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Closed graph property in Alexandroff spaces

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ABSTRACT

In the following text we show if X is an Alexandroff space, then $f : X \to Y$ has closed graph if and only if it has constant closed value on each connected component of X. Moreover, if X is an Alexandroff space and $f : X \to Y$ has closed graph, then $f : X \to Y$ is continuous. As a matter of fact, the number of maps which have closed graph from an Alexandroff space X to a topological space Y depends just on the number of connected components of X and the number of closed points of Y.

RESUMEN

En el siguiente texto, mostramos que si X es un espacio de Alexandroff, entonces $f : X \to Y$ tiene grafo cerrado si y sólo si tiene valor cerrado constante en cada componente conexa de X. Más aún, si X es un espacio de Alexandroff y $f : X \to Y$ tiene grafo cerrado, entonces $f : X \to Y$ es continua. De hecho, el número de funciones que tienen grafo cerrado desde un espacio de Alexandroff X a un espacio topológico Y depende solamente del número de componentes conexas de X y el número de puntos cerrados de Y.

Keywords and Phrases: Alexandroff space, closed graph, connected component.

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1 Introduction

For an arbitrary map $f : X \to Y$, $G_f := \{(x, f(x)) : x \in X\}$ denotes the graph of $f : X \to Y$ [4]. For topological spaces X, Y we say that $f : X \to Y$ has a closed graph if G_f is a closed subset of $X \times Y$. Studying maps with closed graph is the main idea of many texts in different categories, in topological groups [3, 5], topological linear spaces [6], more general spaces like Hausdorff spaces [2]. In this text we study self–maps on Alexandroff spaces with closed graphs. Alexandroff spaces are topological spaces with an extra property, namely, the intersection of every non-empty family of open sets is open. These spaces called Alexandroff spaces in honour of P. Alexandroff's famous paper on 1937 [1].

More details on Alexandroff spaces

As a matter of fact, a topological space X is an Alexandroff space if and only if each $x \in X$ has the smallest open neighbourhood. In Alexandroff space X let us denote the smallest open neighbourhood of $a \in X$ by V_a.

In an Alexandroff space X, $f : X \to \mathcal{P}(X)$ with $f(x) = V_x$ for $x \in X$ (where $\mathcal{P}(X)$ denotes the collection of subsets of X) satisfies the following properties:

(i) $\forall x \in X \, (x \in f(x))$

(ii) $\forall x \in X \, \forall y \in f(x) \, (f(y) \subseteq f(x))$.

Conversely, if there exists a function $f : X \to \mathcal{P}(X)$ which satisfies the above two conditions, then $\{f(x) : x \in X\}$ is a base for an Alexandroff topology on X.

For $f : X \to Y$ and $A \subseteq X$, $f \restriction_A : A \to Y$ denotes the restriction of f to A.

2 Connected components of Alexandroff spaces

In an Alexandroff space X consider the equivalence relation $\Re_X := \{(x, y) \in X \times X :$ there exist $x = x_1, x_2, \ldots, y = x_n \in X$ such that $V_{x_i} \cap V_{x_{i+1}} \neq \emptyset$ for all $i \in \{1, 2, \ldots, n-1\}\ \}$ (see [7, Lemma 2.1] too). Also, let us recall that for each $x \in X$, $[x]_{\Re_X} := \{y \in X : (x, y) \in \Re_X\}$ is the equivalence class of x. In this section we prove that connected components of Alexandroff space X are just equivalence classes of $\Re_X$.

Lemma 2.1. *In an Alexandroff space X for each $x \in X$, $[x]_{\Re_X}$ is open and a subset of connected component of X containing x.*

Proof. Choose an arbitrary $x \in X$. Using the definition of $\Re_X$, it is evident that $V_x \subseteq [x]_{\Re_X}$, which leads us to $[x]_{\Re_X} = \bigcup\{V_z : z \in [x]_{\Re_X}\}$, in particular $[x]_{\Re_X}$ is open.

Note that V_x is connected, otherwise there exist disjoint non-empty open subsets U, V of V_x (hence disjoint non-empty open subsets of X) such that $x \in V_x = U \cup V$, we may suppose $x \in U$ which is a contradiction (since V_x is the smallest open neighbourhood of x and U is a proper open subset of V_x).

