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Class of symmetric $H_{\sqrt{q}}$-Laguerre-Hahn linear forms

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ABSTRACT

The aim of this paper is to study the symmetrized form associated with a H_q-Laguerre-Hahn form, where H_q is the q-derivative operator. Given a H_q-Laguerre-Hahn form u of class s, it is shown that its symmetrized form w is $H_{\sqrt{q}}$-Laguerre-Hahn of class $\tilde{s} \leq 2s+3$. We give the $\sqrt{q}$-Riccati equation satisfied by the Stieltjes formal series $S(w)$ as well as a complete discussion of the class $\tilde{s}$.
As an application of this work, we generate two examples of symmetric $H_{\sqrt{q}}$-Laguerre-Hahn orthogonal polynomials of class two and three.

RESUMEN

El objetivo de este artículo es estudiar la forma simetrizada asociada a una forma H_q-Laguerre-Hahn, donde H_q es el operador q-derivada. Dada una forma H_q-Laguerre-Hahn u de clase s, se muestra que su forma simetrizada w es $H_{\sqrt{q}}$-Laguerre-Hahn de clase $\tilde{s} \leq 2s+3$. Damos la ecuación $\sqrt{q}$-Riccati satisfecha por la serie formal de Stieltjes $S(w)$ y también una discusión completa de la clase $\tilde{s}$.
Como aplicación de este trabajo, generamos dos ejemplos de polinomios ortogonales simétricos $H_{\sqrt{q}}$-Laguerre-Hahn de clases dos y tres.

Keywords and Phrases: Orthogonal q-polynomials, q-derivative operator, q-difference equation, q-Riccati equation, H_q-Laguerre-Hahn character, quadratic decomposition.

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1 Introduction and preliminaries

In [9] a basic theory of H_q-Laguerre-Hahn orthogonal polynomials was introduced, and several characterizations were presented, namely the q-difference equation, the structure relation and the q-Riccati equation. Moreover, a criterion for simplifying the class of a H_q-Laguerre-Hahn form has been established. The paper also provides illustrative examples of these polynomials using standard perturbations (association, co-recursion, inversion) of H_q-classical polynomials [16]. Recently, in [13] we studied the Christoffel and Geronimus transformations in the H_q-Laguerre-Hahn case, and in [12] we proceeded by the addition of a Dirac mass to a H_q-Laguerre-Hahn form. In the two works cited above, a complete discussion of the class of the resulting form was given, and some examples of H_q-Laguerre-Hahn polynomial sequences (in relation with H_q-classical polynomial sequences) of class one and two were highlighted. In addition, in [15], the symmetric H_q-Laguerre-Hahn orthogonal polynomials of class zero were exhaustively described (see also [22]), and in [14], the class one case were also completely described.

Note that our works were a continuation of studies done in the field of D-Laguerre-Hahn polynomials. In fact, in the literature there are several contributions devoted to the study of these polynomials by different processes. In this area you can see [1, 3, 4, 6, 7, 20].

Let u be a regular form, we can define a new form w whose moments are given in terms of that of u such that $(w)_{2n} = (u)_n$, $(w)_{2n+1} = 0$, $n \geq 0$. The form w is said to be the symmetrized form associated with the form u [2, 20]. A necessary and sufficient condition for the regularity of w was given in [5, 19]. In this contribution, we study the symmetrized form associated with a H_q-Laguerre-Hahn form; for the D-semi-classical case (see [2]) and for the D-Laguerre-Hahn case (see [23]). In fact, let u be a regular form whose Stieltjes formal series $S(u)$ satisfies a q-Riccati equation. We show that the formal series $S(w)$ associated with the symmetrized form w satisfies a $\sqrt{q}$-Riccati equation. If we denote by s the class of u and by $\tilde{s}$ that of w, it turns out that: $\tilde{s} \leq 2s+3$ and we specify the exact conditions for which $\tilde{s} = 2s$, $2s+1$, $2s+2$, $2s+3$. Finally, starting from two familes of H_q-Laguerre-Hahn polynomials of class 0, we derived two families of symmetric H_q-Laguerre-Hahn polynomials of classes 2 and 3.

We will now recall some useful results.

We denote by $\mathcal{P}$ the vector space of polynomials with coefficients in $\mathbb{C}$ and by $\mathcal{P}'$ its dual space. The action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ is denoted as $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the moments of u. For any form u, any polynomial g and any $(a, c) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$, let $H_q u$, gu, $h_a u$, Du, $(x-c)^{-1}u$, δ_c and σu, be the forms defined as in [16, 20]

$$\langle H_q u, f\rangle := -\langle u, H_q f\rangle, \quad \langle gu, f\rangle := \langle u, gf\rangle, \quad \langle h_a u, f\rangle := \langle u, h_a f\rangle,$$

$$\langle Du, f\rangle := -\langle u, f'\rangle, \quad \langle (x-c)^{-1}u, f\rangle := \langle u, \theta_c f\rangle, \quad \langle \delta_c, f\rangle := f(c), \quad \langle \sigma u, f\rangle = \langle u, \sigma f\rangle,$$

where for all $f \in \mathcal{P}$ and $q \in \widetilde{\mathbb{C}} := \{z \in \mathbb{C},\, z \neq 0,\, z^n \neq 1,\, n \geq 1\}$ [16]

$$(H_q f)(x) = \begin{cases} \dfrac{f(qx) - f(x)}{(q-1)x}, & x \neq 0, \\ f^{'}(0), & x = 0, \end{cases} \tag{1.1}$$

$$(\theta_c f)(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c}, & x \neq c, \\ f^{'}(c), & x = c, \end{cases} \tag{1.2}$$

$$(h_a f)(x) = f(ax). \tag{1.3}$$

$$(\sigma f)(x) = f\left(x^2\right) \tag{1.4}$$

In particular, we have

$$(H_q u)_n = -[n]_q (u)_{n-1}\,, \quad (\sigma u)_n = (u)_{2n}\,, \quad n \geq 0, \tag{1.5}$$

where $(u)_{-1} = 0$ and $[n]_q := \dfrac{q^n - 1}{q - 1}$, $n \geq 0$ [10].

When $q \to 1$, H_q converges to the derivative operator D.

