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# Persistence of a tumor spheroid with an almost periodic nutrient supply

Homero G. Díaz-Marín[1]
Osvaldo Osuna[2,✉]
Geiser Villavicencio-Pulido[3]

[1] *Facultad de Ciencias Físico-Matemáticas, Universidad Michoacana, Ciudad Universitaria, C.P. 58040. Morelia, Michoacán, México.*
homero.diaz@umich.mx

[2] *Instituto de Física y Matemáticas, Universidad Michoacana Ciudad Universitaria, C.P. 58040. Morelia, Michoacán, México.*
osvaldo.osuna@umich.mx[✉]

[3] *División de Ciencias Biológicas y de la Salud, Depto. de Ciencias Ambientales, Universidad Autónoma Metropolitana Unidad Lerma, Av. Hidalgo Poniente No. 46, Col. La Estación, 52006 Lerma de Villada, Edo. de México, México.*
j.villavicencio@correo.ler.uam.mx

**ABSTRACT**

We prove that a spherical tumor with free boundary furnished with an almost periodic nutrient supply has a twofold long term time evolution: either it vanishes or it tends towards a persistent tumor which oscillates almost periodically. This is determined by a relation of the mean of the nutrient supply and a threshold value meaning the minimal nutrient supply enabling the tumor to live. In each case, global stability is proved for the almost periodic solution $(\sigma_\star(t,x), P_\star(t,x))$ of the corresponding reaction-diffusion equation.

**RESUMEN**

Demostramos que un tumor esférico con frontera libre dotado con un suministro casi periódico de nutrientes tiene una evolución a largo tiempo doble: o bien desaparece o tiende a un tumor persistente que oscila casi periódicamente. Esto está determinado por una relación del promedio del suministro de nutrientes y un valor umbral, es decir, el suministro mínimo de nutrientes que le permite vivir al tumor. En cada caso, se demuestra la estabilidad global para la solución casi periódica $(\sigma_\star(t,x), P_\star(t,x))$ de la ecuación de reacción-difusión correspondiente.

**Keywords and Phrases:** Tumor growth, spheroid tumor, almost periodic function, reaction-diffusion equation.

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# 1 Introduction: A simple spheroid model for tumor-growth

Works [3, 14] are seminal works of an abundant production of mathematical models describing tumor growth, see for instance [6, 9, 10, 13, 15]. The fundamental tool is the reaction-diffusion equation which models the volume growth under the presence of nutrients and inhibitors. The time evolution of the volume of the multicellular tumor spheroid living in a fluid containing nutrients is then given by the solutions to a free boundary problem. Complex models consider a necrotic core in the description of the tumor. In our considerations we avoid such element of the model. A common feature of all models is the prediction a twofold scenario depending on a nutrient supply threshold: either the tumor shrinks and vanishes or the tumor persists.

Typically, under *in vitro* conditions, nutrient as well as inhibitor supplies remain constant. Nevertheless, in more realistic conditions tumors grow up upon varying tissue conditions. With this motivation, experimental data having periodic supply have been obtained for instance in [8]. More recently, experimental designs with oscillatory nutrient and inhibitor supplies have been reported in [11]. Accordingly, mathematical modeling with time-dependent external environment have arisen. For periodic continuous nutrient and inhibitor supplies, there are results in [12,15,16]. This works describe again conditions under which either the tumor vanishes or the tumor remains periodically changes size.

Our main contribution in this work, is considering a more general time-dependent oscillatory nutrient supply. Instead of discussing a constant or periodic external nutrient concentration, we introduce in our model an *almost periodic continuous* function. We can mention two reasons to use this space of functions: (1) Almost periodic functions incorporate variations in experimental conditions that not necessarily are periodic but only approximately periodic. (2) Almost periodic functions permit different factors which may not necessarily be synchronized. This may be interesting when we deal with two growth factors such as nutrient supply and inhibitors, where there is no requirement of rationally dependent frequencies. The case where almost periodic nutrient supply and almost periodic inhibitory factor are non-synchronized will be treated elsewhere. In this work we treat the inhibitor-free case.

We review some of the main concepts and results about almost periodic functions in the Appendix of Section 4. The interested reader may also consult well known references such as [1, 2, 4, 7].

Now, we describe our setting, see for instance [3, 6, 10, 15] for further explanations. Let $\Omega(t) \subset \mathbb{R}^3$ be a bounded region with smooth boundary, $\partial\Omega(t)$, evolving in time. This region is supposed to model the tumor inside a continuum media. We designate by $\sigma(t, x)$ the nutrient concentration in a time-space domain $(t, x) \in \mathbb{R} \times \mathbb{R}^3$. When there are no inhibitors in the continuum media, then the nutrient supply concentration $\sigma$ is proportional to the intensity of the mitosis,

$$S = \mu(\sigma - \tilde{\sigma}).$$

The proportionality constant, $\mu$ measures the intensity of the cell division. The parameter $\tilde{\sigma}$ is a threshold level which divides two regimes: either the tumor grows‘ due to mitosis or the tumor shrinks, due to apoptosis.

In a vascularization free environment, we suppose that $\sigma$ satisfies a reaction-diffusion equation,

$$c\frac{d\sigma}{dt} = \Delta\sigma - \lambda\sigma,$$

where, $\lambda > 0$ is the nutrient consumption rate. The small time-scale ratio between the nutrient diffusion $T_\sigma$ compared to the tumor volume evolution $T_R$, leads to a quasi-stationary evolution where

$$c = \frac{T_\sigma}{T_R} = \frac{1\,\text{min}}{1\,\text{day}} \approx 0.$$

Thus, steady solutions of the reaction-diffusion equation become relevant. Assuming a velocity, $\vec{v}(t,x)$, for the cell-flow inside the tumor, Darcy's law describes this flow as generated by the pressure gradient, $\vec{v} = -\nabla P$, where $P(t,x)$ is the pressure inside the tumor. Since the flow has the mitosis process as source, then $\nabla \cdot \vec{v} = S$. Therefore,

$$-\Delta P = \nabla \cdot (-\nabla P) = \nabla \cdot \vec{v} = S.$$

Thus, the stationary reaction-diffusion equation describing the concentration of the nutrient and the pressure are the following,

$$\Delta\sigma(t,x) = \lambda\sigma(t,x), \qquad x \in \Omega(t), \quad t > 0, \tag{1.1}$$

$$-\Delta P(t,x) = \mu(\sigma(t,x) - \tilde{\sigma}), \qquad x \in \Omega(t), \quad t > 0. \tag{1.2}$$

For an external nutrient supply, $\Phi(t) \geq 0$, homogeneous along the membrane, we have the following free-boundary value conditions

$$\sigma(t,x) = \Phi(t), \qquad x \in \partial\Omega(t), \quad t > 0, \tag{1.3}$$

$$P(t,x) = \gamma H(t,x), \qquad x \in \partial\Omega(t), \quad t > 0, \tag{1.4}$$

where $\gamma > 0$ is a constant representing the cell adhesiveness and $H(t,x)$ designates the mean curvature of the boundary surface, $\partial\Omega(t,x)$.

