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Uniquely bi-embeddable bipartite 2-regular graphs

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ABSTRACT

A bipartite 2-regular graph is a bipartite graph G such that $G = C_{2n_1} \cup C_{2n_2} \cup \cdots \cup C_{2n_k}$ with $k \geq 1$, *i.e.*, G is a vertex-disjoint union of bipartite cycles. In this paper, we completely characterize the bipartite 2-regular graphs which can be uniquely bi-embedded into their bipartite complements.

RESUMEN

Un grafo bipartito 2-regular es un grafo bipartito G tal que $G = C_{2n_1} \cup C_{2n_2} \cup \cdots \cup C_{2n_k}$ con $k \geq 1$, *i.e.*, G es una unión disjunta por vértices de ciclos bipartitos. En este artículo, caracterizamos completamente los grafos bipartitos 2-regulares que pueden ser únicamente bi-incrustados en sus complementos bipartitos.

Keywords and Phrases: Bi-embeddings, bipartite graphs, disjoint union of cycles.

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1 Introduction

In this paper, we only discuss finite simple bipartite graphs and use standard terminology and notation from [1] unless indicated otherwise. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. From [2], an embedding of G is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. A graph G is said to be uniquely embeddable if $G\cup\sigma(G)$ is isomorphic to $G\cup\tau(G)$ for any two embeddings σ and τ of G. In this paper, we will investigate uniquely bi-embeddable bipartite 2-regular graphs. This work is motivated by the result in [2]. Grzelec, Pilśniak and Woźniak characterized all the uniquely embeddable 2-regular graphs in [2] as in the following theorem:

Theorem 1.1 ([2, Theorem 5]). *Let G be a union of vertex-disjoint cycles. Then G is uniquely embeddable if and only if G is one of C_5, C_6, $C_3\cup C_4$, $C_3\cup C_5$, $3C_3$ and $4C_3$.*

Woźniak characterized all uniquely embeddable graph G with $|E(G)| = n-2$ in [8] as in the following theorem:

Theorem 1.2 ([8, Theorem 2]). *Let G be a graph of order n and size $e(G) = n-2$. Then either G is not uniquely embeddable or G is isomorphic to one of the seven following graphs: $K_2\cup K_1$, $2K_2$, $K_3\cup 2K_1$, $K_3\cup K_2\cup K_1$, $C_4\cup 2K_1$, $K_3\cup 2K_2$, $2K_3\cup 2K_1$.*

The unique embedding problem has also been studied for forests, as in the following theorem:

Theorem 1.3 ([3, Theorem 6]). *Let F be a non-star forest of order n having at least one edge. Then either F is not uniquely embeddable or F is isomorphic to one of the following graphs: $K_2\cup K_1$, $2K_2$, $3K_2$, $S(2,2)$ or S'_n for $n\geq 4$.*

For other embedding and packing problems, we refer readers to [4,7,9,11] .

In this paper, we use $G(X,Y)$ to denote a bipartite graph G with (X,Y) as its bipartition. If H is a subgraph of $G(X,Y)$, then the bipartition of H is given by $(V(H)\cap X, V(H)\cap Y)$. For $Z\subseteq V(G)$, we define $G[Z]$ to be the subgraph induced by Z. For convenience, we might use G to denote $G(X,Y)$ in the proofs. We say that a bipartite graph $G(X,Y)$ is bi-embeddable if there is a bijection $\sigma : V(G)\to V(G)$ such that $\sigma(X) = X$, $\sigma(Y) = Y$ and $E(G)\cap E(\sigma(G)) = \varnothing$. Furthermore, a bipartite graph $G(X,Y)$ is said to be uniquely bi-embeddable if G is bi-embeddable and $G\cup\sigma(G)$ is isomorphic to $G\cup\tau(G)$ for any two such bijective mappings σ and τ. We use $B(X,Y)$ to denote the complete bipartite graph with bipartition sets X and Y. Also, we use $G(X,Y)''$ to denote the bipartite complement of $G(X,Y)$ in $B(X,Y)$, where $V(G(X,Y)'') = V(G(X,Y))$ and $E(G(X,Y)'') = E(B(X,Y)) - E(G(X,Y))$. Note that for any two bipartite 2-regular graphs G_1 and G_2, G_1 is isomorphic to G_2 if and only if G_1'' is isomorphic to G_2''.

Let $G_1(X,Y)$ and $G_2(X,Y)$ be two bipartite graphs. We denote the edge sum of G_1 and G_2 by the bipartite graph $G_1\cup G_2$ with (X,Y) as its bipartition and $E(G_1\cup G_2) = E(G_1)\cup E(G_2)$ as its

edge set. If a graph L is the union of m disjoint copies of a graph N where $m \geq 2$, then we denote L by mN.

In this paper, we will prove the following theorem:

Theorem 1.4. *Let $G(X,Y) = C_{2n_1} \cup C_{2n_2} \cup \cdots \cup C_{2n_k}$ with $k \geq 1$ be a bipartite 2-regular graph. $G(X,Y)$ is uniquely bi-embeddable if and only if $G(X,Y)$ is one of C_8, C_{10}, $2C_4$ or $3C_4$. Moreover, $G(X,Y)$ is not bi-embeddable if and only if $G(X,Y)$ is one of C_4, C_6 or $C_4 \cup C_6$.*

For more bi-embedding and bi-packing problems, we refer readers to [5, 6, 10] .

2 Proof of Theorem 1.4

To prove Theorem 1.4, we need the following results.

Lemma 2.1. *If a bipartite graph $G = C_{2n_1} \cup C_{2n_2} \cup \cdots \cup C_{2n_k}$ with $k \geq 1$ has a bi-embedding σ such that the bipartite graph $G \cup \sigma(G)$ is not connected, then G has another bi-embedding σ' such that the bipartite graph $G \cup \sigma'(G)$ is connected. In particular, the bipartite graph G is not uniquely bi-embeddable.*

Proof. Among all bi-embeddings of bipartite graph $G(X,Y)$, let γ be a bi-embedding such that $G \cup \gamma(G)$ has the minimum number of connected components. Suppose, for a contradiction, that $G \cup \gamma(G)$ is disconnected and let G_1 and G_2 be two components of $G \cup \gamma(G)$. Let $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$ be two vertices both in X and both x_1 and x_2 are not cut vertices of G_1 and G_2, respectively. Such x_1 and x_2 exist since G_1 and G_2 are both 4-regular bipartite subgraphs in $G \cup \gamma(G)$ with both G_1 and G_2 formed by two edge-disjoint cycles, respectively. Say $\gamma(u) = x_1$ and $\gamma(v) = x_2$. Let τ be a bijection of $V(G)$ such that

$$\begin{aligned} &\tau(u) = \gamma(v); \\ &\tau(v) = \gamma(u); \\ &\tau(x) = \gamma(x) \quad \text{for all} \quad x \in V(G) - \{u, v\}. \end{aligned}$$

Then it is easy to see that τ is a new bi-embedding of G and $G \cup \tau(G)$ has fewer connected components than $G \cup \gamma(G)$, a contradiction. □