Suppose C_x is the connected component of X which contains x. Now using the definition of $\Re_X$ if $y \in [x]_{\Re_X}$, then there exist $x = x_1, \ldots, x_n = y \in X$ such that $V_{x_i} \cap V_{x_{i+1}} \neq \varnothing$ for each $i \in \{1, \ldots, n-1\}$. Using the fact that union of two connected sets with non-empty intersection is connected too, we have the following chain of connected subsets:

$$V_{x_1} \subseteq V_{x_1} \cup V_{x_2} \subseteq \cdots \subseteq V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_n} .$$

Hence $C = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_n}$ is a connected subset of X containing $x = x_1, y = x_n$, therefore $y \in C \subseteq C_x$ which leads to $[x]_{\Re_X} \subseteq C_x$. □

The following theorem shows that $[x]_{\Re_X}$ is just the connected component of X containing x, for each x in an Alexandroff space X. Also equivalence classes of X with respect to $\Re_X$ are exactly its connected components.

Theorem 2.2. *In an Alexandroff space X for each $x \in X$, the connected component of X which contains x is $[x]_{\Re_X}$.*

Proof. Suppose that C_x is the connected component of X which contains x. If $[x]_{\Re_X} = X$, then $[x]_{\Re_X} = X = C_x$. Otherwise, by Lemma 2.1, $[x]_{\Re_X}$ and $\bigcup\{[y]_{\Re_X} : y \in X \setminus [x]_{\Re_X}\} = X \setminus [x]_{\Re_X}$ are two disjoint open sets, hence $[x]_{\Re_X}$, $X \setminus [x]_{\Re_X}$ is a separation of X, thus either $C_x \subseteq [x]_{\Re_X}$ or $C_x \subseteq X \setminus [x]_{\Re_X}$, considering $x \in C_x \cap [x]_{\Re_X}$ leads to $C_x \cap [x]_{\Re_X} \neq \varnothing$ and $C_x \subseteq [x]_{\Re_X}$. By Lemma 2.1, $[x]_{\Re_X} \subseteq C_x$, which completes the proof. □

3 Maps between Alexandroff spaces and closed graph property

In this section, we show if X is an Alexandroff space and $f : X \to Y$ has closed graph, then $f : X \to Y$ is continuous. Moreover, in an Alexandroff space X the number of maps from X to Y which have closed graph just depends on the number of connected components of X and the number of closed points of Y. As a matter of fact, for Alexandroff space X a map $f : X \to Y$ has closed graph if and only if f maps each connected component of X to a closed point of Y.

Remark 3.1. *In topological spaces X, Y suppose that $x \in X$ and $f : X \to Y$ have a closed graph, then for each $x \in X$ we have:*

(i) $\{x\} \times \overline{\{f(x)\}} \subseteq \overline{\{(x, f(x))\}} \subseteq \overline{G_f} = G_f$*, thus* $\overline{\{f(x)\}} = \{f(x)\}$ *and* $f(x)$ *is a closed point of* Y*,*

(ii) $\overline{\{x\}} \times \{f(x)\} \subseteq \overline{\{(x, f(x))\}} \subseteq \overline{G_f} = G_f$*, thus* $f \restriction_{\overline{\{x\}}} : \overline{\{x\}} \to Y$ *is the constant map with value* $f(x)$*,*

(iii) if $z \in \bigcap\{V : V$ *is an open neighbourhood of* $x\} =: W$*, then* $x \in \overline{\{z\}}$*, thus* $f(x) = f(z)$ *by item (ii) hence* $f \restriction_W : W \to Y$ *is the constant map with value* $f(x)$ *too,*

(iv) in particular, if X *is an Alexandroff space, then in item (iii),* $W = V_x$ *and* $f \restriction_{\overline{\{x\}} \cup V_x} : \overline{\{x\}} \cup V_x \to Y$ *is the constant map with value* $f(x)$*.*

Theorem 3.2. *Let X be an Alexandroff space and let $f : X \to Y$ have a closed graph, then $f : X \to Y$ is continuous.*

Proof. By item (iv) in Remark 3.1 for each $x \in X$, $V_x \subseteq f^{-1}(f(x))$ thus for each $D \subseteq Y$, $f^{-1}(D) = \bigcup\{V_x : f(x) \in D\}$ is an open subset of X. □

Lemma 3.3. *Let X be an Alexandroff space, $x \in X$ and $f : X \to Y$ have a closed graph, then $f \restriction_{[x]_{\Re}} : [x]_{\Re} \to Y$ is a constant map.*

Proof. By Remark 3.1, for each $z \in X$, $f \restriction_{V_z} : V_z \to Y$ is constant. If $y \in [x]_{\Re_X}$, then there exist $x = x_1, \ldots, x_n = y$ in X such that $V_{x_i} \cap V_{x_{i+1}} \neq \emptyset$ for each $i = 1, \ldots, n-1$. So, $f(x) = f(x_1) = f(x_2) = \cdots = f(x_n) = f(y)$. Therefore, $f \restriction_{[x]_{\Re}} : [x]_{\Re} \to Y$ is a constant map. □

Lemma 3.4. *Let X be an Alexandroff space. Then $f : X \to Y$ has closed graph if and only if for every connected component C of X, $f(C)$ is a singleton closed subset of Y.*

Proof. Suppose $f : X \to Y$ has closed graph and C is a connected component of X. By Theorem 2.2, for each $x \in C$, $C = [x]_{\Re_X}$. So Lemma 3.3 shows that f is constant on C, by Remark 3.1, $f(C) = \{f(x)\}$ is closed too.

