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On characteristic foliations of metric contact-symplectic structures

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ABSTRACT

We study compatible and associated metrics for a contact-symplectic pair (η, ω) on a manifold. We show that the integral curves of the Reeb vector field are geodesics for any compatible metric. We prove that all associated metrics share a common volume element, which we give explicitly. When the characteristic foliations of η and ω are orthogonal with respect to an associated metric, their leaves, as well as those of the characteristic foliation of $d\eta$, are minimal. We construct explicit examples on nilpotent Lie groups and nilmanifolds where the characteristic foliations are not both totally geodesic.

RESUMEN

Estudiamos métricas compatibles y asociadas para un par contacto-simpléctico (η, ω) en una variedad diferenciable. Mostramos que las curvas integrales del campo de vectores de Reeb son geodésicas para cualquier métrica compatible. Demostramos que todas las métricas asociadas comparten un elemento de volumen común, el que entregamos explícitamente. Cuando las foliaciones características de η y ω son ortogonales con respecto a una métrica asociada, sus hojas, y también las de la foliación característica de $d\eta$, son minimales. Construimos ejemplos explícitos de grupos de Lie nilpotentes y nilvariedades donde las foliaciones características no son ambas totalmente geodésicas.

Keywords and Phrases: Metric contact-symplectic structure, minimal submanifold, almost contact metric manifold, foliation.

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1 Introduction

Bande [3] introduced the notion of *contact-symplectic pairs*, which he later studied together with Ghiggini and Kotschick [1,2]. Subsequently, he and the present author adapted associated metrics to this structure in order to construct contact pair structures on Boothby–Wang fibrations [4].

A contact-symplectic pair on a manifold M consists of a 1-form η and a 2-form ω such that the triple (M, η, ω) is locally the product of a contact manifold and a symplectic manifold. These manifolds are naturally foliated by the characteristic foliation $\mathcal{S}$ of η and the characteristic foliation $\mathcal{C}$ of ω, which are transverse and complementary. The form η induces contact forms on the leaves of $\mathcal{C}$, while ω induces symplectic forms on the leaves of $\mathcal{S}$. The Reeb vector fields on the contact leaves together define a global vector field ξ, called the Reeb vector field of the contact-symplectic pair.

Compatible and associated metrics are defined analogously to those in contact geometry, beginning with a $(1,1)$-tensor field ϕ that vanishes on ξ and restricts to an almost complex structure on the kernel of η. Thus, the triple (η, ξ, ϕ) determines an almost contact structure, with ω omitted. For an associated metric g, the triple (η, ω, g) is called a *metric contact-symplectic structure* on M.

In this paper, we prove that the integral curves of the Reeb vector field of a contact-symplectic pair are geodesics with respect to any compatible metric. We also show that all associated metrics share the same volume element, and we provide an explicit expression for it in terms of the pair (η, ω). For associated metrics for which the foliations $\mathcal{C}$ and $\mathcal{S}$ are orthogonal, we prove that all their leaves are minimal. In addition, we show that the leaves of the characteristic foliation of $d\eta$ are also minimal.

We provide examples of nilpotent Lie groups and nilmanifolds endowed with metric contact-symplectic structures whose characteristic foliations are orthogonal but not both totally geodesic.

All differential objects considered are assumed to be smooth. For exterior calculus, we adopt the convention of Kobayashi and Nomizu [6]: if $\omega_1, \dots, \omega_r$ are 1-forms and $X_1, \dots, X_r$ are vector fields, then $(\omega_1 \wedge \cdots \wedge \omega_r)(X_1, \dots, X_r) = \dfrac{1}{r!} \det \Big((\omega_i (X_j))_{1 \leq i,j \leq r} \Big)$.

2 Preliminaries

A pair (η, ω) consisting of a 1-form η and a closed 2-form ω on a manifold M is called a *contact-symplectic pair of type* (m, n) if the following conditions are satisfied:

$$\eta \wedge (d\eta)^m \wedge \omega^n \quad \text{is a volume form,} \quad (d\eta)^{m+1} = 0 \quad \text{and} \quad \omega^{n+1} = 0.$$

Such a manifold is necessarily orientable and of odd dimension $2n+2m+1$. Since the cases $n = 0$ or $m = 0$ correspond to classical contact and cosymplectic manifolds, respectively, we will henceforth assume $m \geq 1$ and $n \geq 1$.