Lemma 1.1 ([16,20])**.** *For $f, g \in \mathcal{P}$, $u \in \mathcal{P}'$, $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, we have*

$$h_a(gu) = (h_{a^{-1}} g)(h_a u), \tag{1.6}$$

$$h_{q^{-1}} \circ H_q = H_{q^{-1}}\,, \quad H_q \circ h_{q^{-1}} = q^{-1} H_{q^{-1}}\,, \quad \textit{in } \mathcal{P}, \tag{1.7}$$

$$H_q(fg)(x) = (h_q f)(x)(H_q g)(x) + g(x)(H_q f)(x), \tag{1.8}$$

$$H_q(gu) = (h_{q^{-1}} g) H_q u + q^{-1}(H_{q^{-1}} g) u, \tag{1.9}$$

$$(\theta_b fg)(x) = g(x)(\theta_b f)(x) + f(b)(\theta_b g)(x). \tag{1.10}$$

$$f(x)\sigma u = \sigma\left(f\left(x^2\right) u\right). \tag{1.11}$$

For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product uf is the polynomial [20]

$$(uf)(x) := \left\langle u, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta} \right\rangle = \sum_{i=0}^{n} \left(\sum_{j=i}^{n} (u)_{j-i}\, f_j \right) x^i,$$

where $f(x) = \sum_{i=0}^{n} f_i x^i$. This allows us to define the Cauchy's product of two forms:

$$\langle uv, f \rangle := \langle u, vf \rangle, \quad f \in \mathcal{P}.$$

The Stieltjes formal series of $u \in \mathcal{P}'$ is defined by [20]

$$S(u)(z) := -\sum_{n\geq 0} \frac{(u)_n}{z^{n+1}}. \tag{1.12}$$

The quantum factorial symbol is defined by [8]

$$(x;q)_0 = 1, \quad (x;q)_n = \prod_{k=0}^{n-1}\left(1 - xq^k\right), \quad x, q \in \mathbb{C},\ n \geq 1. \tag{1.13}$$

The q-binomial coefficient is defined by [8]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \tag{1.14}$$

A form u is said regular if there exists a sequence of monic polynomials $\{P_n\}_{n\geq 0}$, $\deg P_n = n$,$n \geq 0$ MPS, such that $\langle u, P_n P_m\rangle = r_n \delta_{n,m}$ with $r_n \neq 0$ for any $n, m \geq 0$ where $\delta_{n,m}$ denotes the Kronecker symbol. In this case, $\{P_n\}_{n\geq 0}$ is called a monic orthogonal polynomial sequence MOPS with respect to the form u and it is unique. Given a MOPS $\{P_n\}_{n\geq 0}$ with respect to the form u, there exists a unique sequence $\{u_n\}_{n\geq 0}$, $u_n \in \mathcal{P}'$, called the dual sequence of $\{P_n\}_{n\geq 0}$, such that $\langle u_n, P_m\rangle = \delta_{n,m}$, $n, m \geq 0$. It holds $u = \lambda u_0$, $\lambda = (u)_0 \neq 0$. Furthermore, if $(u)_0 = 1$, then $u = u_0$. Hence, there exists a unique form u with $(u)_0 = 1$, where $\{P_n\}_{n\geq 0}$ is its corresponding MOPS.

The MOPS $\{P_n\}_{n\geq 0}$ is characterized by the following three-term recurrence relation (Favard's theorem) (TTRR in short) [5, 20]

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ &P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \end{aligned} \tag{1.15}$$

where

$$\beta_n = \frac{\langle u, xP_n^2\rangle}{\langle u, P_n^2\rangle} \in \mathbb{C}, \quad \gamma_{n+1} = \frac{\langle u, P_{n+1}^2\rangle}{\langle u, P_n^2\rangle} \in \mathbb{C}\setminus\{0\}, \quad n \geq 0. \tag{1.16}$$

The shifted MOPS $\left\{\widehat{P}_n := a^{-n}\big(h_a P_n\big)\right\}_{n\geq 0}$ is then orthogonal with respect to $\widehat{u} = h_{a^{-1}}u$ and satisfies (1.15) with [20]

$$\widehat{\beta}_n = \frac{\beta_n}{a}, \quad \widehat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0.$$

The form u is said to be normalized if $(u)_0 = 1$. In this paper, we suppose that any regular form will be normalized. The form u is said to be positive definite if and only if $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} > 0$, for all $n \geq 0$. When u is regular, $\{P_n\}_{n\geq 0}$ is a symmetric MOPS if and only if $\beta_n = 0$, $n \geq 0$, or equivalently $(u)_{2n+1} = 0$, $n \geq 0$ [5].

Given a regular form u and the corresponding MOPS $\{P_n\}_{n\geq 0}$, we define the associated sequence of the first kind $\{P_n^{(1)}\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ by [20]

$$P_n{}^{(1)}(x) = \left\langle u, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x-\xi} \right\rangle = (u\theta_0 P_{n+1})(x), \quad n \geq 0.$$

Proposition 1.2 ([20]). *Let $\{P_n\}_{n\geq 0}$ be a MOPS satisfying the TTRR* (1.15)*, then its associated sequence $\left\{P_n^{(1)}\right\}_{n\geq 0}$ satisfies the TTRR*

$$\begin{aligned} &P_0^{(1)}(x) = 1, \quad P_1^{(1)}(x) = x - \beta_1, \\ &P_{n+2}^{(1)}(x) = (x-\beta_{n+2})P_{n+1}^{(1)}(x) - \gamma_{n+2}P_n^{(1)}(x), \quad n \geq 0. \end{aligned} \tag{1.17}$$

In addition, we have the following fundamental result

$$P_{n+1}^{(1)}(x)P_{n+1}(x) - P_{n+2}(x)P_n^{(1)}(x) = \prod_{\nu=0}^{n} \gamma_{\nu+1}, \quad n \geq 0. \tag{1.18}$$

We recall the definition of the so-called kernel polynomials with K-parameter introduced by Chihara [5].

Let $\{P_n\}_{n>0}$ be a MOPS and c a complex number such that $P_n(c) \neq 0$, for all $n \geq 0$. The sequence of monic kernel polynomials of K-parameter c, $\{P_n^*(c;\cdot)\}_{n\geq 0}$, associated with $\{P_n\}_{n\geq 0}$ is defined by

$$P_n^*(c;x) := \frac{1}{x-c}\left[P_{n+1}(x) - \frac{P_{n+1}(c)}{P_n(c)}P_n(x)\right], \quad n \geq 0.$$

In these conditions, if $\{P_n\}_{n\geq 0}$ is a MOPS associated with the form u, then $\{P_n^*(c;\cdot)\}_{n\geq 0}$ is a MOPS associated with $u^* = (x-c)u$ [5].

Lemma 1.3 ([21]). *Let $(b_n)_{n\geq 0}$ with $b_n \neq 0$, $n \geq 0$, $(c_n)_{n\geq 0}$ be two sequences of complex numbers and $(x_n)_{n\geq 0}$ the sequence satisfying the recurrence relation:*

$$x_{n+1} = b_n x_n + c_n, \quad n \geq 0, \quad x_0 = a \in \mathbb{C} - \{0\}.$$

We have

$$x_{n+1} = \left(\prod_{k=0}^{n} b_k\right)\left\{a + \sum_{k=0}^{n}\left(\prod_{l=0}^{k} b_l\right)^{-1} c_k\right\}, \quad n \geq 0.$$

We will give now some basic facts about the H_q-Laguerre-Hahn character.

Definition 1.4 ([9]). *A form u is called H_q-Laguerre-Hahn when it is regular and satisfies the q-difference equation*

$$H_q(\Phi u) + \Psi u + B\left(x^{-1}u\left(h_q u\right)\right) = 0, \tag{1.19}$$

where Φ, Ψ, B are polynomials, with Φ monic. The corresponding monic orthogonal sequence $\{P_n\}_{n\geq 0}$ is called H_q-Laguerre-Hahn.

