

New numerical radius inequalities for products of Hilbert space operators

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ABSTRACT

In this paper, we prove some inequalities of the operator norm and the numerical radius for Hilbert spaces operators. Among other inequalities, it is shown that if $A, B \in \mathcal{B}(\mathcal{H})$ and $\operatorname{Re}(A) > 0$, then

$$\omega(BA) \leq \left(\frac{\omega(B) + \|B\|}{2} + D_B \right) \omega(A) \leq \frac{7}{2} \omega(A) \omega(B),$$

where $D_B = \inf_{\lambda \in \mathbb{C}} \|B - \lambda I\|$. Moreover, we prove that if $A, B \in \mathcal{B}(\mathcal{H})$ and $A > 0$, then

$$\omega(AB + BA) \leq \frac{3\|A\| - \min_{\lambda \in \sigma(A)} \lambda}{2} \omega(B) < \frac{3}{2} \|A\| \omega(B).$$

RESUMEN

En este artículo demostramos algunas desigualdades para la norma de operadores y el radio numérico para operadores en espacios de Hilbert. Entre otras desigualdades, mostramos que si $A, B \in \mathcal{B}(\mathcal{H})$ y $\operatorname{Re}(A) > 0$, entonces

$$\omega(BA) \leq \left(\frac{\omega(B) + \|B\|}{2} + D_B \right) \omega(A) \leq \frac{7}{2} \omega(A) \omega(B),$$

donde $D_B = \inf_{\lambda \in \mathbb{C}} \|B - \lambda I\|$. Más aún, demostramos que si $A, B \in \mathcal{B}(\mathcal{H})$ y $A > 0$, entonces

$$\omega(AB + BA) \leq \frac{3\|A\| - \min_{\lambda \in \sigma(A)} \lambda}{2} \omega(B) < \frac{3}{2} \|A\| \omega(B).$$

Keywords and Phrases: Bounded linear operator, Hilbert space, norm inequality, numerical radius.

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1 Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators on Hilbert space $\mathcal{H}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be self-adjoint if $A^* = A$, where A^* denotes the adjoint of A . Moreover, $A > 0$ if $\langle Ax, x \rangle > 0$ for all non-zero vectors $x \in \mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, let

$$\omega(A) = \sup \{ |\langle Ax, x \rangle| : \|x\| = 1 \}$$

and

$$\|A\| = \sup \{ \|Ax\| : \|x\| = 1 \},$$

denote the numerical radius and operator norm of A , respectively. In [20], Yamazaki proved that for any $A \in \mathcal{B}(\mathcal{H})$

$$\omega(A) = \sup_{\theta \in \mathbb{R}} \|\operatorname{Re}(e^{i\theta} A)\|. \quad (1.1)$$

It is well-known that for every $A \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \quad (1.2)$$

The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A is normal. Several numerical radius inequalities improving the inequalities in (1.2) have been recently given in [2, 5–7, 11–13, 15, 16, 18, 21] and [22]. It is well-known to all that the submultiplicative property does not hold for the numerical radius. Holbrook in [9] showed that, for any $A, B \in \mathcal{B}(\mathcal{H})$,

$$\omega(AB) \leq 4\omega(A)\omega(B). \quad (1.3)$$

In the case $AB = BA$, then

$$\omega(AB) \leq 2\omega(A)\omega(B).$$

If A is an isometry and $AB = BA$, or a unitary operator that commutes with another operator B , then

$$\omega(AB) \leq \omega(B).$$

The question about the best constant k such that the inequality

$$w(AB) \leq k\|A\|\omega(B) \quad (1.4)$$

holds for all operators $A, B \in \mathcal{B}(\mathcal{H})$ is still open.

Let $D_A = \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|$ (the distance of A from scalar operators), and let R_A denote the radius of the smallest disk in the complex plane containing $\sigma(A)$ (the spectrum of A). It is not hard to check that there exists $\lambda_0 \in \mathbb{C}$ such that $D_A = \|A - \lambda_0 I\|$. It is known (see, *e.g.*, [19]) that $D_A = R_A$

for any normal operator A . The following result from [10] may be stated as well: if $A, B \in \mathcal{B}(\mathcal{H})$, then

$$w(AB) \leq \omega(A)\omega(B) + D_A D_B. \quad (1.5)$$

On the other hand, the authors in [14] proved that if $A, B \in \mathcal{B}(\mathcal{H})$ and A is self-adjoint, then

$$w(AB) \leq D_B \|A\|. \quad (1.6)$$

Concerning the inequality (1.4), it is shown in [1] that if $A, B \in \mathcal{B}(\mathcal{H})$, then

$$\omega(AB) \leq (D_A + \|A\|)\omega(B). \quad (1.7)$$

Also, if $A > 0$, then

$$\omega(AB) \leq \frac{3}{2} \|A\| \omega(B)$$

and

$$\omega(AB) \leq \|A\| \left(\frac{\omega(B) + \|B\|}{2} \right). \quad (1.8)$$

It is shown in [8] that, for any $A, B \in \mathcal{B}(\mathcal{H})$,

$$\omega(A^*B \pm BA) \leq 2\|A\|\omega(B). \quad (1.9)$$

In Section 2, we introduce some numerical radius inequalities for products of two operators. Some applications of these inequalities are considered as well. More precisely, we obtain a generalization of inequality (1.8).

