) \in S$. Hence $s(a, b) \in I \times R_2$ or $s(a', b') \in I \times R_2$, then $I \times R_2$ is an S -almost prime ideal of R .

Conversely, suppose that $I \times R_2$ is an S -almost prime ideal of R , and $a, b \in R_1$ such that $ab \in I - I^2$. Then $(a, 0)(b, 0) \in I \times R_2 - (I \times R_2)^2$ and there exists $(s_1, s_2) \in S$ such that $(s_1, s_2)(a, 0) \in I \times R_2$ or $(s_1, s_2)(b, 0) \in I \times R_2$, hence $s_1a \in I$ or $s_1b \in I$. Thus, I is an S_1 -almost prime ideal of R_1 .

- (2) Similar to (1). □

Proof of Theorem 2.4. It is sufficient to demonstrate the claim for $n = 2$

Assume that $R = R_1 \times R_2$ is an S -AP ring. Let P_1 be an ideal of R_1 disjoint with S_1 . Then $P_1 \times R_2$ is an S -almost prime ideal of R , and so P_1 is S_1 -almost prime by Lemma 2.5. Therefore, R_1 is an S_1 -AP ring. Similar to the argument above, we conclude that R_2 is an S_2 -AP ring. The rest of the proof is clear by using the mathematical induction on n . □

The following example is given to demonstrate that the converse of Theorem 2.4 is not true in general.

Example 2.6. *Let (R, M) be a local ring which is not a field where $M^2 = 0$ and $S = \{1\}$. Then R is S -AP ring by [6, Theorem 17], but $R \times R$ is not necessarily $S \times S$ -AP ring.*

Proof. We only need to show that $M \times M$ is not S -almost prime, let $0 \neq x \in M$ so $(1, x)(x, 1) = (x, x) \in (M \times M) - (M \times M)^2$ but neither $(1, 1)(1, x) \in M \times M$ nor $(1, 1)(x, 1) \in M \times M$, as desired. □

3 Extensions of S -AP rings

In this section, we discuss behavior of S -AP rings under some ring extensions such as ring homomorphism and localization. Moreover, a characterization for S -Noetherian ring is obtained.

We note that the second part of [4, Theorem 2.9] does not hold in general. By the following theorem, we give a generalized and corrected version of [4, Theorem 2.9].

Proposition 3.1. *Let $f : R_1 \rightarrow R_2$ be a ring epimorphism, S be an m.s of R_1 . Then the following assertions hold.*

- (1) *If I is an S -almost prime ideal of R_1 containing $\ker f$, then $f(I)$ is an $f(S)$ -almost prime ideal of R_2 .*
- (2) *If J is an $f(S)$ -almost prime ideal of R_2 and $\ker f \subseteq f^{-1}(J)^2$, then $f^{-1}(J)$ is an S -almost prime ideal of R_1 .*

Proof. (1) First, we show that $f(S)$ is an m.s of R_2 . Clearly, for any $f(s_1), f(s_2) \in f(S)$, $f(s_1)f(s_2) = f(s_1s_2) \in f(S)$. Now, assume that $0 \in f(S)$. Then $f(s) = 0$ for some $s \in S$. This implies that $s \in \ker f$. Since $\ker f \subseteq I$ and $I \cap S = \emptyset$, we have $\ker f \cap S = \emptyset$, a contradiction. Thus, $0 \notin f(S)$ and $f(S)$ is an m.s of R_2 . Assume that $f(s) \in f(I) \cap f(S)$ for some $s \in S$. Then $f(a) = f(s)$ for some $a \in I$ which yields $a - s \in \ker f \subseteq I$, and so $s \in I \cap S$, a contradiction. Hence, $f(I) \cap f(S) = \emptyset$. Suppose that $xy \in f(I) \setminus f(I)^2$ for some $x = f(a)$, $y := f(b)$ in R_2 . From $\ker f \subseteq I$, we have $ab \in I$ and clearly $ab \notin I^2$. Since I is an S -almost prime ideal of R_1 , there exists $s \in S$ such that $sa \in I$ or $sb \in I$. Thus, there exists $f(s) \in f(S)$ satisfying $f(s)f(a) \in f(I)$ or $f(s)f(b) \in f(I)$, as required.

