

First elements for partial-order actions in $R\text{-Mod}$

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ABSTRACT

We study notions of primeness in module theory through the lens of order actions on lattices. Given a poset $\mathcal{P}$ acting on a bounded lattice $\mathcal{L}$, we introduce $\mathcal{P}$ -first (and $\mathcal{P}$ -prime) elements and establish basic permanence properties and examples arising from module-theoretic lattices. Specializing to the natural action of the lattice $R\text{-pr}$ of preradicals on the submodule lattice of a left R -module, we define $R\text{-pr}$ -first submodules and $R\text{-pr}$ -first modules and prove that, for nonzero modules, they coincide with the BJKN notion of prime module (equivalently, every nonzero submodule cogenerates the module). We then extend the theory to $\mathcal{A}$ -first and $\mathcal{A}$ -fully first modules for subclasses $\mathcal{A} \subseteq R\text{-pr}$. As applications, we characterize rings for which every nonzero module is $R\text{-pr}$ -first, and we describe left semiartinian rings, left local rings, and semisimple homogeneous rings (equivalently, left semiartinian left local V -rings) in terms of firstness relative to suitable subclasses of preradicals.

RESUMEN

Estudiamos nociones de primalidad en teoría de módulos a través del lente de acciones de orden en reticulados. Dado un poset $\mathcal{P}$ actuando en un reticulado acotado $\mathcal{L}$, introducimos elementos $\mathcal{P}$ -primeros (y $\mathcal{P}$ -primos) y establecemos propiedades de permanencia básicas y ejemplos que provienen de reticulados de teoría de módulos. Especializando a la acción natural del reticulado $R\text{-pr}$ de prerradicales del reticulado de submódulos de un R -módulo izquierdo, definimos submódulos $R\text{-pr}$ -primeros y módulos $R\text{-pr}$ -primeros y demostramos que, para módulos no cero, estos coinciden con la noción BJKN de módulo primo (equivalentemente, todo submódulo no cero cogenera el módulo). Luego, extendemos la teoría a módulos $\mathcal{A}$ -primeros y $\mathcal{A}$ -completamente primeros para subclases $\mathcal{A} \subseteq R\text{-pr}$. Como aplicaciones, caracterizamos anillos para los cuales todo módulo no cero es $R\text{-pr}$ -primero, y describimos anillos semiartinianos izquierdos, anillos locales izquierdos, y anillos semisimples homogéneos (equivalentemente, V -anillos semiartinianos izquierdos locales izquierdos) en términos de primeridad relativa a subclases apropiadas de prerradicales.

Keywords and Phrases: $R\text{-pr}$; $\mathcal{A}$ -first modules; prime modules; V -rings; left local rings.

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1 Introduction

In this paper we study several notions in module theory that originate in the idea of primeness (in particular, first modules) and extend them to a more general order-theoretic setting.

Our starting point is an order-theoretic viewpoint: given a poset $\mathcal{P}$ acting on a bounded lattice $\mathcal{L}$, we introduce the notion of $\mathcal{P}$ -first elements (a related notion of $\mathcal{P}$ -prime elements is discussed mainly in Remark 2.13). When $\mathcal{L} = \mathcal{L}({}_R M)$ is the submodule lattice of a module and $\mathcal{P} = R\text{-pr}$ is the lattice of preradicals (ordered by $\preceq$) acting by $\alpha \mapsto N := \alpha(N)$, this yields the notion of $R\text{-pr}$ -first modules. One of our main results shows that, for nonzero modules, $R\text{-pr}$ -first modules coincide with the BJKN-prime modules of [4] (Proposition 3.10). We then study firstness relative to subclasses $\mathcal{A} \subseteq R\text{-pr}$ and apply this framework to characterize several ring-theoretic classes, including left semiartinian rings (Proposition 5.9), left local rings (Theorem 5.11), and rings for which every nonzero module is $R\text{-pr}$ -first (Theorem 6.5).

The paper is organized as follows. In the remainder of the introduction we recall the background on prime modules, cogeneration, torsion-free classes, and preradicals. In Section 2 we develop $\mathcal{P}$ -first elements for general poset actions on lattices. In Section 3 we specialize to the $R\text{-pr}$ -action and prove the characterizations of $R\text{-pr}$ -first modules. In Sections 4–6 we introduce $\mathcal{A}$ -first and $\mathcal{A}$ -fully first modules and derive further ring-theoretic applications.

Recall that a left R -module M is called *left semiartinian* if every nonzero factor module of M has nonzero socle, *i.e.*, if $\text{soc}(M/N) \neq 0$ for all proper submodules $N \subsetneq M$. A ring R is called *left semiartinian* if ${}_R R$ is a left semiartinian module. A ring R is called *simple* if R has no nonzero proper two-sided ideals. In particular, if R is simple and left semiartinian, then $\text{soc}({}_R R) \neq 0$, hence $\text{soc}({}_R R) = R$; thus R is semisimple (equivalently, simple artinian), and by the Wedderburn–Artin theorem $R \cong M_n(D)$ for some division ring D and some $n \geq 1$ (see, *e.g.*, [2, 10]).

It is a well-known fact (see, *e.g.*, [15, Proposition VIII.2.8]) that the following conditions are equivalent:

- (1) ${}_R R$ is a left semiartinian module;
- (2) every left R -module is left semiartinian;
- (3) $\text{soc}(M) \neq 0$ for every nonzero left R -module M .

In [9], Johnson calls a module ${}_R M$ *prime* if $\text{ann}_R(M) = \text{ann}_R(N)$ for each nonzero submodule $0 \neq N \leq M$, where $\text{ann}_R(M)$ denotes the (left) annihilator of M in R ; see also [3, 6].

If R is commutative, an ideal I is *prime* if $ab \in I$ implies $a \in I$ or $b \in I$. In particular, if (0) is a prime ideal, then R is an integral domain.

If R is not assumed commutative, a (two-sided) ideal I is *prime* if, for all two-sided ideals J, K of

R , the inclusion $JK \subseteq I$ implies $J \subseteq I$ or $K \subseteq I$. Thus (0) is prime precisely when R is a prime ring, that is, $IJ = 0$ implies $I = 0$ or $J = 0$ for all two-sided ideals I, J of R .

This idea extends to modules by declaring that a module M is prime if, for every nonzero submodule $0 \neq K \leq M$ and every two-sided ideal I of R , $IK = 0$ implies $IM = 0$; this is equivalent to Johnson's definition in [9].

A class of modules is called *pretorsion-free* if it is closed under taking submodules ($\leq$) and direct products (Π). A pretorsion-free class ζ is a *torsion-free class* if it is also closed under extensions (*ext*). Furthermore, a torsion-free class ζ is *hereditary torsion-free* if it is closed under injective envelopes (or injective hulls): if $M \in \zeta$, then $E(M) \in \zeta$, where $E(M)$ denotes the injective envelope of M .

We denote by $\mathcal{L}_{\leq, \Pi, E, \text{ext}}$ the collection of hereditary torsion-free classes (*i.e.*, classes closed under $\leq$, Π , E , and ext), ordered by inclusion; this is a lattice (see, *e.g.*, [8]). Similarly, we denote by $\mathcal{L}_{\leq, \Pi}$ the collection of *pretorsion-free* classes (closed under $\leq$ and Π).

In particular, if $\{\mathcal{A}_i\}_{i \in I}$ is a family of elements of $\mathcal{L}_{\leq, \Pi}$, then its join in the lattice $\mathcal{L}_{\leq, \Pi}$ is given by

$$\bigvee_{i \in I} \mathcal{A}_i = \xi_{\leq, \Pi} \left(\bigcup_{i \in I} \mathcal{A}_i \right),$$

i.e., it is the least pretorsion-free class $\mathcal{C} \in \mathcal{L}_{\leq, \Pi}$ such that $\mathcal{A}_i \subseteq \mathcal{C}$ for all $i \in I$.

For a module $M \in R\text{-Mod}$, we write $\xi_{\leq, \Pi}(M)$ instead of $\xi_{\leq, \Pi}(\{M\})$, where $\{M\}$ denotes the singleton *class* consisting of M . Notice that $\xi_{\leq, \Pi}(M)$ is precisely the class of left R -modules cogenerated by ${}_R M$; in general, this class need not be closed under extensions, and hence need not be a torsion-free class (for instance, the class of $\mathbb{Z}/4\mathbb{Z}$ -modules cogenerated by $\mathbb{Z}/2\mathbb{Z}$ is not extension-closed).

