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Szpiro's conjecture when the denominator of the j-invariant is small

Hector Pasten[1,✉]

[1] *Departamento de Matemáticas, Pontificia Universidad Católica de Chile. Facultad de Matemáticas, 4860 Av. Vicuña Mackenna, Macul, RM, Chile.*
`hector.pasten@uc.cl`✉

ABSTRACT

We prove Szpiro's conjecture for elliptic curves over the rationals having j-invariant with denominator of logarithmic size with respect to its numerator.

RESUMEN

Demostramos la conjetura de Szpiro para curvas elípticas sobre los racionales que tienen j-invariante con denominador de tamaño logarítmico con respecto a su numerador.

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1 Introduction

We begin by fixing some notation. For an elliptic curve E over $\mathbb{Q}$ we write j_E for its j-invariant, Δ_E for the absolute value of its minimal discriminant, and N_E for its conductor.

In the early 80's, Szpiro proposed the following conjecture [14]:

Conjecture 1.1 (Szpiro's conjecture). *Let $\epsilon > 0$. There is a number $c_\epsilon > 0$ depending only on ϵ such that for all elliptic curves E over $\mathbb{Q}$ we have $\Delta_E \leq c_\epsilon \cdot N_E^{6+\epsilon}$.*

This conjecture is very deep. Even a weaker version with the exponent $6+\epsilon$ replaced by some fixed constant would have tremendous consequences such as a version of the *abc*-conjecture –in fact, Szpiro's conjecture was the main motivation for the formulation of the *abc*-conjecture, see [4].

Szpiro's conjecture is known for elliptic curves of prime power discriminant by work of Mestre and Oesterlé [5] and for elliptic curves of integral j-invariant by work of Pesenti and Szpiro [10].

At present, the strongest unconditional result valid for all elliptic curves over $\mathbb{Q}$ is the following (effective) estimate by the author [7, Theorem 1.8] valid for any $\epsilon > 0$:

$$\log \Delta_E \leq (1/4 + \epsilon) \cdot N_E \log N_E + O_\epsilon(1).$$

This improves the earlier bound

$$\log \Delta_E \leq N_E \log N_E + O(N_E \log\log N_E)$$

by Murty and the author [6]; both results use the theory of modular forms.

We mention that an upper bound for the number of (potential) exceptions to Szpiro's conjecture with Δ_E less than a given bound is proved in [1] by Fouvry, Nair, and Tenenbaum.

Our goal is to prove Szpiro's conjecture for elliptic curves whose j-invariant has small denominator. We write $\mathrm{num}(q)$ and $\mathrm{den}(q)$ for the absolute value of the numerator and denominator of a rational number q in reduced form. With this notation, our main result is:

Theorem 1.2 (Main result). *Let $A, B > 0$. For all elliptic curves E over $\mathbb{Q}$ with $\mathrm{den}(j_E) \leq A(\log \mathrm{num}(j_E))^B$ we have*

$$\Delta_E \leq A \cdot 16^{B+1} N_E^{B+5} (\log N_E)^B.$$

In particular, by setting $B = 1$, Szpiro's conjecture holds when $\mathrm{den}(j_E)$ has logarithmic size with respect to $\mathrm{num}(j_E)$:

Corollary 1.3 (Szpiro's conjecture when $\mathrm{den}(j_E)$ is small)**.** *Let* $A > 0$*. For all elliptic curves* E *over* $\mathbb{Q}$ *with* $\mathrm{den}(j_E) \leq A \cdot \log \mathrm{num}(j_E)$ *we have*

$$\Delta_E \leq 256A \cdot N_E^6 \log N_E.$$

Note that this generalizes the Pesenti–Szpiro result [10] on Szpiro's conjecture when $j_E \in \mathbb{Z}$.

Finally, let us mention an application. Theorem 1.2 together with [2, Theorem 0.7] by Hindry and Silverman yield a rather uniform bound for the number of S-integral points on elliptic curves E over $\mathbb{Q}$ whenever $\mathrm{den}(j_E) \leq (\log \mathrm{num}(j_E))^B$ for a fixed B and S a finite set of primes. (Nevertheless, it is likely that the latter condition can be weakened for this application by revisiting ideas from Silverman's thesis.)

About number fields: This note is about elliptic curves over $\mathbb{Q}$, but it is conceivable that the same ideas work whenever (potential) modularity is established in a geometric sense, that is, modular parameterizations from Shimura curves to elliptic curves. See [7, Theorem 1.17] for a concrete bound as the one needed. We do not aim for that level of generality here.