Upon radial symmetry assumption, let $R(t)$ be the outer radius of the sphere $\partial\Omega(t)$, then the PDE problem (1.1) with boundary conditions (1.3) becomes a couple of ODE boundary problems

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\sigma(t,r)\right) &= \lambda\sigma, & 0<r<R(t),\\
\sigma(t,R(t)) &= \Phi(t), & \frac{\partial}{\partial r}\sigma(t,0)=0,\\
-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}P(t,r)\right) &= \mu(\sigma-\tilde{\sigma}), & 0<r<R(t),\\
P(t,R(t)) &= \frac{\gamma}{R(t)}, & \frac{\partial}{\partial r}P(t,0)=0,\\
t &> 0, & R(0)=R_0.
\end{aligned} \tag{1.5}$$

For $P$ we have a second order ODE with boundary conditions both, on $P$ and on its derivative. Moreover, the Neumann boundary condition $\dfrac{\partial P(t,R(t))}{\partial r} = -R'(t)$, arises from the restriction of Darcy's law, $\nabla P = -\vec{v}$, along the normal component of the spherical surface $\partial\Omega(t)$, *i.e.*

$$\frac{\partial P}{\partial \vec{n}} = -\vec{v}|_{\partial\Omega(t)}.$$

This makes the problem a free-boundary one with velocity displacement of the boundary given by $\vec{v}$.

Length rescaling allows us to take $\lambda = 1$. Then, equations (1.5) are solved as follows,

$$\begin{aligned}
\sigma_\star(t,r) &= \Phi(t)\frac{R(t)}{\sinh R(t)}\frac{\sinh r}{r},\\
P_\star(t,r) &= \frac{\mu\tilde{\sigma}}{6}\left(r^2-(R(t))^2\right)+\mu(\Phi(t)-\sigma_\star(t,r))+\frac{\gamma}{R(t)}.
\end{aligned}$$

Notably, the flow, $\displaystyle\iint_{\partial\Omega(t)} \vec{v}\cdot d\mathrm{S}$, of $\vec{v}$ along the boundary $\partial\Omega(t)$ equals the rate of volume change. Mass and volume conservation inside cells imply

$$\frac{d}{dt}\mathrm{vol}(\Omega(t)) = \iiint_{\Omega(t)} \nabla\cdot\vec{v}\, d\,\mathrm{vol}.$$

Hence,

$$\frac{d}{dt}\left(\frac{4\pi R(t)^3}{3}\right) = 4\pi\int_0^{R(t)} \mu(\sigma(t,r)-\tilde{\sigma})r^2\,dr,$$

or

$$R'(t) = \frac{1}{R(t)^2}\int_0^{R(t)} \mu(\sigma(t,r)-\tilde{\sigma})r^2\,dr.$$

The PDE problem (1.5) can be solved in this way by solving the ODE

$$R' = \mu R\left(p(R)\Phi(t) - \frac{\tilde{\sigma}}{3}\right)$$
$$R(t_0) = R_0 \geq 0, \quad \Phi(t) \geq 0, \tag{1.6}$$

where

$$p(x) = \frac{1}{x\tanh x} - \frac{1}{x^2}$$

is strictly decreasing in $x > 0$, and $0 < p(x) < \frac{1}{3}$, see [6, Lemma 3.2].

We propose the following global stability assertion which depends on a condition on the *mean value,*

$$\overline{\Phi} = M[\Phi]$$

of the nutrient supply. See Appendix in Section 4 for the definition of the mean value, $M[\phi]$ of an almost periodic function $\phi(t)$.

**Theorem 1.1.** *Let $R(t)$ be any solution of (1.6) with positive initial condition $R(t_0) > 0$ and an almost periodic nutrient supply $\phi(t) \geq 0$, then we have two possible limits:*

*(1) If $\overline{\Phi} \leq \tilde{\sigma}$, then $\lim_{t\to\infty} R(t) = 0$.*

*(2) If $\overline{\Phi} > \tilde{\sigma}$, then there exists a unique almost periodic solution $R^{\star}(t)$ of (1.6) such that $\lim_{t\to\infty} |R(t) - R^{\star}(t)| = 0$. Furthermore, there is an inclusion of the modules of frequencies, $\widehat{\varphi} \subset \widehat{\Phi}$.*

For the reader's convenience the definition of *module of frequencies* can also be consulted in Section 4. This generalizes the result for the periodic case obtained in [12]. Referring to the problem (1.1) with boundary conditions (1.3), we deduce the following consequence.

**Corollary 1.2.** *Upon radial symmetry, when the mean value of the supply $\overline{\Phi}$ surpasses the threshold value, $\tilde{\sigma}$, the free boundary problem (1.1) has a globally asymptotically stable almost periodic solution $(\sigma_\star, P_\star)$. Otherwise, the free tumor equilibrium is attained in the long term by any initial condition.*

## 2 Proof of the main result

This Section is devoted to demonstrate our main result stated in Theorem 1.1. The main idea is to prove boundedness, above and below, in the open interval $(0, \infty)$ of every solution $\phi(t)$ whose initial condition $\phi(t_0)$ is positive. The main idea then reduces to show that the relative compacity or normal property of the family of translated solutions. A limit in this family provides an asymptotically almost periodic solution. We first review some technical Lemmas.

**Lemma 2.1.** *Let $\phi(t), \varphi(t)$ be any couple of solutions corresponding to initial conditions $\phi(t_0) > \varphi(t_0) \geq 0$ of the ODE*

$$x' = g(t, x).$$

*Suppose that $g(t, x)$ is continuous and $C^1$ with respect to $x$. Suppose that solutions are well-defined for all $t > t_0$. Then $\phi(t) > \varphi(t)$. In particular, for $\phi(t_0) > 0$ we have $\phi(t) > 0$ for all $t \geq t_0$, whenever $\varphi(t) \equiv 0$ is a solution.*

Lemma 2.1 is well known and follows immediately from the property of uniqueness of solutions of an ODE. From boundedness of solutions claimed in Lemma 2.3 below, it can be applied to (1.6).