- (2) If $s \in f^{-1}(J) \cap S$, then $f(s) \in J \cap f(S)$, a contradiction. Hence, $f^{-1}(J) \cap S = \emptyset$. Let $f(s) \in f(S)$, an element associated with J . Let $a, b \in R_1$ such that $ab \in f^{-1}(J) \setminus f^{-1}(J)^2$. It is clear that $f(a)f(b) \in J$ and $ab \notin f^{-1}(J)^2$. Now, we show that $f(a)f(b) \notin J^2$. If $f(ab) \in J^2$, then there is $c \in f^{-1}(J^2)$ such that $f(ab) = f(c)$, hence $ab - c \in \ker f \subseteq f^{-1}(J^2)$ and

$ab \in f^{-1}(J^2) = f^{-1}(J)^2$, which is a contradiction. Hence, $f(a)f(b) \in J \setminus J^2$, and therefore, either $f(s)f(a) \in J$, hence $as \in f^{-1}(J)$, or $f(s)f(b) \in J$, hence $bs \in f^{-1}(J)$, and $f^{-1}(J) \subset R_1$ is S -almost prime. \square

Let R be a ring, $S \subseteq R$ be m.s and I an ideal of R such that $I \cap S = \emptyset$. Let $s \in S$. By \bar{s} , we denote the class of s in R/I and set $\bar{S} = \{\bar{s} : s \in S\}$. It is clear that \bar{S} is a m.s of R/I .

Now, we are ready to give the following relationship between S -almost prime ideals of a ring and those of their quotient rings.

Corollary 3.2 (The corrected version of [4, Theorem 2.9]). *Let S be an m.s of a ring R and $P, I \in I(R)$ such that $P \subseteq I^2$ and $S \cap P = \emptyset$. Then $I \in I(R)$ is S -almost prime if and only if $I/P \in I(R/P)$ is \bar{S} -almost prime.*

In view of Corollary 3.2, we conclude the following result.

Theorem 3.3. *Let S be an m.s of a ring R and $P \in I(R)$ such that $S \cap P = \emptyset$. Then R is a ring of which every ideal I satisfying $P \subseteq I^2$ is S -almost prime if and only if R/P is a \bar{S} -AP ring.*

Proof. The claim is clear by Corollary 3.2. \square

Let R and T be rings, let S a multiplicative subset of R and $f : R \rightarrow T$ a ring homomorphism. Then it is easy to see that whenever $\ker f \cap S = \emptyset$, $f(S)$ is a multiplicative subset of T .

Theorem 3.4. *Let $f : R \rightarrow T$ be a surjective ring homomorphism and S a multiplicative subset of R with $\ker f \cap S = \emptyset$. If R is an S -AP ring, then T is an $f(S)$ -AP ring.*

Proof. Let J be a proper ideal of T disjoint with $f(S)$. Since f is a surjective ring homomorphism, there exists an ideal $f^{-1}(J)$ of R such that $f(f^{-1}(J)) = J$. Note that $J \cap f(S) = \emptyset$ if and only if $f^{-1}(J) \cap S = \emptyset$. Since R is an S -AP ring, $f^{-1}(J)$ is an S -almost prime ideal of R and clearly $\ker f \subseteq f^{-1}(J)$. Hence, J is an $f(S)$ -almost prime ideal of T by Proposition 3.1(1), and thus, T is an $f(S)$ -AP ring. \square

The next proposition studies the S -AP property under the ring extension $R \subseteq T$, where (R, T) is a pair of rings.

