

Inequalities, asymptotic expansions and complete monotonicity properties related to the gamma and digamma functions

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ABSTRACT

The quotient of two gamma functions has the following known asymptotic expansion:

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b} \sum_{k=0}^{\infty} \binom{a-b}{k} B_k^{(a-b+1)}(a) x^{-k} \quad (x \rightarrow \infty),$$

where $B_k^{(\nu)}(x)$ denotes the generalized Bernoulli polynomials. In this article, without using these generalized Bernoulli polynomials $B_k^{(\nu)}(x)$, we present two asymptotic expansions and a recurrence relation for determining the coefficients of each asymptotic expansion related to the quotient $\frac{\Gamma(x+a)}{\Gamma(x+b)}$ of two gamma functions. Let

$$\Omega_n := \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad n \in \mathbb{N}$$

be the volume of the unit ball in $\mathbb{R}^n$. We establish a sharp two-sided inequality for the quotient $\frac{\Omega_{n-1}}{\Omega_n}$. We also present a complete monotonicity property for the Nielsen's β -function, which extends a known result due to Matejíčka.

RESUMEN

El cociente de dos funciones gamma tiene la siguiente expansión asintótica conocida

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b} \sum_{k=0}^{\infty} \binom{a-b}{k} B_k^{(a-b+1)}(a) x^{-k} \quad (x \rightarrow \infty),$$

donde $B_k^{(\nu)}(x)$ denota los polinomios generalizados de Bernoulli. En este artículo, sin usar estos polinomios generalizados de Bernoulli $B_k^{(\nu)}(x)$, presentamos dos expansiones asintóticas y una relación de recurrencia para determinar los coeficientes de cada expansión asintótica relacionada con el cociente $\frac{\Gamma(x+a)}{\Gamma(x+b)}$ de dos funciones gamma. Sea

$$\Omega_n := \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad n \in \mathbb{N}$$

el volumen de la bola unitaria en $\mathbb{R}^n$. Establecemos una desigualdad óptima a ambos lados para el cociente $\frac{\Omega_{n-1}}{\Omega_n}$. También presentamos una propiedad de monotonía completa para la función β de Nielsen, que extiende un resultado conocido de Matejíčka.

Keywords and Phrases: Gamma function, Digamma (or psi) function, Asymptotic expansions, Inequalities, Completely monotonic functions.

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1 Introduction and preliminaries

The gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad (x > 0).$$

The logarithmic derivative of $\Gamma(x)$, given by

$$\psi(x) := \frac{d}{dx} \{\ln \Gamma(x)\} = \frac{\Gamma'(x)}{\Gamma(x)} \quad \text{or} \quad \ln \Gamma(x) = \int_1^x \psi(\tau) d\tau,$$

is called the psi (or digamma) function. The functions $\psi^{(k)}(x)$ ($k \in \mathbb{N} := \{1, 2, 3, \dots\}$) are called polygamma functions (see, for details, [15, 22, 39]).

The quotient of two gamma functions has the following known asymptotic expansion (see, for example, [38, p. 141]):

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b} \sum_{k=0}^{\infty} \binom{a-b}{k} B_k^{(a-b+1)}(a) x^{-k} \quad (x \rightarrow \infty), \quad (1.1)$$

where $B_k^{(\nu)}(x)$ denotes the generalized Bernoulli polynomial defined by

$$\left(\frac{t}{e^t - 1} \right)^\nu e^{xt} = \sum_{k=0}^{\infty} B_k^{(\nu)}(x) \frac{t^k}{k!} \quad (|t| < 2\pi).$$

Firstly, in the present paper, we give a recurrence relation for determining the coefficients of x^{-k} in the expansion (1.1) without the help of the generalized Bernoulli polynomials $B_k^{(\nu)}(x)$ (see Theorem 3.2). $B_n := B_n(0)$ are the Bernoulli numbers.

For $n \in \mathbb{N}$, we let

$$\Omega_n := \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}$$

be the volume of the unit ball in $\mathbb{R}^n$. Several interesting properties of the volume Ω_n of the unit ball in $\mathbb{R}^n$ have been reported by a number of researchers. For example, the monotonicity properties of Ω_n can be found in [4, 5, 8, 20, 21]. For various sharp inequalities involving Ω_n , the reader may consult [1, 2, 6, 9, 11, 13, 23, 26–28, 43, 44]. In particular, Alzer [1] proved that, for $n \in \mathbb{N}$,

$$\sqrt{\frac{n+a_2}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+b_2}{2\pi}}, \quad (1.2)$$

with the best possible constants a_2 and b_2 given by

$$a_2 = \frac{1}{2} \quad \text{and} \quad b_2 = \frac{\pi}{2} - 1 = 0.5707 \dots$$