2 The height of the j-invariant

The Faltings height of an elliptic curve E over $\mathbb{Q}$ is denoted by $h(E)$ (here we really mean the Faltings height over $\mathbb{Q}$, not the stable one). In [6], Murty and the author used the theory of modular forms to prove the following explicit bound for all E over $\mathbb{Q}$:

$$h(E) < 0.1 \cdot N_E \log N_E + 11.$$

For a rational number q we recall that its logarithmic height is $h(q) = \log \max\{\mathrm{num}(q), \mathrm{den}(q)\}$. It turns out that $h(E)$ is related to $h(j_E)$ in a very explicit way; Silverman [12] proved

$$h(j_E) \leq 12h(E) + O(\log(2 + h(j_E)))$$

where the error term has an effective implicit constant. This has been made explicit by Pellarin [9] and, in a sharper form, by Pazuki [8]. For our purposes [9, Lemme 5.2] is enough; this gives $h(j_E) \leq 24 \max\{1, h(E)\} + 94.3$. Putting these results together we obtain:

Corollary 2.1. *For all elliptic curves* E *over* $\mathbb{Q}$ *we have* $h(j_E) \leq 16 \cdot N_E \log N_E$*.*

Proof. The previous discussion leads to

$$h(j_E) \leq 94.3 + 24 \cdot (0.1 \cdot N_E \log N_E + 11) = 2.4 \cdot N_E \log N_E + 358.3.$$

The result follows from the well-known fact that $N_E \geq 11$ for all elliptic curves over $\mathbb{Q}$ —this is classical, but a simple way to see it is that there are no rational Hecke newforms of weight 2 for $\Gamma_0(N)$ when $N < 11$. $\square$

3 An application of Tate's algorithm

For a prime number p we denote by $v_p : \mathbb{Q}^\times \to \mathbb{Z}$ the p-adic valuation.

Let E be an elliptic curve over $\mathbb{Q}$. The primes p that divide Δ_E are the same as the ones that divide N_E. We split these primes p into three types:

- *Type 1:* $v_p(j_E) \geq 0$.
- *Type 2:* $v_p(j_E) < 0$ and E has multiplicative reduction at p.
- *Type 3:* $v_p(j_E) < 0$ and E has additive reduction at p.

The following is proved in [10].

Lemma 3.1 (Primes of Type 1)**.** *If p is of Type 1, then $v_p(\Delta_E) \leq 5v_p(N_E)$.*

As noted in [10], this immediately implies Szpiro's conjecture whenever $j_E \in \mathbb{Z}$.

Let us now consider p of Type 2. The Kodaira type of the special fibre of the minimal regular model of E over $\mathbb{Z}_p$ is I_n for some $n \geq 1$. The output of Tate's algorithm summarized in [13, p. 365, Table 4.1] (which refined Tate's table in [15]) shows that $n = -v_p(j_E) = v_p(\Delta_E)$. So we find:

Lemma 3.2 (Primes of Type 2)**.** *If p is of Type 2, then $v_p(\Delta_E) = -v_p(j_E)$.*

Finally we deal with the primes p of Type 3. In this case the Kodaira type of E at p is I_n^* for a certain integer $n \geq 1$, see the discussion in [3, p. 42]. Let us first deal with $p = 3$ and then with $p = 2$.

If $p \geq 3$, from the data in the I_n^* column of [13, p. 365, Table 4.1] we get

$$v_p(\Delta_E) = 6 + n = 6 - v_p(j_E) = 3v_p(N_E) - v_p(j_E).$$

Let us now consider the case $p = 2$. The number m of geometrically irreducible components of the special fibre of the minimal regular model at $p = 2$ is $m = n + 5$ (see the table in [15].) By the Saito–Ogg formula we have $v_2(N_E) = v_2(\Delta_E) - m + 1 = v_2(\Delta_E) - n - 4$ which gives $v_2(\Delta_E) = v_2(N_E) + 4 + n$.

We need some control on the integer n. In [3, Theorem 2.8] gives the existence of a suitable quadratic extension $L/\mathbb{Q}$ such that if $s+1$ is the valuation of its different ideal over 2, then $n = -v_2(j_E) + 4s$. By [11, p. 58, Remark 1], we have $s \leq 2$ so we get $n \leq -v_2(j_E) + 8$. Therefore

$$v_2(\Delta_E) \leq v_2(N_E) - v_2(j_E) + 12 \leq 3v_2(N_E) - v_2(j_E) + 8$$

because $v_2(N_E) \geq 2$ (additive reduction). Let us summarize our findings:

Lemma 3.3 (Primes of Type 3). *Let p be a prime of Type 3 and write $\delta_p = 8$ if $p = 2$ and $\delta_p = 0$ if $p \geq 3$. Then $v_p(\Delta_E) \leq 3v_p(N_E) - v_p(j_E) + \delta_p$.*

From these three lemmas we deduce the following result, which can be of independent interest:

Corollary 3.4. *Let E be an elliptic curve over $\mathbb{Q}$. Then Δ_E divides $16 \cdot \mathrm{den}(j_E) N_E^5$.*

Perhaps an explanation is needed for the factor 16. This is only necessary when $p = 2$ is a prime of Type 3. In this case

$$v_2(\Delta_E) \leq 3v_2(N_E) - v_2(j_E) + \delta_2 \leq 5v_2(N_E) - 2 \cdot 2 - v_2(j_E) + 8 \leq 5v_2(N_E) + 4 + v_2(\mathrm{den}(j_E)),$$

where we used $v_2(N_E) \geq 2$ as $p = 2$ is of additive reduction.

4 Proof of the main result

Proof of Theorem 1.2. By Corollary 2.1 we have

$$\mathrm{den}(j_E) \leq A(\log \mathrm{num}(j_E))^B \leq Ah(j_E)^B \leq A \cdot 16^B N_E^B (\log N_E)^B.$$

Putting this estimate together with Corollary 3.4, we find $\Delta_E \leq A \cdot 16^{B+1} N_E^{B+5} (\log N_E)^B$. $\square$

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