Proposition 3.5. *Let $R \subseteq T$ be a ring extension such that $IT \cap R = I$ for each ideal I of R and $S \subseteq R$ a multiplicative set. If T is an S -AP ring, then so is R .*

Proof. Let I be a proper ideal of R disjoint with S . Now we will show that IT is a proper ideal of T disjoint with S . Indeed, if $s \in IT \cap S$, then $s \in IT \cap R = I$, a contradiction. Since T is an S -AP ring, then IT is an S -almost prime ideal of T . The rest of the proof is obtained by [4, Theorem 2.17]. \square

Let R be a ring and S an m.s of R . The saturation of S is defined as

$$S^* = \left\{ x \in R : \frac{x}{1} \text{ is a unit of } S^{-1}R \right\}.$$

Note that S^* is an m.s containing S .

Proposition 3.6. *Let R be a ring and S be an m.s of R . Then the following assertions are equivalent.*

- (1) R is an S -AP ring.
- (2) R is an S^* -AP ring.

Proof. For any ideal $P \subset R$, we have $P \cap S = \emptyset$ if, and only if, $P \cap S^* = \emptyset$ because $S^* = \{k \in R : \text{there exists } s \in S \text{ such that } ks \in S\}$. On the other hand, if $P \subset R$ is S -almost prime with associated element $s \in S$, so it is S^* -almost prime with associated element $s \in S^*$. Conversely, if $P \subset R$ is S^* -almost prime with associated element $k \in S^*$, there exists $s \in S$ such that $ks \in S$, so it is S -almost prime with associated element ks . □

The next result gives the closed relationship between S -AP rings and AP-rings concluded from [4, Theorem 2.12].

Proposition 3.7. *If R is an S -AP ring, then $S^{-1}R$ is a AP ring.*

Anderson and Dumitrescu’s introduction of the concept of S -Noetherian rings constitutes a generalisation of Noetherian rings. As previously defined [7], an S -Noetherian ring is one in which any ideal I is S -finite, *i.e.* there exists $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$. The following result characterises the S -Noetherian ring property using the notion of S -almost prime.

Proposition 3.8. *The following statements are equivalent.*

- (1) R is an S -Noetherian ring.
- (2) Any S -almost prime ideal is S -finite.
- (3) Any almost prime ideal is S -finite.
- (4) Any prime ideal is S -finite.

Proof. (1) \implies (2) Suppose that R is S -Noetherian. Consequently, it can be deduced that every ideal is S -finite. Moreover, it is evident that every S -almost prime is S -finite.

- (2) \implies (3) Assume the hypothesis that every S -almost prime ideal is S -finite, it can be deduced that if P is an almost prime ideal of R and $S \cap P \neq \emptyset$, $s \in S \cap P$, then $sP \subseteq sR \subseteq I$. Consequently, P is an S -principal ideal of R , so P is S -finite. On the other hand, if $S \cap P = \emptyset$, then P is S -almost prime ideal of R , so by hypothesis, P is S -finite.
- (3) \implies (4) Assume that every almost prime ideal is S -finite. Let P be a prime ideal of R . By definition, P is an almost prime ideal of R , and thus, by hypothesis P is S -finite.
- (4) \implies (1) This is obtained by [7, Corollary 5]. □

4 Applications in idealization and amalgamation rings

Let R be a ring and L an R -module. The trivial ring extension of R by L (also termed the idealization of L over R) is a commutative ring

$$R \times L := \{(a, l) \mid a \in R \text{ and } l \in L\}$$

under the usual addition and the multiplication defined by $(a, l)(b, m) = (ab, am + bl)$ for all $(a, l), (b, m) \in R \times L$. It is clear that $(1, 0)$ is the identity of $R \times L$, and if S is a m.s of R , then $S \times L$, and $S \times 0$ are m.s of $R \times L$. In the field of commutative ring theory, trivial ring extensions have been shown to play a pivotal role. This is due to the effectiveness of this method in producing new classes of examples and counter-examples of rings subject to various ring theoretic properties. For a more detailed exposition of this topic, the reader is referred to the following source: [2, 3, 9, 10, 17, 18].

