

Coincidence theory and maps with an inclusion property

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ABSTRACT

In this paper we consider a class of maps (whose convex hull has a nice selection property) with another general class of maps and we present coincidence results between these two classes.

RESUMEN

En este artículo, consideramos una clase de aplicaciones (cuya clausura convexa tiene una propiedad de selección agradable) con otra clase general de aplicaciones, y presentamos resultados de coincidencia entre estas dos clases.

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1 Introduction

In this paper motivated by ideas in the literature [6, 10, 12, 17, 18] we present a variety of coincidence results between maps (whose convex hull has a upper semicontinuous selection property) and other classes of maps, namely, *HLPY* (Horvath, Lin, Park and Yu [11, 13]), *Ad* (the admissible maps of Gorniewicz [9]) and *KKM* (Knaster, Kuratowski and Mazurkiewicz [9, 19]) maps. The maps considered will be compact (or coercive) in one of the classes and our arguments are based on a fixed point theorem of the author [15] and a fixed point theorem in the literature [4]. The results obtained in this paper automatically complement and improve equilibrium results in game theory [17, 22].

We begin this section by recalling a technical result of Deguire and Lassonde [6, 22] which will be needed in Section 2.

Theorem 1.1. *Let X be a compact Hausdorff topological space and let $\{Y_i\}_{i \in J}$ (J an index set) be a family of convex sets each in a Hausdorff topological vector space. For each $i \in J$ suppose $T_i : X \rightarrow Y_i$ and assume the following hold:*

(1.1) $T_i(x)$ has convex values for each $x \in X$ and

(1.2) $T_i^{-1}(y_i)$ is open (in X) for each $y_i \in Y_i$.

Also assume for each $x \in X$ that there exists a $j \in J$ with $T_j(x) \neq \emptyset$. Then there exists a subset $D \equiv \prod_{i \in J} D_i$ of $Y \equiv \prod_{i \in J} Y_i$ such that for each $x \in X$ there exists a $i \in J$ with $T_i(x) \cap D_i \neq \emptyset$. Moreover, for each $j \in J$, $D_j \subseteq Y_j$ is a polytope and all of these polytopes, except a finite number, consist of a single point.

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q -dimensional Čech homology group with compact carriers of X . For a continuous map $f : X \rightarrow X$, $H(f)$ is the induced linear map $f_* = \{f_{*q}\}$ where $f_{*q} : H_q(X) \rightarrow H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X , Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \rightarrow X$ is called a Vietoris map (written $p : \Gamma \rightrightarrows X$) if the following two conditions are satisfied:

- (i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic
- (ii) p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \rightarrow Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

- (i) p is a Vietoris map
- (ii) $q(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Next we define the admissible maps of Gorniewicz [9]. A upper semicontinuous map $\phi : X \rightarrow Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ . An example of an admissible map is a Kakutani map. A upper semicontinuous map $\phi : X \rightarrow CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y .

We also discuss the following classes of maps in this paper. Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z, W)$ [11, 13] if W is convex and there exists a map $S : Z \rightarrow W$ with $co(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup \{int S^{-1}(w) : w \in W\}$; here $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. These maps are related to the DKT (Ding, Kim and Tan [7]) maps in the literature and $F \in DKT(Z, W)$ [7] if W is convex and there exists a map $S : Z \rightarrow W$ with $co(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note these maps were motivated from the Φ^* maps. We say $G \in \Phi^*(Z, W)$ [2] if W is convex and there exists a map $S : Z \rightarrow W$ with $S(x) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ and has convex values for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$.

Next we consider a general class of maps, namely the PK (Park [19, 20]) maps (which include some of the maps described above and below). Let X and Y be Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \rightarrow 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : Fix F \neq \emptyset \text{ for all } F \in \mathcal{X}(Z, Z)\}$$

where $Fix F$ denotes the set of fixed points of F .

The class $\mathcal{U}$ of maps is defined by the following properties:

- (i) $\mathcal{U}$ contains the class C of single valued continuous functions;
- (ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and
- (iii) $B^n \in \mathcal{F}(\mathcal{U}_c)$ for all $n \in \{1, 2, \dots\}$; here $B^n = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$.

We say $F \in PK(X, Y)$ if for any compact subset K of X there is a $G \in \mathcal{U}_c(K, G(K))$ with $G(x) \subseteq F(x)$ for each $x \in K$. Recall PK is closed under compositions.

For a subset K of a topological space X , we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given two maps $F, G : X \rightarrow 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α -close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

Let Q be a class of topological spaces. A space Y is an extension space [9] for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \rightarrow Y$ extends to a continuous function $f : X \rightarrow Y$. A space Y is an approximate extension space [9] for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \rightarrow Y$ there exists a continuous function $f : X \rightarrow Y$ such that $f|_K$ is α -close to f_0 .

Let V be a subset of a Hausdorff topological vector space E . Then we say V is Schauder admissible if for every compact subset K of V and every covering $\alpha \in Cov_V(K)$ there exists a continuous function $\pi_\alpha : K \rightarrow V$ such that

- (i) π_α and $i : K \rightarrow V$ are α -close;
- (ii) $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES(\text{compact})$.

Theorem 1.2 ([15]). *Let X be a Schauder admissible subset of a Hausdorff topological vector space and $T \in PK(X, X)$ a compact upper semicontinuous map with closed values. Then there exists a $x \in X$ with $x \in T(x)$.*

Remark 1.3. (i) *Other variations of Theorem 1.1 can be found in [16].*

- (ii) *Every convex subset of a Hausdorff locally convex linear topological space is $AES(\text{compact})$ (see [3]).*

Next we describe some recent maps from the literature [10, 12] which are special cases of maps considered in Section 2. Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological vector space. We say $T : X \rightarrow 2^Y$ has the strong continuous inclusion property (SCIP) [12] at $x \in X$ if there exists an open set $U(x)$ in X containing x and a $F^x : U(x) \rightarrow 2^Y$ such that $F^x(w) \subseteq T(w)$ for all $w \in U(x)$ and $co F^x : U(x) \rightarrow 2^Y$ is compact valued and upper semicontinuous. We write $T \in K MU(X, Y)$ (Khan, McLean and Uyanik [12]) if T has the SCIP at every $x \in X$.