Remark 1.5. *(1) When $B = 0$ and the form u is regular, then u is H_q-semiclassical [17].*

(2) When u satisfies (1.19)*, then $\widehat{u} = h_{a^{-1}} u$ fulfills the q-difference equation [9]*

$$H_q \left(a^{-\deg \Phi} \Phi(ax)\widehat{u}\right) + a^{1-\deg \Phi} \Psi(ax)\widehat{u} + a^{-\deg \Phi} B(ax) \left(x^{-1}\widehat{u}(h_q \widehat{u})\right) = 0. \tag{1.20}$$

(3) If $t = \deg \Phi$, $p = \deg \Psi$, $r = \deg B$ and $d = \max(t, r)$, then we define the class of u the nonnegative integer s [9]

$$s = \min \max(p - 1, d - 2), \tag{1.21}$$

where the minimum is taken over all triples (Φ, Ψ, B) satisfying (1.19)*. Moreover, the H_q-Laguerre-Hahn form u satisfying* (1.19) *is of class $s = \max(p - 1, d - 2)$ if and only if*

$$\prod_{c \in \mathcal{Z}_\Phi} \Big\{ \left|q(h_q \Psi)(c) + (H_q \Phi)(c)\right| + \left|q(h_q B)(c)\right| + \left| \Big\langle u, q(\theta_{cq}\Psi) + (\theta_{cq} \circ \theta_c \Phi) + q\big(h_q u(\theta_0 \circ \theta_{cq} B)\big) \Big\rangle \right| \Big\} > 0, \tag{1.22}$$

where $\mathcal{Z}_\Phi$ is the set of roots of Φ [9]. When $c \in \mathcal{Z}_\Phi$ and (1.19) *may be simplified by $x - c$, then* (1.19) *becomes*

$$H_q((\theta_c \Phi)u) + (q\theta_{cq}\Psi + \theta_{cq} \circ \theta_c \Phi)u + q(\theta_{cq} B) \left(x^{-1}u(h_q u)\right) = 0. \tag{1.23}$$

Proposition 1.6 ([9])**.** *Let u be a regular form, the following statements are equivalent:*

(1) u belongs to the H_q-Laguerre-Hahn class, i.e. satisfying (1.19)*.*

(2) The Stieltjes formal series $S(u)$ satisfies the q-Riccati equation

$$\left(h_{q^{-1}}\Phi\right)(z) \left(H_{q^{-1}} S(u)\right)(z) = B(z)S(u)(z) \left(h_{q^{-1}} S(u)\right)(z) + C(z)S(u)(z) + D(z), \tag{1.24}$$

where Φ and B are polynomials defined in (1.19) *and*

$$\begin{cases} C(z) = -\left(H_{q^{-1}}\Phi\right)(z) - q\Psi(z), \\ D(z) = -\left\{H_{q^{-1}} \left(u\theta_0 \Phi\right)(z) + q\left(u\theta_0 \Psi\right)(z) + q\left(u h_q u\right)\left(\theta_0^2 B\right)(z)\right\}. \end{cases} \tag{1.25}$$

Moreover, u is of class s if and only if

$$\prod_{c \in \mathcal{Z}_\Phi} \{|B(cq)| + |C(cq)| + |D(cq)|\} > 0, \tag{1.26}$$

and one may write

$$s = \max\left(\deg B - 2, \deg C - 1, \deg D\right). \tag{1.27}$$

Let u be a regular form and $\{P_n\}_{n\geq 0}$ its corresponding MOPS. By linear extension, we can define a new form w whose moments are given in terms of that of u such that

$$(w)_{2n} = (u)_n, \quad (w)_{2n+1} = 0, \quad n \geq 0, \tag{1.28}$$

equivalent to saying that

$$\sigma w = u, \quad \sigma(xw) = 0. \tag{1.29}$$

In the following theorem the author gave a necessary and sufficient condition for a MPS $\{Q_n\}_{n\geq 0}$ to be a MOPS with respect to a form w satisfying (1.28), this form will be unique since $(w)_0 = (u)_0 = 1$.

Theorem 1.7 ([19]). *Let $\{P_n\}_{n\geq 0}$ be MOPS and $\{Q_n\}_{n\geq 0}$ a MPS such that*

$$Q_1(x) = x, \quad Q_{2n}(x) = P_n\left(x^2\right), \quad n \geq 0. \tag{1.30}$$

Then $\{Q_n\}_{n\geq 0}$ is a MOPS if and only if $P_n(0) \neq 0$, $Q_{2n+1}(x) = xP_n^*\left(0; x^2\right)$, $n \geq 0$.

In such conditions, if $\{P_n\}_{n\geq 0}$ satisfies the three-term recurrence relation

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0^P, \\ &P_{n+2}(x) = \left(x - \beta_{n+1}^P\right) P_{n+1}(x) - \gamma_{n+1}^P P_n(x), \quad n \geq 0, \end{aligned}$$

then the coefficients $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ for the corresponding three-term recurrence relation satisfied by $\{Q_n\}_{n\geq 0}$ are given by

$$\begin{gathered} \tilde{\beta}_n = 0, \quad n \geq 0, \\ \tilde{\gamma}_{2n+1} = -\frac{P_{n+1}(0)}{P_n(0)}, \quad \tilde{\gamma}_{2n+2} = -\frac{P_n(0)}{P_{n+1}(0)}\gamma_{n+1}^P, \quad n \geq 0. \end{gathered} \tag{1.31}$$

Moreover, if $\{P_n\}_{n\geq 0}$ is orthogonal with respect to the form u, then $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to a form w defined on the basis $\{x^n\}_{n\geq 0}$ of $\mathcal{P}$ by means of (1.28).

Proposition 1.8 ([5]). *Let w be a symmetric and regular form. Let $\{Q_n\}_{n\geq 0}$ be the corresponding MOPS. It satisfies a three-term recurrence relation*

$$\begin{aligned} &Q_0(x) = 1, \quad Q_1(x) = x \\ &Q_{n+2}(x) = x, \quad Q_{n+1}(x) - \tilde{\gamma}_{n+1} Q_n(x), \quad n \geq 0, \end{aligned}$$

It is very well known that [5]

$$Q_{2n}(x) = P_n\left(x^2\right), \quad Q_{2n+1}(x) = xR_n\left(x^2\right),$$

where $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are MOPSs related to regular forms $u = \sigma w$ and $\gamma_1^{-1} x\sigma w$, respectively. The form w is said to be the symmetrized form associated with the form u. Furthermore, *if*

$$P_0(x)=1,\quad P_1(x)=x-\beta_0^P,$$
$$P_{n+2}(x)=\left(x-\beta_{n+1}^P\right)P_{n+1}(x)-\gamma_{n+1}^P P_n(x),\quad n\geq 0,$$

and

$$R_0(x)=1,\quad R_1(x)=x-\beta_0^R,$$
$$R_{n+2}(x)=\left(x-\beta_{n+1}^R\right)R_{n+1}(x)-\gamma_{n+1}^R R_n(x),\quad n\geq 0,$$

we get

$$\begin{cases}\beta_0^P=\tilde{\gamma}_1,\\ \beta_{n+1}^P=\tilde{\gamma}_{2n+2}+\tilde{\gamma}_{2n+3},\quad n\geq 0,\\ \gamma_{n+1}^P=\tilde{\gamma}_{2n+1}\tilde{\gamma}_{2n+2},\quad n\geq 0,\end{cases}$$

and

$$\begin{cases}\beta_{n+1}^R=\tilde{\gamma}_{2n+1}+\tilde{\gamma}_{2n+2},\quad n\geq 0,\\ \gamma_{n+1}^R=\tilde{\gamma}_{2n+2}\tilde{\gamma}_{2n+3},\quad n\geq 0.\end{cases}$$

