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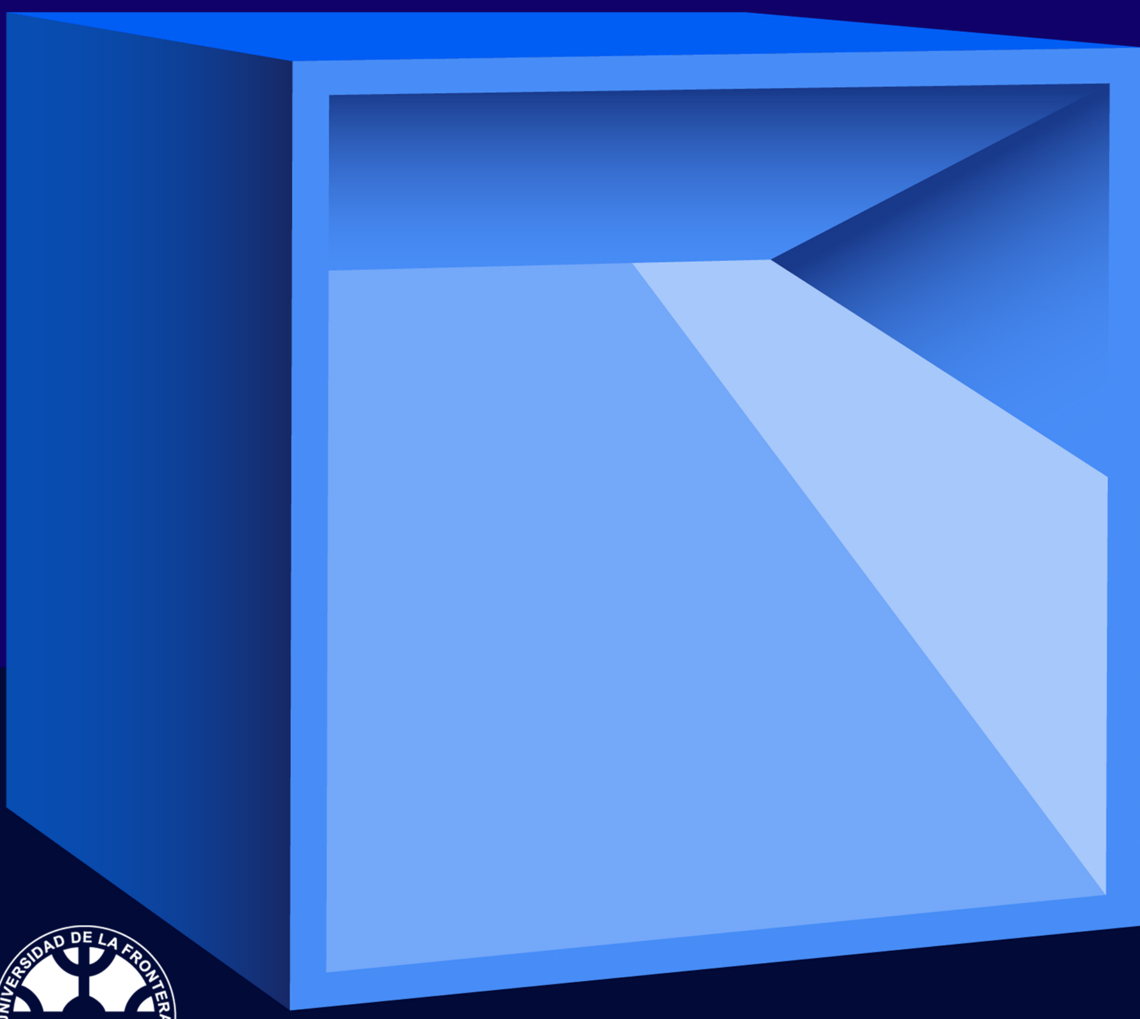

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SUMMARY

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Approximation by Shift Invariant Univariate Sublinear-Shilkret Operators

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ABSTRACT

A very general positive sublinear Shilkret integral type operator is given through a convolution-like iteration of another general positive sublinear operator with a scaling type function. For it sufficient conditions are given for shift invariance, preservation of global smoothness, convergence to the unit with rates. Additionally, two examples of very general specialized operators are presented fulfilling all the above properties, the higher order of approximation of these operators is also considered.

RESUMEN

Un operador muy general positivo sublineal de tipo integral de Shilkret es dado a través de un iteración de tipo convolución de otro operador general positivo sublineal con una función de tipo escalamiento. Para estos operadores, se entregan condiciones suficientes para invariancia por shifts, conservación de la suavidad global y convergencia a la unidad con tasas. Adicionalmente, se presentan dos ejemplos de operadores muy generales especializados que satisfacen todas las propiedades anteriores, también considerando el alto orden de aproximación de estos operadores.

Keywords and Phrases: Jackson type inequality, Shilkret integral, modulus of continuity, shift invariant, global smoothness preservation, quantitative approximation.

2010 AMS Mathematics Subject Classification: 41A17, 41A25, 41A35, 41A36.

1 Introduction

Let X, Y be function spaces of functions from $\mathbb{R}$ into $\mathbb{R}_+$. Let $L_N : X \to Y$, $N \in \mathbb{N}$, be a sequence of operators with the following properties:

(i) (positive homogeneous)

$$L_N(\alpha f) = \alpha L_N(f), \ \ \forall \ \alpha \geq 0, \forall \ f \in X.$$

(ii) (Monotonicity) if $f, g \in X$ satisfy $f \leq g$, then $L_N(f) \leq L_N(g)$, $\forall \ N \in \mathbb{N}$,

and

(iii) (Subadditivity)

$$L_N(f + g) \leq L_N(f) + L_N(g), \ \ \forall \ f, g \in X.$$

We call L_N positive sublinear operators.

In this article we deal with sequences of Shilkret positive sublinear operators that are constructed, with the help of Shilkret integral ([5]). Our functions spaces are continuous functions from $\mathbb{R}$ into $\mathbb{R}_+$. The sequence of operators is generated by a basic operator via dilated translations of convolution type using the Shilkret integral. We prove that our operators possess the following properties: of shift invariance of global smoothness preservation, of convergence to the unit operator with rates. Then we apply our results to two specific families of such Shilkret type operators.

We continue with the higher order of approximation study of these specific operators, and all results are quantitative.

Earlier similar studies have been done by the author, see [3], Chapters 10-17, and [2], Chapters 16, 17. These serve as motivation and inspiration to this work.

2 Background

Here we follow [5].

Let $\mathcal{F}$ be a σ-field of subsets of an arbitrary set Ω. An extended non-negative real valued function μ on $\mathcal{F}$ is called maxitive if $\mu(\varnothing) = 0$ and

$$\mu\left(\cup_{i \in I} E_i\right) = \sup_{i \in I} \mu(E_i), \tag{1}$$

where the set I is of cardinality at most countable, where $\{E_i\}_{i \in I}$ is a disjoint collection of sets from $\mathcal{F}$. We notice that μ is monotone and (1) is true even $\{E_i\}_{i \in I}$ are not disjoint. For more properties of μ see [5]. We also call μ a maxitive measure. Here f stands for a non-negative measurable function on Ω. In [5], Niel Shilkret developed his non-additive integral defined as follows:

$$(N^*)\int_D f d\mu := \sup_{y\in Y}\{y\cdot\mu(D\cap\{f\geq y\})\}, \tag{2}$$

where $Y=[0,m]$ or $Y=[0,m)$ with $0<m\leq\infty$, and $D\in\mathcal{F}$. Here we take $Y=[0,\infty)$.

It is easily proved that

$$(N^*)\int_D f d\mu = \sup_{y>0}\{y\cdot\mu(D\cap\{f>y\})\}. \tag{3}$$

The Shilkret integral takes values in $[0,\infty]$.

The Shilkret integral ([5]) has the following properties:

$$(N^*)\int_\Omega \chi_E d\mu = \mu(E), \tag{4}$$

where χ_E is the indicator function on $E\in\mathcal{F}$,

$$(N^*)\int_D cf d\mu = c\,(N^*)\int_D f d\mu, \quad c\geq 0, \tag{5}$$

$$(N^*)\int_D \sup_{n\in\mathbb{N}} f_n d\mu = \sup_{n\in\mathbb{N}}(N^*)\int_D f_n d\mu, \tag{6}$$

where f_n, $n\in\mathbb{N}$, is an increasing sequence of elementary (countably valued) functions converging uniformly to f. Furthermore we have

$$(N^*)\int_D f d\mu \geq 0, \tag{7}$$

$$f\geq g \text{ implies } (N^*)\int_D f d\mu \geq (N^*)\int_D g d\mu, \tag{8}$$

where $f,g:\Omega\to[0,\infty]$ are measurable.

Let $a\leq f(\omega)\leq b$ for almost every $\omega\in E$, then

$$a\mu(E)\leq(N^*)\int_E f d\mu\leq b\mu(E); \tag{9}$$

$$(N^*)\int_E 1 d\mu = \mu(E); \tag{10}$$

$f>0$ almost everywhere and $(N^*)\int_E f d\mu=0$ imply $\mu(E)=0$;

$(N^*)\int_\Omega f d\mu=0$ if and only $f=0$ almost everywhere;

$(N^*)\int_\Omega f d\mu<\infty$ implies that

$$\overline{N}(f):=\{\omega\in\Omega | f(\omega)\neq 0\} \text{ has } \sigma\text{-finite measure;}$$

$$(N^*)\int_D (f+g)\,d\mu \le (N^*)\int_D f d\mu + (N^*)\int_D g d\mu; \quad (11)$$

and

$$\left|(N^*)\int_D f d\mu - (N^*)\int_D g d\mu\right| \le (N^*)\int_D |f-g|\,d\mu. \quad (12)$$

From now on in this article we assume that $\mu : \mathcal{F} \to [0, +\infty)$.

3 Univariate Theory

This section is motivated and inspired by [3] and [4].

Let $\mathcal{L}$ be the Lebesgue $\sigma-$ algebra on $\mathbb{R}$, and the set function $\mu : \mathcal{L} \to [0, +\infty]$, which is assumed to be maxitive. Let $C_U(\mathbb{R}, \mathbb{R}_+)$ be the space of uniformly continuous functions from $\mathbb{R}$ into $\mathbb{R}_+$, and $C(\mathbb{R}, \mathbb{R}_+)$ the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}_+$. For any $f \in C_U(\mathbb{R}, \mathbb{R}_+)$ we have $\omega_1(f, \delta) < +\infty$, $\delta > 0$, where

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in\mathbb{R}:\\ |x-y|\le\delta}} |f(x) - f(y)|, \quad \delta > 0,$$

is the first modulus of continuity.

Let $\{t_k\}_{k\in\mathbb{Z}}$ be a sequence of positive sublinear operators that map $C_U(\mathbb{R}, \mathbb{R}_+)$ into $C(\mathbb{R}, \mathbb{R}_+)$ with the property

$$(t_k(f))(x) := l_0\left(f\left(2^{-k}\cdot\right)\right)(x), \quad \forall\, x \in \mathbb{R}, \forall\, f \in C_U(\mathbb{R}, \mathbb{R}_+). \quad (13)$$

For a fixed $a > 0$ we assume that

$$\sup_{\substack{u,y\in\mathbb{R}:\\ |u-y|\le a}} |t_0(f,u) - f(y)| \le \omega_1\left(f, \frac{ma+n}{2^r}\right), \ \forall\, f \in C_U(\mathbb{R}, \mathbb{R}_+), \quad (14)$$

where $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $r \in \mathbb{Z}$.

Let $\psi : \mathbb{R} \to \mathbb{R}_+$ which is Lebesgue measurable, such that

$$(N^*)\int_{-a}^{a} \psi(u)\,d\mu(u) = 1. \quad (15)$$

We define the positive sublinear-Shilkret operators

$$(T_0(f))(x) := (N^*)\int_{-a}^{a} (t_0 f)(x-u)\,\psi(u)\,d\mu(u), \quad (16)$$

and

$$(T_k(f))(x) := \left(T_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k x\right), \quad \forall\, k \in \mathbb{Z}, \forall\, x \in \mathbb{R}. \quad (17)$$

Therefore it holds

$$(T_k(f))(x) = (N^*)\int_{-a}^{a} \left(t_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k x - u\right)\psi(u)\,d\mu(u) = \tag{18}$$

$$(N^*)\int_{-a}^{a} (t_k(f))\left(2^k x - u\right)\psi(u)\,d\mu(u),$$

$\forall\, x \in \mathbb{R}$, $\forall\, k \in \mathbb{Z}$.

Indeed here we have

$$(T_k(f))(x) \overset{(8)}{\leq} (N^*)\int_{-a}^{a} \left\|t_k(f)\left(2^k x - \cdot\right)\right\|_{\infty,[-a,a]}\psi(u)\,d\mu(u) \overset{(5)}{=}$$

$$\left\|t_k(f)\left(2^k x - \cdot\right)\right\|_{\infty,[-a,a]}\left((N^*)\int_{-a}^{a}\psi(u)\,d\mu(u)\right) = \tag{19}$$

$$\left\|t_k(f)\left(2^k x - \cdot\right)\right\|_{\infty,[-a,a]} < +\infty.$$

Hence $(T_k(f))(x) \in \mathbb{R}_+$ is well-defined.

Let $f, g \in \mathcal{M}(\mathbb{R}, \mathbb{R}_+)$ (Lebesgue measurable functions) where $X \in A$, $A \subset \mathbb{R}$ is a Lebesgue measurable set.

We derive that

$$\left|(N^*)\int_A f(x)\,d\mu(x) - N^*\int_A g(x)\,d\mu(x)\right| \overset{(12)}{\leq} (N^*)\int_A |f(x) - g(x)|\,d\mu(x). \tag{20}$$

We need

Definition 3.1. *Let* $f_\alpha(\cdot) := f(\cdot + \alpha)$, $\alpha \in \mathbb{R}$, *and* Φ *be an operator. If* $\Phi(f_\alpha) = (\Phi f)_\alpha$, *then* Φ *is called a shift invariant operator.*

We give

Theorem 3.2. *Assume that*

$$\left(t_0\left(f\left(2^{-k}\cdot + \alpha\right)\right)\right)\left(2^k u\right) = \left(t_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k(u + \alpha)\right), \tag{21}$$

for all $k \in \mathbb{Z}$, $\alpha \in \mathbb{R}$ *fixed, all* $u \in \mathbb{R}$ *and any* $f \in C_U(\mathbb{R}, \mathbb{R}_+)$. *Then* T_k *is a shift invariant operator for all* $k \in \mathbb{Z}$.

Proof. We have that

$$(T_k(f(\cdot + \alpha)))(x) = (T_k(f_\alpha))(x) \overset{(18)}{=}$$

$$(N^*)\int_{-a}^{a}\left(t_0\left(f_\alpha\left(2^{-k}\cdot\right)\right)\right)\left(2^k x - u\right)\psi(u)\,d\mu(u) =$$

$$(N^*)\int_{-a}^{a}\left(t_0\left(f\left(2^{-k}\cdot + \alpha\right)\right)\right)\left(2^k x - u\right)\psi(u)\,d\mu(u) =$$

$$(N^*)\int_{-a}^{a}\left(t_0\left(f\left(2^{-k}\cdot+\alpha\right)\right)\right)\left(2^k\left(x-2^{-k}u\right)\right)\psi(u)\,d\mu(u)\overset{(21)}{=}$$
$$(N^*)\int_{-a}^{a}\left(t_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k\left(x-2^{-k}u+\alpha\right)\right)\psi(u)\,d\mu(u)=\tag{22}$$
$$(N^*)\int_{-a}^{a}\left(t_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k(x+\alpha)-u\right)\psi(u)\,d\mu(u)\overset{(18)}{=}\left(T_k(f)\right)(x+\alpha),$$

that is

$$T_k(f_\alpha)=\left(T_k(f)\right)_\alpha,\tag{23}$$

proving the claim. □

It follows the global smoothness of the operators T_k.

Theorem 3.3. *For any* $f\in C_U(\mathbb{R},\mathbb{R}_+)$ *assume that, for all* $u\in\mathbb{R}$,

$$\left|(t_0(f))(x-u)-(t_0(f))(y-u)\right|\le\omega_1(f,|x-y|),\tag{24}$$

for any $x,y\in\mathbb{R}$. *Then*

$$\omega_1(T_kf,\delta)\le\omega_1(f,\delta),\quad\forall\,\delta>0.\tag{25}$$

Proof. We observe that

$$\left|(T_0(f))(x)-(T_0(f))(y)\right|=$$
$$\left|(N^*)\int_{-a}^{a}(t_0f)(x-u)\psi(u)\,d\mu(u)-(N^*)\int_{-a}^{a}(t_0f)(y-u)\psi(u)\,d\mu(u)\right|\overset{(20)}{\le}\tag{26}$$
$$(N^*)\int_{-a}^{a}\left|(t_0f)(x-u)-(t_0f)(y-u)\right|\psi(u)\,d\mu(u)\overset{(\text{by }(24),\,(5))}{\le}$$
$$\omega_1(f,|x-y|)\left((N^*)\int_{-a}^{a}\psi(u)\,d\mu(u)\right)\overset{(15)}{=}\omega_1(f,|x-y|).$$

So that

$$\left|(T_0(f))(x)-(T_0(f))(y)\right|\le\omega_1(f,|x-y|).\tag{27}$$

From (17), (27) we get

$$\left|(T_k(f))(x)-(T_k(f))(y)\right|\overset{(17)}{=}$$
$$\left|\left(T_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^kx\right)-\left(T_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^ky\right)\right|\le\tag{28}$$
$$\omega_1\left(f\left(2^{-k}\cdot\right),2^k|x-y|\right)=\omega_1(f,|x-y|),$$

i.e. global smoothness for T_k has been proved. □

The convergence of T_k to the unit operator, as $k\to+\infty$, k with rates follows:

Theorem 3.4. *For* $f \in C_U(\mathbb{R}, \mathbb{R}_+)$, *under the assumption* (14), *we have*

$$|(T_k(f))(x) - f(x)| \leq \omega_1\left(f, \frac{ma+n}{2^{k+r}}\right), \tag{29}$$

where $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $k, r \in \mathbb{Z}$.

