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On the analytical solution of the Cauchy problem for a linear set-valued differential equation with a Hukuhara derivative

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ABSTRACT

The article considers the Cauchy problem for a linear set-valued differential equation with the Hukuhara derivative and derives an analytical formula for its solution.

RESUMEN

Este artículo considera el problema de Cauchy para una ecuación diferencial lineal con valores en conjuntos con la derivada de Hukuhara y obtiene una fórmula analítica para su solución.

Keywords and Phrases: Differential equation, linear, set-valued mapping, Hukuhara derivative.

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1 Introduction

In 1967, M. Hukuhara introduced the integral and the derivative for set-valued mappings and considered how they are related to each other [8]. These derivative and integral generalize the ordinary derivative and Riemann integral for a single-valued function to the set-valued case. In 1969, F. S. de Blasi and F. Iervolino considered differential equations with the Hukuhara derivative [3]. Subsequently, many authors introduced other derivatives and integrals for set-valued mappings and studied the properties of solutions to various set-valued equations (see [10, 12–18, 21–26] and references therein). Such equations are similar in appearance to the corresponding classical equations, but their study and solutions must account for their set-valued nature. Hence, traditional methods and approaches used for single-valued systems are not always applicable to set-valued systems, necessitating new or alternative methods. Furthermore, the set-valued nature introduces new properties that require exploration.

The article considers the Cauchy problem for a linear set-valued differential equation with the Hukuhara derivative

$$\begin{cases} D_H X(t) = AX(t) + F(t), \\ X(0) = X_0, \end{cases} \tag{1.1}$$

where $X : [0,T] \to conv(\mathbb{R}^n)$ is the unknown set-valued mapping, $D_H X(t)$ is the Hukuhara derivative, $A \in \mathbb{R}^{n\times n}$ is a non-singular matrix, $F : [0,T] \to conv(\mathbb{R}^n)$ is a continuous set-valued mapping, and $X_0 \in conv(\mathbb{R}^n)$ is the initial set. An analytical solution formula for problem (1.1) is obtained, and its difference from the single-valued case is demonstrated.

Previously in [19, 22], it was proven that Cauchy problem (1.1) has a unique solution if A is a non-singular matrix. However, unlike in the single-valued case, an explicit solution to such an equation was not provided.

Subsequently, an explicit form of the solution was obtained for some special cases.

If $A = aI_n$, where I_n is the identity matrix of size n and $a > 0$, then Cauchy problem (1.1) takes the form

$$D_H X(t) = aX(t) + F(t),\ X(0) = X_0, \tag{1.2}$$

and, according to [3, 22], has the following solution:

$$X(t) = e^{at} X_0 + \int_0^t e^{a(t-s)} F(s)\, ds \tag{1.3}$$

and if $F(t) \equiv F$ for all $t \in [0,T]$, then, according to [9], has the following solution:

$$X(t) = e^{at} X_0 + \frac{e^{at}-1}{a} F.$$

If $A \in GL(n, \mathbb{R})$ and $F(t) \equiv F$, then Cauchy problem (1.1) takes the form

$$D_H X(t) = AX(t) + F, \quad X(0) = X_0,$$

and, according to [11], has the following solution:

$$X(t) = X_0 + \sum_{i=1}^{\infty} \left\{ \frac{t^i}{i!} A^i G \right\},$$

where $F \in conv(\mathbb{R}^n)$, $G = X_0 + A^{-1}F$.

We also note that the results of this work can be used to extend the research begun for linear homogeneous set-valued differential equations with generalized derivative [9, 14], for linear homogeneous set-valued differential equations with conformal fractional derivative [10, 12] and for linear homogeneous set-valued differential equations with conformal fractional-fractal derivative [13] to a more general class of problems - linear inhomogeneous set-valued differential equations.

2 Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and let $conv(\mathbb{R}^n)$ be the space of nonempty convex compact subsets of $\mathbb{R}^n$, equipped with the Pompeiu-Hausdorff metric:

$$h(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} \|x - y\|, \sup_{y \in Y} \inf_{x \in X} \|x - y\| \right\},$$

where $X, Y \in conv(\mathbb{R}^n)$.

In the space $conv(\mathbb{R}^n)$, in addition to standard set operations, we consider the following:

- sum of sets X and Y: $X + Y = \bigcup\limits_{x \in X,\, y \in Y} \{x + y\}$.
- scalar multiplication of $\lambda \in \mathbb{R}$ with the set X: $\lambda X = \bigcup\limits_{x \in X} \{\lambda x\}$.

The following properties hold [15, 22, 24].

Properties A. For all $X, Y, Z, W \in conv(\mathbb{R}^n)$ and $\alpha, \beta, \lambda \in \mathbb{R}$:

(1) $(conv(\mathbb{R}^n), h)$ is a complete metric space;

(2) $X + Y = Y + X \in conv(\mathbb{R}^n)$;

(3) if $X + Z = Y + Z$, then $X = Y$;

(4) if $X \subset Z,\ Y \subset W$, then $X + Y \subset Z + W$;

(5) $\lambda X \in conv(\mathbb{R}^n)$;

(6) $\alpha(\beta X) = (\alpha\beta)X$;

(7) if $\alpha\beta \geq 0$, then $(\alpha + \beta)X = \alpha X + \beta X$;

(8) $\lambda(X+Y)=\lambda X+\lambda Y$;

(9) $h(X+Z,Y+Z)=h(X,Y)$;

(10) $h(\lambda X,\lambda Y)=|\lambda|h(X,Y)$.