In this paper our map T will usually be a compact map so T has the SCIP is equivalent (see [1, p. 465]) to T has the continuous inclusion property (CIP) [10].

Now we recall a selection theorem [12].

Theorem 1.4. *Let X be a paracompact subset of a Hausdorff topological space, Y a subset of a Hausdorff topological vector space and $T \in KMU(X, Y)$. Then there exists an upper semicontinuous map $G : X \rightarrow CK(Y)$ with $G(w) \subseteq coT(w)$ for all $w \in X$.*

Next we describe a class of maps more general than the PK maps in our setting. Let X be a convex subset of a Hausdorff topological vector space and Y a Hausdorff topological space. If $S, T : X \rightarrow 2^Y$ are two set valued maps such that $T(co(A)) \subseteq S(A)$ for each finite subset A of X then we call S a generalized KKM mapping w.r.t. T . Now the set valued map $T : X \rightarrow 2^Y$ is said to have the KKM property if for any generalized KKM map $S : X \rightarrow 2^Y$ w.r.t. T the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property (the intersection of each finite subfamily is nonempty). We let

$$KKM(X, Y) = \{T : X \rightarrow 2^Y \mid T \text{ has the } KKM \text{ property}\}.$$

Note $PK(X, Y) \subset KKM(X, Y)$ (see [5]).

A product rule for the KKM class is unknown (the argument given in [18] was incorrect). However a product rule is known for most subclasses of KKM maps (so the results in [18] are correct for these subclasses). As a result in Section 2 we consider subclasses of KKM maps (for simplicity in notation we call these KKM also) where the following Property A holds. We also assume in Theorem 2.24 (repectively, Theorem 2.26) that the analogue of Theorem 1.7 (respectively, Theorem 1.8) and properties (i) and (ii) below hold (in the relevant setting) for this subclass of KKM maps (see Remark 1.5 (i)). Alternatively we could consider Remark 1.5 (ii) in Theorem 2.24 and Theorem 2.26.

Property A. *Let I be a index set, X be a convex set in a Hausdorff topological vector space and $\{Y_i\}_{i \in I}$ be a family of Hausdorff topological spaces. For each $i \in I$ suppose $T_i \in KKM(X, Y_i)$ is an upper semicontinuous map with compact values and let $T : X \rightarrow 2^Y$ (here $Y = \prod_{i \in I} Y_i$) be defined by $T(x) = \prod_{i \in I} T_i(x)$ for $x \in X$. Also assume for each $i \in I$ that there exists a compact set $K_i \subseteq Y_i$ with $T_i(X) \subseteq K_i$. Then $T \in KKM(X, K)$ where $K = \prod_{i \in I} K_i$.*

Remark 1.5. (i) Note we also have $T \in KKM(X, Y)$; consider $i \circ T$ where $i : K \rightarrow Y$ is the inclusion map. Note Property A holds (in the relevant setting) for the usual classes of maps in the literature, for example the maps admissible with respect to Gorniewicz (here I is finite), selectable maps, approximable maps etc

(ii) The results in Theorem 2.24 and Theorem 2.26 are also true for the original KKM maps (see [5]) if we assume the product map is a KKM map instead of assuming that the individual maps are KKM maps (see Remark 2.28).

In Section 2 we will make use of the following two properties [18]. Let C and X be convex subsets of a Hausdorff topological vector space E with $C \subseteq X$ and Y a Hausdorff topological space.

- (i) If $T \in KKM(X, Y)$ then $G \equiv T|_C \in KKM(C, Y)$.
- (ii) If $T \in KKM(X, Y)$, $T(X) \subseteq Z \subseteq Y$ and Z is closed in Y then $T \in KKM(X, Z)$.

Next we recall the following fixed point result for KKM maps. Recall a nonempty subset W of a Hausdorff topological vector space E is said to be admissible if for any nonempty compact subset K of W and every neighborhood V of 0 in E there exists a continuous map $h : K \rightarrow W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E (for example every nonempty convex subset of a locally convex space is admissible).

Theorem 1.6 ([4]). *Let X be an admissible convex set in a Hausdorff topological vector space E and $T \in KKM(X, X)$ be a closed compact map. Then T has a fixed point in X .*

Theorem 1.7 ([14]). *Let X be a convex admissible subset of a Hausdorff topological vector space, Y a subset of a Hausdorff topological vector space, $T \in KKM(X, Y)$ a upper semicontinuous compact map with compact values and $G \in Kak(Y, X)$. Then $GT \in KKM(X, X)$.*

Theorem 1.8 ([14]). *Let X be a convex admissible subset of a Hausdorff topological vector space, Y a convex subset of a Hausdorff topological vector space and Y normal, $T \in KKM(X, Y)$ a upper semicontinuous compact map with compact values and $G \in Kak(Y, X)$. Then $TG \in KKM(Y, Y)$.*

2 Coincidence results

In this section we present a number of collectively coincidence results between two classes of set-valued maps.

2.1 Coincidence results between Kakutani (the compact and coercive cases) and Φ^* type map

We begin when one family consists of compact sets and then we extend when the compact family is replaced by either compact or coercive type maps.

Theorem 2.1. *Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i \in J}$ be a family of compact sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$ for $x \in X$. For each $j \in I$, suppose $G_j : Y \equiv \prod_{i \in J} Y_i \rightarrow X_j$ and in addition there exists a map $S_j : Y \rightarrow X_j$*

with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Finally, suppose for each $y \in Y$ there exists a $j \in I$ with $S_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in co F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$ (here x_i (respectively, y_j) is the projection of x (respectively, y) on X_i (respectively, Y_j)).

Proof. From Theorem 1.1 (note Y is compact) there exists a subset $C = \prod_{i \in I} C_i$ of X (as described in Theorem 1.1 so C is homeomorphic to a convex compact set of finite dimension) such that for each $y \in Y$ there exists a $j \in I$ with $S_j(y) \cap C_j \neq \emptyset$.