Proof. We notice that

$$|(T_k(f))(x) - f(x)| \overset{(17)}{=} \left|\left(T_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k x\right) - f(x)\right| \overset{(18)}{=}$$

$$\left|(N^*)\int_{-a}^{a}\left(t_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k x - u\right)\psi(u)\,d\mu(u) - f(x)\right| \overset{(15)}{=}$$

$$\left|(N^*)\int_{-a}^{a}\left(t_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k x - u\right)\psi(u)\,d\mu(u) - (N^*)\int_{-a}^{a} f(x)\psi(u)\,d\mu(u)\right| \overset{(20)}{\leq}$$

$$(N^*)\int_{-a}^{a}\left|\left(t_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k x - u\right) - f(x)\right|\psi(u)\,d\mu(u) = \tag{30}$$

$$(N^*)\int_{-a}^{a}\left|\left(t_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k x - u\right) - f\left(2^{-k}\cdot\right)\left(2^k x\right)\right|\psi(u)\,d\mu(u) \overset{(14)}{\leq}$$

(here $\left|\left(2^k x - u\right) - 2^k x\right| = |u| \leq a$)

$$\omega_1\left(f\left(2^{-k}\cdot\right), \frac{ma+n}{2^r}\right)\left((N^*)\int_{-a}^{a}\psi(u)\,d\mu(u)\right) \overset{(15)}{=}$$

$$\omega_1\left(f\left(2^{-k}\cdot\right), \frac{ma+n}{2^r}\right)\cdot 1 = \omega_1\left(f, \frac{ma+n}{2^{k+r}}\right), \tag{31}$$

proving the claim. □

We give some applications.

For each $k \in \mathbb{Z}$, we define

(i)

$$(B_k f)(x) := (N^*)\int_{-a}^{a} f\left(x - \frac{u}{2^k}\right)\psi(u)\,d\mu(u), \tag{32}$$

i.e., here

$$(t_k(f))(u) = f\left(\tfrac{u}{2^k}\right), \quad \text{and} \quad (t_0(f))(u) = f(u), \tag{33}$$

are continuous in $u \in \mathbb{R}$.

Also for $k \in \mathbb{Z}$, we define

(ii)

$$(\Gamma_k(f))(x) := (N^*)\int_{-a}^{a}\gamma_k^f\left(2^k x - u\right)\psi(u)\,d\mu(u), \tag{34}$$

where

$$(t_k(f))(u) = \gamma_k^f(u) := \sum_{j=0}^{n} w_j f\left(\frac{u}{2^k} + \frac{j}{2^k n}\right), \tag{35}$$

$n \in \mathbb{N}$, $w_j \geq 0$, $\sum_{j=0}^{n} w_j = 1$,

is continuous in $u \in \mathbb{R}$.

Notice here that

$$(t_0(f))(u) = \gamma_0^f(u) = \sum_{j=0}^{n} w_j f\left(u + \frac{j}{n}\right) \tag{36}$$

is also continuous in $u \in \mathbb{R}$.

Indeed we have

$$(\Gamma_k(f))(x) = (N^*)\int_{-a}^{a}\left[\sum_{j=0}^{n} w_j f\left(\left(x - \frac{u}{2^k}\right) + \frac{j}{2^k n}\right)\right]\psi(u)\,d\mu(u). \tag{37}$$

Clealry here we have

$$\begin{gathered}(B_k(f))(x) = \left(B_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k x\right), \\ \text{and} \\ (\Gamma_k(f))(x) = \left(\Gamma_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k x\right),\end{gathered} \tag{38}$$

$\forall\, k \in \mathbb{Z}$, $\forall\, x \in \mathbb{R}$.

We give

Proposition 3.5. B_k, Γ_k *are shift invariant operators.*

Proof. (i) For B_k operators: Here $t_0 f = f$. Hence

$$\begin{gathered}\left(t_0\left(f\left(2^{-k}\cdot+\alpha\right)\right)\right)\left(2^k u\right) = f\left(2^{-k}2^k u + \alpha\right) = f(u+\alpha) = \\ \left(t_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k(u+\alpha)\right).\end{gathered} \tag{39}$$

(ii) For Γ_k operators:

$$(t_0(f))(u) = \sum_{j=0}^{n} w_j f\left(u + \frac{j}{n}\right).$$

Hence

$$\begin{gathered}\left(t_0\left(f\left(2^{-k}\cdot+\alpha\right)\right)\right)\left(2^k u\right) = \sum_{j=0}^{n} w_j f\left(2^{-k}\left(2^k u + \frac{j}{n}\right) + \alpha\right) = \\ \sum_{j=0}^{n} w_j f\left(2^{-k}\left(2^k(u+\alpha) + \frac{j}{n}\right)\right) = \left(t_0\left(f\left(2^{-k}\cdot\right)\right)\right)\left(2^k(u+\alpha)\right),\end{gathered} \tag{40}$$

proving the claim. ∎

Next we show that the operators B_k, Γ_k possess the property of global smoothness preservation.

Theorem 3.6. *For all* $f \in C_U(\mathbb{R}, \mathbb{R}_+)$ *and all* $\delta > 0$ *we have*

$$\omega_1(B_k f, \delta) \leq \omega_1(f, \delta), \quad \text{and} \quad \omega_1(\Gamma_k f, \delta) \leq \omega_1(f, \delta). \tag{41}$$

Proof. (i) For B_k operators: Here $t_0 f = f$, therefore

$$|(t_0(f))(x-u) - (t_0(f))(y-u)| = |f(x-u) - f(y-u)| \leq \omega_1(f, |x-y|). \tag{42}$$

(ii) For Γ_k operators: We observe that

$$|(t_0(f))(x-u) - (t_0(f))(y-u)| = \left|\gamma_0^f(x-u) - \gamma_0^f(y-u)\right| =$$

$$\left|\sum_{j=0}^{n} w_j \left(f\left(x-u+\frac{j}{n}\right) - f\left(y-u+\frac{j}{n}\right)\right)\right| \leq$$

$$\sum_{j=0}^{n} w_j \left|f\left(x-u+\frac{j}{n}\right) - f\left(y-u+\frac{j}{n}\right)\right| \leq$$

$$\omega_1(f, |x-y|)\left(\sum_{j=0}^{n} w_j\right) = \omega_1(f, |x-y|), \tag{43}$$

proving the claim. ☐

The operators B_k, Γ_k, $k \in \mathbb{Z}$, converge to the unit operator with rates presented next.

Theorem 3.7. *For* $k \in \mathbb{Z}$,

$$|(B_k(f))(x) - f(x)| \leq \omega_1\left(f, \tfrac{a}{2^k}\right), \quad \text{and} \quad |(\Gamma_k(f))(x) - f(x)| \leq \omega_1\left(f, \tfrac{a+1}{2^k}\right). \tag{44}$$

Proof. (i) For B_k operators: Here $(t_0(f))(u) = f(u)$ and

$$\sup_{\substack{u,y \in \mathbb{R} \\ |u-y| \leq a}} |(t_0(f))(u) - f(y)| = \sup_{\substack{u,y \in \mathbb{R} \\ |u-y| \leq a}} |f(u) - f(y)| = \omega_1(f, a), \tag{45}$$

and we use Theorem 3.4.

(ii) For Γ_k operators: Here we see that

$$\sup_{\substack{u,y \in \mathbb{R} \\ |u-y| \leq a}} |(t_0(f))(u) - f(y)| = \sup_{\substack{u,y \in \mathbb{R} \\ |u-y| \leq a}} \left|\sum_{j=0}^{n} w_j f\left(u+\frac{j}{n}\right) - f(y)\right| \leq$$

$$\sup_{\substack{u,y\in\mathbb{R}\\|u-y|\le a}}\sum_{j=0}^{n}w_j\left|f\left(u+\frac{j}{n}\right)-f(y)\right|\le\sup_{\substack{u,y\in\mathbb{R}\\|u-y|\le a}}\sum_{j=0}^{n}w_j\omega_1\left(f,\left|u+\frac{j}{n}-y\right|\right)\le \tag{46}$$

$$\sup_{\substack{u,y\in\mathbb{R}\\|u-y|\le a}}\sum_{j=0}^{n}w_j\omega_1\left(f,\frac{j}{n}+|u-y|\right)\le\left(\sum_{j=0}^{n}w_j\right)\omega_1(f,1+\alpha)=\omega_1(f,\alpha+1).$$

By (29) we are done. □

4 Higher order of Approximation

Here all are as in Section 3. See also earlier our work [1], and [2], Chapter 16.

We give

Theorem 4.1. *Let* $f\in C^N(\mathbb{R},\mathbb{R}_+)$, $N\ge 1$. *Consider the Shilkret-sublinear operators*

$$(B_kf)(x)=(N^*)\int_{-a}^{a}f\left(x-\frac{u}{2^k}\right)\psi(u)\,d\mu(u),$$

$\forall\, k\in\mathbb{Z}$, $\forall\, x\in\mathbb{R}$. *Then*

$$|(B_kf)(x)-f(x)|\le\sum_{i=1}^{N}\frac{\left|f^{(i)}(x)\right|}{i!}\frac{a^i}{2^{ki}}+\frac{a^N}{2^{kN}N!}\omega_1\left(f^{(N)},\frac{a}{2^k}\right). \tag{47}$$

If $f^{(N)}$ *is uniformly continuous or bounded and continuous, then as* $k\to+\infty$ *we obtain that* $(B_kf)(x)\to f(x)$ *pointwise with rates.*

Proof. Since $f\in C^N(\mathbb{R},\mathbb{R}_+)$, $N\ge1$, by Taylor's formula we have

$$f\left(x-\frac{u}{2^k}\right)-f(x)=\sum_{i=1}^{N}\frac{f^{(i)}(x)}{i!}\left(-\frac{u}{2^k}\right)^i+ \tag{48}$$

$$\int_x^{x-\frac{u}{2^k}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(x-\frac{u}{2^k}-t\right)^{N-1}}{(N-1)!}dt.$$

Call

$$\Gamma_u(x):=\left|\int_x^{x-\frac{u}{2^k}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(x-\frac{u}{2^k}-t\right)^{N-1}}{(N-1)!}dt\right|. \tag{49}$$

Next we estimate $\Gamma_u(x)$, where $u\in[-a,a]$.

i) Case of $-a\le u\le 0$, then $x\le x-\frac{u}{2^k}$. Then

$$\Gamma_u(x)\le\int_x^{x-\frac{u}{2^k}}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(x-\frac{u}{2^k}-t\right)^{N-1}}{(N-1)!}dt\le$$

$$\int_{x}^{x-\frac{u}{2^k}} \omega_1\left(f^{(N)}, |t-x|\right) \frac{\left(x-\frac{u}{2^k}-t\right)^{N-1}}{(N-1)!} dt \le$$

$$\omega_1\left(f^{(N)}, \frac{|u|}{2^k}\right) \int_{x}^{x-\frac{u}{2^k}} \frac{\left(x-\frac{u}{2^k}-t\right)^{N-1}}{(N-1)!} dt \le \tag{50}$$

$$\omega_1\left(f^{(N)}, \frac{a}{2^k}\right) \frac{\left(-\frac{u}{2^k}\right)^N}{N!} \le \omega_1\left(f^{(N)}, \frac{a}{2^k}\right) \frac{a^N}{2^{kN} N!}.$$

That is, when $-a \le u \le 0$, then

$$\Gamma_u(x) \le \omega_1\left(f^{(N)}, \frac{a}{2^k}\right) \frac{a^N}{2^{kN} N!}. \tag{51}$$

ii) Case of $0 \le u \le a$, then $x \ge x - \frac{u}{2^k}$. Then

$$\Gamma_u(x) = \left| \int_{x-\frac{u}{2^k}}^{x} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(t-x+\frac{u}{2^k}\right)^{N-1}}{(N-1)!} dt \right| \le$$

$$\int_{x-\frac{u}{2^k}}^{x} \left|f^{(N)}(t) - f^{(N)}(x)\right| \frac{\left(t-x+\frac{u}{2^k}\right)^{N-1}}{(N-1)!} dt \le \tag{52}$$

$$\int_{x-\frac{u}{2^k}}^{x} \omega_1\left(f^{(N)}, |t-x|\right) \frac{\left(t-x+\frac{u}{2^k}\right)^{N-1}}{(N-1)!} dt \le$$

$$\omega_1\left(f^{(N)}, \frac{|u|}{2^k}\right) \int_{x-\frac{u}{2^k}}^{x} \frac{\left(t-x+\frac{u}{2^k}\right)^{N-1}}{(N-1)!} dt \le$$

$$\omega_1\left(f^{(N)}, \frac{a}{2^k}\right) \frac{\left(\frac{u}{2^k}\right)^N}{N!} \le \omega_1\left(f^{(N)}, \frac{a}{2^k}\right) \frac{a^N}{2^{kN} N!}. \tag{53}$$

That is, when $0 \le u \le a$, then

$$\Gamma_u(x) \le \omega_1\left(f^{(N)}, \frac{a}{2^k}\right) \frac{a^N}{2^{kN} N!}. \tag{54}$$

We proved that

$$\Gamma_u(x) \le \omega_1\left(f^{(N)}, \frac{a}{2^k}\right) \frac{a^N}{2^{kN} N!} := \rho \ge 0, \tag{55}$$

$\forall\, k \in \mathbb{Z}$, $\forall\, x \in \mathbb{R}$, $|u| \le a$.

By (48) we get that ($|u| \le a$)

$$\left| f\left(x - \frac{u}{2^k}\right) - f(x) \right| \le \sum_{i=1}^{N} \frac{\left|f^{(i)}(x)\right|}{i!} \frac{a^i}{2^{ki}} + \rho. \tag{56}$$

We observe that

$$|(B_k f)(x) - f(x)| =$$

$$\left|(N^*)\int_{-a}^{a} f\left(x-\frac{u}{2^k}\right)\psi(u)\,d\mu(u)-(N^*)\int_{-a}^{a} f(x)\psi(u)\,d\mu(u)\right| \overset{(20)}{\leq} \tag{57}$$

$$(N^*)\int_{-a}^{a}\left|f\left(x-\frac{u}{2^k}\right)-f(x)\right|\psi(u)\,d\mu(u)\leq$$

$$\left(\sum_{i=1}^{N}\frac{\left|f^{(i)}(x)\right|}{i!}\frac{a^i}{2^{ki}}+\rho\right)\left((N^*)\int_{-a}^{a}\psi(u)\,d\mu(u)\right)\overset{(15)}{=}$$

$$\left(\sum_{i=1}^{N}\frac{\left|f^{(i)}(x)\right|}{i!}\frac{a^i}{2^{ki}}+\rho\right)\cdot 1= \tag{58}$$

$$\sum_{i=1}^{N}\frac{\left|f^{(i)}(x)\right|}{i!}\frac{a^i}{2^{ki}}+\frac{a^N}{2^{kN}N!}\omega_1\left(f^{(N)},\frac{a}{2^k}\right),$$

proving the claim. □

Corollary 4.2. *Let* $f\in C^1(\mathbb{R},\mathbb{R}_+)$. *Then*

$$|(B_k f)(x)-f(x)|\leq\frac{a}{2^k}\left(|f'(x)|+\omega_1\left(f',\frac{a}{2^k}\right)\right), \tag{59}$$

$\forall\, k\in\mathbb{Z}$, $\forall\, x\in\mathbb{R}$.

Proof. By (47) for $N=1$. □

We also present

Theorem 4.3. *Let* $f\in C^N(\mathbb{R},\mathbb{R}_+)$, $N\geq 1$. *Consider the Shilkret-sublinear operators*

$$(\Gamma_k(f))(x)=(N^*)\int_{-a}^{a}\left[\sum_{j=0}^{n}w_j f\left(\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}\right)\right]\psi(u)\,d\mu(u), \tag{60}$$

$\forall\, k\in\mathbb{Z}$, $\forall\, x\in\mathbb{R}$. *Then*

$$|(\Gamma_k f)(x)-f(x)|\leq\sum_{i=1}^{N}\frac{\left|f^{(i)}(x)\right|}{i!}\frac{(a+1)^i}{2^{ki}}+\frac{(a+1)^N}{N!2^{kN}}\omega_1\left(f^{(N)},\frac{a+1}{2^k}\right). \tag{61}$$

If $f^{(N)}$ *is uniformly continuous or bounded and continuous, then as* $k\to+\infty$ *we obtain that* $(\Gamma_k f)(x)\to f(x)$, *pointwise with rates.*

Corollary 4.4. *Let* $f\in C^1(\mathbb{R},\mathbb{R}_+)$. *Then*

$$|(\Gamma_k f)(x)-f(x)|\leq\frac{(a+1)}{2^k}\left[|f'(x)|+\omega_1\left(f',\frac{a+1}{2^k}\right)\right], \tag{62}$$

$\forall\, k\in\mathbb{Z}$, $\forall\, x\in\mathbb{R}$.

