

Characterizations of kites as graceful graphs II

MIROSLAV HAVIAR^{1,2,3,✉} JAKUB IVAN¹ 

¹ *Department of Mathematics, Faculty of
Natural Sciences, Matej Bel University,
Banská Bystrica, Slovakia.*

miroslav.haviar@umb.sk✉

kuboo.ivan@gmail.com

² *Mathematical Institute, Slovak Academy of
Sciences, Bratislava, Slovakia.*

³ *Department of Mathematics and Applied
Mathematics, University of Johannesburg,
Auckland Park, South Africa.*

ABSTRACT

Vertex and edge labellings have played a significant role in the study of graphs since the foundation of the theory of graph labellings in the 1960s. This paper is focused on the specific case of vertex labellings called graceful labellings. It is a companion to a paper published in this journal in 2024, in which we characterized “kites” formed by the cycles known to be graceful. It was left there as a possible further research direction to characterize the remaining kites formed by the non-graceful cycles. The present paper completes this task. The kites formed by the non-graceful cycles are characterized among all simple graphs via Sheppard’s labelling sequences, and via labelling relations and graph chessboards. While the labelling relations are rather closely related to the labelling sequences, the graph chessboards provide a nice visualization of the gracefulness of the graphs they represent.

RESUMEN

Los etiquetados de vértices y ejes han jugado un rol significativo en el estudio de grafos desde la fundación de la teoría de etiquetados de grafos en los 1960s. Este artículo se enfoca en el caso específico de los etiquetados de vértices conocidos como etiquetados agradados. Este es un artículo complementario a uno publicado en esta revista en 2024, en el cual caracterizamos “cometas” formadas por los ciclos que se sabe que son agradados. Ahí se dejó como una posible línea de investigación futura caracterizar las cometas faltantes formadas por los ciclos no agradados. El presente artículo completa esta tarea. Las cometas formadas por ciclos no agradados se caracterizan entre todos los grafos simples vía las secuencias de etiquetados de Sheppard y vía relaciones de etiquetado y grafos tablero de ajedrez. Mientras las relaciones de etiquetado están relacionadas con cierta cercanía a las secuencias de etiquetados, los grafos tablero de ajedrez entregan una visualización agradable de lo agraciado de los grafos que representan.

Keywords and Phrases: Graph, graceful labelling, graph chessboard, labelling sequence, labelling relation.

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1 Introduction

The development in the study of vertex and edge labellings of graphs has been in the last decades covered by a rather well-known electronic book *A Dynamic Survey of Graph Labeling* updated each year by Gallian [1]. It contains a chapter on graceful labellings and in total over four thousand bibliographic sources on vertex and edge labellings including hundreds of papers on studying various graceful graphs.

It is appropriate to mention that the interest on graph labellings had been much accelerated by the famous and still open *Graceful Tree Conjecture*. This was first stated in 1965 by Rosa in his dissertation [8], and then became known from his paper [9] in 1967. Its original version says that every tree can be labelled by a so-called β -labelling, which soon was popularised by Golomb [2] as a *graceful labelling*.

By a *graceful labelling* of a simple graph of size m is meant a labelling of its vertices by numbers from the set $\{0, 1, \dots, m\}$ in the way that no two vertices share the same label and the edge labels cover all values of the set $\{1, 2, \dots, m\}$, provided the edge label is the absolute value of the difference of the vertex labels. By a *graceful graph* is meant a graph, for which there exists a graceful labelling.

Yet we want to emphasize that the study of graceful graphs, which has naturally been focused in part on various classes of trees, has equally dealt with various cyclic simple graphs. This is well witnessed by [1]. The study of graceful labellings of special cyclic graphs called “kites” was presented in our recent paper [5], and the present paper is a natural continuation and completion of that paper. So we want to emphasize that our results, both in [5] and here, belong to the broader program of classifying graceful graphs, without implying a direct bearing on the Graceful Tree Conjecture.

In Section 2 we present fundamental concepts and facts needed later. We also recall basic tools used throughout this paper, which are Sheppard’s labelling sequences introduced in [10] as well as simple chessboards and labelling relations introduced and studied by the first author of this paper and Ivaška in [4]. These tools were already presented in this journal in [3].

In our companion paper [5] we introduced and studied an infinite family of graceful graphs, called *kites*. The kites $K_n(G)$ are graphs where a path P_n is joined with an arbitrary graph G . Then we say that the graph G “forms” the kite. We focused in [5] on “fan” kites and “lantern” kites, and on the kites $K_n(C_m)$ formed by the cycles C_m known to be graceful, which correspond to arbitrary non-negative integer values $m \equiv 0, 3 \pmod{4}$, where $m \geq 4$. The kites $K_n(C_m)$ formed by the non-graceful cycles C_m , which correspond to the values $m \equiv 1, 2 \pmod{4}$ for $m \geq 5$ were not studied in [5] as we expected an increased complexity of this task. The increased complexity of the task is demonstrated in this paper, where we complete the study of the kites $K_n(C_m)$ for

the values $m \equiv 1, 2 \pmod{4}$ with $m \geq 5$. We characterize them via the simple chessboards, the labelling sequences and the labelling relations. The descriptions are more complicated than the ones in [5]: the value $m \equiv 1 \pmod{4}$ requires two subcases, while the value $m \equiv 2 \pmod{4}$ requires even five subcases.

2 Preliminaries

We recall from [9] and [4] basic terms and facts regarding the vertex labellings, labelling sequences, labelling relations and simple chessboards of graphs. In this paper we consider only finite *simple graphs*, i.e. finite unoriented graphs without loops and multiple edges.

Definition 2.1 ([4,9]). *A vertex labelling f of a simple graph $G = (V, E)$ is a one-to-one mapping of its vertex set V into the set of non-negative integers assigning so-called vertex labels to the vertices of G .*

By a *labelling* in this paper we will mean a vertex labelling. The *induced label* of an edge uv in a labelling f is the number $|f(u) - f(v)|$, where $f(u)$, $f(v)$ are the labels of the vertices u , v respectively.

Rosa in his dissertation [8] in 1965 introduced several types of labellings: the main labellings were α -, β -, σ - and ρ -labellings, which soon appeared also in his paper [9]. In this paper we entirely focus on Rosa's β -labellings, renamed as *graceful labellings* by Golomb [2] in 1972.

Definition 2.2 ([4,9]). *A graceful labelling (or β -labelling) of a graph $G = (V, E)$ of size m is a vertex labelling with the following properties:*

- (1) $f(V_G) \subseteq \{0, 1, \dots, m\}$, and
- (2) $f(E_G) = \{1, 2, \dots, m\}$.

Labelled graphs can be represented by simple chessboards. This idea was introduced and developed in [4] in 2015. A *simple chessboard* of a labelled graph of order n is a table with n rows and n columns corresponding to the vertices, where each edge uv of the graph corresponds to the dots with coordinates $[u, v]$ and $[v, u]$ (a dot with coordinates $[i, j]$ is meant to be the dot in the i -th row and the j -th column of the table). The r -th *diagonal* in the table is the set of all cells of the chessboard with coordinates $[i, j]$ where $i - j = r$ and $i \geq j$. The *main diagonal* (or the 0-th diagonal) has no dots since we consider only simple graphs, the other diagonals are said to be *associate*. The simple chessboard is symmetric with respect to the main diagonal and so we focus only on the diagonals “below” the main diagonal.

A simple chessboard is said to be *graceful* if there is exactly one dot on each of its associate diagonals. Obviously, this corresponds to the fact that the graph labelling is graceful. The simple

chessboards provide a helpful visualization of labelled graphs because they reflect useful properties of labelled graphs such as degrees of vertices or gracefulness. We can see the simple chessboard of the labelled kite $K_4(C_5)$ formed by the cycle C_5 in Figure 1. The graph has size 8 and its labelling is graceful as there is exactly one dot on each associate diagonal.

