

Geometric structure of generalized Fibonacci vectors

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ABSTRACT

Consecutive-term vectors associated with second-order linear recurrences have been studied in several settings, including generalized Fibonacci vectors, Fibonacci/Lucas vectors, Horadam vectors, and Pell/Pell–Lucas vectors. In this paper, we give a unified geometric formulation for length- m vectors generated by the recurrence $U_n = pU_{n-1} + qU_{n-2}$ under the hypothesis of two distinct real characteristic roots. Using the intrinsic two-dimensional structure of the solution space, we derive Binet-type vector formulas and show that every exterior product of two consecutive-term vectors is a scalar multiple of one fixed bivector. This exterior-algebra formulation explains the planar confinement observed in three dimensions and yields closed-form inner products in arbitrary dimension, including mixed products involving the normalized Lucas-type companion sequence. The Pell specialization recovers the consistent Pell/Pell–Lucas vector identities and makes their rank-two origin explicit.

RESUMEN

Los vectores con términos consecutivos asociados a recurrencias lineales de segundo orden han sido estudiados en variados contextos, incluyendo vectores generalizados de Fibonacci, vectores de Fibonacci/Lucas, vectores de Horadam y vectores de Pell/Pell–Lucas. En este artículo, entregamos una formulación geométrica unificada para vectores de longitud m generados por la recurrencia $U_n = pU_{n-1} + qU_{n-2}$, bajo la hipótesis de dos raíces características reales distintas. Usando la estructura intrínseca dos dimensional del espacio de soluciones, derivamos fórmulas de tipo Binet vectoriales y mostramos que todo producto exterior de dos vectores con términos consecutivos es un múltiplo escalar de un bivector fijo. Esta formulación usando álgebra exterior explica el confinamiento planar observado en tres dimensiones y entrega productos interiores con fórmula cerrada en dimensión arbitraria, incluyendo productos mixtos que involucran la sucesión de acompañamiento de tipo Lucas normalizada. La especialización de Pell recupera las identidades vectoriales consistentes de Pell/Pell–Lucas y hace explícito su origen de rango dos.

Keywords and Phrases: linear recurrence, Lucas sequences, consecutive-term vectors, inner product, exterior product, Pell numbers.

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1 Introduction

Second-order linear recurrences are ubiquitous in discrete mathematics and number theory; see, for example, [3] for general background on linear difference equations and Horadam's classical paper [5] for generalized Fibonacci sequences. Classical instances include the Fibonacci/Lucas family, for which many identities and applications are compiled in [7], and the Pell/Pell–Lucas family, which is closely tied to Pell-type quadratic Diophantine equations and continued-fraction methods [8]. Many other second-order families, such as Jacobsthal-type sequences, fit the same framework.

A useful geometric viewpoint is obtained by grouping consecutive terms of a recurrence into a fixed-length vector. This idea predates several recent vector-specific studies. Jaiswal [6] studied geometric properties of generalized Fibonacci sequences, and Fuller [4] extended that viewpoint to higher-dimensional vectors whose components belong to a generalized Fibonacci sequence; in particular, Fuller explicitly obtained a rank-two phenomenon for matrices formed from such vectors. In the Fibonacci/Lucas setting, Salter [9] introduced length- m Fibonacci/Lucas vectors and derived closed forms for dot products for arbitrary m . More recently, Çetinberk and Yüce [2] revisited Fibonacci vectors from a vector-algebraic standpoint, while Bingöl and Gökbaş [1] developed Horadam-vector formulas involving inner products, norms, and vector products. For the Pell/Pell–Lucas sequences, Uysal and Özkan [10] studied analogous consecutive-term vectors in dimensions 2 and 3, and for arbitrary m ; they obtained explicit inner products and cross products, noting a fixed-plane phenomenon in $\mathbb{R}^3$.

Building upon the context discussed above, the present paper provides a unifying geometric framework. Rather than analyzing specific parameter families in isolation, our purpose is to capture the intrinsic rank-two nature of these recurrences by isolating the underlying root-plane mechanism and expressing it uniformly through exterior algebra. This formulation treats planar confinement, cross products in $\mathbb{R}^3$, all 2×2 minors, and inner products in arbitrary dimension as consequences of one fixed two-dimensional eigenspace. Concretely, we study the length- m consecutive-term vectors associated with the general second-order recurrence

$$U_n = pU_{n-1} + qU_{n-2} \quad (n \geq 2), \quad (1.1)$$

under the hypothesis that the characteristic polynomial has two distinct real roots (equivalently, $\Delta = p^2 + 4q > 0$). We consider

$$\mathbf{U}_n^{(m)} = (U_n, U_{n+1}, \dots, U_{n+m-1})^\top \in \mathbb{R}^m,$$

together with a companion sequence $\{V_n\}$ whose Binet formula is the symmetrized expression $V_n = \frac{r_1^n + r_2^n}{2}$. Note that this normalization aligns with the Pell–Lucas convention in [10] and corresponds to half the standard Lucas numbers $(L_n/2)$ in the Fibonacci case. The Pell/Pell–

Lucas case is recovered by taking $(p, q) = (2, 1)$.

A central point of this work is that the relevant vector identities are controlled by the fixed two-dimensional root plane generated by

$$\mathbf{r}_i = (1, r_i, \dots, r_i^{m-1})^\top \quad (i = 1, 2),$$

where r_1, r_2 are the distinct characteristic roots of $x^2 - px - q = 0$. We first record Binet-type vector formulas for $\mathbf{U}_n^{(m)}$ and $\mathbf{V}_n^{(m)}$ in Theorem 3.1, together with the canonical shift matrix $\mathbf{T}_m$ satisfying $\mathbf{U}_{n+1}^{(m)} = \mathbf{T}_m \mathbf{U}_n^{(m)}$ and its companion analogue in Proposition 3.3. We then give the rank-two containment and the $m-2$ coordinate constraints in Theorem 3.5. The exterior-product result, Theorem 4.2, shows that every bivector $\mathbf{U}_n^{(m)} \wedge \mathbf{U}_k^{(m)}$ is a scalar multiple of $\mathbf{r}_1 \wedge \mathbf{r}_2$, and the $\mathbb{R}^3$ cross-product identities are obtained from this statement by applying the Hodge dual. Finally, Theorem 5.3 gives closed forms for $\langle \mathbf{U}_n^{(m)}, \mathbf{U}_k^{(m)} \rangle$, $\langle \mathbf{V}_n^{(m)}, \mathbf{V}_k^{(m)} \rangle$, and $\langle \mathbf{U}_n^{(m)}, \mathbf{V}_k^{(m)} \rangle$ in arbitrary dimension through the geometric factor $S_m(q) = \sum_{j=0}^{m-1} (-q)^j$. The Pell specialization $(p, q) = (2, 1)$ recovers the consistent Pell/Pell–Lucas identities and clarifies the role of the normalization $V_n = Q_n$.

Organization. Section 2 collects definitions and known facts on second-order recurrences needed later. Section 3 establishes the rank-two structure and the shift-matrix representation. Section 4 develops the exterior-product direction invariant and its $\mathbb{R}^3$ cross-product consequence. Finally, Section 5 derives the closed-form inner products and specializes them to Pell/Pell–Lucas vectors.