Theorem 4.1. *Let R be a ring and L an R -module, and let $R \times L$ be trivial ring extension of R by L and S be an m.s of R . Then the following statements hold.*

- (1) *If $R \times L$ is a $(S \times L)$ -AP ring, then R is an S -AP ring.*
- (2) *Let R be an integral domain with quotient field Q and L be a Q -vector space. Then the following assertions are shown to be equivalent.*
- i) $R \times L$ is an $(S \times L)$ -AP ring.*
 - ii) R is an S -AP ring.*

To prove this theorem, we need the following lemmas.

Lemma 4.2 ([9, Corollary 3.4]). *Let R be an integral domain and L an R -module. Then the following conditions are equivalent.*

- (1) *Any ideal of $R \times L$ is comparable to $0 \times L$.*
- (2) *Any ideal of $R \times L$ is of the form $I \times L$ for some ideal I of R or $0 \times N$ for some submodule N of L .*
- (3) *Any ideal of $R \times L$ is homogeneous.*
- (4) *L is divisible.*

Lemma 4.3. *Let R be an integral domain with quotient field Q , L be a Q -vector space and N be a Q -vector subspace of L . Then $0 \times N$ is a weakly prime ideal of $R \times L$.*

Proof. Note that if $a \in R \setminus \{0\}$ and $l \in L \setminus N$, then $al \notin N$. Hence for every $a \in R$ and $l \in L \setminus N$, we get $al = 0$ or $al \notin N$. Therefore by [19, Corollary 3.2] we have $0 \times N$ is a weakly prime ideal of $R \times L$, as desired. □

Proof of Theorem 4.1.

- (1) Clear by [4, Theorem 3.1].
- (2) (i) \implies (ii) From (1).
 (ii) \implies (i) Assume that R is an S -AP ring. By Lemma 4.2, every ideal of $R \times L$ has the form $P \times L$ for some ideal P of R or $0 \times N$ for some submodule N of L .

Case 1: Suppose that $J = P \times L$ is an ideal of $R \times L$ disjoint with $S \times L$. Then clearly $P \cap S = \emptyset$. Let $(x, l_1), (y, l_2) \in R \times L$ such that $(x, l_1)(y, l_2) \in P \times L - (P \times L)^2$. Then $(xy, xl_2 + yl_1) \in P \times L - (P \times L)^2$, and so $xy \in P - P^2$. Since P is S -almost prime, there exists $s \in S$ with $sx \in P$ or $sy \in P$. Hence $(s, 0)(x, l_1) \in P \times L$ or $(s, 0)(y, l_2) \in P \times L$. Thus, $P \times L$ is an $(S \times L)$ -almost prime ideal of $R \times L$.

Case 2: Assume that $J = 0 \times N$ is an ideal of $R \times L$. Then $J = 0 \times N$ is a weakly prime ideal of $R \times L$ by Lemma 4.3, and so $0 \times N$ is an $(S \times L)$ -almost prime ideal of $R \times L$.

Thus $R \times L$ is an $(S \times L)$ -AP ring. □

Example 4.4. *Consider the ring \mathbb{Z} and the m.s $S = \mathbb{Z} \setminus \{0\}$ of \mathbb{Z} . Then $\mathbb{Z} \times \mathbb{Q}$ is an $(S \times \mathbb{Q})$ -AP ring by Theorem 4.1 (2).*

Theorem 4.5. *Let (R, M) be a local ring, $S \subseteq R$ an m.s and L an R -module such that $ML = 0$. Then $R \rtimes L$ is a $(S \rtimes L)$ -AP ring if and only if R is an S -AP ring.*

Proof. Let R be an S -AP ring. Suppose that $S \not\subseteq U(R)$. Let I be an ideal of $R \rtimes L$ disjoint with $S \rtimes L$. Set $I_0 = \{a \in R : (a, 0) \in I\}$. It is clear to see that I_0 is an ideal of R disjoint with S . Now let $(a, l), (b, m) \in R \rtimes L$ such that $(a, l)(b, m) = (ab, am + bl) \in I - I^2$. There are three possible cases:

Case 1: If $a \notin M$, then $a \in U(R)$, and so $(a, l) \in U(R \rtimes L) = U(R) \rtimes L$. Hence $(b, m) = (a, l)^{-1}(a, l)(b, m) \in I$, and so $(s, 0)(b, m) \in I$ for all $s \in S$.