Recall that a left R -module C cogenerates a left R -module M if there exist a set X and a monomorphism $M \rightarrow C^X$. More generally, C cogenerates a class $\mathcal{X} \subseteq R\text{-Mod}$ if C cogenerates every module in $\mathcal{X}$.

We recall next the notion of prime module from [4], since it will be shown to coincide with ***R-pr***-firstness. The concept of *prime module* was introduced by L. Bican, P. Jambor, T. Kepka, and P. Němec in Section 2 of [4]. A module M is defined as *prime* if every nonzero submodule ${}_R N$ of ${}_R M$ cogenerates the class $\xi_{\leq, \Pi}({}_R M)$.

Definition 1.1. *A module M is BJKN-prime if each nonzero submodule of M cogenerates ${}_R M$.*

In the same section of [4], the authors introduce a product of two submodules A and B of a given module M .

Definition 1.2. Let $A, B \leq M$ be submodules of a left R -module ${}_R M$. The product of A and B in M is defined by

$$A \bullet_M B := \sum \{f(A) \mid f \in \text{Hom}_R(M, B)\},$$

where for $f \in \text{Hom}_R(M, B)$ we view $f(A)$ as the image of A under the restriction $f|_A : A \rightarrow B$. In particular, $A \bullet_M B \leq B \leq M$, and the definition depends on M through the homomorphisms $M \rightarrow B$.

Proposition 1.3 ([4, Proposition 2.1]). Let M be a nonzero R -module. Then M is *BJKN-prime* if and only if, for all nonzero submodules $0 \neq L, N \leq M$, the product $L \bullet_M N$ is a nonzero submodule of M .

In particular, if there exist nonzero submodules $0 \neq L, N \leq M$ such that $L \bullet_M N = 0$, then M is not *BJKN-prime*.

A ring R is a *V-ring* if every simple left R -module is injective. The ring R is *left semiartinian* if every nonzero left R -module has nonzero socle. The ring R is *left local* if any two simple left R -modules are isomorphic. A left R -module ${}_R M$ is *semisimple* if it is the sum of its simple submodules. Finally, R is *semisimple* (Artinian) if it is semisimple as a left R -module.

If ${}_R M$ is a left R -module, it is well known that the set of submodules of M , which we denote $\mathcal{L}({}_R M)$, is a complete lattice; see [2, Proposition 2.5]. We say that a submodule ${}_R N$ of ${}_R M$ is *fully invariant* in M if $f(N) \leq N$ for all $f \in \text{End}({}_R M)$. It is straightforward to verify that the fully invariant submodules of M form a complete sublattice of $\mathcal{L}({}_R M)$. We denote by $_{f.i.}\mathcal{L}({}_R M)$ the lattice of fully invariant submodules of M , and we write $N_{f.i.} \leq_R M$ to indicate that N is a fully invariant submodule of M .

Before turning to our main definitions, we recall basic facts on preradicals, since they provide the acting poset in the $R\text{-pr}$ -action on submodule lattices used throughout the paper.

1.1 Preradicals in $R\text{-Mod}$.

We recall the basic definitions and results on preradicals in $R\text{-Mod}$. For further background on preradicals, see [5, 7, 11, 12, 14, 15]. A preradical σ on $R\text{-Mod}$ is a functor $\sigma : R\text{-Mod} \rightarrow R\text{-Mod}$ such that:

- (1) $\sigma(M) \leq M$, for each $M \in R\text{-Mod}$.
- (2) For each R -morphism $f : M \rightarrow N$, the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \uparrow & & \uparrow \\ \sigma(M) & \xrightarrow{f|} & \sigma(N) \end{array}$$

Recall that for each $\beta \in R\text{-}\mathbf{pr}$ and each family $\{M_i\}_{i \in I}$ of R -modules one has $\beta(\bigoplus_{i \in I} M_i) = \bigoplus_{i \in I} \beta(M_i)$; see [5, Proposition I.1.2]. $R\text{-}\mathbf{pr}$ denotes the collection of all preradicals in $R\text{-}Mod$. By [14, Theorem 7], $R\text{-}\mathbf{pr}$ is a big lattice with smallest element $\underline{0}$ and largest element $\underline{1}$, where:

- (1) the order in $R\text{-}\mathbf{pr}$ is given by $\alpha \preceq \beta$ if $\alpha(M) \leq \beta(M)$, for every $M \in R\text{-}Mod$;
- (2) for any family of preradicals $\{\sigma_i\}_{i \in I}$ in $R\text{-}Mod$, the supremum and infimum of the family are given, respectively, by

$$\begin{aligned} \text{(a)} \quad & \left(\bigvee_{i \in I} \sigma_i \right) (M) = \sum_{i \in I} \sigma_i(M) \text{ and} \\ \text{(b)} \quad & \left(\bigwedge_{i \in I} \sigma_i \right) (M) = \bigcap_{i \in I} \sigma_i(M). \end{aligned}$$

In addition to the lattice operations, $R\text{-}\mathbf{pr}$ is endowed with two further classical operations: the product and the coproduct of preradicals. Following [14, Definition 2], for $\sigma, \tau \in R\text{-}\mathbf{pr}$ and $M \in R\text{-}Mod$, their product is defined by $(\sigma \circ \tau)(M) = \sigma(\tau(M))$, whereas their coproduct $(\sigma : \tau)$ is determined by

$$\frac{(\sigma : \tau)(M)}{\sigma(M)} = \tau \left(\frac{M}{\sigma(M)} \right).$$

Equivalently, if

$$\pi_\sigma : M \longrightarrow M/\sigma(M)$$

denotes the canonical projection, then

$$(\sigma : \tau)(M) = \pi_\sigma^{-1}(\tau(M/\sigma(M))).$$

By [14, Remark 1], these four classical operations satisfy

$$\sigma \circ \tau \preceq \sigma \wedge \tau \preceq \sigma \preceq \sigma \vee \tau \preceq (\sigma : \tau).$$

Moreover, the operations $\wedge$, $\vee$, $\circ$, and $:$ are associative and order-preserving. The lattice operations $\wedge$ and $\vee$ are commutative, whereas the product and coproduct need not be commutative.

Recall that $\sigma \in R\text{-}\mathbf{pr}$ is idempotent if $\sigma \circ \sigma = \sigma$. The preradical σ is radical if $\sigma(M/\sigma(M)) = 0$ for each $M \in R\text{-}Mod$. Equivalently, by the definition of the coproduct of preradicals, $(\sigma : \sigma) = \sigma$. We say that σ is a left exact preradical if it is a left exact functor.

The preradical σ is a t -radical (also known as a cohereditary preradical) if $\sigma(M) = \sigma(R)M$ for each $M \in R\text{-}Mod$. Recall that σ is a t -radical if and only if σ preserves epimorphisms (see [15, Chapter VI, Exercise 5]).

Moreover, σ is left exact if and only if, for each submodule $N \leq M$, one has $\sigma(N) = \sigma(M) \cap N$ (see [15, Chapter VI, Proposition 1.7]). In particular, the socle preradical soc is left exact, that is, $\text{soc} \in R\text{-lep}$.

We will denote by $R\text{-pid}$, $R\text{-rad}$, $R\text{-lep}$, $R\text{-trad}$ and $R\text{-ler}$ the collections of idempotent preradicals, radicals, left exact preradicals, t -radicals, and left exact radicals, respectively. For $\sigma \in R\text{-pr}$ we denote by $\widehat{\sigma}$ the largest idempotent preradical smaller than σ and by $\overline{\sigma}$ the least radical greater than σ (see [15, Chapter VI, Proposition 1.5]).

1.2 Relationships between preradicals and fully invariant submodules.

Recall for a fully invariant submodule ${}_R N$ of ${}_R M$, the preradicals α_N^M and ω_N^M defined in [14, Definition 4].

Definition 1.4. Let ${}_R N$ be a fully invariant submodule of ${}_R M$ and $U \in R\text{-Mod}$.