On the range of indices and hypotheses. The results in Section 3 are formulated for the forward orbit of the recurrence and therefore only require indices $n \geq 0$. By contrast, the exterior-product and inner-product identities in Sections 4 and 5 are stated for arbitrary integer shifts and therefore use the extension of the sequences to all of $\mathbb{Z}$; in those sections we accordingly assume $q \neq 0$.

2 Preliminaries

2.1 Second-order linear recurrences and companion sequence

Fix parameters $p, q \in \mathbb{R}$ and consider the recurrence (1.1) with initial conditions

$$U_0 = 0, \quad U_1 = 1. \quad (2.1)$$

Let r_1, r_2 be the roots of the characteristic polynomial

$$x^2 - px - q = 0. \quad (2.2)$$

Throughout we assume that (2.2) has two distinct real roots $r_1 \neq r_2$. Equivalently, the discriminant

$$\Delta := (r_1 - r_2)^2 = p^2 + 4q \quad (2.3)$$

is positive, *i.e.* $\Delta > 0$. Note also that

$$r_1 + r_2 = p, \quad r_1 r_2 = -q. \quad (2.4)$$

(For background on such recurrences and their asymptotics, see, *e.g.*, [3, Ch. 2, Ch. 8]; for Pell-specific material see [8].)

Definition 2.1 (Normalized Lucas-type companion). *Define the companion sequence $\{V_n\}_{n \geq 0}$ by*

$$\begin{aligned} V_0 &= 1, \quad V_1 = \frac{p}{2}, \\ V_n &= pV_{n-1} + qV_{n-2} \quad (n \geq 2). \end{aligned} \quad (2.5)$$

The normalization in Definition 2.1 matches the Pell–Lucas convention used by Uysal and Ozkan ($Q_n = (a^n + b^n)/2$ in the Pell case) [10]. In many texts, the (unnormalized) Lucas sequence of the second kind is $L_n = 2V_n$; we keep the factor $1/2$ so that our Pell specialization reproduces the formulas in [10] without rescaling.

Lemma 2.2 (Binet formulas). *Let $r_1 \neq r_2$ be the roots of (2.2) and set $D := r_1 - r_2$. Then for all $n \geq 0$,*

$$U_n = \frac{r_1^n - r_2^n}{D}, \quad (2.6)$$

and for all $n \geq 0$,

$$V_n = \frac{r_1^n + r_2^n}{2}. \quad (2.7)$$

Proof. We provide a detailed derivation for completeness.

Step 1: General form. Because (1.1) is a homogeneous linear recurrence of order 2 with distinct characteristic roots, its general solution has the form $U_n = Ar_1^n + Br_2^n$ for constants A, B depending on the initial values.

Step 2: Determine A, B for U_n . Using (2.1):

$$U_0 = 0 = A + B, \quad U_1 = 1 = Ar_1 + Br_2.$$

From $A + B = 0$ we get $B = -A$, hence $1 = A(r_1 - r_2) = AD$, so $A = 1/D$ and $B = -1/D$, giving (2.6).

Step 3: Determine A, B for V_n . Similarly, $V_n = Ar_1^n + Br_2^n$ for some constants. Using (2.5):

$$V_0 = 1 = A + B, \quad V_1 = \frac{p}{2} = Ar_1 + Br_2.$$

Because $r_1 + r_2 = p$ by (2.4), we see that choosing $A = B = \frac{1}{2}$ satisfies both equations: $A + B = 1$ and $Ar_1 + Br_2 = \frac{1}{2}(r_1 + r_2) = \frac{p}{2}$. Therefore $V_n = \frac{1}{2}(r_1^n + r_2^n)$, proving (2.7). $\square$

2.2 Extension to negative indices

In several statements below (especially in Sections 4–5) it is convenient to allow integer shifts $n - k$ without restricting $n \geq k$. This requires extending the sequences $\{U_n\}$ and $\{V_n\}$ to all integers.

Definition 2.3 (Extension to $\mathbb{Z}$). Assume $q \neq 0$ (equivalently $r_1 r_2 \neq 0$). We extend $\{U_n\}$ and $\{V_n\}$ uniquely to sequences indexed by all $n \in \mathbb{Z}$ by requiring that (1.1) and (2.5) hold for every $n \in \mathbb{Z}$. Equivalently, for $n \leq 1$ we define backwards by

$$U_{n-2} := \frac{U_n - pU_{n-1}}{q}, \quad V_{n-2} := \frac{V_n - pV_{n-1}}{q}. \quad (2.8)$$

Lemma 2.4 (Negative-index formulas). Assume $q \neq 0$ and $r_1 \neq r_2$. Then the Binet formulas (2.6)–(2.7) hold for all $n \in \mathbb{Z}$, and moreover, for every integer $n \geq 1$,

$$U_{-n} = -(r_1 r_2)^{-n} U_n = -(-q)^{-n} U_n, \quad V_{-n} = (r_1 r_2)^{-n} V_n = (-q)^{-n} V_n. \quad (2.9)$$

Proof. Since $q \neq 0$, we have $r_1 r_2 = -q \neq 0$, hence $r_1, r_2 \neq 0$. For $n \in \mathbb{Z}$, define U_n and V_n by the right-hand sides of (2.6) and (2.7). These expressions satisfy the recurrence because r_1^n and r_2^n do, and they match the given initial values at $n = 0, 1$. By uniqueness of solutions to a second-order recurrence, this agrees with the backward extension in Definition 2.3.

For $n \geq 1$,

$$U_{-n} = \frac{r_1^{-n} - r_2^{-n}}{D} = \frac{r_2^n - r_1^n}{D(r_1 r_2)^n} = -(r_1 r_2)^{-n} \frac{r_1^n - r_2^n}{D} = -(r_1 r_2)^{-n} U_n,$$

and similarly

$$V_{-n} = \frac{r_1^{-n} + r_2^{-n}}{2} = \frac{r_1^n + r_2^n}{2(r_1 r_2)^n} = (r_1 r_2)^{-n} V_n.$$

Using $r_1 r_2 = -q$ gives (2.9). $\square$

2.3 Consecutive-term vectors and basic linear-algebra notation

Fix an integer $m \geq 1$. We equip $\mathbb{R}^m$ with the standard Euclidean inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^m x_j y_j$, and norm $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

Definition 2.5 (Consecutive-term vectors (length m)). *Fix an integer $m \geq 1$. For each $n \geq 0$ define the consecutive-term vectors associated with $\{U_n\}$ and $\{V_n\}$ by*

$$\mathbf{U}_n^{(m)} := (U_n, U_{n+1}, \dots, U_{n+m-1})^\top \in \mathbb{R}^m, \quad \mathbf{V}_n^{(m)} := (V_n, V_{n+1}, \dots, V_{n+m-1})^\top \in \mathbb{R}^m. \quad (2.10)$$

If, in addition, $q \neq 0$ and the sequences are extended to $\mathbb{Z}$ as in Definition 2.3, then the same formulas define $\mathbf{U}_n^{(m)}$ and $\mathbf{V}_n^{(m)}$ for all $n \in \mathbb{Z}$.