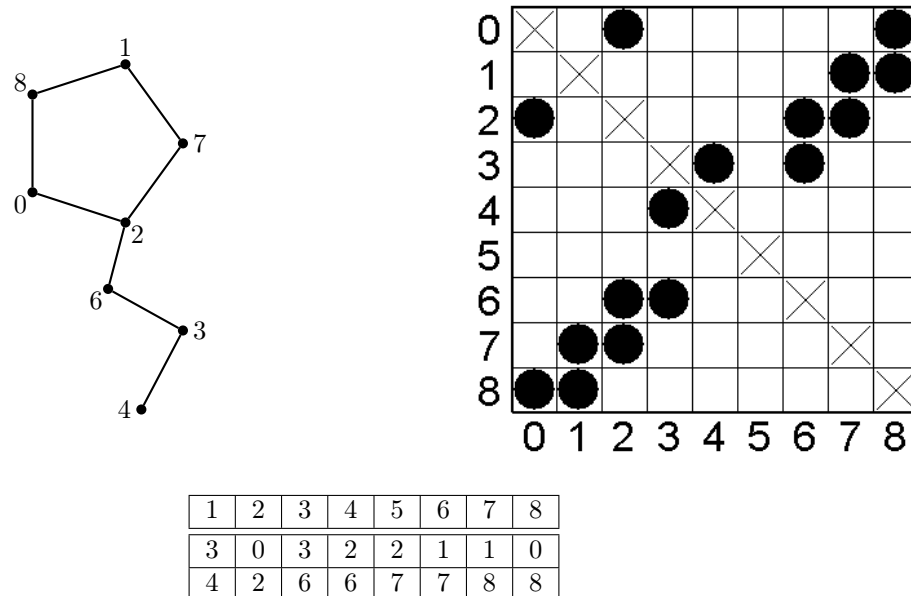


Figure 1: Representations of the kite $K_4(C_5)$

Each gracefully labelled graph can also be represented by its labelling sequence and its labelling relation. Sheppard in [10] introduced the concept of a labelling sequence and proved that there is a unique correspondence between gracefully labelled graphs and the labelling sequences.

Definition 2.3 ([4, 10]). For a positive integer m , a labelling sequence (henceforth we sometimes write briefly *LS*) is the sequence of non-negative integers $(j_1, j_2, \dots, j_m)$, denoted (j_i) , where

$$0 \leq j_i \leq m - i \quad \text{for all } i \in \{1, 2, \dots, m\}. \quad (\text{LS})$$

Labelling sequences can be understood as encoding graceful labellings of graphs. More precisely, the connection is described in the following theorem.

Theorem 2.4 ([4, 10]). There exists a one-to-one correspondence between graphs of size m having a graceful labelling f and between labelling sequences (j_i) of m terms. The correspondence is given by

$$j_i = \min\{f(u), f(v)\}, \quad i \in \{1, 2, \dots, m\},$$

where u, v are the end-vertices of the edge labelled i .

The concept of a labelling relation is closely related to the concept of the labelling sequence. It was introduced and studied in [4].

Definition 2.5 ([4]). *Let $L = (j_1, j_2, \dots, j_m)$ be a labelling sequence. Then the relation $A(L) = \{[j_i, j_i + i], i \in \{1, 2, \dots, m\}\}$ is called a labelling relation assigned to the labelling sequence L .*

The concept of a *labelling table* serves to visualize a labelling relation and also a labelling sequence of a graceful graph. The labelling table consists of a header and two rows. (See the last table in Figure 1.) In the header there are numbers $\{1, 2, \dots, m\}$ and in the first row there is the labelling sequence (j_i) of the graceful graph. The numbers in the second row are sums of the numbers from the header and from the first row. Hence the pairs of numbers from the first and second rows in each column of the labelling table are the elements of the labelling relation. They also represent the edges of the given graceful graph.

In the labelling table of the graceful kite $K_4(C_5)$ in Figure 1, in the first row is seen the LS $(3, 0, 3, 2, 2, 1, 1, 0)$ of the graph. The labelling relation is formed by the pairs of numbers from the first and second rows of each column and represents the edges of the labelled kite $K_4(C_5)$ (the last edge of the path, for example, is represented by the pair $[3, 4]$).

3 Kites formed by the cycles C_m for $m \equiv 1 \pmod{4}$

In this and the subsequent section we study two classes of the kites $K_n(C_m)$ formed by non-graceful cycles, that is, the cycles C_m for $m \equiv 1 \pmod{4}$ and $m \equiv 2 \pmod{4}$, where $m \geq 5$. In our companion paper [5] this study was left as a further research direction as only the kites formed by the graceful cycles C_m for $m \equiv 0, 3 \pmod{4}$ with $m \geq 4$ were investigated there. In these two sections we fully complete this unfulfilled task of [5] by characterizing the kites $K_n(C_m)$ for $m \equiv 1, 2 \pmod{4}$, where $m \geq 5$ and $n \geq \lfloor \frac{m}{2} \rfloor$, via Sheppard's labelling sequences, the simple chessboards and the labelling relations. These results are based on a recent thesis [6] of the second author supervised by the first author.

In this section we present our characterizations of the kites $K_n(C_m)$ formed by the cycles C_m for $m \equiv 1 \pmod{4}$, where $m \geq 5$ and $n \geq \lfloor \frac{m}{2} \rfloor$. We differentiate two subclasses of the kites $K_n(C_m)$ according to the parity of the order of their path P_n : n is even and n is odd. We note that for n even and $m \equiv 1 \pmod{4}$ the size $s = m + n - 1$ of the kite $K_n(C_m)$ is an even number.

Example 3.1. (a) *We consider the gracefully labelled kite $K_{12}(C_9)$ presented in Figure 2. Its simple chessboard can be divided into three parts: the first part is a “stairway” from column 0 to column 8, the second part is the “single dot” with coordinates $[4, 0]$ and the third part is a “stairway” starting with two horizontal dots in the row 11.*

The $LS(j_i)$ is $(9, 9, 8, 0, 8, 7, 7, 6, 6, 5, 5, 4, 4, 3, 3, 2, 2, 1, 1, 0)$. We can divide it into four parts. The first part $(9, 9, 8)$ is the LS of one part of the path P_{12} , then $(8, 7, 7, 6, 6, 5, 5, 4)$ represents the remaining part of the path P_{12} . The third part of the LS is the number $0 = j_i$ for $i = \lfloor \frac{m}{2} \rfloor = \lfloor \frac{9}{2} \rfloor = 4$ representing one edge of the cycle C_9 and the last part $(4, 3, 3, 2, 2, 1, 1, 0)$ represents the rest of the cycle C_9 .

- (b) We also consider the gracefully labelled kite $K_8(C_{13})$ presented in Figure 3. We leave it to the reader to see how its simple chessboard is divided into three parts and its labelling sequence is divided into four parts.

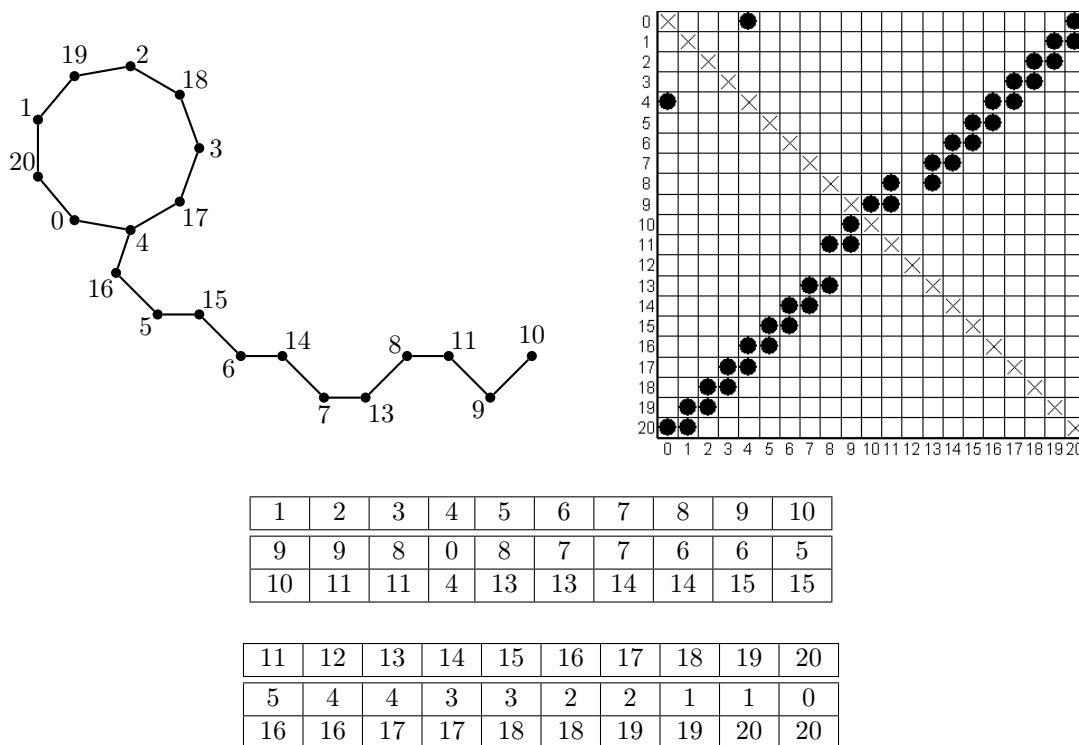
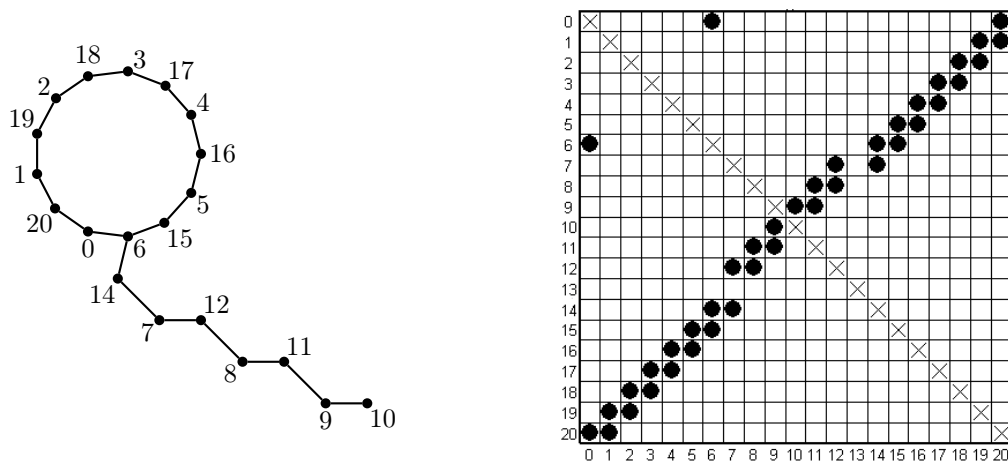