2 Main results

In order to derive our main results, we need the following lemma, which can be found in [4].

Lemma 2.1. *Let a, b and $e \in \mathcal{H}$. Then*

$$|\langle a, e \rangle \langle e, b \rangle| \leq \frac{\|e\|^2}{2} (\|a\| \|b\| + |\langle a, b \rangle|).$$

The following result may be as well.

Lemma 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$. If $A > 0$, then*

$$|\langle Ax, y \rangle| \leq \frac{\|A\|}{2} (\|x\| \|y\| + |\langle x, y \rangle|),$$

for every $x, y \in \mathcal{H}$.

Proof. Putting $a = x$, $e = Ax$ and $b = y$, where $x, y \in \mathcal{H}$, in Lemma 2.1 gives

$$|\langle x, Ax \rangle \langle Ax, y \rangle| \leq \frac{\|Ax\|^2}{2} (\|x\| \|y\| + |\langle x, y \rangle|).$$

Hence,

$$\begin{aligned} |\langle Ax, y \rangle| &\leq \frac{\|Ax\|^2}{2\langle x, Ax \rangle} (\|x\| \|y\| + |\langle x, y \rangle|) \leq \frac{\|A^{\frac{1}{2}} A^{\frac{1}{2}} x\|^2}{2\|A^{\frac{1}{2}} x\|} (\|x\| \|y\| + |\langle x, y \rangle|) \\ &\leq \frac{\|A^{\frac{1}{2}}\|^2}{2} (\|x\| \|y\| + |\langle x, y \rangle|) \leq \frac{\|A\|}{2} (\|x\| \|y\| + |\langle x, y \rangle|). \end{aligned}$$

Therefore,

$$|\langle Ax, y \rangle| \leq \frac{\|A\|}{2} (\|x\| \|y\| + |\langle x, y \rangle|).$$

This completes the proof. $\square$

Remark 2.3. The inequality in Lemma 2.2 has already been proven by a different method in [17]. We use this lemma here as an auxiliary tool to derive the main inequalities of the paper.

The following theorem provides a generalization of inequality (1.8) for Hilbert space operators.

Theorem 2.4. Let $A, B, C \in \mathcal{B}(\mathcal{H})$. If $A > 0$, then

$$\omega(BAC) \leq \left(\frac{\omega(BC) + \|B\| \|C\|}{2} \right) \|A\|.$$

Proof. Suppose that $x \in \mathcal{H}$ with $\|x\| = 1$. By Lemma 2.2,

$$\begin{aligned} |\langle ACx, B^*x \rangle| &\leq \frac{\|A\|}{2} (\|Cx\| \|B^*x\| + |\langle Cx, B^*x \rangle|) \leq \frac{\|A\|}{2} (\|B\| \|C\| + |\langle BCx, x \rangle|) \\ &\leq \frac{\|A\|}{2} \left(\|B\| \|C\| + \sup_{\|x\|=1} |\langle BCx, x \rangle| \right) = \frac{\|A\|}{2} (\|B\| \|C\| + \omega(BC)). \end{aligned}$$

Consequently,

$$|\langle BACx, x \rangle| \leq \frac{\|A\|}{2} (\|B\| \|C\| + \omega(BC)).$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ gives

$$\omega(BAC) \leq \frac{\|A\|}{2} (\|B\| \|C\| + \omega(BC)),$$

which is exactly the desired result. $\square$

Remark 2.5. In particular, choosing $B = I$ (the identity operator) in Theorem 2.4, we obtain inequality (1.8). Moreover, if $A > 0$, then

$$\left\| \frac{2}{\|A\|} A - I \right\| \leq 1,$$

and hence Theorem 2.4 follows as a particular case of Theorem 4.4 in [3].