It is known that the space $conv(\mathbb{R}^n)$ is not a linear space with respect to the given operations, because in the general case, there is no opposite element for $X \in conv(\mathbb{R}^n)$, that is, no set $-X$ such that $X+(-X)=\{\mathbf{0}\}$, the opposite element exists only in the case when $X \in \mathbb{R}^n$. The absence of an opposite element in the space $conv(\mathbb{R}^n)$ leads to the ambiguity in defining the concept of set difference and the conditions for its existence.

In this paper, we will use the difference of Hukuhara [8].

Definition 2.1 ([8]). *Let $X,Y \in conv(\mathbb{R}^n)$. A set $Z \in conv(\mathbb{R}^n)$ such that $X=Y+Z$ is called the Hukuhara difference of the sets X and Y, denoted $X\overset{H}{-}Y$.*

The Hukuhara difference has the following properties [8,15,22,24,25].

Properties B.

(1) If the Hukuhara difference $X\overset{H}{-}Y$ exists, then it is unique and $(X\overset{H}{-}Y)+Y=X$.

(2) $X\overset{H}{-}X=\{\mathbf{0}\}$ for all $X \in conv(\mathbb{R}^n)$.

(3) $(X+Y)\overset{H}{-}Y=X$ for all $X,Y \in conv(\mathbb{R}^n)$.

Also, let us add one more operation: the product of a matrix with a set $AX=\bigcup\limits_{x\in X}\{Ax\}$, where $A \in \mathbb{R}^{n\times n}$ is a real matrix of size $n\times n$ and $X \in conv(\mathbb{R}^n)$.

We will list some properties of this operation [4,7].

Properties C.

(1) If $A \in \mathbb{R}^{n\times n}$ and $X \in conv(\mathbb{R}^n)$, then $AX \in conv(\mathbb{R}^k)$, where $k=\operatorname{rank}(A)$;

(2) If $A \in \mathbb{R}^{n\times n}$ and $X,Y \in conv(\mathbb{R}^n)$, then $A(X+Y)=AX+AY$;

(3) If $A,B \in \mathbb{R}^{n\times n}$ and $X \in conv(\mathbb{R}^n)$, then $(A+B)X \subseteq AX+BX$;

(4) If $A \in \mathbb{R}^{n\times n}$, $X,Y \in conv(\mathbb{R}^n)$ and $X \subseteq Y$, then $AX \subseteq AY$.

Remark 2.2. *For practical computation of the set $Y=AX$, either the singular value decomposition (SVD) of the matrix A [4, 6, 12] or the mathematical apparatus of support functions of sets [1, 7, 15, 19, 25] is typically used.*

Let $X : [0, T] \to conv(\mathbb{R}^n)$ be a set-valued mapping.

Definition 2.3 ([8])**.** *Let* $t \in (0, T)$*. If for all sufficiently small* $\varepsilon > 0$ *such, that* $(t - \varepsilon, t + \varepsilon) \subset (0, T)$*, the Hukuhara differences* $X(t + \varepsilon) \overset{H}{-} X(t)$ *and* $X(t) \overset{H}{-} X(t - \varepsilon)$ *exist, and there exists* $Z \in conv(\mathbb{R}^n)$ *such that the following equality holds:*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \left(X(t + \varepsilon) \overset{H}{-} X(t) \right) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \left(X(t) \overset{H}{-} X(t - \varepsilon) \right) = Z, \tag{2.1}$$

we will say that the set-valued mapping $X(\cdot)$ *has the Hukuhara derivative at the point* $t \in (0, T)$ *and* $D_H X(t) = Z$*.*

If $D_H X(t)$ exists for all $t \in (0, T)$, and the limits $\lim_{t \downarrow 0} D_H X(t)$ and $\lim_{t \uparrow T} D_H X(t)$ exist, we will assume that $D_H X(0) = \lim_{t \downarrow 0} D_H X(t)$ and $D_H X(T) = \lim_{t \uparrow T} D_H X(t)$.

Definition 2.4. *If the Hukuhara derivative* $D_H X(t)$ *exists for all* $t \in [0, T]$*, we will say that the set-valued mapping* $X(\cdot)$ *is differentiable in the Hukuhara sense on* $[0, T]$*.*

The Hukuhara derivative has the following properties [12, 15, 19, 22, 24].

Properties D.

(1) If the set-valued mapping $X(t) \equiv X$ for all $t \geq 0$, then $DX(t) \equiv \{\mathbf{0}\}$;

(2) if the set-valued mappings $X(\cdot)$ and $Y(\cdot)$ are differentiable at $t > 0$, then

$$D_H(\alpha X(t) + \beta Y(t)) = \alpha D_H X(t) + \beta D_H Y(t);$$

(3) if the set-valued mapping $X(\cdot)$ is continuous on $\mathbb{R}_+$, then

$$D_H \left(\int_0^t X(s) ds \right) = X(t), \quad t > 0,$$

where $\alpha, \beta \geq 0$, the integral is understood in the sense of the Riemann-Hukuhara integral [20, 25].

The Riemann-Hukuhara integral is defined analogously to the Riemann integral for single-valued functions, taking into account the set-valued nature of the integrand mapping [20, 25] and possesses the following properties [8, 19, 20, 22, 24, 25].