The two-sided inequality (1.2) refines a result due to Borgwardt [9, p. 253], who proved (1.2) for $a_2 = 0$ and $b_2 = 1$. Mortici [27, Theorem 3] obtained the following inequality:

$$\sqrt{\frac{n + \frac{1}{2}}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n + \frac{1}{2}}{2\pi} + \frac{1}{16\pi n}} \quad (n \in \mathbb{N}), \quad (1.3)$$

which improves the right-hand side of (1.2). Ban and Chen [6, Theorem 3.1] improved (1.3) and obtained the following two-sided inequality:

$$\sqrt{\frac{n + \frac{1}{2}}{2\pi} + \frac{1}{16\pi(n + \vartheta_1)}} \leq \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n + \frac{1}{2}}{2\pi} + \frac{1}{16\pi(n + \vartheta_2)}} \quad (n \in \mathbb{N}), \quad (1.4)$$

with the best possible constants given by

$$\vartheta_1 = \frac{13 - 4\pi}{4\pi - 12} = 0.7656283\dots \quad \text{and} \quad \vartheta_2 = \frac{1}{2}.$$

Secondly, in the present paper, we establish a sharp two-sided inequality for $\frac{\Omega_{n-1}}{\Omega_n}$ (see Theorem 4.1), which improves the inequalities (1.4).

Gautschi [19] proved that the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ is greater than or equal to 1, that is,

$$\frac{2\Gamma(x)\Gamma(1/x)}{\Gamma(x) + \Gamma(1/x)} \geq 1 \quad \text{for } x > 0, \quad (1.5)$$

with equality if $x = 1$. Similarly, Alzer and Jameson [3] proved that

$$\frac{2\psi(x)\psi(1/x)}{\psi(x) + \psi(1/x)} \geq -\gamma \quad \text{for } x > 0, \quad (1.6)$$

with equality if $x = 1$. Here γ is the Euler-Mascheroni constant given by

$$\gamma = 0.5772156649\dots$$

The Nielsen's β -function $\beta(x)$ is defined in [37, p. 16] by

$$\beta(x) = \frac{1}{2} \left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) \right] = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} dt \quad (x > 0).$$

Many properties of the Nielsen's β -function $\beta(x)$ have been reported by several works (see, for example, [7, 29–35, 37, 45, 46]). Inspired by the inequalities in (1.5) and (1.6), Nantomah [34] conjectured that

$$\frac{2\beta(x)\beta(1/x)}{\beta(x) + \beta(1/x)} \leq \ln 2 \quad (x > 0), \quad (1.7)$$

with equality if $x = 1$. Subsequently, Matejíčka [25] and Nantomah [36] proved this conjecture (1.7) to be true. Matejíčka's proof used the following inequality (see [25, Lemma 1]):

$$2(\beta'(x))^2 - \beta''(x)\beta(x) > 0 \quad (x > 0). \quad (1.8)$$

By using (1.8), Nantomah [36] provided a simple proof of the inequality (1.7). In fact, in his above-cited work, Matejíčka [25, Eq. (12)] had pointed out that, in order to establish (1.8), it is sufficient to show that

$$-\frac{2}{x^2}\beta(x) - \frac{4}{x}\beta'(x) - \beta''(x) > 0 \quad (x > 0). \quad (1.9)$$

Thirdly, in the present paper, we prove that the function $M(x)$ given by

$$M(x) := -\frac{2}{x^2}\beta(x) - \frac{4}{x}\beta'(x) - \beta''(x) \quad (1.10)$$

is completely monotonic on $(0, \infty)$ (see Theorem 5.1). As a consequence of Theorem 5.1, the inequality (1.9) is obtained.

A function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for } x \in I \quad \text{and} \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \quad (1.11)$$

Dubourdieu [18, p. 98] pointed out that, if a non-constant function f is completely monotonic on $I = (a, \infty)$, then strict inequality holds true in (1.11) (see also [42] for a simpler proof of this result).

The numerical values, which we have given in this article, were computed by using the computer program *MAPLE* 11.

2 A set of lemmas

In this section, we present the following lemmas which will be needed in the derivations of our main results.