Throughout the rest of this paper, let $0<q<1$ or $q>1$ and we write

$$S(u)(z)=S_u(z),\quad S(w)(z)=S_w(z).$$

2 The $H_{\sqrt{q}}$-Laguerre–Hahn character of the form w

Lemma 2.1. *The following equality holds*

$$z\left(H_{\sqrt{q}^{-1}}S_w\right)(z)-S_w(z)=\sqrt{q}^{-1}(\sqrt{q}^{-1}+1)z^3\left(H_{q^{-1}}S_u\right)(z^2).\tag{2.1}$$

Proof. From (1.12) and (1.28), we have

$$zS_u\left(z^2\right)=S_w(z)\tag{2.2}$$

Therefore,

$$\begin{aligned}\left(H_{\sqrt{q}^{-1}}S_w\right)(z)&\underset{\text{by (1.1)}}{=}\frac{S_w(\sqrt{q}^{-1}z)-S_w(z)}{(\sqrt{q}^{-1}-1)z}\underset{\text{by (2.2)}}{=}\frac{\sqrt{q}^{-1}zS_u(q^{-1}z^2)-zS_u(z^2)}{(\sqrt{q}^{-1}-1)z}\\&=\sqrt{q}^{-1}(\sqrt{q}^{-1}+1)\frac{S_u(q^{-1}z^2)-\sqrt{q}S_u(z^2)}{(q^{-1}-1)}\\&=\sqrt{q}^{-1}(\sqrt{q}^{-1}+1)z^2\left\{\frac{S_u(q^{-1}z^2)-S_u(z^2)+(1-\sqrt{q})S_u(z^2)}{(q^{-1}-1)z^2}\right\}\\&\underset{\text{by (1.1)-(2.2)}}{=}\sqrt{q}^{-1}(\sqrt{q}^{-1}+1)z^2\left(H_{q^{-1}}S_u\right)(z^2)+S_u(z^2).\end{aligned}\tag{2.3}$$

Multiplying (2.3) by z yields (2.1). □

Proposition 2.2. *Under the conditions of Theorem 1.7, if u is a H_q-Laguerre-Hahn form of class s such that its Stieltjes formal series S_u satisfies the q-Riccati equation* (1.24)*, then the form w defined by* (1.28) *is $H_{\sqrt{q}}$-Laguerre-Hahn and its Stieltjes formal series S_w satisfies the $\sqrt{q}$-Riccati equation*

$$\left(h_{\sqrt{q}^{-1}}\tilde{\Phi}\right)(z)\left(H_{\sqrt{q}^{-1}}S_w\right)(z)=\tilde{B}(z)S_w(z)\left(h_{\sqrt{q}^{-1}}S_w\right)(z)+\tilde{C}(z)S_w(z)+\tilde{D}(z), \tag{2.4}$$

where

$$\begin{cases}\tilde{\Phi}(z)=z\Phi\left(z^2\right),\\ \tilde{B}(z)=q^{-1}\left(\sqrt{q}+1\right)zB\left(z^2\right),\\ \tilde{C}(z)=q^{\frac{-3}{2}}\left(\sqrt{q}+1\right)z^2C\left(z^2\right)+\sqrt{q}^{-1}\Phi\left(q^{-1}z^2\right),\\ \tilde{D}(z)=q^{\frac{-3}{2}}\left(\sqrt{q}+1\right)z^3D\left(z^2\right).\end{cases} \tag{2.5}$$

Moreover, w is of class $\tilde{s}\leq 2s+3$.

Proof. From (1.3), the q-Riccati equation satisfied by the formal Stieltjes series S_u can be written in this form

$$\Phi\left(q^{-1}z\right)\left(H_{q^{-1}}S_u\right)(z)=B(z)S_u(z)S_u\left(q^{-1}z\right)+C(z)S_u(z)+D(z) \tag{2.6}$$

We make the change of variable $z\longleftarrow z^2$ in (2.6),

$$\Phi\left(q^{-1}z^2\right)\left(H_{q^{-1}}S_u\right)\left(z^2\right)=B\left(z^2\right)S_u\left(z^2\right)S_u\left(q^{-1}z^2\right)+C(z^2)S_u\left(z^2\right)+D\left(z^2\right). \tag{2.7}$$

Then we multiply (2.7) by $q^{-1}(\sqrt{q}+1)z^3$, we get

$$\begin{aligned}q^{-1}\left(\sqrt{q}+1\right)z^3\Phi\left(q^{-1}z^2\right)\left(H_{q^{-1}}S_u\right)\left(z^2\right)&=q^{-1}\left(\sqrt{q}+1\right)z^3B\left(z^2\right)S_u\left(z^2\right)S_u\left(q^{-1}z^2\right)+\\ &q^{-1}\left(\sqrt{q}+1\right)z^3C\left(z^2\right)S_u\left(z^2\right)+q^{-1}\left(\sqrt{q}+1\right)z^3D\left(z^2\right),\end{aligned} \tag{2.8}$$

with

$$\left(H_{q^{-1}}S_u\right)(z)=\frac{S_u\left(q^{-1}z\right)-S_u(z)}{\left(q^{-1}-1\right)z},\quad \left(H_{q^{-1}}S_u\right)\left(z^2\right)=\frac{S_u\left(q^{-1}z^2\right)-S_u\left(z^2\right)}{\left(q^{-1}-1\right)z^2}. \tag{2.9}$$

In an equivalent way

$$\begin{aligned}&\Phi\left(q^{-1}z^2\right)\left(q^{-1}\left(\sqrt{q}+1\right)\ z^3\left(H_{q^{-1}}S_u\right)\left(z^2\right)\right)\\ &\quad=\sqrt{q}^{-1}\left(\sqrt{q}+1\right)zB(z^2)\left(zS_u(z^2)\right)\left(\sqrt{q}^{-1}zS_u\left(\left(\sqrt{q}^{-1}z\right)^2\right)\right)\\ &\quad+q^{-1}\left(\sqrt{q}+1\right)z^2C(z^2)\left(zS_u(z^2)\right)+q^{-1}\left(\sqrt{q}+1\right)z^3D\left(z^2\right).\end{aligned} \tag{2.10}$$