**Lemma 2.2.** *If $\overline{\Phi} < \tilde{\sigma}$, then for any solution of* (1.6) *with positive initial condition $\phi(t_0) > 0$, we have $\lim_{t\to\infty} \phi(t) = 0$.*

*Proof.* We observe that $(\ln \phi(t))' = \dfrac{\phi'(t)}{\phi(t)}$. We proceed by contradiction and suppose that,

$$\phi(t_k) \searrow \Theta = \limsup_{t\geq 0} \phi(t) > 0$$

for an increasing sequence $t_k \nearrow \infty$. We recall that

$$\frac{\phi'(t)}{\phi(t)} = \mu \left[ p(\phi(t))\Phi(t) - \frac{\tilde{\sigma}}{3} \right] \leq \frac{\mu}{3}(\Phi(t) - \tilde{\sigma}).$$

Therefore,

$$\limsup_{k\to\infty} \frac{\ln \phi(t_k) - \ln \phi(t_0)}{t_k - t_0} \leq \limsup_{k\to\infty} \frac{\mu}{3(t_k - t_0)} \int_{t_0}^{t_k} (\Phi(t) - \tilde{\sigma})\, dt = \frac{\mu}{3} \left( \overline{\Phi} - \tilde{\sigma} \right).$$

or,

$$\limsup_{k\to\infty} \frac{\ln \dfrac{\phi(t_k)}{\phi(t_0)}}{t_k - t_0} \leq -\frac{\mu(\tilde{\sigma} - \overline{\Phi})}{3} < 0.$$

Thus, $0 \leq \limsup_{k\to\infty} \ln \phi(t_k) = -\infty$ or $\limsup_{k\to\infty} \phi(t_k) = 0$. □

**Lemma 2.3.** *Any solution, $\phi(t)$, of* (1.6) *with initial $\phi(t_0) > 0$ is bounded above, i.e.*

$$\phi(t) \leq \sup\{\phi(t) \,:\, t \geq t_0\} < \infty.$$

*Proof.* For the upper bound suppose that there exists an increasing sequence $t_k > t_{k-1} > t_0$ such that $t_k \to \infty$ when $k \to \infty$.

By contradiction, let us suppose that $\lim_{k\to\infty} \phi(t_k) = \infty$. Without loss of generality, by taking subsequence if necessary, we can suppose that $\phi(t_k) \nearrow \infty$ is monotone increasing and that $\phi'(t_k) > 0$.

Then $\lim_{k\to\infty} p(\phi(t_k)) = 0$, and

$$(\ln \phi(t_k))' = \frac{\phi'(t_k)}{\phi(t_k)} \geq 0.$$

Hence,

$$\mu\frac{\tilde{\sigma}}{3} = \lim_{k\to\infty} \mu p(\phi(t_k))\Phi(t_k) - (\ln \phi(t_k))' \leq 0.$$

However, $\tilde{\sigma} > 0$. This finishes the proof of Lemma 2.3. □

**Lemma 2.4.** *If* $\overline{\Phi} > \tilde{\sigma} > 0$, *then any solution,* $\phi(t)$, *of* (1.6) *with positive initial condition,* $\phi(t_0) > 0$ *is bounded below, i.e.*

$$\phi(t) \geq \inf\{\phi(t) \,:\, t \geq t_0\} = \phi_* > 0.$$

*Proof.* Suppose that $\phi_* = 0$. Recalling Lemma 2.1 we have that $\phi(t) > 0$ for every $t \geq t_0$, and by continuity there is no interval $[t_0, t_2]$ where

$$\inf\{\phi(t) \,:\, t_0 \leq t \leq t_2\} = 0.$$

Therefore, there exists an increasing sequence $t_k \to \infty$ such that

$$\phi(t) \geq \phi(t_k) \searrow 0, \quad p(\phi(t)) \leq p(\phi(t_k)) \nearrow 1/3, \quad \forall t \in [t_0, t_k],$$

because $p(x)$ is a decreasing function. Then $\ln \phi(t_k)$ would tend towards $-\infty$. Therefore,

$$0 \geq \liminf_{k\to\infty} \frac{\ln \phi(t_k) - \ln(\phi(t_0))}{t_k - t_0}.$$

Moreover, if $\overline{\Phi} > \tilde{\sigma}$ we consider $1/3 > \varepsilon > 0$ such that

$$(1 - 3\varepsilon)\overline{\Phi} > \tilde{\sigma}.$$

Due to the convergence $p(\phi(t_k)) \nearrow 1/3$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\varepsilon \geq \frac{1}{3} - p(\phi(t)) \geq \frac{1}{3} - p(\phi(t_k)), \quad \forall t \in [t_k, t_j], \quad j \geq k \geq N_\varepsilon.$$

Therefore,

$$\begin{aligned}
0 &\geq \liminf_{k\to\infty} \frac{\ln(\phi(t_k)/\phi(t_0))}{t_k - t_0} = \sup_{k\geq 0} \inf_{j\geq k} \frac{\mu}{t_j - t_0} \int_{t_0}^{t_j} p(\phi(t))\Phi(t) - \frac{1}{3}\tilde{\sigma}\, dt \\
&= \sup_{k\geq 0} \inf_{j\geq k} \left\{ \frac{\mu}{t_j - t_0} \int_{t_k}^{t_j} p(\phi(t))\Phi(t) - \frac{1}{3}\tilde{\sigma}\, dt + \frac{\mu}{t_j - t_0} \int_{t_0}^{t_k} p(\phi(t))\Phi(t) - \frac{1}{3}\tilde{\sigma}\, dt \right\} \\
&\geq \sup_{k\geq 0} \inf_{j\geq k} \left\{ \frac{\mu}{t_j - t_0} \int_{t_k}^{t_j} p(\phi(t))\Phi(t) - \frac{1}{3}\tilde{\sigma}\, dt \right\} + 0
\end{aligned}$$

$$\begin{aligned}
&\geq \sup_{k\geq N_\varepsilon} \inf_{j\geq k} \frac{\mu}{t_j - t_0}\int_{t_k}^{t_j} p(\phi(t))\Phi(t) - \frac{1}{3}\tilde{\sigma}\,dt \\
&\geq \mu \cdot \sup_{k\geq N_\varepsilon} \inf_{j\geq k} \frac{1}{t_j - t_0}\int_{t_k}^{t_j} \left(\frac{1}{3} - \varepsilon\right)\Phi(t) - \frac{1}{3}\tilde{\sigma}\,dt \\
&\geq \left(\frac{1}{3} - \varepsilon\right)\mu \cdot \sup_{k\geq N_\varepsilon} \inf_{j\geq k} \frac{t_j - t_k}{t_j - t_0}\frac{1}{t_j - t_k}\int_{t_k}^{t_j} \Phi(t) - \frac{1}{1-3\varepsilon}\tilde{\sigma}\,dt \\
&= \left(\frac{1}{3} - \varepsilon\right)\mu \cdot \sup_{k\geq N_\varepsilon} \inf_{j\geq k} \frac{1}{t_j - t_k}\int_{t_k}^{t_j} \Phi(t) - \frac{1}{1-3\varepsilon}\tilde{\sigma}\,dt \\
&= \left(\frac{1}{3} - \varepsilon\right)\mu\left(\overline{\Phi} - \frac{1}{1-3\varepsilon}\tilde{\sigma}\right) = \frac{1}{3}\mu\left((1-3\varepsilon)\overline{\Phi} - \tilde{\sigma}\right) > 0.
\end{aligned}$$

We reach a contradiction. So $\phi_* > 0$. □

**Lemma 2.5.** *If $\overline{\Phi} = \tilde{\sigma} > 0$, then for any solution, $\phi(t)$, of* (1.6) *with positive initial condition, $\phi(t_0) > 0$, we have $\lim_{t\to\infty} \phi(t) = 0$.*

*Proof.* Define

$$\Theta := \limsup_{T\to\infty} \phi(T), \quad \theta := \liminf_{T\to\infty} \phi(T) \geq 0.$$

Notice that $0 \leq \theta \leq \Theta < +\infty$. We shall prove by contradiction that $\Theta = \theta = 0$.