Lemma 2.2. *For a path P_n, P_n is bi-embeddable if and only if $n \geq 7$.*

Proof. Note that P_n is bipartite for any $n \geq 2$. We say $P_n = P_n(X, Y)$ where X and Y are the partition sets of P_n. Let $t = \lfloor n/2 \rfloor$. Let $P_n = x_1y_1 \cdots x_ty_t$ if n is even and $P_n = x_1y_1 \cdots x_ty_tx_{t+1}$ if n is odd with $y_i \in Y$ for all $i \in \{1, \ldots, t\}$. It is easy to check that P_n is not bi-embeddable if $n \leq 6$. So suppose $n \geq 7$. If $n = 7$, let ϕ be a bijection of $V(P_7)$ such that

$$\phi(x_1, y_1, x_2, y_2, x_3, y_3, x_4) = (x_2, y_3, x_1, y_2, x_4, y_1, x_3).$$

Then it is easy to check that ϕ is a bi-embedding of P_7. For $n \geq 8$, let ρ be a bijection of $V(P_n)$ such that

$$\begin{aligned} \rho(x_i) &= x_i, \quad \text{with} \quad 1 \leq i \leq t \quad \text{if } n \text{ is even and} \quad 1 \leq i \leq t+1 \quad \text{if } n \text{ is odd;} \\ \rho(y_i) &= y_{i+2}, \quad i \in \{1, \ldots, t-2\}; \\ \rho(y_{t-1}, y_t) &= (y_1, y_2). \end{aligned}$$

Then it is easy to see that ρ is a bi-embedding of P_n. □

Lemma 2.3. *Let $G(X, Y)$ be a bipartite graph such that $G = C_m \cup P_n$ with $m \in \{4, 6\}$, $n \geq 3$, i.e., G is a disjoint union of a bipartite 4-cycle and a path with order at least 3 or G is a disjoint union of a bipartite 6-cycle and a path of order at least 3. Then G is bi-embeddable.*

Proof. Let $C_m = a_1a_2 \cdots a_ma_1$ with $m \in \{4, 6\}$ and $a_1 \in X$. Let $P_n = x_1y_1 \cdots x_ty_t$ if n is even and $P_n = x_1y_1 \cdots x_ty_tx_{t+1}$ if n is odd, with $y_i \in Y$ for all $i \in \{1, \ldots, t\}$, where $t = \lfloor n/2 \rfloor$. Consider $m = 4$. It is easy to check that there is a bi-embedding of G if P_n has $3 \leq n \leq 6$. Let $n \geq 7$. By Lemma 2.2, there is a bi-embedding ρ of P_n. Then we can extend the bi-embedding ρ in Lemma 2.2 to τ by letting

$$\begin{aligned} \tau(a_1, a_2, a_3, a_4) &= (x_1, a_2, x_2, a_4); \\ \tau(x_1, x_2) &= (a_1, a_3); \\ \tau(u) &= \rho(u), \quad \text{for all } u \in V(P_n) - \{x_1, x_2\}. \end{aligned}$$

Then it is easy to check that τ is a bi-embedding of G. Similarly, one can show that there is a bi-embedding of $G = C_6 \cup P_n$ with $n \geq 3$. □

Lemma 2.4. *Let $G(X,Y)$ be a bipartite graph such that $G = C_4 \cup C_6 \cup P_n$ with $n \geq 1$, i.e., G is a disjoint union of a 4-cycle, a 6-cycle and a path of order at least 1. Then G is bi-embeddable.*

Proof. Let $C_4 = a_1a_2a_3a_4a_1$, $C_6 = b_1b_2b_3b_4b_5b_6b_1$ with $\{a_1, b_1\} \subseteq X$. Let $P_n = x_1y_1 \cdots x_ty_t$ if n is even and $P_n = x_1y_1 \cdots x_ty_tx_{t+1}$ if n is odd, with $y_i \in Y$ for all $i \in \{1, \ldots, t\}$, where $t = \lfloor n/2 \rfloor$. It is easy to check that there is a bi-embedding of G if $1 \leq n \leq 6$. If $n = 7$, we can extend the bi-embedding ϕ in Lemma 2.2 to τ where

$$\begin{aligned} \tau(a_1, a_2, a_3, a_4) &= (b_1, a_2, b_3, a_4); \\ \tau(b_1, b_2, b_3, b_4, b_5, b_6) &= (a_1, b_2, a_3, b_4, x_1, b_6); \\ \tau(x_2) &= b_5; \\ \tau(u) &= \phi(u), \quad \text{for all } u \in V(P_7) - \{x_2\}. \end{aligned}$$

Then it is easy to see that τ is a bi-embedding of $C_4 \cup C_6 \cup P_7$. For $n \geq 8$, we can extend the bi-embedding ρ in Lemma 2.2 to τ where

$$\begin{aligned} \tau(a_1, a_2, a_3, a_4) &= (b_1, a_2, b_3, a_4); \\ \tau(b_1, b_2, b_3, b_4, b_5, b_6) &= (a_1, b_2, a_3, b_4, x_1, b_6); \\ \tau(x_1) &= b_5; \\ \tau(u) &= \rho(u), \quad \text{for all } u \in V(P_n) - \{x_1\}. \end{aligned}$$

Then it is easy to see that τ is a bi-embedding of G. □

Now we are ready to prove Theorem 1.4. We will divide the proof of Theorem 1.4 into several cases according to the number of disjoint bipartite cycles in $G(X,Y)$. For convenience, in the following proof, we denote the bipartite graph $G(X,Y)$ as G. Note that $|X| = |Y|$.

Case $k = 1$

As $k = 1$, we let $n_1 = n$, *i.e.*, $G = C_{2n}$ with partition sets X and Y such that $|X| = |Y| = n$. It is easy to check that C_4 and C_6 are not bi-embeddable.

It is easy to see that each of C_8 and C_{10} is bi-embeddable. Clearly, for every bi-embedding σ of C_8 we have $C_8 \cup \sigma(C_8) \cong K_{4,4}$ since every vertex has degree 4 in $C_8 \cup \sigma(C_8)$. So C_8 is uniquely bi-embeddable. For every bi-embedding σ of C_{10}, $C_{10} \cup \sigma(C_{10})$ is a 4-regular bipartite subgraph of $K_{5,5}$. Furthermore, the bi-complement of every $C_{10} \cup \sigma(C_{10})$ is a bipartite 1-factor of $K_{5,5}$ which is a perfect matching and all perfect matchings in $K_{5,5}$ are isomorphic. This implies that C_{10} is uniquely bi-embeddable.

For $n = 6$, let $G = v_1v_2\cdots v_{12}$ with $v_1 \in X$ and let σ and τ be two bijections of $V(G)$ such that

$$\sigma(v_1, v_2, \ldots, v_{12}) = (v_1, v_4, v_9, v_{12}, v_7, v_{10}, v_5, v_2, v_{11}, v_8, v_3, v_6);$$
$$\tau(v_1, v_2, \ldots, v_{12}) = (v_1, v_6, v_{11}, v_4, v_9, v_2, v_7, v_{12}, v_5, v_{10}, v_3, v_8).$$

Then it is easy to check that σ and τ are two bi-embeddings of G. So $(G \cup \sigma(G))'' \cong C_6 \cup C_6$ with $v_1v_{10}v_3v_{12}v_5v_8v_1$ and $v_2v_7v_4v_{11}v_6v_9v_2$ as those two disjoint 6-cycles. But $(G \cup \tau(G))'' \cong 3C_4$ with $v_1v_4v_7v_{10}v_1$, $v_3v_6v_9v_{12}v_3$ and $v_5v_2v_{11}v_8v_5$ as those three disjoint 4-cycles. So $(G \cup \sigma(G))'' \ncong (G \cup \tau(G))''$. Thus C_{12} is not uniquely bi-embeddable.