We also use the vectors built from powers of a scalar:

$$\mathbf{r}_i := (1, r_i, r_i^2, \dots, r_i^{m-1})^\top \in \mathbb{R}^m \quad (i = 1, 2). \quad (2.11)$$

Finally, we recall the exterior (wedge) product on $\mathbb{R}^m$: for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, the bivector $\mathbf{x} \wedge \mathbf{y}$ is bilinear, alternating ($\mathbf{x} \wedge \mathbf{x} = 0$), and satisfies $\mathbf{y} \wedge \mathbf{x} = -\mathbf{x} \wedge \mathbf{y}$. When $m = 3$, the Hodge dual identifies bivectors with vectors and $\mathbf{x} \wedge \mathbf{y}$ corresponds to the cross product $\mathbf{x} \times \mathbf{y}$.

3 Rank-two structure and shift-matrix representation

In this section we work with the forward orbit of the recurrence only, so all indices are taken with $n \geq 0$. No extension to negative indices is needed here.

3.1 Binet vector formulas

We begin by lifting Lemma 2.2 to the vector setting.

Theorem 3.1 (Binet vector formulas). *Let $m \geq 1$ and $D = r_1 - r_2 \neq 0$. Then for all $n \geq 0$,*

$$\mathbf{U}_n^{(m)} = \frac{r_1^n \mathbf{r}_1 - r_2^n \mathbf{r}_2}{D}, \quad (3.1)$$

and for all $n \geq 0$,

$$\mathbf{V}_n^{(m)} = \frac{r_1^n \mathbf{r}_1 + r_2^n \mathbf{r}_2}{2}. \quad (3.2)$$

If $q \neq 0$ and the sequences are extended to $\mathbb{Z}$ as in Definition 2.3, then (3.1) and (3.2) hold for every $n \in \mathbb{Z}$.

Proof. We prove (3.1); the proof of (3.2) is analogous. For $n \geq 0$, the argument is as follows.

By Definition 2.5,

$$\mathbf{U}_n^{(m)} = (U_n, U_{n+1}, \dots, U_{n+m-1})^\top.$$

Using the scalar Binet formula (2.6) for each component,

$$U_{n+j} = \frac{r_1^{n+j} - r_2^{n+j}}{D} \quad (j = 0, 1, \dots, m-1).$$

Therefore

$$\mathbf{U}_n^{(m)} = \left(\frac{r_1^n - r_2^n}{D}, \frac{r_1^{n+1} - r_2^{n+1}}{D}, \dots, \frac{r_1^{n+m-1} - r_2^{n+m-1}}{D} \right)^\top.$$

Factor out the common $\frac{1}{D}$ and rewrite powers:

$$\mathbf{U}_n^{(m)} = \frac{1}{D} (r_1^n(1, r_1, \dots, r_1^{m-1})^\top - r_2^n(1, r_2, \dots, r_2^{m-1})^\top).$$

By (2.11), this is exactly (3.1). If $q \neq 0$ and $n \in \mathbb{Z}$, the same componentwise proof applies using Lemma 2.4, which extends the scalar Binet formulas to all integer indices. $\square$

Remark 3.2. For Fibonacci/Lucas vectors, a vector form of Binet's formula (with eigenvectors built from $(1, \alpha, \dots, \alpha^{m-1})^\top$ and $(1, \beta, \dots, \beta^{m-1})^\top$) is already implicit/explicit in [9] (see also [2]). Theorem 3.1 keeps this eigenvector viewpoint but formulates it uniformly for arbitrary real distinct roots $r_1 \neq r_2$ and for the normalized companion sequence V_n .

3.2 Shift matrix for consecutive-term vectors

The consecutive-term vectors evolve by a fixed linear map.

Proposition 3.3 (Shift matrix). Fix $m \geq 2$. Define $\mathsf{T}_m \in \mathbb{R}^{m \times m}$ by

$$\mathsf{T}_m := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & q & p \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & q & p \end{pmatrix}. \quad (3.3)$$

Then for all $n \geq 0$,

$$\mathbf{U}_{n+1}^{(m)} = \mathsf{T}_m \mathbf{U}_n^{(m)}, \quad \mathbf{V}_{n+1}^{(m)} = \mathsf{T}_m \mathbf{V}_n^{(m)}. \quad (3.4)$$

Proof. Write $\mathbf{U}_n^{(m)} = (U_n, U_{n+1}, \dots, U_{n+m-1})^\top$. Then multiplying by $\mathbf{T}_m$ as in (3.3) yields the vector whose first $m-1$ components are shifted:

$$\left(\mathbf{T}_m \mathbf{U}_n^{(m)}\right)_1 = U_{n+1}, \quad \left(\mathbf{T}_m \mathbf{U}_n^{(m)}\right)_2 = U_{n+2}, \quad \dots, \quad \left(\mathbf{T}_m \mathbf{U}_n^{(m)}\right)_{m-1} = U_{n+m-1}.$$

The last component uses the last row of $\mathbf{T}_m$:

$$\left(\mathbf{T}_m \mathbf{U}_n^{(m)}\right)_m = q U_{n+m-2} + p U_{n+m-1}.$$

By the defining recurrence (1.1) with index $n+m$, $U_{n+m} = p U_{n+m-1} + q U_{n+m-2}$, so $\left(\mathbf{T}_m \mathbf{U}_n^{(m)}\right)_m = U_{n+m}$. Hence $\mathbf{T}_m \mathbf{U}_n^{(m)} = (U_{n+1}, U_{n+2}, \dots, U_{n+m})^\top = \mathbf{U}_{n+1}^{(m)}$, proving the first identity in (3.4). The proof for $\mathbf{V}_n^{(m)}$ is identical, since V_n satisfies the same recurrence (2.5). $\square$

Remark 3.4. In the Fibonacci/Lucas setting, the existence of a fixed shift (companion) matrix realizing $\mathbf{f}_{n+1}^{(m)} = \mathbf{T}_m \mathbf{f}_n^{(m)}$ (and similarly for Lucas vectors) already appears in [9]. Proposition 3.3 records the same mechanism in the general (p, q) -recurrence and fixes a canonical choice of $\mathbf{T}_m$ valid for all m .

3.3 Rank-two structure

The next theorem generalizes the planar phenomenon observed for $m=3$ in [10].

Theorem 3.5 (Rank-two structure). *Fix $m \geq 2$ and assume $\Delta > 0$ (equivalently, $r_1 \neq r_2$ are real).*

(i) *Every consecutive-term vector lies in a fixed two-dimensional subspace:*

$$\mathbf{U}_n^{(m)}, \mathbf{V}_n^{(m)} \in \text{span}\{\mathbf{r}_1, \mathbf{r}_2\} \subset \mathbb{R}^m, \quad \text{for all } n \geq 0. \quad (3.5)$$

Consequently, the set $\{\mathbf{U}_n^{(m)} : n \geq 0\}$ (and likewise $\{\mathbf{V}_n^{(m)}\}$) has rank at most 2.

(ii) *The consecutive-term vectors satisfy $m-2$ explicit linear constraints: for any $n \geq 0$ and each $j = 0, 1, \dots, m-3$,*

$$\left(\mathbf{U}_n^{(m)}\right)_{j+3} - p \left(\mathbf{U}_n^{(m)}\right)_{j+2} - q \left(\mathbf{U}_n^{(m)}\right)_{j+1} = 0, \quad (3.6)$$

and the same holds with $\mathbf{V}_n^{(m)}$ in place of $\mathbf{U}_n^{(m)}$.