Properties E. If $\lambda : [0,T] \to \mathbb{R}$ is a continuous function such that $\lambda(t)\lambda(s) \geq 0$ for all $t, s \in [0,T]$, $A : [0,T] \to \mathbb{R}^{n\times n}$ is a continuous matrix function and $A(t) \in GL(n,\mathbb{R})$ for all $t \in [0,T]$, $X \in conv(\mathbb{R}^n)$ and $F, G : [0,T] \to conv(\mathbb{R}^n)$ are continuous set-valued mappings, then

(1) $\int\limits_0^t F(s)\,ds \in conv(\mathbb{R}^n)$ for all $t \in (0,T]$;

(2) $\int\limits_0^t F(s)\,ds + \int\limits_t^T F(s) = \int\limits_0^T F(s)\,ds$ for all $t \in [0,T]$;

(3) $\int\limits_0^t (F(s)+G(s))\,ds = \int\limits_0^t F(s)\,ds + \int\limits_0^t G(s)\,ds$ for all $t \in [0,T]$;

(4) if the set-valued mapping $F(\cdot)$ is continuously differentiable in the sense of Hukuhara on $[0,T]$, then $\int\limits_0^t D_H F(s)\,ds = F(t) \overset{H}{-} F(0)$ for all $t \in [0,T]$;

(5) $\int\limits_0^t \lambda(s)X\,ds = \int\limits_0^t \lambda(s)\,ds\,X$ for all $t \in [0,T]$;

(6) $\int\limits_0^t A(s)\,ds\,X \subseteq \int\limits_0^t A(s)X\,ds$ for all $t \in [0,T]$;

(7) if $F(t) \subset G(t)$ for all $t \in [0,T]$, then $\int\limits_0^t F(s)\,ds \subseteq \int\limits_0^t G(s)\,ds$ for all $t \in [0,T]$;

(8) $h\left(\int\limits_0^t F(s)\,ds, \int\limits_0^t G(s)\,ds\right) \leq \int\limits_0^t h(F(s),G(s))\,ds$ for all $t \in [0,T]$.

3 Linear set-valued differential equation with the Hukuhara derivative.

Now we will consider the Cauchy problem (1.1).

Definition 3.1. *A set-valued mapping $X : [0,T] \to conv(\mathbb{R}^n)$ is called a solution of Cauchy problem* (1.1) *if it is Hukuhara differentiable and satisfies system* (1.1) *for all $t \in [0,T]$.*

Suppose that equation (1.1) has the following solution:

$$X(t) = X_0 + \sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}U^i Z\right\} + \sum_{i=0}^{\infty}\left\{\int\limits_0^t \left[\frac{(t-s)^i V^i}{i!}Y(s)\right]ds\right\}, \tag{3.1}$$

where $t \in [0,T]$, $U, V \in \mathbb{R}^{n\times n}$ are non-singular matrices, $Y : [0,T] \to conv(\mathbb{R}^n)$ is a continuous set-valued mapping, and $Z \in conv(\mathbb{R}^n)$.

It is easy to verify that $X(0) = X_0$.

First, let us prove some properties of (3.1). For this purpose, we introduce the following notations:

$$\eta_U = \|U\| = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} u_{ij}^2}, \quad \eta_V = \|V\|,$$

$$\gamma_X = \frac{1}{2}diam(X_0) = \frac{1}{2}\max_{x,y\in X_0}\|x-y\|, \quad \gamma_Z = \frac{1}{2}diam(Z), \quad \gamma_Y(t) = \frac{1}{2}diam(Y(t)),$$

$$X_{1,0}(t) = X_0,\ X_{2,0}(t) = \int_0^t Y(s)\,ds, \quad X_{1,i}(t) = \frac{t^i}{i!}U^i Z,\ X_{2,i}(t) = \int_0^t \frac{(t-s)^i}{i!}V^i Y(s)\,ds,\ i \geq 1,$$

$$\xi^j(t) - \text{the Lebesgue measure of the set } X^j(t) = \sum_{i=0}^{j} X_{1,i}(t) + \sum_{i=0}^{j} X_{2,i}(t),\ j \geq 0,$$

$$\vartheta(t) - \text{the Lebesgue measure of the set } X(t).$$

Since $X(0) = X^0(0) = X^1(0) = \cdots = X^j(0) = \cdots$, we have $\vartheta(0) = \xi^0(0) = \xi^1(0) = \cdots = \xi^j(0) = \cdots$.

Let the vectors $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^n$, and $\mathbf{y}(t) \in \mathbb{R}^n$ be such that $X_0 \subseteq B_{\gamma_X}(\mathbf{x})$, $Z \subseteq B_{\gamma_Z}(\mathbf{z})$, and $Y(t) \subseteq B_{\gamma_{Y(t)}}(\mathbf{y}(t))$, where $B_r(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}_0\| \leq r\}$ is n-ball of radius r and center $\mathbf{x}_0$.

Then, for all $t \in [0, T]$ and $i \geq 1$:

$$\begin{aligned} X_{1,i}(t) &= \frac{t^i}{i!}U^i Z \subseteq \frac{t^i}{i!}U^i B_{\gamma_Z}(\mathbf{z}) = \frac{t^i}{i!}U^i \mathbf{z} + \frac{t^i}{i!}U^i B_{\gamma_Z}(\mathbf{0}) \subseteq \frac{t^i}{i!}U^i \mathbf{z} + \frac{t^i}{i!}\eta_U^i B_{\gamma_Z}(\mathbf{0}) \\ &= \frac{t^i U^i}{i!}\mathbf{z} + \frac{t^i \eta_U^i \gamma_Z}{i!}B_1(\mathbf{0}), \end{aligned}$$

$$\begin{aligned} X_{2,i}(t) &= \int_0^t \frac{(t-s)^i}{i!}V^i Y(s)\,ds \subseteq \int_0^t \frac{(t-s)^i}{i!}V^i B_{\gamma_{Y(s)}}(\mathbf{y}(s))\,ds \\ &= \int_0^t \left[\frac{(t-s)^i}{i!}V^i \mathbf{y}(s) + \frac{(t-s)^i}{i!}V^i B_{\gamma_{Y(s)}}(\mathbf{0})\right] ds \\ &\subseteq \int_0^t \frac{(t-s)^i}{i!}V^i \mathbf{y}(s)\,ds + \int_0^t \frac{(t-s)^i}{i!}\eta_V^i B_{\gamma_{Y(s)}}(\mathbf{0})\,ds \\ &= \int_0^t \frac{(t-s)^i}{i!}V^i \mathbf{y}(s)\,ds + \int_0^t \frac{(t-s)^i \eta_V^i}{i!}\,\gamma_Y(s)\,ds\,B_1(\mathbf{0}). \end{aligned}$$