For $G = C_{2n}$ with $n \geq 7$, we shall show that there are two distinct bi-embeddings σ and τ of G such that $G \cup \sigma(G)$ contains $K_{3,4}$ as a subgraph but $G \cup \tau(G)$ does not.

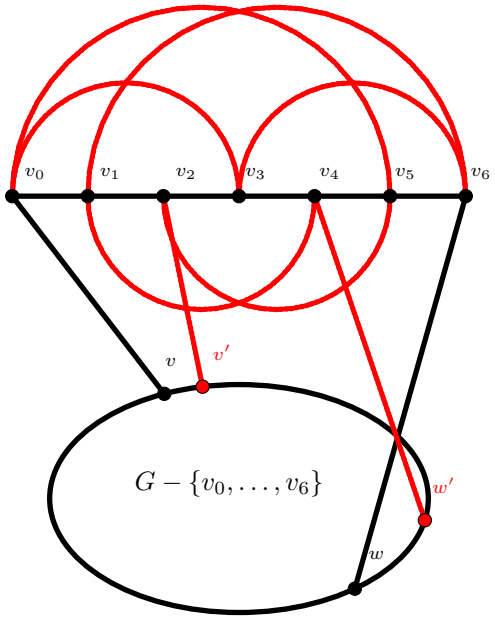


Figure 1: $G \cup \sigma(G)$ contains a $K_{3,4}$.

Let v, v_0, v_1, v_2, v_3, v_4, v_5, v_6, w be nine consecutive vertices of G and let P be the path from v to w in $G[V(G) - \{v_0, \ldots, v_6\}]$, *i.e.*, $P = G - \{v_0, \ldots, v_6\}$. Since $|V(G)| \geq 14$, $|V(P)| \geq 7$. By Lemma 2.2, there is a bi-embedding of P into its bi-complement, say σ. We may say $\sigma(v) = v'$ and $\sigma(w) = w'$. By letting

$$\sigma(v_0, v_1, v_2, v_3, v_4, v_5, v_6) = (v_2, v_5, v_0, v_3, v_6, v_1, v_4),$$

we extend σ to G such that $G \cup \sigma(G)$ contains a subgraph $K_{3,4}$ on $\{v_0, \ldots, v_6\}$ and it is easy to see that σ is a bi-embedding of G. See Figure 1. Since we will use this result for later arguments,

we state this result as Lemma 2.5.

Lemma 2.5. *Let $P = v_0v_1v_2v_3v_4v_5v_6$ be a path of order 7 in a bipartite 2-regular graph G. If $G-P$ is bi-embeddable, then there is a bi-embedding σ such that $G \cup \sigma(G)$ contains $K_{3,4}$ on $\{v_0, \ldots, v_6\}$.*

We now construct another bi-embedding ξ such that $G \cup \xi(G)$ is $K_{3,4}$-free. Since we will use this construction later, we state it as the following Lemma.

Lemma 2.6. *There is a bi-embedding ξ of $G = C_{2n}$ with $n \geq 5$ such that $G \cup \xi(G)$ is $K_{3,4}$-free.*

Proof. For convenience, let $C_{2n} = x_1y_1x_2y_2 \cdots x_ny_nx_1$ with $x_1 \in X$. As $n \geq 5$, let ξ be a bijection of $V(G)$ such that

$$\begin{cases} \xi(x_i) = x_i, & \text{with} \quad 1 \leq i \leq n; \\ \xi(y_i) = y_{i+2}, & i \in \{1, \ldots, n-2\}; \\ \xi(y_{n-1}, y_n) = (y_1, y_2). \end{cases} \tag{2.1}$$

Then it is easy to check that ξ is a bi-embedding of G. Let $H = G \cup \xi(G)$. By the construction of ξ, we see that $H = G \cup \xi(G)$ is a highly symmetric 4-regular bipartite graph. For any $i \in \{1, \ldots, n\}$, $N_H(x_i) = \{y_{i-1}, y_i, y_{i+1}, y_{i+2}\}$ and $N_H(y_i) = \{x_{i-2}, x_{i-1}, x_i, x_{i+1}\}$ where the subscripts are taken modulo n. For $n \geq 5$, two cyclic intervals of length 4 in $\mathbb{Z}_n$ are equal only if they coincide as intervals. So if $i \neq j$ with $i, j \in \{1, \ldots, n\}$, $|N_H(x_i) \cap N_H(x_j)| \leq 3$ and $|N_H(y_i) \cap N_H(y_j)| \leq 3$. Thus, it follows that

$$|N_H(u) \cap N_H(v)| \leq 3, \quad \text{for any two distinct vertices} \quad u, v \in V(G). \tag{2.2}$$

Thus, ξ is a $K_{3,4}$-free bi-embedding of G. □

Thus, $G \cup \sigma(G)$ and $G \cup \tau(G)$ are not isomorphic. This shows that $G = C_{2n}$ is not uniquely bi-embeddable for $n \geq 7$.

Case $k = 2$

Let $G = C_{2n_1} \cup C_{2n_2}$. Assume, without loss of generality, that $n_1 \leq n_2$. From the work in Case $k = 1$, if $n_1 \geq 4$, each of C_{2n_1} and C_{2n_2} has a bi-embedding and so G has a bi-embedding σ such that $G \cup \sigma(G)$ is disconnected. By Lemma 2.1, G is not uniquely bi-embeddable. Therefore, $n_1 = 2$ or $n_1 = 3$. So we have the following two subcases: $G = C_4 \cup C_{2n_2}$ with $n_2 \geq 2$ or $G = C_6 \cup C_{2n_2}$ with $n_2 \geq 3$.

Subcase 1.1. $G = C_4 \cup C_{2n_2}$ **with** $n_2 \geq 2$

For $n_2 = 2$, $G = C_4 \cup C_4$ is bi-embeddable and it is easy to check that for any bi-embedding σ of G, $G \cup \sigma(G) \cong K_{4,4}$ since the degree of any vertex in $G \cup \sigma(G)$ is equal to 4. Thus, $C_4 \cup C_4$ is uniquely bi-embeddable.

For $n_2 = 3$, we will show that $G = C_4 \cup C_6$ is not bi-embeddable. Let $C_4 = x_1x_2x_3x_4x_1$ and $C_6 = v_1v_2v_3v_4v_5v_6v_1$ with $\{x_1, v_1\} \subseteq X$. Suppose there is a bi-embedding σ of G. Then $\sigma(G)$ is 2-regular. Say, without loss of generality, that $\{x_1v_2, x_1v_4\} \subseteq E(\sigma(G))$. Note that $N_{\sigma(G)}(x_3) \subseteq \{v_2, v_4, v_6\}$. Suppose that $\{x_3v_2, x_3v_4\} \subseteq E(\sigma(G))$ for the moment. Then $G[V(G) - \{x_1, x_3, v_2, v_4\}]$ is the 6-cycle in $\sigma(G)$. Since σ is a bi-embedding, $\{v_1x_2, v_1x_4\} \subseteq E(\sigma(G))$. This implies that v_5 must be adjacent to v_6 in $\sigma(G)$ which is impossible. So, $|N_{\sigma(G)}(x_1) \cap N_{\sigma(G)}(x_3)| = 1$. Assume, without loss of generality, that $\{x_3v_2, x_3v_6\} \subseteq E(\sigma(G))$. But this implies that both v_4 and v_6 must be adjacent to u in $\sigma(G)$ where $u \in \{v_1, v_3, v_5\}$, which is also impossible since σ is a bi-embedding. Thus $C_4 \cup C_6$ is not bi-embeddable.