The forms η and ω have constant Élie Cartan classes, equal to $2m+1$ and $2n$, respectively. As a consequence, the distributions

$$\begin{aligned} T\mathcal{S} &= \{X \in TM \mid \eta(X) = 0, d\eta(X,Y) = 0, \ \forall Y \in TM\} \\ T\mathcal{C} &= \{X \in TM \mid \omega(X,Y) = 0, \ \forall Y \in TM\} \end{aligned}$$

are integrable and determine two transverse and complementary foliations on M, denoted by $\mathcal{S}$ and $\mathcal{C}$, respectively. The leaves of $\mathcal{S}$ are symplectic manifolds of dimension $2n$, with symplectic form induced by ω, while the leaves of $\mathcal{C}$ are contact manifolds of dimension $2m+1$, with contact form induced by η (see [3]).

The vector field ξ defined uniquely by the equations

$$\eta(\xi) = 1, \quad i_\xi d\eta = 0, \quad i_\xi \omega = 0$$

is called the *Reeb vector field* of the contact-symplectic pair. It is tangent to the foliation $\mathcal{C}$ and restricts to the classical Reeb vector field on each contact leaf (see [3]). Its flow preserves both forms η and ω:

$$\mathcal{L}_\xi \eta = 0, \quad \mathcal{L}_\xi \omega = 0,$$

where $\mathcal{L}_X$ denotes the Lie derivative along a vector field X.

The characteristic subbundle of $d\eta$ is $\mathbb{R}\xi \oplus T\mathcal{S}$, which is integrable. The corresponding foliation will be denoted by $\mathcal{K}$. Each of its leaves is a $(2n+1)$-dimensional cosymplectic manifold foliated by leaves of $\mathcal{S}$ (see [3]).

We also define the *horizontal subbundle* $\mathcal{H}$, of dimension $2n+2m$, to be the kernel of η, and the subbundle $\mathcal{D} = \mathcal{H} \cap T\mathcal{C}$, of dimension $2m$, which corresponds to the contact distribution along the leaves of $\mathcal{C}$. The tangent bundle of M thus splits as:

$$TM = T\mathcal{S} \oplus T\mathcal{C} = T\mathcal{S} \oplus \mathbb{R}\xi \oplus \mathcal{D} = T\mathcal{K} \oplus \mathcal{D} = \mathcal{H} \oplus \mathbb{R}\xi.$$

A basic example is given by the product of a $(2m+1)$-dimensional contact manifold with a $2n$-dimensional symplectic manifold. In fact, any contact-symplectic pair of type (m,n) is locally isomorphic to such a product.

An *almost contact-symplectic structure* on a manifold M is a triple (η, ω, ϕ) where (η, ω) is a contact-symplectic pair with Reeb vector field ξ, and ϕ is a $(1,1)$-tensor field satisfying

$$\phi^2 = -I + \eta \otimes \xi.$$

This condition ensures that the triple (ϕ, ξ, η) defines an almost contact structure on M, and in particular:

$$\phi\xi = 0, \quad \eta \circ \phi = 0$$

and ϕ induces an almost complex structure on the horizontal subbundle $\mathcal{H}$.

On manifolds endowed with almost contact-symplectic structures, one can define Riemannian metrics adapted to the structure, as follows:

Definition 2.1. *Let (η, ω, ϕ) be an almost contact-symplectic structure on a manifold M with Reeb vector field ξ. A Riemannian metric g on M is said to be:*

(i) compatible *if, for all vector fields X, Y,*

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

(ii) associated *if, for all vector fields X, Y,*

$$g(X, \phi Y) = (d\eta + \omega)(X, Y),$$
$$g(X, \xi) = \eta(X).$$

A metric g is compatible with the almost contact-symplectic structure (η, ω, ϕ) if and only if the quadruple (ϕ, ξ, η, g) is an almost contact metric structure (see [5] for a definition). So an almost contact-symplectic pair structure (η, ω, ϕ) always admits a compatible metric. In particular, for any compatible metric g, we have

$$g(\xi, \xi) = 1 \quad \text{and} \quad g(X, \xi) = \eta(X),$$

so that the subbundles $\mathcal{H}$ and $\mathbb{R}\xi$ are orthogonal, ξ being the Reeb vector field of the contact-symplectic pair.

Proposition 2.2. *For an almost contact-symplectic structure, every associated metric is compatible. However, the converse does not hold in general.*

Proof. Let g be an associated metric. Then:

$$\begin{aligned} g(\phi X, \phi Y) = (d\eta + \omega)(\phi X, Y) = -(d\eta + \omega)(Y, \phi X) = -g(Y, \phi^2 X) \\ = -g(Y, -X + \eta(X)\xi) = g(X, Y) - \eta(X)\eta(Y), \end{aligned}$$

which shows that g is compatible.