Consequently,

$$\begin{aligned}
h\left(X^j(t),\{\mathbf{0}\}\right) &\leq \|\mathbf{x}\| + h\left(\gamma_X B_1(\mathbf{0}),\{\mathbf{0}\}\right) \left\| \left[\sum_{i=1}^{j} \frac{t^i}{i!} U^i\right] \mathbf{z} \right\| + h\left(\left[\sum_{i=1}^{j} \frac{t^i}{i!} \eta_U^i \gamma_Z\right] B_1(\mathbf{0}),\{\mathbf{0}\}\right) \\
&\quad + \left\| \sum_{i=0}^{j} \int_0^t \left[\frac{(t-s)^i}{i!} V^i\right] \mathbf{y}(s)\, ds \right\| + h\left(\sum_{i=0}^{j} \int_0^t \left[\frac{(t-s)^i}{i!} \eta_V^i \gamma_Y(s)\right] ds\, B_1(\mathbf{0}),\{\mathbf{0}\}\right) \\
&\leq \|\mathbf{x}\| + \gamma_X + \left[\sum_{i=1}^{j} \frac{t^i}{i!} \eta_U^i\right] \|\mathbf{z}\| + \sum_{i=1}^{j} \frac{t^i}{i!} \eta_U^i \gamma_Z + \int_0^t \left[\sum_{i=0}^{j} \frac{(t-s)^i}{i!} \eta_V^i\right] \|\mathbf{y}(s)\|\, ds \\
&\quad + \int_0^t \sum_{i=0}^{j} \frac{(t-s)^i}{i!} \eta_V^i \gamma_Y(s)\, ds.
\end{aligned}$$

Because

$$\sum_{i=1}^{\infty} \frac{t^i}{i!} \eta_U^i \gamma_Z = \left(e^{t\eta_U} - 1\right)\gamma_Z, \quad \sum_{i=0}^{\infty} \frac{(t-s)^i}{i!} \eta_V^i \gamma_Y(s) = e^{(t-s)\eta_V} \gamma_Y(s),$$

$$\sum_{i=1}^{\infty} \frac{t^i}{i!} U^i = e^{tU} - I_n, \quad \sum_{i=0}^{\infty} \frac{(t-s)^i}{i!} V^i = e^{(t-s)V},$$

then

$$\begin{aligned}
\lim_{j\to\infty} X^j(t) \subseteq \mathbf{x} + \gamma_X B_1(\mathbf{0}) + \left[e^{tU} - I_n\right]\mathbf{x} + \left[\left(e^{t\eta_U} - 1\right)\gamma_Z\right] B_1(\mathbf{0}) + \\
+ \int_0^t e^{(t-s)V} \mathbf{y}(s)\, ds + \left[\int_0^t e^{(t-s)\eta_V} \gamma_Y(s)\, ds\right] B_1(\mathbf{0}) \quad (3.2)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{j\to\infty} h\left(X^j(t),\{\mathbf{0}\}\right) \leq \|\mathbf{x}\| + \gamma_X + \left[e^{t\eta_U} - 1\right]\|\mathbf{z}\| + \left(e^{t\eta_U} - 1\right)\gamma_Z + \\
+ \int_0^t e^{(t-s)\eta_V} \|\mathbf{y}(s)\|\, ds + \int_0^t e^{(t-s)\eta_V} \gamma_Y(s)\, ds. \quad (3.3)
\end{aligned}$$

Since for $p > q,\ p, q \in \mathbb{N}$

$$\begin{aligned}
h(X^p(t), X^q(t)) \leq h\Bigg(& X_0 + \sum_{i=1}^{p} \left\{\frac{t^i}{i!} U^i Z\right\} + \sum_{i=0}^{p} \int_0^t \left\{\frac{(t-s)^i}{i!} V^i Y(s)\right\} ds, \\
& X_0 + \sum_{i=1}^{q} \left\{\frac{t^i}{i!} U^i Z\right\} + \sum_{i=0}^{q} \int_0^t \left\{\frac{(t-s)^i}{i!} V^i Y(s)\right\} ds\Bigg) \\
\leq h\Bigg(& \sum_{i=q+1}^{p} \left\{\frac{t^i}{i!} U^i Z\right\} + \sum_{i=q+1}^{p} \int_0^t \left\{\frac{(t-s)^i}{i!} V^i Y(s)\right\} ds, \{\mathbf{0}\}\Bigg)
\end{aligned}$$