**Suppose that $\theta > 0$:** We have, $\phi_* > 0$, then $p(\phi_*) < 1/3$. Therefore, for every increasing sequence $t_0 < t_k < t_j$, such that $\phi(t_k) \leq \phi(t_j)$ we have $p(\phi(t_k)) \leq p(\phi(t_j)) \leq p(\phi_*)$. Whence,

$$\begin{aligned}
\limsup_{j\to\infty} \frac{\ln\phi(t_j) - \ln\phi(t_k)}{t_j - t_k} &= \limsup_{j\to\infty} \frac{\mu}{t_j - t_k}\int_{t_j}^{t_k} p(\phi(t))\Phi(t) - \frac{\tilde{\sigma}}{3}\,dt \\
&\leq \limsup_{j\to\infty} \frac{\mu}{t_j - t_k}\int_{t_j}^{t_k} p(\phi_*)\Phi(t) - \frac{\tilde{\sigma}}{3}\,dt = \mu\left[p(\phi_*) - \frac{1}{3}\right]\overline{\Phi} < 0.
\end{aligned}$$

In particular, for a sequence $\phi(t_k) \nearrow \theta > 0$ we reach a contradiction. Namely,

$$0 \leq \limsup_{j\to\infty} \frac{\ln\phi(t_j) - \ln\phi(t_k)}{t_j - t_k} < 0.$$

Therefore $\theta = 0$.

**Suppose that $\Theta > \theta = 0$:** Since $\theta = 0$, there exists an increasing sequence $t'_k \nearrow \infty$, such that $\phi(t'_k) \searrow 0$. Thus,

$$p(\phi(t'_k)) \leq p(\phi(t'_j)) \nearrow 1/3, \quad \forall j \geq k.$$

For every $\varepsilon > 0$, due to the convergence $p(\phi(t'_k)) \nearrow 1/3$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\varepsilon \geq \frac{1}{3} - p(\phi(t)) \geq \frac{1}{3} - p(\phi(t'_k)), \quad \forall t \in [t'_k, t'_j], \quad j \geq k \geq N_\varepsilon.$$

Therefore,

$$
\begin{aligned}
-\liminf_{k\to\infty}\frac{\ln(\phi(t'_k)/\phi(t_0))}{t'_k-t_0} &= \limsup_{k\to\infty}-\frac{\ln(\phi(t'_k)/\phi(t_0))}{t'_k-t_0}\\
&= \inf_{k\geq 0}\sup_{j\geq k}\frac{\mu}{t'_j-t_0}\left[\int_{t'_k}^{t'_j}\frac{\tilde{\sigma}}{3}-p(\phi(t))\Phi(t)\,dt+\int_{t'_0}^{t'_k}\frac{\tilde{\sigma}}{3}-p(\phi(t))\Phi(t)\,dt\right]\\
&= \inf_{k\geq 0}\sup_{j\geq k}\frac{\mu}{t'_j-t_0}\int_{t'_k}^{t'_j}\frac{\tilde{\sigma}}{3}-p(\phi(t))\Phi(t)\,dt+0\\
&\leq \inf_{k\geq N_\varepsilon}\sup_{j\geq k}\frac{\mu}{t'_j-t_0}\int_{t'_k}^{t'_j}\frac{\tilde{\sigma}}{3}-p(\phi(t))\Phi(t)\,dt\\
&\leq \mu\cdot\inf_{k\geq N_\varepsilon}\sup_{j\geq k}\frac{t'_j-t'_k}{t'_j-t_0}\frac{1}{t'_j-t'_k}\int_{t'_k}^{t'_j}\frac{\tilde{\sigma}}{3}-p(\phi(t))\Phi(t)\,dt\\
&\leq \mu\cdot\inf_{k\geq N_\varepsilon}\sup_{j\geq k}\frac{1}{t'_j-t'_k}\int_{t'_k}^{t'_j}\frac{\tilde{\sigma}}{3}+\left(\varepsilon-\frac{1}{3}\right)\Phi(t)\,dt\\
&\leq \mu\left[\frac{\tilde{\sigma}}{3}+\left(\varepsilon-\frac{1}{3}\right)\overline{\Phi}\right]=\mu\varepsilon\overline{\Phi}
\end{aligned}
$$

Hence, $\liminf_{k\to\infty}\frac{\ln(\phi(t'_k)/\phi(t_0))}{t'_k-t_0} > -\mu\varepsilon\overline{\Phi}$ for each $\varepsilon>0$, *i.e.*

$$
\liminf_{k\to\infty}\frac{\ln(\phi(t'_k)/\phi(t_0))}{t'_k-t_0}\geq 0. \tag{2.1}
$$

Now, suppose that $\Theta>0$, then $p(\Theta)<1/3$. Therefore, if we take a sequence $t_k\nearrow\infty$ such that $\phi(t_k)\searrow\Theta>0$, we have $p(\phi(t_j)<p(\phi(t_k))<p(\Theta)<1/3$, for every $j>k$ and

$$
\begin{aligned}
\limsup_{j\to\infty}\frac{\ln\phi(t_j)-\ln\phi(t_k)}{t_j-t_k} &= \limsup_{j\to\infty}\frac{\mu}{t_j-t_k}\int_{t_j}^{t_k}p(\phi(t))\Phi(t)-\frac{\tilde{\sigma}}{3}\,dt\\
&\leq \mu\left[p(\Theta)\overline{\Phi}-\frac{\tilde{\sigma}}{3}\right]=\overline{\Phi}\mu\left[p(\Theta)-\frac{1}{3}\right]<0.
\end{aligned}
$$

Then,

$$
\limsup_{j\to\infty}\frac{\ln\phi(t_j)-\ln\phi(t_k)}{t_j-t_k}<0.
$$

Supposing, without loss of generality, that $\phi(t_0)\geq\phi(t_k)$

$$
\limsup_{j\to\infty}\frac{\ln\phi(t_j)-\ln\phi(t_0)}{t_j-t_0}=\limsup_{j\to\infty}\frac{\ln\phi(t_j)-\ln\phi(t_k)}{t_j-t_k}\frac{t_j-t_k}{t_j-t_0}\leq\limsup_{j\to\infty}\frac{\ln\phi(t_j)-\ln\phi(t_k)}{t_j-t_k}.
$$