For $i \in J$, let $\Phi_i^* = \Phi_i|_C$ (the restriction of Φ_i to C , which we will also denote as Φ_i) and note $\Phi_i \in Kak(C, Y_i)$. Now $i \in I$, let $L_i : Y \rightarrow C_i$ and $G_i^* : Y \rightarrow C_i$ be given by

$$L_i(y) = S_i(y) \cap C_i \quad \text{and} \quad G_i^*(y) = G_i(y) \cap C_i \quad \text{for } y \in Y.$$

For $i \in I$, note $L_i(y) \subseteq G_i^*(y)$ for $y \in Y$, $L_i(y)$ has convex values for each $y \in Y$ (note S_i is convex valued and C_i is convex) and for $x \in C_i$ note

$$L_i^{-1}(x) = \{z \in Y : x \in L_i(z) = S_i(z) \cap C_i\} = \{z \in Y : x \in S_i(z)\} = S_i^{-1}(x)$$

which is open in Y . Next note for each $y \in Y$ there exists (see the beginning of the proof) a $j \in I$ with $S_j(y) \cap C_j \neq \emptyset$ i.e. $L_j(y) \neq \emptyset$.

Note $E_i = \{y \in Y : L_i(y) \neq \emptyset\}$, $i \in I$ is an open covering of Y (recall the fibres of L_i are open in Y). Now from [8, Lemma 5.1.6, p. 301] there exists a covering $\{Q_i\}_{i \in I}$ of Y where Q_i is closed in Y and $Q_i \subseteq E_i$ for all $i \in I$. For each $i \in I$ let $N_i : Y \rightarrow C_i$ and $P_i : Y \rightarrow C_i$ be given by

$$N_i(y) = \begin{cases} G_i^*(y), & y \in Q_i \\ C_i, & y \in Y \setminus Q_i \end{cases} \quad \text{and} \quad P_i(y) = \begin{cases} L_i(y), & y \in Q_i \\ C_i, & y \in Y \setminus Q_i \end{cases}$$

and we claim $N_i \in \Phi^*(Y, C_i)$. To see this, first note for $i \in I$ that $P_i(y) \neq \emptyset$ for $y \in Y$ (if $y \in Q_i$, then $P_i(y) = L_i(y) \neq \emptyset$, since $Q_i \subseteq E_i$; whereas if $y \in Y \setminus Q_i$, then $P_i(y) = C_i$). Next we note for $i \in I$ that $P_i(y) \subseteq N_i(y)$, for $y \in Y$ (if $y \in Q_i$, then $P_i(y) = L_i(y) = S_i(y) \cap C_i \subseteq G_i(y) \cap C_i = G_i^*(y) = N_i(y)$; whereas if $y \in Y \setminus Q_i$, then $P_i(y) = C_i = N_i(y)$). Also note for $i \in I$ that $P_i(y)$ is convex valued for $y \in Y$ and for $x \in C_i$ note

$$\begin{aligned} P_i^{-1}(x) &= \{z \in Y : x \in P_i(z)\} = \{z \in Y \setminus Q_i : x \in P_i(z) = C_i\} \cup \{z \in Q_i : x \in L_i(z)\} \\ &= (Y \setminus Q_i) \cup \{z \in Q_i : x \in L_i(z)\} = (Y \setminus Q_i) \cup [Q_i \cap \{z \in Y : x \in L_i(z)\}] \\ &= (Y \setminus Q_i) \cup [Q_i \cap L_i^{-1}(x)] = Y \cap [(Y \setminus Q_i) \cup L_i^{-1}(x)] = (Y \setminus Q_i) \cup L_i^{-1}(x) \end{aligned}$$

which is open in Y (note $L_i^{-1}(x)$ is open in Y and Q_i is closed in Y). Thus $N_i \in \Phi^*(Y, C_i)$ for each

$i \in I$. Now for each $i \in I$ from [2] there exists a continuous (single valued) selection $g_i : Y \rightarrow C_i$ of N_i with $g_i(y) \in P_i(y) \subseteq N_i(y)$ for $y \in Y$.

Let $\Phi(x) = \prod_{i \in J} \Phi_i(x)$, for $x \in C$, and note (recall the product of upper semicontinuous maps with compact values is upper semicontinuous with compact values [1]) $\Phi \in Kak(C, Y)$ is an upper semicontinuous compact map with compact values. Now let $g(y) = \prod_{i \in I} g_i(y)$, for $y \in Y$, and note $g : Y \rightarrow C$ is continuous. Combine the above (note the composition of two admissible with respect to Gorniewicz maps is admissible with respect to Gorniewicz) and we have $g\Phi \in Ad(C, C)$ and note C is a convex compact set of finite dimension so Theorem 1.2 guarantees a $x \in C$ with $x \in g\Phi(x)$. Now let $y \in \Phi(x)$ with $x = g(y)$. Note $y \in \Phi(x)$ so $y_j \in \Phi_j(x) \subseteq co F_j(x)$, for all $j \in J$ (note $x \in C$). Also we have $x_i = g_i(y) \in P_i(y) \subseteq N_i(y)$, for $i \in I$. Now since $\{Q_i\}_{i \in I}$ is a covering of Y , then there exists a $i_0 \in I$, with $y \in Q_{i_0}$, so $x_{i_0} \in N_{i_0}(y) = G_{i_0}^*(y) = G_{i_0}(y) \cap C_{i_0}$ i.e. $x_{i_0} \in G_{i_0}(y)$. $\square$

Remark 2.2. If $F_i \in KMU(X, Y_i)$, for $i \in J$ and X is paracompact, then Theorem 1.4 guarantees that there exists an upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$.

Now we replace the compactness of the spaces with compactness of the maps.

Theorem 2.3. Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$, and in addition there exists a compact set K_i with $\Phi_i(X) \subseteq K_i \subseteq Y_i$. For each, $j \in I$, suppose $G_j : Y \equiv \prod_{i \in J} Y_i \rightarrow X_j$ and in addition there exists a map $S_j : Y \rightarrow X_j$ with $S_j(y) \subseteq G_j(y)$, for $y \in Y$, $S_j(y)$ has convex values for $y \in Y$ and $S_j^{-1}(w)$ is open (in Y), for each $w \in X_j$. Finally, suppose for each $y \in Y$ there exists a $j \in I$ with $S_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.