Lemma 2.1 (see [12, p. 520, Corollary 1]). *Let $m, n \in \mathbb{N}$. Then for $x > 0$,*

$$\begin{aligned} \sum_{j=1}^{2m} \left(1 - \frac{1}{2^{2j}}\right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}} &< (-1)^n \left(\psi^{(n-1)}(x+1) - \psi^{(n-1)}\left(x + \frac{1}{2}\right) \right) + \frac{(n-1)!}{2x^n} \\ &< \sum_{j=1}^{2m-1} \left(1 - \frac{1}{2^{2j}}\right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}}. \end{aligned} \quad (2.1)$$

In particular, we deduce from (2.1) that, for $x > 0$,

$$\begin{aligned} \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8} - \frac{31}{2048x^{10}} \\ < \psi(x+1) - \psi\left(x + \frac{1}{2}\right) < \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8}. \end{aligned} \quad (2.2)$$

We obtain from the right-hand side of inequality (2.2) that for $x \geq 1$,

$$-\psi(x+1) + \psi\left(x + \frac{1}{2}\right) + \frac{2}{4x+1} > -\left(\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8}\right) + \frac{2}{4x+1} \quad (2.3)$$

$$= \frac{256x^6 - 128x^5 - 32x^4 + 64x^3 + 16x^2 - 68x - 17}{2048x^8(4x+1)} > 0. \quad (2.4)$$

Lemma 2.2. For $x > 0$, let

$$V(x) = \frac{\frac{\Omega_{2x-1}}{\Omega_{2x}}}{\sqrt{\frac{2x+\frac{1}{2}}{2\pi}}} = \frac{\Gamma(x+1)}{\Gamma\left(x + \frac{1}{2}\right) \sqrt{x + \frac{1}{4}}}, \quad (2.5)$$

where

$$\Omega_x = \frac{\pi^{x/2}}{\Gamma\left(\frac{x}{2} + 1\right)}.$$

Then, for $x \geq 2$,

$$1 + \frac{1}{64x^2} - \frac{1}{128x^3} < V(x) < 1 + \frac{1}{64x^2} - \frac{1}{128x^3} + \frac{5}{8192x^4} + \frac{11}{8192x^5} + \frac{31}{524288x^6}. \quad (2.6)$$

Proof. Here we only prove the right-hand side of inequality (2.6), the proof of the left-hand side of inequality (2.6) is analogous. The right-hand side of inequality (2.6) is obtained by considering the function $U(x)$ defined, for $x \geq 2$, by

$$\begin{aligned} U(x) &= \ln \Gamma(x+1) - \ln \Gamma\left(x + \frac{1}{2}\right) - \frac{1}{2} \ln\left(x + \frac{1}{4}\right) \\ &\quad - \ln\left(1 + \frac{1}{64x^2} - \frac{1}{128x^3} + \frac{5}{8192x^4} + \frac{11}{8192x^5} + \frac{31}{524288x^6}\right). \end{aligned}$$

Differentiating $U(x)$ and applying the left-hand side of (2.2), we obtain

$$\begin{aligned} U'(x) &= \psi(x+1) - \psi\left(x + \frac{1}{2}\right) - \frac{2}{4x+1} + \frac{2(8192x^4 - 6144x^3 + 640x^2 + 1760x + 93)}{x(524288x^6 + 8192x^4 - 4096x^3 + 320x^2 + 704x + 31)} \\ &> \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8} - \frac{31}{2048x^{10}} - \frac{2}{4x+1} \\ &\quad + \frac{2(8192x^4 - 6144x^3 + 640x^2 + 1760x + 93)}{x(524288x^6 + 8192x^4 - 4096x^3 + 320x^2 + 704x + 31)} \\ &= \frac{P_9(x-2)}{2048x^{10}(4x+1)(524288x^6 + 8192x^4 - 4096x^3 + 320x^2 + 704x + 31)}, \end{aligned}$$

where

$$P_9(x) = 11058246099 + 58467842760x + 131458931639x^2 + 167085606012x^3 \\ + 133287700304x^4 + 69526128064x^5 + 23785330912x^6 + 5156054912x^7 + 643418368x^8 + 35237888x^9.$$

We then obtain $U'(x) > 0$ for $x \geq 2$. Hence, $U(x)$ is strictly increasing for $x \geq 2$, and we have

$$U(x) < \lim_{t \rightarrow \infty} U(t) = 0 \quad (x \geq 2).$$

This means that the inequality (2.6) holds true for $x \geq 2$. The proof of Lemma 2.2 is thus completed. $\square$

Remark 2.3. *Computer experiments suggest that the two-sided inequality (2.6) is valid for $x > 0$.*

3 Asymptotic expansions for the quotient of two gamma functions

Our first main result is stated as Theorem 3.1 below.

