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The Maschke octic contains 96 pairwise disjoint lines

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ABSTRACT

Miyaoka proved that a smooth surface of degree d in $\mathbf{P}^3(\mathbb{C})$ contains at most $2d(d-2)$ pairwise disjoint lines. In this note, we verify that the Maschke octic contains 96 pairwise disjoint lines, thereby proving that Miyaoka's bound is optimal for $d = 8$.

RESUMEN

Miyaoka demostró que una superficie suave de grado d en $\mathbf{P}^3(\mathbb{C})$ contiene a lo más $2d(d-2)$ líneas disjuntas dos a dos. En esta nota, verificamos que la óctica de Maschke contiene 96 líneas disjuntas dos a dos, demostrando de este modo que la cota de Miyaoka es óptima para $d = 8$.

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1 Introduction and result

It is reasonable to expect that surfaces in $\mathbf{P}^3(\mathbb{C})$ containing many singular points, lines, conics, or pairwise disjoint lines have an interesting symmetry group, finite but likely quite large. Starting from this principle, one can approach the problem in reverse: consider an interesting finite subgroup G of $\mathrm{GL}_4(\mathbb{C})$, and attempt to find, among its homogeneous polynomial invariants, those that define surfaces which are exceptional in various respects. This method has been successfully applied in numerous papers (see, for example, [1,3,4,12]: this list is far from being exhaustive). Here, we study a specific example: the Maschke octic, for which Boissière and Sarti proved that it contains 352 lines. We show that, among these 352 lines, one can find 96 that are pairwise disjoint, proving that Miyaoka's bound [9, §2.2] for this problem is optimal in degree 8. This is done using Magma [7], and the (easy) codes are freely available [6].

Let i be a square root of -1 in the field $\mathbb{C}$ of complex numbers. For better readability, we use dots to represent zeroes in sparse matrices (as is standard for character tables of finite groups). Let

$$s_1 = \begin{pmatrix} . & 1 & . & . \\ 1 & . & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 1 & . & . & . \\ . & . & 1 & . \\ . & 1 & . & . \\ . & . & . & 1 \end{pmatrix}, \qquad s_3 = \begin{pmatrix} . & -i & . & . \\ i & . & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix},$$

$$s_4 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \qquad \text{and} \qquad s_5 = \begin{pmatrix} -1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}$$

be elements of $\mathrm{GL}_4(\mathbb{C})$. Then s_1, s_2, s_3, s_4 and s_5 are reflections of order 2: they generate a subgroup of $\mathbf{GL}_4(\mathbb{C})$ which is known to be the group denoted by G_{31} in Shephard-Todd classification of finite complex reflection groups [13, p. 287] (note that the construction of the reflection group G_{31} is based on an earlier paper by Maschke [8]): it is finite of order 46 080. For its natural action on $V = \mathbb{C}^4$, it is irreducible.

We denote by (x, y, z, t) the dual basis of the canonical basis of V: the algebra $\mathbb{C}[V]$ of polynomial functions on V is the polynomial algebra $\mathbb{C}[x, y, z, t]$. We set

$$f = x^8 + y^8 + z^8 + t^8 + 14\,(x^4y^4 + x^4z^4 + x^4t^4 + y^4z^4 + y^4t^4 + z^4t^4) + 168\,x^2y^2z^2t^2.$$

Then, f is, up to a scalar, the unique invariant polynomial of G_{31} of degree 8 (the uniqueness follows from [13, Table VII]). We denote by $\mathcal{M}$ the surface in $\mathbf{P}(V) = \mathbf{P}^3(\mathbb{C})$ defined as the zero locus of f. It is straightforward to verify with Magma that $\mathcal{M}$ is smooth, which implies that the polynomial f and the surface $\mathcal{M}$ are irreducible. The surface $\mathcal{M}$ is called the *Maschke octic.*

It was proved by Boissière and Sarti that $\mathcal{M}$ contains 352 lines [2, Theo 3.1]. It was checked by Sarti and the author [5, §6.5] that these 352 lines are divided into two G_{31}-orbits Ω_{160} and Ω_{192} of respective cardinality 160 and 192. More precisely, let L_{160} and L_{192} denote the lines in $\mathbf{P}^3(\mathbb{C})$ defined by

$$L_{160}:\begin{cases}2x+(i+1)(\sqrt{3}+1)y=0,\\ 2z+(i+1)(\sqrt{3}+1)t=0\end{cases}\qquad\text{and}\qquad L_{192}:\begin{cases}(i+1)(\sqrt{5}+1)x-2(y+z)=0,\\ 2(y-z)+(i-1)(\sqrt{5}+1)t=0.\end{cases}$$

Then L_{160} and L_{192} are contained in $\mathcal{M}$ and Ω_k is the orbit of L_k (for $k\in\{160,192\}$). A computer calculation with Magma allows one to check the next result:

Theorem 1.1. *The orbit Ω_{192} contains a family of 96 pairwise disjoint lines.*

Proof. Let us give a few details about how this computation is made and some precision on the statement (everything is checked with Magma: see [6]). First, let

$$a=\frac{1-i}{2}\begin{pmatrix}1&.&.&1\\ .&1&1&.\\ .&-1&1&.\\ 1&.&.&-1\end{pmatrix}\quad\text{and}\quad b=\begin{pmatrix}.&.&-1&.\\ -1&.&.&.\\ .&1&.&.\\ .&.&.&1\end{pmatrix}.$$

Then $a,\,b\in G_{31}$. Let $G=\langle a,b\rangle$ and let Ω denote the G-orbit of the line L_{192}. Then $|G|=1152$ and $|\Omega|=96$. One can then check that, if $L,\,L'\in\Omega$ are such that $L\neq L'$, then $L\cap L'=\varnothing$. This shows that one can choose for the family of 96 pairwise distinct lines in Ω_{192} the orbit of L_{192} under the action of G. □

Remark 1.2. *(1) For $d\geq 5$, it is known that there exists a smooth surface of degree d in $\mathbf{P}^3(\mathbb{C})$ containing $d(d-2)+2$ pairwise disjoint lines [11]*[1]*. For $d=8$, this gives a smooth octic surface with 50 pairwise disjoint lines: the above theorem improves this record drastically.*

On the other hand, Miyaoka [9, §2.2] proved that a smooth surface of degree $d\geq 3$ in $\mathbf{P}^3(\mathbb{C})$ contains at most $2d(d-2)$ pairwise disjoint lines. The above theorem shows that Miyaoka's bound is optimal for $d=8$.

(2) Miyaoka's bound is optimal for $d\in\{2,3,4,8\}$ and there is no value of d for which it is known that Miyaoka's bound is not optimal.

(3) It is natural to ask whether the Maschke octic is the unique octic surface of $\mathbf{P}^3(\mathbb{C})$ (up to the action of $\mathbf{GL}_4(\mathbb{C})$) containing 96 pairwise disjoint lines.

[1] For d odd, this can be improved to $d(d-2)+4$; see [10] for $d=5$ and [2, Theo. 5.1] for $d\geq 7$.

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