Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 29, no. 1, pp. 1–6, January 2027
DOI: 10.56754/0719-0646.2901.001

On the identity $D^3 + D^2 = G$ in semiprime rings

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ABSTRACT

The purpose of this paper is to prove the following result. Let R be a 2−torsion free semiprime ring and let $D, G : R \to R$ be derivations. Suppose that $D^3(x) + D^2(x) = G(x)$ holds for all $x \in R$. In this case $D = 0$. The result is related to the classical theory of derivations initiated by Posner.

RESUMEN

El propósito de este artículo es demostrar el siguiente resultado. Sea R un anillo semiprimo libre de 2−torsión y sean $D, G : R \to R$ derivaciones. Supongamos que $D^3(x) + D^2(x) = G(x)$ es válido para todo $x \in R$. En este caso $D = 0$. El resultado se relaciona con la teoría clásica de derivaciones iniciada por Posner.

Keywords and Phrases: Prime ring, semiprime ring, Banach algebra, derivation.

2020 AMS Mathematics Subject Classification: 16N60, 16W25.

Published: 03 November, 2026
Accepted: 12 September, 2026
Received: 06 May, 2026

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1 Introduction and the main theorem

In what follows, R denotes an associative ring with center $Z(R)$. We say that R is n-torsion free, for an integer $n > 1$, if $nx = 0$ for $x \in R$ forces $x = 0$. The commutator $xy - yx$ is written as $[x, y]$. An additive mapping $D : R \to R$ is a derivation when

$$D(xy) = D(x)y + xD(y)$$

for every $x, y \in R$. Such a derivation is called inner if there is a fixed $a \in R$ such that $D(x) = [a, x]$ for every $x \in R$. Recall also that R is prime when $aRb = (0)$ implies $a = 0$ or $b = 0$, and that R is semiprime when $aRa = (0)$ implies $a = 0$.

A fundamental theorem of Posner [3], usually referred to as Posner's first theorem, says that the composition of two nonzero derivations of a prime ring of characteristic different from two cannot itself be a derivation. This result does not extend to semiprime rings in general. A related result that will be used below is the following theorem from [2].

Theorem 1.1 ([2, Lemma 1.1.9])**.** *Let R be a $2-$torsion free semiprime ring and let $D, G : R \to R$ be derivations satisfying*

$$D^2(x) = G(x)$$

for every $x \in R$. Then $D = 0$.

The conclusion changes when the second iterate is replaced by the third iterate. One cannot prove that $D = 0$ in case we have $D^3(x) = G(x)$, for all $x \in R$, where D and G are derivations on R. To see this, choose $a \notin Z(R)$ with $a^2 = 0$. For the inner derivation $D(x) = [a, x]$, a direct computation gives $D^3(x) = 0$ for every $x \in R$, whereas $D \neq 0$ because $a \notin Z(R)$. In a different setting, Vukman [4] obtained the following result.

Theorem 1.2 ([4, Theorem 3])**.** *Let A be a complex semisimple Banach algebra, and let $D : A \to A$ be a linear derivation. Suppose that the mapping $x \to \alpha D^3(x) + D^2(x)$ is a derivation, where α is a fixed complex number. Then $D = 0$.*

Semisimple Banach algebras are, in particular, semiprime. The preceding theorem suggests considering the corresponding identity over 2-torsion free semiprime rings. The main statement established here is the following.

Theorem 1.3. *Let R be a $2-$torsion free semiprime ring and let $D, G : R \to R$ be derivations satisfying*

$$D^3(x) + D^2(x) = G(x)$$

for all $x \in R$. In this case $D = 0$.