For $n_2 = 4$, we will show that there are two distinct bi-embeddings of $G = C_4 \cup C_8$. Denote the 4-cycle and 8-cycle by $x_1x_2x_3x_4x_1$ and $v_1v_2v_3v_4v_5v_6v_7v_8$, respectively. We may say that $\{x_1, v_1\} \subseteq X$. Let σ and τ be two bijections of $V(G)$ such that

$$\begin{aligned}
\sigma(x_1, x_2, x_3, x_4) &= (x_1, v_2, x_3, v_4);\\
\sigma(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8) &= (v_1, v_6, v_3, v_8, v_5, x_2, v_7, x_4);\\
\tau(x_1, x_2, x_3, x_4) &= (x_1, v_4, x_3, v_8);\\
\tau(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8) &= (v_1, x_2, v_7, v_2, v_5, x_4, v_3, v_6).
\end{aligned}$$

Then it is easy to see that σ and τ are two bi-embeddings of G. Furthermore, $(G \cup \sigma(G))'' \cong C_4 \cup C_8$ where $C_4 = x_1v_6x_3v_8x_1$ and $C_8 = v_1x_2v_3x_4v_5v_2v_7v_4v_1$. But $(G \cup \tau(G))'' \cong 3C_4$ with $x_1v_2x_3v_6x_1$, $v_3x_2v_5v_8v_3$ and $v_1x_4v_7v_4v_1$ as those three disjoint 4-cycles. So $(G \cup \sigma(G))'' \ncong (G \cup \tau(G))''$. Thus, $C_4 \cup C_8$ is not uniquely bi-embeddable.

Now, let $G = C_4 \cup C_{2n_2}$ with $n_2 \geq 5$. We will show that there are two distinct bi-embeddings σ and τ such that $G \cup \sigma(G)$ contains $K_{3,4}$ but $G \cup \tau(G)$ does not. For convenience, we denote $C_4 = v_1v_2v_3v_4v_1$ and $C_{2n_2} = x_1y_1x_2y_2 \cdots x_{n_2}y_{n_2}x_1$ with $\{x_1, v_1\} \subseteq X$ where $n_2 \geq 5$. Let P be a path of order 7 in G. So $P \subseteq C_{2n_2}$. By Lemma 2.3 and Lemma 2.5, there is a bi-embedding σ of G such that $K_{3,4} \subseteq G \cup \sigma(G)$. Now, we will construct another bi-embedding τ such that $G \cup \tau(G)$ is $K_{3,4}$-free. We can extend the bi-embedding ξ in Lemma 2.6 equation (2.1) to τ by letting

$$\begin{aligned}
\tau(v_1, v_2, v_3, v_4) &= (v_1, y_3, v_3, y_4);\\
\tau(y_1, y_2) &= (v_2, v_4);\\
\tau(u) &= \xi(u) \quad \text{for all} \quad u \in V(G) - \{v_1, v_2, v_3, v_4, y_1, y_2\}.
\end{aligned}$$

Then it is easy to check that τ is a bi-embedding of G. Similar to (2.2) in Lemma 2.6, let $H = G \cup \tau(G)$ and we get for any two distinct vertices $u, v \in V(G) - \{v_1, v_3\}$, $|N_H(u) \cap N_H(v)| \leq 3$. For v_1 and v_3, $N_H(v_1) = N_H(v_3) = \{v_2, v_4, y_3, y_4\}$ and $N_H(v_2) \cap N_H(v_4) \cap N_H(y_3) \cap N_H(y_4) = \{v_1, v_3\}$ which forms exactly one $K_{2,4}$ on $\{v_1, v_3, v_2, v_4, y_3, y_4\}$. So $G \cup \tau(G)$ is $K_{3,4}$-free. Thus, $G \cup \sigma(G)$ and $G \cup \tau(G)$ are not isomorphic. And this shows that $G = C_4 \cup C_{2n_2}$ is not uniquely bi-embeddable for $n_2 \geq 5$.

Subcase 1.2. $G = C_6 \cup C_{2n_2}$ **with** $n_2 \geq 3$

For $n_2 = 3$, we will show that there are two distinct bi-embeddings of $G = C_6 \cup C_6$. Denote two 6-cycles in G by $x_1x_2x_3x_4x_5x_6x_1$ and $v_1v_2v_3v_4v_5v_6v_1$, respectively, with $\{x_1, v_1\} \subseteq X$. Let σ and τ be two bijections of $V(G)$ such that

$$\begin{aligned}
\sigma(x_1, x_2, x_3, x_4, x_5, x_6) &= (x_1, v_2, x_3, v_4, x_5, v_6);\\
\sigma(v_1, v_2, v_3, v_4, v_5, v_6) &= (v_1, x_2, v_3, x_4, v_5, x_6);\\
\tau(x_1, x_2, x_3, x_4, x_5, x_6) &= (x_1, x_4, v_1, v_4, x_3, v_2);\\
\tau(v_1, v_2, v_3, v_4, v_5, v_6) &= (x_5, v_6, v_3, x_6, v_5, x_2).
\end{aligned}$$

Then it is easy to check that σ and τ are two bi-embeddings of G. Moreover, $(G \cup \sigma(G))'' \cong 3C_4$ with $x_1x_4v_1v_4x_1$, $x_3x_6v_3v_6x_3$ and $x_5x_2v_5v_2x_5$ as those three disjoint 4-cycles. But $(G \cup \tau(G))'' \cong C_{12}$. So $(G \cup \sigma(G))'' \ncong (G \cup \tau(G))''$. Thus $C_6 \cup C_6$ is not uniquely bi-embeddable.

For $n_2 = 4$, we will show that there are two distinct bi-embeddings of $G = C_6 \cup C_8$. Denote two cycles in G by $C_6 = x_1x_2x_3x_4x_5x_6x_1$ and $C_8 = v_1v_2v_3v_4v_5v_6v_7v_8v_1$ with $\{x_1, v_1\} \subseteq X$. Let σ and τ be two bijections of $V(G)$ such that

$$\begin{aligned}
\sigma(x_1, x_2, x_3, x_4, x_5, x_6) &= (x_1, v_2, x_3, v_4, x_5, v_6);\\
\sigma(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8) &= (v_1, x_2, v_7, x_4, v_3, v_8, v_5, x_6);\\
\tau(x_1, x_2, x_3, x_4, x_5, x_6) &= (x_1, x_4, v_1, v_4, x_3, v_6);\\
\tau(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8) &= (x_5, v_8, v_5, v_2, v_7, x_6, v_3, x_2).
\end{aligned}$$