Proof. By (61) for $N=1$. □

Proof. of Theorem 4.3.

Since $f \in C^N(\mathbb{R})$, $N \geq 1$, by Taylor's formula we get

$$\sum_{j=0}^{n} w_j f\left(\left(x - \frac{u}{2^k}\right) + \frac{j}{2^k n}\right) - f(x) =$$

$$\sum_{i=1}^{N} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{n} w_j \left(-\frac{u}{2^k} + \frac{j}{2^k n}\right)^i + \tag{63}$$

$$\sum_{j=0}^{n} w_j \int_x^{\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\left(x - \frac{u}{2^k}\right) + \frac{j}{2^k n} - t\right)^{N-1}}{(N-1)!} dt.$$

Call

$$\varepsilon(x, u, j) := \int_x^{\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\left(x - \frac{u}{2^k}\right) + \frac{j}{2^k n} - t\right)^{N-1}}{(N-1)!} dt. \tag{64}$$

We estimate $\varepsilon(x, u, j)$. Here $|u| \leq a$.

i) case of $u \leq \frac{j}{n}$, iff $\frac{u}{2^k} \leq \frac{j}{2^k n}$, iff $x \leq x - \frac{u}{2^k} + \frac{j}{2^k n}$.

Hence

$$|\varepsilon(x, u, j)| \leq \int_x^{\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}} \left|f^{(N)}(t) - f^{(N)}(x)\right| \frac{\left(\left(x - \frac{u}{2^k}\right) + \frac{j}{2^k n} - t\right)^{N-1}}{(N-1)!} dt \leq \tag{65}$$

$$\int_x^{\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}} \omega_1\left(f^{(N)}, |t - x|\right) \frac{\left(\left(x - \frac{u}{2^k}\right) + \frac{j}{2^k n} - t\right)^{N-1}}{(N-1)!} dt \leq$$

$$\omega_1\left(f^{(N)}, \left[\frac{j}{2^k n} - \frac{u}{2^k}\right]\right) \int_x^{\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}} \frac{\left(\left(x - \frac{u}{2^k}\right) + \frac{j}{2^k n} - t\right)^{N-1}}{(N-1)!} dt \leq$$

$$\omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \frac{\left(\frac{j}{2^k n} - \frac{u}{2^k}\right)^N}{N!} \leq \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \frac{(a+1)^N}{2^{kN} N!}. \tag{66}$$

For $u \leq \frac{j}{n}$, we hve proved that

$$|\varepsilon(x, u, j)| \leq \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right) \frac{(a+1)^N}{2^{kN} N!}. \tag{67}$$

ii) case of $u \geq \frac{j}{n}$, iff $\frac{u}{2^k} \geq \frac{j}{2^k n}$, iff $x \geq x - \frac{u}{2^k} + \frac{j}{2^k n}$.

We observe that

$$|\varepsilon(x, u, j)| =$$

$$\left|\int_{\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}}^{x}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(t-\left[\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}\right]\right)^{N-1}}{(N-1)!}dt\right|\leq \tag{68}$$

$$\int_{\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}}^{x}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(t-\left[\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}\right]\right)^{N-1}}{(N-1)!}dt\leq$$

$$\int_{\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}}^{x}\omega_1\left(f^{(N)},|t-x|\right)\frac{\left(t-\left[\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}\right]\right)^{N-1}}{(N-1)!}dt\leq$$

$$\omega_1\left(f^{(N)},\frac{u}{2^k}-\frac{j}{2^k n}\right)\int_{\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}}^{x}\frac{\left(t-\left[\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}\right]\right)^{N-1}}{(N-1)!}dt\leq$$

$$\omega_1\left(f^{(N)},\frac{a+1}{2^k}\right)\frac{\left(\frac{u}{2^k}-\frac{j}{2^k n}\right)^{N}}{N!}\leq\omega_1\left(f^{(N)},\frac{a+1}{2^k}\right)\frac{(a+1)^N}{2^{kN}N!}. \tag{69}$$

So when $u\geq\frac{j}{n}$, we proved that

$$|\varepsilon(x,u,j)|\leq\omega_1\left(f^{(N)},\frac{a+1}{2^k}\right)\frac{(a+1)^N}{2^{kN}N!}. \tag{70}$$

Therefore it always holds

$$|\varepsilon(x,u,j)|\leq\omega_1\left(f^{(N)},\frac{a+1}{2^k}\right)\frac{(a+1)^N}{2^{kN}N!}. \tag{71}$$

Consequently we derive

$$\sum_{j=0}^{n}w_j|\varepsilon(x,u,j)|\leq\omega_1\left(f^{(N)},\frac{a+1}{2^k}\right)\frac{(a+1)^N}{2^{kN}N!}:=\overline{\psi}. \tag{72}$$

By (63) we find

$$\left|\sum_{j=0}^{n}w_j f\left(\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}\right)-f(x)\right|\leq\sum_{i=1}^{N}\frac{\left|f^{(i)}(x)\right|}{i!}\frac{(a+1)^i}{2^{ki}}+\overline{\psi}. \tag{73}$$

Therefore we get

$$|(\Gamma_k(f))(x)-f(x)|=$$

$$\left|(N^*)\int_{-a}^{a}\left[\sum_{j=0}^{n}w_j f\left(\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}\right)\right]\psi(u)\,d\mu(u)-(N^*)\int_{-a}^{a}f(x)\psi(u)\,d\mu(u)\right|\overset{(20)}{\leq} \tag{74}$$

$$(N^*)\int_{-a}^{a}\left|\sum_{j=0}^{n}w_j f\left(\left(x-\frac{u}{2^k}\right)+\frac{j}{2^k n}\right)-f(x)\right|\psi(u)\,d\mu(u)\overset{(73)}{\leq}$$

$$\left[\sum_{i=1}^{N} \frac{\left|f^{(i)}(x)\right|}{i!} \frac{(a+1)^i}{2^{ki}} + \overline{\psi}\right] (N^*) \int_{-a}^{a} \psi(u) \, d\mu(u) \overset{(15)}{=}$$

$$\left[\sum_{i=1}^{N} \frac{\left|f^{(i)}(x)\right|}{i!} \frac{(a+1)^i}{2^{ki}} + \overline{\psi}\right] \cdot 1 =$$

$$\sum_{i=1}^{N} \frac{\left|f^{(i)}(x)\right|}{i!} \frac{(a+1)^i}{2^{ki}} + \frac{(a+1)^N}{2^{kN} N!} \omega_1 \left(f^{(N)}, \frac{a+1}{2^k}\right), \tag{75}$$

proving the claim. □

We finish with

Corollary 4.5. *Let* $f \in C^N(\mathbb{R}, \mathbb{R}_+)$, $N \geq 1$, $f^{(i)}(x) = 0$, $i = 1, ..., N$. *Then*

i)

$$|(B_k(f))(x) - f(x)| \leq \frac{a^N}{2^{kN} N!} \omega_1 \left(f^{(N)}, \frac{a}{2^k}\right), \tag{76}$$

and

ii)

$$|(\Gamma_k(f))(x) - f(x)| \leq \frac{(a+1)^N}{N! 2^{kN}} \omega_1 \left(f^{(N)}, \frac{a+1}{2^k}\right), \tag{77}$$

$\forall\, k \in \mathbb{Z}$, $\forall\, x \in \mathbb{R}$.

Proof. By (47) and (61). □

Corollary 4.6. *Let* $f \in C^1(\mathbb{R}, \mathbb{R}_+)$, $f'(x) = 0$. *Then*

i)

$$|(B_k(f))(x) - f(x)| \leq \frac{a}{2^k} \omega_1 \left(f', \frac{a}{2^k}\right), \tag{78}$$

and

ii)

$$|(\Gamma_k(f))(x) - f(x)| \leq \left(\frac{a+1}{2^k}\right) \omega_1 \left(f', \frac{a+1}{2^k}\right), \tag{79}$$

$\forall\, k \in \mathbb{Z}$, $\forall\, x \in \mathbb{R}$.

Proof. By (59) and (62). □

In inequalities (76)-(79) observe the high speed of convergence and approximation.

5 Appendix

Let $f \in C_U(\mathbb{R}, \mathbb{R}_+)$, and the positive sublinear Shilkret operator

$$(M(f))(x) := (N^*)\int_{-a}^{a} f(x+u)\,\psi(u)\,d\mu(u), \ \ \forall\, x \in \mathbb{R}. \tag{80}$$

We observe the following (for any $x, y \in \mathbb{R}$):

$$|(M(f))(x) - (M(f))(y)| =$$

$$\left| (N^*)\int_{-a}^{a} f(x+u)\,\psi(u)\,d\mu(u) - (N^*)\int_{-a}^{a} f(y+u)\,\psi(u)\,d\mu(u) \right| \overset{(20)}{\leq}$$

$$(N^*)\int_{-a}^{a} |f(x+u) - f(y+u)|\,\psi(u)\,d\mu(u) \leq$$

$$\omega_1(f, |x-y|)\left((N^*)\int_{-a}^{a} \psi(u)\,d\mu(u) \right) \overset{(15)}{=} \omega_1(f, |x-y|)\cdot 1 = \omega_1(f, |x-y|). \tag{81}$$

Therefore it holds the global smoothness preservation property:

$$\omega_1(M(f), \delta) \leq \omega_1(f, \delta), \ \ \forall\, \delta > 0. \tag{82}$$

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W_2-Curvature Tensor on Generalized Sasakian Space Forms

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ABSTRACT

In this paper, we study W_2-pseudosymmetric, W_2-locally symmetric, W_2-locally ϕ-symmetric and W_2-ϕ-recurrent generalized Sasakian space form. Further, illustrative examples are given.

RESUMEN

En este artículo, estudiamos formas espaciales Sasakianas generalizadas W_2-seudosimétricas, W_2-localmente ϕ-simétricas y W_2-ϕ-recurrentes. Ejemplos ilustrativos son dados.

Keywords and Phrases: Generalized Sasakian space form, W_2-curvature tensor, pseudosymmetric, ϕ-recurrent, Einstein manifold.

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1 Introduction

The nature of a Riemannian manifold depends on the curvature tensor R of the manifold. It is well known that the sectional curvatures of a manifold determine its curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as a real space form and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$

Representation for these spaces are hyperbolic spaces ($c < 0$), spheres ($c > 0$) and Euclidean spaces ($c = 0$).

The ϕ-sectional curvature of a Sasakian space form is defined by Sasakian manifold and it has a specific form of its curvature tensor. Same notion also holds for Kenmotsu and cosymplectic space forms. In order to generalize such space forms in a common frame Alegre, Blair and Carriazo [1] introduced and studied generalized Sasakian space forms.

A generalized Sasakian space form is an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, whose curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y \\ &- g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1.1}$$

The Riemanian curvature tensor of a generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ is simply given by

$$R = f_1R_1 + f_2R_2 + f_3R_3,$$

where f_1, f_2, f_3 are differential functions on $M^{2n+1}(f_1, f_2, f_3)$ and

$$\begin{aligned} R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y, \\ R_2(X,Y)Z &= g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \quad and \\ R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \end{aligned}$$

where $f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}$. Here c denotes the constant ϕ-sectional curvature. The properties of generalized Sasakian space form was studied by many geometers such has [2, 9, 10, 14, 17, 18, 19, 21, 26]. The concept of local symmetry of a Riemanian manifold has been studied by many authors in several ways to a different extent. The locally ϕ-symmetry of Sasakian manifold was introduce by Takahashi in [28]. De and et al generalize this to the notion of ϕ-symmetry and then introduced the notion of ϕ-recurrent Sasakian manifold in [11]. Further ϕ-recurrent condition was studied on Kenmotsu manifold [8], LP-Sasakian manifold [29] and $(LCS)_n$-manifold [20].

In[16], Pokhariyal and Mishra have defined the W_2-curvature tensor, given by

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{2n}\{g(X,Z)QY - g(Y,Z)QX\}, \tag{1.2}$$

here R and Q are the Riemanian curvature tensor and Ricci operator of Riemanian manifold respectively.

In a generalized Sasakian space forms, the W_2-curvature tensor satisfies the condition

$$\eta(W_2(X,Y)Z) = 0. \tag{1.3}$$

Many Geometers studied the W_2 curvature tensor studied on different manifolds such has generalized Sasakian space forms [13], Lorentzian para Sasakian manifolds [30] and Kenmotsu manifolds [25]

Motivated by these ideas, we made an attempt to study the properties of generalized Sasakian space form. The present paper is organized as follows: In section 2, we review some preliminary results. In section 3, we study W_2-pseudosymmetric generalized Sasakian space form. Section 4, deals with the W_2-locally symmetric generalized Sasakian space forms and it is shown that a generalized Sasakian space form of dimension greater than three is W_2-locally symmetric if and only if it is conformally flat. Section 5, is devoted to the study of W_2-locally ϕ-symmetric generalized Sasakian space forms. Finally in last section, we discus the W_2-ϕ-recurrent generalized Sasakian space form and found to be Einstein manifold.

2 Generalized Sasakian space-forms

The Riemannian manifold M^{2n+1} is called an almost contact metric manifold if the following result holds [5, 6]:

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$
$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(X,\xi) = \eta(X), \tag{2.2}$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$
$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0 \tag{2.4}$$
$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y), \qquad \forall\ X, Y \in (T_p M). \tag{2.5}$$

A almost contact metric manifold is said to be Sasakian if and only if [5, 23]

$$\begin{aligned} (\nabla_X \phi)Y &= g(X,Y)\xi - \eta(Y)X, && (2.6)\\ \nabla_X \xi &= -\phi X. && (2.7) \end{aligned}$$

Again we know that [1] in $(2n+1)$-dimensional generalized Sasakian space form:

$$
\begin{aligned}
S(X,Y) &= (2nf_1+3f_2-f_3)g(X,Y) \\
&- (3f_2+(2n-1)f_3)\eta(X)\eta(Y), && (2.8)\\
S(\phi X,\phi Y) &= S(X,Y)+2n(f_1-f_3)\eta(X)\eta(Y), && (2.9)\\
QX &= (2nf_1+3f_2-f_3)X \\
&- (3f_2+(2n-1)f_3)\eta(X)\xi, && (2.10)\\
r &= 2n(2n+1)f_1+6nf_2-4nf_3, && (2.11)\\
R(X,Y)\xi &= (f_1-f_3)\{\eta(Y)X-\eta(X)Y\}, && (2.12)\\
R(\xi,X)Y &= (f_1-f_3)\{g(X,Y)\xi-\eta(Y)X\}, && (2.13)\\
\eta(R(X,Y)Z) &= (f_1-f_3)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}, && (2.14)\\
S(X,\xi) &= 2n(f_1-f_3)\eta(X). && (2.15)
\end{aligned}
$$

Here R, S, Q and r are the Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature tensor of generalized Sasakian space forms in that order.

3 W_2-pseudosymmetric generalized Sasakian space forms

The concept of a pseudosymmetric manifold was introduced by Chaki [7] and Deszcz [12]. In this article we shall study properties of pseudosymmetric manifold according to Deszcz. Semisymmetric manifolds satisfies the condition $R\cdot R=0$ and were categorized by Szabo in [27]. Every pseudosymmetric manifold is semisymmetric but semisymmetric manifold need not be pseudosymmetric.