Corollary 2.6. Let $A, B \in \mathcal{B}(\mathcal{H})$. If $\operatorname{Re}(A) > 0$, then

$$\omega(BA) \leq (\omega(B) + \|B\|)\omega(A) + \omega(BA^*).$$

Proof. For every $\theta \in \mathbb{R}$, we have

$$\begin{aligned} \omega(BA) &= \omega(B(A + A^*) - BA^*) \leq \omega(B(A + A^*)) + \omega(BA^*) \\ &\leq \omega(A + A^*) \frac{\|B\| + \omega(B)}{2} + \omega(BA^*) \quad (\text{by Theorem 2.4}) \\ &= \|\operatorname{Re}(A)\|(\|B\| + \omega(B)) + \omega(BA^*) \leq \sup_{\theta \in \mathbb{R}} \|\operatorname{Re}(e^{i\theta} A)\|(\|B\| + \omega(B)) + \omega(BA^*) \\ &= \omega(A)(\|B\| + \omega(B)) + \omega(BA^*). \end{aligned}$$

Consequently,

$$\omega(BA) \leq \omega(A)(\|B\| + \omega(B)) + \omega(BA^*).$$

This completes the proof. $\square$

Remark 2.7. Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\operatorname{Re}(A) > 0$. If $BA^* = 0$, then by Corollary 2.6 we gives $\omega(BA) \leq (\omega(B) + \|B\|)\omega(A) \leq 3\omega(B)\omega(A)$, which is an improvement for inequality (1.3).

Theorem 2.8. If $A, B \in \mathcal{B}(\mathcal{H})$ and $\operatorname{Re}(A) > 0$, then

$$\omega(BA) \leq \left(\frac{\omega(B) + \|B\|}{2} + D_B \right) \omega(A)$$

Proof. Clearly, $\|\operatorname{Re}(BA)\| = \omega(\operatorname{Re}(BA))$. Then

$$\begin{aligned} \|\operatorname{Re}(BA)\| &= \omega\left(\frac{BA + A^*B^*}{2}\right) \leq \frac{1}{2}\omega(B(A + A^*) - BA^* + A^*B^*) \\ &\leq \frac{1}{2}(\omega(B(A + A^*)) + \omega(BA - AB^*)) \\ &\leq \frac{1}{2}\left(\frac{\|B\| + \omega(B)}{2}\|A + A^*\| + \omega(BA - AB^*)\right) \quad (\text{by Theorem 2.4}) \\ &= \frac{\|B\| + \omega(B)}{2}\|\operatorname{Re}(A)\| + \frac{1}{2}\omega(BA - AB^*) \\ &\leq \frac{\|B\| + \omega(B)}{2} \sup_{\theta \in \mathbb{R}} \|e^{i\theta} A\| + \frac{1}{2}\omega(BA - AB^*) = \frac{\|B\| + \omega(B)}{2}\omega(A) + \frac{1}{2}\omega(BA - AB^*). \end{aligned}$$

Hence,

$$\|\operatorname{Re}(BA)\| \leq \frac{\|B\| + \omega(B)}{2} \omega(A) + \frac{1}{2} \omega(BA - AB^*) \quad (2.1)$$

Let $z_0 \in \mathbb{C}$ be such that

$$\|B - z_0 I\| = D_B.$$

If $z_0 = 0$, then

$$\|B - z_0 I\| = \|B\| = D_B,$$

and the result follows from (1.9) and (2.1). On the other hand, if $z_0 \neq 0$, we set

$$\alpha_0 = \frac{\overline{z_0}}{|z_0|}.$$

Replacing B by $\alpha_0 B$ in the inequality (2.1) gives

$$\begin{aligned} \|\operatorname{Re}(\alpha_0 BA)\| &\leq \frac{\|B\| + \omega(B)}{2} \omega(A) + \frac{1}{2} \omega(\alpha_0 BA - \bar{\alpha}_0 AB^*) \\ &\leq \frac{\|B\| + \omega(B)}{2} \omega(A) + \frac{1}{2} \omega(\alpha_0 (B - z_0 I)A - \bar{\alpha}_0 A(B - z_0 I)^*) \\ &\leq \frac{\|B\| + \omega(B)}{2} \omega(A) + \|B - z_0 I\| \omega(A). \quad (\text{by (1.9)}) \end{aligned}$$

Consequently,

$$\|\operatorname{Re}(\alpha_0 BA)\| \leq \frac{\|B\| + \omega(B)}{2} \omega(A) + D_B \omega(A). \quad (2.2)$$

Suppose that $\theta \in \mathbb{R}$ and $\theta_0 \in \mathbb{R}$ such that $\alpha_0 = e^{i\theta_0}$. Replacing B by $e^{i\theta} B$ in the inequality (2.2) gives

$$\left\| \operatorname{Re} \left(e^{i(\theta+\theta_0)} BA \right) \right\| \leq \frac{\|B\| + \omega(B)}{2} \omega(A) + D_B \omega(A).$$