Figure 2: Representations of the kite $K_{12}(C_9)$

In the next definition the notation “C1K” stands for “kite formed by cycle C_m for $m \equiv 1 \pmod{4}$ ”.

Definition 3.2. Let $s = m + n - 1$ for $m \equiv 1 \pmod{4}$, where $m \geq 5$, $n \geq \lfloor \frac{m}{2} \rfloor$ and n is even. By an “even C1K-graph chessboard” of size s we mean the simple chessboard such as in Figures 2 and 3, which has three parts: the first part is a “stairway” from column 0 to column $\frac{s}{2} - \frac{m-1}{4}$, the second part is the “single” dot with coordinates $[\lfloor \frac{m}{2} \rfloor, 0]$ and the third part is again a “stairway” starting with two horizontal dots in the row $\frac{s}{2} + \frac{m-1}{4} - 1$.

The following theorem characterizes the kites $K_n(C_m)$ for $m \equiv 1 \pmod{4}$ and n even, where $m \geq 5$ and $n \geq \lfloor \frac{m}{2} \rfloor$, via their graph chessboards, labelling sequences and labelling relations.

Figure 3: Representations of the kite $K_8(C_{13})$

Theorem 3.3. Let G be a graph of size $s = m + n - 1$ for $m \equiv 1 \pmod{4}$, where $m \geq 5$, $n \geq \lfloor \frac{m}{2} \rfloor$ and n is even. Then the following are equivalent:

- (1) G is the kite $K_n(C_m)$.
- (2) G has a graceful labelling and an even $C1K$ -graph chessboard of size s .
- (3) There exists a labelling sequence $L = (j_1, j_2, \dots, j_s)$ of G such that

$$j_i = \begin{cases} \left\lfloor \frac{s-i}{2} \right\rfloor, & \text{if } i < \left\lfloor \frac{m}{2} \right\rfloor; \\ 0, & \text{if } i = \left\lfloor \frac{m}{2} \right\rfloor; \\ \left\lfloor \frac{s-i+1}{2} \right\rfloor, & \text{if } i > \left\lfloor \frac{m}{2} \right\rfloor. \end{cases} \quad (LSC1K\text{-even})$$

- (4) There exists a labelling sequence L of G with the labelling relation

$$A(L) = \left\{ \left[\left\lfloor \frac{s-i}{2} \right\rfloor, \left\lfloor \frac{s-i}{2} \right\rfloor + i \right] \mid i < \left\lfloor \frac{m}{2} \right\rfloor \right\} \cup \left\{ \left[0, \left\lfloor \frac{m}{2} \right\rfloor \right] \right\} \cup \left\{ \left[\left\lfloor \frac{s-i+1}{2} \right\rfloor, \left\lfloor \frac{s-i+1}{2} \right\rfloor + i \right] \mid i > \left\lfloor \frac{m}{2} \right\rfloor \right\}.$$

Proof. (1) $\implies$ (2): Let G be the kite $K_n(C_m)$. We label it such that we start labelling the cycle C_m : we label the vertex joining the cycle with the path (the “joining vertex”) by number $\lfloor \frac{m}{2} \rfloor$ and we follow in the clockwise direction with numbers $0, s, 1, s-1, \dots, s - \lfloor \frac{m}{2} \rfloor + 1$. Next we label the path P_n . We start from number $\lfloor \frac{m}{2} \rfloor$ and label every second vertex with numbers $\lfloor \frac{m}{2} \rfloor + 1, \lfloor \frac{m}{2} \rfloor + 2, \dots$. In the remaining part of the path we start to label by $s - \lfloor \frac{m}{2} \rfloor$ the vertex adjacent to the joining vertex and we then label every second vertex by numbers $s - (\lfloor \frac{m}{2} \rfloor + 1), s - (\lfloor \frac{m}{2} \rfloor + 2), \dots$ but we skip the number $\frac{s}{2} + \frac{m-1}{4}$. (We notice that this is then the number of the “empty row” in the corresponding graph chessboards, see Figures 2 and 3; indeed, we say in Definition 3.2 that the “stairway” in the third part of the “even C1K-graph chessboard” starts with two horizontal dots in the row $\frac{s}{2} + \frac{m-1}{4} - 1$, hence the “stairway” is interrupted in the “empty row” $\frac{s}{2} + \frac{m-1}{4}$.) To show the gracefulness of this labelling, we use the mentioned corresponding simple chessboards of G (see again Figures 2 and 3). The cycle C_m of the kite is in the corresponding chessboard represented by the “stairway” in rows s to $s - \lfloor \frac{m}{2} \rfloor + 1$ and by the “single” dot with coordinates $[\lfloor \frac{m}{2} \rfloor, 0]$. The path P_n is in the chessboard represented by part of the previous “stairway” starting in the row $s - \lfloor \frac{m}{2} \rfloor$, then the row $\frac{s}{2} + \frac{m-1}{4}$ is without any dots (the “empty row”) and the path P_n continues in the chessboard with a “stairway” starting with two horizontal dots in the row $\frac{s}{2} + \frac{m-1}{4} - 1$. So we obtain an even C1K-graph chessboard of our labelling, thus the labelling is graceful.

(2) $\implies$ (3): Assume we have a graceful labelling of the graph G with an even C1K-graph chessboard. We are going to show that the corresponding LS satisfies the formula (*LSC1K-even*). The “stairway” in the chessboard starting with two horizontal dots in the row $\frac{s}{2} + \frac{m-1}{4} - 1$ represents in the corresponding LS the numbers $\lfloor \frac{s-i}{2} \rfloor$ for $i < \lfloor \frac{m}{2} \rfloor$. The “separated single” dot in the chessboard represents the number 0 in the LS. The remaining “stairway” in the chessboard from column 0 to column $\frac{s}{2} - \frac{m-1}{4}$ represents the numbers $\lfloor \frac{s-i+1}{2} \rfloor$ for $i > \lfloor \frac{m}{2} \rfloor$. So the formula (*LSC1K-even*) holds.

(3) $\implies$ (4): Assume now that (3) holds, *i.e.* there exists a LS $L = (j_1, j_2, \dots, j_s)$ of G that satisfies the formula (*LSC1K-even*). We show that this LS has the labelling relation $A(L)$ as described in (4). The numbers $\lfloor \frac{s-i}{2} \rfloor$ in the LS obviously correspond in $A(L)$ to the pairs $[\lfloor \frac{s-i}{2} \rfloor, \lfloor \frac{s-i}{2} \rfloor + i]$ for $i < \lfloor \frac{m}{2} \rfloor$. The number 0 clearly corresponds to the pair $[0, \lfloor \frac{m}{2} \rfloor]$. The numbers $\lfloor \frac{s-i+1}{2} \rfloor$ in the LS correspond in $A(L)$ to the pairs $[\lfloor \frac{s-i+1}{2} \rfloor, \lfloor \frac{s-i+1}{2} \rfloor + i]$ for $i > \lfloor \frac{m}{2} \rfloor$.