An $(2n+1)$-dimensional Riemannian manifold M^{2n+1} is said to be pseudosymmetric, if

$$(R(X,Y)\cdot R)(U,V)W = L_R\{((X\wedge Y)\cdot R)(U,V)W)\}. \tag{3.1}$$

where L_R is some smooth function on $U_R=\{x\in M^{2n+1}|R-\frac{r}{n(n-1)}G\neq 0 \text{ at } x\}$, where G is the $(0,4)$-tensor defined by $G(X_1,X_2,X_3,X_4)=g((X_1\wedge X_2)X_3,X_4)$ and $(X\wedge Y)Z$ is the endomorphism and it is defined as,

$$(X\wedge Y)Z = g(Y,Z)X-g(X,Z)Y \tag{3.2}$$

An $(2n+1)$-dimensional generalized Sasakian space form M^{2n+1} is said to be W_2-pseudosymmetric, if

$$(R(X,Y)\cdot W_2)(U,V)Z = L_{W_2}\{(X\wedge Y)\cdot W_2)(U,V)Z\}, \tag{3.3}$$

holds on the set $U_{W_2}=\{x\in M^{2n+1}|W_2\neq 0 \text{ at } x\}$, where L_{W_2} is some function on U_{W_2}.
Suppose that generalized Sasakian space form is W_2-pseudosymmetric.
Now the left- hand side of (3.3) is

$$
\begin{aligned}
&R(\xi,Y)W_2(U,V)Z-W_2(R(\xi,Y)U,V)Z \\
&- W_2(U,R(\xi,Y)V)Z-W_2(U,V)R(\xi,Y)Z=0. && (3.4)
\end{aligned}
$$

In the view of (2.12) the above expression becomes

$$\begin{aligned} & (f_1 - f_3)\{g(Y, W_2(U, V)Z)\xi - \eta(W_2(U, V)Z)Y \\ - & g(Y, U)W_2(\xi, V)Z + \eta(U)W_2(Y, V)Z \\ - & g(Y, V)W_2(U, \xi)Z + \eta(V)W_2(U, Y)Z \\ - & g(Y, Z)W_2(U, V)\xi + \eta(Z)W_2(U, V)Y\} = 0. \end{aligned} \tag{3.5}$$

Next the right hand side of (3.3) is

$$\begin{aligned} & L_{W_2}\{(\xi \wedge Y)W_2(U, V)Z - W_2((\xi \wedge Y)U, V)Z \\ - & W_2(U, (\xi \wedge Y)V)Z - W_2(U, V)(\xi \wedge Y)Z\} = 0. \end{aligned} \tag{3.6}$$

By virtue of (3.2), (3.6) becomes

$$\begin{aligned} & L_{W_2}\{g(Y, W_2(U, V)Z)\xi - \eta(W_2(U, V)Z)Y \\ - & g(Y, U)W_2(\xi, V)Z + \eta(U)W_2(Y, V)Z \\ - & g(Y, V)W_2(U, \xi)Z + \eta(V)W_2(U, Y)Z \\ - & g(Y, Z)W_2(U, V)\xi + \eta(Z)W_2(U, V)Y\} = 0. \end{aligned} \tag{3.7}$$

Using the expressions (3.5) and (3.7) in (3.3) and taking inner product with ξ, we obtain

$$\begin{aligned} & \{L_{W_2} - (f_1 - f_3)\}\{W_2(U, V, Z, Y) - \eta(W_2(U, V)Z)\eta(Y) \\ - & g(Y, U)\eta(W_2(\xi, V)Z) + \eta(U)\eta(W_2(Y, V)Z) \\ - & g(Y, V)\eta(W_2(U, \xi)Z) + \eta(V)\eta(W_2(U, V)Z) \\ - & g(Y, Z)\eta(W_2(U, V)\xi) + \eta(Z)\eta(W_2(U, V)Z)\} = 0, \end{aligned} \tag{3.8}$$

where $W_2(U, V, Z, Y) = g(Y, W_2(U, V)Z)$ and using(1.3) we get either

$$L_{W_2} = (f_1 - f_3) \text{ or } W_2(U, V, Z, Y) = 0. \tag{3.9}$$

Thus we have following:

Theorem 3.1. *If $M^{2n+1}(f_1, f_2, f_3)$ is W_2-pseudosymmetric generalized Sasakian space form, then $M^{2n+1}(f_1, f_2, f_3)$ is either W_2-flat, or $L_{W_2} = (f_1 - f_3)$ if $(f_1 \neq f_3)$.*

Also in a generalized Sasakian space form, Singh and Pandey [24] proved the following,

Theorem 3.2. *A (2n+1)-dimensional $(n > 1)$ generalized Sasakian space form satisfying $W_2 = 0$ is an η-Einstein manifolds.*

In view of theorem (3.1) and theorem (3.2) we can state the following corollary.

Corolary 1. *If $M^{2n+1}(f_1, f_2, f_3)$ is a W_2-pseudosymmetric generalized Sasakian space forms then M^{2n+1} is either η-Einstein manifold or $L_{W_2} = (f_1 - f_3)$ if $(f_1 \neq f_3)$.*

4 W_2-locally symmetric generalized Sasakian space forms

Definition 1. *A* $(2n+1)$ *dimensional* $(n > 1)$ *generalized Sasakian space form is called projectively locally symmetric if it satisfies [18].*

$$(\nabla_W P)(X, Y)Z = 0.$$

for all vector fields X, Y, Z *orthogonal to* ξ *and an arbitrary vector field* W.

Analogous to this definition, we define a $(2n + 1)$ dimensional $(n > 1)$ W_2-locally symmetric generalized Sasakian space form if

$$(\nabla_W W_2)(X, Y)Z = 0,$$

for all vector fields X, Y, Z orthogonal to ξ and an arbitrary vector field W.
From (1.1) and (1.2), we have

$$\begin{aligned} W_2(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\} + \frac{1}{2n}\{g(X,Z)QY - g(Y,Z)QX\}. \end{aligned} \tag{4.1}$$

Taking covariant differentiation of (4.1) with respect to an arbitrary vector field W, we get

$$\begin{aligned} (\nabla_W W_2)(X,Y)Z &= df_1(W)\{g(Y,Z)X - g(X,Z)Y\} \\ &+ df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + f_2\{g(X,\phi Z)(\nabla_W \phi)Y \\ &+ g(X,(\nabla_W \phi)Z)\phi Y - g(Y,\phi Z)(\nabla_W \phi)X \\ &- g(Y,(\nabla_W \phi)Z)\phi X + 2g(X,\phi Y)(\nabla_W \phi)Z \\ &+ 2g(X,(\nabla_W \phi)Y)\phi Z\} + df_3(W)\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\ &+ f_3\{(\nabla_W \eta)(X)\eta(Z)Y + \eta(X)(\nabla_W \eta)(Z)Y \\ &- (\nabla_W \eta)(Y)\eta(Z)X - \eta(Y)(\nabla_W \eta)\eta(Z)X \\ &+ g(X,Z)(\nabla_W \eta)(Y)\xi + g(X,Z)\eta(Y)\nabla_W \xi \\ &- g(Y,Z)(\nabla_W \eta)(X)\xi - g(Y,Z)\eta(X)\nabla_W \xi\} \\ &+ \frac{1}{2n}\{g(X,Z)(\nabla_W Q)(Y) - g(Y,Z)(\nabla_W Q)(X)\}. \end{aligned} \tag{4.2}$$

where ∇ denotes the Riemannian connection on the manifold.
Differentiating (2.10) covariantly with respect to a W, one can get

$$\begin{aligned} (\nabla_W Q)(Y) &= d(2nf_1 + 3f_2 - f_3)(W)Y - d(3f_2 + (2n-1)f_3)(W)\eta(Y)\xi \\ &- (3f_2 + (2n-1)f_3)[(\nabla_W \eta)(Y)\xi + \eta(Y)(\nabla_W \xi)]. \end{aligned} \tag{4.3}$$

In view of (4.3) and (4.2), it follows that

$$\begin{aligned}
(\nabla_W W_2)(X,Y)Z &= df_1(W)\{g(Y,Z)X - g(X,Z)Y\} \\
&+ df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\
&+ 2g(X,\phi Y)\phi Z\} + f_2\{g(X,\phi Z)(\nabla_W \phi)Y \\
&+ g(X,(\nabla_W\phi)Z)\phi Y - g(Y,\phi Z)(\nabla_W\phi)X \\
&- g(Y,(\nabla_W\phi)Z)\phi X + 2g(X,\phi Y)(\nabla_W\phi)Z \\
&+ 2g(X,(\nabla_W\phi)Y)\phi Z\} + df_3(W)\{\eta(X)\eta(Z)Y \\
&- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\
&+ f_3\{(\nabla_W\eta)(X)\eta(Z)Y + \eta(X)(\nabla_W\eta)(Z)Y \\
&- (\nabla_W\eta)(Y)\eta(Z)X - \eta(Y)(\nabla_W\eta)\eta(Z)X \\
&+ g(X,Z)(\nabla_W\eta)(Y)\xi + g(X,Z)\eta(Y)\nabla_W\xi \\
&- g(Y,Z)(\nabla_W\eta)(X)\xi - g(Y,Z)\eta(X)\nabla_W\xi\} \\
&+ \frac{1}{2n}[g(X,Z)\{d(2nf_1 + 3f_2 - f_3)(W)Y - d(3f_2 \\
&+ (2n-1)f_3)(W)\eta(Y)\xi - (3f_2 + (2n-1)f_3)[(\nabla_W\eta)(Y)\xi \\
&+ \eta(Y)(\nabla_W\xi)]\} - g(Y,Z)\{d(2nf_1 + 3f_2 - f_3)(W)X \\
&- d(3f_2 + (2n-1)f_3)(W)\eta(X)\xi \\
&- (3f_2 + (2n-1)f_3)[(\nabla_W\eta)(X)\xi + \eta(X)(\nabla_W\xi)]\}]. \qquad (4.4)
\end{aligned}$$

Taking X, Y, Z orthogonal to ξ in (4.4) and then taking the inner product of the resultant equation with V, followed by setting $V = Z = e_i$ in the above equation, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i, i = 1, 2, \ldots\ldots, 2n+1$, we get

$$\begin{aligned}
& f_2\{-g(\phi X,(\nabla_W\phi)Y) + \sum_{i=1}^{n} g(X,(\nabla_W\phi)e_i)g(\phi Y, e_i) \\
+ \;& g(\phi Y,(\nabla_W\phi)X) - \sum_{i=1}^{n} g(Y,(\nabla_W\phi)e_i)g(\phi X, e_i) \\
+ \;& 2\sum_{i=1}^{n} g(X,\phi Y)g((\nabla_W\phi)e_i, e_i)\} = 0. \qquad (4.5)
\end{aligned}$$

For Levi Civita connection ∇,

$$(\nabla_W g)(X,Y) = 0,$$

which gives

$$(\nabla_W g)(X,Y) - g(\nabla_W X, Y) - g(X, \nabla_W Y) = 0.$$

Putting $X = e_i$ and $Y = \phi e_i$ in the above equation, we obtain

$$-g(\nabla_W e_i, \phi e_i) - g(e_i, (\nabla_W\phi)e_i) = 0,$$

which can be written as

$$g(e_i, \phi(\nabla_W e_i)) - g(e_i, (\nabla_W \phi)e_i) = 0.$$

Thus we have

$$g(e_i, (\nabla_W \phi)e_i) = 0. \tag{4.6}$$

By the virtue of (4.5) and (4.6) takes the form

$$\begin{aligned} & f_2\{-g(\phi X, (\nabla_W \phi)Y) + \sum_{i=1} g(X, (\nabla_W \phi)e_i) g(\phi Y, e_i) \\ + \quad & g(\phi Y, (\nabla_W \phi)X) - \sum_{i=1} g(Y, (\nabla_W \phi)e_i) g(\phi X, e_i)\} = 0. \end{aligned} \tag{4.7}$$

The above equation yields $f_2 = 0$. It is known that a generalized Sasakian space form of dimension greater than three is conformally flat if and only if $f_2 = 0$ [14]. Hence the manifold under consideration is conformally flat. Conversely, suppose that the manifold is conformally flat. Then $f_2 = 0$. In addition, if we consider X, Y, Z orthogonal to ξ then (1.1) yields

$$R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\}.$$

The above equation gives,

$$r = 2n(2n + 1)f_1. \tag{4.8}$$

In view of (2.11) and (4.8), we obtain $f_3 = 0$. Hence from (4.4), we get

$$(\nabla_W W_2)(X, Y)Z = 0.$$

Therefore, the manifold is W_2-locally symmetric.
Thus we have the following assertion.

Theorem 4.1. *A $(2n + 1)$ dimensional $(n > 1)$ generalized Sasakian space form is W_2-locally symmetric if and only if it is conformally flat.*

or

Theorem 4.2. *A $(2n + 1)$ dimensional $(n > 1)$ generalized Sasakian space form is W_2-locally symmetric if and only if f_1 is constant.*

5 W_2-Locally ϕ-symmetric generalized Sasakian space forms

Definition 2. *A generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ of dimension greater than three is called W_2-locally ϕ-symmetric if it satisfies*

$$\phi^2((\nabla_W W_2)(X, Y)Z) = 0, \tag{5.1}$$

for all vector fields X, Y, Z orthogonal to ξ on M^{2n+1}. Let us consider a W_2-locally ϕ-symmetric generalized Sasakian space form of dimension greater than three. Then from the definition and (2.1), we have

$$-((\nabla_W W_2)(X,Y)Z)+\eta(\nabla_W W_2)(X,Y)Z)\xi=0, \tag{5.2}$$

Taking the inner product g in both sides of the above equation with respect to W, we get

$$-g((\nabla_W W_2)(X,Y)Z,W)+\eta(\nabla_W W_2)(X,Y)Z)\eta(W)=0, \tag{5.3}$$

If we take orthogonal to W, then the above equation yields,

$$g((\nabla_W W_2)(X,Y)Z,W)=0, \tag{5.4}$$

The above equation is true for all W orthogonal to ξ. If we choose $W\neq 0$ and not orthogonal to $(\nabla_W W_2)(X,Y)Z$, then it follows that

$$(\nabla_W W_2)(X,Y)Z=0 \tag{5.5}$$

Hence, the manifold is W_2-locally symmetric and hence by theorem 4.3, it is conformally flat. Conversely, let the manifold is conformally flat and hence $f_2\neq 0$. Again, for X,Y,Z orthogonal to ξ, we have applying ϕ^2 on both side to equation (4.4), one can get

$$\begin{aligned}\phi^2(\nabla_W W_2)(X,Y)Z &= -df_2(W)\{g(X,\phi Z)\phi X-g(Y,\phi Z)+2g(X,\phi Y)\phi Z\}\\ &- \frac{1}{2n}\{d(3f_2-f_3)(W)[g(X,Z)Y-g(Y,Z)X]\}.\end{aligned} \tag{5.6}$$

if $f_2=f_3=0$, the above equation yields

$$\phi^2(\nabla_W W_2)(X,Y)Z=0$$

for all X,Y,Z are orthogonal to ξ, therefore the manifold is W_2-locally ϕ-symmetric.
Now we are in a position to state the following statement,

Theorem 5.1. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian space form M^{2n+1} is W_2-locally ϕ-symmetric if and only if it is conformally flat.*

6 W_2-ϕ-recurrent generalized Sasakian Space form

Definition 3. *A generalized Sasakian space form is said to be ϕ-recurrent if there exists a non-zero 1-form A such that,(see[11])*

$$\phi^2((\nabla_W R)(X,Y)Z)=A(W)R(X,Y)Z,$$

for arbitrary vector fields X,Y,Z,W. If the 1-form A vanishes, then the manifold reduces to a ϕ-symmetric manifold.

According to the definition of ϕ-recurrent generalized Sasakian space form, we define W_2-ϕ-recurrent generalized sasakian space form by

$$\phi^2((\nabla_W W_2)(X,Y)Z) = A(W)W_2(X,Y)Z. \tag{6.1}$$

Then by (2.1) and (6.1), we have

$$-(\nabla_W W_2)(X,Y)Z + \eta((\nabla_W W_2)(X,Y)Z)\xi = A(W)W_2(X,Y)Z, \tag{6.2}$$

for arbitrary vector fields X, Y, Z, W. From the above equation it follows that

$$\begin{aligned} &- \quad g((\nabla_W W_2)(X,Y)Z, U) + \eta((\nabla_W W_2)(X,Y)Z)\eta(U) \\ &= \quad A(W)g(W_2(X,Y)Z, U). \end{aligned} \tag{6.3}$$

Let $\{e_i\}, i = 1, 2, ...2n+1$, be an orthogonal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (6.3) and taking summation over $i, 1 \leq i \leq 2n+1$, we get

$$\begin{aligned} &- \quad (\nabla_W S)(Y,Z) - \frac{1}{2n}[(\nabla_W S(Y,Z)) - g(Y,Z)dr(W)] \\ &+ \quad \sum_{i=1}^{2n+1} \eta((\nabla_W W_2)(e_i,Y)Z)\eta(e_i) = A(W)\{(\nabla_W S)(Y,Z) \\ &- \quad \frac{1}{2n}[(\nabla_W S)(Y,Z) - g(Y,Z)dr(W)]\}. \end{aligned} \tag{6.4}$$

Setting $Z = \xi$ in (6.4) then using (2.5), (2.13) and (2.7) and then replace Y by ϕY in (6.4), we get

$$S(Y,W) = 2n(f_1 - f_3)g(Y,W). \tag{6.5}$$

Hence we can state following theorem:

Theorem 6.1. *Let generalized Sasakian space forms M^{2n+1} is W_2-ϕ-recurrent, then it is an Einstein manifold, provided $(f_1 - f_3) \neq 0$.*

7 Example

In [1], generalized complex space-form of dimension two is $N(a,b)$ and the warped product $M = R \times N$ endowed with the almost contact metric structure is a three dimensional generalized Sasakian-space-form whose smooth functions $f_1 = \frac{a-(f')^2}{f^2}$, $f_2 = \frac{b}{f^2}$ and $f_3 = \frac{a-(f')^2}{f^2} + \frac{f''}{f}$. Here $f = f(t)$, $t \in R$ and f' indicates the derivative of f with respect to t. Suppose we set $a = 2$, $b = 0$ and $f(t) = t$ with $t \neq 0$, then $f_1 = \frac{1}{t^2}$, $f_2 = 0$ and $f_3 = \frac{1}{t^2}$, we have from (1.2)

$$\begin{aligned} W_2(X,Y)Z \quad &= \quad \frac{1}{t^2}\{g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \quad g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} + \frac{1}{2t^2}\{g(X,Z)Y - g(Y,Z)X \\ &- \quad g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{7.1}$$

Now differentiating covariantly with respect to W and taking X, Y, Z are orthogonal to ξ and then apply ϕ^2 on both side of the above equation

$$\phi^2(\nabla_W W_2(X,Y)Z) = -\frac{3}{2}d(\frac{1}{t^2})\{g(X,Z)Y - g(Y,Z)X\}. \tag{7.2}$$

By the virtue of (7.2) we can easily say generalized Sasakian space forms is W_2-locally ϕ-symmetric if and only if $\frac{1}{t^2}$ is constant or both f_1 and f_2 are constants.