To prove that the converse fails in general, suppose that (η, ω, ϕ) is an almost contact-symplectic structure. Then $(\eta, -\omega, \phi)$ also defines such a structure. Take a metric g compatible with the common almost contact structure (ϕ, ξ, η). It cannot be simultaneously associated to both, since this would imply

$$(d\eta + \omega)(X, Y) = g(X, \phi Y) = (d\eta - \omega)(X, Y),$$

which leads to

$$\omega(X, Y) = 0$$

for all X, Y, contradicting the fact that ω is symplectic on $T\mathcal{S}$. $\square$

A *metric contact-symplectic structure* is a 4-tuple (η, ω, ϕ, g) where the triple (η, ω, ϕ) is an almost contact-symplectic structure and g a Riemannian metric associated to it. Note that the required equations use the objects ξ and ϕ that are uniquely defined by (η, ω, g). Hence, the definition can equivalently be reformulated as follows:

Definition 2.3. *A* metric contact-symplectic structure *on a manifold is a triple (η, ω, g) where (η, ω) is a contact-symplectic pair with Reeb vector field ξ, and g a Riemannian metric such that:*

(i) $g(X, \xi) = \eta(X)$, and

(ii) the endomorphism field ϕ defined by $g(X, \phi Y) = (d\eta + \omega)(X, Y)$ satisfies $\phi^2 = -I + \eta \otimes \xi$.

3 Examples

We begin with the following straightforward construction.

Example 3.1. *The product of a symplectic manifold endowed with an associated metric and a contact metric manifold (see [5] for definitions) yields a manifold with a metric contact-symplectic structure. In this case, the two characteristic foliations are orthogonal and, moreover, both totally geodesic.*

It is important to note that although any contact-symplectic manifold is locally a product of a contact manifold and a symplectic manifold, there exist metric contact-symplectic manifolds with orthogonal characteristic foliations that are not locally such products. The following examples illustrate this well.

Example 3.2. *Let G^5 be the simply connected 5-dimensional nilpotent Lie group defined by the structure equations*

$$d\alpha_2 = \alpha_1 \wedge \alpha_3, \quad d\alpha_4 = \alpha_1 \wedge \alpha_5, \quad d\alpha_1 = d\alpha_3 = d\alpha_5 = 0.$$

The pair (η, ω), where $\eta = \alpha_2$ and $\omega = \alpha_4 \wedge \alpha_5$, is a contact-symplectic pair of type $(1,1)$ on G^5. The Reeb vector field is $\xi = X_2$, where the X_i's are dual to the α_i's. The characteristic distribution $T\mathcal{S}$ of η is spanned by X_4 and X_5, while the characteristic distribution $T\mathcal{C}$ of ω is spanned by X_1, X_2 and X_3. Consider the metric

$$g = \alpha_2^2 + \frac{1}{2}\left(\alpha_1^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2\right),$$

which becomes an associated metric for the contact-symplectic pair when equipped with the endomorphism field ϕ defined by

$$\phi X_2 = 0, \quad \phi X_3 = X_1, \quad \phi X_1 = -X_3, \quad \phi X_5 = X_4, \quad \phi X_4 = -X_5.$$

With respect to this metric, the two characteristic distributions $T\mathcal{S}$ and $T\mathcal{C}$ are orthogonal. The foliation $\mathcal{C}$ is easily seen to be totally geodesic. However, denoting by ∇ the Levi-Civita connection, we have:

$$-2g(\nabla_{X_4} X_5, X_1) = g([X_5, X_1], X_4) + g([X_4, X_1], X_5) + g([X_5, X_4], X_1) = g(X_4, X_4) \neq 0.$$

This shows that the foliation $\mathcal{S}$ is not totally geodesic.

Since the structure constants of the Lie algebra of G^5 are rational, the group admits cocompact lattices Γ. The left-invariant metric contact-symplectic structure (η, ω, g) then descends to any compact nilmanifold G^5/Γ, yielding a metric contact-symplectic structure of type $(1,1)$ with orthogonal characteristic foliations, where only one of the two is totally geodesic.