Proof. Note $K \equiv \prod_{i \in J} K_i (\subseteq Y)$ is compact. For $i \in I$, let $G_i^{**} = G_i|_K$ and $S_i^{**} = S_i|_K$, and note $G_i^{**} : K \rightarrow X_i$ and $S_i^{**} : K \rightarrow X_i$, $S_i^{**}(y) \subseteq G_i^{**}(y)$ for $y \in K$, $S_i^{**}(y)$ has convex values for $y \in K$; and for $x \in X_i$ note

$$(S_i^{**})^{-1}(x) = \{z \in K : x \in S_i^{**}(z)\} = K \cap \{z \in Y : x \in S_i(z)\} = K \cap S_i^{-1}(x),$$

which is open in $K \cap Y = K$. Next note if $y \in K$, then there exists a $i_0 \in I$ with $S_{i_0}^{**}(y) \neq \emptyset$, since for any $w \in Y$ there exists a $j \in I$ with $S_j(w) \neq \emptyset$. Finally, note for $i \in J$ that $\Phi_i \in Kak(X, K_i)$ since $\Phi_i(X) \subseteq K_i$. Now apply Theorem 2.1 with $\{X_i\}_{i \in I}$, $\{K_i\}_{i \in J}$, F_i , Φ_i , G_j^{**} and S_j^{**} so there exists a $x \in X$, a $y \in K$, a $i_0 \in I$ with $y_j \in co F_j(y)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}^{**}(y) (= G_{i_0}(y))$. $\square$

Now we present a result for coercive type maps.

Theorem 2.4. *Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$ for $x \in X$. For each $j \in I$, suppose $G_j : Y \equiv \prod_{i \in J} Y_i \rightarrow X_j$ and in addition there exists a map $S_j : Y \rightarrow X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Also suppose for each $y \in Y$ there exists a $j \in I$ with $S_j(y) \neq \emptyset$. Finally, assume there is a compact subset K of Y and for each $i \in I$ a convex compact subset Z_i of X_i such that for each $y \in Y \setminus K$ there exists a $j \in I$ with $S_j(y) \cap Z_j \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in co F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. From Theorem 1.1 (note K is compact) there exists a subset $C = \prod_{i \in I} C_i$ of X (as described in Theorem 1.1 so C is homeomorphic to a convex compact set of finite dimension) such that for each $y \in K$ there exists a $j \in I$ with $S_j(y) \cap C_j \neq \emptyset$. Also (see the statement of Theorem 2.4) for each $y \in Y \setminus K$ there exists a $i \in I$ with $S_i(x) \cap Z_i \neq \emptyset$. Let

$$\Omega_i = co(Z_i \cup C_i) \quad \text{for } i \in I,$$

which is a convex compact (see [1]) subset of X_i . Let $\Omega = \prod_{i \in I} \Omega_i$ which is a convex compact subset of X . For each $i \in I$, let $S_i^* : Y \rightarrow \Omega_i$ and $G_i^* : Y \rightarrow \Omega_i$ be given by

$$S_i^*(y) = S_i(y) \cap \Omega_i \quad \text{and} \quad G_i^*(y) = G_i(y) \cap \Omega_i \quad \text{for } y \in Y.$$

For $i \in I$, note $S_i^*(y) \subseteq G_i^*(y)$ for $y \in Y$, $S_i^*(y)$ has convex values for each $y \in Y$ and for $x \in \Omega_i$ note

$$(S_i^*)^{-1}(x) = \{z \in Y : x \in S_i^*(z)\} = \{z \in Y : x \in S_i(z) \cap \Omega_i\} = \{z \in Y : x \in S_i(z)\} = S_i^{-1}(x),$$

which is open in Y . Next note for each $y \in Y$, there exists a $i_0 \in I$ with $S_{i_0}^*(y) \neq \emptyset$. To see this, let $y \in Y$. If $y \in K$, then there exists a $i \in I$ with $S_i(y) \cap C_i \neq \emptyset$ so $S_i(y) \cap \Omega_i \neq \emptyset$ (since $C_i \subseteq \Omega_i$); whereas if $y \in Y \setminus K$, then there exists a $j \in I$ with $S_j(y) \cap Z_j \neq \emptyset$ so $S_j(y) \cap \Omega_j \neq \emptyset$ (since $Z_j \subseteq \Omega_j$).

For $i \in J$, note $\Phi_i \in Kak(\Omega, Y_i)$. Also note $\Phi_i(\Omega)$ is compact since Ω is compact and Φ_i is upper semicontinuous with compact values. Let $K_i = \Phi_i(\Omega)$. Now we apply Theorem 2.3 with $\{\Omega_i\}_{i \in I}$, $\{Y_i\}_{i \in J}$, F_i , Φ_i , S_i^* and G_i^* (and $K_i = \Phi_i(\Omega)$), so there exists a $x \in \Omega$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in co F_j(y)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}^*(y) (= G_{i_0}(y) \cap \Omega_{i_0})$. $\square$

2.2 Coincidence results between Kakutani (the compact and coercive cases) and DKT type maps

Note Theorem 2.1, Theorem 2.3 and Theorem 2.4 can easily be extended to DKT type maps. To see this, we will extend Theorem 2.3 and Theorem 2.4

Theorem 2.5. *Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$, and in addition there exists a compact set K_i with $\Phi_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in I$, suppose $G_j : Y \equiv \prod_{i \in J} Y_i \rightarrow X_j$ and in addition there exists a map $T_j : Y \rightarrow X_j$ with $co(T_j(y)) \subseteq G_j(y)$, for $y \in Y$, and $T_j^{-1}(w)$ is open (in Y), for each $w \in X_j$. Finally, suppose for each $y \in Y$ there exists a $j \in I$ with $T_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For $j \in I$, let $S_j(y) = co(T_j(y))$ for $y \in Y$, note $S_j(y)$ has convex values for each $y \in Y$, $S_j(y) \subseteq G_j(y)$ for $y \in Y$, and from [21, Lemma 5.1] we have that $S_j^{-1}(w)$ is open (in Y), for each $w \in X_j$. Also note if $y \in Y$, then there exists a $i_0 \in I$ with $T_{i_0}(y) \neq \emptyset$ and so $\emptyset \neq T_{i_0}(y) \subseteq co(T_{i_0}(y)) = S_{i_0}(y)$. Now apply Theorem 2.3. $\square$

Remark 2.6. *From the proof note in the statement of Theorem 2.5 we could replace “for each $y \in Y$, there exists a $j \in I$ with $T_j(y) \neq \emptyset$ ” with “for each, $y \in Y$ there exists a $j \in I$ with $co(T_j(y)) \neq \emptyset$ ”.*