Case 2: If $b \notin M$. Similarly we get $(a, l)(b, m)(b, m)^{-1} = (a, l) \in I$, and so $(s, 0)(a, l) \in I$ for all $s \in S$.

Case 3: If $a, b \in M$, we have $(a, l)(b, m) = (ab, 0) \in I - I^2$, so $ab \in I_0$ and $ab \notin I_0^2$. Since if $ab = \sum_{i=1}^n a_i b_i \in I_0^2$ with $a_i, b_i \in I_0$ for every $i \in \{1, \dots, n\}$. Thus $(a_i, 0), (b_i, 0) \in I$ for every $i \in \{1, \dots, n\}$. Then $\sum_{i=1}^n (a_i, 0)(b_i, 0) = (ab, 0) \in I^2$ which is absurd. So $ab \in I_0 - I_0^2$. Hence there exists $s \in S$ such that $sa \in I_0$ or $sb \in I_0$ as I_0 is S -almost prime. We have already assumed that $S \not\subseteq U(R)$. On the other hand, since (R, M) is a local ring, $U(R) = R \setminus M$. In conclusion, $S \cap M \neq \emptyset$. Let $s' \in S \cap M$. If $sa \in I_0$, then $(ss', 0)(a, l) = (ss'a, ss'l) = (ss'a, 0) = (s', 0)(sa, 0) \in I$. If $sb \in I_0$, then similarly we get $(ss', 0)(b, m) \in I$. Thus, I is $(S \rtimes L)$ -almost prime and $R \rtimes L$ is an $(S \rtimes L)$ -AP ring.

If $S \subseteq U(R)$, then R is a local ring with every ideal is almost prime so by [6, Theorem 17] $M^2 = 0$ and hence, $R \rtimes L$ is a local ring with maximal ideal $M \rtimes L$. Now $(M \rtimes L)^2 = M^2 \rtimes ML = 0$, and hence every ideal of $R \rtimes L$ disjoint with $(S \rtimes L)$ is $(S \rtimes L)$ -almost prime by [6, Theorem 17] as desired. The converse part follows from Theorem 4.1. \square

Example 4.6. *Let $S = \mathbb{Z} \setminus p\mathbb{Z}$ where p is a prime number. Consider $R = \mathbb{Z}_{(p)} = S^{-1}\mathbb{Z}$ which is a local domain with maximal ideal $M = p\mathbb{Z}_{(p)}$. Let E be an R/M -vector space. Then $R \rtimes E$ is a $(S \rtimes E)$ -AP ring.*

Let (R, R') be a pair of rings, J be an ideal of R' and $f : R \rightarrow R'$ be a homomorphism. In this section, we consider the following subring of $R \times R'$

$$R \rtimes^f J = \{(a, f(a) + j) : a \in R \text{ and } j \in J\}$$

is called the amalgamation of R and R' along J with respect to f . If f is the identity homomorphism on R , then we get the amalgamated duplication of R along an ideal J , $R \rtimes J = \{(a, a + j) : a \in R, j \in J\}$. As a natural generalization of the duplication construction in [14], the

amalgamation ring was initiated by D'Anna, Finocchiaro and Fontana. For more details regarding amalgamation rings, we refer the reader to [12, 13, 15, 21].

Let S be an m.s of a ring R . Notice that $S' = \{(s, f(s)) : s \in S\}$ is an m.s of $R \bowtie^f J$. Also, if $0 \notin f(S)$ then $f(S)$ is an m.s of R' . Set $T = \{I \bowtie^f J : I \text{ ideal of } R\}$.