- (1) $\alpha_N^M(U) = \sum \{f(N) \mid f \in \text{Hom}_R(M, U)\}$ and
- (2) $\omega_N^M(U) = \bigcap \{f^{-1}(N) \mid f \in \text{Hom}_R(U, M)\}.$

More generally, for ${}_R N$ a submodule of ${}_R M$, we can define the preradical β_N^M by

$$\beta_N^M(U) = \sum \{f(N) \mid f \in \text{Hom}_R(M, U)\}.$$

With Definition 1.2, we have

$$A \bullet_M B = \beta_A^M(B) \quad \text{for all submodules } A, B \leq M,$$

since both sides equal $\sum \{f(A) \mid f \in \text{Hom}_R(M, B)\}.$

Remark 1.5. Let $M \in R\text{-Mod}$ and let N be a fully invariant submodule of M . By [14, Proposition 5], $\alpha_N^M : R\text{-Mod} \rightarrow R\text{-Mod}$ is the least preradical ρ such that $\rho(M) = N$, and $\omega_N^M : R\text{-Mod} \rightarrow R\text{-Mod}$ is the largest preradical ρ such that $\rho(M) = N$. Thus

$$\{\rho \in R\text{-pr} \mid \rho(M) = N\} = [\alpha_N^M, \omega_N^M],$$

an interval in $R\text{-pr}$.

Remark 1.6. Let $M \in R\text{-Mod}$. Since $\omega_0^M(M) = 0$, we have

$$\frac{(\omega_0^M : \omega_0^M)(M)}{\omega_0^M(M)} = \omega_0^M \left(\frac{M}{\omega_0^M(M)} \right) = \omega_0^M \left(\frac{M}{0} \right) = \omega_0^M(M) = 0.$$

Therefore, $(\omega_0^M : \omega_0^M)(M) = 0$. By Remark 1.5, ω_0^M is the largest preradical ρ such that $\rho(M) = 0$. Hence $(\omega_0^M : \omega_0^M) \preceq \omega_0^M$. On the other hand, by the general inequalities for the coproduct, $\omega_0^M \preceq (\omega_0^M : \omega_0^M)$. Consequently, $(\omega_0^M : \omega_0^M) = \omega_0^M$, and therefore ω_0^M is a radical.

Note that $\alpha_M^M(L)$ is the trace of M in L , that is, $\alpha_M^M = \text{tr}_M$. Furthermore, $\omega_0^M(L)$ is the reject of M in L ; it is the smallest submodule of L such that $L/\omega_0^M(L)$ embeds into a direct product of copies of M .

Remark 1.7. If $L_{f.i.} \leq_R K_{f.i.} \leq_R M$, then $L_{f.i.} \leq_R M$.

Proof. If $f : M \rightarrow M$ is an R -morphism, then f restricts to an R -endomorphism of K , which in turn restricts to an endomorphism of L , see [7, Exercises 6.31.(1) and (2)] . $\square$

Since each simple left R -module is isomorphic to R/I for some maximal left ideal I of R , we may fix a set $R\text{-simp}$ of representatives of the isomorphism classes of simple left R -modules. Recall that $\text{soc} : R\text{-Mod} \rightarrow R\text{-Mod}$ is an idempotent preradical such that $\text{soc}(M)$ is the largest semisimple submodule of M . Furthermore, $\text{soc} = \bigvee \{\alpha_S^S \mid S \in R\text{-simp}\}$. Similarly, $\text{rad} = \bigwedge \{\omega_0^S \mid S \in R\text{-simp}\}$. A submodule $N \leq M$ is superfluous if $N + K = M$ implies $K = M$, and it is essential if $N \cap K \neq 0$ for every $0 \neq K \leq M$. Moreover, $\text{soc}(M)$ is the intersection of all essential submodules of M , while $\text{rad}(M)$ is the sum of all superfluous submodules of M ; see [2, Section 9].

2 $\mathcal{P}$ -first elements in a lattice.

Definition 2.1. A lattice $\mathcal{L}$ is bounded if elements $0, 1 \in \mathcal{L}$ exist such that $0 \leq x \leq 1$ for each $x \in \mathcal{L}$.

The following definition appears in [1].

Definition 2.2. Let $\mathcal{L} = (L, \leq, \vee, \wedge)$ be a lattice and let $\mathcal{P} = (P, \leq')$ be a poset. A $\mathcal{P}$ -action on $\mathcal{L}$ is a function $\rightarrow : \mathcal{P} \times \mathcal{L} \rightarrow \mathcal{L}$ satisfying the following conditions, for all $s, t \in P$ and $x, y \in L$:

$$i) \ s \leq' t \implies s \rightarrow x \leq t \rightarrow x.$$

$$ii) \ x \leq y \implies s \rightarrow x \leq s \rightarrow y.$$

$$iii) \ s \rightarrow x \leq x.$$

Since lattices are particular posets, actions of lattices on lattices may be regarded as special cases of the preceding notion of a poset action.

In what follows we mainly consider actions of certain posets (in particular, the lattices $\mathcal{L}({}_R M)$, $R\text{-pr}$, and ${}_f \mathcal{L}({}_R M)$) on lattices, as in Definition 2.6 and Examples 2.3 and 2.4 below.

We distinguish $\mathcal{L}({}_R R)$, the lattice of left ideals of R , from $\text{Lat}(R)$, which will denote the lattice of two-sided ideals of R (i.e., ideals $I \triangleleft R$ with $RI \subseteq I$ and $IR \subseteq I$).

Example 2.3. The lattice $\mathcal{L}({}_R R)$ acts on $\mathcal{L}({}_R M)$.

$$\begin{aligned} \mathcal{L}({}_R R) \times \mathcal{L}({}_R M) &\xrightarrow{\quad} \mathcal{L}({}_R M) \\ (I, N) &\longmapsto IN. \end{aligned}$$

Example 2.4. The lattice $R\text{-}\mathbf{pr}$ acts on $\mathcal{L}({}_R M)$.

$$\begin{aligned} R\text{-}\mathbf{pr} \times \mathcal{L}({}_R M) &\xrightarrow{\quad} \mathcal{L}({}_R M) \\ (r, N) &\longmapsto r(N). \end{aligned}$$

Remark 2.5. Let $\mathcal{P} = (P, \leq')$ be a poset, $\mathcal{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a $\mathcal{P}$ -action $\rightarrow: \mathcal{P} \times \mathcal{L} \rightarrow \mathcal{L}$, and let $\mathcal{P}_0$ be a subposet of $\mathcal{P}$. Then the restriction of $\rightarrow$ to $\mathcal{P}_0 \times \mathcal{L}$ is a $\mathcal{P}_0$ -action.

Definition 2.6. Let $\mathcal{P} = (P, \leq')$ be a poset, $\mathcal{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a $\mathcal{P}$ -action $\rightarrow: \mathcal{P} \times \mathcal{L} \rightarrow \mathcal{L}$ and $x \in L \setminus \{0\}$. We say that:

i) x is $\mathcal{P}$ -first if for each $0 \neq z \leq x$ and $s \in P$

$$s \rightarrow z = 0 \implies s \rightarrow x = 0.$$

ii) x is $\mathcal{P}$ -prime if for each $z \in L$ and each $s \in P$

$$s \rightarrow z \leq x \implies (s \rightarrow 1 \leq x \text{ or } z \leq x).$$

Remark 2.7. When we apply these definitions to the action of Example 2.3, we note that the $\mathcal{L}(R)$ -first modules are precisely the nonzero prime modules.

We omit the easy proof of the following proposition.

Proposition 2.8. Let $f: \mathcal{P} \rightarrow \mathcal{Q}$ be an order-preserving map. Every $\mathcal{Q}$ -action on $\mathcal{L}$ induces a $\mathcal{P}$ -action on $\mathcal{L}$ via f .

Explicitly, $\mathcal{Q} \times \mathcal{L} \xrightarrow{\quad} \mathcal{L} \xrightarrow{(_)_f} \mathcal{P} \times \mathcal{L} \xrightarrow{\quad} \mathcal{L}$, where, for each $a \in \mathcal{P}$ and $x \in \mathcal{L}$, $a \rightarrow_f x := f(a) \rightarrow x$.

Lemma 2.9. Let $\mathcal{P} = (P, \leq')$ be a poset, $\mathcal{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a $\mathcal{Q}$ -action $\rightarrow: \mathcal{Q} \times \mathcal{L} \rightarrow \mathcal{L}$ and let $f: \mathcal{P} \rightarrow \mathcal{Q}$ be an order-preserving function. Then $\mathcal{Q}$ -first elements of $\mathcal{L}$ with respect to $\rightarrow$ are $\mathcal{P}$ -first elements of $\mathcal{L}$ with respect to $\rightarrow_f$.