Substituting (2.1) into (2.10) gives

$$\begin{aligned}\Phi\left(q^{-1}z^2\right)\left(z\left(H_{\sqrt{q}^{-1}}S_w\right)(z)-S_w(z)\right)&=\sqrt{q}^{-1}\left(\sqrt{q}+1\right)zB\left(z^2\right)S_w(z)S_w\left(\sqrt{q}^{-1}z\right)\\&+q^{-1}\left(\sqrt{q}+1\right)z^2C\left(z^2\right)S_w(z)+q^{-1}\left(\sqrt{q}+1\right)z^3D\left(z^2\right),\end{aligned}\tag{2.11}$$

equivalent to

$$\begin{aligned}z\Phi\left(q^{-1}z^2\right)\left(H_{\sqrt{q}^{-1}}S_w\right)(z)&=\sqrt{q}^{-1}\left(\sqrt{q}+1\right)zB\left(z^2\right)S_w(z)S_w\left(\sqrt{q}^{-1}z\right)\\&+\left(q^{-1}\left(\sqrt{q}+1\right)z^2C\left(z^2\right)+\Phi\left(q^{-1}z^2\right)\right)S_w(z)+q^{-1}\left(\sqrt{q}+1\right)z^3D\left(z^2\right).\end{aligned}\tag{2.12}$$

This amounts to

$$\begin{aligned}\sqrt{q}\left(h_{\sqrt{q}^{-1}}\tilde{\Phi}\right)(z)\left(H_{\sqrt{q}^{-1}}S_w\right)(z)&=\sqrt{q}^{-1}\left(\sqrt{q}+1\right)zB\left(z^2\right)S_w(z)\left(h_{\sqrt{q}^{-1}}S_w\right)(z)\\&+\left(q^{-1}\left(\sqrt{q}+1\right)z^2C\left(z^2\right)+\Phi\left(q^{-1}z^2\right)\right)S_w(z)+q^{-1}\left(\sqrt{q}+1\right)z^3D\left(z^2\right),\end{aligned}\tag{2.13}$$

where $\tilde{\Phi}(z)=z\Phi(z^2)$. Thus, we obtain (2.4)-(2.5), by dividing the previous equation by $\sqrt{q}$.
It follows from (1.8), (1.25) and (2.5) that

$$\tilde{\Psi}(z)=-\sqrt{q}^{-1}\left(H_{\sqrt{q}^{-1}}\tilde{\Phi}\right)(z)-\sqrt{q}^{-1}\tilde{C}(z)=q^{-1}\left(\sqrt{q}+1\right)\left(z^2\Psi\left(z^2\right)-\Phi\left(z^2\right)\right).\tag{2.14}$$

Since $\sqrt{q}\in\widetilde{\mathbb{C}}$ when $q\in\widetilde{\mathbb{C}}$, then according to the equivalence in Proposition 1.6 with $q\leftarrow\sqrt{q}$, the form w satisfies the $\sqrt{q}$-difference equation

$$H_{\sqrt{q}}(\tilde{\Phi}w)+\tilde{\Psi}w+\tilde{B}\left(x^{-1}w\left(h_{\sqrt{q}}w\right)\right)=0,\tag{2.15}$$

with

$$\begin{cases}\tilde{\Phi}(x)=x\Phi\left(x^2\right),\\ \tilde{B}(x)=q^{-1}\left(\sqrt{q}+1\right)xB\left(x^2\right),\\ \tilde{\Psi}(x)=q^{-1}\left(\sqrt{q}+1\right)\left(x^2\Psi\left(x^2\right)-\Phi\left(x^2\right)\right).\end{cases}$$

As u is of class s, we deduce from the third point of Remark 1.5 that

$$s=\max\left(\deg(\Psi)-1,\max\left(\deg(\Phi),\deg(B)\right)-2\right),$$

which gives

$$\deg(\Phi)\leq s+2,\quad \deg(\Psi)\leq s+1,\quad \deg(B)\leq s+2.$$

According to (2.5) and (2.14), we have

$$\deg\left(\tilde{\Phi}\right) \leq 2s+5, \quad \deg\left(\tilde{\Psi}\right) \leq 2s+4, \quad \deg\left(\tilde{B}\right) \leq 2s+5.$$

Finally, by the definition of the class given in (1.21), we have

$$\tilde{s} \leq \max\left(\deg\left(\tilde{\Psi}\right)-1, \max\left(\deg\left(\tilde{\Phi}\right), \deg\left(\tilde{B}\right)\right)-2\right) \leq 2s+3. \qquad \square$$

Lemma 2.3. *The class of w depends only on the zero $z=0$ of $\tilde{\Phi}$.*

Proof. u is a H_q-Laguerre-Hahn form of class s and its Stieltjes formal series $S_u(z)$ satisfies (2.4), therefore the polynomials $h_{q^{-1}}\Phi$, B, C and D are coprime. Let $\tilde{\Phi}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ be as in Proposition 2.2 and let c be a non-zero root of $\tilde{\Phi}$, which gives $\Phi\left(c^2\right)=0$. From (1.26) since u is of class s, we have $\left|B\left(c^2q\right)\right|+\left|C\left(c^2q\right)\right|+\left|D\left(c^2q\right)\right| \neq 0$.

- If $B\left(c^2q\right) \neq 0$, then $\tilde{B}\left(c\sqrt{q}\right) \neq 0$.
- If $B\left(c^2q\right)=0$ and $C\left(c^2q\right) \neq 0$, then $\tilde{C}\left(c\sqrt{q}\right) \neq 0$.
- If $B\left(c^2q\right)=C\left(c^2q\right)=0$, then $D\left(c^2q\right) \neq 0$ and this implies that $\tilde{D}\left(c\sqrt{q}\right) \neq 0$.

Consequently, for any non-zero root of $\tilde{\Phi}$, we have $|\tilde{B}\left(c\sqrt{q}\right)|+|\tilde{C}\left(c\sqrt{q}\right)|+|\tilde{D}\left(c\sqrt{q}\right)| \neq 0$. $\square$

Proposition 2.4. *Taking into account the conditions of Proposition 2.2, we have the following different cases for the class of w.*

(1) If $\Phi(0) \neq 0$, then $\tilde{s}=2s+3$. In addition the Stieltjes formal series S_w satisfies the $\sqrt{q}$-Riccati equation (2.4) with (2.5).

(2) If $\Phi(0)=0$ and $B(0) \neq 0$, then $\tilde{s}=2s+2$. In addition the Stieltjes formal series S_w satisfies the $\sqrt{q}$-Riccati equation (2.4) with

$$\begin{cases} \tilde{\Phi}(z)=\Phi\left(z^2\right), \\ \tilde{B}(z)=q^{-1}\left(\sqrt{q}+1\right)B\left(z^2\right), \\ \tilde{C}(z)=q^{\frac{-3}{2}}z\left\{\left(\sqrt{q}+1\right)C\left(z^2\right)+\left(\theta_0\Phi\right)\left(q^{-1}z^2\right)\right\}, \\ \tilde{D}(z)=q^{\frac{-3}{2}}\left(\sqrt{q}+1\right)z^2D\left(z^2\right). \end{cases} \tag{2.16}$$