Therefore,

$$
\limsup_{j\to\infty}\frac{\ln\phi(t_j)-\ln\phi(t_0)}{t_j-t_0}<0. \tag{2.2}
$$

By summarizing (2.1) and (2.2), we finally reach a contradiction as follows. Since $\Theta>0$ we can, without loss of generality, suppose that $\phi(t_j)\geq\Theta\geq\phi(t'_l)$. We can also suppose, without loss of generality, that $t_j < t'_j$. Hence,

$$0 \leq \liminf_{j\to\infty} \frac{\ln \phi(t'_j) - \ln \phi(t_0)}{t'_j - t_0} \leq \limsup_{j\to\infty} \frac{\ln \phi(t_j) - \ln \phi(t_0)}{t_j - t_0} < 0. \qquad \square$$

Finding *asymptotically almost periodic* solutions, see Definition 4.4 in Section 4, allows us to find almost periodic solutions too, according to Theorem 4.5 in Section 4. Among the characterizations of asymptotically almost periodic functions we have the Bochner's property described in Theorem 4.6 in Section 4 which will be useful to prove our main result given in Theorem 1.1.

Now we complete the proof of Theorem 1.1.

*Proof of existence.* We suppose that $\overline{\Phi} > \tilde{\sigma}$ and take a solution $\phi(t)$ of (1.6). Due to Lemmas 2.2 and 2.3, $\phi(t)$ is contained in a compact $K = [\phi_*, \phi^*] \subset (0, \infty)$ for $t \geq t_0$. For an increasing divergent sequence $h = \{h_k \geq t_0\}_{k\in\mathbb{N}}$, $\lim_{k\to\infty} h_k = \infty$, we define $\phi_k(t) = \phi(t + h_k)$. Then

$$|(\ln \phi_k(t) - \ln \phi_m(t))'| = \left| \frac{\phi'_k(t)}{\phi_k(t)} - \frac{\phi'_m(t)}{\phi_m(t)} \right| \leq \frac{\mu}{3} \left| \Phi(t + h_k) - \Phi(t + h_m) \right|.$$

Where the second inequality follows from substitution

$$\frac{\phi'_k(t)}{\phi_k(t)} - \frac{\phi'_m(t)}{\phi_m(t)} = \mu \left[ p(\phi(t + h_k))\Phi(t + h_k) - p(\phi(t + h_m))\Phi(t + h_m) \right],$$

which in turn yields due to the bound $p(t) \leq 1/3$.

Invoking Bochner's property in Theorem 4.3, Section 4, modulo a subsequence we can suppose that $\lim_{k\to\infty} \Phi(t + h_k)$ uniformly along $\mathbb{R}$. Hence

$$\psi_\sharp(t) = \lim_{k\to\infty} \psi_k(t), \quad \psi_k(t) = (\ln \phi_{n_k}(t))',$$

converges uniformly along $[t_0, \infty)$. This proves that both $(\ln \phi(t))'$ and $\psi_\sharp(t)$ are asymptotically almost periodic, according to Theorem 4.6 in the Appendix. This means that we can decompose

$$\varphi(t) := (\ln \phi(t))' = \varphi_\flat(t) + q(t), \quad \lim_{t\to\infty} q(t) = 0,$$

where $\varphi_\flat(t)$ is almost periodic.

Notably, it can be deduced that $y = \ln \phi(t)$ is a solution of

$$\frac{dy}{dt} = \mu \left( p\left(e^y\right) \Phi - \frac{\tilde{\sigma}}{3} \right). \tag{2.3}$$

Hence,

$$y = \tilde{\varphi}(t) := \int_{t_0}^{t} \varphi(s)\, ds + \ln \phi(t_0),$$

is a solution of (2.3). From boundedness of solutions proven in Lemmas 2.3, we can see that a solution $\exp \tilde{\varphi}(t)$ of (2.3) is bounded in $(0, \infty)$ as $t \to \infty$. Therefore $\tilde{\varphi}(t)$ remains bounded above and below as $t$ grows.

On the other hand, if

$$\tilde{\varphi}_\flat(t) := \int_{t_0}^{t} \varphi_\flat(s)\, ds + \ln \phi(t_0) + c_1, \quad \tilde{q}(t) = \int_{t_0}^{t} q(s)\, ds - c_1$$

then $\tilde{\varphi} = \tilde{\varphi}_\flat + \tilde{q}$ and

$$\tilde{\varphi}_\flat' + \tilde{q}' = \tilde{\varphi}_\flat' + q = \mu \left( p \left( e^{\tilde{\varphi}_\flat} e^{\tilde{q}} \right) \Phi - \frac{\tilde{\sigma}}{3} \right),$$

Remark that $\tilde{\varphi}_\flat$ is defined up to an arbitrary constant $c_1 \in \mathbb{R}$. Notice that we do not claim that $\tilde{\varphi}_\flat$ is the almost periodic component of an almost periodic function $\tilde{\varphi}$. Nonetheless, we claim that $\tilde{\varphi}_\flat$ is a solution of the differential equation (2.4) below

$$\frac{dy}{dt} = \mu \left( p \left( e^{\tilde{q}+y} \right) \Phi - \frac{\tilde{\sigma}}{3} \right) - q. \tag{2.4}$$

Now let us consider an increasing sequence $\{h_n \geq 0\}_{n=1}^{\infty}$, $h_n \nearrow \infty$, then modulo extracting a subsequence, we get the following uniform limits which are asymptotically almost periodic and almost periodic respectively:

$$\tilde{\varphi}_\sharp(t) = \lim_{k \to \infty} \tilde{\varphi}(t + h_{n_k}), \quad 0 = \lim_{k \to \infty} q(t + h_{n_k}), \quad \Phi_\sharp(t) = \lim_{k \to \infty} \Phi(t + h_{n_k}).$$

We claim that there exists also the (asymptotically almost periodic) limit

$$\tilde{\varphi}_{\flat,\sharp}(t) = \lim_{k \to \infty} \tilde{\varphi}_\flat(t + h_{n_k}),$$

and that $y = \tilde{\varphi}_{\flat,\sharp}(t)$ as well as $y = \tilde{\varphi}_\sharp(t)$ solve

$$\frac{dy}{dt} = \mu \left( p \left( e^{y} \right) \Phi_\sharp - \frac{\tilde{\sigma}}{3} \right). \tag{2.5}$$