Then it is easy to see that σ and τ are two bi-embeddings of G. Let $H_1 = G \cup \sigma(G)$ and $H_2 = G \cup \tau(G)$. We will show that $H_1'' \ncong H_2''$. Clearly, $x_1x_4v_1v_4x_1$, $x_3x_6v_3v_6x_3$ and $x_5x_2v_5v_2x_5$ are three disjoint 4-cycles in H_1''. Suppose for a contradiction that there are three disjoint 4-cycles in H_2'' and denote those three 4-cycles as L_1, L_2 and L_3, respectively. Suppose $x_4 \in V(L_i)$ for some $i \in \{1, 2, 3\}$. Clearly, $x_4v_5x_2v_7x_4$ is the only 4-cycle in H_2'' containing x_4 and we denote it as L_1. Then it is easy to check that $x_6 \notin V(L_j)$ for any $j \in \{2, 3\}$. This implies that $\{v_2, v_4, v_6, v_8\} \subseteq V(L_2 \cup L_3)$. But it is easy to see that v_6 is not in any 4-cycle in $H_2''[V(H_2'') -$

$\{x_4, v_5, x_2, v_7, x_6\}]$, a contradiction. So we assume that $x_4 \notin L_i$ for any $i \in \{1,2,3\}$. This implies that $\{x_2, x_6, v_2, v_4, v_6, v_8\} \subseteq V(L_1 \cup L_2 \cup L_3)$. For x_6, $x_6v_1x_2v_5x_6$ is the only 4-cycle in $H_2''[V(H_2'') - \{x_4\}]$ contains x_6 and we denote it as L_1. But it is easy to see that $v_6 \notin L_j$ for any $j \in \{2,3\}$, a contradiction. Thus, $3C_4 \subseteq (G \cup \sigma(G))''$ and $(G \cup \tau(G))''$ is $3C_4$-free and this implies that $C_6 \cup C_8$ is not uniquely bi-embeddable.

Now, let $G = C_6 \cup C_{2n_2}$ with $n_2 \geq 5$. We will show that there are two distinct bi-embeddings σ and τ such that $G \cup \sigma(G)$ contains $K_{3,4}$ but $G \cup \tau(G)$ does not. For convenience, we denote $C_6 = v_1v_2v_3v_4v_5v_6v_1$ and $C_{2n_2} = x_1y_1x_2y_2 \cdots x_{n_2}y_{n_2}x_1$ with $\{x_1, v_1\} \subseteq X$ where $n_2 \geq 5$. Let P be a path of order 7 in G. So $P \subseteq C_{2n_2}$. By Lemma 2.3 and Lemma 2.5, there is a bi-embedding σ of G such that $K_{3,4} \subseteq G \cup \sigma(G)$. Now, we will construct another bi-embedding τ such that $G \cup \tau(G)$ is $K_{3,4}$-free. We can extend the bi-embedding ξ in Lemma 2.6 equation (2.1) to τ by letting

$$\begin{aligned} \tau(v_1, v_2, v_3, v_4, v_5, v_6) &= (v_1, y_3, v_3, y_4, v_5, y_5); \\ \tau(y_1, y_2, y_3) &= (v_2, v_4, v_6); \\ \tau(u) &= \xi(u) \quad \text{for all} \quad u \in V(G) - \{v_1, v_2, v_3, v_4, v_5, v_6, y_1, y_2, y_3\}. \end{aligned}$$

Then it is easy to check that τ is a bi-embedding of G. Similar to (2.2) in Lemma 2.6, let $H = G \cup \tau(G)$ and we get for any two distinct vertices $u, v \in V(G)$, $|N_H(u) \cap N_H(v)| \leq 3$. So $G \cup \tau(G)$ is $K_{3,4}$-free. Thus, $G \cup \sigma(G)$ and $G \cup \tau(G)$ are not isomorphic. And this shows that $G = C_6 \cup C_{2n_2}$ is not uniquely bi-embeddable for $n_2 \geq 5$.

Case $k = 3$

Let $G = C_{2n_1} \cup C_{2n_2} \cup C_{2n_3}$. Assume, without loss of generality, that $n_1 \leq n_2 \leq n_3$. So we can treat G as two subgraphs G_1 and G_2 with $G_1 = C_{2n_1} \cup C_{2n_2}$ and $G_2 = C_{2n_3}$. By Case $k = 1$ and Case $k = 2$ we have the following arguments. If $n_1 \geq 4$, each of G_1 and G_2 has a bi-embedding and so G has a bi-embedding σ such that $G \cup \sigma(G)$ is disconnected. By Lemma 2.1, G is not uniquely bi-embeddable. So $n_1 \in \{2,3\}$. If $n_2 \geq 4$, each of G_1 and G_2 also has a bi-embedding and G has a bi-embedding σ such that $G \cup \sigma(G)$ is disconnected. By Lemma 2.1, G is not uniquely bi-embeddable. So $n_2 \in \{2,3\}$. If $n_1 = n_2 = 2$ and $n_3 \geq 4$, each of G_1 and G_2 also has a bi-embedding and G has a bi-embedding σ such that $G \cup \sigma(G)$ is disconnected. By Lemma 2.1, G is not uniquely bi-embeddable. If $n_1 = n_2 = 3$ and $n_3 \geq 4$, each of G_1 and G_2 also has a bi-embedding and G has a bi-embedding σ such that $G \cup \sigma(G)$ is disconnected. By Lemma 2.1, G is not uniquely bi-embeddable. Thus, G is isomorphic to one of the following graphs: $3C_4$, $2C_4 \cup C_6$, $C_4 \cup C_6 \cup C_{2n_3}$ with $n_3 \geq 3$ and $3C_6$. We divide our argument into the following subcases.

Subcase 2.1. $G = 3C_4$

Denote three 4-cycles in G by $x_1x_2x_3x_4x_1$, $v_1v_2v_3v_4v_1$ and $z_1z_2z_3z_4z_1$, respectively. Assume that $\{x_1, v_1, z_1\} \subseteq X$. We will show that for any bi-embedding ψ of G, $(G \cup \psi(G))'' \cong 3C_4$. It is easy to check that G is bi-embeddable and for any bi-embedding ψ of G, $(G \cup \psi(G))''$ is a 2-factor. Let σ be a bi-embedding of G. By the definition of bi-embedding, $\sigma(G) \cong G$ and $E(\sigma(G)) \cap E(G) = \varnothing$. It follows that $\sigma(G) \cong 3C_4$, *i.e.*, $\sigma(G)$ is still a disjoint union of three bipartite 4-cycles. Let $\{x, y\} \subseteq V(G)$. First note that $G = 3C_4$ and so $d_G(x) = d_G(y) = 2$. Moreover, $N_G(x) \cap N_G(y) = \varnothing$ or $N_G(x) = N_G(y)$ as G is a disjoint union of three bipartite 4-cycles. Now, we claim that for any $\{u, v\} \subseteq \sigma(G)$, $N_{\sigma(G)}(u) \cap N_{\sigma(G)}(v) = \varnothing$ or $N_{\sigma(G)}(u) = N_{\sigma(G)}(v)$ with $d_{\sigma(G)}(u) = d_{\sigma(G)}(v) = 2$. Suppose that $N_{\sigma(G)}(u) \cap N_{\sigma(G)}(v) \neq \varnothing$. Let $w_1 \in N_{\sigma(G)}(u) \cap N_{\sigma(G)}(v)$. As $\sigma(G) \cong 3C_4$, u, v and w_1 lie in the same component in $\sigma(G)$, *i.e.*, they must lie in a bipartite 4-cycle in $\sigma(G)$. Denote this bipartite 4-cycle as C. Then $C = uw_1vw_2u$, where w_2 is the remaining vertex of C. It follows that $N_{\sigma(G)}(u) = N_{\sigma(G)}(v) = \{w_1, w_2\}$ in $\sigma(G)$. Thus the claim holds.