2 Proof of the main theorem

Proof. Set

$$F(x) = D^3(x) + D^2(x) - G(x).$$

The hypothesis is therefore equivalent to

$$F(x) = 0, \tag{2.1}$$

for all $x \in R$. Expanding $F(xy)$ by means of the derivation rule gives

$$F(xy) = F(x)y + xF(y) + H(x)D(y) + D(x)H(y), \tag{2.2}$$

for all $x, y \in R$, where

$$H(x) = 3D^2(x) + D(x).$$

Now apply (2.2) with xy in place of x in (2.1), and use (2.1) to eliminate the terms involving F. This yields

$$H(x)D(y) + D(x)H(y) = 0, \tag{2.3}$$

for all $x, y \in R$. The product formula for H is

$$H(xy) = H(x)y + xH(y) + 6D(x)D(y), \tag{2.4}$$

valid for all $x, y \in R$.

Next, replace y by yx in (2.3) and expand $H(yx)$ by (2.4). We obtain

$$\begin{aligned} 0 = H(x)D(yx) + D(x)H(yx) = H(x)D(y)x + H(x)yD(x) + \\ D(x)H(y)x + D(x)yH(x) + 6D(x)D(y)D(x). \end{aligned}$$

Using (2.3) to cancel the corresponding terms leaves

$$H(x)yD(x) + D(x)yH(x) + 6D(x)D(y)D(x) = 0, \tag{2.5}$$

for all $x, y \in R$.

In (2.5), substitute $yD(x)$ for y. This gives

$$H(x)yD(x)^2 + D(x)yD(x)H(x) + 6D(x)D(y)D(x)^2 + 6D(x)yD^2(x)D(x) = 0,$$

for all $x, y \in R$. Multiplying (2.5) on the right by $D(x)$ and subtracting the resulting identity from the preceding one, we obtain

$$D(x)y\,[D(x), H(x)] + 6D(x)yD^2(x)D(x) = 0,$$

for all $x, y \in R$. Since

$$[D(x), H(x)] = 3\left(D(x)D^2(x) - D^2(x)D(x)\right),$$

the preceding identity can be written as

$$D(x)y3(D^2(x)D(x) + D(x)D^2(x)) = 0, \tag{2.6}$$

for all $x, y \in R$.

Taking $y = x$ in (2.3) gives

$$3(D^2(x)D(x) + D(x)D^2(x)) + 2D(x)^2 = 0$$

for every $x \in R$. Consequently, the first factor in (2.6) can be replaced by $-2D(x)^2$. Hence

$$D(x)yD(x)^2 = 0,$$

for all $x, y \in R$, where the 2-torsion-free assumption has been used. Multiplying this identity on the left by $D(x)$ gives

$$D(x)^2yD(x)^2 = 0$$

for all $x, y \in R$. Semiprimeness of R then implies

$$D(x)^2 = 0$$

for every $x \in R$.

The Giambruno–Herstein theorem [1, Theorem 3] now finishes the argument. □

We close by recording the following conjectural extension.

Conjecture 2.1. *Let $n \geq 3$ be an integer, and let $D, G : R \to R$ be derivations, where R is a semiprime ring with suitable torsion restrictions. Suppose that*

$$D^n(x) + D^2(x) = G(x)$$

holds for all $x \in R$. In this case $D = 0$.

Acknowledgment

Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Ethics declarations (Conflicts of interest)

All authors declare that they have no conflicts of interest that are relevant to the content of this article.

References

[1] A. Giambruno and I. N. Herstein, "Derivations with nilpotent values," *Rend. Circ. Mat. Palermo*, vol. 30, pp. 199–206, 1981, doi: 10.1007/BF02844306.

[2] I. N. Herstein, *Rings with Involution*, ser. Chicago Lectures in Mathematics. Chicago and London: University of Chicago Press, 1976.

[3] E. C. Posner, "Derivations in prime rings," *Proc. Am. Math. Soc.*, vol. 8, pp. 1093–1100, 1958, doi: 10.2307/2032686.

[4] J. Vukman, "A result concerning derivations in Banach algebras," *Proc. Am. Math. Soc.*, vol. 116, no. 4, pp. 971–975, 1992, doi: 10.2307/2159476.