In particular, note that one may restrict a $\mathcal{Q}$ -action on $\mathcal{L}$ to any subposet $\mathcal{P} \subseteq \mathcal{Q}$.

The following corollary is immediate.

Corollary 2.10. Let $\mathcal{P} = (P, \leq')$ be a poset, $\mathcal{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a $\mathcal{P}$ -action $\rightarrow: \mathcal{P} \times \mathcal{L} \rightarrow \mathcal{L}$, and let $\mathcal{P}_0$ be a subposet of $\mathcal{P}$. Then any $\mathcal{P}$ -first element of $\mathcal{L}$ is $\mathcal{P}_0$ -first.

Definition 2.11. Let $\mathcal{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and $y, x \in L$ such that $y \leq x$. We denote $[y, x]$ the bounded lattice of all $z \in L$ such that $y \leq z \leq x$.

Remark 2.12. Let $\mathcal{P} = (P, \leq')$ be a poset, $\mathcal{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a $\mathcal{P}$ -action $\rightarrow: \mathcal{P} \times \mathcal{L} \rightarrow \mathcal{L}$ and $x \in L$. Then $\rightarrow|_{\mathcal{P} \times [0, x]}: \mathcal{P} \times [0, x] \rightarrow [0, x]$ is well-defined and is a $\mathcal{P}$ -action.

Recall [1, Remarks 1.8(6) and 1.8(7)].

Remark 2.13. Let $\mathcal{P} = (P, \leq')$ be a poset, $\mathcal{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a $\mathcal{P}$ -action $\rightarrow: \mathcal{P} \times \mathcal{L} \rightarrow \mathcal{L}$ and $x \in L \setminus \{0\}$. Then x is $\mathcal{P}$ -first in $\mathcal{L}$ if and only if 0 is $\mathcal{P}$ -prime in $[0, x]$.

Remark 2.13 gives us a way to relate the concept of $\mathcal{P}$ -first with the concept of $\mathcal{P}$ -prime.

Definition 2.14. Let $\mathcal{L}$ be a lattice with least element 0 , that is, $0 \leq x$ for every $x \in \mathcal{L}$. An element $a \in \mathcal{L}$ is called an atom if $a \neq 0$ and, whenever $x \in \mathcal{L}$ satisfies $0 \leq x \leq a$, then $x = 0$ or $x = a$.

Lemma 2.15. Let $\mathcal{P} = (P, \leq')$ be a poset, $\mathcal{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a $\mathcal{P}$ -action $\rightarrow: \mathcal{P} \times \mathcal{L} \rightarrow \mathcal{L}$. All atoms of $\mathcal{L}$ are $\mathcal{P}$ -first.

Proof. Let a be an atom of $\mathcal{L}$. Let $s \in P$ and $0 \neq z \leq a$. If $s \rightarrow z = 0$, then $z = a$. From this it follows that $s \rightarrow a = 0$. Therefore, a is $\mathcal{P}$ -first. $\square$

3 R -pr-first modules.

We now apply Definition 2.6 to the action described in Example 2.4 to define first modules with respect to preradicals.

Definition 3.1. Let $M \in R\text{-Mod}$ and $N \in \mathcal{L}(M) \setminus \{0\}$. We say that N is an R -**pr**-first submodule of M if, for all $K \in \mathcal{L}(M) \setminus \{0\}$ and all $\alpha \in R\text{-pr}$, the conditions $K \leq N$ and $\alpha(K) = 0$ imply $\alpha(N) = 0$.

Example 3.2. The nonzero submodules of ${}_Z\mathbb{Z}$ are $\mathbb{Z}$ -**pr**-first submodules of $\mathbb{Z}$, since for each $N, K \in \mathcal{L}(\mathbb{Z}) \setminus \{0\}$ we have $N \cong K$.

Lemma 3.3. Let $M \in R\text{-Mod}$. Every simple submodule of M is R -**pr**-first.

Proof. This follows from Lemma 2.15. $\square$

The following lemma is a consequence of [2, Lemma 9.2], and of the fact that for any $S, M \in R\text{-Mod}$ with S simple, $\alpha_S^S(M)$ is a semisimple homogeneous R -module.

Lemma 3.4. *Let $M \in R\text{-Mod}$ and let N be a nonzero semisimple submodule of M . Then N is a $R\text{-pr}$ -first submodule of M if and only if $N \cong S^{(I)}$ for some set I and some simple left R -module S .*

Proof. ($\implies$) Since N is semisimple, write $N = \bigoplus_{i \in I} S_i$ with S_i simple. If there exist $i, j \in I$ such that $S_i \not\cong S_j$, set $S := S_i$ and consider the preradical α_S^S . Then $\alpha_S^S(S) = S \neq 0$, while $\alpha_S^S(S_j) = 0$ because $\text{Hom}_R(S, S_j) = 0$. Taking the nonzero submodule $K := S_j \leq N$, we get $\alpha_S^S(K) = 0$ but $\alpha_S^S(N) \neq 0$, contradicting that N is $R\text{-pr}$ -first. Hence all S_i are isomorphic, so $N \cong S^{(I)}$ for some simple module S .

($\impliedby$) Assume $N \cong S^{(I)}$ for a simple S . Let $0 \neq K \leq N$ and let $\alpha \in R\text{-pr}$ with $\alpha(K) = 0$. Since K is semisimple, it contains a simple submodule $0 \neq T \leq K$ with $T \cong S$. Then $\alpha(T) \leq \alpha(K) = 0$, so $\alpha(T) = 0$, and hence $\alpha(S) = 0$. As preradicals commute with coproducts, we obtain

$$\alpha(N) = \alpha(S^{(I)}) = (\alpha(S))^{(I)} = 0.$$

Therefore N is $R\text{-pr}$ -first. $\square$

Remark 3.5. *We find that if N is a submodule of M , then N is $R\text{-pr}$ -first in $\mathcal{L}(M)$ if and only if N is $R\text{-pr}$ -first in $\mathcal{L}(N)$. In this sense, we will say that an R -module M is $R\text{-pr}$ -first if it is $R\text{-pr}$ -first as a submodule of itself.*

Example 3.6. ${}_Z\mathbb{Z}$ is $\mathbb{Z}\text{-pr}$ -first, since for any $N \in \mathcal{L}(\mathbb{Z}) \setminus \{0\}$ we have $N \cong \mathbb{Z}$.

Recall [13, Definition 42].

Definition 3.7. *An R -module M is diuniform if any nonzero fully invariant submodule N of M is essential in M .*

Proposition 3.8. *Let M be an $R\text{-pr}$ -first R -module. Then M is diuniform.*

Proof. Let ${}_R M$ be $R\text{-pr}$ -first, $N \in \mathcal{L}_{fi}(M) \setminus \{0\}$ and $K \in \mathcal{L}(M) \setminus \{0\}$. We have $\alpha_N^M(M) = N \neq 0$ and M is $R\text{-pr}$ -first, so $\alpha_N^M(K) \neq 0$. Moreover, $\alpha_N^M(K) \leq N$ and $\alpha_N^M(K) \leq K$, since α_N^M is a preradical, and N is fully invariant, so $0 \neq \alpha_N^M(K) \leq N \cap K$, then $N \cap K \neq 0$. Therefore, N is essential in M . $\square$

The converse of Proposition 3.8 does not hold, as the following example shows.

Example 3.9. Any two nonzero submodules of ${}_Z\mathbb{Z}_{p^\infty}$ have nonzero intersection. Hence, every nonzero fully invariant submodule of ${}_Z\mathbb{Z}_{p^\infty}$ is essential, and therefore ${}_Z\mathbb{Z}_{p^\infty}$ is diuniform. However, $\text{Hom}_Z(\mathbb{Z}_{p^\infty}, \mathbb{Z}_p) = 0$, so $\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_{p^\infty}}(\mathbb{Z}_p) = 0$ and $\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_{p^\infty}}(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_p \neq 0$. Thus, ${}_Z\mathbb{Z}_{p^\infty}$ is not $\mathbb{Z}$ -pr-first.

The following proposition characterizes the R -pr-first modules.