For instance, we choose $u = x_1$. If $v = x_3$, then $N_G(x_1) = N_G(x_3) = \{x_2, x_4\}$. If $v \neq x_3$, say, without loss of generality, that $v = v_1$, then it is easy to see that $N_G(x_1) \cap N_G(v_1) = \varnothing$. So if $v = x_3$, then $\sigma(\{x_1, x_2, x_3, x_4\})$ must form a bipartite 4-cycle in $\sigma(G)$ since $\sigma(G) \cong 3C_4$. It follows that $N_{\sigma(G)}(\sigma(x_1)) = N_{\sigma(G)}(\sigma(x_3)) = \{\sigma(x_2), \sigma(x_4)\}$. Assume that $v \neq x_3$, say, without loss of generality, that $v = v_1$. We suppose that $N_{\sigma(G)}(\sigma(x_1)) \cap N_{\sigma(G)}(\sigma(v_1)) \neq \varnothing$. As $N_{\sigma(G)}(\sigma(x_1)) = N_{\sigma(G)}(\sigma(x_3)) = \{\sigma(x_2), \sigma(x_4)\}$, $N_{\sigma(G)}(\sigma(v_1)) \cap \{\sigma(x_2), \sigma(x_4)\} \neq \varnothing$ and so we have either $d_{\sigma(G)}(\sigma(x_2)) \geq 3$ or $d_{\sigma(G)}(\sigma(x_4)) \geq 3$ which contradicts to $\sigma(G) \cong 3C_4$. Thus $N_{\sigma(G)}(x_1) \cap N_{\sigma(G)}(v_1) = \varnothing$.

Now, let $H = G \cup \sigma(G)$. By the structure of G and $\sigma(G)$, it is easy to see that $H'' \cong 3C_4$. Thus $G = 3C_4$ is uniquely bi-embeddable.

Subcase 2.2. $G = 2C_4 \cup C_6$

We will show that there are two distinct bi-embeddings of G. For convenience, denote one C_4 by $x_1x_2x_3x_4x_1$, the other C_4 by $v_1v_2v_3v_4v_1$ and C_6 by $z_1z_2z_3z_4z_5z_6z_1$ in G with $\{x_1, v_1, z_1\} \subseteq X$. Let σ and τ be two bijections of $V(G)$ such that

$$\begin{aligned}
\sigma(x_1, x_2, x_3, x_4) &= (x_1, v_2, x_3, v_4);\\
\sigma(v_1, v_2, v_3, v_4) &= (v_3, z_2, z_5, x_4);\\
\sigma(z_1, z_2, z_3, z_4, z_5, z_6) &= (z_1, z_4, v_1, z_6, z_3, x_2);\\
\tau(x_1, x_2, x_3, x_4) &= (x_1, v_2, z_1, v_4);\\
\tau(v_1, v_2, v_3, v_4) &= (v_1, z_2, z_5, x_2);\\
\tau(z_1, z_2, z_3, z_4, z_5, z_6) &= (x_3, z_4, v_3, x_4, z_3, z_6).
\end{aligned}$$

Then it is easy to see that σ and τ are two bi-embeddings of G. Let $H_1 = G \cup \sigma(G)$ and $H_2 = G \cup \tau(G)$. We will show that $H_1'' \ncong H_2''$. For H_1'', $H_1''[\{x_1, x_3, z_2, z_4, z_6\}]$ contains a $K_{2,3}$ and $H_1''[\{v_2, v_4, z_1, z_3, z_5\}]$ contains another $K_{2,3}$. So $2K_{2,3} \subseteq H_1''$. For H_2'', $|N_{H_2''}(u) \cap N_{H_2''}(v)| \leq 2$ for any two distinct vertices $u, v \in V(H_2'') - \{v_2, v_4\}$ and $N_{H_2''}(v_2) = N_{H_2''}(v_4) = \{x_3, z_3, z_5\}$. This implies that H_2'' contains exactly one $K_{2,3}$ on $\{v_2, v_4, x_3, z_3, z_5\}$. So H_2'' is $2K_{2,3}$-free. Thus $2C_4 \cup C_6$ is not uniquely bi-embeddable.

Subcase 2.3. $G = C_4 \cup C_6 \cup C_{2n_3}$ **with** $n_3 \geq 3$

First, we show that there are two distinct bi-embeddings when $n_3 = 3$, *i.e.*, $G = C_4 \cup 2C_6$. For convenience, denote C_4 by $x_1x_2x_3x_4x_1$, one C_6 by $v_1v_2v_3v_4v_5v_6v_1$ and the other C_6 by $z_1z_2z_3z_4z_5z_6z_1$ in G with $\{x_1, v_1, z_1\} \subseteq X$. Let σ and τ be two bijections of $V(G)$ such that

$$\begin{aligned}
\sigma(x_1, x_2, x_3, x_4) &= (x_1, v_2, x_3, v_4);\\
\sigma(v_1, v_2, v_3, v_4, v_5, v_6) &= (v_1, x_2, v_3, x_4, z_1, z_4);\\
\sigma(z_1, z_2, z_3, z_4, z_5, z_6) &= (z_5, v_6, z_3, z_6, v_5, z_2);\\
\tau(x_1, x_2, x_3, x_4) &= (x_1, v_2, z_1, v_4);\\
\tau(v_1, v_2, v_3, v_4, v_5, v_6) &= (v_1, z_2, z_5, v_6, v_3, x_2);\\
\tau(z_1, z_2, z_3, z_4, z_5, z_6) &= (x_3, z_4, v_5, x_4, z_3, z_6).
\end{aligned}$$

Then it is easy to check that σ and τ are two bi-embeddings of G. Let $H_1 = G \cup \sigma(G)$ and $H_2 = G \cup \tau(G)$. We will show that $H_1'' \ncong H_2''$. For H_1'', $H_1''[\{x_1, x_3, v_3, v_6, z_2, z_4, z_6\}]$ contains a $K_{3,4}$ But for any two distinct vertices $u, v \in V(H_2'')$, $|N_{H_2''}(u) \cap N_{H_2''}(v)| \leq 3$. So H_2'' is $K_{3,4}$-free. Thus $C_4 \cup 2C_6$ is not uniquely bi-embeddable.