(4) $\implies$ (1): Let (4) hold, *i.e.* there exists a LS L of G with the labelling relation $A(L)$ as described in (4). We are going to show that the pairs in $A(L)$ represent the edges of the kite $K_n(C_m)$. The pairs $[\lfloor \frac{s-i}{2} \rfloor, \lfloor \frac{s-i}{2} \rfloor + i]$ for $i < \lfloor \frac{m}{2} \rfloor$ represent in the graph G the ending part of a path. The pairs $[\lfloor \frac{s-i+1}{2} \rfloor, \lfloor \frac{s-i+1}{2} \rfloor + i]$ for $i = \lfloor \frac{m}{2} \rfloor + 1, \dots, s - m + 1$ represent the remaining edges of the path P_n . The pairs $[\lfloor \frac{s-i+1}{2} \rfloor, \lfloor \frac{s-i+1}{2} \rfloor + i]$ for $i = s - m + 2, \dots, s$

represent in G the edges of the cycle C_m starting from the joining vertex in the counter-clockwise direction and skipping the last edge of the cycle. This last edge of the cycle C_m is represented in $A(L)$ by the pair $[0, \lfloor \frac{m}{2} \rfloor]$. Hence the graph G is the kite $K_n(C_m)$. $\square$

The second subcase differs in the parity of the order of the path P_n , which is now an odd number. We illustrate it in an example and define the concept of an *odd C1K-graph chessboard*. We present a theorem characterizing the kites $K_n(C_m)$ formed by the cycles C_m for $m \equiv 1 \pmod{4}$ with odd n , where $m \geq 5$ and $n \geq \lfloor \frac{m}{2} \rfloor$.

Example 3.4. In Figure 4 we see the representations of the kite $K_{13}(C_9)$. The simple chessboard can be divided into three parts: the first part is a “stairway” from column 0 to column 8, the second part is the “single” dot with coordinates $[4, 0]$ and the third part is again a “stairway” starting with two vertical dots in the column 10.

The LS is $(11, 10, 10, 0, 8, 8, 7, 7, 6, 6, 5, 5, 4, 4, 3, 3, 2, 2, 1, 1, 0)$. We can divide it into four parts: the first part $(11, 10, 10)$ is the LS of a part of the path P_{13} , then $(8, 8, 7, 7, 6, 6, 5, 5, 4)$ is the LS of the remaining part of the path P_{13} . The number 0 and the last part of the LS , which is $(4, 3, 3, 2, 2, 1, 1, 0)$, together represent the cycle C_9 .

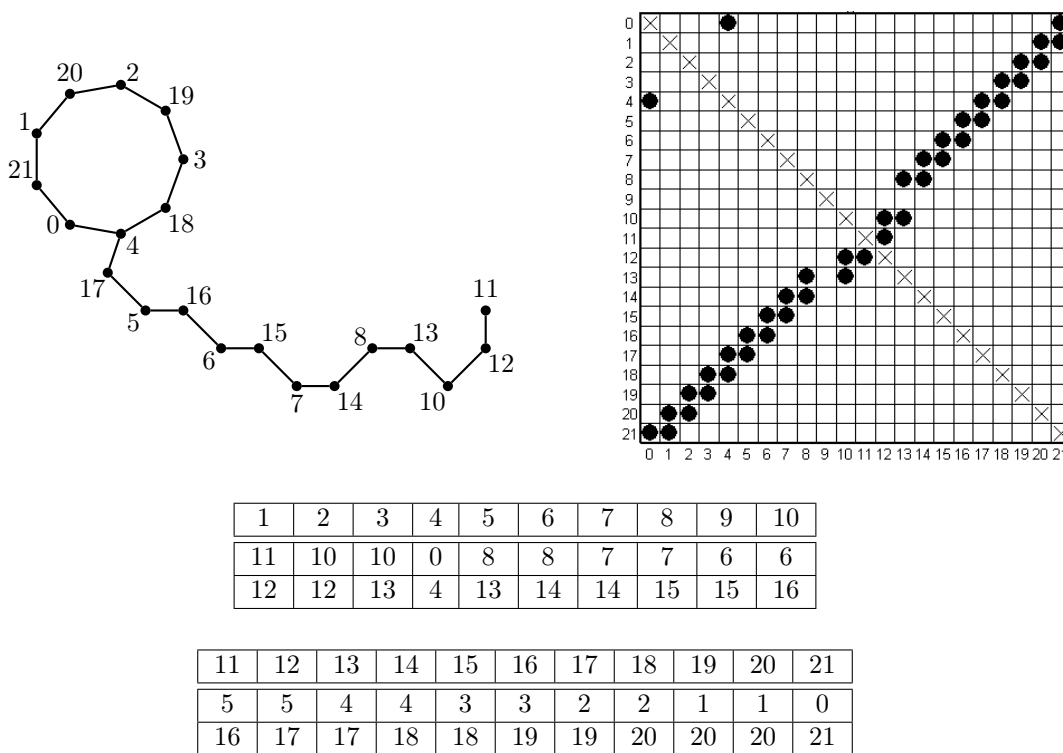


Figure 4: Representations of the kite $K_{13}(C_9)$

Definition 3.5. Let $s = m + n - 1$ for $m \equiv 1 \pmod{4}$, where $m \geq 5$, $n \geq \lfloor \frac{m}{2} \rfloor$ and n is odd. By an “odd C1K-graph chessboard” of size s we mean the simple chessboard (such as in Figure 4), which has three parts: the first part is a “stairway” from column 0 to column $\lfloor \frac{s}{2} \rfloor - \frac{m-1}{4}$, the second part is the “single” dot with coordinates $[\lfloor \frac{m}{2} \rfloor, 0]$ and the third part is again a “stairway” starting with two vertical dots in the column $\lfloor \frac{s}{2} \rfloor - \frac{m-1}{4} + 2$.

Theorem 3.6. Let G be a graph of size $s = m + n - 1$ for $m \equiv 1 \pmod{4}$, where $m \geq 5$, $n \geq \lfloor \frac{m}{2} \rfloor$ and n is odd. Then the following are equivalent:

- (1) G is the kite $K_n(C_m)$.
- (2) G has a graceful labelling and an odd C1K-graph chessboard of size s .
- (3) There exists a labelling sequence $L = (j_1, j_2, \dots, j_s)$ of G such that

$$j_i = \begin{cases} \left\lceil \frac{s-i+1}{2} \right\rceil, & \text{if } i < \lfloor \frac{m}{2} \rfloor; \\ 0, & \text{if } i = \lfloor \frac{m}{2} \rfloor; \\ \left\lfloor \frac{s-i}{2} \right\rfloor, & \text{if } i > \lfloor \frac{m}{2} \rfloor. \end{cases} \quad (LSC1K\text{-odd})$$

- (4) There exists a labelling sequence L of G with the labelling relation

$$A(L) = \left\{ \left[\left\lceil \frac{s-i+1}{2} \right\rceil, \left\lceil \frac{s-i+1}{2} \right\rceil + i \right] \mid i < \lfloor \frac{m}{2} \rfloor \right\} \cup \left\{ \left[0, \left\lfloor \frac{m}{2} \right\rfloor \right] \right\} \cup \left\{ \left[\left\lfloor \frac{s-i}{2} \right\rfloor, \left\lfloor \frac{s-i}{2} \right\rfloor + i \right] \mid i > \lfloor \frac{m}{2} \rfloor \right\}.$$

Proof. (1) $\implies$ (2): Let G be the kite $K_n(C_m)$. We label it such that we start labelling the cycle C_m : we label the vertex joining the cycle with the path (the “joining vertex”) by number $\lfloor \frac{m}{2} \rfloor$ and we follow in the clockwise direction by numbers $0, s, 1, s-1, \dots, s - \lfloor \frac{m}{2} \rfloor + 1$. Next we label the path P_n . We start from number $\lfloor \frac{m}{2} \rfloor$ and label every second vertex by numbers $\lfloor \frac{m}{2} \rfloor + 1, \lfloor \frac{m}{2} \rfloor + 2, \dots$, but we skip the number $\lceil \frac{1}{2}(s - \lfloor \frac{m}{2} \rfloor) \rceil$ (the “empty column”). In the remaining part of the path we start with the vertex next to the joining vertex and we label every second vertex by numbers $s - \lfloor \frac{m}{2} \rfloor, s - (\lfloor \frac{m}{2} \rfloor + 1), \dots$. To show the gracefulness of this labelling, we use the corresponding simple chessboard of G (see Figure 4). The cycle C_m of the kite is in the corresponding chessboard represented by the “stairway” in columns 0 to $\lfloor \frac{m}{2} \rfloor$ and by the “single” dot with coordinates $[\lfloor \frac{m}{2} \rfloor, 0]$. The path P_n is in the chessboard represented by part of the “stairway” starting with the upper dot in column $\lfloor \frac{m}{2} \rfloor$, but the column $\lceil \frac{1}{2}(s - \lfloor \frac{m}{2} \rfloor) \rceil$ is without any dots (the “empty column”). So we obtain an odd C1K-graph chessboard, which means that our labelling is graceful, since every diagonal of the simple chessboard has exactly one dot.