Proposition 3.10. Let M be a nonzero R -module. The following statements are equivalent:

- (1) ${}_RM$ is R -pr-first.
- (2) M is $BJKN$ -prime.
- (3) Each nonzero cyclic submodule of M cogenerates M .
- (4) For any $x, y \in M \setminus \{0\}$, $\exists f_x : M \rightarrow Ry$ such that $f_x(x) \neq 0$.

Proof. (1) $\implies$ (2): Let $A, B \leq M$ be nonzero submodules. Since $\text{id}_M \in \text{Hom}_R(M, M)$, we have $0 \neq A \leq \beta_A^M(M)$. As M is R -pr-first and $B \neq 0$, it follows that $0 \neq \beta_A^M(B) = \sum \{f(A) \mid f \in \text{Hom}_R(M, B)\} = A \bullet_M B$. Therefore M is $BJKN$ -prime by Proposition 1.3.

(2) $\implies$ (3): This is immediate, since M is $BJKN$ -prime means that every nonzero submodule of M cogenerates M , hence in particular every nonzero cyclic submodule cogenerates M .

(3) $\implies$ (4): Let $x, y \in M \setminus \{0\}$. Since Ry cogenerates M , there exists a monomorphism $\varphi : M \rightarrow (Ry)^Z$ for some set Z . Hence there exists $z \in Z$ such that, for the canonical projection $\pi_z : (Ry)^Z \rightarrow Ry$, we have $(\pi_z \circ \varphi)(x) \neq 0$.

(4) $\implies$ (1): Let σ be a preradical with $\sigma(M) \neq 0$ and let $0 \neq B \leq M$. Choose $0 \neq x \in \sigma(M)$ and $0 \neq y \in B$. By (4), there exists $f : M \rightarrow Ry$ such that $f(x) \neq 0$. Consider the induced map $f|_{\sigma(M)} : \sigma(M) \rightarrow \sigma(Ry)$; then $f(x) \in \sigma(Ry)$ and $f(x) \neq 0$, so $\sigma(Ry) \neq 0$. Since $Ry \leq B$, we have $\sigma(Ry) \leq \sigma(B)$, and therefore $\sigma(B) \neq 0$. Hence M is R -pr-first. $\square$

4 First modules relative to a subclass of R -pr.

Let $\mathcal{A} \subseteq R\text{-pr}$. Via the inclusion $\mathcal{A} \hookrightarrow R\text{-pr}$ and the R -pr-action of Example 2.4, we obtain an induced action of $\mathcal{A}$ on the submodule lattice of a module, as in Proposition 2.8. This leads to the following definition.

Definition 4.1. Let $M \in R\text{-Mod}$ and $\mathcal{A} \subseteq R\text{-pr}$. We say that:

- i) M is $\mathcal{A}$ -first if $0 \neq M$ and $\alpha(K) = 0$ imply $\alpha(M) = 0$, for all $K \in \mathcal{L}(M) \setminus \{0\}$, and for all $\alpha \in \mathcal{A}$.
- ii) M is $\mathcal{A}$ -fully first if $\alpha(K) \neq 0$, for all $K \in \mathcal{L}(M) \setminus \{0\}$ and for all $\alpha \in \mathcal{A}$.

Note that each t -radical σ can be written as $\sigma(R) \cdot _$. Moreover, for every preradical σ , the value $\sigma(R)$ is a two-sided ideal of R . Thus, there is a lattice isomorphism between $R\text{-trad}$ and $\text{Lat}(R)$ sending σ to $\sigma(R)$. The following proposition follows from Lemma 2.9.

Proposition 4.2. *For each nonzero $M \in R\text{-Mod}$ the following statements are equivalent.*

- (1) M is a prime module.
- (2) M is a $R\text{-trad}$ -first module.

Theorem 4.3. *For a ring R , the following statements are equivalent:*

- (1) R is a simple ring.
- (2) Every R -module is prime.

Proof. (1) $\implies$ (2): Let R be simple and let M be an R -module. If $M = 0$, then M is prime. Assume that $M \neq 0$. Let I be a two-sided ideal of R and let $N \leq M$ be such that $IN = 0$. Since R is simple, either $I = 0$ or $I = R$. If $I = R$, then $N = RN = IN = 0$. Hence, $I = 0$ or $N = 0$, and therefore M is prime.

(2) $\implies$ (1): Assume that every R -module is prime and let I be a proper two-sided ideal of R . Consider $M := R/I \oplus R$. Then M is prime and $I(R/I) = 0$, so $0 = IM = I(R/I \oplus R) = I(R/I) \oplus IR$. Thus $IR = 0$, and therefore $I = 0$. Hence R has no nonzero proper two-sided ideals, *i.e.*, R is simple. $\square$

Proposition 4.4. *Let $M \in R\text{-Mod}$ be a nonzero module. The following statements are equivalent.*

- (1) M is a $R\text{-pid}$ -first module.
- (2) For each pair of nonzero submodules N and K of M , there exists a nonzero morphism $f : N \rightarrow K$.

Proof. (1) $\implies$ (2): Let $N, K \in \mathcal{L}(M) \setminus \{0\}$. Then $tr_N \in R\text{-pid}$ and $0 \neq N \leq tr_N(M)$. Moreover, M is $R\text{-pid}$ -first and $0 \neq K \leq M$, so $tr_N(K) \neq 0$, which implies that there exists a nonzero morphism $f : N \rightarrow K$.

(2) $\implies$ (1): Let $N \in \mathcal{L}(M) \setminus \{0\}$ and $\sigma \in R\text{-pid}$ such that $\sigma(M) \neq 0$. Then there exists a nonzero morphism $f : \sigma(M) \rightarrow N$. Since $\sigma \in R\text{-pid}$, the following diagram commutes:

$$\begin{array}{ccc} \sigma(M) & \xrightarrow{f} & N \\ \uparrow & & \uparrow \\ \sigma(\sigma(M)) & \xrightarrow{f|} & \sigma(N) \end{array}$$

Thus, $0 \neq f(\sigma(M)) = f(\sigma(\sigma(M))) \leq \sigma(N)$. Therefore, M is $R\text{-pid}$ -first. $\square$

Definition 4.5. A nonzero left R -module M is called retractable if for every nonzero submodule $0 \neq N \leq M$, there exists a nonzero homomorphism $M \rightarrow N$ (equivalently, $\text{Hom}_R(M, N) \neq 0$).

Proposition 4.6. Let $M \in R\text{-Mod}$ be a nonzero module. If M is retractable and $\text{End}_R(M)$ is a prime ring, then M is a R -pid-first module.

Proof. Let $\sigma \in R\text{-pid}$ such that $\sigma(M) \neq 0$. Let N be a nonzero submodule of M . Define:

$$\begin{aligned} I &= \{f \in \text{End}_R(M) \mid f(M) \subseteq \sigma(M)\}, \\ X &= \{f \in \text{End}_R(M) \mid f(M) \subseteq N\}, \\ J &= \left\{ \sum_{i=1}^n s_i f_i \mid n \in \mathbb{N}, s_i \in \text{End}_R(M), f_i \in X \right\}. \end{aligned}$$

It is clear that I and J are ideals of $\text{End}_R(M)$. Since M is retractable, there exist nonzero morphisms $g : M \rightarrow N$ and $f : M \rightarrow \sigma(M)$. Let $\bar{g} : M \rightarrow M$ and $\bar{f} : M \rightarrow M$ denote the morphisms induced by the inclusions $N \hookrightarrow M$ and $\sigma(M) \hookrightarrow M$, respectively. Then $0 \neq \bar{f} \in I$ and $0 \neq \bar{g} \in X \subseteq J$, so I and J are nonzero ideals of $\text{End}_R(M)$. Since $\text{End}_R(M)$ is prime, it follows that $JI \neq 0$. Thus, there exist $h \in \text{End}_R(M)$, $l \in X$, and $k \in I$ such that $h l k \neq 0$, hence $l k \neq 0$. Moreover, $0 \neq \text{Im}(l k) = l(\text{Im}(k)) \subseteq l(\sigma(M))$. Since $l(\sigma(M)) \subseteq l(M) \subseteq N$, the restriction $\bar{l} = l|_{\sigma(M)} : \sigma(M) \rightarrow N$ is a nonzero morphism. Since $\sigma \in R\text{-pid}$, the following diagram commutes:

$$\begin{array}{ccc} \sigma(M) & \xrightarrow{\bar{l}} & N \\ \uparrow & & \uparrow \\ \sigma(M) = \sigma(\sigma(M)) & \xrightarrow{\bar{l}|} & \sigma(N) \end{array}$$

Thus, $0 \neq \bar{l}(\sigma(M)) = \bar{l}(\sigma(\sigma(M))) \leq \sigma(N)$. Therefore, M is R -pid-first. $\square$

Example 4.7. Let $R = \mathbb{Z}_{p^2} = \mathbb{Z}/(p^2\mathbb{Z})$. ${}_R R$ has only three submodules: R , $\langle p + p^2\mathbb{Z} \rangle$, and 0 . Therefore R is a R -pid-first module by Proposition 4.4. On the other hand, $\text{End}({}_R R) \cong R$, and R is not a prime ring. Hence the converse of Proposition 4.6 does not hold.