$$\begin{aligned}
&\leq \left\{ \sum_{i=q+1}^{p} \frac{t^i}{i!} \eta_U^i \right\} \|\mathbf{z}\| + \left\{ \sum_{i=q+1}^{p} \frac{t^i}{i!} \eta_U^i \right\} \gamma_Z + \sum_{i=q+1}^{p} \int_0^t \left\{ \frac{(t-s)^i}{i!} \eta_V^i \right\} \|\mathbf{y}(s)\| \, ds \\
&\quad + \sum_{i=q+1}^{p} \int_0^t \frac{(t-s)^i}{i!} \eta_V^i \gamma_Y(s) \, ds \\
&\leq \left\{ \sum_{i=q+1}^{p} \frac{T^i}{i!} \eta_U^i \right\} (\|\mathbf{z}\| + \gamma_Z) \\
&\quad + \left\{ \sum_{i=q+1}^{p} \int_0^T \frac{(T-s)^i}{i!} \eta_V^i \, ds \right\} \left(\max_{t\in[0,T]} \|\mathbf{y}(t)\| + \max_{t\in[0,T]} \gamma_Y(t) \right) \\
&= \left\{ \sum_{i=q+1}^{p} \frac{T^i}{i!} \eta_U^i \right\} (\|\mathbf{z}\| + \gamma_Z) \\
&\quad + \left\{ \sum_{i=q+1}^{p} \frac{T^{i+1}}{(i+1)!} \eta_V^i \right\} \left(\max_{t\in[0,T]} \|\mathbf{y}(t)\| + \max_{t\in[0,T]} \gamma_Y(t) \right).
\end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that for all $p > q > N$ the inequality $h(X^p(t), X^q(t)) < \varepsilon$ holds, meaning that the sequence $\{X^k(t)\}_{k=0}^{\infty}$ converges uniformly to $X(t)$ on the interval $[0, T]$.

Also, we have for all $t \in [0, T]$

$$\xi^0(t) \leq \xi^1(t) \leq \cdots \leq \xi^j(t) \leq \cdots$$

and

$$\lim_{j\to\infty} \xi^j(t) \leq \left(\gamma_X + \left(e^{t\eta_U} - 1\right) \gamma_Z + \int_0^t e^{(t-s)\eta_V} \gamma_Y(s) \, ds \right)^n \varrho, \tag{3.4}$$

where $\varrho = \dfrac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$ is the Lebesgue measure of the set $B_1(\mathbf{0})$ [5], and $\Gamma(n)$ is the Gamma function.

Thus, $\lim\limits_{j\to\infty} \xi^j(t)$ exists and equals $\vartheta(t)$, for all $t \in [0, T]$, and the sequence $\{\xi^j(t)\}_{j=0}^{\infty}$ converges uniformly to $\vartheta(t)$ on the interval $[0, T]$.

Substitute $X(\cdot)$ into the equation (1.1) and check the identity:

$$D_H X(t) = AX(t) + F(t),$$

or find the conditions for its validity.

Since $D_H(X_0)=\{\mathbf{0}\}$, $D_H\left(\int\limits_0^t Y(s)\,ds\right)=Y(t)$, $D_H X_{1,i}(t)=D_H\left(\frac{t^i}{i!}U^iZ\right)=\frac{t^{i-1}}{(i-1)!}U^iZ=UX_{1,i-1}(t)$,

$$D_H X_{2,i}(t)=D_H\left(\int\limits_0^t\left\{\frac{(t-s)^i\,V^i}{i!}Y(s)\right\}ds\right)=\int\limits_0^t\left\{\frac{(t-s)^{i-1}\,V^i}{(i-1)!}Y(s)\right\}ds=VX_{2,i-1}(t)$$

for all $i=1,2,\ldots$, and the series

$$\sum_{i=0}^{\infty}X_{1,i}(t)=\sum_{i=0}^{\infty}\left\{\frac{t^i}{i!}U^iZ\right\},\quad \sum_{i=0}^{\infty}D_HX_{1,i}(t)=U\sum_{i=1}^{\infty}X_{1,i-1}(t)=U\sum_{i=0}^{\infty}X_{1,i}(t),$$

$$\sum_{i=0}^{\infty}X_{2,i}(t)=\sum_{i=0}^{\infty}\int\limits_0^t\left\{\frac{(t-s)^i\,V^i}{i!}Y(s)\right\}ds,$$

$$\sum_{i=0}^{\infty}D_HX_{2,i}(t)=\sum_{i=0}^{\infty}D_H\left(\int\limits_0^t\left\{\frac{(t-s)^i\,V^i}{i!}Y(s)\right\}ds\right)=Y(t)+V\sum_{i=0}^{\infty}X_{2,i}(t)$$

converge uniformly on $[0,T]$, then

$$D_HX(t)=UZ+U\sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}U^iZ\right\}+Y(t)+V\sum_{i=0}^{\infty}\int\limits_0^t\left[\frac{(t-s)^i\,V^i}{i!}Y(s)\right]ds.$$

Also,

$$AX(t)+F(t)=AX_0+A\sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}U^iZ\right\}+A\sum_{i=0}^{\infty}\left\{\int\limits_0^t\left[\frac{(t-s)^i\,V^i}{i!}Y(s)\right]ds\right\}+F(t).$$

Then we obtain the following equality:

$$UZ+U\sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}U^iZ\right\}+Y(t)+V\sum_{i=0}^{\infty}\int\limits_0^t\left[\frac{(t-s)^i\,V^i}{i!}Y(s)\right]ds=$$
$$=AX_0+A\sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}U^iZ\right\}+A\sum_{i=0}^{\infty}\int\limits_0^t\left[\frac{(t-s)^i\,V^i}{i!}Y(s)\right]ds+F(t),$$

which will hold if $U=V=A$, $Z=X_0$ and $Y(t)=F(t)$.

Thus, the following theorem can be formulated.