Taking the supremum over $\theta \in \mathbb{R}$ gives

$$\omega(BA) \leq \frac{\|B\| + \omega(B)}{2} \omega(A) + D_B \omega(A),$$

which is exactly the desired result. $\square$

Corollary 2.9. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\operatorname{Re}(A) > 0$. Suppose that there exists a positive constant k such that $D_B \leq k\|B\|$, then*

$$\omega(BA) \leq \frac{(4k+3)}{2} \omega(A) \omega(B).$$

Proof. From the hypothesis $D_B \leq k\|B\|$ and Theorem 2.8,

$$\begin{aligned} \omega(BA) &\leq \left(\frac{\omega(B) + \|B\|}{2} + D_B \right) \omega(A) \leq \left(\frac{\omega(B) + (2k+1)\|B\|}{2} \right) \omega(A) \\ &\leq \frac{(4k+3)}{2} \omega(A) \omega(B). \quad (\text{LHS of (1.2)}) \end{aligned}$$

Consequently,

$$\omega(BA) \leq \frac{(4k+3)}{2} \omega(A) \omega(B),$$

which is exactly the desired result. $\square$

Remark 2.10. Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\operatorname{Re}(A) > 0$. Since

$$D_B = \inf_{\lambda \in \mathbb{C}} \|B - \lambda I\| \leq \|B\|,$$

by taking $k = 1$ in Corollary 2.9, we obtain

$$\omega(BA) \leq \frac{7}{2} \omega(A) \omega(B),$$

which is an improvement of inequality (1.3).

More generally, for certain operators one can choose $k < 1$, which yields a sharper estimate in Corollary 2.9. For example, consider the nontrivial orthogonal projection

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since

$$\|P - \lambda I\| = \max\{|\lambda|, |1 - \lambda|\},$$

we have

$$D_P = \inf_{\lambda \in \mathbb{C}} \max\{|\lambda|, |1 - \lambda|\} = \frac{1}{2} = \frac{1}{2} \|P\|.$$

Thus, $k = \frac{1}{2} < 1$ can be chosen in Corollary 2.9 for $B = P$, and consequently

$$\omega(PA) \leq \frac{5}{2} \omega(A) \omega(P).$$

On the other hand, $k < 1$ cannot be chosen uniformly for all operators. Indeed, for

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we have

$$D_N = \|N\| = 1.$$

Theorem 2.11. *Let $A, B \in \mathcal{B}(\mathcal{H})$. If A is self-adjoint, then*

$$\omega(AB + BA) \leq (D_A + \|A\|) \omega(B). \quad (2.3)$$

Proof. Clearly, $\|\operatorname{Re}(AB)\| = \omega(\operatorname{Re}(AB))$. Then

$$\begin{aligned} \|\operatorname{Re}(AB + BA)\| &= \omega\left(\frac{AB + BA + B^*A + AB^*}{2}\right) \leq \frac{\omega(A(B + B^*))}{2} + \frac{\omega((B + B^*)A)}{2} \\ &\leq \frac{D_{B+B^*}\|A\|}{2} + \frac{D_A\|B + B^*\|}{2} \quad (\text{by (1.6)}) \\ &\leq \frac{\|B + B^*\|}{2}(\|A\| + D_A) = \|\operatorname{Re}(B)\|(\|A\| + D_A) \\ &\leq \sup_{\theta \in \mathbb{R}} \|\operatorname{Re}(e^{i\theta}B)\|(\|A\| + D_A) = \omega(B)(\|A\| + D_A). \end{aligned}$$

Consequently,

$$\|\operatorname{Re}(AB + BA)\| \leq \omega(B)(\|A\| + D_A). \quad (2.4)$$

Let $\theta \in \mathbb{R}$ and replacing B by $e^{i\theta}B$ in the inequality (2.4) gives

$$\|\operatorname{Re}(e^{i\theta}(AB + BA))\| \leq \omega(B)(\|A\| + D_A).$$

Taking the supremum over $\theta \in \mathbb{R}$ gives

$$\omega(AB + BA) \leq (\|A\| + D_A)\omega(B). \quad \square$$

Remark 2.12. *Let $A, B \in \mathcal{B}(\mathcal{H})$ satisfy the condition in Theorem 2.11. Since $D_A \leq \|A\|$, inequality (2.3) is a considerable improvement of inequality (1.9).*

As a direct consequence of Theorem 2.11, we obtain the following refinement of the well-known inequality (1.9).

Corollary 2.13. *Let $A, B \in \mathcal{B}(\mathcal{H})$. If $A > 0$, then*

$$\omega(AB + BA) \leq \frac{3\|A\| - \min_{\lambda \in \sigma(A)} \lambda}{2} \omega(B) < \frac{3}{2} \|A\| \omega(B).$$

Proof. Since $D_A = R_A = \frac{\max_{\lambda \in \sigma(A)} \lambda - \min_{\lambda \in \sigma(A)} \lambda}{2}$ and $\max_{\lambda \in \sigma(A)} \lambda = \|A\|$, the result follows from Theorem 2.11. $\square$

Declarations

Competing interests

The authors declare that they do not have any competing interests.

Conflicts of interest

All authors declare that they have no conflict of interest.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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