Proposition 4.8. For a nonzero ${}_R M$, the following statements are equivalent:

- (1) M is R -rad-first.
- (2) M is $BJKN$ -prime.

Proof. (1) $\implies$ (2): Let $N \in \mathcal{L}(M) \setminus \{0\}$. Then $\omega_0^N \in R\text{-rad}$ and $\omega_0^N(N) = 0$. Since M is R -rad-first and $0 \neq N \leq M$, then $\omega_0^N(M) = 0$, which implies that M is cogenerated by N . Therefore, M is $BJKN$ -prime.

(2) $\implies$ (1): Suppose M is $BJKN$ -prime. Then, by Proposition 3.10, M is R -**pr**-first, and by Lemma 2.9, M is R -rad-first. $\square$

5 The classes of $\mathcal{A}$ -first modules and the class of $\mathcal{A}$ -fully first modules

Let $\sigma \in R\text{-pr}$. $\mathbb{T}_\sigma = \{M \in R\text{-Mod} \mid \sigma(M) = M\}$ is the class of σ -pretorsion modules. $\mathbb{F}_\sigma = \{M \in R\text{-Mod} \mid \sigma(M) = 0\}$ is the class of σ -pretorsion free modules. For $\mathcal{A} \subseteq R\text{-pr}$, we define $\mathbb{T}_\mathcal{A} := \bigcap_{r \in \mathcal{A}} \mathbb{T}_r$ and $\mathbb{F}_\mathcal{A} := \bigcap_{r \in \mathcal{A}} \mathbb{F}_r$.

Now, we denote

- $\mathcal{P}_\mathcal{A}$ is the class of all $\mathcal{A}$ -fully first modules;
- $\mathbb{P}_\mathcal{A}$ is the class of all $\mathcal{A}$ -first modules and the zero module.

We write $\mathcal{P}_\sigma$ and $\mathbb{P}_\sigma$ instead of $\mathcal{P}_{\{\sigma\}}$ and $\mathbb{P}_{\{\sigma\}}$, respectively.

Remark 5.1. For any $\sigma, \beta \in R\text{-pr}$ and $\mathcal{A}, \mathcal{B} \subseteq R\text{-pr}$, it follows that:

- (1) $\mathcal{P}_\mathcal{A} = \bigcap_{r \in \mathcal{A}} \mathcal{P}_r$ and $\mathbb{P}_\mathcal{A} = \bigcap_{r \in \mathcal{A}} \mathbb{P}_r$.
- (2) $\mathbb{P}_\sigma = \mathcal{P}_\sigma \cup \mathbb{F}_\sigma$.
- (3) $\mathcal{P}_\mathcal{A} \subseteq \mathbb{P}_\mathcal{A}$ and $\mathbb{F}_\mathcal{A} \subseteq \mathbb{P}_\mathcal{A}$.
- (4) If $\sigma \preceq \beta$, then $\mathcal{P}_\sigma \subseteq \mathcal{P}_\beta$.
- (5) If $\mathcal{B} \subseteq \mathcal{A}$ then $\mathcal{P}_\mathcal{A} \subseteq \mathcal{P}_\mathcal{B}$ and $\mathbb{P}_\mathcal{A} \subseteq \mathbb{P}_\mathcal{B}$.

To see point (2) of Remark 5.1, note that $M \in \mathbb{P}_\sigma$ if and only if for each $0 \neq N \leq M$ we have $\sigma(N) = 0 \implies \sigma(M) = 0$. This is equivalent to $\sigma(M) = 0$ or $\sigma(N) \neq 0$ for each $0 \neq N \leq M$.

Remark 5.2. Let $M \in R\text{-Mod}$ and let $0 \neq N \in \mathcal{L}_{fi}(M)$. If $M \in \mathcal{P}_{\alpha_N^M}$ (equivalently, M is α_N^M -fully first), then $N \leq_e M$.

Indeed, $\alpha_N^M(M) = N \neq 0$. Hence, for every nonzero submodule $0 \neq K \leq M$, we have $\alpha_N^M(K) \neq 0$. Since always $\alpha_N^M(K) \leq N \cap K$, it follows that $N \cap K \neq 0$, i.e., N is essential in M .

In particular, Proposition 3.8 shows that if M is $R\text{-pr}$ -first, then M is diuniform, i.e., every nonzero fully invariant submodule of M is essential in M .

Example 5.3. $\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}$ -first modules in $\mathbb{Z}\text{-mod}$.

Let $p \in \mathbb{Z}$ be a prime number, and let $\sigma = \alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}$. For a nonzero $\mathbb{Z}$ -module M the following hold:

- (1) M is σ -fully first if and only if M is a p -group.

If $N \in \mathcal{P}_\sigma$ and $0 \neq x \in N$, then $\sigma(\langle x \rangle) \neq 0$, so there exists $0 \neq f \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, N)$. From this we find that, for some $r \in \mathbb{Z}$, $0 \neq f(1 + p\mathbb{Z}) = x^r$ is an element of order p of N . We

conclude that x has an order that is a multiple of p . If $o(x) = p^n s$ with $n \in \mathbb{N}$ and p does not divide s , then $o(x^{p^n}) = s$, so $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, \langle x^{p^n} \rangle) = 0$. Thus $\sigma(\langle x^{p^n} \rangle) = 0$, which implies that $\langle x^{p^n} \rangle = 0$. Therefore $o(x) = p^n$, so N is a p -group.

Conversely, if N is a p -group and $0 \neq K \leq N$, then there exists $x \in K \setminus \{0\}$, so there exists $n \in \mathbb{N} \setminus \{0\}$ such that $o(x) = p^n$. Moreover, $o(x^{p^{n-1}}) = p$, which implies $\langle x^{p^{n-1}} \rangle \cong \mathbb{Z}_p$ and thus $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, K) \neq 0$, from which it follows that $\sigma(K) \neq 0$. Consequently, $N \in \mathcal{P}_{\sigma}$ if and only if N is a p -group.

(2) M is σ -first if and only if M has no elements of order p or M is a p -group.

If $N \in \mathbb{Z}\text{-Mod}$ has an element a of order p , then $K = \langle a \rangle \cong \mathbb{Z}_p$, so $\sigma(N) \neq 0$. On the other hand, if $\sigma(N) \neq 0$, then there exists $0 \neq f \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, N)$, so $f(1 + p\mathbb{Z})$ is an element of order p of N . Therefore, we have $N \in \mathbb{F}_{\sigma}$ if and only if N does not have elements of order p .

Thus, by Remark 5.1(2) and Example 5.3(1), $N \in \mathbb{P}_{\sigma}$ if and only if N has no elements of order p or N is a p -group.

Example 5.4. $\{\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p} \mid p \text{ is prime}\}$ -first $\mathbb{Z}$ -modules.

Let $\mathcal{A} = \{\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p} \mid p \text{ is prime}\}$. The following statements hold:

- (1) There are no nonzero $\mathcal{A}$ -fully first $\mathbb{Z}$ -modules.
- (2) For a non-zero $\mathbb{Z}$ -module M we have the following statements that are equivalent:
 - (a) M is $\mathcal{A}$ -first.
 - (b) M has no nonzero elements of finite order, or M is a p -group for some prime p .