Theorem 3.2. *If the matrix* $A \in \mathbb{R}^{n\times n}$ *is a non-degenerate matrix and the set-valued map* $F(\cdot) : [0,T] \to conv(\mathbb{R}^n)$ *is continuous on* $[0,T]$*, then the system* (1.1) *has the following solution:*

$$X(t) = X_0 + \sum_{i=1}^{\infty} \left\{ \frac{t^i}{i!} A^i X_0 \right\} + \sum_{i=0}^{\infty} \int_0^t \frac{(t-s)^i\, A^i}{i!} F(s)\, ds. \tag{3.5}$$

Remark 3.3. *Accordingly, considering* (3.2), (3.3), (3.4) *and* (3.5)*, we obtain*

$$X(t) \subseteq e^{tA}\mathbf{x} + e^{t\|A\|}\gamma_X B_1(\mathbf{0}) + \int_0^t e^{(t-s)A}\mathbf{f}(s)\, ds + \left[\int_0^t e^{(t-s)\|A\|}\gamma_F(s)\, ds \right] B_1(\mathbf{0}),$$

$$h\left(X(t), \{\mathbf{0}\}\right) \le e^{t\|A\|}\|\mathbf{x}\| + e^{t\|A\|}\gamma_X + \int_0^t e^{(t-s)\|A\|}\|\mathbf{f}(s)\|\, ds + \int_0^t e^{(t-s)\|A\|}\gamma_F(s)\, ds$$

and

$$\vartheta(t) \le \left(e^{t\|A\|}\gamma_X + \int_0^t e^{(t-s)\|A\|}\gamma_F(s)\, ds \right)^n \varrho\,,$$

where $\gamma_F(t) = \dfrac{1}{2} \max\limits_{f_1, f_2 \in F(t)} \|f_1 - f_2\|$*, the vector* $\mathbf{f}(t) \in \mathbb{R}^n$ *such that* $F(t) \subseteq B_{\gamma_F(t)}(\mathbf{f}(t))$.

Here are some corollaries of the theorem and remarks.

Corollary 3.4. *If* $X_0 \in \mathbb{R}^n$ *and* $F : [0,T] \to \mathbb{R}^n$*, then*

$$\sum_{i=0}^{\infty} \left\{ \frac{t^i}{i!} A^i X_0 \right\} = \left\{ \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i \right\} X_0 = e^{tA} X_0,$$

$$\sum_{i=0}^{\infty} \left\{ \int_0^t \frac{(t-s)^i A^i}{i!} F(s)\, ds \right\} = \int_0^t \sum_{i=0}^{\infty} \left\{ \frac{(t-s)^i A^i}{i!} \right\} F(s)\, ds = \int_0^t e^{(t-s)A} F(s)\, ds,$$

and accordingly, (3.5) *can be rewritten in the following form, yielding the well-known formula for ordinary linear differential equations:*

$$X(t) = e^{tA} X_0 + \int_0^t e^{(t-s)A} F(s)\, ds.$$

Corollary 3.5. *If* $X_0 \in conv(\mathbb{R}^n)$ *and* $F : [0,T] \to \mathbb{R}^n$*, then in this case,* (3.5) *will take the form*

$$X(t) = \sum_{i=0}^{\infty} \left\{ \frac{t^i}{i!} A^i X_0 \right\} + \int_0^t e^{(t-s)A} F(s)\, ds.$$

Remark 3.6. *Note that from Property C (3), we have*

$$\sum_{i=0}^{\infty}\left\{\frac{t^i}{i!}A^iX_0\right\} \supseteq \left\{\sum_{i=0}^{\infty}\frac{t^i}{i!}A^i\right\}X_0 = e^{At}X_0,$$

that is, in the general case we have

$$e^{At}X_0 + \int\limits_0^t e^{(t-s)A}F(s)\,ds \subseteq X(t).$$

However, if, for example,

(1) *the matrix A is such that its singular values $\sigma_1,\dots,\sigma_n$ satisfy the condition $\sigma_1 = \cdots = \sigma_n = \sigma$ and the set X_0 is such that $AX_0 = \sigma X_0$, then $A^2X_0 = AAX_0 = A\sigma X_0 = \sigma AX_0 = \sigma^2X_0,\dots,A^kX_0 = \sigma^kX_0,\dots$ and, accordingly,*

$$\sum_{i=0}^{\infty}\left\{\frac{t^i}{i!}A^iX_0\right\} = \sum_{i=0}^{\infty}\left\{\frac{t^i}{i!}\sigma^iX_0\right\} = \left\{\sum_{i=0}^{\infty}\frac{t^i\sigma^i}{i!}\right\}X_0 = e^{\sigma t}X_0 = e^{At}X_0.$$

Then the system (1.1) *has the following solution [12, 14, 21]:*

$$X(t) = e^{\sigma t}X_0 + \int\limits_0^t e^{(t-s)A}F(s)\,ds = e^{At}X_0 + \int\limits_0^t e^{(t-s)A}F(s)\,ds.$$

For example, the condition $AX_0 = \sigma X_0$ holds for

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad \text{and} \quad X_0 = B_1(\mathbf{0}) = \{\, x \in \mathbb{R}^2 \,:\, \|x\| \le 1\},$$

where $\sigma = \sqrt{a^2+b^2}$ [10, 12, 21].