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Pre-regular *sp*-Open Sets in Topological Spaces

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ABSTRACT

In this paper, a new class of generalized open sets in a topological space, called pre-regular *sp*-open sets, is introduced and studied. This class is contained in the class of semi-preclopen sets and cotains all pre-clopen sets. We obtain decompositions of regular open sets by using pre-regular sp-open sets.

RESUMEN

En este artículo se introduce y estudia una nueva clase de conjuntos abiertos generalizados en un espacio topológico, llamados conjuntos pre-regulares sp-abiertos. Esa clase está contenida en la clase de conjuntos semi-preclopen y contiene todos los conjuntos pre-clopen. Obtenemos descomposiciones de conjuntos abiertos regulares usando conjuntos pre-regulares sp-abiertos.

Keywords and Phrases: Generalized open sets, preopen, regular open, pre-regular *sp*-open, decompositions of complete continuity.

2010 AMS Mathematics Subject Classification: 54A05.

1 Introduction

In general topology, by repeated applications of interior (int) and closure (cl) operators several different new classes of sets are defined in the following way.

Definition 1. *A subset* A *of a space* X *is said to be*

i) *semi-open [10] if* $A \subseteq cl(intA)$.

ii) *preopen [11] if* $A \subseteq int(clA)$.

iii) *semi-preopen [2] or* β*-open [1] if* $A \subseteq cl(int(clA))$.

iv) α*-open [12] if* $A \subseteq int(cl(intA))$.

v) *regular open [13] if* $A = int(clA)$.

vi) *b-open [3] if* $A \subseteq cl(intA) \cup int(clA)$.

vii) *pre-regular p-open [9] if* $A = pint(pclA)$.

The complements of the above open sets are called their respective closed sets.

Definition 2. *A subset* A *of a space* X *is called a q-set [14] or* δ*-set [5] if* $int(clA) \subseteq cl(intA)$.

In this paper, we introduce and study a new class of sets, called pre-regular sp-open sets using pre-interior and semi-preclosure operators. This class is contained in the class of semi-preclopen sets and cotains all pre-clopen sets. Moreover, we investigate the relationship between this class of sets and other class of open sets. By using pre-regular sp-open sets, we obtain decompositions of regular open sets. In the last section, we obtain decompositions of complete continuity. Throughout this paper (X, τ) (briefly X) denotes a topological space on which no separation axioms are assumed, unless explicity stated.

We recollect some of the relations that, together with their duals, we shall use in the sequel.

Proposition 1. [2] *Let* A *be a subset of a space* X*. Then*

i) $pclA = A \cup cl(intA)$ *and* $pintA = A \cap int(clA)$.

ii) $spclA = A \cup int(cl(intA))$ *and* $spintA = A \cap cl(int(clA))$.

iii) $pint(spclA) = (A \cap int(clA)) \cup int(cl(intA))$.

iv) $pcl(spintA) = (A \cup cl(intA)) \cap cl(int(clA))$.

Definition 3. *A function* $f : X \to Y$ *is called completely continuous* [4] *(resp.* α*-continuous [8],semi-continuous [10], q-continuous [14]) if the inverse image of every open subset of* Y *is a regular open (resp.* α*-open, semi-open, a q-set) subset of* X.

2 pre-regular *sp*-open sets

In this section, we define and characterize pre-regular *sp*-open sets and study some of their properties.

Definition 4. *A subset* A *of a topological space* (X, τ) *is said to be pre-regular sp-open if* $A = \mathrm{pint}(\mathrm{spcl}A)$. *The complement of a pre-regular sp-open set is said to be pre-regular sp-closed.*

We denote the collection of all pre-regular *sp*-open (resp. preopen, preclosed, pre-semiopen, pre-semiclosed, pre-clopen, pre-semiclopen) sets of (X, τ) by $\mathrm{PRSPO}(X)$ (resp. $\mathrm{PO}(X)$, $\mathrm{PC}(X)$, $\mathrm{PSO}(X)$, $\mathrm{PSC}(X)$, $\mathrm{PCO}(X)$, $\mathrm{PSCO}(X)$).

Theorem 2.1. *Let* (X, τ) *be a topological space and* A, B *subsets of* X. *Then the following hold:*

i) *If* $A \subseteq B$, *then* $\mathrm{pint}(\mathrm{spcl}A) \subseteq \mathrm{pint}(\mathrm{spcl}B)$.

ii) *If* $A \in \mathrm{PO}(X, \tau)$, *then* $A \subseteq \mathrm{pint}(\mathrm{spcl}A)$.

iii) *If* $A \in \mathrm{SPC}(X, \tau)$, *then* $\mathrm{pint}(\mathrm{spcl}A) \subseteq A$.

iv) *We have* $\mathrm{pint}(\mathrm{spcl}(\mathrm{pint}(\mathrm{spcl}A))) = \mathrm{pint}(\mathrm{spcl}A)$.

v) *If* $A \in \mathrm{SPC}(X, \tau)$, *then* $\mathrm{pint}A$ *is a pre-regular sp-open set.*

Proof. i) Suppose that $A \subseteq B$. Then $\mathrm{pint}(\mathrm{spcl}A) \subseteq \mathrm{pint}(\mathrm{spcl}B)$.

ii) Suppose that $A \in \mathrm{PO}(X, \tau)$. Since $A \subseteq \mathrm{spcl}A$, we have $A \subseteq \mathrm{pint}(\mathrm{spcl}A)$.

iii) Suppose that $A \in \mathrm{SPC}(X, \tau)$. Since $\mathrm{pint}A \subseteq A$, we have $\mathrm{pint}(\mathrm{spcl}A) \subseteq A$.

iv) We have $\mathrm{pint}(\mathrm{spcl}(\mathrm{pint}(\mathrm{spcl}A))) \subset \mathrm{pint}(\mathrm{spcl}(\mathrm{spcl}A)) = \mathrm{pint}(\mathrm{spcl}A)$ and $\mathrm{pint}(\mathrm{spcl}(\mathrm{pint}(\mathrm{spcl}A))) \supset \mathrm{pint}(\mathrm{pint}(\mathrm{spcl}A)) = \mathrm{pint}(\mathrm{spcl}A)$. Hence $\mathrm{pint}(\mathrm{spcl}(\mathrm{pint}(\mathrm{spcl}A))) = \mathrm{pint}(\mathrm{spcl}A)$.

v) Suppose that $A \in \mathrm{SPC}(X, \tau)$. By (i), we have $\mathrm{pint}(\mathrm{spcl}(\mathrm{pint}A)) \subseteq \mathrm{pint}(\mathrm{spcl}A) = \mathrm{pint}A$. On the other hand, we have $\mathrm{pint}A \subseteq \mathrm{spcl}(\mathrm{pint}A)$. Therefore $\mathrm{pint}A \subseteq \mathrm{pint}(\mathrm{spcl}(\mathrm{pint}A))$ and hence $\mathrm{pint}(\mathrm{spcl}(\mathrm{pint}A)) = \mathrm{pint}A$.

□

Remark 2.2. *The family of pre-regular sp-open sets is not closed under finite union as well as finite intersection. It will be shown in the following example.*

Example 2.3. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{\emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. *Then* $\{a\}$ *and* $\{b\}$ *are pre-regular sp-open sets but their union* $\{a, b\}$ *is not a pre-regular sp-open set. Moreover,* $\{a, c, d\}$ *and* $\{b, c, d\}$ *are pre-regular sp-open but their intersection* $\{c, d\}$ *is not a pre-regular sp-open set.*

Theorem 2.5 and 2.6 give the characterizations of pre-regular *sp*-open sets.

Theorem 2.4. *Let (X, τ) be a topological space. For a subset A of X, the following are equivalent:*

i) A *is pre-regular sp-open.*

ii) $A = \mathrm{spcl}A \cap \mathrm{int}(\mathrm{cl}A)$.

iii) $A = \mathrm{pint}A \cup \mathrm{int}(\mathrm{cl}(\mathrm{int}A))$.

Proof. It follows form Proposition 1.3. □

Theorem 2.5. *Let (X, τ) be a topological space. A subset A of X is pre-regular sp-open if and only if it is preopen and semi-preclosed .*

Proof. Let A be pre-regular *sp*-open. Then $A = \mathrm{pint}(\mathrm{spcl}A)$. Hence $\mathrm{pint}A = \mathrm{pint}(\mathrm{pint}(\mathrm{spcl}A)) = \mathrm{pint}(\mathrm{spcl}A) = A$. Thus A is preopen. By Theorem 2.5, $A = \mathrm{pint}A \cup \mathrm{int}(\mathrm{cl}(\mathrm{int}A))$ and $\mathrm{int}(\mathrm{cl}(\mathrm{int}A)) \subseteq A$. Therefore, A is semi-preclosed. Conversely assume that A is both preopen and semi-preclosed. Then $A = \mathrm{pint}A$ and $A = \mathrm{spcl}A$. Now $\mathrm{pint}(\mathrm{spcl}A) = \mathrm{pint}A = A$. Hence A is pre-regular *sp*-open. □

Corolary 1. *For a topological space (X, τ), we have* $PO(X) \cap PC(X) \subseteq PRSPO(X) \subseteq SPO(X) \cap SPC(X)$.

Proof. This is obvious. □

Remark 2.6. *The converse inclusions in Corollary 2.7 need not be true as the following examples show.*

Example 2.7. *Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$.*
Then $\{a, d\}$ is semi-preclopen but not pre-regular sp-open.

Example 2.8. *Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{a, b, c\}, \{b, c, d\}, X\}$. Then $\{c\}$ is pre-regular sp-open but it is not pre-clopen.*

Theorem 2.9. *In any space (X, τ), the empty set is the only subset which is nowhere dense and pre-regular sp-open.*

Proof. Suppose A is nowhere dense and pre-regular *sp*-open. Then by Theorem 2.5, $A = \mathrm{pint}(\mathrm{spcl}A) = \mathrm{spcl}A \cap \mathrm{int}(\mathrm{cl}A) = \mathrm{spcl}A \cap \emptyset = \emptyset$. □

Remark 2.10. *The notions of pre-regular sp-open sets and open sets (hence α-open sets, semi-open sets, q-sets) are independent of each other. It is shown in [5] and [14] that every semi-open set is a q-set, that is, a δ-set.*

Example 2.11. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, \{a, b\}, X\}$*. Then* $\{a, b\}$ *is open hence* α*-open, semi-open, a q-set but it is not pre-regular sp-open. Also,* $\{a\}$ *is pre-regular sp-open but it is not a q-set.*

Theorem 2.12. *Every regular open set is pre-regular sp-open.*

Proof. Let A be regular open. Then $A = int(clA)$. By Proposition 1.3, $pint(spclA) = (spclA) \cap int(cl(spclA)) = spclA \cap int(cl[A \cup int(cl(intA))]) = spclA \cap int(clA) = spclA \cap A = A$. This shows that A is pre-regular *sp*-open. □

The above disscusion can be summarized in the following diagram:

DIAGRAM

regular open ⇒ open ⇒ α-open ⇒ semi-open ⇒ q-set

⇓ ⇓ ⇓

pre-regular *sp*-open ⇒ preopen ⇒ *b*-open ⇒ semi-preopen

Remark 2.13. *A q-set and a semi-preopen set are independent by Example 2.13 and the following example.*

Example 2.14. *Let* R *be the real numbers with the usual topology. Then for each* $x \in R$*,* $cl(int(cl\{x\})) = \emptyset$ *and it does not contain* $\{x\}$*. Hence* $\{x\}$ *is not semi-preopen. But* $int(cl\{x\}) = cl(int\{x\}) = \emptyset$ *and* $\{x\}$ *is a q-set.*

Theorem 2.15. *Every pre-regular p-open set is pre-regular sp-open.*

Proof. Let A be pre-regular *p*-open. Then $A = pint(pclA)$ and A is preopen. Since $spclA \subseteq pclA$, we have $pint(spclA) \subseteq pint(pclA) = A$. On the other hand, we have $A \subseteq spclA$. Since A is preopen, $A = pintA \subseteq pint(spclA)$. Hence $A = pint(spclA)$. □

Theorem 2.16. *For a subset* A *of a space* X*, the following are equivalent:*

i) A *is regular open.*

ii) A *is pre-regular sp-open and a q-set.*

iii) A *is* α*-open and semi-preclosed.*

Proof. i) ⇒ ii). Let A be regular open. Then, by Theorem 2.14 A is pre-regular *sp*-open and also by Diagram, A is a *q*-set.

ii) ⇒ i). Since A is a *q*-set, $int(clA) \subset cl(intA)$ and $int(clA) \subset int(cl(intA)) \subset int(clA)$. Therefore, we have $int(clA) = int(cl(intA))$. By using Theorem 2.5, we obtain $int(clA) = [A \cup int(clA)] \cap int(clA) = [A \cup int(cl(intA))] \cap int(clA) = spclA \cap int(clA) = A$.

i) ⇒ iii). Let A be regular open. Then A is open and $A = int(clA) = int(cl(intA))$. Therefore, every regular open set is α-open and semi-preclosed.

iii) $\Rightarrow$ i). Let A be α-open and semi-preclosed. Then $\mathrm{int}(\mathrm{cl}(\mathrm{int}A)) \subset A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}A))$. Therefore, $A = \mathrm{int}(\mathrm{cl}(\mathrm{int}A))$ and hence $\mathrm{int}(\mathrm{cl}A) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}A)))) = \mathrm{int}(\mathrm{cl}(\mathrm{int}A)) = A$. Hence A is regular open. □

Corolary 2. *Suppose A is pre-regular sp-open. Then the following are hold:*

i) *If A is open, then A is regular open.*

ii) *If A is closed, then A is clopen.*

iii) *If A is semi-open, then A is regular open.*

iv) *If A is semi-closed, then A is α-open and semi-preclosed.*

Proof. Since A is pre-regular *sp*-open, by Theorem 2.5 $A = \mathrm{spcl}A \cap \mathrm{int}(\mathrm{cl}A) = \mathrm{pint}A \cup \mathrm{int}(\mathrm{cl}(\mathrm{int}A))$.

i) Suppose A is open. Then by Diagram, A is a *q*-set and by Theorem 2.18, we have A is regular open.

ii) Suppose A is closed. Now $A = \mathrm{spcl}A \cap \mathrm{int}(\mathrm{cl}A) = \mathrm{spcl}A \cap \mathrm{int}A = \mathrm{int}A$. Hence A is open and hence clopen.

iii) Since every semi-open set is a *q*-set, by Theorem 2.18 A is regular open.

iv) Suppose A is semi- closed. Then $\mathrm{int}(\mathrm{cl}A) \subseteq A$. This implies $\mathrm{int}(\mathrm{cl}A) \subset \mathrm{int}A \subset \mathrm{cl}(\mathrm{int}A)$. Hence A is a *q*-set and by Theorem 2.18, A is α-open and semi-preclosed.

□

Remark 2.17. *In a partition space (X, τ), a subset A of X is preopen if and only if A is pre-regular sp-open.*

Theorem 2.18. *If a space (X, τ) is submaximal, then any finite intersection of pre-regular sp-open sets is pre-regular sp-open.*

Proof. Let $\{A_i | i \in I\}$ be a finite family of pre-regular *sp*-open sets. Then $\{A_i | i \in I\}$ is a finite family of preopen sets. Since X is submaximal, $\bigcap_{i\in I} A_i$ is pre open. Therefore by Theorem 2.2 (ii), $\bigcap_{i\in I} A_i \subseteq \mathrm{pint}(\mathrm{spcl}(\bigcap_{i\in I} A_i)$. On the other hand, for each $i \in I$, we have $\bigcap_{i\in I} A_i \subseteq A_i$ and by Theorem 2.2 (i) $\mathrm{pint}(\mathrm{spcl}(\bigcap_{i\in I} A_i))) \subseteq \mathrm{pint}(\mathrm{spcl}A_i)$. Since $\mathrm{pint}(\mathrm{spcl}A_i) = A_i$, we have $\mathrm{pint}(\mathrm{spcl}(\bigcap_{i\in I} A_i))) \subseteq \bigcap_{i\in I} A_i$. Hence $\mathrm{pint}(\mathrm{spcl}(\bigcap_{i\in I} A_i))) = \bigcap_{i\in I} A_i$. □

Theorem 2.19. *If A is pre-regular sp-closed and a rare set of a space (X, τ), then A is semi-preopen.*

Proof. Since A is pre-regular *sp*-closed, by Theorem 2.5 $A = pcl(spintA) = spintA \cup cl(intA)$. Since A is a rare set, $intA = \emptyset$. Thus $A = spintA$. Hence A is semi-preopen. □

Recall that a space (X, τ) is said to be an extremally disconnected if the closure of every open subset of X is open. Moreover, it is shown in [7] (X, τ) is extremally disconnected if and only if $SPO(X) = PO(X)$.

Theorem 2.20. *For an extremally disconnected space (X, τ), the following are equivalent:*

i) A *is pre-regular sp-open.*

ii) A *is pre-regular sp-closed.*

iii) A *is pre-clopen.*

iv) A *is semi-preclopen.*

Proof. $(i) \Leftrightarrow (iii)$. Suppose that A is pre-regular *sp*-open. Then by Theorem 2.6, A is preopen and semi-preclosed. Since X is extremally disconnected, A is pre-clopen. Hence A is pre-closed. The converse is obvious by Theorem 2.6.
$(ii) \Leftrightarrow (iv)$. Let A be pre-regular *sp*-closed. Then $X \backslash A$ is pre-regular *sp*-open and by $(i) \Leftrightarrow (iii)$ $X \backslash A$ is pre-clopen. Therefore, A is semi-preclopen. The converse is obvious.
$(iii) \Leftrightarrow (iv)$. This is obvious. □

Recall that a space (X, τ) has the property Q [10] if $int(clA) = cl(intA)$ for all subset A of X.