- (2) $\implies$ (3): Assume we have a graceful labelling of the graph G with an odd C1K-graph chessboard. We show that the corresponding LS satisfies the formula (*LSC1K-odd*). The “stairway” in the direction from the main diagonal represents in the corresponding LS numbers $\lceil \frac{s-i+1}{2} \rceil$ for $i < \lfloor \frac{m}{2} \rfloor$. The “single” dot represents in the LS the number 0. The remaining “stairway” represents numbers $\lceil \frac{s-i}{2} \rceil$ for $i > \lfloor \frac{m}{2} \rfloor$. So the formula (*LSC1K-odd*) holds.
- (3) $\implies$ (4): Assume now that (3) holds, *i.e.* there exists a LS $L = (j_1, j_2, \dots, j_s)$ that satisfies the formula (*LSC1K-odd*). We show that this LS has the labelling relation $A(L)$ as described in (4). The numbers $\lceil \frac{s-i+1}{2} \rceil$ obviously correspond in $A(L)$ to the pairs $[\lceil \frac{s-i+1}{2} \rceil, \lceil \frac{s-i+1}{2} \rceil + i]$ for $i < \lfloor \frac{m}{2} \rfloor$. The number 0 clearly corresponds to the pair $[0, \lfloor \frac{m}{2} \rfloor]$. The numbers $\lceil \frac{s-i}{2} \rceil$ correspond in $A(L)$ to the pairs $[\lceil \frac{s-i}{2} \rceil, \lceil \frac{s-i}{2} \rceil + i]$ for $i > \lfloor \frac{m}{2} \rfloor$. The first coordinates of these pairs are members of the LS and the second coordinates are the result of adding the number i to them.
- (4) $\implies$ (1): Let (4) hold, *i.e.* there exists a LS L with the labelling relation $A(L)$ as described in (4). We show that the pairs in $A(L)$ represent the edges of the kite $K_n(C_m)$. The pairs $[\lceil \frac{s-i+1}{2} \rceil, \lceil \frac{s-i+1}{2} \rceil + i]$ for $i < \lfloor \frac{m}{2} \rfloor$ represent the ending part of a path. The pairs $[\lceil \frac{s-i}{2} \rceil, \lceil \frac{s-i}{2} \rceil + i]$ for $i = \lfloor \frac{m}{2} \rfloor + 1, \dots, s - m + 1$ represent the remaining edges of the path P_n . The pairs $[\lceil \frac{s-i}{2} \rceil, \lceil \frac{s-i}{2} \rceil + i]$ for $i = s - m + 2, \dots, s$ represent the edges of the cycle C_m starting from the joining vertex in the counter-clockwise direction and skipping the last edge of the cycle ending in the joining vertex. This last edge of the cycle is represented by the pair $[0, \lfloor \frac{m}{2} \rfloor]$ in $A(L)$. Hence G is the kite $K_n(C_m)$. $\square$

4 Kites formed by the cycles C_m for $m \equiv 2 \pmod{4}$

In this section we characterize the kites $K_n(C_m)$ formed by the cycles C_m for $m \equiv 2 \pmod{4}$, where $m \geq 6$. We differentiate five subclasses according to the length of the path P_n and so we talk about labelling sequences and the corresponding *C2K-graph chessboards* of five types. We firstly present in a theorem and prove the result for $n = 3$, where the formulas for the characterizing labelling sequence, which we associate with *type 4*, only have four parts. Then we gradually associate the other characterizing labelling sequences, where the formulas for these labelling sequences have more than four parts, and their corresponding C2K-graph chessboards, with *types 3, 0, 1, 2* in cases $n \equiv 3 \pmod{4}$, $n \equiv 0 \pmod{4}$, $n \equiv 1 \pmod{4}$ and $n \equiv 2 \pmod{4}$, respectively, where $n \geq 2$ and $n \neq 3$. In all cases we present explicitly the labelling sequence of a given type. The characterizations of the kites of all four types 0 – 3 via the labelling sequences, labelling relations and the simple chessboards is presented in a single theorem covering all four subcases.

Example 4.1. We consider the kite $K_3(C_{10})$ presented in Figure 5. Its labelling sequence $(8, 4, 4, 3, 3, 0, 3, 2, 2, 1, 1, 0)$ consists of four parts: the first is number 8, which represents an edge of the path P_3 , then $(4, 4, 3)$ is the LS of a part of the cycle C_{10} . The number 3 after that represents the remaining edge of the path. The number 0 and the last part of the LS, which is $(3, 2, 2, 1, 1, 0)$ together represent the rest of the cycle C_{10} .

The corresponding C2K-graph chessboard can accordingly be divided into four parts: the first part is a “stairway” from column 0 to column 3, the second part is the “single” dot with coordinates $[6, 0]$ and the third part is again a “stairway” starting with two vertical dots in the column 3 and ending in the column 4. The fourth part is the “single” dot with coordinates $[9, 8]$.

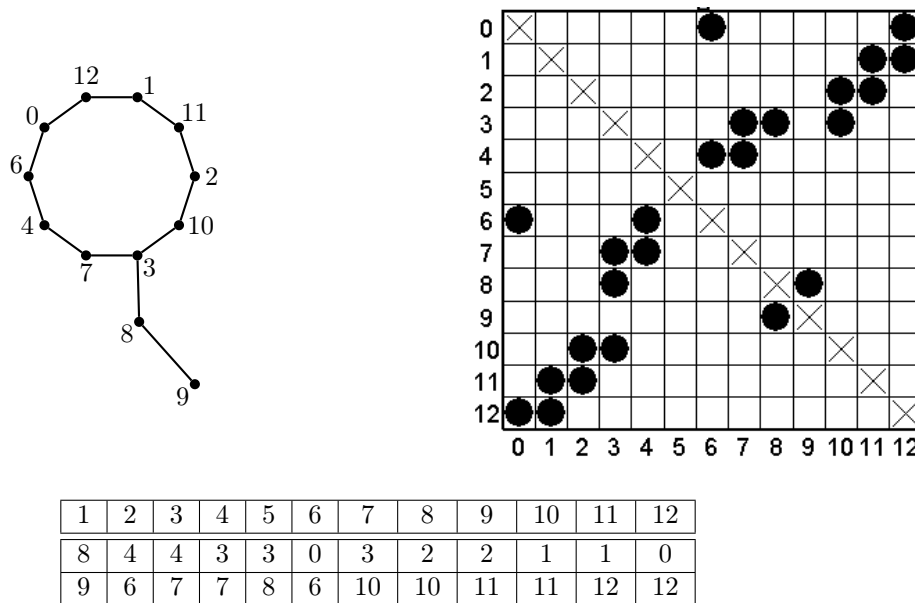


Figure 5: Representations of the kite $K_3(C_{10})$

Definition 4.2. Let $s = m + 2$ for $m \equiv 2 \pmod{4}$, where $m \geq 6$. By a labelling sequence of type 4 we mean the labelling sequence $L = (j_i)$ of s terms such that

$$j_i = \begin{cases} s - \frac{m+2}{4} - 1, & \text{if } i = 1; \\ \left\lfloor \frac{s-i-1}{2} \right\rfloor, & \text{if } 1 < i < s - \frac{m}{2} - 1; \\ 0, & \text{if } i = s - \frac{m}{2} - 1; \\ \left\lfloor \frac{s-i+1}{2} \right\rfloor, & \text{if } i > s - \frac{m}{2} - 1. \end{cases} \quad (LSC2K_4)$$

By a C2K-graph chessboard of type 4 of size $s = m + 2$ we mean a simple chessboard (such as in Figure 5), corresponding to a labelling sequence of type 4, which can be divided into four parts: the first part is a “stairway” from column 0 to column $\frac{m+2}{4}$, the second part is the “single” dot with

coordinates $[\frac{s}{2}, 0]$ and the third part is again a “stairway” starting with two vertical dots in the column $\frac{m+2}{4}$ and ending in the column $\frac{m}{2} - 1$. The fourth part is the “single” dot with coordinates $[s - \frac{m+2}{4}, s - \frac{m+2}{4} - 1]$.

Now we formulate a theorem characterizing the kites $K_n(C_m)$ formed by the cycles C_m for $m \equiv 2 \pmod{4}$, where $n = 3$.