Proof. (i) By Theorem 3.1,

$$\mathbf{U}_n^{(m)} = \frac{r_1^n \mathbf{r}_1 - r_2^n \mathbf{r}_2}{D}, \quad \mathbf{V}_n^{(m)} = \frac{r_1^n \mathbf{r}_1 + r_2^n \mathbf{r}_2}{2}.$$

Both are linear combinations of $\mathbf{r}_1$ and $\mathbf{r}_2$, hence lie in their span, proving (3.5). Because the span is two-dimensional (since $r_1 \neq r_2$ implies $\mathbf{r}_1, \mathbf{r}_2$ are linearly independent), any set of such vectors has rank at most 2.

(ii) Fix $n \geq 0$ and $j \in \{0, 1, \dots, m-3\}$. By Definition 2.5,

$$\left(\mathbf{U}_n^{(m)}\right)_{j+1} = U_{n+j}, \quad \left(\mathbf{U}_n^{(m)}\right)_{j+2} = U_{n+j+1}, \quad \left(\mathbf{U}_n^{(m)}\right)_{j+3} = U_{n+j+2}.$$

Apply the recurrence (1.1) at index $n+j+2$:

$$U_{n+j+2} = pU_{n+j+1} + qU_{n+j}.$$

Rearranging gives $U_{n+j+2} - pU_{n+j+1} - qU_{n+j} = 0$, which is exactly (3.6). The proof for $\mathbf{V}_n^{(m)}$ is identical since V_n satisfies (2.5). $\square$

Remark 3.6. *In the Fibonacci/Lucas case, it was observed that all length- m consecutive-term vectors lie in a fixed plane in $\mathbb{R}^m$ [9]. Theorem 3.5 shows that this “planarity” is not special: it is forced by the intrinsic two-dimensional solution space of any second-order recurrence, and we additionally record the $m-2$ explicit linear constraints in coordinates.*

Corollary 3.7 (Pell case: all 3D vectors lie in a fixed plane). *Let $(p, q) = (2, 1)$ (Pell case) and $m = 3$. Then every Pell consecutive-term vector $\mathbf{P}_n^{(3)} = (P_n, P_{n+1}, P_{n+2})^\top$ satisfies $P_{n+2} - 2P_{n+1} - P_n = 0$. Consequently, all such vectors lie in the plane orthogonal to $(1, 2, -1)$ (equivalently, to $(-1, -2, 1)$). Likewise, the normalized Pell–Lucas consecutive-term vectors $\mathbf{Q}_n^{(3)}$ lie in the plane orthogonal to $(-1, -2, 1)$ (equivalently, to $(1, 2, -1)$).*

Proof. This is Theorem 3.5(ii) with $(p, q) = (2, 1)$, and the observation that the linear constraint $x_3 - 2x_2 - x_1 = 0$ defines a plane with normal $(1, 2, -1)$ (equivalently $(-1, -2, 1)$). The Pell–Lucas statement is the same constraint applied to $\mathbf{V}_n^{(3)}$; the change of sign corresponds to the choice of orientation (compare [10, Cor. 2.1.9]). $\square$

4 Exterior-product invariants

The preceding section identified the fixed root plane containing all consecutive-term vectors. We now pass from containment in that plane to orientation and area data by studying exterior products. The point is that a two-dimensional subspace of $\mathbb{R}^m$ determines a one-dimensional line in $\bigwedge^2 \mathbb{R}^m$, and all bivectors generated by the recurrence must lie on this line.

Throughout this section and Section 5, we work with integer shifts and therefore assume $q \neq 0$ (equivalently, $r_1 r_2 \neq 0$), so that the sequences $\{U_n\}$ and $\{V_n\}$ are extended to all integers via Definition 2.3 and Lemma 2.4. In particular, all statements in these two sections are understood for indices in $\mathbb{Z}$ under this standing assumption.

4.1 A constant bivector direction

The next theorem shows that all bivectors $\mathbf{U}_n^{(m)} \wedge \mathbf{U}_k^{(m)}$ are collinear in the exterior space $\bigwedge^2 \mathbb{R}^m$.

Lemma 4.1. *For all integers n, k ,*

$$r_2^n r_1^k - r_1^n r_2^k = -(r_1 r_2)^k (r_1 - r_2) U_{n-k}. \quad (4.1)$$

Proof. Let $D = r_1 - r_2$. Starting from the left-hand side,

$$r_2^n r_1^k - r_1^n r_2^k = (r_1 r_2)^k (r_2^{n-k} - r_1^{n-k}) = (r_1 r_2)^k (-(r_1^{n-k} - r_2^{n-k})).$$

By the Binet formula (2.6) applied to index $n - k$, $r_1^{n-k} - r_2^{n-k} = D U_{n-k}$. Substituting yields (4.1). $\square$

Theorem 4.2 (Exterior-product invariant). *Fix $m \geq 2$, assume $\Delta > 0$, and assume $q \neq 0$. Let $\mathbf{r}_1, \mathbf{r}_2$ be as in (2.11) and set $D = r_1 - r_2$. Then for all integers n, k ,*

$$\mathbf{U}_n^{(m)} \wedge \mathbf{U}_k^{(m)} = -\frac{(r_1 r_2)^k}{D} U_{n-k} (\mathbf{r}_1 \wedge \mathbf{r}_2), \quad (4.2)$$

$$\mathbf{V}_n^{(m)} \wedge \mathbf{V}_k^{(m)} = \frac{(r_1 r_2)^k D}{4} U_{n-k} (\mathbf{r}_1 \wedge \mathbf{r}_2). \quad (4.3)$$

Using $r_1 r_2 = -q$, the factors $(r_1 r_2)^k$ may be written as $(-q)^k$.

Proof. Let $D = r_1 - r_2$.

Proof of (4.2). By Theorem 3.1,

$$\mathbf{U}_n^{(m)} = \frac{r_1^n \mathbf{r}_1 - r_2^n \mathbf{r}_2}{D}, \quad \mathbf{U}_k^{(m)} = \frac{r_1^k \mathbf{r}_1 - r_2^k \mathbf{r}_2}{D}.$$

Using bilinearity and alternation of $\wedge$,

$$\begin{aligned} \mathbf{U}_n^{(m)} \wedge \mathbf{U}_k^{(m)} &= \frac{1}{D^2} (r_1^n \mathbf{r}_1 - r_2^n \mathbf{r}_2) \wedge (r_1^k \mathbf{r}_1 - r_2^k \mathbf{r}_2) = \frac{1}{D^2} (-r_1^n r_2^k (\mathbf{r}_1 \wedge \mathbf{r}_2) + r_2^n r_1^k (\mathbf{r}_1 \wedge \mathbf{r}_2)) \\ &= \frac{r_2^n r_1^k - r_1^n r_2^k}{D^2} (\mathbf{r}_1 \wedge \mathbf{r}_2). \end{aligned}$$

Now apply Lemma 4.1: $r_2^n r_1^k - r_1^n r_2^k = -(r_1 r_2)^k D U_{n-k}$. Substituting gives (4.2).