To see this we just notice the uniform convergence of the l.h.s. of the differential equation

$$\tilde{\varphi}'(t + h_{n_k}) = \mu \left( p \left( \tilde{\varphi}(t + h_{n_k}) \right) \Phi(t + h_{n_k}) - \frac{\tilde{\sigma}}{3} \right).$$

and then apply the uniform convergence of the derivative criterion given in Theorem 4.8 in Appendix 4. Therefore, the difference

$$\tilde{q}_\flat(t) := \tilde{\varphi}_\sharp(t) - \tilde{\varphi}_{\flat,\sharp}(t)$$

remains monotone and being asymptotic almost periodic necessarily goes to 0 as $t \nearrow \infty$. We conclude that $\tilde{\varphi}_{\flat,\sharp}(t)$ is an almost periodic solution of (2.5) and that it is almost periodic component of

$$\tilde{\varphi}_\sharp(t) = \tilde{\varphi}_{\flat,\sharp}(t) + \tilde{q}_\flat(t).$$

Finally, the limit of the reversed translations yields almost periodic limits

$$\Phi(t) = \lim_{k\to\infty} \Phi_\sharp(t - h_{n_k}),$$

and

$$\tilde{\varphi}_{\flat,\natural}(t) := \lim_{k\to\infty} \tilde{\varphi}_{\flat,\sharp}(t - h_{n_k}).$$

Indeed, if we denote $\Phi_k(t) := \Phi(t + h_{n_k})$, uniform convergence $\Phi_k \to \Phi_\sharp$, implies that for every $\varepsilon > 0$, there exists $\tilde{N}_\varepsilon \in \mathbb{N}$, such that for every $k \geq \tilde{N}_\varepsilon$ and $t \geq 0$,

$$\begin{aligned}\varepsilon > \|\Phi_\sharp - \Phi_k\|_\infty &\geq |\Phi_\sharp(t - h_{n_k}) - \Phi_k(t - h_{n_k})| \\ &= |\Phi_\sharp(t - h_{n_k}) - \Phi((t + h_{n_k}) - h_{n_k})| = |\Phi_\sharp(t - h_{n_k}) - \Phi\,(t)|\,.\end{aligned}$$

Hence, $y = \tilde{\varphi}_{\flat,\natural}(t)$ defines a solution of (2.3). At last,

$$\phi_\flat(t) := \exp \tilde{\varphi}_{\flat,\natural}(t)$$

is a solution of (1.6). Such solution $\phi_\flat(t)$ is almost periodic because exp, is uniformly continuous in the closed interval $\left[(\tilde{\varphi}_\flat)_*, \tilde{\varphi}_\flat^*\right]$ and we apply Theorem 4.9 in Section 4. □

The stability property

$$\lim_{t\to\infty} |\phi\,(t) - \phi_\flat(t)| = 0,$$

follows from $\lim_{t\to\infty} |\tilde{\varphi}\,(t) - \tilde{\varphi}_\flat(t)| = 0$ and from being asymptotic almost periodic.

*Proof of uniqueness.* Suppose that $\varphi_1(t)$ and $\varphi_2(t)$ are two different almost periodic solutions, with $0 < \varphi_1(t_0) \leq \varphi_2(t_0)$, then

$$\left(\ln \frac{\varphi_1(t)}{\varphi_2(t)}\right)' = \mu\Phi(t)\,(p(\varphi_1(t)) - p(\varphi_2(t)) \leq 0.$$

Since $(\ln(x))' \geq 0$, then $\dfrac{\varphi_1(t)}{\varphi_2(t)}$ is a non-increasing function. Moreover, the quotient of almost periodic functions is almost periodic by Theorem 4.7. The only almost periodic functions that are non-decreasing are constant. Therefore, $\dfrac{\varphi_1(t)}{\varphi_2(t)}$ is constant, and by having the same initial condition this constant is 1. Hence, $\varphi_1 = \varphi_2$. $\square$

# 3 Numerical examples

We consider $\mu = 1$, $\Phi = \cos(2\pi t/7)+\cos(2\sqrt{2}\pi t/7)+2.5$ with three different values of the threshold, $\tilde{\sigma} = 3, 1$ and $0.5$. See Figures 1, 2 and 3, respectively. All images were programmed in Mathematica.

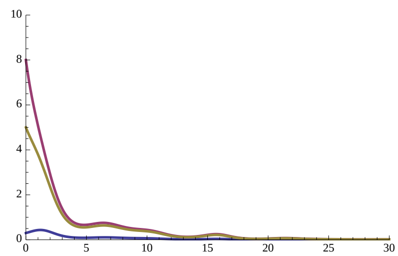


Figure 1: Case $\tilde{\sigma} = 3 > \overline{\Phi} = 2.5$ with initial conditions $R_0 = 0.5, 5, 8$, respectively. We observe an exponential decay towards 0 of the solutions.

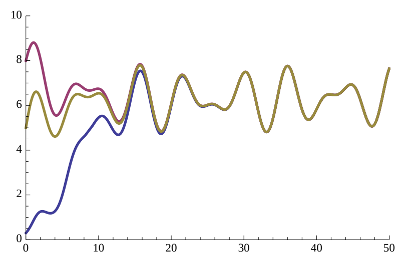


Figure 2: Case $\tilde{\sigma} = 1 < \overline{\Phi} = 2.5$ with initial conditions $R_0 = 0.5, 5, 8$, respectively. We observe asymptotic convergence towards an almost periodic solution.

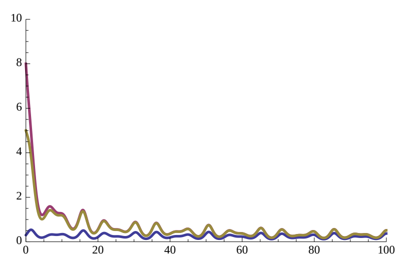


Figure 3: Case $\tilde{\sigma} = \overline{\Phi} = 2.5$ with initial conditions $R_0 = 0.5, 5, 8$, respectively. We observe slow convergence towards 0.

# 4 Appendix: Almost periodic functions

Along this section, we state the main results of the theory of almost periodic functions. Proofs and further information can be found in [1, 2, 4, 7]. For a more recent account of the theory, see also [5].

**Definition 4.1.** *The space of* almost periodic functions *is the closure* $\overline{\mathcal{T}} = \mathcal{AP}(\mathbb{R}, \mathbb{C})$ *of the algebra* $\mathcal{T}$ *of all trigonometric polynomials*

$$c_0 + c_1 e^{\mathrm{i}\lambda_1 t} + \cdots + c_n e^{\mathrm{i}\lambda_n t}$$

*whose frequency set,* $\{\lambda_1 \dots, \lambda_n\} \subset \mathbb{R}$ *is* arbitrary *and* $c_k \in \mathbb{C}$ *for* $k = 1, \dots, n$. *We consider* $\mathcal{T}$ *as a subspace of the space of bounded continuous functions* $\mathcal{CB}(\mathbb{R}, \mathbb{C})$ *with the* sup-*norm.*

We just write down the main properties of the space $\mathcal{AP}(\mathbb{R}, \mathbb{C})$:

I. Every $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$ is uniformly continuous.

II. $\mathcal{AP}(\mathbb{R}, \mathbb{C})$ is a Banach algebra.

III. For every $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$, there exists a numerable collection of frequencies $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$ whose corresponding *Fourier coefficients*:

$$c[\phi, \lambda_k] = \lim_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} \phi(s) \cdot e^{-\mathrm{i}\lambda_k s}\, ds$$

which do not vanish and do not depend on $t_0$. There exists an associated Fourier series

$$\phi(t) \sim \sum_{k=1}^{\infty} c[\phi, \lambda_k] e^{\mathrm{i}\lambda_k t}.$$

IV. For every $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$, there exists the *mean value*,

$$\overline{\phi} = M[\phi] = \lim_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} \phi(s)\, ds,$$

which is a well-defined positive linear continuous functional, $M : \mathcal{AP}(\mathbb{R}, \mathbb{C}) \to \mathbb{R}$, regardless of $t_0 \in \mathbb{R}$.