Theorem 2.21. *Let (X, τ) be a space with the property Q. For a subset $A \subseteq X$, the following properties are equivalent:*

i) A *is pre-regular sp-open.*

ii) A *is pre-regular sp-closed.*

iii) A *is regular open.*

iv) A *is regular closed.*

Proof. $(i) \Leftrightarrow (iii)$. By Proposition 1.3, $pint(spclA) = [A \cap int(clA)] \cup int(cl(intA)) = [A \cap int(clA)] \cup int(int(clA)) = int(clA)$. This completes the proof.
$(ii) \Leftrightarrow (iv)$. By Proposition 1.3, $pcl(spintA) = [A \cup cl(intA)] \cap cl(int(clA)) = [A \cup cl(intA)] \cap cl(cl(intA)) = cl(intA)$. This completes the proof.
$(iii) \Leftrightarrow (iv)$. This is obvious. □

3 Decompositions of complete continuity

In this section, the notion of pre-regular *sp*-continuous functions is introduced and the decompositions of complete continuity are discussed.

Definition 5. *A function* $f : X \to Y$ *is said to be pre-regular sp-continuous (briefly, prsp-continuous) if* $f^{-1}(V)$ *is pre-regular sp-open in* X *for each open subset* V *of* Y.

By Theorems 2.18 and Daigram, we have the following main theorem

Theorem 3.1. *For a function* $f : X \to Y$*, the following properties are equivalent:*

i) f *is completely continuous.*

ii) f *is prsp-continuous and continuous.*

iii) f *is prsp-continuous and* α*-continuous.*

iv) f *is prsp-continuous and semi-continuous.*

v) f *is prsp-continuous and q-continuous.*

Remark 3.2. *As shown by the following examples, prsp-continuity and continuity (hence* α*-continuity, semi-continuity, q-continuity) are independent of each other.*

Example 3.3. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ *and* $\sigma = \{\emptyset, \{a, b\}, X\}$*. Then*

i) The identity function $f : (X, \tau) \to (X, \tau)$ *is continuous but it is not prsp-continuous since* $f^{-1}(\{a\}) = \{a\}$ *is open but it is not pre-regular sp-open.*

ii) Consider the function $f : (X, \sigma) \to (X, \tau)$ *defined by* $f(a) = a$, $f(b) = c$ *and* $f(c) = b$*. Then* f *is prsp-continuous but it is not q-continuous, since* $f^{-1}(\{a\}) = \{a\}$ *is pre-regular sp-open but it is not a q-set in* (X, σ).

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Common Fixed Point Results in C^*-Algebra Valued b-Metric Spaces Via Digraphs

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ABSTRACT

We discuss the existence and uniqueness of points of coincidence and common fixed points for a pair of self-mappings defined on a C^*-algebra valued b-metric space endowed with a graph. Our results extend and supplement several recent results in the literature. Strength of hypotheses made in the first result have been weighted through illustrative examples.

RESUMEN

Discutimos la existencia y unicidad de puntos de coincidencia y puntos fijos comumes para un par de aplicaciones definidas en un b-espacio métrico a valores en una álgebra C^* dotado de un grafo en sí mismo. Nuestros resultados extienden y suplementan diversos resultados recientes en la literatura. La fuerza de las hipótesis impuestas al primer resultado se evalúa a través de ejemplos ilustrativos.

Keywords and Phrases: C^*-algebra, C^*-algebra valued b-metric, directed graph, C^*-algebra valued G-contraction, common fixed point.

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1 Introduction

In 1922 [5], Polish mathematician S. Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle. This fundamental principle was largely applied in many branches of mathematics. Several authors generalized this interesting theorem in different ways(see [1, 2, 6, 13, 18, 25, 26, 27]). In this context, Bakhtin [4] and Czerwik [11] developed the notion of $\mathfrak{b}$-metric spaces and proved some fixed point theorems for single-valued and multi-valued mappings in the setting of $\mathfrak{b}$-metric spaces. In 2014, Z. Ma et.al.[22] introduced the concept of C^*-algebra valued metric spaces by using the set of all positive elements of an unital C^*-algebra instead of the set of real numbers. In [21], the authors introduced another new concept, known as C^*-algebra valued $\mathfrak{b}$-metric spaces as a generalization of C^*-algebra valued metric spaces and $\mathfrak{b}$-metric spaces.

In recent investigations, the study of fixed point theory endowed with a graph plays an important role in many aspects. In 2005, Echenique [15] studied fixed point theory by using graphs. After that, Espinola and Kirk [16] applied fixed point results in graph theory. Recently, combining fixed point theory and graph theory, a series of articles(see [3, 8, 9, 10, 20, 24] and references therein) have been dedicated to the improvement of fixed point theory.

The idea of common fixed point was initially given by Junck [19]. In fact, the author introduced the concept of weak compatibility and obtained a common fixed point result. Several authors have obtained coincidence points and common fixed points for various classes of mappings on a metric space by using this concept. Motivated by some recent works on the extension of Banach contraction principle to metric spaces with a graph, we reformulated some important common fixed point results in metric spaces to C^*-algebra valued $\mathfrak{b}$-metric spaces endowed with a graph. As some consequences of this study, we deduce several related results in fixed point theory. Finally, some examples are provided to illustrate the results.

2 Some basic concepts

We begin with some basic notations, definitions and properties of C^*-algebras. Let $\mathbb{A}$ be an unital algebra with the unit I. An involution on $\mathbb{A}$ is a conjugate linear map $\mathfrak{a} \mapsto \mathfrak{a}^*$ on $\mathbb{A}$ such that $\mathfrak{a}^{**} = \mathfrak{a}$ and $(\mathfrak{a}\mathfrak{b})^* = \mathfrak{b}^*\mathfrak{a}^*$ for all $\mathfrak{a}, \mathfrak{b} \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a $*$-algebra. A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\| \mathfrak{a}^* \| = \| \mathfrak{a} \|$ for all $\mathfrak{a} \in \mathbb{A}$. A C^*-algebra is a Banach $*$-algebra such that $\| \mathfrak{a}^*\mathfrak{a} \| = \| \mathfrak{a} \|^2$ for all $\mathfrak{a} \in \mathbb{A}$. Let H be a Hilbert space and $\mathsf{B}(\mathsf{H})$, the set of all bounded linear operators on H. Then, under the norm topology, $\mathsf{B}(\mathsf{H})$ is a C^*-algebra.

Throughout this discussion, by $\mathbb{A}$ we always denote an unital C^*-algebra with the unit I and the zero element θ. Set $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$, if $x \in \mathbb{A}_h$ and $\sigma(x) \subset [0, \infty)$, where $\sigma(x)$ is the spectrum of x. Using positive elements, one can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows:

$$x \preceq y \text{ if and only if } y - x \succeq \theta.$$

We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$.

From now on, by $\mathbb{A}_+$, we denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ and by $\mathbb{A}^{'}$, we denote the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$.

Lemma 2.1. *[14, 23] Suppose that $\mathbb{A}$ is an unital C^*-algebra with a unit I.*

(i) For any $x \in \mathbb{A}_+$, we have $x \preceq I \Leftrightarrow \| x \| \leq 1$.

(ii) If $a \in \mathbb{A}_+$ with $\| a \| < \frac{1}{2}$, then $I - a$ is invertible and $\| a(I-a)^{-1} \| < 1$.

(iii) Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$.

(iv) Let $a \in \mathbb{A}^{'}$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$, and $I - a \in \mathbb{A}_+^{'}$ is an invertible operator, then $(I-a)^{-1}b \succeq (I-a)^{-1}c$.

Remark 2.2. *It is worth mentioning that $x \preceq y \Rightarrow \| x \| \leq \| y \|$ for $x, y \in \mathbb{A}_+$. In fact, it follows from Lemma 2.1 (i).*

Definition 2.3. *[22] Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:*

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a C^*-algebra valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space.

Definition 2.4. *[4] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:*

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

Definition 2.5. [21] *Let X be a nonempty set and $A \in \mathbb{A}'_+$ such that $A \succeq I$. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:*

(i) $\theta \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \preceq A\,(d(x, z) + d(z, y))$ *for all* $x, y, z \in X$.

Then d is called a C^*-algebra valued b-metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued b-metric space.

It seems important to note that if $\mathbb{A} = \mathbb{C}, A = 1$, then the C^*-algebra valued b-metric spaces are just the ordinary metric spaces. Moreover, it is obvious that C^*-algebra valued b-metric spaces generalize the concepts of C^*-algebra valued metric spaces and b-metric spaces.

Definition 2.6. [26] *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued b-metric space, $x \in X$ and (x_n) be a sequence in X. Then*

(i) (x_n) *converges to* x *with respect to* $\mathbb{A}$ *if for any* $\epsilon > 0$ *there is* n_0 *such that for all* $n > n_0$, $\| d(x_n, x) \| \leq \epsilon$. *We denote it by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x (n \to \infty)$.

(ii) (x_n) *is Cauchy with respect to* $\mathbb{A}$ *if for any* $\epsilon > 0$ *there is* n_0 *such that for all* $n, m > n_0$, $\| d(x_n, x_m) \| \leq \epsilon$.

(iii) $(X, \mathbb{A}, d)$ *is a complete* C^**-algebra valued* b*-metric space if every Cauchy sequence with respect to* $\mathbb{A}$ *is convergent.*

Example 2.7. *If X is a Banach space, then $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued b-metric space with $A = 2^{p-1} I$ if we set*

$$d(x, y) = \| x - y \|^p \; I$$

where $p > 1$ is a real number. But $(X, \mathbb{A}, d)$ is not a C^*-algebra valued metric space because if $X = \mathbb{R}$, then $| x - y |^p \leq | x - z |^p + | z - y |^p$ is impossible for all $x > z > y$.

Definition 2.8. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued b-metric space with the coefficient $A \succeq I$. We call a mapping $f : X \to X$ a C^*-algebra valued contraction mapping on X if there exists $B \in \mathbb{A}$ with $\| B \|^2 < \frac{1}{\|A\|}$ such that*

$$d(fx, fy) \preceq B^* \, d(x, y) B$$

for all $x, y \in X$.

Definition 2.9. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued b-metric space with the coefficient $A \succeq I$. A mapping $f : X \to X$ is called a C^*-algebra valued Fisher contraction if there exists $B \in \mathbb{A}'_+$ with $\| BA \| < \frac{1}{\|A\|+1}$ such that*

$$d(fx, fy) \preceq B\,[d(fx, y) + d(fy, x)]$$

for all $x, y \in X$.

Definition 2.10. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space with the coefficient* $A \succeq I$*. A mapping* $f : X \to X$ *is called a* C^**-algebra valued Kannan operator if there exists* $B \in \mathbb{A}'_+$ *with* $\| B \| < \frac{1}{\|A\|+1}$ *such that*

$$d(fx, fy) \preceq B\,[d(fx, x) + d(fy, y)]$$

for all $x, y \in X$.

Definition 2.11. *[2] Let* T *and* S *be self mappings of a set* X*. If* $y = Tx = Sx$ *for some* x *in* X*, then* x *is called a coincidence point of* T *and* S *and* y *is called a point of coincidence of* T *and* S.

Definition 2.12. *[19] The mappings* $T, S : X \to X$ *are weakly compatible, if for every* $x \in X$*, the following holds:*

$$T(Sx) = S(Tx) \text{ whenever } Sx = Tx.$$

Proposition 2.13. *[2] Let* S *and* T *be weakly compatible selfmaps of a nonempty set* X*. If* S *and* T *have a unique point of coincidence* $y = Sx = Tx$*, then* y *is the unique common fixed point of* S *and* T.

Definition 2.14. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space with the coefficient* $A \succeq I$*. A mapping* $f : X \to X$ *is called* C^**-algebra valued expansive if there exists* $B \in \mathbb{A}$ *with* $0 < \| B \|^2 < \frac{1}{\|A\|}$ *such that*

$$B^* d(fx, fy) B \succeq d(x, y)$$

for all $x, y \in X$.

We next review some basic notions in graph theory.

Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued b-metric space. Let G be a directed graph (digraph) with a set of vertices $V(G) = X$ and a set of edges $E(G)$ contains all the loops, i.e., $E(G) \supseteq \Delta$, where $\Delta = \{(x, x) : x \in X\}$. We also assume that G has no parallel edges and so we can identify G with the pair $(V(G), E(G))$. G may be considered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the graph obtained from G by reversing the direction of edges i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [7, 12, 17]. If x, y are vertices of the digraph G, then a path in G from x to y of length n $(n \in \mathbb{N})$ is a sequence $(x_i)_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \cdots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected.

Definition 2.15. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space with the coefficient* $A \succeq I$ *and let* $G = (V(G), E(G))$ *be a graph. A mapping* $f : X \to X$ *is called a* C^**-algebra valued* G*-contraction if there exists a* $B \in \mathbb{A}$ *with* $\| B \|^2 < \frac{1}{\|A\|}$ *such that*

$$d(fx, fy) \preceq B^* d(x, y) B,$$

for all $x, y \in X$ *with* $(x, y) \in E(G)$.

Any C^*-algebra valued contraction mapping on X is a G_0-contraction, where G_0 is the complete graph defined by $(X, X \times X)$. But it is worth mentioning that a C^*-algebra valued G-contraction need not be a C^*-algebra valued contraction (see Remark 3.23).

Definition 2.16. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space with the coefficient* $A \succeq I$ *and let* $G = (V(G), E(G))$ *be a graph. A mapping* $f : X \to X$ *is called* C^**-algebra valued Fisher* G*-contraction if there exists* $B \in \mathbb{A}_+'$ *with* $\| BA \| < \frac{1}{\|A\|+1}$ *such that*

$$d(fx, fy) \preceq B \left[d(fx, y) + d(fy, x) \right]$$

for all $x, y \in X$ *with* $(x, y) \in E(G)$.

It is easy to observe that a C^*-algebra valued Fisher contraction is a C^*-algebra valued Fisher G_0-contraction. But it is important to note that a C^*-algebra valued Fisher G-contraction need not be a C^*-algebra valued Fisher contraction. The following example supports the above remark.

Example 2.17. *Let* $X = [0, \infty)$ *and* $B(H)$ *be the set of all bounded linear operators on a Hilbert space* H*. Define* $d : X \times X \to B(H)$ *by* $d(x, y) = \mid x - y \mid^2 I$ *for all* $x, y \in X$*. Then* $(X, B(H), d)$ *is a* C^**-algebra valued* b*-metric space with the coefficient* $A = 2I$*. Let* G *be a digraph such that* $V(G) = X$ *and* $E(G) = \Delta \cup \{(3^t x, 3^t(x+1)) : x \in X \text{ with } x \geq 2,\ t = 0, 1, 2, \cdots\}$.

Let $f : X \to X$ *be defined by* $fx = 3x$ *for all* $x \in X$.

For $x = 3^t z,\ y = 3^t(z+1),\ z \geq 2$*, we have*

$$\begin{aligned}
d(fx, fy) &= d\left(3^{t+1} z, 3^{t+1}(z+1)\right) \\
&= 3^{2t+2} I \\
&\preceq \frac{9}{58} 3^{2t}(8z^2 + 8z + 10) I \\
&= B \left[d\left(3^{t+1} z, 3^t(z+1)\right) + d\left(3^{t+1}(z+1), 3^t z\right) \right] \\
&= B \left[d(fx, y) + d(fy, x) \right],
\end{aligned}$$

where $B = \frac{9}{58} I \in B(H)_+'$ *with* $\| BA \| < \frac{1}{\|A\|+1}$*. Thus,* f *is a* C^**-algebra valued Fisher* G*-contraction.*

We now verify that f *is not a* C^**-algebra valued Fisher contraction. In fact, if* $x = 3,\ y = 0,$

then for any arbitrary $B \in B(H)_+^{'}$ *with* $\| BA \| < \frac{1}{\|A\|+1} = \frac{1}{3}$*(which implies* $3BA \prec I$*), we have*

$$\begin{aligned} B\left[d(fx,y)+d(fy,x)\right] &= B\left[d(f3,0)+d(f0,3)\right] \\ &= 90BI \\ &= 45BA \\ &= \frac{5}{27}(3BA)(81I) \\ &\prec 81I \\ &= d(fx,fy). \end{aligned}$$

Definition 2.18. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space with the coefficient* $A \succeq I$ *and let* $G = (V(G), E(G))$ *be a graph. A mapping* $f : X \to X$ *is called* C^**-algebra valued* G*-Kannan if there exists* $B \in \mathbb{A}_+^{'}$ *with* $\| B \| < \frac{1}{\|A\|+1}$ *such that*

$$d(fx,fy) \preceq B\left[d(fx,x)+d(fy,y)\right]$$

for all $x, y \in X$ *with* $(x, y) \in E(G)$.

Note that any C^*-algebra valued Kannan operator is C^*-algebra valued G_0-Kannan. However, a C^*-algebra valued G-Kannan operator need not be a C^*-algebra valued Kannan operator (see Remark 3.28).