(3) If $\Phi(0) = B(0) = 0$ *and* $\left(\sqrt{q}+1\right)C(0) + \Phi'(0) \neq 0$*, then* $\tilde{s} = 2s+1$*. In addition the Stieltjes formal series* S_w *satisfies the* $\sqrt{q}$*-Riccati equation* (2.4) *with*

$$\begin{cases} \tilde{\Phi}(z) = z\left(\theta_0\Phi\right)\left(z^2\right), \\ \tilde{B}(z) = q^{-1}\left(\sqrt{q}+1\right)z\left(\theta_0 B\right)\left(z^2\right), \\ \tilde{C}(z) = q^{\frac{-3}{2}}\left\{\left(\sqrt{q}+1\right)C\left(z^2\right) + \left(\theta_0\Phi\right)\left(q^{-1}z^2\right)\right\}, \\ \tilde{D}(z) = q^{\frac{-3}{2}}\left(\sqrt{q}+1\right)zD\left(z^2\right). \end{cases} \tag{2.17}$$

(4) If $\Phi(0) = B(0) = \left(\sqrt{q}+1\right)C(0) + \Phi'(0) = 0$*, then* $\tilde{s} = 2s$*. In addition the Stieltjes formal series* S_w *satisfies the* $\sqrt{q}$*-Riccati equation* (2.4) *with*

$$\begin{cases} \tilde{\Phi}(z) = \left(\theta_0\Phi\right)\left(z^2\right), \\ \tilde{B}(z) = q^{-1}\left(\sqrt{q}+1\right)\left(\theta_0 B\right)\left(z^2\right), \\ \tilde{C}(z) = q^{\frac{-3}{2}}z\left\{\left(\sqrt{q}+1\right)\left(\theta_0 C\right)\left(z^2\right) + q^{-1}\left(\theta_0^2\Phi\right)\left(q^{-1}z^2\right)\right\}, \\ \tilde{D}(z) = q^{\frac{-3}{2}}\left(\sqrt{q}+1\right)D\left(z^2\right). \end{cases} \tag{2.18}$$

Proof. (1) If $\Phi(0) \neq 0$, then from (2.5) we have $\tilde{C}(0) \neq 0$. Thus, equation (2.4) cannot be simplified by z. Let $t = \deg(\Phi)$, $r = \deg(B)$ and $p = \deg(\Psi)$. From (2.5) and (2.14) we obtain $\deg\left(\tilde{\Phi}\right) = 2t+1$, $\deg\left(\tilde{B}\right) = 2r+1$ and $\tilde{p} = \deg\left(\tilde{\Psi}\right) \leq \max(2p+2, 2t)$. Therefore, $\tilde{s} = \max(2t-1, 2r-1, 2p+1) = 2s+3$, since $\max(t-2, r-2, p-1) = s$.

(2) If $\Phi(0) = 0$, then the equation (2.4) with (2.5) are divisible by z and thus the order of the class of w decreases by one. Then, S_w fulfills (2.4) with (2.16).

Here, we have $\tilde{B}(0) \underset{\text{by (2.16)}}{=} q^{-1}\left(\sqrt{q}+1\right)B(0)$ and $\tilde{C}(0) = \tilde{D}(0) \underset{\text{by (2.16)}}{=} 0$. Consequently, if $B(0) \neq 0$, we cannot simplify (2.4)-(2.16), which implies that the class of w is $\tilde{s} = 2s+2$.

(3) If $\Phi(0) = B(0) = 0$, equation (2.4) with (2.16) can be simplified by z. Hence, S_w fulfills (2.4) with (2.17).

In this case $\tilde{B}(0) = \tilde{D}(0) \underset{\text{by (2.17)}}{=} 0$ and $\tilde{C}(0) \underset{\text{by (1.2)-(2.17)}}{=} q^{\frac{-3}{2}}\left\{\left(\sqrt{q}+1\right)C(0) + \Phi'(0)\right\}$.

Therefore, if $\left(\sqrt{q}+1\right)C(0) + \Phi'(0) \neq 0$, (2.4) with (2.17) cannot be simplified by z, which means that the class of w is $\tilde{s} = 2s+1$.

(4) If $\Phi(0) = B(0) = \left(\sqrt{q}+1\right)C(0) + \Phi'(0) = 0$, then (2.4) with (2.17) can be simplified by z. Thus, S_w satisfies (2.4) with (2.18).

If $\tilde{\Phi}(0) \underset{\text{by (1.2)-(2.18)}}{=} \Phi'(0) = 0$, then we have $C(0) = 0$ since $\left(\sqrt{q}+1\right)C(0) + \Phi'(0) = 0$. In view of u is H_q-Laguerre-Hahn of class s, we obtain from (1.26) that $D(0) \neq 0$. Therefore, $\tilde{D}(0) \underset{\text{by (2.18)}}{=} q^{\frac{-3}{2}}\left(\sqrt{q}+1\right)D(0) \neq 0$. Then, it is not possible to simplify (2.4)-(2.18), which means the class of w is $\tilde{s} = 2s$. □

3 Two illustrative examples

3.1 Example 1: $u = \mathcal{W}^{(1)}$

Let $u = \mathcal{W}^{(1)}$ be the first associated of Stieltjes-Wigert form (see Table 1). Let $\left\{S_n^{(1)}\right\}_{n\geq 0}$ be its (MOPS). It is H_q-Laguerre-Hahn of class $s = 0$ fulfilling [13]

$$\begin{cases} \beta_n^{(1)} = q^{-n-\frac{5}{2}}\left\{(1+q)q^{-n-1} - q\right\}, \quad \gamma_{n+1}^{(1)} = q^{-4n-8}\left(1 - q^{n+2}\right), \quad n \geq 0, \\ H_q\left(x\left(x + q^{-\frac{5}{2}}(q-1)\right)\mathcal{W}^{(1)}\right) - (q-1)^{-1}\left\{x - q^{-\frac{7}{2}} - q^{-\frac{5}{2}} + q^{-\frac{3}{2}}\right\}u \\ \qquad\qquad -q^{-5}\left(x^{-1}\mathcal{W}^{(1)}\left(h_q\mathcal{W}^{(1)}\right)\right) = 0, \\ q^{-1}z\left(q^{-1}z + q^{-\frac{5}{2}}(q-1)\right)H_{q^{-1}}\left(S_{\mathcal{W}^{(1)}}\right)(z) = -q^{-5}S_{\mathcal{W}^{(1)}}(z) \\ \left(h_{q^{-1}}S_{\mathcal{W}^{(1)}}(z) + q^{-1}(q-1)^{-1}\left(z + q^{-\frac{1}{2}} - 2q^{-\frac{3}{2}}\right)\right)S_{\mathcal{W}^{(1)}}(z) + (q-1)^{-1}. \end{cases} \tag{3.1}$$

Lemma 3.1 ([13]). *Let $0 < q < 1$. The following equalities hold*

$$S_n(0) = (-1)^n\, q^{-\frac{n(2n+1)}{2}}, \quad n \geq 0, \tag{3.2}$$

$$S_n^{(1)}(0) = (-1)^n\, q^{-\frac{(n+2)(2n+1)}{2}}\, \left(1 - (q;q)_{n+1}\right) \neq 0, \quad n \geq 0. \tag{3.3}$$