V. For every $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$, the Parseval's equality holds:

$$M\left[|\phi|^2\right] = \sum_{k=1}^{\infty} |c[\phi, \lambda_k]|^2 .$$

**Definition 4.2.** *A continuous function $f : \mathbb{R} \times D \to \mathbb{R}$ is said to be* uniformly almost periodic *with respect to $x \in D \subset \mathbb{R}^n$ if for every compact $K \subset D$,*

$$|f(t+\tau, x) - f(t, x)| < \varepsilon, \quad \forall \in \mathbb{R}, \quad \forall x \in D,$$

*for each translation number, $\tau \in T(\varepsilon, f, K)$, and any length $\ell(\varepsilon, f, K) > 0$, not depending on a particular choice $x$ remaining the same on compact set $K \subset D$.*

More specifically, if $f$ has *real* Fourier expansion,

$$f(t, x) \sim \overline{f}(x) + \sum_{n=0}^{\infty} a\,[f, \lambda_n] \cos(\lambda_n t) + b[f, \lambda_n] \sin(\lambda_n t)\,,$$

then $f$ is uniformly almost periodic, whenever the coefficients $a[\cdot, \lambda_n], b[\cdot, \lambda_n]$ do not depend on $x$, see [4, Chapter VI] .

An important characterization of the space $\mathcal{AP}(\mathbb{R}, \mathbb{C})$ is given by the following assertion, whose proof is given for instance in [5, Propositions 3.6 and 3.7] and [17, Lemmas I.2.1, Theorem I.2.3 ]

**Theorem 4.3.** *The following properties are equivalent:*

*(1)* $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$.

*(2) (Bohr's property) Given $\phi \in \mathcal{CB}(\mathbb{R}, \mathbb{C})$, for every $\varepsilon > 0$ there exists a set of real numbers $T(\phi, \varepsilon) \subset \mathbb{R}$ and an $\varepsilon$-length, $\ell = \ell(\phi, \varepsilon) > 0$, such that each interval $(a, a + \ell)$ of length $\ell$ contains at least one* $\varepsilon$-almost period $\tau \in T(\phi, \varepsilon)$, *such that*

$$|\phi(t+\tau) - \phi(t)| < \varepsilon, \quad \forall t \in \mathbb{R}$$

*(3) (Bochner's property) For every sequence $\{h_n\}_{n=1}^{\infty} \subset \mathbb{R}$, the family of translations*

$$\mathcal{F} = \{\phi_n(t) := \phi(t + h_n)\ :\ n \in \mathbb{N}\} \subset \mathcal{CB}(\mathbb{R}, \mathbb{C}),$$

*is relatively compact, i.e. there exists a subsequence $\{h_{n_k}\}_{k=1}^{\infty} \subset \{h_n\}_{n=1}^{\infty}$ such that $\phi_{n_k}(t)$ converges uniformly to*

$$\phi_\sharp(t) = \lim_{k\to\infty} \phi(t + h_{n_k}) \in \mathcal{CB}(\mathbb{R}, \mathbb{C}).$$

In Theorem 4.3, the limit function $\phi \in \mathcal{CB}(\mathbb{R}, \mathbb{C})$ described in Bochner's property necessarily belong to $\mathcal{AP}(\mathbb{R}, \mathbb{C})$ and also $\phi_\sharp \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$.

**Definition 4.4.** *A continuous function $\phi : [0, \infty) \to \mathbb{R}$ is said to be* asymptotically almost periodic *if it can be decomposed (uniquely) as*

$$\psi(t) = \psi_\flat(t) + r(t), \quad \psi_\flat \in \mathcal{AP}(\mathbb{R}, \mathbb{C}), \quad r \in \mathcal{CB}(\mathbb{R}, \mathbb{C})$$

*where $\lim_{t\to\infty} r(t) = 0$. The space of asymptotically almost periodic functions will be denoted as $\mathcal{AAP}(\mathbb{R}, \mathbb{C})$, so that*

$$\mathcal{AP}(\mathbb{R}, \mathbb{C}) \subset \mathcal{AAP}(\mathbb{R}, \mathbb{C}) \subset \mathcal{CB}(\mathbb{R}, \mathbb{C}).$$

Such decomposition happens to be unique.

**Theorem 4.5** ([7, Theorem 9.2])**.** *If $f(t, x)$ is almost periodic in $t$, uniformly with respect to $x$ in compact subsets of $\mathbb{R}^n$ (see Section 4), and if $\phi(t)$ is an asymptotically almost periodic solution of the ODE, $x' = f(t, x)$, $x \in D \subset \mathbb{R}^n$. Then the almost periodic part $\phi_\flat(t)$ of $\phi(t)$ is also a solution of this ODE.*

As in the case of almost periodic functions, there are characterizations of asymptotic almost periodic functions in terms of Bohr's and Bochner's type properties. Specifically the following assertion holds.

**Theorem 4.6.** *The following properties are equivalent:*

*(1)* $\psi \in \mathcal{AAP}(\mathbb{R}, \mathbb{C})$.