Theorem 4.3. *Let G be a graph of size $s = m + 2$ for $m \equiv 2 \pmod{4}$, where $m \geq 6$. Then the following are equivalent:*

- (1) G is the kite $K_3(C_m)$.
- (2) G has a graceful labelling and a C2K-graph chessboard of type 4 of size s .
- (3) There exists a labelling sequence $L = (j_i)$ of s terms of G of type 4.
- (4) There exists a labelling relation $A(L)$ of G corresponding to the labelling sequence $L = (j_i)$ of s terms of G of type 4.

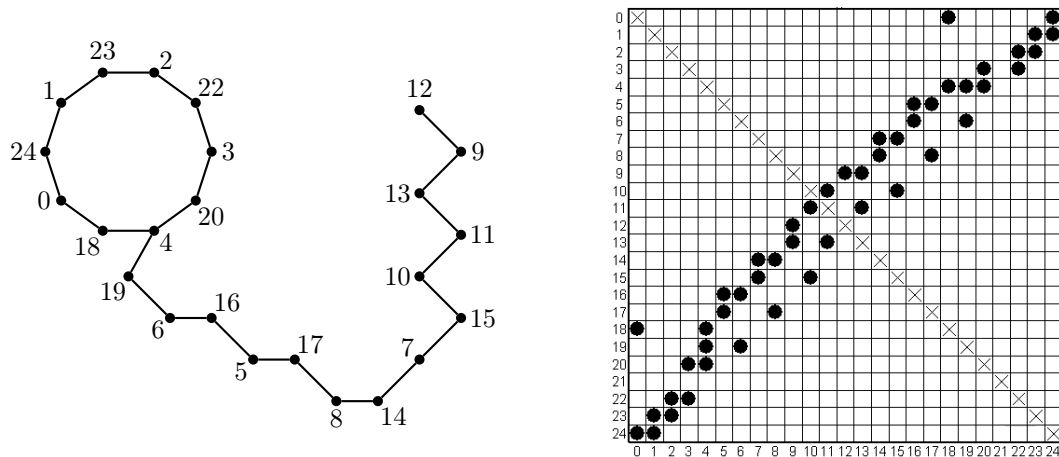
Proof. (1) $\implies$ (2): Let G be the kite $K_3(C_m)$. We label it such that we start labelling the cycle C_m : we label the vertex joining the cycle with the path (the “joining vertex”) by number $\frac{m+2}{4}$ and we follow in the clockwise direction with numbers $s - \frac{m+2}{4} - 2, \frac{m+2}{4} + 1, s - \frac{m+2}{4} - 3, \dots, \frac{s}{2}$ for the first part of the cycle. Then for the rest of the cycle we continue with numbers $0, s, 1, s - 1, \dots, \frac{m+2}{4}$. Next we label the path P_3 . We label by number $s - \frac{m+2}{4} - 1$ the vertex adjacent to the joining vertex and we label the remaining (ending) vertex of the path P_3 by number $s - \frac{m+2}{4}$. To show the gracefulness of this labelling, we use the corresponding simple chessboard of G (see Figure 5). The cycle C_m of the kite is in the corresponding chessboard represented by the first “stairway” in the columns 0 to $\frac{m+2}{4}$, by the “single” dot with coordinates $[\frac{s}{2}, 0]$ and by the second “stairway” starting with two vertical dots in the column $\frac{m+2}{4}$, but the lower dot of these two dots is representing an edge of the path P_3 . The remaining edge of the path P_3 is in the chessboard represented by the “single” dot with coordinates $[s - \frac{m+2}{4}, s - \frac{m+2}{4} - 1]$. So we obtain a C2K-graph chessboard of type 4, which means that our labelling is graceful.

(2) $\implies$ (3): Assume we have a graceful labelling of the graph G with a C2K-graph chessboard of type 4. We show that the corresponding LS satisfies the formula $(LSC2K4)$. The “single” dot in the chessboard closer to the main diagonal represents the number $s - \frac{m+2}{4} - 1$. The second “stairway” closer to the main diagonal represents in the corresponding LS numbers $[\frac{s-i-1}{2}]$ for $1 < i < s - \frac{m}{2} - 1$. The “single” dot in the chessboard in the column 0 represents in the LS number 0. The remaining “stairway” in the chessboard starting in the column 0 represents numbers $[\frac{s-i+1}{2}]$ for $i > s - \frac{m}{2} - 1$. So the LS satisfies the formula $(LSC2K4)$.

- (3) $\implies$ (4): Assume now that (3) holds, *i.e.* there exists a LS $L = (j_1, j_2, \dots, j_s)$ that satisfies the formula $(LSC2K4)$. We show that this LS has the labelling relation $A(L)$ as given in (4). The number $s - \frac{m+2}{4} - 1$ in the LS corresponds in $A(L)$ to the pair $[s - \frac{m+2}{4} - 1, s - \frac{m+2}{4}]$. The numbers $\lfloor \frac{s-i-1}{2} \rfloor$ obviously correspond to the pairs $[\lfloor \frac{s-i-1}{2} \rfloor, \lfloor \frac{s-i-1}{2} \rfloor + i]$ for $1 < i < s - \frac{m}{2} - 1$. Number 0 in the LS clearly corresponds in $A(L)$ to the pair $[0, \frac{s}{2}]$. Finally, the numbers $\lfloor \frac{s-i+1}{2} \rfloor$ in the LS correspond in $A(L)$ to the pairs $[\lfloor \frac{s-i+1}{2} \rfloor, \lfloor \frac{s-i+1}{2} \rfloor + i]$ for $i > s - \frac{m}{2} - 1$.
- (4) $\implies$ (1) : Let (4) hold, *i.e.* there exists a labelling relation $A(L)$ of G corresponding to the LS $L = (j_i)$ of s terms of G of type 4. We show that the pairs in $A(L)$ represent the edges of the kite $K_3(C_m)$. The two pairs $[s - \frac{m+2}{4} - 1, s - \frac{m+2}{4}]$ and $[\lfloor \frac{s-i-1}{2} \rfloor, \lfloor \frac{s-i-1}{2} \rfloor + i]$ for $i = s - \frac{m}{2} - 2$ represent in G the path P_3 . The pairs $[\lfloor \frac{s-i-1}{2} \rfloor, \lfloor \frac{s-i-1}{2} \rfloor + i]$ in $A(L)$ for $i = 2, \dots, s - \frac{m}{2} - 3$ represent in G the edges of a part of the cycle C_m . The next edge of the cycle C_m (in the clockwise direction) is represented in $A(L)$ by the pair $[0, \frac{s}{2}]$. The pairs $[\lfloor \frac{s-i+1}{2} \rfloor, \lfloor \frac{s-i+1}{2} \rfloor + i]$ in $A(L)$ for $i = s - \frac{m}{2}, \dots, s$ represent in G the edges of the remaining part of the cycle C_m . Hence G is the kite $K_3(C_m)$. $\square$

Next we illustrate in an example the labelling sequence and the corresponding *C2K-graph chessboard* of type 3, which correspond to the subcase $n \equiv 3 \pmod{4}$, where $n \geq 4$. (The case $n = 3$ was just treated separately above.) After the illustrating example, we present explicitly the definition of the labelling sequence and the corresponding C2K-graph chessboard of type 3. For the types 0–2 of the labelling sequence and of the corresponding C2K-graph chessboard, which correspond to the subcases $n \equiv 0 \pmod{4}$, $n \equiv 1 \pmod{4}$, and $n \equiv 2 \pmod{4}$, respectively, we only present the definitions since these three types are rather similar to the type 3. We note that in all four subcases below we have $n \geq 2$ and $n \neq 3$.

Example 4.4. In Figure 6 we see the representations of the kite $K_{15}(C_{10})$. The LS is $(10, 11, 9, 9, 10, 8, 7, 7, 8, 6, 5, 5, 6, 4, 4, 4, 3, 0, 3, 2, 2, 1, 1, 0)$. One can divide it into four parts: the first part $(10, 11, 9, 9, 10, 8, 7, 7, 8, 6, 5, 5, 6)$ is the LS of the path P_{15} except for one edge of P_{15} , then the first number 4 represents an edge from the cycle C_{10} . The second number 4 represents the remaining edge of the path P_{15} . The rest of the LS, which is $(4, 3, 0, 3, 2, 2, 1, 1, 0)$, represents the rest of the cycle C_{10} .

Figure 6: Representations of the kite $K_{15}(C_{10})$

Definition 4.5. Let $m, n \geq 4$ and $s = m + n - 1$. Let $m \equiv 2 \pmod{4}$ and $n \equiv 3 \pmod{4}$.