Theorem 2.7. *Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$. For each $j \in I$, suppose $G_j : Y \equiv \prod_{i \in J} Y_i \rightarrow X_j$ and in addition there exists a map $T_j : Y \rightarrow X_j$ with $co(T_j(y)) \subseteq G_j(y)$, for $y \in Y$ and $T_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Also, suppose for each $y \in Y$ there exists a $j \in I$ with $T_j(y) \neq \emptyset$. Finally, assume there is a compact subset K of Y and for each $i \in I$ a convex compact subset Z_i of X_i , such that for each $y \in Y \setminus K$ there exists a $j \in I$ with $T_j(y) \cap Z_j \neq \emptyset$ (or, alternatively $co(T_j(y)) \cap Z_j \neq \emptyset$). Then, there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For $i \in I$ let S_i be as in Theorem 2.5 and note S_i has the same properties as in Theorem 2.4 (note if $y \in Y \setminus K$, then there exists a $j \in I$ with $\emptyset \neq co(T_j(y)) \cap Z_j = S_j(y) \cap Z_j$). Now apply Theorem 2.4 $\square$

2.3 Coincidence results between Kakutani (the compact and coercive cases) and HLPY type maps

Next we consider the analogue of Theorem 2.3 and Theorem 2.4 for HLPY maps.

Theorem 2.8. *Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists an upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$, and in addition there exists a compact set K_i with $\Phi_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in I$, suppose $G_j : Y \equiv \prod_{i \in J} Y_i \rightarrow X_j$ and in addition there exists a map $T_j : Y \rightarrow X_j$ with $co(T_j(y)) \subseteq G_j(y)$, for $y \in Y$ and $Y = \bigcup_{j \in I} \bigcup \{int T_j^{-1}(w) : w \in X_j\}$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For $j \in I$, let $R_j : Y \rightarrow X_j$ be given by

$$R_j(y) = \{z_j : y \in int T_j^{-1}(z_j)\}, \quad y \in Y$$

and let $S_j : Y \rightarrow X_j$ be given by $S_j(y) = co(R_j(y))$, for $y \in Y$. For $j \in I$, note $R_j(y) \subseteq T_j(y)$ for $y \in Y$ since if $z_j \in R_j(y)$ then $y \in int T_j^{-1}(z_j) \subseteq T_j^{-1}(z_j)$ so $z_j \in T_j(y)$. Thus, for $j \in I$, we have $S_j(y) = co(R_j(y)) \subseteq co(T_j(y)) \subseteq G_j(y)$, for $y \in Y$.

Next for $j \in I$, notice for $x_j \in X_j$ that $R_j^{-1}(x_j) = \{z : x_j \in R_j(z)\} = int T_j^{-1}(x_j)$, so $R_j^{-1}(x_j)$ is open (in Y) and so from [21, Lemma 5.1], we have that $S_j^{-1}(x_j)$ is open (in Y).

Finally, let $y \in Y$ and since $Y = \bigcup_{j \in I} \bigcup \{int T_j^{-1}(w) : w \in X_j\}$, there exists a $j \in I$ with $y \in int T_j^{-1}(w)$, for some $w \in X_j$ and so $w \in R_j(y)$ (i.e. $R_j(y) \neq \emptyset$) and as a result $\emptyset \neq R_j(y) \subseteq co(R_j(y)) = S_j(y)$. Now apply Theorem 2.3 $\square$

Theorem 2.9. *Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists an upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$. For each $j \in I$, suppose $G_j : Y \equiv \prod_{i \in J} Y_i \rightarrow X_j$ and in addition there exists a map $T_j : Y \rightarrow X_j$ with $co(T_j(y)) \subseteq G_j(y)$, for $y \in Y$, and $Y = \bigcup_{j \in I} \bigcup \{int T_j^{-1}(w) : w \in X_j\}$. Also assume there is a compact subset K of Y and for each $i \in I$ a convex compact subset Z_i of X_i , such that for each $y \in Y \setminus K$ there exists a $j \in I$ with $R_j(y) \cap Z_j \neq \emptyset$ (or, alternatively $co(R_j(y)) \cap Z_j \neq \emptyset$), where $R_j : Y \rightarrow X_j$ is given by $R_j(y) = \{z_j : y \in int T_j^{-1}(z_j)\}$, for $y \in Y$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For $i \in I$, let S_i be as in Theorem 2.8 (note S_i has the same properties as in Theorem 2.8). Let $y \in Y \setminus K$ and note there exists a $j \in I$ with $\text{co}(R_j(y)) \cap Z_j \neq \emptyset$, so $\emptyset \neq \text{co}(R_j(y)) \cap Z_j = S_j(y) \cap Z_j$. Now apply Theorem 2.4 $\square$

2.4 Coincidence results between Kakutani and compact HLPY type maps

In our results so far we assumed the maps Φ_i were compact type maps. In our next results we will assume compactness on the Φ^* (or DKT or HLPY) type maps.

Theorem 2.10. *Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space Z_i and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i with $Y \equiv \prod_{i \in J} Y_i$ a paracompact subset of $E \equiv \prod_{i \in J} E_i$. For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq \text{co} F_i(x)$, for $x \in X$. For each $j \in I$, suppose $G_j : Y \rightarrow X_j$ and there exists a convex compact set K_j with $G_j(Y) \subseteq K_j \subseteq X_j$ and also there exists a map $S_j : Y \rightarrow X_j$ with $S_j(y) \subseteq G_j(y)$, for $y \in Y$ and $S_j^{-1}(w)$ is open (in Y), for each $w \in X_j$. Also assume $K \equiv \prod_{i \in I} K_i$ is a Schauder admissible subset of $Z \equiv \prod_{i \in I} Z_i$. Finally suppose for each $y \in Y$, there exists a $j \in I$ with $S_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in \text{co} F_j(x)$, for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. Note $C_i = \{y \in Y : S_i(y) \neq \emptyset\}$, $i \in I$ is an open covering of Y and since Y is paracompact [8, Lemma 5.1.6, p. 301], there exists a covering $\{D_i\}_{i \in I}$ of Y , where D_i is closed in Y and $D_i \subset C_i$, for all $i \in I$. Now for each $i \in I$, let $M_i : Y \rightarrow K_i$ and $L_i : Y \rightarrow K_i$ be given by

$$M_i(y) = \begin{cases} G_i(y), & y \in D_i \\ K_i, & y \in Y \setminus D_i \end{cases} \quad \text{and} \quad L_i(y) = \begin{cases} S_i(y), & y \in D_i \\ K_i, & y \in Y \setminus D_i. \end{cases}$$