*(2) (Bohr's property in $[0, \infty)$) Given $\psi \in \mathcal{CB}(\mathbb{R}, \mathbb{C})$, for every $\varepsilon > 0$, there exists a set of real numbers $T(\psi, \varepsilon) \subset \mathbb{R}$ and an $\varepsilon$-length, $\ell = \ell(\psi, \varepsilon) > 0$, such that each interval $(a, a+\ell) \subset [0, \infty)$ of length $\ell$ contains at least one $\varepsilon$-*almost period *$\tau \in T(\psi, \varepsilon)$, such that*

$$|\psi(t + \tau) - \psi(t)| < \varepsilon, \quad \forall t \geq t_\varepsilon$$

*for certain $t_\varepsilon \geq 0$.*

*(3) (Bochner's property in $[0, \infty)$) Given $\psi \in \mathcal{CB}(\mathbb{R}, \mathbb{C})$, for every sequence $\{h_n\}_{n=1}^\infty \subset \mathbb{R}$, such that $h_n > 0$ and $\lim_{n\to\infty} h_n = \infty$, the family of translations*

$$\mathcal{F} = \{\psi_n(t) := \psi(t + h_n) \, : \, n \in \mathbb{N}\} \subset \mathcal{CB}(\mathbb{R}, \mathbb{C}),$$

*is relatively compact, i.e. there exists a subsequence $\{h_{n_k}\}_{k=1}^\infty \subset \{h_n\}_{n=1}^\infty$ such that $\psi_{n_k}(t)$ converges uniformly along $[0, \infty)$ to*

$$\psi_\sharp(t) = \lim_{k\to\infty} \psi(t + h_{n_k}) \in \mathcal{CB}(\mathbb{R}, \mathbb{C}).$$

In Theorem 4.6, the limit function $\psi \in \mathcal{CB}(\mathbb{R},\mathbb{C})$ described in Bochner's property necessarily belongs to $\mathcal{AAP}(\mathbb{R},\mathbb{C})$ and also $\psi_\sharp \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$. See in [7, Theorem 9.3] and [17, Theorems I.3.2, I.3.4, I.3.5, I.3.9, I.3.10] for a complete discussion of the proof of Theorem 4.6.

Other properties of the space of almost periodic functions that will be useful for our purposes can be summarized in the following assertions.

**Theorem 4.7** ([4, Theorems 1.5, 2.1]). *Let $\phi(t), \varphi(t)$ be almost periodic functions, and $a, c \in \mathbb{R}$ constants, then the following functions are also almost periodic*

$$\phi(t) + c\varphi(t),\ \phi(t+a),\ \phi(at),\ \phi(t)\cdot\varphi(t),\ 1/\phi(t)\ \textit{when}\ \phi(t) \geq \phi_* > 0.$$

**Theorem 4.8** ([4, Theorem 4.1], [17, Theorem 3.3]). *The primitive of an (asymptotically) almost periodic function is (asymptotically) almost periodic if and only if it is bounded on the real line. If the derivative $\psi'(t)$ of a derivable asymptotically almost periodic function $\psi(t)$ is also asymptotically almost periodic, then the associated decomposition*

$$\psi(t) = \psi_\flat(t) + r(t), \quad \psi_\flat \in \mathcal{AP}(\mathbb{R},\mathbb{C}), \quad \lim_{t\to\infty} r(t) = 0,$$

*induces the corresponding decomposition associated to $\psi' \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ as follows*

$$\psi'(t) = \psi_\flat'(t) + r'(t), \quad \psi_\flat' = \frac{d\psi_\flat}{dt} \in \mathcal{AP}(\mathbb{R},\mathbb{C}), \quad r' = \frac{dr}{dt}, \quad \lim_{t\to\infty} r'(t) = 0.$$

**Theorem 4.9** ([4, Theorem 1.7]). *Let $G(z)$ be a uniformly continuous function of $s \in R \subset \mathbb{C}$. If for $f(x)$ an almost periodic function $f(x) \in R$ for every $x \in \mathbb{R}$, then the function*

$$F(x) = G(f(x))$$

*is almost periodic.*

## 5 Aknowlegements

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## References

[1] A. S. Besicovitch, *Almost Periodic Functions.* Cambridge University Press, 1932.

[2] H. Bohr, *Almost Periodic Functions.* Chelsea Publishing Company, New York, 1947.

[3] H. M. Byrne and M. A. J. Chaplain, "Growth of non-necrotic tumors in the presence and absence of inhibitors," *Math. Biosci.*, vol. 130, no. 2, pp. 151–181, 1995, doi: 10.1016/0025-5564(94)00117-3.

[4] C. Corduneanu, *Almost Periodic Functions*, 2nd ed. Chelsea Publishing Company, New York, 1989.

[5] C. Corduneanu, *Almost Periodic Oscillations and Waves.* Springer–Verlag, 2009.

[6] S. Cui and A. Friedman, "Analysis of a mathematical model of the effect of inhibitors on the growth of tumors," *Math. Biosci.*, vol. 164, no. 2, pp. 103–137, 2000, doi: 10.1016/S0025-5564(99)00063-2.

[7] A. M. Fink, *Almost Periodic Differential Equations*, ser. Lecture Notes in Mathematics. Springer–Verlag, 1974, vol. 377.

[8] J. Folkman and M. Hochberg, "Self-regulation of growth in three dimensions," *J. of Experimental Medicine*, vol. 138, pp. 745–753, 1973, doi: 10.1084/jem.138.4.745.

[9] A. Friedman and F. Reitich, "Analysis of a mathematical model for the growth of tumors," *J. Math. Biol.*, vol. 38, pp. 262–284, 1999, doi: 10.1007/s002850050149.

[10] A. Friedman and B. Hu, "Asymptotic stability for a free boundary problem arising in a tumor model," *J. Differential Equations*, vol. 227, pp. 598–639, 2006, doi: 10.1016/j.jde.2005.09.008.

[11] S. Grist, S. Nasseri, L. Laplatine, J. Schmol, D. Tao, J. Hua, L. Chrostowski, and K. Cheung, "Long-term monitoring in a microfluidic system to study tumour spheroid response to chronic and cycling hypoxia," *Nature Scientific Reports*, vol. 9, 2019, Art. ID 17782, doi: 10.1038/s41598-019-54001-8.

[12] W. He and R. Xing, "The existence and linear stability of periodic solution for free boundary problem modeling tumor growth with a periodic supply of external nutrients," *Nonlinear Analysis: Real World Applications*, vol. 60, 2021, Art. ID 103290, doi: https://doi.org/10.1016/j.nonrwa.2021.103290.

[13] Y. Huang, Z. Zhang, and B. Hu, "Linear stability for a free boundary tumor model with a periodic supply of external nutrients," *Math. Meth. Appl. Sci.*, vol. 42, pp. 1039–1054, 2019, doi: 10.1002/mma.5412.

[14] S. A. Maggelakis and J. A. Adam, "Mathematical model of prevascular growth of a spherical carcinoma," *Math. Comput. Modelling*, vol. 13, no. 5, pp. 23–38, 1990, doi: 10.1016/0895-7177(90)90040-T.

[15] S. Xu, "Analysis of a free boundary problem for tumor growth in a periodic external environment," *Boundary Value Problems*, 2015, Art. ID 140,(2015), doi: 10.1186/s13661-015-0399-0.

[16] S. Xu, T. Chen, and M. Bai, "Analysis of a free boundary problem for avascular tumor growth with a periodic supply of nutrients," *Discrete. Contin. Dyn. Syst. Ser. B*, vol. 21, pp. 997–1008, 2016, doi: 10.3934/dcdsb.2016.21.997.

[17] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, ser. Applied Mathematical Sciences. Springer–Verlag, 1975, vol. 14.