Now, suppose for every connected component C of X, $f(C)$ is a singleton closed subset of Y. If $(z, w) \in \overline{G_f}$, then there exists a net $\{x_\alpha\}_{\alpha\in\Gamma}$ in X such that $\{(x_\alpha, f(x_\alpha))\}_{\alpha\in\Gamma}$ converges to (z, w). Hence $\{x_\alpha\}_{\alpha\in\Gamma}$ converges to z and there exists $\beta \in \Gamma$ such that $x_\alpha \in V_z$ for each $\alpha \geq \beta$. Since V_z is a subset of connected component containing z, f is constant on V_z, so $f(x_\alpha) = f(z)$ for each $\alpha \geq \beta$. Since $\{f(x_\alpha)\}_{\alpha\in\Gamma}$ converges to w, $\{f(z)\}_{\alpha\in\Gamma,\alpha\geq\beta} = \{f(x_\alpha)\}_{\alpha\in\Gamma,\alpha\geq\beta}$ converges to w too, hence $w \in \overline{\{f(z)\}} = \{f(z)\}$. Therefore, $(z, w) = (z, f(z)) \in G_f$ and G_f is a closed subset of $X \times Y$. □

Theorem 3.5 (main)**.** *Consider an Alexandroff space X and arbitrary map $f : X \to Y$. The following statements are equivalent:*

- *$f : X \to Y$ has closed graph,*
- *for every connected component C of X, $f(C)$ is a singleton closed subset of Y,*
- *for every $x \in X$, $f\left([x]_{\Re_X}\right)$ is a singleton closed subset of Y.*

Moreover, in the above case $f : X \to Y$ is continuous.

Proof. Use Theorems 2.2, 3.2 and Lemma 3.4. □

Corollary 3.6. *Suppose X is an Alexandroff space and Y is an arbitrary topological space, let:*

$$\alpha := card\{C : C \text{ is a connected component of } X\} \left(= card\left(\frac{X}{\Re_X}\right)\right),$$

$$\beta := card\{y : y \text{ is a closed point of } Y\}.$$

Then by Theorem 3.5 we have (by $\mathcal{C}(X, Y)$ we mean the collection of continuous maps from X to Y, also Y^X denotes the collection of all maps from X to Y):

$$card\left\{f \in Y^X : f \text{ has closed graph}\right\} = \beta^\alpha \leq card(\mathcal{C}(X, Y)).$$

Compare the following (counter)example by Theorem 3.2. The following example carries a family of Alexandroff spaces have functions with closed graph these spaces have also constant functions without closed graph.

Example 3.7. *Suppose X is a connected Alexandroff space, then by Lemma 3.4, $f : X \to X$ has closed graph if and only if it is a constant map with closed point value. Equip $\mathbb{Z} = \{0, \pm1, \pm2, \dots\}$ with the topology generated by the base consisting of sets of the form $\{2m+1\}$ and $\{2m-1, 2m, 2m+1\}$, in which m varies arbitrarily in $\mathbb{Z}$. We call this space Khalimsky line and denote it by $\mathcal{K}$. For every $n \in \mathbb{N}$, $\mathcal{K}^n$ with product topology is called n dimensional Khalimsky space. $\mathcal{K}^n$ is a connected Alexandroff space and its closed points are $M = \{(2\lambda_1, \dots, 2\lambda_n) : \lambda_1, \dots, \lambda_n \in \mathbb{Z}\}$, so $f : \mathcal{K}^n \to \mathcal{K}^n$ has closed graph if and only if there exist integers $\lambda_1, \dots, \lambda_n$ such that f is the map with constant* *value* $(2\lambda_1, \ldots, 2\lambda_n)$. *So there are infinite countable self-map on* $\mathcal{K}^n$ *which have closed graph. However* $g : \mathcal{K}^n \to \mathcal{K}^n$ *with constant value* $(1, \ldots, 1)$ *is continuous and it does not have closed graph [7].*

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