Proof of (4.3). Using (3.2),

$$\mathbf{V}_n^{(m)} = \frac{r_1^n \mathbf{r}_1 + r_2^n \mathbf{r}_2}{2}, \quad \mathbf{V}_k^{(m)} = \frac{r_1^k \mathbf{r}_1 + r_2^k \mathbf{r}_2}{2}.$$

Then

$$\begin{aligned} \mathbf{V}_n^{(m)} \wedge \mathbf{V}_k^{(m)} &= \frac{1}{4} (r_1^n \mathbf{r}_1 + r_2^n \mathbf{r}_2) \wedge (r_1^k \mathbf{r}_1 + r_2^k \mathbf{r}_2) = \frac{1}{4} (r_1^n r_2^k (\mathbf{r}_1 \wedge \mathbf{r}_2) + r_2^n r_1^k (\mathbf{r}_2 \wedge \mathbf{r}_1)) \\ &= \frac{1}{4} (r_1^n r_2^k - r_2^n r_1^k) (\mathbf{r}_1 \wedge \mathbf{r}_2). \end{aligned}$$

By Lemma 4.1 (equivalently, multiplying that identity by -1), $r_1^n r_2^k - r_2^n r_1^k = (r_1 r_2)^k D U_{n-k}$. Substituting yields (4.3). $\square$

Note 4.3. For Fibonacci 3-vectors, constant-direction cross product identities (hence a fixed normal direction) are derived in [2] by a vector/anti-symmetric-matrix approach. Theorem 4.2 may be viewed as a dimension-free strengthening: it produces a single bivector direction in $\wedge^2 \mathbb{R}^m$ for all m , from which the $\mathbb{R}^3$ cross-product formulas follow as a special case.

4.2 Coordinate form: 2×2 minors and Cassini-type identities

The bivector statement of Theorem 4.2 can be unpacked into explicit determinant identities.

Proposition 4.4 (2×2 minors in closed form). Assume $q \neq 0$ and $r_1 \neq r_2$, and extend $\{U_n\}$ to $\mathbb{Z}$ as in Definition 2.3. Then for all integers n, k, i, j ,

$$U_{n+i} U_{k+j} - U_{n+j} U_{k+i} = (-q)^{k+i} U_{n-k} U_{j-i}. \quad (4.4)$$

Proof. Let $D = r_1 - r_2 \neq 0$. Using the Binet formula (2.6),

$$U_{n+i} U_{k+j} - U_{n+j} U_{k+i} = \frac{(r_1^{n+i} - r_2^{n+i})(r_1^{k+j} - r_2^{k+j}) - (r_1^{n+j} - r_2^{n+j})(r_1^{k+i} - r_2^{k+i})}{D^2}.$$

Expand the numerator completely. The “pure” terms cancel:

$$r_1^{n+i} r_1^{k+j} - r_1^{n+j} r_1^{k+i} = 0, \quad r_2^{n+i} r_2^{k+j} - r_2^{n+j} r_2^{k+i} = 0.$$

Hence only mixed terms remain, giving

$$\begin{aligned} U_{n+i}U_{k+j} - U_{n+j}U_{k+i} &= \frac{-r_1^{n+i}r_2^{k+j} - r_2^{n+i}r_1^{k+j} + r_1^{n+j}r_2^{k+i} + r_2^{n+j}r_1^{k+i}}{D^2} \\ &= \frac{(r_1^{j-i} - r_2^{j-i})(r_1^{n+i}r_2^{k+i} - r_2^{n+i}r_1^{k+i})}{D^2}, \end{aligned}$$

where the second equality follows by factoring $r_1^{n+i}r_2^{k+i}$ and $r_2^{n+i}r_1^{k+i}$ from the first and third, and from the second and fourth terms, respectively.

Now use (2.6) twice:

$$r_1^{j-i} - r_2^{j-i} = D U_{j-i}, \quad r_1^{n+i}r_2^{k+i} - r_2^{n+i}r_1^{k+i} = (r_1r_2)^{k+i} (r_1^{n-k} - r_2^{n-k}) = (r_1r_2)^{k+i} D U_{n-k}.$$

Substitute into the previous expression and cancel D^2 :

$$U_{n+i}U_{k+j} - U_{n+j}U_{k+i} = (r_1r_2)^{k+i} U_{n-k}U_{j-i}.$$

Finally, $r_1r_2 = -q$ by (2.4), so $(r_1r_2)^{k+i} = (-q)^{k+i}$, yielding (4.4). $\square$

Remark 4.5. For Fibonacci/Lucas parameters, (4.4) is a Vajda–Cassini determinant identity; related special cases are recorded in [2]. The proof above emphasizes that these 2×2 minors are coordinate shadows of the wedge-product collinearity in Theorem 4.2, and therefore persist for general (p, q) .

Corollary 4.6 (Generalized Cassini identity). For all integers $n \geq 1$,

$$U_n^2 - U_{n-1}U_{n+1} = (-q)^{n-1}. \quad (4.5)$$

Proof. Apply Proposition 4.4 with $k = n - 1$, $i = 0$, $j = 1$. Then

$$U_{n+0}U_{(n-1)+1} - U_{n+1}U_{(n-1)+0} = (-q)^{(n-1)+0}U_{n-(n-1)}U_{1-0}.$$

That is, $U_n^2 - U_{n+1}U_{n-1} = (-q)^{n-1}U_1U_1$. Since $U_1 = 1$, we obtain (4.5). $\square$

4.3 The 3-dimensional cross-product form

When $m = 3$ the bivector direction corresponds to a constant normal vector.

Corollary 4.7 (Cross products in $\mathbb{R}^3$). *Let $m = 3$, assume $\Delta > 0$, and assume $q \neq 0$. Then for all integers n, k ,*

$$\mathbf{U}_n^{(3)} \times \mathbf{U}_k^{(3)} = (-q)^k U_{n-k}(-q, -p, 1), \quad (4.6)$$

$$\mathbf{V}_n^{(3)} \times \mathbf{V}_k^{(3)} = \frac{\Delta}{4} (-q)^k U_{n-k}(q, p, -1). \quad (4.7)$$

In particular, all $\mathbf{U}_n^{(3)}$ lie in the plane orthogonal to $(-q, -p, 1)$.

Proof. We give a direct computation from Theorem 4.2 and the explicit form of $\mathbf{r}_1 \times \mathbf{r}_2$.

Step 1: Compute $\mathbf{r}_1 \times \mathbf{r}_2$. For $m = 3$, $\mathbf{r}_1 = (1, r_1, r_1^2)^\top$ and $\mathbf{r}_2 = (1, r_2, r_2^2)^\top$. Using the determinant formula for the cross product,

$$\mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & r_1 & r_1^2 \\ 1 & r_2 & r_2^2 \end{vmatrix} = (r_1 r_2^2 - r_1^2 r_2, \quad r_1^2 - r_2^2, \quad r_2 - r_1).$$

Factor $(r_2 - r_1)$:

$$r_1 r_2^2 - r_1^2 r_2 = r_1 r_2 (r_2 - r_1), \quad r_1^2 - r_2^2 = (r_1 - r_2)(r_1 + r_2) = -(r_2 - r_1)(r_1 + r_2),$$

so

$$\mathbf{r}_1 \times \mathbf{r}_2 = (r_2 - r_1)(r_1 r_2, \quad -(r_1 + r_2), \quad 1).$$

Using (2.4), $r_1 r_2 = -q$ and $r_1 + r_2 = p$, hence

$$\mathbf{r}_1 \times \mathbf{r}_2 = (r_2 - r_1)(-q, -p, 1) = -(r_1 - r_2)(-q, -p, 1). \quad (4.8)$$

Step 2: Use the wedge formula. In $\mathbb{R}^3$, the Hodge dual identifies $\mathbf{x} \wedge \mathbf{y}$ with $\mathbf{x} \times \mathbf{y}$. Thus (4.2) becomes

$$\mathbf{U}_n^{(3)} \times \mathbf{U}_k^{(3)} = -\frac{(r_1 r_2)^k}{r_1 - r_2} U_{n-k}(\mathbf{r}_1 \times \mathbf{r}_2).$$

Substitute (4.8):

$$\mathbf{U}_n^{(3)} \times \mathbf{U}_k^{(3)} = -\frac{(r_1 r_2)^k}{r_1 - r_2} U_{n-k}(-(r_1 - r_2)(-q, -p, 1)) = (r_1 r_2)^k U_{n-k}(-q, -p, 1).$$

Finally, $(r_1 r_2)^k = (-q)^k$ by (2.4), giving (4.6).