Theorem 3.1. *As $x \rightarrow \infty$, the following asymptotic expansion holds true:*

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b} \exp \left(\sum_{k=1}^{\infty} \lambda_k x^{-k} \right) \quad (x \rightarrow \infty), \quad (3.1)$$

where the coefficients λ_k are given by the following recurrence relation:

$$\lambda_1 = \frac{(a-b)(a+b-1)}{2} \quad \text{and} \\ \lambda_k = \frac{(-1)^{k+1}}{k} \left\{ \frac{a^{k+1} - b^{k+1} - (a-b)}{k+1} + \sum_{j=1}^{k-1} (-1)^j \lambda_j \binom{k}{k-j+1} \right\} \quad \text{for } k \geq 2. \quad (3.2)$$

Proof. Upon setting

$$P(x) = \ln \left(\frac{\Gamma(x+a)}{x^{a-b} \Gamma(x+b)} \right) \quad \text{and} \quad Q(x) = \sum_{k=1}^{\infty} \frac{\lambda_k}{x^k},$$

noting that

$$\Delta P = P(x+1) - P(x) = -\frac{a^2 - a - b^2 + b}{2x^2} + \dots, \quad \Delta Q = Q(x+1) - Q(x) = -\frac{\lambda_1}{x^2} + \dots,$$

we can let $\Delta P \sim \Delta Q$ as $x \rightarrow \infty$, where λ_k are real numbers to be determined.

By direct computation, we get

$$\begin{aligned}\Delta P &= \ln\left(1 + \frac{a}{x}\right) - \ln\left(1 + \frac{b}{x}\right) - (a-b)\ln\left(1 + \frac{1}{x}\right) \\ &= \sum_{k=2}^{\infty} \frac{(-1)^{k-1} [a^k - b^k - (a-b)]}{k} x^{-k}.\end{aligned}\quad (3.3)$$

Again, by direct computation, we get

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\lambda_k}{(x+1)^k} &= \sum_{k=1}^{\infty} \frac{\lambda_k}{x^k} \left(1 + \frac{1}{x}\right)^{-k} = \sum_{k=1}^{\infty} \frac{\lambda_k}{x^k} \sum_{j=0}^{\infty} \binom{-k}{j} \frac{1}{x^j} \\ &= \sum_{k=1}^{\infty} \frac{\lambda_k}{x^k} \sum_{j=0}^{\infty} (-1)^j \binom{k+j-1}{j} \frac{1}{x^j} = \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^k \lambda_j (-1)^{k-j} \binom{k-1}{k-j} \right\} x^{-k}.\end{aligned}$$

We thus find that

$$\Delta Q = \sum_{k=2}^{\infty} \left\{ \sum_{j=1}^k (-1)^{k-j} \lambda_j \binom{k-1}{k-j} - \lambda_k \right\} x^{-k}.\quad (3.4)$$

Now, upon equating the coefficients of x^{-k} on the right-hand sides of (3.3) and (3.4), we have

$$\frac{(-1)^{k-1} [a^k - b^k - (a-b)]}{k} = \sum_{j=1}^k (-1)^{k-j} \lambda_j \binom{k-1}{k-j} - \lambda_k = \sum_{j=1}^{k-1} (-1)^{k-j} \lambda_j \binom{k-1}{k-j} \quad (k \geq 2)$$

and

$$\begin{aligned}-\frac{a^k - b^k - (a-b)}{k} &= \sum_{j=1}^{k-1} (-1)^j \lambda_j \binom{k-1}{k-j} = \sum_{j=1}^{k-2} (-1)^j \lambda_j \binom{k-1}{k-j} + (-1)^{k-1} (k-1) \lambda_{k-1} \\ &\quad (k \geq 2),\end{aligned}$$

where (and elsewhere in this paper) an empty sum is understood to be zero.

For $k = 2$, we obtain

$$\lambda_1 = \frac{(a-b)(a+b-1)}{2}.$$

Also, for $k \geq 3$, we have

$$\lambda_{k-1} = \frac{(-1)^k}{k-1} \left\{ \frac{a^k - b^k - (a-b)}{k} + \sum_{j=1}^{k-2} (-1)^j \lambda_j \binom{k-1}{k-j} \right\},$$

which can be written precisely as (3.2). This evidently completes the proof of Theorem 3.1. $\square$

By making use of (3.2), we list the first few coefficients λ_k as follows:

$$\begin{aligned}\lambda_1 &= \frac{1}{2} [(a-b)(a+b-1)], \\ \lambda_2 &= -\frac{1}{12} [(a-b)(2a^2-3a+2ab-3b+2b^2+1)], \\ \lambda_3 &= \frac{1}{12} [(a-b)(a+b-1)(a^2-a-b+b^2)], \\ \lambda_4 &= -\frac{1}{120} [(a-b)(6a^4-15a^3+6a^3b+10a^2-15a^2b+6a^2b^2+10ab-15ab^2+6ab^3+10b^2 \\ &\quad -15b^3+6b^4-1)], \\ \lambda_5 &= \frac{1}{60} [(a-b)(a+b-1)(2a^4-4a^3+a^2-2a^2b+2a^2b^2+2ab-2ab^2+a+b+b^2-4b^3+2b^4)],\end{aligned}$$

and so on. In particular, by setting $(a, b) = (\frac{1}{2}, 1)$ in (3.1), we get

$$\frac{\Gamma(x + \frac{1}{2})}{\Gamma(x+1)} \sim \frac{1}{\sqrt{x}} \exp\left(-\frac{1}{8x} + \frac{1}{192x^3} - \frac{1}{640x^5} + \dots\right) \quad (x \rightarrow \infty). \quad (3.5)$$