We claim for each $i \in I$, that $M_i \in \Phi^*(Y, K_i)$. Let $i \in I$. First note $L_i(y) \neq \emptyset$, for $y \in Y$, since if $y \in D_i$, then $L_i(y) = S_i(y) \neq \emptyset$ since $D_i \subset C_i$; whereas if $y \in Y \setminus D_i$, then $L_i(y) = K_i$. Also if $y \in D_i$ then $L_i(y) = S_i(y) \subseteq G_i(y) = M_i(y)$; whereas if $y \in Y \setminus D_i$, we have $L_i(y) = K_i = M_i(y)$. Combining gives $L_i(y) \subseteq M_i(y)$, for $y \in Y$. Finally note if $x \in K_i$, we have

$$\begin{aligned} L_i^{-1}(x) &= \{z \in Y : x \in L_i(z)\} = \{z \in Y \setminus D_i : x \in L_i(z) = K_i\} \cup \{z \in D_i : x \in L_i(z)\} \\ &= (Y \setminus D_i) \cup \{z \in D_i : x \in S_i(z)\} = (Y \setminus D_i) \cup [D_i \cap \{z \in Y : x \in S_i(z)\}] \\ &= (Y \setminus D_i) \cup [D_i \cap S_i^{-1}(x)] = Y \cap [(Y \setminus D_i) \cup S_i^{-1}(x)] = (Y \setminus D_i) \cup S_i^{-1}(x), \end{aligned}$$

which is open in Y (note $S_i^{-1}(x)$ is open in Y and D_i is closed in Y). Thus for each $i \in I$, we have $M_i \in \Phi^*(Y, K_i)$.

Now for $i \in I$ (since Y is paracompact), from [2], there exists a continuous (single valued) selection $g_i : Y \rightarrow K_i$ of L_i i.e. $g_i(y) \in L_i(y) \subseteq M_i(y)$, for $y \in Y$. Now let $g(y) = \prod_{i \in I} g_i(y)$, for $y \in Y$ and note $g \in C(Y, K)$. Also note $\Phi_i \in Kak(K, Y_i)$, for $i \in J$ and let $\Phi(x) = \prod_{i \in J} \Phi_i(x)$, for $x \in K$ and note $\Phi \in Kak(K, Y)$. Combining and we have $g\Phi \in Ad(K, K)$. Now Theorem 1.2 guarantees a $x \in K$ with $x \in g\Phi(x)$. Now let $y \in \Phi(x)$ with $x = g(y)$. Thus $y_j \in \Phi_j(x) \subseteq co F_j(x)$, for all $j \in J$. Also note $x_i = g_i(y) \in L_i(y) \subseteq M_i(y)$, for all $i \in I$. Next since $\{D_i\}_{i \in I}$ is a covering of Y there exists a $i_0 \in I$ with $y \in D_{i_0}$ so $x_{i_0} \in M_{i_0}(y) = G_{i_0}(y)$. $\square$

Remark 2.11. *It is possible in the statement of Theorem 2.10 to replace “suppose for each $y \in Y$, there exists a $j \in I$ with $S_j(y) \neq \emptyset$ ” with “suppose $G_i \in \Phi^*(Y, X_i)$, for each $i \in I$ ” and we can argue differently and obtain another result. In this case note for $i \in I$, that $G_i \in \Phi^*(Y, K_i)$ (since in particular $S_i^{-1}(w)$ is open (in Y), for each $w \in K_i$ and also note $S_i : Y \rightarrow K_i$ since $S_i(y) \subseteq G_i(y) \subseteq K_i$, for each $y \in Y$) so from [2], there exists a continuous (single valued) selection $g_i : Y \rightarrow K_i$ of G_i . Now let $g(y) = \prod_{i \in I} g_i(y)$, for $y \in Y$ and note $g \in C(Y, K)$. Also note $\Phi_i \in Kak(K, Y_i)$, for $i \in J$ and let $\Phi(x) = \prod_{i \in J} \Phi_i(x)$, for $x \in K$ and note $\Phi \in Kak(K, Y)$. Combining and we have $g\Phi \in Ad(K, K)$. Now Theorem 1.2 guarantees a $x \in K$ with $x \in g\Phi(x)$. Now let $y \in \Phi(x)$ with $x = g(y)$. Thus $y_j \in \Phi_j(x) \subseteq co F_j(x)$ for all $j \in J$. Also note $x_i = g_i(y) \in G_i(y)$ for all $i \in I$.*

Remark 2.12. *There is also an analogue of Remark 2.11, for all the other theorems in this paper (with appropriate assumptions).*

Theorem 2.13. *Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space Z_i and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i with $Y \equiv \prod_{i \in J} Y_i$ a paracompact subset of $E \equiv \prod_{i \in J} E_i$. For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$. For each $j \in I$, suppose $G_j : Y \rightarrow X_j$ and there exists a convex compact set K_j with $G_j(Y) \subseteq K_j \subseteq X_j$, and also, there exists a map $T_j : Y \rightarrow K_j$ with $co(T_j(y)) \subseteq G_j(y)$, for $y \in Y$ and $T_j^{-1}(w)$ is open (in Y), for each $w \in K_j$. Also assume $K \equiv \prod_{i \in I} K_i$ is a Schauder admissible subset of $Z \equiv \prod_{i \in I} Z_i$. Finally, suppose for each $y \in Y$, there exists a $j \in I$ with $T_j(y) \neq \emptyset$ (or, alternatively $co T_j(y) \neq \emptyset$). Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For $i \in I$ let $S_i(y) = co(T_j(y))$, for $y \in Y$, and note S_i has the same properties as in Theorem 2.10. Now apply Theorem 2.10. $\square$

Theorem 2.14. *Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space Z_i and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i with $Y \equiv \prod_{i \in J} Y_i$ a paracompact subset of $E \equiv \prod_{i \in J} E_i$. For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e.*

$\Phi_i \in Kak(X, Y_i)$ with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$. For each $j \in I$, suppose $G_j : Y \rightarrow X_j$ and there exists a convex compact set K_j with $G_j(Y) \subseteq K_j \subseteq X_j$, and also, there exists a map $T_j : Y \rightarrow K_j$ with $co(T_j(y)) \subseteq G_j(y)$, for $y \in Y$ and $Y = \bigcup_{j \in I} \bigcup \{int T_j^{-1}(w) : w \in K_j\}$. Also assume $K \equiv \prod_{i \in I} K_i$ is a Schauder admissible subset of $Z \equiv \prod_{i \in I} Z_i$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.