Explanation. (1) Let $M \in \mathcal{P}_{\mathcal{A}}$ and let $p, q \in \mathbb{Z}$ be two distinct primes. If $M \neq 0$, then $\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}(M) \neq 0$. Consequently, $\alpha_{\mathbb{Z}_q}^{\mathbb{Z}_q}(\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}(M)) \neq 0$, which is a contradiction since $\alpha_{\mathbb{Z}_q}^{\mathbb{Z}_q} \alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p} = \underline{0}$. Therefore, $M = 0$.

Note that $N \in \mathbb{F}_{\mathcal{A}}$ if and only if N has no nontrivial elements of finite order. Equivalently, N is a torsion-free abelian group.

(2) Let $M \in \mathbb{Z}\text{-Mod}$.

(a) $\implies$ (b): If $M \in \mathbb{P}_{\mathcal{A}}$ and $M \notin \mathbb{F}_{\mathcal{A}}$, then there exists a prime $p \in \mathbb{Z}$ such that $M \notin \mathbb{F}_{\sigma}$, where $\sigma = \alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}$. Consequently, we have $\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}(M) \neq 0$. This implies that for each $0 \neq N \leq M$, $\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}(N) \neq 0$. Hence, each nonzero submodule of M contains an element of order p , indicating that M is a p -group.

If $M \in \mathbb{F}_{\mathcal{A}}$ then $M \in \mathbb{P}_{\mathcal{A}}$ by Remark 5.1(3).

(b) $\implies$ (a): If $p \in \mathbb{Z}$ is a prime and M is a nonzero p -group, then, by Lagrange's Theorem, M has no elements of order q , for each prime $q \neq p$, so that $\alpha_{\mathbb{Z}_q}^{\mathbb{Z}_q}(M) = 0$. If N is a

nonzero submodule of M , then there exists $x \in N \setminus \{0\}$ with order power of p , so, by Cauchy's Theorem, there exists $y \in \mathbb{Z}x$ with order p , thus

$$f : \mathbb{Z}_p \longrightarrow N$$

$$a + p\mathbb{Z} \longmapsto ay$$

f is a well-defined nonzero $\mathbb{Z}$ -morphism, which implies $\alpha_{\mathbb{Z}_p}^{\mathbb{Z}_p}(N) \neq 0$. Then M is an $\mathcal{A}$ -first module.

Recall that the module C is called a cogenerator of $R\text{-Mod}$ if it cogenerates every left R -module.

Lemma 5.5. *For every (nonzero) left R -module N there exist a set I and simple left R -modules S_i ($i \in I$) such that N embeds into a direct product $\prod_{i \in I} E(S_i)$ of injective envelopes of simple modules.*

Proof. For each $0 \neq n \in N$, choose a maximal submodule $K_n \subsetneq Rn$ and set $S_n := Rn/K_n$ (simple). Let $\pi_n : Rn \rightarrow S_n$ be the natural epimorphism. Since $E(S_n)$ is injective, π_n extends to some R -morphism $f_n : N \rightarrow E(S_n)$. Then $f_n(n) \neq 0$. Hence the product map

$$\prod_{0 \neq n \in N} f_n : N \longrightarrow \prod_{0 \neq n \in N} E(S_n)$$

is injective. □

Proposition 5.6 ([2, Proposition 18.15]). *Let E be an injective left R -module. Then the following statements are equivalent:*

- (1) E is a cogenerator of $R\text{-Mod}$;
- (2) $\text{Hom}_R(T, E) \neq 0$ for every simple left R -module T ;
- (3) E cogenerates every simple left R -module.

Proposition 5.7. *For a ring R , we have $\mathcal{P}_{\text{soc}} = R\text{-Mod}$ if and only if $\mathbb{P}_{\text{soc}} = R\text{-Mod}$.*

Proof. By Remark 5.1(2), it is clear that if $\mathcal{P}_{\text{soc}} = R\text{-Mod}$, then $\mathbb{P}_{\text{soc}} = R\text{-Mod}$. Conversely, suppose that $\mathbb{P}_{\text{soc}} = R\text{-Mod}$, and let $N \in R\text{-Mod} \setminus \{0\}$. Set

$$C_0 = \bigoplus_{S \in R\text{-simp}} S.$$

Every simple left R -module is isomorphic to a direct summand of C_0 , and hence embeds into C_0 . Since C_0 embeds into its injective envelope $E(C_0)$, the module $E(C_0)$ cogenerates every simple left R -module. Therefore, by Proposition 5.6, $E(C_0)$ is a cogenerator of $R\text{-Mod}$.

Since $E(C_0)$ is a cogenerator, it cogenerates N . Equivalently (by the definition of “cogenerates”), there exist a set X and a monomorphism

$$N \longrightarrow E(C_0)^X.$$

Moreover, $\text{soc}(E(C_0)^X) \neq 0$ and, by hypothesis, $E(C_0)^X \in \mathbb{P}_{\text{soc}}$. Thus, by Remark 5.1(2), $E(C_0)^X \in \mathcal{P}_{\text{soc}}$.

Now, let $0 \neq L \leq N$. Identifying N with its image in $E(C_0)^X$, we have $0 \neq L \leq E(C_0)^X$. Since $E(C_0)^X \in \mathcal{P}_{\text{soc}}$, it follows that $\text{soc}(L) \neq 0$. Hence, $N \in \mathcal{P}_{\text{soc}}$.

Therefore, $\mathcal{P}_{\text{soc}} = R\text{-Mod}$. □

Remark 5.8. *If $N \in R\text{-Mod}$ is semisimple, then $N \in \mathbb{P}_{R\text{-lep}}$ if and only if N is semisimple homogeneous. Indeed, since α_S^S is left exact for every $S \in R\text{-simp}$, one can adapt the proof of Lemma 3.4 to $R\text{-lep}$.*

Proposition 5.9. $\mathcal{P}_{\text{soc}} = R\text{-Mod}$ if and only if R is a left semiartinian ring.

Proof. The ring R is left semiartinian if and only if $\text{soc}(M) \neq 0$ for all $0 \neq M \in R\text{-Mod}$. Therefore, $\mathcal{P}_{\text{soc}} = R\text{-Mod}$ if and only if R is left semiartinian. □

Remark 5.10. *If $\sigma, \tau \in R\text{-lep}$, then $\sigma(\tau(M)) = \sigma(M) \cap \tau(M) = \tau(\sigma(M))$ for all $M \in R\text{-Mod}$. In particular, $\sigma \circ \tau = \tau \circ \sigma$.*

Theorem 5.11. *For a ring R , the following statements are equivalent:*

(1) R is a left semiartinian left local ring,

(2) $\mathbb{P}_{R\text{-lep}} = R\text{-Mod}$.

Proof. (1) $\implies$ (2): Let $M \in R\text{-Mod}$, $0 \neq N \leq M$, and let $\sigma \in R\text{-lep}$ with $\sigma(M) \neq 0$. Since R is left semiartinian, we have $\text{soc}(\sigma(M)) \neq 0$ and $\text{soc}(N) \neq 0$. Moreover, both σ and soc are left exact preradicals, so by Remark 5.10 we have $\sigma(\text{soc}(M)) = \text{soc}(\sigma(M)) \neq 0$.

Since R is left local, all simple left R -modules are isomorphic; let S be such a simple module. From $\sigma(\text{soc}(M)) \neq 0$ we get $\sigma(S) = S$ for some simple submodule $S \leq \text{soc}(M)$, and thus S embeds in N . Applying σ , we obtain an embedding $S \cong \sigma(S) \hookrightarrow \sigma(N)$, so $\sigma(N) \neq 0$. Therefore $M \in \mathbb{P}_{R\text{-lep}}$.

(2) $\implies$ (1): By Remark 5.1(1) we have $\mathbb{P}_{\text{soc}} = R\text{-Mod}$. Hence, by Proposition 5.9 and Proposition 5.7, the ring R is left semiartinian. Moreover, by Remark 5.8, the ring R is left local. □

6 Rings for which every nonzero module is $R\text{-pr}$ -first.

Proposition 6.1. *Let $S \in R\text{-simp}$ such that $S \leq E(S)$. Then S is superfluous in $E(S)$.*

Proof. Let $N \leq E(S)$ such that $S + N = E(S)$. Since $S \leq E(S)$, we have $N \neq 0$. As S is essential in $E(S)$, it follows that $0 \neq S \cap N \leq S$. Since S is simple, we get $S \cap N = S$, hence $S \leq N$. Therefore $N = S + N = E(S)$, and thus S is superfluous in $E(S)$. $\square$

Remark 6.2. $\mathbb{P}_{R\text{-pr}} = R\text{-Mod}$ implies $\mathbb{P}_{R\text{-lep}} = R\text{-Mod}$. This is because, by Remark 5.1(5), we have $\mathbb{P}_{R\text{-pr}} \subseteq \mathbb{P}_{R\text{-lep}}$.