Theorem 3.2 below gives a recurrence relation for determining the coefficients of x^{-k} in the expansion (1.1) without the help of the generalized Bernoulli polynomials $B_k^{(\nu)}(x)$. The proof of Theorem 3.2 is based instead upon the Bell polynomials. The Bell polynomials, named in honor of Eric Temple Bell (1883–1960), are a triangular array of polynomials given by (see, for example, Comtet [16, pp. 133–134], Cvijović [17] and Masjed-Jamei *et al.* [24])

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \dots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \dots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

where the sum is taken over all non-negative integers $j_1, j_2, j_3, \dots, j_{n-k+1}$ such that

$$j_1 + j_2 + \dots + j_{n-k+1} = k \quad \text{and} \quad j_1 + 2j_2 + \dots + (n-k+1)j_{n-k+1} = n.$$

The following sum:

$$B_n(x_1, x_2, x_3, \dots, x_n) = \sum_{k=1}^n B_{n,k}(x_1, x_2, x_3, \dots, x_{n-k+1})$$

is sometimes called the n th complete Bell polynomial. These complete Bell polynomials satisfy the following identity:

$$B_n(x_1, x_2, x_3, \dots, x_n) = \begin{vmatrix} x_1 & \binom{n-1}{1}x_2 & \binom{n-1}{2}x_3 & \binom{n-1}{3}x_4 & \cdots & \cdots & x_n \\ -1 & x_1 & \binom{n-2}{1}x_2 & \binom{n-2}{2}x_3 & \cdots & \cdots & x_{n-1} \\ 0 & -1 & x_1 & \binom{n-3}{1}x_2 & \cdots & \cdots & x_{n-2} \\ 0 & 0 & -1 & x_1 & \cdots & \cdots & x_{n-3} \\ 0 & 0 & 0 & -1 & \cdots & \cdots & x_{n-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & x_1 \end{vmatrix}. \quad (3.6)$$

In order to contrast them with the complete Bell polynomials, the polynomials $B_{n,k}$ defined above are sometimes called *partial* Bell polynomials. The complete Bell polynomials appear in the exponential of a formal power series:

$$\exp\left(\sum_{n=1}^{\infty} \frac{x_n}{n!} u^n\right) = \sum_{n=0}^{\infty} \frac{B_n(x_1, \dots, x_n)}{n!} u^n. \quad (3.7)$$

The Bell polynomials are quite general in nature and they have been found in many applications in combinatorics. In his monograph, Comtet [16] devoted much to a thorough presentation of the Bell polynomials in the chapter on identities and expansions. For more results, see the work by Charalambides [10], which is cited in [17, Section 1], and the work of Riordan [40], which is used in [24].

Theorem 3.2. *As $x \rightarrow \infty$, the following asymptotic expansion holds true:*

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b} \sum_{j=0}^{\infty} \mu_j x^{-j} \quad (3.8)$$

with the coefficients μ_j given by the recursion formula:

$$\mu_0 = 1, \quad \mu_j = \sum_{\ell=0}^{j-1} \frac{j-\ell}{j} \lambda_{j-\ell} \mu_\ell \quad (j \in \mathbb{N}), \quad (3.9)$$

where the coefficients λ_k are given in (3.2).

Proof. It follows from (3.1) and (3.7) that

$$\frac{\Gamma(x+a)}{x^{a-b} \Gamma(x+b)} \sim \exp\left(\sum_{j=1}^{\infty} \frac{j! \lambda_j}{j!} \frac{1}{x^j}\right) \sim \sum_{j=0}^{\infty} \frac{\mu_j}{x^j} \quad (x \rightarrow \infty),$$

where

$$\mu_j = \frac{B_j(1! \lambda_1, 2! \lambda_2, \dots, j! \lambda_j)}{j!}. \quad (3.10)$$

Bulò *et al.* [41, Theorem 1] proved that the complete Bell polynomials can be expressed by using the following recurrence relation:

$$B_n(x_1, x_2, \dots, x_n) = \begin{cases} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x_{n-\ell} B_\ell(x_1, x_2, \dots, x_\ell) & (n > 0) \\ 1 & (\text{otherwise}). \end{cases} \quad (3.11)$$