The identity (4.7) is obtained similarly from (4.3):

$$\mathbf{V}_n^{(3)} \times \mathbf{V}_k^{(3)} = \frac{(r_1 r_2)^k (r_1 - r_2)}{4} U_{n-k}(\mathbf{r}_1 \times \mathbf{r}_2),$$

then substitute (4.8) and use $(r_1 - r_2)^2 = \Delta$ from (2.3), which yields $\frac{\Delta}{4} (r_1 r_2)^k U_{n-k}(q, p, -1)$, equivalent to (4.7).

The final statement follows because for any fixed n , all vectors $\mathbf{U}_n^{(3)}$ satisfy the linear constraint $x_3 - px_2 - qx_1 = 0$ (Theorem 3.5(ii)), which has normal vector $(-q, -p, 1)$. $\square$

Corollary 4.8 (Recovery of the Pell/Pell–Lucas cross products). *Let $(p, q) = (2, 1)$ and $m = 3$. Then for all integers n, k ,*

$$\mathbf{P}_n^{(3)} \times \mathbf{P}_k^{(3)} = (-1)^k P_{n-k}(-1, -2, 1), \quad \mathbf{Q}_n^{(3)} \times \mathbf{Q}_k^{(3)} = 2(-1)^k P_{n-k}(1, 2, -1),$$

which matches [10, Thm. 2.1.6 and Cor. 2.1.7].

Proof. In the Pell case, $U_n = P_n$ and $V_n = Q_n$ (normalized Pell–Lucas) and $(-q)^k = (-1)^k$. Also $\Delta = p^2 + 4q = 8$; therefore (4.6) gives the first formula, and (4.7) yields

$$\mathbf{Q}_n^{(3)} \times \mathbf{Q}_k^{(3)} = \frac{8}{4} (-1)^k P_{n-k}(1, 2, -1) = 2(-1)^k P_{n-k}(1, 2, -1),$$

as claimed. $\square$

5 Closed-form inner products

We next compute metric invariants of the same consecutive-term vectors. The only new scalar factor that enters is the finite geometric sum arising from the mixed root products $r_1^j r_2^j = (-q)^j$.

5.1 A geometric-series factor

The cross terms in inner products involve the geometric sum $\sum_{j=0}^{m-1} (-q)^j$.

Definition 5.1. For $m \geq 1$, define

$$S_m(q) := \sum_{j=0}^{m-1} (-q)^j. \tag{5.1}$$

Equivalently,

$$S_m(q) = \begin{cases} m, & q = -1, \\ \frac{1 - (-q)^m}{1 + q}, & q \neq -1. \end{cases}$$

Lemma 5.2. For all integers n, k ,

$$r_1^n r_2^k + r_2^n r_1^k = 2(r_1 r_2)^k V_{n-k}. \quad (5.2)$$

Proof. Factor $(r_1 r_2)^k$:

$$r_1^n r_2^k + r_2^n r_1^k = (r_1 r_2)^k (r_1^{n-k} + r_2^{n-k}).$$

By the Binet formula (2.7), $r_1^{n-k} + r_2^{n-k} = 2V_{n-k}$, proving (5.2). $\square$

5.2 Inner products between consecutive-term vectors

We now derive explicit expressions for all inner products in $\mathbb{R}^m$.

Theorem 5.3 (Closed-form inner products). Fix $m \geq 1$, assume $\Delta > 0$ (equivalently, $r_1 \neq r_2$ are real), and assume $q \neq 0$. Let $D = r_1 - r_2$ and $S_m(q)$ be as in Definition 5.1. Then for all integers n, k ,

$$\langle \mathbf{U}_n^{(m)}, \mathbf{U}_k^{(m)} \rangle = \frac{2}{D^2} \left(\sum_{j=0}^{m-1} V_{n+k+2j} - (r_1 r_2)^k V_{n-k} S_m(q) \right), \quad (5.3)$$

$$\langle \mathbf{V}_n^{(m)}, \mathbf{V}_k^{(m)} \rangle = \frac{1}{2} \left(\sum_{j=0}^{m-1} V_{n+k+2j} + (r_1 r_2)^k V_{n-k} S_m(q) \right), \quad (5.4)$$

$$\langle \mathbf{U}_n^{(m)}, \mathbf{V}_k^{(m)} \rangle = \frac{1}{2} \left(\sum_{j=0}^{m-1} U_{n+k+2j} + (r_1 r_2)^k U_{n-k} S_m(q) \right). \quad (5.5)$$

Using $r_1 r_2 = -q$, the factors $(r_1 r_2)^k$ may be written as $(-q)^k$.

Proof. Proof of (5.3). By Definition 2.5,

$$\langle \mathbf{U}_n^{(m)}, \mathbf{U}_k^{(m)} \rangle = \sum_{j=0}^{m-1} U_{n+j} U_{k+j}.$$

Using (2.6),

$$U_{n+j} U_{k+j} = \frac{(r_1^{n+j} - r_2^{n+j})(r_1^{k+j} - r_2^{k+j})}{D^2}.$$

Expand:

$$\left(r_1^{n+j} - r_2^{n+j}\right) \left(r_1^{k+j} - r_2^{k+j}\right) = r_1^{n+k+2j} + r_2^{n+k+2j} - \left(r_1^{n+j} r_2^{k+j} + r_2^{n+j} r_1^{k+j}\right).$$

Hence

$$\left\langle \mathbf{U}_n^{(m)}, \mathbf{U}_k^{(m)} \right\rangle = \frac{1}{D^2} \sum_{j=0}^{m-1} \left(r_1^{n+k+2j} + r_2^{n+k+2j} - \left(r_1^{n+j} r_2^{k+j} + r_2^{n+j} r_1^{k+j} \right) \right).$$

Now use $r_1^t + r_2^t = 2V_t$ to get $\sum_{j=0}^{m-1} \left(r_1^{n+k+2j} + r_2^{n+k+2j} \right) = 2 \sum_{j=0}^{m-1} V_{n+k+2j}$. For the mixed terms,

$$r_1^{n+j} r_2^{k+j} + r_2^{n+j} r_1^{k+j} = \left(r_1^n r_2^k + r_2^n r_1^k \right) (r_1 r_2)^j,$$

so

$$\sum_{j=0}^{m-1} \left(r_1^{n+j} r_2^{k+j} + r_2^{n+j} r_1^{k+j} \right) = \left(r_1^n r_2^k + r_2^n r_1^k \right) \sum_{j=0}^{m-1} (r_1 r_2)^j.$$

Using $r_1 r_2 = -q$ this sum is $S_m(q) = \sum_{j=0}^{m-1} (-q)^j$, and by Lemma 5.2, $r_1^n r_2^k + r_2^n r_1^k = 2(r_1 r_2)^k V_{n-k}$.