Example 3.3. *Let G^7 be the simply connected 7-dimensional nilpotent Lie group defined by the structure equations:*

$$d\alpha_{j-1} = \alpha_1 \wedge \alpha_j \quad \text{for} \quad j = 3,4,5,7 \quad \text{and} \quad d\alpha_1 = d\alpha_5 = d\alpha_7 = 0.$$

Set $\eta = \alpha_2$ and $\omega = \alpha_4 \wedge \alpha_5 + \alpha_6 \wedge \alpha_7$. Then (η, ω) is a contact-symplectic pair of type $(1,2)$ on G^7, with Reeb vector field $\xi = X_2$, where the X_i's are dual to the α_i's. The characteristic distribution $T\mathcal{S}$ of η is spanned by X_4, X_5, X_6 and X_7, while the characteristic distribution $T\mathcal{C}$ of ω is spanned by X_1, X_2 and X_3. Consider the metric

$$g = \alpha_2^2 + \frac{1}{2}\left(\alpha_1^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2 + \alpha_7^2\right),$$

which is associated to the contact-symplectic pair via the endomorphism field ϕ defined to be zero on X_2 and:

$$\phi X_3 = X_1, \quad \phi X_1 = -X_3, \quad \phi X_5 = X_4, \quad \phi X_4 = -X_5, \quad \phi X_7 = X_6, \quad \phi X_6 = -X_7.$$

The characteristic distributions $T\mathcal{S}$ and $T\mathcal{C}$ are again orthogonal with respect to this metric. We now show that neither foliation is totally geodesic. Letting ∇ denote the Levi-Civita connection, we compute:

$$-2g(\nabla_{X_3}X_1, X_4) = g([X_1, X_4], X_3) + g([X_3, X_4], X_1) + g([X_1, X_3], X_4) = -g(X_3, X_3) \neq 0$$

and

$$-2g(\nabla_{X_4}X_5, X_1) = g([X_5, X_1], X_4) + g([X_4, X_1], X_5) + g([X_5, X_4], X_1) = g(X_4, X_4) \neq 0,$$

thus proving that both foliations $\mathcal{C}$ and $\mathcal{S}$ fail to be totally geodesic.

As in the previous case, the rationality of the structure constants guarantees the existence of cocompact lattices Γ in G^7. The left-invariant metric contact-symplectic structure (η, ω, g) then descends to each compact nilmanifold G^7/Γ, yielding a metric contact-symplectic structure of type $(1, 2)$ with orthogonal characteristic foliations, neither of which is totally geodesic.

4 Minimal foliations

We first turn our attention to the integral curves of the Reeb vector field of an almost contact-symplectic structure endowed with a compatible Riemannian metric. The following result actually holds for the larger class of almost contact metric manifolds satisfying $\mathcal{L}_\xi\eta = 0$, which includes all contact metric manifolds, for which the result is well known.

Theorem 4.1. *Let (η, ω, ϕ, g) be an almost contact-symplectic structure on a manifold, with a compatible Riemannian metric g. Then the integral curves of its Reeb vector field ξ are geodesics.*

Proof. Let ∇ denote the Levi-Civita connection of g. Since η is invariant under the flow of ξ *i.e.* $\mathcal{L}_\xi\eta = 0$, it follows that for all vector fields X:

$$\begin{aligned} 0 &= (\mathcal{L}_\xi\eta)(X) = \xi(\eta(X)) - \eta(\mathcal{L}_\xi X) = \xi g(X, \xi) - g(\xi, \nabla_\xi X - \nabla_X\xi) \\ &= g(\nabla_\xi X, \xi) + g(X, \nabla_\xi\xi) - g(\xi, \nabla_\xi X - \nabla_X\xi) = g(X, \nabla_\xi\xi) + g(\xi, \nabla_X\xi). \end{aligned}$$

But since $g(\xi, \xi) = 1$, we differentiate both sides to obtain:

$$0 = Xg(\xi, \xi) = 2g(\xi, \nabla_X\xi).$$

Hence, from the earlier computation, $g(X, \nabla_\xi\xi) = 0$ for all X, and thus $\nabla_\xi\xi = 0$. Therefore, the integral curves of ξ are geodesics. $\square$

Almost contact-symplectic and contact structures are special cases of the broader class of almost contact structures. The following result provides an explicit expression for the volume element of a Riemannian metric associated with a contact-symplectic pair, generalizing a classical result from contact and symplectic geometry (see [5]).

Theorem 4.2. *For a contact-symplectic pair (η, ω) of type (m, n) on a manifold, all associated metrics have the same volume element given by*

$$dV = \frac{(-1)^{m+n}}{2^{m+n} m!\ n!}\ \eta \wedge (d\eta)^m \wedge \omega^n.$$

Proof. We know that for a metric contact-symplectic structure (η, ω, g, ϕ) of type (m, n) with Reeb vector field ξ on a manifold M, (ϕ, ξ, η, g) is an almost contact metric structure on M. Consequently, one can locally construct a ϕ-basis, that is, an orthonormal frame $\{\xi, X_1, \ldots, X_{m+n}, X_{1^*}, \ldots, X_{(m+n)^*}\}$ where $X_{i^*} = \phi X_i$. Let $\{\eta, \theta^1, \ldots, \theta^{m+n}, \theta^{1^*}, \ldots, \theta^{(m+n)^*}\}$ be its dual basis. Then

$$dV = \eta \wedge \theta^1 \wedge \theta^{1^*} \wedge \cdots \wedge \theta^{m+n} \wedge \theta^{(m+n)^*}.$$