Theorem 4.7. *Let R and R' be two rings, S be an m.s of R , J an ideal of R' and $f : R \rightarrow R'$ be a ring homomorphism. Then the following statements hold.*

- (1) *If $R \bowtie^f J$ is an S' -AP ring, then R is an S -AP ring.*
- (2) *Let $f^{-1}(J) = \{0\}$. Then $R \bowtie^f J$ is an S' -AP ring if and only if $f(R) + J$ is an $f(S)$ -AP ring.*
- (3) *Let $f(a)J = 0$ for every nonunit $a \in R$. Then R is an S -AP ring if and only if every ideal in T is an S' -almost prime ideal of $R \bowtie^f J$.*

Proof. (1) Assume that $R \bowtie^f J$ is a S' -AP ring. We prove that R is a ring in which every ideal is S -almost prime. Let I be an ideal of R and $a, b \in R$ with $ab \in I - I^2$, then $(a, f(a))(b, f(b)) \in I \bowtie^f J - (I \bowtie^f J)^2$. So, there exists $s \in S$ such that $(s, f(s))(a, f(a)) \in I \bowtie^f J$ or $(s, f(s))(b, f(b)) \in I \bowtie^f J$. Then $sa \in I$ or $sb \in I$, hence I is S -almost prime in R , so R is an S -AP ring.

(2) As $f^{-1}(J) = 0$, from the claim [12, Proposition 5.1(3)] we have the isomorphism $R \bowtie^f J \cong f(R) + J$. Let $\psi : R \bowtie^f J \rightarrow f(R) + J$ be the natural projection of $R \bowtie^f J \subseteq R \times (f(R) + J)$ into $f(R) + J$. Then ψ is a surjective ring homomorphism and its kernel is $\ker(\psi) = f^{-1}(J) \times \{0\} = 0$. Thus ψ is an isomorphism ring homomorphism with $\psi(S') = f(S)$.

(3) Assume that R is an S -AP ring. Let $L = I \bowtie^f J$ be an ideal in T . We prove that $I \bowtie^f J$ is S' -almost prime. Let $(a, f(a) + i), (b, f(b) + j) \in R \bowtie^f J$ such that $(a, f(a) + i)(b, f(b) + j) = (ab, f(ab) + f(a)j + f(b)i + ij) \in I \bowtie^f J - (I \bowtie^f J)^2$, then two cases are possible:

Case 1: If $ab \in I - I^2$, then there exists $s \in S$ such that $sa \in I$ or $sb \in I$. Hence $(s, f(s))(a, f(a) + i) \in I \bowtie^f J$ or $(s, f(s))(b, f(b) + j) \in I \bowtie^f J$. Hence, $I \bowtie^f J$ is a S' -almost prime ideal of T .

Case 2: If $ab \in I^2$, then $ab = \sum_{i=1}^n a_i b_i$ with $a_i, b_i \in I$ so $(ab, f(ab) + ij) = \sum_{i=1}^n (a_i b_i, f(a_i b_i)) + (ab, f(ab) + ij)$
 $(a_1 b_1, f(a_1 b_1) + ij)$ hence $(ab, f(ab) + ij) = \sum_{i=2}^n (a_i, f(a_i))(b_i, f(b_i)) + (a_1, f(a_1) + i)$
 $(b_1, f(b_1) + j) \in (I \bowtie^f J)^2$ a contradiction, as desired. □

As a conclusion of Theorem 4.7 (3), we have the following corollary.

Corollary 4.8. *Let (R, M) be a local ring with $f(M)J = 0$. Then R is an S -AP ring if and only if every ideal in T is S' -almost prime.*

Let I be a proper ideal of a ring R . The (amalgamated) duplication of R along I is a special amalgamation given by

$$R \bowtie I := \{(a, a + i) \mid a \in R, i \in I\}$$

Note that if S is an m.s of R , then $S' = \{(s, s) \mid s \in S\}$ is an m.s of $R \bowtie I$. Set, $T' = \{K \bowtie I : K \text{ ideal of } R\}$.

Corollary 4.9. *Let R be a ring, S a m.s of R and $I \in I(R)$ such that $aI = 0$ for every nonunit $a \in R$. Then R is an S -AP ring if and only if every ideal in T' is S' -almost prime.*

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