By virtue of Lemma 3.1 and Proposition 2.2, for $0 < q < 1$, w is a $H_{\sqrt{q}}$-Laguerre-Hahn form. By the second point of Proposition 2.4 and (3.1), we have $\Phi(0) = 0$ and $B(0) = -q^{-5} \neq 0$. Therefore, w is of class $\tilde{s} = 2$. Its Stieltjes formal series S_w satisfies (2.4) with

$$\begin{cases} \tilde{\Phi}(z) = z^2\left(z^2 + q^{-\frac{5}{2}}(q-1)\right), \\ \tilde{B}(z) = -q^{-6}\left(\sqrt{q} + 1\right), \\ \tilde{C}(z) = q^{-2}\left(\sqrt{q} - 1\right)^{-1} z\left\{z^2 + q^{-2}\left(-1 + \sqrt{q}(q-1)\right)\right\}, \\ \tilde{D}(z) = q^{\frac{-3}{2}}\left(\sqrt{q} - 1\right)^{-1} z^2. \end{cases} \tag{3.4}$$

From Theorem 1.7 and (3.3), we obtain

$$\begin{cases} \tilde{\gamma}_{2n+1} = q^{-2n-\frac{7}{2}}\,\dfrac{1 - (q;q)_{n+2}}{1 - (q;q)_{n+1}}, \quad n \geq 0, \\ \tilde{\gamma}_{2n+2} = q^{-2n-\frac{9}{2}}\,\left(1 - q^{n+2}\right)\dfrac{1 - (q;q)_{n+1}}{1 - (q;q)_{n+2}}, \quad n \geq 0. \end{cases} \tag{3.5}$$

Furthermore, for $0 < q < 1$, the form w is positive definite.

3.2 Example 2: $u = \mathcal{U}^{(1)}$

Let $\mathcal{U}^{(1)}$ be the first associated of the Al-Salam-Carlitz form of the first kind $\mathcal{U}(a,q)$ for $a>0,\ q>1$ (see Table 2). Let $\left\{U_n^{(1)}\right\}_{n\geq 0}$ be its (MOPS). $\{\mathcal{U}^{(1)}\}$ is H_q-Laguerre-Hahn of class $s=0$ fulfilling [13]

$$\begin{cases} \beta_n^{(1)} = (1+a)q^{n+1}, \quad \gamma_{n+1}^{(1)} = -aq^{n+1}\left(1-q^{n+2}\right), \quad n\geq 0, \\ H_q\left(\mathcal{U}^{(1)}\right) + a^{-1}q^{-1}(q-1)^{-1}\{x-q(1+a)\}\mathcal{U}^{(1)} - q\left(x^{-1}\mathcal{U}^{(1)}\left(h_q\mathcal{U}^{(1)}\right)\right) = 0, \\ H_{q^{-1}}(S_{\mathcal{U}^{(1)}})(z) = -qS_{\mathcal{U}^{(1)}}(z)(h_{q^{-1}}S_{\mathcal{U}^{(1)}})(z) - a^{-1}(q-1)^{-1}\left(z-q(1+a)\right)S_{\mathcal{U}^{(1)}}(z) \\ \qquad\qquad -a^{-1}(q-1)^{-1}. \end{cases} \tag{3.6}$$

Lemma 3.2. *Let $a>0$ and $q>1$. The following results hold*

$$U_n(0) = (-1)^n q^{\frac{n(n-1)}{2}} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j q^{-j(n-j)} \neq 0, \quad n\geq 0, \tag{3.7}$$

$$\begin{gathered} U_n^{(1)}(0) = (-1)^n q^{\frac{n(n+1)}{2}} \varpi_{n+1}(a,q) \sum_{k=0}^{n} \frac{(-a)^k q^{\frac{-k(k+1)}{2}}(q;q)_k}{\varpi_k(a,q)\varpi_{k+1}(a,q)} \neq 0, \quad n\geq 0, \\ \varpi_n(a,q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j q^{-j(n-j)}. \end{gathered} \tag{3.8}$$

Proof. The Al-Salam-Carlitz polynomials of the first kind are given by [11]

$$U_n(x) = \sum_{k=0}^{n} \frac{(q;q)_n(-a)^{n-k}}{(q;q)_k(q;q)_{n-k}} q^{(n-k)(n-k-1)/2} x^k (1/x;q)_k, \tag{3.9}$$

with

$$x^k(1/x;q)_k = \prod_{j=0}^{k-1}\left(x-q^j\right) = (-1)^k q^{k(k-1)/2}\left(q^{1-k}x;q\right)_k. \tag{3.10}$$

By substituting (3.10) into (3.9) and letting $x\to 0$, we obtain

$$\begin{aligned} U_n(0) &= \sum_{k=0}^{n} \frac{(q;q)_n(-a)^{n-k}}{(q;q)_k(q;q)_{n-k}} q^{(n-k)(n-k-1)/2}(-1)^k q^{k(k-1)/2} \\ &= (-1)^n (q;q)_n \sum_{k=0}^{n} \frac{a^{n-k}}{(q;q)_k(q;q)_{n-k}} q^{\frac{(n-k)(n-k-1)+k(k-1)}{2}}. \end{aligned} \tag{3.11}$$

Let us perform the index change $j=n-k$

$$\begin{aligned} U_n(0) &= (-1)^n(q;q)_n \sum_{j=0}^{n} \frac{a^j}{(q;q)_{n-j}(q;q)_j} q^{\frac{j(j-1)+(n-j)(n-j-1)}{2}} \\ &\underset{\text{by } (1.14)}{=} (-1)^n q^{\frac{n(n-1)}{2}} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j q^{-j(n-j)}. \end{aligned} \tag{3.12}$$

Since $\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j q^{-j(n-j)} > 0$, for $a > 0$ and $q > 1$, it follows that $\forall n \in \mathbb{N},\ U_n(0) \neq 0$. Now, since $U_n(0) \neq 0,\ n \geq 0,\ a > 0,\ q > 1$, and according to (1.18), we can write

$$U_{n+1}^{(1)}(0) = \frac{U_{n+2}(0)}{U_{n+1}(0)} U_n^{(1)}(0) + \frac{\prod_{\nu=0}^{n} \gamma_{\nu+1}}{U_{n+1}(0)}, \quad n \geq 0. \tag{3.13}$$

Next, by applying Lemma 1.3, we get for (3.13)

$$U_{n+1}^{(1)}(0) = U_{n+2}(0) \left(\frac{U_0^{(1)}(0)}{U_1(0)} + \sum_{k=0}^{n} \frac{(-a)^{k+1} q^{\frac{k(k+1)}{2}} (q;q)_{k+1}}{U_{k+1}(0) U_{k+2}(0)} \right), \quad n \geq 0. \tag{3.14}$$

Since $U_0^{(1)}(0) = 1$ and $U_1(0) = -(1+a)$, we have

$$U_{n+1}^{(1)}(0) = \frac{-U_{n+2}(0)}{1+a} \left(1 - (a+1) \sum_{k=0}^{n} \frac{(-a)^{k+1} q^{\frac{1}{2}k(k+1)} (q;q)_{k+1}}{U_{k+1}(0) U_{k+2}(0)} \right), \quad n \geq 0. \tag{3.15}$$