By a labelling sequence of type 3 is meant the labelling sequence $L = (j_i)$ of s terms such that

$$j_i = \begin{cases} \frac{s}{2} - 2, & \text{if } i = 1; \\ \frac{s}{2} - 1, & \text{if } i = 2; \\ \left\lfloor \frac{s-i}{2} \right\rfloor - 1, & \text{if } 2 < i \leq n-3 \wedge i \equiv 0, 2, 3 \pmod{4}; \\ \frac{s}{2} - \frac{i-1}{2}, & \text{if } 2 < i \leq n-2 \wedge i \equiv 1 \pmod{4}; \\ \frac{m}{2} - 1, & \text{if } i \in \{n, n-1\}; \\ \left\lfloor \frac{s-i}{2} \right\rfloor, & \text{if } n < i < s - \frac{m}{2} - 1; \\ 0, & \text{if } i = s - \frac{m}{2} - 1; \\ \left\lfloor \frac{s-i+1}{2} \right\rfloor, & \text{if } i > s - \frac{m}{2} - 1. \end{cases} \quad (LSC2K3)$$

By a $C2K$ -graph chessboard of type 3 of size s is meant the simple chessboard corresponding to a labelling sequence of type 3.

Definition 4.6. Let $m, n \geq 4$ and $s = m + n - 1$. Let $m \equiv 2 \pmod{4}$ and $n \equiv 0 \pmod{4}$.

By a labelling sequence of type 0 we mean the labelling sequence $L = (j_i)$ of s terms such that

$$j_i = \begin{cases} \left\lfloor \frac{s-i}{2} \right\rfloor - 1, & \text{if } 1 \leq i \leq n-3 \wedge i \equiv 0, 1, 3 \pmod{4}; \\ \left\lfloor \frac{s}{2} \right\rfloor - \frac{i-2}{2}, & \text{if } 1 \leq i \leq n-2 \wedge i \equiv 2 \pmod{4}; \\ \frac{m}{2} - 1, & \text{if } i \in \{n, n-1\}; \\ \left\lfloor \frac{s-i}{2} \right\rfloor, & \text{if } n < i < s - \frac{m}{2} - 1; \\ 0, & \text{if } i = s - \frac{m}{2} - 1; \\ \left\lfloor \frac{s-i+1}{2} \right\rfloor, & \text{if } i > s - \frac{m}{2} - 1. \end{cases} \quad (LSC2K0)$$

By a C2K-graph chessboard of type 0 of size s we mean the simple chessboard corresponding to a labelling sequence of type 0.

Definition 4.7. Let $m, n \geq 4$ and $s = m + n - 1$. Let $m \equiv 2 \pmod{4}$ and $n \equiv 1 \pmod{4}$.

By a labelling sequence of type 1 is meant the labelling sequence $L = (j_i)$ of s terms such that

$$j_i = \begin{cases} \left\lfloor \frac{s-i}{2} \right\rfloor - 1, & \text{if } 1 \leq i \leq n-3 \wedge i \equiv 0, 1, 2 \pmod{4}; \\ \frac{s}{2} - \frac{i-1}{2}, & \text{if } 1 \leq i \leq n-2 \wedge i \equiv 3 \pmod{4}; \\ \frac{m}{2} - 1, & \text{if } i \in \{n, n-1\}; \\ \left\lfloor \frac{s-i}{2} \right\rfloor, & \text{if } n < i < s - \frac{m}{2} - 1; \\ 0, & \text{if } i = s - \frac{m}{2} - 1; \\ \left\lfloor \frac{s-i+1}{2} \right\rfloor, & \text{if } i > s - \frac{m}{2} - 1. \end{cases} \quad (LSC2K1)$$

By a C2K-graph chessboard of type 1 of size s is meant the simple chessboard corresponding to a labelling sequence of type 1.

Definition 4.8. Let $m \geq 4$, $n \geq 2$ and $s = m + n - 1$. Let $m \equiv 2 \pmod{4}$ and $n \equiv 2 \pmod{4}$.

By a labelling sequence of type 2 we mean the labelling sequence $L = (j_i)$ of s terms such that

$$j_i = \begin{cases} \left\lfloor \frac{s-i}{2} \right\rfloor - 1, & \text{if } 1 \leq i \leq n-3 \wedge i \equiv 1, 2, 3 \pmod{4}; \\ \left\lfloor \frac{s}{2} \right\rfloor - \frac{i-2}{2}, & \text{if } 1 \leq i \leq n-2 \wedge i \equiv 0 \pmod{4}; \\ \frac{m}{2} - 1, & \text{if } i \in \{n, n-1\}; \\ \left\lfloor \frac{s-i}{2} \right\rfloor, & \text{if } n < i < s - \frac{m}{2} - 1; \\ 0, & \text{if } i = s - \frac{m}{2} - 1; \\ \left\lfloor \frac{s-i+1}{2} \right\rfloor, & \text{if } i > s - \frac{m}{2} - 1. \end{cases} \quad (LSC2K2)$$

By a $C2K$ -graph chessboard of type 2 of size s we mean the simple chessboard corresponding to a labelling sequence of type 2.

Now we present a theorem that covers all four types 0–3 and characterizes the kites $K_n(C_m)$ via the labelling sequences, labelling relations and the simple chessboards.

Theorem 4.9. Let $m \geq 4$, $n \geq 2$ with $n \neq 3$ and $s = m + n - 1$. Let $m \equiv 2 \pmod{4}$ and $n \equiv j \pmod{4}$ for $j \in \{0, 1, 2, 3\}$.

Let G be a graph of size s . Then the following are equivalent:

- (1) G is the kite $K_n(C_m)$.
- (2) G has a graceful labelling and a $C2K$ -graph chessboard of type j of size s .
- (3) There exists a labelling sequence $L = (j_i)$ of s terms of G of type j .
- (4) There exists a labelling relation $A(L)$ of G corresponding to the labelling sequence $L = (j_i)$ of s terms of G of type j .

Proof. (1) $\implies$ (2) : Let G be the kite $K_n(C_m)$. We label it such that we start labelling the cycle C_m : we label the vertex joining the cycle with the path (the “joining vertex”) by number $\frac{m}{2} - 1$ and we follow in the clockwise direction first by number $s - \frac{m}{2} - 1$, then by numbers $0, s, 1, s-1, \dots, s - \frac{m}{2} + 1$, but we skip the number $s - \frac{m+2}{4}$. Next we label the path P_n . We will do it in several steps. Firstly, we label the vertex next to the joining vertex by $s - \frac{m}{2}$ and we label every fourth vertex of the path P_n (if exists) by number $s - \frac{m}{2} - 2k$ for $k = 1, \dots, \lfloor \frac{n-2}{4} \rfloor$, except for type $j = 3$ (see Figure 6), where we do not number the last vertex fitting this sequence (the second last vertex of the path). Secondly, we start with the vertex next to the starting vertex of the previous part (if exists) and we label it by $\frac{m}{2} + 1$, then we label every

fourth vertex (if exists) by number $(\frac{m}{2} + 1) + 2k$ for $k = 1, \dots, \lfloor \frac{n-3}{4} \rfloor$. Thirdly, we label the vertex next to the starting vertex of the previous part (if exists) by $s - (\frac{m}{2} + 3)$ and we label then every fourth vertex (if exists) by number $s - (\frac{m}{2} + 3) - 2k$ for $k = 1, \dots, \lfloor \frac{n-4}{4} \rfloor$, but for types $j \in \{1, 3\}$ we do not number the last vertex fitting this sequence (the second last vertex of the path for $j = 1$ and the fourth last vertex for $j = 3$). Finally, we label the vertex next to the starting vertex of the previous part (if exists) by $\frac{m}{2}$, and then we label every fourth vertex (if exists) by number $\frac{m}{2} + 2k$ $k = 1, \dots, \lfloor \frac{n-5}{4} \rfloor$, but for types $j \in \{1, 3\}$ we do not number the last vertex fitting this sequence (last vertex of the path for $j = 1$ and third last for $j = 3$). Now for $j = 1$, we label the second last vertex of the path by $\frac{s}{2} - 2$ and the last vertex by $\frac{s}{2}$. In the case that $j = 3$ (see Figure 6), we label the remaining vertices of the path from the fourth last vertex to the second last vertex with numbers $\frac{s}{2} - 1$, $\frac{s}{2} + 1$ and $\frac{s}{2} - 3$. To show the gracefulness of this labelling, we use the corresponding simple chessboard of G . The cycle C_m of the kite is in the corresponding chessboard represented by the “stairway” in columns 0 to $\frac{m+2}{4}$, by the “single” dot with coordinates $[s - \frac{m}{2} - 1, 0]$, by another “stairway” in columns $\frac{m+2}{4}$ to $\frac{m}{2} - 1$ except the dot with coordinates $[s - \frac{m}{2}, \frac{m}{2} - 1]$, and by the dot with coordinates $[s - \frac{m}{2} - 1, \frac{m}{2} - 1]$. The path P_n is in the chessboard represented by the “single” dot with coordinates $[s - \frac{m}{2}, \frac{m}{2} - 1]$, by the “incomplete stairway” starting with two vertical dots in column $\frac{m}{2}$ and ending in column $\lfloor \frac{s}{2} \rfloor - 1$ and the “single” dots below it. So we obtain a C2K-graph chessboard of type j , thus our labelling is graceful, since every diagonal of the simple chessboard has exactly one dot.