Remark 6.3. A semisimple ring R is said to be homogeneous if all simple left R -modules are isomorphic (equivalently, ${}_R R$ is a direct sum of copies of a single simple left R -module).

Lemma 6.4. *Let $0 \neq \alpha \in R\text{-pr}$. Then there exists $S \in R\text{-simp}$ such that $S \leq \alpha(E(S))$. In particular, $\alpha_S^{E(S)} \preceq \alpha$.*

Proof. Choose an R -module M with $\alpha(M) \neq 0$, and pick $0 \neq x \in \alpha(M)$. Let Rx be the cyclic submodule generated by x , choose a maximal submodule $K \subsetneq Rx$, and set $S := Rx/K$ (simple). Let $\pi : Rx \twoheadrightarrow S$ be the canonical epimorphism. Since $E(S)$ is injective, π extends to an R -morphism $f : M \rightarrow E(S)$. Then $f(x) \neq 0$, and since $x \in \alpha(M)$ we have $0 \neq f(x) = \alpha(f)(x) \in \alpha(E(S))$, so $\alpha(E(S)) \neq 0$.

As $S \leq E(S)$ is essential and simple, every nonzero submodule of $E(S)$ meets S nontrivially; hence $0 \neq \alpha(E(S)) \cap S = S$, i.e., $S \leq \alpha(E(S))$. $\square$

Theorem 6.5. *For a ring R , the following statements are equivalent:*

- (1) R is a left semiartinian left local V -ring.
- (2) $\mathbb{P}_{R\text{-pr}} = R\text{-Mod}$.
- (3) R is a semisimple homogeneous ring.

Proof. (1) $\implies$ (2): Let $0 \neq \alpha \in R\text{-pr}$. By Lemma 6.4, there exists $S \in R\text{-simp}$ such that $\alpha_S^{E(S)} \preceq \alpha$. Since R is a left local V -ring, we have

$$\text{soc} = \alpha_S^S = \alpha_S^{E(S)}.$$

As R is left semiartinian, for every nonzero $M \in R\text{-Mod}$ we have $\text{soc}(M) \neq 0$. Hence

$$0 \neq \text{soc}(M) = \alpha_S^{E(S)}(M) \leq \alpha(M).$$

Therefore every nonzero R -module is $R\text{-pr}$ -first, i.e., $\mathbb{P}_{R\text{-pr}} = R\text{-Mod}$.

(2) $\implies$ (1): Assume $\mathbb{P}_{R\text{-}\mathbf{pr}} = R\text{-Mod}$. By Remark 6.2, we have $\mathbb{P}_{R\text{-}lep} = R\text{-Mod}$, and then by Theorem 5.11 the ring R is left semiartinian and left local.

Let $S \in R\text{-simp}$. If $S \not\leq E(S)$, then S is superfluous in $E(S)$ by Proposition 6.1. In particular, $\text{rad}(E(S)) \neq 0$ while $\text{rad}(S) = 0$, so $E(S) \notin \mathbb{P}_{R\text{-}\mathbf{pr}}$, a contradiction. Hence $S = E(S)$ for all $S \in R\text{-simp}$, and therefore R is a V -ring.

(3) $\implies$ (1): If R is a semisimple homogeneous ring, then (up to isomorphism) R has a unique simple left module, hence R is left local. Moreover, since R is semisimple, every left R -module is injective; in particular, every simple left R -module is injective, so R is a V -ring. Clearly R is left semiartinian.

(1) $\implies$ (3): Assume that R is a left semiartinian left local V -ring. Then R contains a simple submodule $S \leq R$. Since R is a V -ring, S is injective, so S is a direct summand of R ; in particular, S is projective.

Suppose $\text{soc}(R) \subsetneq R$. Choose a maximal left ideal M of R such that $\text{soc}(R) \leq M$. Since R is left local, we have $R/M \cong S$, and because S is projective the short exact sequence

$$0 \rightarrow M \rightarrow R \rightarrow R/M \rightarrow 0$$

splits. Hence there exists a submodule $K \leq R$ with $R = M \oplus K$; in particular, $K \cap M = 0$ and $K \cong R/M \cong S$, so K is simple. But then $0 \neq K \leq \text{soc}(R) \leq M$, contradicting $K \cap M = 0$. Therefore $\text{soc}(R) = R$, so R is semisimple. Since R is left local, it follows that R is homogeneous. $\square$

Example 6.6. Let $F = \text{End}_{\mathbb{R}}(\mathbb{R}^{(\mathbb{N})})$, where $\mathbb{R}^{(\mathbb{N})}$ denotes the direct sum of countably many copies of $\mathbb{R}$. Let

$$I := \{f \in F \mid \text{rank}(f) < \infty\}$$

be the ideal of finite-rank endomorphisms and set $S := F/I$. It is well known that S is a simple ring (see, e.g., [16]); hence, by Theorem 4.3, every left S -module is prime.

Let $S\text{-}\mathbf{pr}$ denote the collection of all preradicals in $S\text{-Mod}$ (that is, $S\text{-}\mathbf{pr}$ is the analogue of $R\text{-}\mathbf{pr}$ with R replaced by S), and let $\mathbb{P}_{S\text{-}\mathbf{pr}}$ be the class of all $S\text{-}\mathbf{pr}$ -first left S -modules together with 0.

On the other hand, S is not a semisimple ring. Therefore $\mathbb{P}_{S\text{-}\mathbf{pr}} \neq S\text{-Mod}$ by Theorem 6.5 (applied with $R = S$, and with $R\text{-}\mathbf{pr}$ replaced by its analogue $S\text{-}\mathbf{pr}$ in $S\text{-Mod}$), and so there exists an S -module M which is not $S\text{-}\mathbf{pr}$ -first. Since every S -module is prime, this M is prime but not $S\text{-}\mathbf{pr}$ -first. In particular, a module can be prime without being BJKN-prime.

Definition 6.7. Let R be a ring. We say that R is a BKN ring if $\text{Hom}_R(M, N) \neq 0$ for any nonzero $M, N \in R\text{-Mod}$ (see [5, Proposition VI.2.3]).

Proposition 6.8. *Let R be a ring. If $\mathbb{P}_{R\text{-pr}} = R\text{-Mod}$, then R is a BKN ring.*

Proof. Assume $\mathbb{P}_{R\text{-pr}} = R\text{-Mod}$. By Theorem 6.5, R is a semisimple homogeneous ring. In particular, there is (up to isomorphism) a unique simple left R -module; fix one and denote it by S .

Let $M, N \in R\text{-Mod}$ be nonzero. Since R is semisimple, M and N are semisimple and hence $\text{soc}(M) = M \neq 0$ and $\text{soc}(N) = N \neq 0$; in particular, each of M and N contains a submodule isomorphic to S .

Choose a monomorphism $f : S \hookrightarrow N$. Also choose a monomorphism $i : S \hookrightarrow M$. Since R is semisimple, S is injective, so i splits and there exists $0 \neq g \in \text{Hom}_R(M, S)$.

Then $0 \neq f \circ g \in \text{Hom}_R(M, N)$, so $\text{Hom}_R(M, N) \neq 0$ for all nonzero $M, N \in R\text{-Mod}$. Hence R is a BKN ring. $\square$

Remark 6.9. *A ring R is called a left MAX ring if every nonzero left R -module has a maximal submodule.*

Example 6.10. *Consider the ring $\mathbb{Z}_{p^2}$ with p a prime. Then $\mathbb{Z}_{p^2}$ is a left local left semiartinian left MAX ring and it is a BKN ring. Moreover, $E(\mathbb{Z}_p) = \mathbb{Z}_{p^2}$, so $\mathbb{Z}_{p^2}$ is not a V-ring. Hence, by Theorem 6.5, we have $\mathbb{P}_{R\text{-pr}} \neq R\text{-Mod}$. In particular, the converse implication of Proposition 6.8 fails.*

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