Therefore, by employing (3.11), the formula (3.10) can be rewritten as follows:

$$\begin{aligned} \mu_0 &= 1 \quad \text{and} \\ \mu_j &= \frac{1}{j!} \sum_{\ell=0}^{j-1} \binom{j-1}{\ell} (j-\ell)! \lambda_{j-\ell} B_\ell(1! \lambda_1, 2! \lambda_2, \dots, \ell! \lambda_\ell) \\ &= \frac{1}{j!} \sum_{\ell=0}^{j-1} \binom{j-1}{\ell} (j-\ell)! \lambda_{j-\ell} \ell! \mu_\ell = \sum_{\ell=0}^{j-1} \frac{j-\ell}{j} \lambda_{j-\ell} \mu_\ell, \quad j \in \mathbb{N}. \end{aligned}$$

This completes the proof of Theorem 3.2. $\square$

Remark 3.3. We can calculate the coefficients μ_j in (3.8) by using the formulas (3.10) and (3.6). We thus find that

$$\mu_n = \frac{1}{n!} \begin{vmatrix} 1! \lambda_1 & \binom{n-1}{1} 2! \lambda_2 & \binom{n-1}{2} 3! \lambda_3 & \binom{n-1}{3} 4! \lambda_4 & \cdots & \cdots & n! \lambda_n \\ -1 & 1! \lambda_1 & \binom{n-2}{1} 2! \lambda_2 & \binom{n-2}{2} 3! \lambda_3 & \cdots & \cdots & (n-1)! \lambda_{n-1} \\ 0 & -1 & 1! \lambda_1 & \binom{n-3}{1} 2! \lambda_2 & \cdots & \cdots & (n-2)! \lambda_{n-2} \\ 0 & 0 & -1 & 1! \lambda_1 & \cdots & \cdots & (n-3)! \lambda_{n-3} \\ 0 & 0 & 0 & -1 & \cdots & \cdots & (n-4)! \lambda_{n-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1! \lambda_1 \end{vmatrix}. \quad (3.12)$$

The representation using a recursive algorithm for the coefficients μ_j in (3.9) is more practical for numerical evaluation than the expression in (3.12).

By using (3.9), we give the first few coefficients μ_j as follows:

$$\begin{aligned} \mu_0 &= 1, \\ \mu_1 &= \frac{(a-b)(a+b-1)}{2}, \end{aligned}$$

$$\begin{aligned}\mu_2 &= \frac{(a-b)(a+b-1)(3a^2-7a+6ab-5b+3b^2+2)}{24}, \\ \mu_3 &= \frac{(a-b)(a-b-1)(a-b-2)(a+b-1)(a^2-3a+2ab-b+b^2)}{48}, \\ \mu_4 &= \frac{1}{5760}(a-b)(a-b-1)(a-b-2)(a-b-3)(15a^4-90a^3+60a^3b-210a^2b \\ &\quad + 90a^2b^2+125a^2+110ab-18a-150ab^2+60ab^3-30b^3+15b^4+18b-8+5b^2), \\ \mu_5 &= \frac{1}{11520}(a-b)(a-b-1)(a-b-2)(a-b-3)(a-b-4)(a+b-1)(3a^4-22a^3 \\ &\quad + 12a^3b+33a^2-46a^2b+18a^2b^2+6ba-26ab^2+10a+12ab^3-7b^2+6b-2b^3+3b^4).\end{aligned}$$

In particular, upon setting $(a, b) = (\frac{1}{2}, 1)$ in (3.8), we have

$$\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)} \sim \frac{1}{\sqrt{x}} \left(1 - \frac{1}{8x} + \frac{1}{128x^2} + \frac{5}{1024x^3} - \frac{21}{32768x^4} - \frac{399}{262144x^5} + \cdots \right) \quad (3.13)$$

as $x \rightarrow \infty$.

Remark 3.4. The Wallis quotient, which is defined for every integer $n \in \mathbb{N}$ by (see [14])

$$\frac{(2n-1)!!}{(2n)!!} = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)},$$

has many applications in pure and applied mathematics. Herein we employ the special double factorial notation as follows:

$$\begin{aligned}(2n)!! &= 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!, \\ (2n-1)!! &= 1 \cdot 3 \cdot 5 \cdots (2n-1) = \pi^{-1/2} 2^n \Gamma(n+\frac{1}{2}), \\ 0!! &= 1, \quad (-1)!! = 1.\end{aligned}$$

Upon replacement of x by n in (3.5) and (3.13), we get the following asymptotic expansions for the Wallis quotient $\frac{(2n-1)!!}{(2n)!!}$:

$$\frac{(2n-1)!!}{(2n)!!} \sim \frac{1}{\sqrt{\pi n}} \exp \left(-\frac{1}{8n} + \frac{1}{192n^3} - \frac{1}{640n^5} + \cdots \right)$$

and

$$\frac{(2n-1)!!}{(2n)!!} \sim \frac{1}{\sqrt{\pi n}} \left(1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} - \frac{21}{32768n^4} - \frac{399}{262144n^5} + \cdots \right)$$

as $n \rightarrow \infty$.