Therefore the mixed contribution equals $2(r_1 r_2)^k V_{n-k} S_m(q)$, and substituting yields (5.3).

Proof of (5.4). Similarly,

$$\left\langle \mathbf{V}_n^{(m)}, \mathbf{V}_k^{(m)} \right\rangle = \sum_{j=0}^{m-1} V_{n+j} V_{k+j} = \frac{1}{4} \sum_{j=0}^{m-1} \left(r_1^{n+j} + r_2^{n+j} \right) \left(r_1^{k+j} + r_2^{k+j} \right).$$

Expanding gives

$$V_{n+j} V_{k+j} = \frac{1}{4} \left(r_1^{n+k+2j} + r_2^{n+k+2j} + \left(r_1^{n+j} r_2^{k+j} + r_2^{n+j} r_1^{k+j} \right) \right),$$

and the same substitutions as above produce (5.4).

Proof of (5.5). By Definition 2.5,

$$\left\langle \mathbf{U}_n^{(m)}, \mathbf{V}_k^{(m)} \right\rangle = \sum_{j=0}^{m-1} U_{n+j} V_{k+j}.$$

Using Binet,

$$\begin{aligned} U_{n+j} V_{k+j} &= \frac{1}{2D} \left(r_1^{n+j} - r_2^{n+j} \right) \left(r_1^{k+j} + r_2^{k+j} \right) \\ &= \frac{1}{2D} \left(r_1^{n+k+2j} - r_2^{n+k+2j} + r_1^{n+j} r_2^{k+j} - r_2^{n+j} r_1^{k+j} \right). \end{aligned}$$

Summing over j ,

$$\left\langle \mathbf{U}_n^{(m)}, \mathbf{V}_k^{(m)} \right\rangle = \frac{1}{2D} \left(\sum_{j=0}^{m-1} \left(r_1^{n+k+2j} - r_2^{n+k+2j} \right) + \sum_{j=0}^{m-1} \left(r_1^{n+j} r_2^{k+j} - r_2^{n+j} r_1^{k+j} \right) \right).$$

For the first sum, use $r_1^t - r_2^t = DU_t$ to get

$$\sum_{j=0}^{m-1} \left(r_1^{n+k+2j} - r_2^{n+k+2j} \right) = D \sum_{j=0}^{m-1} U_{n+k+2j}.$$

For the mixed sum,

$$r_1^{n+j} r_2^{k+j} - r_2^{n+j} r_1^{k+j} = (r_1^n r_2^k - r_2^n r_1^k) (r_1 r_2)^j,$$

hence

$$\sum_{j=0}^{m-1} \left(r_1^{n+j} r_2^{k+j} - r_2^{n+j} r_1^{k+j} \right) = (r_1^n r_2^k - r_2^n r_1^k) \sum_{j=0}^{m-1} (r_1 r_2)^j = (r_1^n r_2^k - r_2^n r_1^k) S_m(q).$$

Finally, by Lemma 4.1 (again multiplying by -1), $r_1^n r_2^k - r_2^n r_1^k = (r_1 r_2)^k D U_{n-k}$. Substituting into the mixed sum and then into the prefactor $\frac{1}{2D}$ gives

$$\left\langle \mathbf{U}_n^{(m)}, \mathbf{V}_k^{(m)} \right\rangle = \frac{1}{2} \left(\sum_{j=0}^{m-1} U_{n+k+2j} + (r_1 r_2)^k U_{n-k} S_m(q) \right),$$

which is (5.5). □

Remark 5.4. Closed forms for dot products of Fibonacci/Lucas vectors in arbitrary dimension were obtained in [9]; related Fibonacci and Horadam vector formulas appear in [1, 2]. Theorem 5.3 isolates the (p, q) -uniform root-plane calculation for the canonical first-kind sequence U_n and the normalized companion V_n , including the mixed products $\left\langle \mathbf{U}_n^{(m)}, \mathbf{V}_k^{(m)} \right\rangle$ and the factor $S_m(q)$.

Corollary 5.5 (Norms). With the notation of Theorem 5.3,

$$\left\| \mathbf{U}_n^{(m)} \right\|^2 = \frac{2}{D^2} \left(\sum_{j=0}^{m-1} V_{2n+2j} - (r_1 r_2)^n V_0 S_m(q) \right) = \frac{2}{D^2} \left(\sum_{j=0}^{m-1} V_{2n+2j} - (r_1 r_2)^n S_m(q) \right),$$

since $V_0 = 1$, and similarly for $\left\| \mathbf{V}_n^{(m)} \right\|^2$.

Proof. Set $k = n$ in (5.3) and use $V_{n-n} = V_0 = 1$. □

5.3 Pell specialization and recovery of [10]

We now show that Theorem 5.3 contains the key Pell/Pell–Lucas inner products in [10] as special cases.

Lemma 5.6 (Even-step sums in the Pell case). *Let $(p, q) = (2, 1)$, so $U_n = P_n$ is the Pell sequence and $V_n = Q_n$ is the normalized Pell–Lucas sequence with Binet forms*

$$P_n = \frac{a^n - b^n}{a - b}, \quad Q_n = \frac{a^n + b^n}{2}, \quad ab = -1,$$

as in [10]. Then for all integers t and $m \geq 1$,

$$\sum_{j=0}^{m-1} Q_{t+2j} = \frac{Q_{t+2m-1} - Q_{t-1}}{2}, \quad (5.6)$$

$$\sum_{j=0}^{m-1} P_{t+2j} = \frac{P_{t+2m-1} - P_{t-1}}{2}. \quad (5.7)$$

Proof. Let a, b be the roots of $x^2 - 2x - 1 = 0$, so $a - b \neq 0$, $a^2 - 1 = 2a$, $b^2 - 1 = 2b$, and $ab = -1$. By Binet,

$$Q_{t+2j} = \frac{a^{t+2j} + b^{t+2j}}{2}.$$

Summing over j gives

$$\sum_{j=0}^{m-1} Q_{t+2j} = \frac{1}{2} \left(a^t \sum_{j=0}^{m-1} a^{2j} + b^t \sum_{j=0}^{m-1} b^{2j} \right).$$

The geometric sums are

$$\sum_{j=0}^{m-1} a^{2j} = \frac{a^{2m} - 1}{a^2 - 1} = \frac{a^{2m} - 1}{2a} = \frac{a^{2m-1} - a^{-1}}{2},$$

and similarly

$$\sum_{j=0}^{m-1} b^{2j} = \frac{b^{2m} - 1}{b^2 - 1} = \frac{b^{2m} - 1}{2b} = \frac{b^{2m-1} - b^{-1}}{2}.$$

Substituting gives

$$\sum_{j=0}^{m-1} Q_{t+2j} = \frac{1}{4} (a^{t+2m-1} - a^{t-1} + b^{t+2m-1} - b^{t-1}) = \frac{1}{2} (Q_{t+2m-1} - Q_{t-1}),$$

which proves (5.6).