- (2) $\implies$ (3): Assume we have a graceful labelling of the graph G with a C2K-graph chessboard of type j . We show that the corresponding LS satisfies the formula $(LSC2Kj)$. The “incomplete stairway” in the chessboard in the direction from the main diagonal represents in the corresponding LS numbers $\lfloor \frac{s-i}{2} \rfloor - 1$ for $i \leq n - 3$ where $i \equiv j, j + 1, j + 3 \pmod{4}$. However when $j = 3$, we have the number $\frac{s}{2} - 2$ for $i = 1$. The “single” dots below represent, depending on j , the numbers $\lfloor \frac{s}{2} \rfloor - \frac{i-2}{2}$ when $j = 0$ or $j = 2$, and the numbers $\frac{s}{2} - \frac{i-1}{2}$ when $j = 1$ or $j = 3$, all for $i \leq n - 2$ where $i \equiv j + 2 \pmod{4}$. However, when $j = 3$, we have the number $\frac{s}{2} - 1$ for $i = 2$. The two vertical dots on top of the “stairway” in the middle represent the numbers $\frac{m}{2} - 1$ for $i = \{n, n - 1\}$. The “stairway” in the middle represents numbers $\lfloor \frac{s-i}{2} \rfloor$ for $n < i < s - \frac{m}{2} - 1$. The “single” dot above represents the number 0. The remaining “stairway” represents numbers $\lfloor \frac{s-i+1}{2} \rfloor$ for $i > s - \frac{m}{2} - 1$. So $(LSC2Kj)$ holds.
- (3) $\implies$ (4): Assume now that (3) holds, *i.e.* there exists a LS $L = (j_1, j_2, \dots, j_s)$ that satisfies the formula $(LSC2Kj)$. We show that this LS has the labelling relation $A(L)$ as described in (4). Specially for $j = 3$, the number $\frac{s}{2} - 2$ corresponds in $A(L)$ to the pair $[\frac{s}{2} - 2, \frac{s}{2} - 2 + i]$ for $i = 1$ and the number $\frac{s}{2} - 1$ corresponds in $A(L)$ to the pair $[\frac{s}{2} - 1, \frac{s}{2} - 1 + i]$ for $i = 2$. For all j , the numbers $\lfloor \frac{s-i}{2} \rfloor - 1$ obviously correspond in $A(L)$ to the pairs $[\lfloor \frac{s-i}{2} \rfloor - 1, \lfloor \frac{s-i}{2} \rfloor - 1 + i]$ for $i \leq n - 3$ where $i \equiv j, j + 1, j + 3 \pmod{4}$. The numbers $x_j \in \{\lfloor \frac{s}{2} \rfloor - \frac{i-2}{2}, \frac{s}{2} - \frac{i-1}{2}\}$

correspond in $A(L)$ to the pairs $[x_j, x_j + i]$ for $i \leq n - 2$ where $i \equiv j + 2 \pmod{4}$. The numbers $\frac{m}{2} - 1$ correspond in $A(L)$ to the pairs $[\frac{m}{2} - 1, \frac{m}{2} - 1 + i]$ for $i = \{n, n - 1\}$. The numbers $\lfloor \frac{s-i}{2} \rfloor$ correspond in $A(L)$ to the pairs $[\lfloor \frac{s-i}{2} \rfloor, \lfloor \frac{s-i}{2} \rfloor + i]$ for $n < i < s - \frac{m}{2} - 1$. The number 0 clearly corresponds to the pair $[0, s - \frac{m}{2} - 1]$. The numbers $\lfloor \frac{s-i+1}{2} \rfloor$ correspond in $A(L)$ to the pairs $[\lfloor \frac{s-i+1}{2} \rfloor, \lfloor \frac{s-i+1}{2} \rfloor + i]$ for $i > s - \frac{m}{2} - 1$. The first coordinates of these pairs are members of the LS and the second coordinates are the result of adding the number i to them.

(4) $\implies$ (1): Let (4) hold, *i.e.* there exists a LS L with the labelling relation $A(L)$ as described in (4). We show that the pairs in $A(L)$ represent the edges of the kite $K_n(C_m)$. The entire path P_n of the kite except for one edge is represented by the pairs $[\lfloor \frac{s-i}{2} \rfloor - 1, \lfloor \frac{s-i}{2} \rfloor - 1 + i]$ for $i \leq n - 3$ where $i \equiv j, j + 1, j + 3 \pmod{4}$, then by the pairs $[x_j, x_j + i]$ assuming that $x_j \in \{\lfloor \frac{s}{2} \rfloor - \frac{i-2}{2}, \frac{s}{2} - \frac{i-1}{2}\}$ for $i \leq n - 2$ where $i \equiv j + 2 \pmod{4}$, and for $j = 3$ also by the pair $[\frac{s}{2} - 2, \frac{s}{2} - 2 + i]$ if $i = 1$ and by $[\frac{s}{2} - 1, \frac{s}{2} - 1 + i]$ if $i = 2$. The pair $[\frac{m}{2} - 1, \frac{m}{2} - 1 + i]$ for $i = n$ represents the remaining edge of the path P_n . The pair $[\frac{m}{2} - 1, \frac{m}{2} - 1 + i]$ for $i = n - 1$ represents the first edge of the cycle C_m starting from the joining vertex in the clockwise direction. The second edge of the cycle C_m in the clockwise direction is represented by $[0, s - \frac{m}{2} - 1]$. The pairs $[\lfloor \frac{s-i+1}{2} \rfloor, \lfloor \frac{s-i+1}{2} \rfloor + i]$ for $i = s - \frac{m}{2}, \dots, s$ represent the following part of the cycle C_m with skipping the last $\frac{m}{2} - 3$ edges of the cycle ending in the joining vertex. This last part of the cycle is represented by $[\lfloor \frac{s-i}{2} \rfloor, \lfloor \frac{s-i}{2} \rfloor + i]$ for $i = n + 1, \dots, s - \frac{m}{2} - 2$ in $A(L)$. Hence G is the kite $K_n(C_m)$. $\square$

5 Conclusion

We characterized via simple chessboards, labelling sequences, and labelling relations the kites $K_n(C_m)$ formed by the cycles C_m for $m \equiv 1 \pmod{4}$, where $m \geq 5$ and $n \geq \lfloor \frac{m}{2} \rfloor$, and the kites $K_n(C_m)$ formed by the cycles C_m for $m \equiv 2 \pmod{4}$, where $m \geq 6$ and $n \geq 2$. These graphs nicely illustrate that the kites $K_n(G)$ formed even by non-graceful graphs G can be proven graceful. Our results complete the study of the kites $K_n(C_m)$ from the companion paper by Haviar and Kotuřová [5], where the kites $K_n(C_m)$ formed by the graceful cycles C_m for $m \equiv 0, 3 \pmod{4}$ were characterized. Our descriptions turn out to be more complex than those in [5]: while for $m \equiv 1 \pmod{4}$ we need to consider two subcases, for $m \equiv 2 \pmod{4}$ we distinguished five subcases.

We note that the gracefulness of certain similar graphs was studied by Koh, Rogers, Lee and Toh in [7] and by Truszczyński in [11]. Truszczyński's methods for his "dragons" were very technical and laborious, and lacked visualisability. Therefore already the paper [5] presented different methods for the kites $K_n(C_m)$ in the cases $m \equiv 0, 3 \pmod{4}$ and it visualized the gracefulness of these graphs. The paper [5] did not present the cases $m \equiv 1 \pmod{4}$ and $m \equiv 2 \pmod{4}$. Here the

unfulfilled task has been completed. We not only visualized gracefulness of these graphs in this paper via the graph chessboards, but we also characterized these graphs among all simple graphs via the labelling sequences and the labelling relations.

One of possibilities for further research work would be describing, in the same manner as here, other kites $K_n(G)$ (where the path P_n is joined with a graph G) formed by various graphs G , which can even be non-graceful (as were the cycles C_m considered in this paper).

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