Replacing n by $n-1$ in (3.15) gives

$$U_{n}^{(1)}(0) = \frac{-U_{n+1}(0)}{1+a} \left(1 - (a+1) \sum_{k=0}^{n-1} \frac{(-a)^{k+1} q^{\frac{1}{2}k(k+1)} (q;q)_{k+1}}{U_{k+1}(0) U_{k+2}(0)} \right), \quad n \geq 1, \tag{3.16}$$

equivalent to

$$U_{n}^{(1)}(0) = \frac{-U_{n+1}(0)}{1+a} \left(1 - (a+1) \sum_{k=1}^{n} \frac{(-a)^{k} q^{\frac{1}{2}k(k-1)} (q;q)_{k}}{U_{k}(0) U_{k+1}(0)} \right), \quad n \geq 1. \tag{3.17}$$

The equality (3.17) remains true for $n = 0$, hence we have

$$U_{n}^{(1)}(0) = U_{n+1}(0) \sum_{k=0}^{n} \frac{(-a)^{k} q^{\frac{1}{2}k(k-1)} (q;q)_{k}}{U_{k}(0) U_{k+1}(0)}, \quad n \geq 0. \tag{3.18}$$

By setting $\varpi_n(a,q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j q^{-j(n-j)},\ a > 0,\ q > 1,\ n \geq 0$, we obtain

$$U_{n}^{(1)}(0) = (-1)^n q^{\frac{n(n+1)}{2}} \varpi_{n+1}(a,q) \sum_{k=0}^{n} \frac{(-a)^{k} q^{\frac{-k(k+1)}{2}} (q;q)_{k}}{\varpi_k(a,q) \varpi_{k+1}(a,q)}, \quad n \geq 0. \tag{3.19}$$

When $q > 1$ and $n \geq 0$, $(q,q)_n$ has the same sign as $(-1)^n$.

Now, since $a > 0$ and $\forall n \geq 0$, $\varpi_n(a,q) > 0$, we obtain

$$\varpi_{n+1}(a,q) \sum_{k=0}^{n} \frac{(-a)^k q^{\frac{-k(k+1)}{2}} (q;q)_k}{\varpi_k(a,q)\varpi_{k+1}(a,q)} > 0, \quad n \geq 0. \tag{3.20}$$

Therefore, $U_n^{(1)}(0) \neq 0, \quad n \geq 0.$ □

By virtue of Lemma 3.2 and Proposition 2.2, for $a > 0$, $q > 1$, w is a $H_{\sqrt{q}}$-Laguerre–Hahn form. By the first point of Proposition 2.4 and (3.6), we have $\Phi(0) \neq 0$. Therefore, w is of class $\tilde{s} = 3$. Its Stieltjes formal series S_w satisfies (2.4) with

$$\begin{cases} \tilde{\Phi}(z) = z, \\ \tilde{B}(z) = -\left(\sqrt{q}+1\right) z, \\ \tilde{C}(z) = q^{\frac{-3}{2}} a^{-1} \left(1-\sqrt{q}\right)^{-1} z^2 \left(z^2 - q(1+a)\right) + \sqrt{q}^{-1}, \\ \tilde{D}(z) = q^{\frac{-3}{2}} a^{-1} \left(1-\sqrt{q}\right)^{-1} z^3. \end{cases} \tag{3.21}$$

From Theorem 1.7 and (3.8), we obtain for $a > 0,\ q > 1$

$$\begin{cases} \tilde{\gamma}_{2n+1} = q^{n+1} \dfrac{\varpi_{n+2}(a,q)}{\varpi_{n+1}(a,q)} \dfrac{\Lambda_{n+1}(a,q)}{\Lambda_n(a,q)}, \quad n \geq 0, \\ \tilde{\gamma}_{2n+2} = -a(1-q^{n+2}) \dfrac{\varpi_{n+1}(a,q)}{\varpi_{n+2}(a,q)} \dfrac{\Lambda_n(a,q)}{\Lambda_{n+1}(a,q)}, \quad n \geq 0, \\ \Lambda_n(a,q) = \displaystyle\sum_{k=0}^{n} \frac{(-a)^k q^{\frac{-k(k+1)}{2}} (q;q)_k}{\varpi_k(a,q)\varpi_{k+1}(a,q)}, \quad n \geq 0, \\ \varpi_n(a,q) = \displaystyle\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j q^{-j(n-j)}, \quad n \geq 0. \end{cases} \tag{3.22}$$

For $a > 0$, $q > 1$, $\varpi_n > 0$, $n \geq 0$, according to (3.20). Moreover $(-a)^n (q,q)_n = a^n |(q,q)_n| > 0$, $n \geq 0$, therefore $\Lambda_n > 0$, $n \geq 0$. Hence, the form w is positive definite.

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Appendix

Table 1: Stieltjes-Wigert polynomials [24]

The Stieltjes-Wigert form $\mathcal{W}$ with respect to the MOPS $\{S_n\}_{n\geq 0}$
$S_n(x) = (-1)^n q^{-\left(n^2+\frac{n}{2}\right)} {}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix} \middle\vert q; -q^{n+\frac{3}{2}}x\right), \quad n \geq 0.$
$\beta_n = \{(1+q)q^{-n} - q\}\, q^{-n-\frac{3}{2}}, \quad n \geq 0,$
$\gamma_{n+1} = \left(1 - q^{n+1}\right) q^{-4n-4}, \quad n \geq 0.$
$H_q\left(x^2\mathcal{W}\right) - (q-1)^{-1}\left(x - q^{-\frac{3}{2}}\right)\mathcal{W} = 0,$
$(\mathcal{W})_n = q^{-\frac{n(n+2)}{2}}, \quad n \geq 0.$
$\langle \mathcal{W}, f\rangle = \sqrt{\frac{q}{2\pi \ln q^{-1}}} \int_0^{+\infty} \exp\left(-\frac{\ln^2 x}{2\ln q^{-1}}\right) f(x)\, dx,$
$f \in \mathcal{P},\ 0 < q < 1.$

Table 2: Al-Salam-Carlitz polynomials [18]

Al-Salam-Carlitz form $\mathcal{U}(a,q)$ with respect to the MOPS $\{U_n\}_{n\geq 0}$
$U_n(x) = (-a)^n q^{\frac{n(n-1)}{2}}\ {}_2\phi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \middle\vert q; \frac{qx}{a}\right), \quad n \geq 0.$
$\beta_n = (1+a)q^n, \quad n \geq 0,$
$\gamma_{n+1} = -aq^n\left(1 - q^{n+1}\right), \quad n \geq 0.$
$H_q\left(\mathcal{U}(a,q)\right) - a^{-1}(q-1)^{-1}\left(x - (a-1)\right)\mathcal{U}(a,q) = 0,$
$(\mathcal{U}(a,q))_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k, \quad n \geq 0.$
$\langle \mathcal{U}(a,q), f\rangle = \frac{1}{(1-q)\left(q, a^{-1}q, a; q\right)_\infty} \int_a^1 \left(a^{-1}tq, tq; q\right)_\infty f(t)\, d_q t,$
$f \in \mathcal{P},\ a < 0,\ 0 < q < 1.$

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