Remark 2.19. *If* f *is a* C^**-algebra valued* G*-contraction(resp.,* G*-Kannan or Fisher* G*-contraction), then* f *is both a* C^**-algebra valued* G^{-1}*-contraction(resp.,* G^{-1}*-Kannan or Fisher* G^{-1}*-contraction) and a* C^**-algebra valued* $\tilde{G}$*-contraction(resp.,* $\tilde{G}$*-Kannan or Fisher* $\tilde{G}$*-contraction).*

3 Main Results

In this section we always assume that $(X, \mathbb{A}, d)$ is a C^*-algebra valued b-metric space with the coefficient $A \succeq I$ and G is a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$.
Let $f,\ g : X \to X$ be such that $f(X) \subseteq g(X)$. If $x_0 \in X$ is arbitrary, then there exists an element $x_1 \in X$ such that $fx_0 = gx_1$, since $f(X) \subseteq g(X)$. Proceeding in this way, we can construct a sequence (gx_n) such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$.

Definition 3.1. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space endowed with a graph* G *and* $f,\ g : X \to X$ *be such that* $f(X) \subseteq g(X)$*. We define* C_{gf} *the set of all elements* x_0 *of* X *such that* $(gx_n, gx_m) \in E(\tilde{G})$ *for* $m,\ n = 0, 1, 2, \cdots$ *and for every sequence* (gx_n) *such that* $gx_n = fx_{n-1}$.

If $g = I$, the identity map on X, then obviously C_{gf} becomes C_f which is the collection of all elements x of X such that $(f^n x, f^m x) \in E(\tilde{G})$ for $m,\ n = 0, 1, 2, \cdots$.

Theorem 3.2. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued b-metric space endowed with a graph G and the mappings $f,\ g : X \to X$ be such that*

$$d(fx, fy) \preceq B^* \, d(gx, gy) \, B \tag{3.1}$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, where $B \in \mathbb{A}$ and $\| B \|^2 < \frac{1}{\|A\|}$. Suppose $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X with the following property:

$(*)$ If (gx_n) is a sequence in X such that $gx_n \to x$ and $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.
Then f and g have a point of coincidence in X if $C_{gf} \neq \emptyset$. Moreover, f and g have a unique point of coincidence in X if the graph G has the following property:

$(**)$ If $x,\ y$ are points of coincidence of f and g in X, then $(x, y) \in E(\tilde{G})$.
Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Suppose that $C_{gf} \neq \emptyset$. We choose an $x_0 \in C_{gf}$ and keep it fixed. Since $f(X) \subseteq g(X)$, there exists a sequence (gx_n) such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$ and $(gx_n, gx_m) \in E(\tilde{G})$ for $m,\ n = 0, 1, 2, \cdots$.
It is a well known fact that in a C^*-algebra $\mathbb{A}$, if $a, b \in \mathbb{A}_+$ and $a \preceq b$, then for any $x \in \mathbb{A}$ both x^*ax and x^*bx are positive elements and $x^*ax \preceq x^*bx$[23].

For any $n \in \mathbb{N}$, we have by using condition (3.1) that

$$d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n) \preceq B^* d(gx_{n-1}, gx_n) B. \tag{3.2}$$

By repeated use of condition (3.2), we get

$$d(gx_n, gx_{n+1}) \preceq (B^*)^n d(gx_0, gx_1) B^n = (B^n)^* B_0 B^n, \tag{3.3}$$

for all $n \in \mathbb{N}$, where $B_0 = d(gx_0, gx_1) \in \mathbb{A}_+$.

For any $m, n \in \mathbb{N}$ with $m > n$, we have by using condition (3.3) that

$$\begin{aligned}
d(gx_n, gx_m) &\preceq A[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\
&\preceq Ad(gx_n, gx_{n+1}) + A^2 d(gx_{n+1}, gx_{n+2}) + \cdots \\
&\quad + A^{m-n-1} d(gx_{m-2}, gx_{m-1}) + A^{m-n-1} d(gx_{m-1}, gx_m) \\
&\preceq A(B^*)^n B_0 B^n + A^2 (B^*)^{n+1} B_0 B^{n+1} + A^3 (B^*)^{n+2} B_0 B^{n+2} + \cdots \\
&\quad + A^{m-n-1} (B^*)^{m-2} B_0 B^{m-2} + A^{m-n-1} (B^*)^{m-1} B_0 B^{m-1} \\
&\preceq \sum_{k=1}^{m-n-1} A^k (B^*)^{n+k-1} B_0 B^{n+k-1} + A^{m-n} (B^*)^{m-1} B_0 B^{m-1} \\
&= \sum_{k=1}^{m-n} A^k (B^*)^{n+k-1} B_0 B^{n+k-1} \\
&\preceq \sum_{k=1}^{m-n} \| A^k (B^*)^{n+k-1} B_0 B^{n+k-1} \| I \\
&\preceq \| B_0 \| \sum_{k=1}^{m-n} \| A \|^k \| B \|^{2(n+k-1)} I \\
&= \| B_0 \| \| B \|^{2n} \| A \| \sum_{k=1}^{m-n} \left(\| A \| \| B \|^2 \right)^{k-1} I \\
&\preceq \frac{\| B_0 \| \| B \|^{2n} \| A \|}{1 - \| A \| \| B \|^2} I, \ since \ \| B \|^2 < \frac{1}{\| A \|} \\
&\rightarrow \theta \ as \ n \rightarrow \infty.
\end{aligned}$$

Therefore, (gx_n) is a Cauchy sequence with respect to $\mathbb{A}$. Since $g(X)$ is complete, there exists an $u \in g(X)$ such that $\lim_{n\to\infty} gx_n = u = gv$ for some $v \in X$.

As $x_0 \in C_{gf}$, it follows that $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geq 0$, and so by property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$.

Using condition (3.1), we have

$$\begin{aligned}
d(fv, gv) &\preceq A[d(fv, fx_{n_i}) + d(fx_{n_i}, gv)] \\
&\preceq AB^* d(gv, gx_{n_i}) B + Ad(gx_{n_i+1}, gv) \\
&\rightarrow \theta \ as \ i \rightarrow \infty.
\end{aligned}$$

This implies that $d(fv, gv) = \theta$ and hence $fv = gv = u$. Therefore, u is a point of coincidence of f and g.

The next is to show that the point of coincidence is unique. Assume that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property $(**)$, we have

$(u, u^*) \in E(\tilde{G})$. Then,

$$\begin{aligned} d(u, u^*) &= d(fv, fx) \\ &\preceq B^* d(gv, gx) B \\ &= B^* d(u, u^*) B, \end{aligned}$$

which implies that,

$$\begin{aligned} \| d(u, u^*) \| &\leq \| B^* d(u, u^*) B \| \\ &\leq \| B^* \| \| d(u, u^*) \| \| B \| \\ &= \| B \|^2 \| d(u, u^*) \| . \end{aligned}$$

Since $\| B \|^2 < \frac{1}{\|A\|} \leq 1$, it follows that $d(u, u^*) = \theta$ i.e., $u = u^*$. Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.13, f and g have a unique common fixed point in X. □

The following corollary gives fixed point of Banach G-contraction in C^*-algebra valued b-metric spaces.

Corollary 3.3. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued b-metric space endowed with a graph G and the mapping $f : X \to X$ be such that*

$$d(fx, fy) \preceq B^* d(x, y) B \tag{3.4}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $B \in \mathbb{A}$ with $\| B \|^2 < \frac{1}{\|A\|}$. Suppose $(X, \mathbb{A}, d, G)$ has the following property:

$(*)'$ *If (x_n) is a sequence in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.*
Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the following property:

$(**)'$ *If x, y are fixed points of f in X, then $(x, y) \in E(\tilde{G})$.*

Proof. The proof can be obtained from Theorem 3.2 by considering $g = I$, the identity map on X. □

Corollary 3.4. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued b-metric space and the mappings f, $g : X \to X$ be such that (3.1) holds for all $x, y \in X$, where $B \in \mathbb{A}$ with $\| B \|^2 < \frac{1}{\|A\|}$. If $f(X) \subseteq g(X)$ and*

$g(X)$ *is a complete subspace of* X*, then* f *and* g *have a unique point of coincidence in* X*. Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point in* X*.*

Proof. The proof follows from Theorem 3.2 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$.

□

The following corollary is analogue of Banach Contraction Principle.

Corollary 3.5. *Let* $(X, \mathbb{A}, d)$ *be a complete* C^**-algebra valued* b*-metric space and the mapping* $f : X \to X$ *be such that (3.4) holds for all* $x, y \in X$*, where* $B \in \mathbb{A}$ *with* $\| B \|^2 < \frac{1}{\|A\|}$*. Then* f *has a unique fixed point* u *in* X *and* $f^n x \to u$ *for all* $x \in X$*.*

Proof. It follows from Theorem 3.2 by putting $G = G_0$ and $g = I$. □

Remark 3.6. *We observe that Banach contraction theorem in a complete metric space can be obtained from Corollary 3.5 by taking* $\mathbb{A} = \mathbb{C}, A = 1$*. Thus, Theorem 3.2 is a generalization of Banach contraction theorem in metric spaces to* C^**-algebra valued* b*-metric spaces.*

From Theorem 3.2, we obtain the following corollary concerning the fixed point of expansive mapping in C^*-algebra valued b-metric spaces.

Corollary 3.7. *Let* $(X, \mathbb{A}, d)$ *be a complete* C^**-algebra valued* b*-metric space and let* $g : X \to X$ *be an onto mapping satisfying*

$$B^* d(gx, gy) B \succeq d(x, y)$$

for all $x, y \in X$*, where* $B \in \mathbb{A}$ *with* $\| B \|^2 < \frac{1}{\|A\|}$*. Then* g *has a unique fixed point in* X*.*

Proof. The conclusion of the corollary follows from Theorem 3.2 by taking $G = G_0$ and $f = I$. □

Corollary 3.8. *Let* $(X, \mathbb{A}, d)$ *be a complete* C^**-algebra valued* b*-metric space endowed with a partial ordering* $\sqsubseteq$ *and the mapping* $f : X \to X$ *be such that (3.4) holds for all* $x, y \in X$ *with* $x \sqsubseteq y$ *or,* $y \sqsubseteq x$*, where* $B \in \mathbb{A}$ *and* $\| B \|^2 < \frac{1}{\|A\|}$*. Suppose* $(X, \mathbb{A}, d, \sqsubseteq)$ *has the following property:*

(†) *If* (x_n) *is a sequence in* X *such that* $x_n \to x$ *and* x_n*,* x_{n+1} *are comparable for all* $n \geq 1$*, then there exists a subsequence* (x_{n_i}) *of* (x_n) *such that* x_{n_i}*,* x *are comparable for all* $i \geq 1$*.*
If there exists $x_0 \in X$ *such that* $f^n x_0, f^m x_0$ *are comparable for* $m, n = 0, 1, 2, \cdots$*, then* f *has a fixed point in* X*. Moreover,* f *has a unique fixed point in* X *if the following property holds:*

(††) *If* x*,* y *are fixed points of* f *in* X*, then* x, y *are comparable.*

Proof. The proof can be obtained from Theorem 3.2 by taking $g = I$ and $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \sqsubseteq y \text{ or } y \sqsubseteq x\}$. □

Theorem 3.9. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space endowed with a graph* G *and the mappings* $f,\ g : X \to X$ *be such that*

$$d(fx, fy) \preceq B\,[d(fx, gy) + d(fy, gx)] \tag{3.5}$$

for all $x, y \in X$ *with* $(gx, gy) \in E(\tilde{G})$*, where* $B \in \mathbb{A}'_+$ *and* $\| BA \| < \frac{1}{\|A\|+1}$*. Suppose* $f(X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X *with the property* $(*)$*. Then* f *and* g *have a point of coincidence in* X *if* $C_{gf} \neq \emptyset$*. Moreover,* f *and* g *have a unique point of coincidence in* X *if the graph* G *has the property* $(**)$*. Furthermore, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point in* X*.*

Proof. It follows from condition (3.5) that $B(d(fx, gy) + d(fy, gx))$ is a positive element.

Suppose that $C_{gf} \neq \emptyset$. We choose an $x_0 \in C_{gf}$ and keep it fixed. We can construct a sequence (gx_n) such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$. Evidently, $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$.

For any $n \in \mathbb{N}$, we have by using condition (3.5) and Lemma 2.1(iii) that

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\preceq B[d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})] \\ &= B[d(fx_{n-1}, fx_{n-1}) + d(fx_n, fx_{n-2})] \\ &\preceq BA[d(fx_n, fx_{n-1}) + d(fx_{n-1}, fx_{n-2})] \\ &= BA\, d(gx_{n+1}, gx_n) + BA\, d(gx_n, gx_{n-1})] \end{aligned}$$

which implies that,

$$(I - BA)d(gx_n, gx_{n+1}) \preceq BAd(gx_n, gx_{n-1}). \tag{3.6}$$

Now, $A,\ B \in \mathbb{A}'_+$ implies that $BA \in \mathbb{A}'_+$. Since $\| BA \| < \frac{1}{2}$, by Lemma 2.1, it follows that $(I - BA)$ is invertible and $\| BA(I - BA)^{-1} \| = \| (I - BA)^{-1}BA \| < 1$. Moreover, by Lemma 2.1, $BA \preceq I$ i.e., $I - BA \succeq \theta$. Since $BA \in \mathbb{A}'_+$, we have $(I - BA) \in \mathbb{A}'_+$. Furthermore, $(I - BA)^{-1} \in \mathbb{A}'_+$. By using Lemma 2.1(iv), it follows from (3.6) that

$$d(gx_n, gx_{n+1}) \preceq (I - BA)^{-1}BA\, d(gx_n, gx_{n-1}) = td(gx_{n-1}, gx_n), \tag{3.7}$$

where $t = (I - BA)^{-1}BA \in \mathbb{A}'_+$.

By repeated use of condition (3.7), we get

$$d(gx_n, gx_{n+1}) \preceq t^n d(gx_0, gx_1) = t^n B_0, \tag{3.8}$$

for all $n \in \mathbb{N}$, where $B_0 = d(gx_0, gx_1) \in \mathbb{A}_+$.

We now prove that if $\| BA \| < \frac{1}{\|A\|+1}$, then $\| t \| < \frac{1}{\|A\|}$.
We have,

$$\begin{aligned}
\| t \| &= \| (I - BA)^{-1} BA \| \\
&\leq \| (I - BA)^{-1} \| \| BA \| \\
&\leq \frac{1}{1 - \| BA \|} \| BA \| \\
&< \frac{1}{\| A \|}, \text{since } \| BA \| < \frac{1}{\| A \| + 1}.
\end{aligned}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have by using condition (3.8) that

$$\begin{aligned}
d(gx_n, gx_m) &\preceq A[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\
&\preceq Ad(gx_n, gx_{n+1}) + A^2 d(gx_{n+1}, gx_{n+2}) + \cdots \\
&\quad + A^{m-n-1} d(gx_{m-2}, gx_{m-1}) + A^{m-n-1} d(gx_{m-1}, gx_m) \\
&\preceq At^n B_0 + A^2 t^{n+1} B_0 + A^3 t^{n+2} B_0 + \cdots \\
&\quad + A^{m-n-1} t^{m-2} B_0 + A^{m-n-1} t^{m-1} B_0 \\
&\preceq \sum_{k=1}^{m-n} A^k t^{n+k-1} B_0, \text{ since } A \succeq I \text{ and } A \in \mathbb{A}'_+ \\
&\preceq \sum_{k=1}^{m-n} \| A^k t^{n+k-1} B_0 \| I \\
&\preceq \| B_0 \| \| A \| \| t \|^n \sum_{k=1}^{m-n} (\| A \| \| t \|)^{k-1} I \\
&\preceq \| B_0 \| \| A \| \| t \|^n \frac{1}{1 - \| A \| \| t \|} I \\
&\to \theta \text{ as } n \to \infty.
\end{aligned}$$

Therefore, (gx_n) is a Cauchy sequence with respect to $\mathbb{A}$. As $g(X)$ is complete, there exists an $u \in g(X)$ such that $\lim_{n\to\infty} gx_n = u = gv$ for some $v \in X$. By property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$.

Using condition (3.5), we have

$$\begin{aligned}
d(fv, gv) &\preceq A[d(fv, fx_{n_i}) + d(fx_{n_i}, gv)] \\
&\preceq AB[d(fv, gx_{n_i}) + d(fx_{n_i}, gv)] + Ad(gx_{n_i+1}, gv) \\
&\preceq ABA[d(fv, gv) + d(gv, gx_{n_i})] + ABd(gx_{n_i+1}, gv) + Ad(gx_{n_i+1}, gv)
\end{aligned}$$

which implies that,

$$(I - BA^2) d(fv, gv) \preceq BA^2 d(gv, gx_{n_i}) + ABd(gx_{n_i+1}, gv) + Ad(gx_{n_i+1}, gv).$$

Since $\| BA^2 \| < \frac{\|A\|}{\|A\|+1} < 1$, we have $(I - BA^2)^{-1}$ exists. By using Lemma 2.1, it follows that

$$\begin{aligned} d(fv, gv) &\preceq (I - BA^2)^{-1} BA^2 d(gv, gx_{n_i}) + (I - BA^2)^{-1} ABd(gx_{n_i+1}, gv) \\ &\quad +(I - BA^2)^{-1} Ad(gx_{n_i+1}, gv) \\ &\to \theta \text{ as } i \to \infty. \end{aligned}$$

This implies that $d(fv, gv) = \theta$ i.e., $fv = gv = u$ and hence u is a point of coincidence of f and g.