Second, we show that there are two distinct bi-embeddings when $n_3 = 4$, *i.e.*, $G = C_4 \cup C_6 \cup C_8$. We will show that there are two distinct bi-embeddings σ and τ such that $G \cup \sigma(G)$ contains $K_{3,4}$ but $G \cup \tau(G)$ does not. For convenience, denote C_4 by $v_1v_2v_3v_4v_1$, C_6 by $z_1z_2z_3z_4z_5z_6z_1$ and C_8 by $x_1y_1x_2y_2x_3y_3x_4y_4x_1$ in G with $\{x_1, v_1, z_1\} \subseteq X$. Let P be a path of order 7 in G. So $P \subseteq C_8$. By Lemma 2.4 and Lemma 2.5, there is a bi-embedding σ of G such that $K_{3,4} \subseteq G \cup \sigma(G)$. Now, we will construct another bi-embedding τ such that $G \cup \tau(G)$ is $K_{3,4}$-free. Let

$$\begin{aligned}
\tau(v_1, v_2, v_3, v_4) &= (v_1, z_2, v_3, z_4);\\
\tau(z_1, z_2, z_3, z_4, z_5, z_6) &= (z_1, y_1, z_3, y_2, z_5, y_3);\\
\tau(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) &= (x_1, v_2, x_2, y_4, x_3, v_4, x_4, z_6).
\end{aligned}$$

Then it is easy to see that τ is a bi-embedding of G. Let $H = G \cup \tau(G)$ and it is easy to check that for any two distinct vertices $u, v \in V(G) - \{v_1, v_3\}$, $|N_H(u) \cap N_H(v)| \leq 3$. For v_1 and v_3,

$N_H(v_1) = N_H(v_3) = \{v_2, v_4, z_2, z_4\}$ and $N_H(v_2) \cap N_H(v_4) \cap N_H(z_2) \cap N_H(z_4) = \{v_1, v_3\}$ which forms exactly one $K_{2,4}$ on $\{v_1, v_3, v_2, v_4, z_2, z_4\}$. So $G \cup \tau(G)$ is $K_{3,4}$-free. Thus, $G \cup \sigma(G)$ and $G \cup \tau(G)$ are not isomorphic and this shows that $G = C_4 \cup C_6 \cup C_8$ is not uniquely bi-embeddable.

Now, let $G = C_4 \cup C_6 \cup C_{2n_3}$ with $n_3 \geq 5$. We will show that there are two distinct bi-embeddings σ and τ such that $G \cup \sigma(G)$ contains $K_{3,4}$ but $G \cup \tau(G)$ does not. For convenience, denote C_4 by $v_1v_2v_3v_4v_1$, C_6 by $z_1z_2z_3z_4z_5z_6z_1$ and C_{2n_3} by $x_1y_1 \cdots x_{n_3}y_{n_3}x_1$ in G with $\{x_1, v_1, z_1\} \subseteq X$ where $n_3 \geq 5$. Let P be a path of order 7 in G. So $P \subseteq C_{2n_3}$. By Lemma 2.4 and Lemma 2.5, there is a bi-embedding σ of G such that $K_{3,4} \subseteq G \cup \sigma(G)$. Now, we will construct another bi-embedding τ such that $G \cup \tau(G)$ is $K_{3,4}$-free. We can extend the bi-embedding ξ in Lemma 2.6 equation (2.1) to τ by letting

$$\begin{aligned}
\tau(v_1, v_2, v_3, v_4) &= (v_1, y_3, v_3, y_4);\\
\tau(z_1, z_2, z_3, z_4, z_5, z_6) &= (z_1, v_2, z_3, v_4, z_5, y_5);\\
\tau(y_1, y_2, y_3) &= (z_2, z_4, z_6);\\
\tau(u) &= \xi(u) \quad \text{for all} \quad u \in V(G) - \{v_1, v_2, v_3, v_4, z_1, z_2, z_3, z_4, z_5, z_6, y_1, y_2, y_3\}.
\end{aligned}$$

Then it is easy to see that τ is a bi-embedding of G. Similar to (2.2) in Lemma 2.6, let $H = G \cup \tau(G)$ and we get any $u, v \in V(G) - \{v_1, v_3\}$, $|N_H(u) \cap N_H(v)| \leq 3$. For v_1 and v_3, $N_H(v_1) = N_H(v_3) = \{v_2, v_4, y_3, y_4\}$ and $N_H(v_2) \cap N_H(v_4) \cap N_H(y_3) \cap N_H(y_4) = \{v_1, v_3\}$ which forms exactly one $K_{2,4}$ on $\{v_1, v_3, v_2, v_4, y_3, y_4\}$. So $G \cup \tau(G)$ is $K_{3,4}$-free. Thus, $G \cup \sigma(G)$ and $G \cup \tau(G)$ are not isomorphic and this shows that $G = C_4 \cup C_6 \cup C_{2n_3}$ is not uniquely bi-embeddable for $n_3 \geq 5$.

Subcase 2.4. $G = 3C_6$

We will show that there are two distinct bi-embeddings of $G = 3C_6$. For convenience, denote those three 6-cycles by $x_1x_2x_3x_4x_5x_6x_1$, $v_1v_2v_3v_4v_5v_6v_1$ and $z_1z_2z_3z_4z_5z_6z_1$ in G, respectively, where $\{x_1, v_1, z_1\} \subseteq X$. Let σ and τ be two bijections of $V(G)$ such that

$$\begin{aligned}
\sigma(x_1, x_2, x_3, x_4, x_5, x_6) &= (x_1, x_4, v_1, x_6, x_3, v_2);\\
\sigma(v_1, v_2, v_3, v_4, v_5, v_6) &= (x_5, x_2, v_3, z_2, z_5, v_4);\\
\sigma(z_1, z_2, z_3, z_4, z_5, z_6) &= (z_1, z_4, v_5, z_6, z_3, v_6);\\
\tau(x_1, x_2, x_3, x_4, x_5, x_6) &= (x_1, x_4, v_1, v_4, z_1, z_4);\\
\tau(v_1, v_2, v_3, v_4, v_5, v_6) &= (x_5, x_2, v_5, v_2, z_5, z_2);\\
\tau(z_1, z_2, z_3, z_4, z_5, z_6) &= (x_3, x_6, v_3, v_6, z_3, z_6).
\end{aligned}$$

Then it is easy to check that σ and τ are two bi-embeddings of G. Let $H_1 = G \cup \sigma(G)$ and $H_2 = G \cup \tau(G)$. We will show that $H_1 \ncong H_2$. For H_1, $H_1[\{x_1, x_3, x_2, x_4, x_6, v_2\}]$ contains a $K_{2,4}$.

But for any two distinct vertices $u, v \in V(H_2)$, $|N_{H_2}(u) \cap N_{H_2}(v)| \leq 3$. This implies that H_2 is $K_{2,4}$-free. Thus $3C_6$ is not uniquely bi-embeddable.

Case $k \geq 4$

In this case, we will divide the case into two subcases, one is $k = 4$ and another is $k \geq 5$.

Subcase 3.1. $k = 4$

Let $G = C_{2n_1} \cup C_{2n_2} \cup C_{2n_3} \cup C_{2n_4}$. Assume, without loss of generality, that $n_1 \leq n_2 \leq n_3 \leq n_4$. Let $G = G_1 \cup G_2$ where $G_1 = C_{2n_1} \cup C_{2n_2}$ and $G_2 = C_{2n_3} \cup C_{2n_4}$. By Case $k = 2$ and Lemma 2.1, G is not uniquely bi-embeddable unless G is isomorphic to $3C_4 \cup C_6$ or $C_4 \cup 3C_6$. Here are some examples for using Lemma 2.1 and Case $k = 2$. If $n_1 \geq 3$, then each G_1 and G_2 has a bi-embedding and so G has a bi-embedding σ such that $G \cup \sigma(G)$ is disconnected. By Lemma 2.1, G is not uniquely bi-embeddable. So $n_1 = 2$. Similarly, if $n_1 = n_2 = n_3 = n_4 = 2$, G has a disconnected bi-embedding. If $n_1 = n_2 = 2$ and $n_3 = n_4 = 3$, G also has a disconnected bi-embedding. By Lemma 2.1, G is not uniquely bi-embeddable. It therefore remains to consider the two cases $G \cong 3C_4 \cup C_6$ and $G \cong C_4 \cup 3C_6$.