4 A sharp two-sided inequality for $\frac{\Omega_{n-1}}{\Omega_n}$

It is easy to see that

$$\frac{\Omega_{n-1}}{\Omega_n} = \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}. \quad (4.1)$$

The asymptotic expansion of the quotient in (4.1) follows immediately from the well-known expansion (1.1). Hence we obtain the expansion

$$\begin{aligned} \frac{\Omega_{n-1}}{\Omega_n} &= \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)} \sim \sqrt{\frac{n}{2\pi}} \sum_{k=0}^{\infty} \binom{1/2}{k} B_k^{(3/2)}(1) \left(\frac{2}{n}\right)^k \\ &= \sqrt{\frac{n}{2\pi}} \left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{5}{128n^3} - \frac{21}{2048n^4} + \frac{399}{8192n^5} \right. \\ &\quad \left. + \frac{869}{65536n^6} - \frac{39325}{262144n^7} - \frac{334477}{8388608n^8} + \dots \right) \end{aligned} \quad (4.2)$$

as $n \rightarrow \infty$. Formula (4.2) was given by Mortici [28, Theorem 8].

By using the computer program *MAPLE* 11, we find that

$$\frac{\Omega_{n-1}}{\Omega_n} \sim \sqrt{\frac{n + \frac{1}{2}}{2\pi}} \left(1 + \frac{\frac{1}{16}}{n^2 + n + \frac{27}{32}} \right) \quad (n \rightarrow \infty).$$

This fact motivated us to establish Theorem 4.1.

Theorem 4.1. *For $n \in \mathbb{N}$, the following two-sided inequality holds true:*

$$\sqrt{\frac{n + \frac{1}{2}}{2\pi}} \left(1 + \frac{1}{16n^2 + 16n + \alpha} \right) < \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n + \frac{1}{2}}{2\pi}} \left(1 + \frac{1}{16n^2 + 16n + \beta} \right), \quad (4.3)$$

where the constants α and β , given by

$$\alpha = \frac{27}{2} \quad \text{and} \quad \beta = \frac{99 - 32\sqrt{3\pi}}{\sqrt{3\pi} - 3} = 10.86931539 \dots$$

are the best possible.

Proof. If we write (4.3) as follows:

$$\alpha > x_n \geq \beta,$$

where

$$x_n = \frac{1}{\frac{\Omega_{n-1}}{\Omega_n} - 1} - (16n^2 + 16n) = \frac{1}{\frac{\Gamma(\frac{n}{2}+1)}{\sqrt{\frac{n}{2} + \frac{1}{4}}\Gamma(\frac{n}{2} + \frac{1}{2})} - 1} - (16n^2 + 16n),$$

we find that

$$x_1 = \frac{99 - 32\sqrt{3}\pi}{\sqrt{3}\pi - 3}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} \left\{ \frac{1}{\frac{\Gamma(\frac{n}{2}+1)}{\sqrt{\frac{n}{2} + \frac{1}{4}}\Gamma(\frac{n}{2} + \frac{1}{2})} - 1} - (16n^2 + 16n) \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{1}{\frac{1}{16n^2} - \frac{1}{16n^3} + \frac{5}{512n^4} + \frac{11}{256n^5} + \frac{31}{8192n^6} + O(n^{-7})} - (16n^2 + 16n) \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{27}{2} + O\left(\frac{1}{n^2}\right) \right\} = \frac{27}{2}. \end{aligned}$$

This limit is obtained by using the asymptotic expansion (4.2) and some computation.

In order to prove Theorem 4.1, it suffices to show that the sequence $\{x_n\}$ is strictly increasing for $n \geq 1$. The monotonicity property of $\{x_n\}$ is obtained by considering the function $f(x)$ defined by

$$f(x) = \frac{1}{\frac{\Omega_{2x-1}}{\Omega_{2x}} - 1} - (16(2x)^2 + 16(2x)) = \frac{1}{V(x) - 1} - (16(2x)^2 + 16(2x)),$$

where $V(x)$ is given as in (2.5).