(2) *The matrix $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ and $X_0 = \{\, x \in \mathbb{R}^2 : |x_i| \le 1,\ i = 1,2\}$.*

Since the singular value decomposition $U\Sigma V^T$ of the matrix A is

$$A = U\Sigma V^T = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} |a| & 0 \\ 0 & |b| \end{pmatrix}\begin{pmatrix} 0 & \frac{b}{|b|} \\ \frac{a}{|a|} & 0 \end{pmatrix}^T, & \text{if } |a| \ge |b|, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} |b| & 0 \\ 0 & |a| \end{pmatrix}\begin{pmatrix} \frac{b}{|b|} & 0 \\ 0 & \frac{a}{|a|} \end{pmatrix}^T, & \text{if } |a| < |b|, \end{cases}$$

then $AX_0 = U\Sigma V^TX_0 = U\Sigma X_0 = U\Pi_{\sigma_1,\sigma_2} = \Pi_{\sigma_1,\sigma_2} = \{x \in \mathbb{R}^2 : |x_i| \le \sigma_i,\ i = 1,2\}$, $A^2X_0 = A\Pi_{\sigma_1,\sigma_2} = \Pi_{\sigma_1^2,\sigma_2^2},\dots,A^iX_0 = A\Pi_{\sigma_1^{i-1},\sigma_2^{i-1}} = \Pi_{\sigma_1^i,\sigma_2^i},\dots,$ where $\sigma_1 =$

$\max\{|a|,|b|\}$, $\sigma_2=\min\{|a|,|b|\}$. *Accordingly,*

$$X_0+\sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}A^iX_0\right\}=X_0+\sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}\Pi_{\sigma_1^i,\sigma_2^i}\right\}=X_0+\sum_{i=1}^{\infty}\Pi_{\frac{t^i}{i!}\sigma_1^i,\frac{t^i}{i!}\sigma_2^i}=\Pi_{e^{\sigma_1 t},e^{\sigma_2 t}}.$$

Since $\Pi_{e^{\sigma_1 t},e^{\sigma_2 t}}=e^{\Sigma t}X_0=e^{At}X_0$, *the system* (1.1) *has the following solution:*

$$X(t)=e^{\Sigma t}X_0+\int_0^t e^{(t-s)A}F(s)\,ds=e^{At}X_0+\int_0^t e^{(t-s)A}F(s)\,ds.$$

Remark 3.7. *If* $X_0\in conv(\mathbb{R}^n)$ *and* $F:[0,T]\to conv(\mathbb{R}^n)$ *then the system* (1.1) *has the solution* (3.5). *Similarly, as in Remark 3.8, from Property C (3) we have*

$$\sum_{i=0}^{\infty}\left\{\frac{t^i}{i!}A^iX_0\right\}\supseteq\left\{\sum_{i=0}^{\infty}\frac{t^i}{i!}A^i\right\}X_0=e^{At}X_0,$$

$$\sum_{i=0}^{\infty}\int_0^t\frac{(t-s)^i\,A^i}{i!}F(s)\,ds\supseteq\int_0^t\left(\sum_{i=0}^{\infty}\frac{(t-s)^i\,A^i}{i!}\right)F(s)\,ds=\int_0^t e^{A(t-s)}F(s)\,ds,$$

that is, in the general case we have

$$e^{At}X_0+\int_0^t e^{(t-s)A}F(s)\,ds\subseteq X(t).$$

However, if, for example:

(1) *The matrix A, the set X_0 and the set-valued mapping $F(\cdot)$ are such that $AX_0=\sigma X_0$ and $AF(t)=\sigma F(t)$ for all $t\in[0,T]$, then the system* (1.1) *has the following solution*

$$X(t)=e^{At}X_0+\int_0^t e^{(t-s)A}F(s)\,ds=e^{\sigma t}X_0+\int_0^t e^{\sigma(t-s)}F(s)\,ds.$$

For example, the condition $AX_0=\sigma X_0$ and $AF(t)=\sigma F(t)$ for all $t\in[0,T]$ holds for $A=\begin{pmatrix}a & -b\\ b & a\end{pmatrix}$, $X_0=B_1(\mathbf{0})=\{\,x\in\mathbb{R}^2\,:\,\|x\|\le 1\}$, $F(t)=\{\,f(t)\in\mathbb{R}^2:\|f(t)\|\le g(t)\}$, *for all $t\in[0,T]$, and $g(t)>0$, for all $t\in[0,T]$, and $\sigma=\sqrt{a^2+b^2}$.*

(2) *The matrix* $A=\begin{pmatrix}0 & a\\ b & 0\end{pmatrix}$, $X_0=\{\,x\in\mathbb{R}^2:|x_i|\le 1,\,i=1,2\}$ *and* $F(t)=\{\,f(t)\in\mathbb{R}^2:|f_i(t)|\le g(t),\,i=1,2\}$ *for all $t\in[0,T]$ and $g(t)>0$ for all $t\in[0,T]$.*

Then the system (1.1) *has the following solution*

$$X(t) = e^{At}X_0 + \int_0^t e^{(t-s)A}F(s)\,ds = e^{\Sigma t}X_0 + \int_0^t e^{(t-s)\Sigma}F(s)\,ds,$$

where Σ *is the singular value matrix described in Remark 3.6.*

Remark 3.8. *If* $A = aI_n$ $(a < 0)$, $X_0 \in conv(\mathbb{R}^n)$ *and* $F : [0,T] \to conv(\mathbb{R}^n)$, *then the Cauchy problem* (1.1) *takes the form*

$$D_H X(t) = aX(t) + F(t), \quad X(0) = X_0. \tag{3.6}$$

However, since $a < 0$, *using the formula* (1.3) *is not possible. We rewrite the system* (3.6) *in matrix form* (1.1), *where* $A = -|a|I_n$, *and according to Theorem 3.2, the solution of the system* (3.6) *can be written in the form* (3.5).