For $G = 3C_4 \cup C_6$, we will show that there are two distinct bi-embeddings of G. Denote those 4 cycles by $x_1x_2x_3x_4x_1$, $v_1v_2v_3v_4v_1$, $z_1z_2z_3z_4z_1$ and $y_1y_2y_3y_4y_5y_6y_1$, respectively, where $\{x_1, v_1, z_1, y_1\} \subseteq X$. Let σ and τ be two bijections of $V(G)$ such that

$$\begin{aligned}
\sigma(x_1, x_2, x_3, x_4) &= (y_5, y_2, v_3, z_2);\\
\sigma(v_1, v_2, v_3, v_4) &= (x_1, v_2, x_3, v_4);\\
\sigma(z_1, z_2, z_3, z_4) &= (v_1, x_2, z_1, x_4);\\
\sigma(y_1, y_2, y_3, y_4, y_5, y_6) &= (y_1, y_4, z_3, y_6, y_3, z_4);\\
\tau(x_1, x_2, x_3, x_4) &= (x_1, v_4, y_3, v_2);\\
\tau(v_1, v_2, v_3, v_4) &= (v_1, y_2, y_5, z_2);\\
\tau(z_1, z_2, z_3, z_4) &= (z_1, y_6, z_3, x_4);\\
\tau(y_1, y_2, y_3, y_4, y_5, y_6) &= (y_1, x_2, v_3, z_4, x_3, y_4).
\end{aligned}$$

Then it is easy to see that σ and τ are two bi-embeddings of G. Let $H_1 = G \cup \sigma(G)$ and $H_2 = G \cup \tau(G)$. We will show that $H_1 \ncong H_2$. For H_1, $H_1[\{x_1, x_3, v_1, x_2, x_4, v_2, v_4\}]$ contains a $K_{3,4}$. In H_2, it contains exactly two $K_{2,4}$. For z_1 and z_3, $N_{H_2}(z_1) = N_{H_2}(z_3) = \{x_4, z_2, z_4, y_6\}$ and $N_{H_2}(x_4) \cap N_{H_2}(z_2) \cap N_{H_2}(z_4) \cap N_{H_2}(y_6) = \{z_1, z_3\}$ which forms one $K_{2,4}$ on $\{z_1, z_3, x_4, z_2, z_4, y_6\}$. For v_2 and v_4, $N_{H_2}(v_2) = N_{H_2}(v_4) = \{v_1, v_3, x_1, y_3\}$ and $N_{H_2}(v_1) \cap N_{H_2}(v_3) \cap N_{H_2}(x_1) \cap N_{H_2}(y_3) =$

$\{v_2, v_4\}$ which forms another $K_{2,4}$ on $\{v_1, v_2, v_3, v_4, x_1, y_3\}$. And for any two distinct vertices $u, v \in V(G) - \{z_1, z_3, v_2, v_4\}$, $|N_{H_2}(u) \cap N_{H_2}(v)| \leq 3$. It follows that no three vertices of H_2 have four common neighbours and so H_2 is $K_{3,4}$-free. Thus $3C_4 \cup C_6$ is not uniquely bi-embeddable.

For $G = C_4 \cup 3C_6$, we will show that there are two distinct bi-embeddings of G. Denote those cycles by $x_1x_2x_3x_4x_1$, $v_1v_2v_3v_4v_5v_6v_1$, $z_1z_2z_3z_4z_5z_6z_1$ and $y_1y_2y_3y_4y_5y_6y_1$, respectively, where $\{x_1, v_1, z_1, y_1\} \subseteq X$. Let

$$\begin{aligned}
\sigma(x_1, x_2, x_3, x_4) &= (x_1, v_2, v_5, y_2);\\
\sigma(v_1, v_2, v_3, v_4, v_5, v_6) &= (v_1, x_2, v_3, v_6, y_1, v_4);\\
\sigma(z_1, z_2, z_3, z_4, z_5, z_6) &= (x_3, y_4, z_1, x_4, y_5, z_4);\\
\sigma(y_1, y_2, y_3, y_4, y_5, y_6) &= (y_3, z_2, z_5, y_6, z_3, z_6);\\
\tau(x_1, x_2, x_3, x_4) &= (x_1, v_2, z_3, y_4);\\
\tau(v_1, v_2, v_3, v_4, v_5, v_6) &= (v_1, v_4, z_1, z_4, y_3, y_6);\\
\tau(z_1, z_2, z_3, z_4, z_5, z_6) &= (y_1, v_6, v_3, x_2, z_5, z_2);\\
\tau(y_1, y_2, y_3, y_4, y_5, y_6) &= (x_3, z_6, v_5, x_4, y_5, y_2).
\end{aligned}$$

Then it is easy to check that σ and τ are two bi-embeddings of G. Let $H_1 = G \cup \sigma(G)$ and $H_2 = G \cup \tau(G)$. We will show that $H_1 \ncong H_2$. For H_1, $H_1[\{z_1, z_3, z_5, y_3, z_2, z_6\}]$ contains a $K_{2,4}$. But for any two distinct vertices $u, v \in V(H_2)$, $|N_{H_2}(u) \cap N_{H_2}(v)| \leq 3$. This implies that H_2 is $K_{2,4}$-free. Thus $C_4 \cup 3C_6$ is not uniquely bi-embeddable.

Subcase 3.2. $k \geq 5$

Let $G = C_{2n_1} \cup \cdots \cup C_{2n_k}$. Assume, without loss of generality, that $n_1 \leq \cdots \leq n_k$. If $k = 5$, let $G = G_1 \cup G_2$ where $G_1 = C_{2n_1} \cup C_{2n_2}$ and $G_2 = C_{2n_3} \cup C_{2n_4} \cup C_{2n_5}$. By Case $k = 2$, Case $k = 3$ and Lemma 2.1, $G_1 = C_4 \cup C_6$ and $G_2 = C_{2n_3} \cup C_{2n_4} \cup C_{2n_5}$ with $n_5 \geq n_4 \geq n_3 \geq 3$. But by Case $k = 2$ and Case $k = 3$, there is a bi-embedding of $C_4 \cup C_6 \cup C_{2n_3}$ and $C_{2n_4} \cup C_{2n_5}$ with $n_5 \geq n_4 \geq n_3 \geq 3$ which is a disconnected bi-embedding of G. By Lemma 2.1, there is another bi-embedding σ' of G such that $G \cup \sigma'(G)$ is connected, *i.e.*, G is not uniquely bi-embeddable. If $k \geq 6$, we let $G = G_1 \cup G_2$ such that $G_1 = C_{2n_1} \cup C_{2n_2} \cup C_{2n_3}$ and $G_2 = C_{2n_4} \cup \cdots \cup C_{2n_k}$. As $k \geq 6$, by previous cases there is a bi-embedding of G which is disconnected. Thus, by Lemma 2.1, there is another bi-embedding of G, *i.e.*, G is not uniquely bi-embeddable.

This completes the proof of Theorem 1.4.

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