Proof. For $j \in I$, let $R_j : Y \rightarrow K_j$ be given by

$$R_j(y) = \{z_j : y \in int T_j^{-1}(z_j)\}, \quad y \in Y$$

and let $S_j : Y \rightarrow K_j$ be given by $S_j(y) = co(R_j(y))$, for $y \in Y$. Note S_j (see Theorem 2.8) has the same properties as in Theorem 2.10. Now apply Theorem 2.10. $\square$

2.5 Coincidence results between Kakutani and similar classes of maps

Next we present some collectively coincidence results when the Φ^* , DKT or $HLPY$ type maps are replaced by other types of maps (like Kak , Ad or KKM maps).

Theorem 2.15. Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of sets each in a Hausdorff topological vector space Z_i and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$. For each $j \in I$, let K_j be a nonempty compact subset of X_j and suppose $G_j \in Kak(Y, K_j)$ (or, alternatively $G_j \in Kak(Y, X_j)$ and $G_j(Y) \subseteq K_j$) where $Y \equiv \prod_{i \in J} Y_i$. Also assume $K \equiv \prod_{i \in I} K_i$ is a Schauder admissible subset of $Z \equiv \prod_{i \in I} Z_i$. Then there exists a $x \in X$, a $y \in Y$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_i \in G_i(y)$, for all $i \in I$.

Proof. Let $\Phi(x) = \prod_{i \in J} \Phi_i(x)$, for $x \in X$, and $G(y) = \prod_{j \in I} G_j(y)$, for $y \in Y$. Note $\Phi \in Kak(X, Y)$ (so $\Phi \in Kak(K, Y)$) and $G \in Kak(Y, K)$. Combining and note $G\Phi \in Ad(K, K)$. Now Theorem 1.2 guarantees a $x \in K$, with $x \in G\Phi(x)$. Let $y \in \Phi(x)$, with $x \in G(y)$. Thus $y_j \in \Phi_j(x) \subseteq co F_j(x)$, for all $j \in J$ and $x_i \in G_i(y)$, for all $i \in I$. $\square$

Remark 2.16. Note one could also consider the map ΦG in the proof of Theorem 2.15 and note $\Phi G \in Ad(Y, Y)$ is a compact map (note G is a compact map and Φ is upper semicontinuous with compact values). Thus we can apply Theorem 1.2 if “ $K \equiv \prod_{i \in I} K_i$ is a Schauder admissible subset of $Z \equiv \prod_{i \in I} Z_i$ ” is replaced by “ Y is a Schauder admissible subset of $E \equiv \prod_{i \in J} E_i$ ”.

Theorem 2.17. Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of sets each in a Hausdorff topological vector space Z_i and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous

map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$ and in addition there exists a compact set K_i with $\Phi_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in I$, suppose $G_j \in Kak(Y, X_j)$ where $Y \equiv \prod_{i \in J} Y_i$. Also assume $K \equiv \prod_{i \in J} K_i$ is a Schauder admissible subset of $E \equiv \prod_{i \in J} E_i$. Then there exists a $x \in X$, a $y \in Y$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_i \in G_i(y)$, for all $i \in I$.

Proof. Let Φ and G be as in Theorem 2.15. Note $\Phi \in Kak(X, Y)$ (so $\Phi \in Kak(X, K)$ since $\Phi_i(X) \subseteq K_i$ for $i \in J$) and $G \in Kak(Y, X)$ (so $G \in Kak(K, X)$). Combining and note $\Phi G \in Ad(K, K)$ so Theorem 1.2 guarantees a $y \in K$ with $y \in \Phi G(y)$. The result is now immediate. $\square$

Remark 2.18. One could also consider $G \Phi$ in the proof of Theorem 2.17 (with an appropriate assumption).

Theorem 2.19. Let I and J be index sets with I a finite set, $\{X_i\}_{i \in I}$ be a family of sets each in a Hausdorff topological vector space Z_i and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$. For each $j \in I$, let K_j be a nonempty compact subset of X_j and suppose $G_j \in Ad(Y, K_j)$ (or, alternatively $G_j \in Ad(Y, X_j)$ and $G_j(Y) \subseteq K_j$) where $Y \equiv \prod_{i \in J} Y_i$. Also assume $K \equiv \prod_{i \in I} K_i$ is a Schauder admissible subset of $Z \equiv \prod_{i \in I} Z_i$. Then there exists a $x \in X$, a $y \in Y$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_i \in G_i(y)$, for all $i \in I$.

Proof. Let Φ and G be as in Theorem 2.15. Note $\Phi \in Kak(X, Y)$ (so $\Phi \in Kak(K, Y)$). Next note from [9] that the finite product of admissible with respect to Gorniewicz maps is admissible with respect to Gorniewicz (follows from Künneth's theorem), so $G \in Ad(Y, K)$. Combining we have $G \Phi \in Ad(K, K)$ (note the composition of two admissible with respect to Gorniewicz maps is admissible with respect to Gorniewicz), so Theorem 1.2 guarantees a $x \in K$ with $x \in G \Phi(x)$, and the result is now immediate. $\square$

Remark 2.20. Note there is an analogue of Remark 2.16 (with an appropriate assumption).

Theorem 2.21. Let I and J be index sets with I a finite set, $\{X_i\}_{i \in I}$ be a family of sets each in a Hausdorff topological vector space Z_i and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$, and in addition there exists a compact set K_i with $\Phi_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in I$, suppose $G_j \in Ad(Y, X_j)$ where $Y \equiv \prod_{i \in J} Y_i$. Also assume $K \equiv \prod_{i \in J} K_i$ is a Schauder admissible subset of $E \equiv \prod_{i \in J} E_i$. Then there exists a $x \in X$, a $y \in Y$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_i \in G_i(y)$, for all $i \in I$.