Observe that

$$d\eta + \omega = \sum_{i=1}^{m+n} \left(\theta^{i^*} \wedge \theta^i - \theta^i \wedge \theta^{i^*} \right) = -2 \sum_{i=1}^{m+n} \theta^i \wedge \theta^{i^*}$$

which follows from the identity $g(X, \phi Y) = (d\eta + \omega)(X, Y)$ and the orthonormality of the local ϕ-basis. Next we have

$$(d\eta + \omega)^{m+n} = (-2)^{m+n}(m+n)!\ \theta^1 \wedge \theta^{1^*} \wedge \cdots \wedge \theta^{m+n} \wedge \theta^{(m+n)^*}.$$

On the other hand,

$$(d\eta + \omega)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} (d\eta)^k \wedge \omega^{m+n-k} = \frac{(m+n)!}{m!\ n!} (d\eta)^m \wedge \omega^n.$$

Comparing both expressions yields:

$$\eta \wedge (d\eta)^m \wedge \omega^n = (-2)^{m+n} m!\ n!\ \eta \wedge \theta^1 \wedge \theta^{1^*} \wedge \cdots \wedge \theta^{m+n} \wedge \theta^{(m+n)^*} = (-2)^{m+n} m!\ n!\ dV.$$

from which the formula follows. □

We now turn to the study of the characteristic foliations arising from a metric contact-symplectic structure (η, ω, g). Let $\mathcal{S}$, $\mathcal{C}$, and $\mathcal{K}$ denote respectively the characteristic foliations of η, ω, and $d\eta$. Their leaves are, respectively symplectic, contact and cosymplectic manifolds.

Theorem 4.3. *Let (η, ω, g) be a metric contact-symplectic structure on a manifold M, and suppose that the characteristic foliations $\mathcal{S}$ of η and $\mathcal{C}$ of ω are orthogonal. Then:*

- *the leaves of $\mathcal{S}$,*
- *the leaves of $\mathcal{C}$,*
- *and the leaves of $\mathcal{K}$, the characteristic foliation of $d\eta$,*

are all minimal submanifolds of the Riemannian manifold (M, g).

Proof. We apply Rummler's minimality criterion (see [7]), which states that a p-dimensional foliation $\mathcal{F}$ on a Riemannian manifold is minimal if and only if its characteristic form α is relatively closed on $T\mathcal{F}$; that is,

$$d\alpha\,(X_1, \ldots, X_p, Y) = 0$$

for all $X_1, \ldots, X_p$ tangent to $\mathcal{F}$ and Y tangent to the manifold. We recall that α is the p-form which vanishes on vectors orthogonal to $\mathcal{F}$ and whose restriction to $\mathcal{F}$ is the volume form of the induced metric on the leaves.

Let (m, n) be the type numbers of the contact-symplectic pair (η, ω), and let ξ denote its Reeb vector field.

- The foliation $\mathcal{S}$ has dimension $2n$. Since $T\mathcal{C}$ is the orthogonal subbundle of $T\mathcal{S}$, the characteristic form of $\mathcal{S}$ is:
$$\alpha_{\mathcal{S}} = \frac{(-1)^n}{2^n\, n!}\,\omega^n,$$
which is closed, and thus relatively closed on $T\mathcal{S}$. Therefore, the leaves of $\mathcal{S}$ are minimal.

- The foliation $\mathcal{C}$ has dimension $2m+1$, and its characteristic form is:
$$\alpha_{\mathcal{C}} = \frac{(-1)^m}{2^m\, m!}\,\eta \wedge (d\eta)^m,$$
which is again closed. Hence, the leaves of $\mathcal{C}$ are minimal.

- The foliation $\mathcal{K}$ has dimension $2n+1$, and its characteristic form is:
$$\alpha_{\mathcal{K}} = \frac{(-1)^n}{2^n\, n!}\,\eta \wedge \omega^n.$$
As ξ is tangent to $\mathcal{K}$ and since
$$i_\xi d\,(\eta \wedge \omega^n) = i_\xi\,(d\eta \wedge \omega^n) = 0,$$
the form $\alpha_{\mathcal{K}}$ is relatively closed on $T\mathcal{K}$. Thus, the leaves of $\mathcal{K}$ are also minimal. □

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