Now, differentiating $f(x)$ and applying (2.6) as well as the right-hand side of (2.3), we obtain

$$\begin{aligned} (V(x) - 1)^2 f'(x) &= -V'(x) - (128x + 32)(V(x) - 1)^2 \\ &= V(x) \left[-\psi(x+1) + \psi\left(x + \frac{1}{2}\right) + \frac{2}{4x+1} \right] - (128x + 32)(V(x) - 1)^2 \\ &> \left(1 + \frac{1}{64x^2} - \frac{1}{128x^3} \right) \left(\frac{256x^6 - 128x^5 - 32x^4 + 64x^3 + 16x^2 - 68x - 17}{2048x^8(4x+1)} \right) \\ &\quad - (128x + 32) \left(\frac{1}{64x^2} - \frac{1}{128x^3} + \frac{5}{8192x^4} + \frac{11}{8192x^5} + \frac{31}{524288x^6} \right)^2 \\ &= \frac{P_6(x-5)}{8589934592x^{12}(4x+1)}, \end{aligned}$$

where

$$P_6(x) = 79340860039 + 271938974680x + 222261507696x^2 + 81737763840x^3 \\ + 15550863360x^4 + 1501003776x^5 + 58392576x^6.$$

We thus obtain $f'(x) > 0$ for $x \geq 5$. Hence, $f(x)$ is strictly increasing for $x \geq 5$. We then find that the sequence $\{x_n\} = \{f(n/2)\}$ is strictly increasing for $n \geq 10$.

Direct computation would yield

$$x_1 = 10.8693\dots, \quad x_2 = 12.0729\dots, \quad x_3 = 12.6295\dots, \quad x_4 = 12.9222\dots, \\ x_5 = 13.0919\dots, \quad x_6 = 13.1978\dots, \quad x_7 = 13.2680\dots, \quad x_8 = 13.3171\dots, \\ x_9 = 13.3517\dots, \quad x_{10} = 13.3773\dots$$

Hence, the sequence $\{x_n\}$ is strictly increasing for $n \geq 1$. The proof of Theorem 4.1 is thus completed. $\square$

5 Completely monotonic properties and inequalities for Nielsen's beta function

In this section, we establish the following result.

Theorem 5.1. *The function $M(x)$, defined by (1.10), is completely monotonic on $(0, \infty)$.*

Proof. By using the following representations:

$$\beta(x) = \int_0^\infty \frac{e^{-xt}}{1 + e^{-t}} dt \quad (5.1)$$

and

$$\frac{m!}{x^{m+1}} = \int_0^\infty t^m e^{-xt} dt \quad (5.2)$$

for $x > 0$ and $m \in \mathbb{N}_0$ as well as the convolution theorem for the Laplace transform, we have

$$M(x) = -2 \int_0^\infty t e^{-xt} dt \int_0^\infty \omega(t) e^{-xt} dt + 4 \int_0^\infty e^{-xt} dt \int_0^\infty t \omega(t) e^{-xt} dt - \int_0^\infty t^2 \omega(t) e^{-xt} dt \\ = -2 \int_0^\infty \left(\int_0^t \omega(u)(t-u) du \right) e^{-xt} dt + 4 \int_0^\infty \left(\int_0^t u \omega(u) du \right) e^{-xt} dt - \int_0^\infty t^2 \omega(t) e^{-xt} dt \\ = \int_0^\infty \lambda(t) e^{-xt} dt, \quad (5.3)$$

where

$$\begin{aligned}\lambda(t) &= -2 \int_0^t \omega(u)(t-u) \, du + 4 \int_0^t u\omega(u) \, du - t^2\omega(t) \\ &= -2t \int_0^t \omega(u) \, du + 6 \int_0^t u\omega(u) \, du - t^2\omega(t)\end{aligned}$$

and

$$\omega(t) = 1 - \frac{1}{e^t + 1} \quad (t > 0).$$

Differentiation of $\lambda(t)$ yields

$$\lambda'(t) = -2 \int_0^t \omega(u) \, du + 2t\omega(t) - t^2\omega'(t)$$

and

$$\lambda''(t) = -t^2\omega''(t) = \frac{t^2 e^t (e^t - 1)}{(e^t + 1)^3} > 0 \quad (t > 0).$$

Hence, $\lambda'(t)$ is strictly increasing for $t > 0$, and we have

$$\lambda'(t) > \lambda'(0) = 0 \quad (t > 0).$$

Therefore, $\lambda(t)$ is strictly increasing for $t > 0$, which implies that

$$\lambda(t) > \lambda(0) = 0 \quad t > 0.$$

Finally, from (5.3), we obtain

$$(-1)^n M^{(n)}(x) = \int_0^\infty \lambda(t) t^n e^{-xt} \, dt > 0 \quad (x > 0; n \in \mathbb{N}_0),$$

which completes the proof of Theorem 5.1. $\square$

One reviewer suggested that the function M seems to be completely monotonic of order 2, meaning that $x^2 M(x)$ is completely monotonic.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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