Since $A = -|a|I_n$, $A^3 = -|a|^3 I_n, \dots, A^{2i-1} = -|a|^{2i-1}I_n, \dots$ *and* $A^2 = |a|^2 I_n, A^4 = |a|^4 I_n, \dots, A^{2i} = |a|^{2i} I_n, \dots$, *we have*

$$\begin{aligned}\sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}A^iX_0\right\} &= \sum_{i=1}^{\infty}\left\{\frac{t^{2i-1}}{(2i-1)!}A^{2i-1}X_0\right\} + \sum_{i=1}^{\infty}\left\{\frac{t^{2i}}{(2i)!}A^{2i}X_0\right\} \\ &= \sum_{i=1}^{\infty}\left\{\frac{(|a|t)^{2i-1}}{(2i-1)!}(-I_n)X_0\right\} + \sum_{i=1}^{\infty}\left\{\frac{(|a|t)^{2i}}{(2i)!}I_nX_0\right\}\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{\infty}\int_0^t\left\{\frac{(t-s)^i}{i!}A^iF(s)\right\}ds &= \sum_{i=1}^{\infty}\int_0^t\left\{\frac{(t-s)^{2i-1}}{(2i-1)!}A^{2i-1}F(s)\right\}ds + \sum_{i=0}^{\infty}\int_0^t\left\{\frac{(t-s)^{2i}}{(2i)!}A^{2i}F(s)\right\}ds \\ &= \int_0^t\left\{\sum_{i=1}^{\infty}\frac{|a|^{2i-1}(t-s)^{2i-1}}{(2i-1)!}(-I_n)F(s)\right\}ds + \int_0^t\left\{\sum_{i=0}^{\infty}\frac{|a|^{2i}(t-s)^{2i}}{(2i)!}I_nF(s)\right\}ds.\end{aligned}$$

Since the sets $\bar{X}_0 = (-I_n)X_0$ *and* $\bar{F}(t) = (-I_n)F(t)$ *are centrally symmetric to the sets* X_0 *and* $F(t)$ *relative to the point* $\mathbf{0}$ *[2], we can write*

$$\begin{aligned}X(t) = X_0 &+ \sum_{i=1}^{\infty}\left\{\frac{|a|^{2i-1}t^{2i-1}}{(2i-1)!}\bar{X}_0\right\} + \sum_{i=1}^{\infty}\left\{\frac{|a|^{2i}t^{2i}}{(2i)!}X_0\right\} \\ &+ \int_0^t\sum_{i=1}^{\infty}\left\{\frac{|a|^{2i-1}(t-s)^{2i-1}}{(2i-1)!}\bar{F}(s)\right\}ds + \int_0^t\sum_{i=0}^{\infty}\left\{\frac{|a|^{2i}(t-s)^{2i}}{(2i)!}F(s)\right\}ds.\end{aligned}$$

Since

$$\sum_{i=1}^{\infty}\left\{\frac{(\sigma t)^{2i-1}}{(2i-1)!}\bar{X}_0\right\}=\sum_{i=1}^{\infty}\left\{\frac{(\sigma t)^{2i-1}}{(2i-1)!}\right\}\bar{X}_0=\sinh(\sigma t)\bar{X}_0,$$

$$X_0+\sum_{i=1}^{\infty}\left\{\frac{(\sigma t)^{2i}}{(2i)!}X_0\right\}=\sum_{i=0}^{\infty}\left\{\frac{(\sigma t)^{2i}}{(2i)!}\right\}X_0=\cosh(\sigma t)X_0,$$

$$\sum_{i=1}^{\infty}\left\{\frac{|a|^{2i-1}(t-s)^{2i-1}}{(2i-1)!}\bar{F}(s)\right\}=\sum_{i=1}^{\infty}\left\{\frac{|a|^{2i-1}(t-s)^{2i-1}}{(2i-1)!}\right\}\bar{F}(s)=\sinh(|a|(t-s))\bar{F}(s),$$

$$\sum_{i=0}^{\infty}\left\{\frac{|a|^{2i}(t-s)^{2i}}{(2i)!}F(s)\right\}=\sum_{i=0}^{\infty}\left\{\frac{|a|^{2i}(t-s)^{2i}}{(2i)!}\right\}F(s)=\cosh(|a|(t-s))F(s),$$

then the solution of the system (3.6) *can be written in the following form:*

$$X(t)=\sinh(|a|t)\bar{X}_0+\cosh(|a|t)X_0+\int_0^t\sinh(|a|(t-s))\bar{F}(s)\,ds+\int_0^t\cosh(|a|(t-s))F(s)\,ds. \quad (3.7)$$

Remark 3.9. *If $X_0\equiv\bar{X}_0$ and $F(t)\equiv\bar{F}(t)$ for all $t\geq 0$, then*

$$\begin{aligned}\sinh(|a|t)\bar{X}_0+\cosh(|a|t)X_0+\int_0^t\sinh(|a|(t-s))\bar{F}(s)\,ds+\int_0^t\cosh(|a|(t-s))F(s)\,ds\\ =\sinh(|a|t)X_0+\cosh(|a|t)X_0+\int_0^t\sinh(|a|(t-s))F(s)\,ds+\int_0^t\cosh(|a|(t-s))F(s)\,ds\\ =e^{|a|t}X_0+\int_0^t e^{|a|(t-s)}F(s)\,ds.\end{aligned}$$

That is, if the sets X_0 and $F(t)$ are centrally symmetric with respect to the point $\mathbf{0}$ for all $t\geq 0$, then for all $a\neq 0$, the solution of the system (1.2) *has the form* (1.3), *with the replacement of a by $|a|$.*

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