For the Pell sequence, the same computation starts from

$$P_{t+2j} = \frac{a^{t+2j} - b^{t+2j}}{a - b}.$$

Thus

$$\sum_{j=0}^{m-1} P_{t+2j} = \frac{1}{a-b} \left(a^t \sum_{j=0}^{m-1} a^{2j} - b^t \sum_{j=0}^{m-1} b^{2j} \right).$$

Using the two geometric sums already computed, we obtain

$$\sum_{j=0}^{m-1} P_{t+2j} = \frac{1}{2(a-b)} (a^{t+2m-1} - a^{t-1} - b^{t+2m-1} + b^{t-1}) = \frac{1}{2} (P_{t+2m-1} - P_{t-1}),$$

which proves (5.7). $\square$

Corollary 5.7 (Pell inner products in arbitrary dimension). *Let $(p, q) = (2, 1)$ and write $\mathbf{P}_n^{(m)}$ and $\mathbf{Q}_n^{(m)}$ for the Pell and normalized Pell–Lucas consecutive-term vectors of [10]. Then for all integers n, k and $m \geq 1$,*

$$\langle \mathbf{P}_n^{(m)}, \mathbf{P}_k^{(m)} \rangle = \frac{Q_{n+k+2m-1} - Q_{n+k-1}}{8} - \frac{(-1)^k Q_{n-k}}{4} \cdot \frac{1 - (-1)^m}{2}, \quad (5.8)$$

$$\langle \mathbf{Q}_n^{(m)}, \mathbf{Q}_k^{(m)} \rangle = \frac{Q_{n+k+2m-1} - Q_{n+k-1}}{4} + (-1)^k Q_{n-k} \cdot \frac{1 - (-1)^m}{4}, \quad (5.9)$$

$$\langle \mathbf{P}_n^{(m)}, \mathbf{Q}_k^{(m)} \rangle = \frac{P_{n+k+2m-1} - P_{n+k-1}}{4} + (-1)^k P_{n-k} \cdot \frac{1 - (-1)^m}{4}. \quad (5.10)$$

Proof. We start from Theorem 5.3 and specialize carefully.

Step 1: Specialize parameters. In the Pell case, $r_1 = a$, $r_2 = b$, $D = a - b$, and $D^2 = \Delta = 8$.

Also $r_1 r_2 = ab = -1$. Therefore $(r_1 r_2)^k = (-1)^k$.

Step 2: Evaluate the geometric factor $S_m(q)$. Here $q = 1$, so

$$S_m(1) = \sum_{j=0}^{m-1} (-1)^j = \frac{1 - (-1)^m}{2}.$$

Step 3: Convert the even-step sums to differences. Applying Lemma 5.6 with $t = n + k$,

$$\sum_{j=0}^{m-1} Q_{n+k+2j} = \frac{Q_{n+k+2m-1} - Q_{n+k-1}}{2}, \quad \sum_{j=0}^{m-1} P_{n+k+2j} = \frac{P_{n+k+2m-1} - P_{n+k-1}}{2}.$$

Step 4: Substitute into the general formulas. From (5.3) and the substitutions above,

$$\begin{aligned} \langle \mathbf{P}_n^{(m)}, \mathbf{P}_k^{(m)} \rangle &= \frac{2}{8} \left(\sum_{j=0}^{m-1} Q_{n+k+2j} - (-1)^k Q_{n-k} S_m(1) \right) \\ &= \frac{1}{4} \sum_{j=0}^{m-1} Q_{n+k+2j} - \frac{(-1)^k Q_{n-k}}{4} S_m(1), \end{aligned}$$

and Steps 2–3 yield (5.8). Similarly, applying (5.4) gives

$$\left\langle \mathbf{Q}_n^{(m)}, \mathbf{Q}_k^{(m)} \right\rangle = \frac{1}{2} \left(\sum_{j=0}^{m-1} Q_{n+k+2j} + (-1)^k Q_{n-k} S_m(1) \right),$$

and Steps 2–3 yield (5.9). Finally, applying (5.5) gives

$$\left\langle \mathbf{P}_n^{(m)}, \mathbf{Q}_k^{(m)} \right\rangle = \frac{1}{2} \left(\sum_{j=0}^{m-1} P_{n+k+2j} + (-1)^k P_{n-k} S_m(1) \right),$$

and Steps 2–3 yield (5.10). □

Remark 5.8. Under the normalized convention $Q_n = \frac{1}{2}(a^n + b^n)$ used in [10] (hence $Q_0 = 1$ and $Q_1 = 1$), the specialization of Corollary 5.7 to $m = 2$ gives

$$\left\langle \mathbf{P}_n^{(2)}, \mathbf{P}_k^{(2)} \right\rangle = P_{n+k+1}, \quad \left\langle \mathbf{P}_n^{(2)}, \mathbf{Q}_k^{(2)} \right\rangle = Q_{n+k+1}, \quad \left\langle \mathbf{Q}_n^{(2)}, \mathbf{Q}_k^{(2)} \right\rangle = 2P_{n+k+1}.$$

Indeed, since $S_2(1) = 0$, formulas (5.8) and (5.9) reduce to

$$\left\langle \mathbf{P}_n^{(2)}, \mathbf{P}_k^{(2)} \right\rangle = \frac{Q_{t+3} - Q_{t-1}}{8}, \quad \left\langle \mathbf{Q}_n^{(2)}, \mathbf{Q}_k^{(2)} \right\rangle = \frac{Q_{t+3} - Q_{t-1}}{4}, \quad (t = n + k),$$

and the standard Pell identities imply

$$\frac{Q_{t+3} - Q_{t-1}}{8} = P_{t+1}, \quad \frac{Q_{t+3} - Q_{t-1}}{4} = 2P_{t+1}.$$

Moreover, specializing (5.5) yields

$$\left\langle \mathbf{P}_n^{(2)}, \mathbf{Q}_k^{(2)} \right\rangle = \frac{1}{2}(P_t + P_{t+2}) = Q_{t+1}, \quad (t = n + k),$$

since $P_{t+2} = 2P_{t+1} + P_t$, hence

$$\frac{1}{2}(P_t + P_{t+2}) = P_t + P_{t+1} = Q_{t+1}.$$

The first identity above, $\left\langle \mathbf{P}_n^{(2)}, \mathbf{P}_k^{(2)} \right\rangle = P_{n+k+1}$, recovers [10, Thm. 2.1.4(i)]. The same direct calculation indicates that [10, Thm. 2.1.4(ii), (iii)] interchanges the right-hand sides of the mixed and Pell–Lucas dot products under the normalization $Q_n = \frac{1}{2}(a^n + b^n)$. For example, $\mathbf{P}_1^{(2)} = (1, 2)^\top$ and $\mathbf{Q}_1^{(2)} = (1, 3)^\top$, so $\left\langle \mathbf{P}_1^{(2)}, \mathbf{Q}_1^{(2)} \right\rangle = 7 = Q_3$, whereas $\left\langle \mathbf{Q}_1^{(2)}, \mathbf{Q}_1^{(2)} \right\rangle = 10 = 2P_3$.

Note 5.9 (Meaning of the generalization). *Theorems 3.5, 4.2, and 5.3 place the geometric phenomena studied in earlier generalized Fibonacci, Horadam, and Pell/Pell–Lucas vector papers within one root-plane framework. The ambient dimension m changes how this fixed rank-two dynamics is embedded into $\mathbb{R}^m$, while the underlying exterior direction is determined only by the two characteristic roots.*

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Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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