Proof. Let Φ and G be as in Theorem 2.15 and note $\Phi \in Kak(X, K)$ and $G \in Ad(Y, X)$ (so $G \in Ad(K, X)$). Combining and note $\Phi G \in Ad(K, K)$, so Theorem 1.2 guarantees a $y \in K$ with $y \in \Phi G(y)$ and the result is now immediate. $\square$

Remark 2.22. Note there is an analogue of Remark 2.18 (with an appropriate assumption).

Theorem 2.23. Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of sets each in a Hausdorff topological vector space Z_i and $\{Y_i\}_{i \in J}$ be a family of sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$, and in addition there exists a compact set K_i with $\Phi_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in I$, suppose $G_j : Y \equiv \prod_{i \in J} Y_i \rightarrow Z_j$ and there exists a upper semicontinuous map $\Psi_j : Y \rightarrow CK(X_j)$ (i.e. $\Psi_j \in Kak(Y, X_j)$) with $\Psi_j(y) \subseteq co G_j(y)$, for $y \in Y$. Also assume $K \equiv \prod_{i \in J} K_i$ is a Schauder admissible subset of $E \equiv \prod_{i \in J} E_i$. Then there exists a $x \in X$, a $y \in Y$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_i \in co G_i(y)$, for all $i \in I$.

Proof. Let $\Phi(x) = \prod_{i \in J} \Phi_i(x)$, for $x \in X$ and $\Psi(y) = \prod_{j \in I} \Psi_j(y)$, for $y \in Y$. Note $\Phi \in Kak(X, Y)$ (so $\Phi \in Kak(X, K)$) and $\Psi \in Kak(Y, X)$ (so $\Psi \in Kak(K, X)$). Thus $\Phi \Psi \in Ad(K, K)$, so Theorem 1.2 guarantees a $y \in K$ with $y \in \Phi \Psi(y)$ and the result is now immediate. $\square$

Theorem 2.24. Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of sets each in a Hausdorff topological vector space Z_i and $\{Y_i\}_{i \in J}$ be a family of convex sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists a upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$, for $x \in X$, and in addition there exists a convex compact set K_i with $\Phi_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in I$, assume $G_j \in KKM(K, X_j)$ is an upper semicontinuous map with compact values (or, alternatively $G_j \in KKM(Y, X_j)$ is an upper semicontinuous map with compact values) where $K \equiv \prod_{i \in J} K_i$ and $Y \equiv \prod_{i \in J} Y_i$. Also assume K is an admissible subset of $E \equiv \prod_{i \in J} E_i$. Then there exists a $x \in X$, a $y \in Y$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_i \in G_i(y)$, for all $i \in I$.

Proof. Let Φ be as in Theorem 2.15 and note $\Phi \in Kak(X, Y)$ (so $\Phi \in Kak(X, K)$). Let $G(y) = \prod_{j \in I} G_j(y)$, for $y \in K$, and from Property A we have that $G \in KKM(K, A)$ is an upper semicontinuous map with compact values (here $A = \prod_{i \in I} G_i(K)$). Note also that $\Phi \in Kak(A, K)$. Now Theorem 1.7 guarantees that $\Phi G \in KKM(K, K)$, so Theorem 1.6 guarantees a $y \in K$ with $y \in \Phi G(y)$ and the result is now immediate. $\square$

Remark 2.25. In the proof of Theorem 2.24 we could also consider $G \Phi$ and using Theorem 1.8 we see (with appropriate assumptions) that $G \Phi \in KKM(X, X)$ and that then one could apply Theorem 1.6.

Theorem 2.26. *Let I and J be index sets, $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space Z_i and $\{Y_i\}_{i \in J}$ be a family of convex sets each in a Hausdorff topological vector space E_i . For each $i \in J$, suppose $F_i : X \equiv \prod_{j \in I} X_j \rightarrow E_i$ and there exists an upper semicontinuous map $\Phi_i : X \rightarrow CK(Y_i)$ (i.e. $\Phi_i \in Kak(X, Y_i)$) with $\Phi_i(x) \subseteq co F_i(x)$ for $x \in X$. For each $j \in I$, let K_j be a nonempty convex compact subset of X_j and suppose $G_j \in KKM(Y, K_j)$ is an upper semicontinuous map with compact values (or, alternatively $G_j \in KKM(Y, X_j)$ is an upper semicontinuous map with compact values and $G_j(Y) \subseteq K_j$) where $Y \equiv \prod_{i \in J} Y_i$. Also assume $K \equiv \prod_{i \in I} K_i$ is an admissible subset of $Z \equiv \prod_{i \in I} Z_i$, Y is an admissible subset $E \equiv \prod_{i \in J} E_i$ and assume Y is a normal space. Then there exists a $x \in X$, a $y \in Y$ with $y_j \in co F_j(x)$, for all $j \in J$ and $x_i \in G_i(y)$, for all $i \in I$.*

Proof. Let Φ be as in Theorem 2.15 and $G(y) = \prod_{j \in I} G_j(y)$, for $y \in K$. Note $\Phi \in Kak(X, Y)$ (so $\Phi \in Kak(K, Y)$) and $G \in KKM(Y, K)$ is an upper semicontinuous map with compact values. Now Theorem 1.8 guarantees that $G\Phi \in KKM(K, K)$, so Theorem 1.6 guarantees a $x \in K$ with $x \in G\Phi(x)$, and the result is now immediate. $\square$

Remark 2.27. *One could also consider ΦG in the proof of Theorem 2.26 (with appropriate assumptions).*

Remark 2.28. *Note the results in Theorem 2.24 and Theorem 2.26 hold for the original KKM maps (see [5]) if we assume $G \in KKM(Y, X)$ instead of assuming $G_j \in KKM(Y, K_j)$, for each $j \in I$; here $G(y) = \prod_{j \in I} G_j(y)$, for $y \in Y$.*

3 Conclusion

In this paper we use a fixed point result in the literature [4] and properties of maps to present a collectively coincidence theory between similar and different classes of maps. Both the compact and coercive cases are considered. The theory is partly motivated from results in [14, 18, 22], so this paper may be viewed as a companion paper. The paper also motivates potential future directions in coincidence theory, for example considering approximable maps and selectable maps to name but a few.

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