Finally, to prove the uniqueness of point of coincidence, suppose that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property $(**)$, we have $(u, u^*) \in E(\tilde{G})$. Then,

$$\begin{aligned} d(u, u^*) &= d(fv, fx) \\ &\preceq B[d(fv, gx) + d(fx, gv)] \\ &= B[d(u, u^*) + d(u, u^*)] \\ &\preceq AB[d(u, u^*) + d(u, u^*)] \end{aligned}$$

which implies that,

$$d(u, u^*) \preceq (I - AB)^{-1} AB\, d(u, u^*).$$

So, it must be the case that

$$\begin{aligned} \| d(u, u^*) \| &\leq \| (I - AB)^{-1} AB\, d(u, u^*) \| \\ &\leq \| (I - AB)^{-1} AB \| \| d(u, u^*) \| . \end{aligned}$$

Since $\| (I - AB)^{-1} AB \| < 1$, we have $\| d(u, u^*) \| = 0$ i.e., $u = u^*$. Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.13, f and g have a unique common fixed point in X. □

Corollary 3.10. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued b-metric space endowed with a graph G and the mapping $f : X \to X$ be such that*

$$d(fx, fy) \preceq B\,[d(fx, y) + d(fy, x)] \tag{3.9}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $B \in \mathbb{A}'_+$ and $\| BA \| < \frac{1}{\|A\|+1}$. Suppose $(X, \mathbb{A}, d, G)$ has the property $(\acute{*})$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the property $(\acute{**})$.

Proof. The proof can be obtained from Theorem 3.9 by putting $g = I$. □

Corollary 3.11. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space and the mappings* $f, g : X \to X$ *be such that* (3.5) *holds for all* $x, y \in X$*, where* $B \in \mathbb{A}_+^{'}$ *and* $\| BA \| < \frac{1}{\|A\|+1}$*. If* $f(X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X*, then* f *and* g *have a unique point of coincidence in* X*. Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point in* X*.*

Proof. The proof can be obtained from Theorem 3.9 by taking $G = G_0$. □

Corollary 3.12. *Let* $(X, \mathbb{A}, d)$ *be a complete* C^**-algebra valued* b*-metric space and the mapping* $f : X \to X$ *be such that* (3.9) *holds for all* $x, y \in X$*, where* $B \in \mathbb{A}_+^{'}$ *with* $\| BA \| < \frac{1}{\|A\|+1}$*. Then* f *has a unique fixed point in* X*.*

Proof. The proof follows from Theorem 3.9 by taking $G = G_0$ and $g = I$.

□

Remark 3.13. *We observe that Brian Fisher's theorem in a complete metric space can be obtained from Corollary 3.12 by taking* $\mathbb{A} = \mathbb{C}, A = 1$*. Thus, Theorem 3.9 is a generalization of Brian Fisher's theorem in metric spaces to* C^**-algebra valued* b*-metric spaces.*

Corollary 3.14. *Let* $(X, \mathbb{A}, d)$ *be a complete* C^**-algebra valued* b*-metric space endowed with a partial ordering* $\sqsubseteq$ *and the mapping* $f : X \to X$ *be such that* (3.9) *holds for all* $x, y \in X$ *with* $x \sqsubseteq y$ *or,* $y \sqsubseteq x$*, where* $B \in \mathbb{A}_+^{'}$ *with* $\| BA \| < \frac{1}{\|A\|+1}$*. Suppose* $(X, \mathbb{A}, d, \sqsubseteq)$ *has the property* $(\dagger)$*. If there exists* $x_0 \in X$ *such that* $f^n x_0, f^m x_0$ *are comparable for* $m, n = 0, 1, 2, \cdots$*, then* f *has a fixed point in* X*. Moreover,* f *has a unique fixed point in* X *if the property* $(\dagger\dagger)$ *holds.*

Proof. The proof can be obtained from Theorem 3.9 by taking $G = G_2$ and $g = I$. □

Theorem 3.15. *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra valued* b*-metric space endowed with a graph* G *and the mappings* $f, g : X \to X$ *be such that*

$$d(fx, fy) \preceq B\,[d(fx, gx) + d(fy, gy)] \tag{3.10}$$

for all $x, y \in X$ *with* $(gx, gy) \in E(\tilde{G})$*, where* $B \in \mathbb{A}_+^{'}$ *and* $\| B \| < \frac{1}{\|A\|+1}$*. Suppose* $f(X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X *with the property* $(*)$*. Then* f *and* g *have a point of coincidence in* X *if* $C_{gf} \neq \emptyset$*. Moreover,* f *and* g *have a unique point of coincidence in* X *if the graph* G *has the property* $(**)$*. Furthermore, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point in* X*.*

Proof. We observe that $B(d(fx, gx) + d(fy, gy))$ is a positive element.

Suppose that $C_{gf} \neq \emptyset$. We choose an $x_0 \in C_{gf}$ and keep it fixed. We can construct a sequence (gx_n) such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$. Evidently, $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$.

For any $n \in \mathbb{N}$, we have by using condition (3.10) that

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\preceq B[d(fx_{n-1}, gx_{n-1}) + d(fx_n, gx_n)] \\ &= B\, d(gx_n, gx_{n-1}) + B\, d(gx_n, gx_{n+1}) \end{aligned}$$

which implies that,

$$(I - B)d(gx_n, gx_{n+1}) \preceq Bd(gx_n, gx_{n-1}). \tag{3.11}$$

Since $B \in \mathbb{A}'_+$ and $\| B \| < \frac{1}{2}$, by Lemma 2.1, it follows that $B \preceq I$ and $(I - B)$ is invertible with $\| B(I-B)^{-1} \| = \| (I-B)^{-1}B \| < 1$. Furthermore, $(I-B)$, $(I-B)^{-1} \in \mathbb{A}'_+$ and so, $(I-B)^{-1}B \in \mathbb{A}'_+$. Again, by using Lemma 2.1(iv), it follows from condition (3.11) that

$$d(gx_n, gx_{n+1}) \preceq (I - B)^{-1}B\, d(gx_n, gx_{n-1}) = td(gx_{n-1}, gx_n), \tag{3.12}$$

where $t = (I - B)^{-1}B \in \mathbb{A}'_+$.

By repeated use of condition (3.12), we get

$$d(gx_n, gx_{n+1}) \preceq t^n d(gx_0, gx_1) = t^n B_0, \tag{3.13}$$

for all $n \in \mathbb{N}$, where $B_0 = d(gx_0, gx_1) \in \mathbb{A}_+$.

We now prove that if $\| B \| < \frac{1}{\|A\|+1}$, then $\| t \| < \frac{1}{\|A\|}$.
We have,

$$\begin{aligned} \| t \| &= \| (I - B)^{-1}B \| \\ &\leq \| (I - B)^{-1} \| \| B \| \\ &\leq \frac{1}{1 - \| B \|} \| B \| \\ &< \frac{1}{\| A \|}, \text{since } \| B \| < \frac{1}{\| A \| + 1}. \end{aligned}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have by using condition (3.13) that

$$\begin{aligned}
d(gx_n, gx_m) &\preceq A[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\
&\preceq Ad(gx_n, gx_{n+1}) + A^2 d(gx_{n+1}, gx_{n+2}) + \cdots \\
&\quad + A^{m-n-1} d(gx_{m-2}, gx_{m-1}) + A^{m-n-1} d(gx_{m-1}, gx_m) \\
&\preceq At^n B_0 + A^2 t^{n+1} B_0 + A^3 t^{n+2} B_0 + \cdots \\
&\quad + A^{m-n-1} t^{m-2} B_0 + A^{m-n-1} t^{m-1} B_0 \\
&\preceq \sum_{k=1}^{m-n} A^k t^{n+k-1} B_0, \text{ since } A \succeq I \text{ and } A \in \mathbb{A}'_+ \\
&\preceq \sum_{k=1}^{m-n} \| A^k t^{n+k-1} B_0 \| I \\
&\preceq \| B_0 \| \| A \| \| t \|^n \sum_{k=1}^{m-n} (\| A \| \| t \|)^{k-1} I \\
&\preceq \| B_0 \| \| A \| \| t \|^n \frac{1}{1 - \| A \| \| t \|} I \\
&\to \theta \text{ as } n \to \infty.
\end{aligned}$$

Therefore, (gx_n) is a Cauchy sequence with respect to $\mathbb{A}$. By completeness of $g(X)$, there exists an $u \in g(X)$ such that $\lim_{n\to\infty} gx_n = u = gv$ for some $v \in X$. By property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$.

Using condition (3.10), we have

$$\begin{aligned}
d(fv, gv) &\preceq A[d(fv, fx_{n_i}) + d(fx_{n_i}, gv)] \\
&\preceq AB[d(fv, gv) + d(fx_{n_i}, gx_{n_i})] + Ad(gx_{n_i+1}, gv)
\end{aligned}$$

which implies that,

$$(I - AB)d(fv, gv) \preceq ABd(gx_{n_i+1}, gx_{n_i}) + Ad(gx_{n_i+1}, gv).$$

Since $\| AB \| < \frac{\|A\|}{\|A\|+1} < 1$, we have $(I - AB)^{-1}$ exists and $(I - AB) \in \mathbb{A}'_+$. By using Lemma 2.1, it follows that

$$d(fv, gv) \preceq (I - AB)^{-1} ABd(gx_{n_i+1}, gx_{n_i}) + (I - AB)^{-1} Ad(gx_{n_i+1}, gv).$$

Then,

$$\begin{aligned}
\| d(fv, gv) \| &\leq \| (I - AB)^{-1} AB \| \| d(gx_{n_i+1}, gx_{n_i}) \| \\
&\quad + \| (I - AB)^{-1} A \| \| d(gx_{n_i+1}, gv) \| \\
&\leq \| (I - AB)^{-1} AB \| \| t \|^{n_i} \| B_0 \| \\
&\quad + \| (I - AB)^{-1} A \| \| d(gx_{n_i+1}, gv) \| \\
&\to 0 \text{ as } i \to \infty.
\end{aligned}$$

This implies that $d(fv, gv) = \theta$ i.e., $fv = gv = u$ and hence u is a point of coincidence of f and g.

Finally, to prove the uniqueness of point of coincidence, suppose that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property $(**)$, we have $(u, u^*) \in E(\tilde{G})$. Then,

$$\begin{aligned} d(u, u^*) &= d(fv, fx) \\ &\preceq B[d(fv, gv) + d(fx, gx)] \\ &= \theta \end{aligned}$$

which implies that, $d(u, u^*) = \theta$ i.e., $u = u^*$. Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.13, f and g have a unique common fixed point in X. □

Corollary 3.16. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued b-metric space endowed with a graph G and the mapping $f : X \to X$ be such that*

$$d(fx, fy) \preceq B\,[d(fx, x) + d(fy, y)] \tag{3.14}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $B \in \mathbb{A}_+'$ and $\| B \| < \frac{1}{\|A\|+1}$. Suppose $(X, \mathbb{A}, d, G)$ has the property $(\acute{*})$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the property $(*\acute{*})$.

Proof. The proof can be obtained from Theorem 3.15 by putting $g = I$. □

Corollary 3.17. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued b-metric space and the mappings $f, g : X \to X$ be such that (3.10) holds for all $x, y \in X$, where $B \in \mathbb{A}_+'$ and $\| B \| < \frac{1}{\|A\|+1}$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

Proof. The proof can be obtained from Theorem 3.15 by taking $G = G_0$. □

Corollary 3.18. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued b-metric space and the mapping $f : X \to X$ be such that (3.14) holds for all $x, y \in X$, where $B \in \mathbb{A}_+'$ with $\| B \| < \frac{1}{\|A\|+1}$. Then f has a unique fixed point in X.*

Proof. The proof follows from Theorem 3.15 by taking $G = G_0$ and $g = I$.

□

Remark 3.19. *We observe that Kannan's fixed point theorem in a complete metric space can be obtained from Corollary 3.18 by taking* $\mathbb{A} = \mathbb{C}, A = 1$. *Thus, Theorem 3.15 is a generalization of Kannan's fixed point theorem in metric spaces to* C^**-algebra valued* b*-metric spaces.*

Corollary 3.20. *Let* $(X, \mathbb{A}, d)$ *be a complete* C^**-algebra valued* b*-metric space endowed with a partial ordering* $\sqsubseteq$ *and the mapping* $f : X \to X$ *be such that (3.14) holds for all* $x, y \in X$ *with* $x \sqsubseteq y$ *or,* $y \sqsubseteq x$*, where* $B \in \mathbb{A}'_+$ *with* $\| B \| < \frac{1}{\|A\|+1}$. *Suppose* $(X, \mathbb{A}, d, \sqsubseteq)$ *has the property* $(\dagger)$. *If there exists* $x_0 \in X$ *such that* $f^n x_0, f^m x_0$ *are comparable for* $m, n = 0, 1, 2, \cdots$*, then* f *has a fixed point in* X. *Moreover,* f *has a unique fixed point in* X *if the property* $(\dagger\dagger)$ *holds.*

Proof. The proof can be obtained from Theorem 3.15 by taking $G = G_2$ and $g = I$. □

We furnish some examples in favour of our results.

Example 3.21. *Let* $X = \mathbb{R}$ *and* $B(H)$ *be the set of all bounded linear operators on a Hilbert space* H. *Define* $d : X \times X \to B(H)$ *by* $d(x, y) = | x - y |^3 \ I$ *for all* $x, y \in X$*, where* I *is the identity operator on* H. *Then* $(X, B(H), d)$ *is a complete* C^**-algebra valued* b*-metric space with the coefficient* $A = 4I$. *Let* G *be a digraph such that* $V(G) = X$ *and* $E(G) = \Delta \cup \{(\frac{1}{n}, 0) : n = 1, 2, 3 \cdots\}$.

Let $f, g : X \to X$ *be defined by*

$$\begin{aligned} fx &= \frac{x}{5}, \ \text{if } x \neq \frac{4}{5} \\ &= 1, \ \text{if } x = \frac{4}{5} \end{aligned}$$

and $gx = 2x$ *for all* $x \in X$. *Obviously,* $f(X) \subseteq g(X) = X$.

If $x = 0, \ y = \frac{1}{2n}, n = 1, 2, 3, \cdots$*, then* $gx = 0, \ gy = \frac{1}{n}$ *and so* $(gx, gy) \in E(\tilde{G})$.

For $x = 0, \ y = \frac{1}{2n}$*, we have*

$$\begin{aligned} d(fx, fy) &= d\left(0, \frac{1}{10n}\right) \\ &= \frac{1}{10^3.n^3} \, I \\ &\prec \frac{1}{25n^3} \, I \\ &= \frac{1}{25} \, d(gx, gy) \\ &= B^* \, d(gx, gy) B, \end{aligned}$$

where $B = \frac{1}{5} I \in B(H)$.

Therefore,

$$d(fx, fy) \preceq B^* d(gx, gy)B$$

for all $x, y \in X$ *with* $(gx, gy) \in E(\tilde{G})$, *where* $B \in B(H)$ *and* $\| B \|^2 < \frac{1}{\|A\|}$. *We can verify that* $0 \in C_{gf}$. *In fact,* $gx_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$ *gives that* $gx_1 = f0 = 0 \Rightarrow x_1 = 0$ *and so* $gx_2 = fx_1 = 0 \Rightarrow x_2 = 0$. *Proceeding in this way, we get* $gx_n = 0$ *for* $n = 0, 1, 2, \cdots$ *and hence* $(gx_n, gx_m) = (0, 0) \in E(\tilde{G})$ *for* $m, n = 0, 1, 2, \cdots$.

Also, any sequence (gx_n) *with the property* $(gx_n, gx_{n+1}) \in E(\tilde{G})$ *must be either a constant sequence or a sequence of the following form*

$$\begin{aligned} gx_n &= 0, \text{ if } n \text{ is odd} \\ &= \frac{1}{n}, \text{ if } n \text{ is even} \end{aligned}$$

where the words 'odd' and 'even' are interchangeable. Consequently it follows that property $(*)$ *holds. Furthermore,* f *and* g *are weakly compatible. Thus, we have all the conditions of Theorem 3.2 and* 0 *is the unique common fixed point of* f *and* g *in* X.

Remark 3.22. *It is worth mentioning that weak compatibility condition in Theorem 3.2 cannot be relaxed. In Example 3.21, if we take* $gx = 2x - 9$ *for all* $x \in X$ *instead of* $gx = 2x$, *then* $5 \in C_{gf}$ *and* $f(5) = g(5) = 1$ *but* $g(f(5)) \neq f(g(5))$ *i.e.,* f *and* g *are not weakly compatible. However, all other conditions of Theorem 3.2 are satisfied. We observe that* 1 *is the unique point of coincidence of* f *and* g *without being any common fixed point.*

Remark 3.23. *In Example 3.21,* f *is a* C^**-algebra valued* G*-contraction but it is not a* C^**-algebra valued contraction. In fact, for* $x = \frac{4}{5}$, $y = 0$, *we have*

$$\begin{aligned} d(fx, fy) &= d(1, 0) \\ &= I \\ &= \frac{125}{64} \cdot \frac{64}{125} I \\ &= \frac{125}{64} d(x, y) \\ &\succ B^* d(x, y) B, \end{aligned}$$

for any $B \in B(H)$ *with* $\| B \|^2 < \frac{1}{\|A\|}$. *This implies that* f *is not a* C^**-algebra valued contraction.*

The following example shows that property $(*)$ is necessary in Theorem 3.2.

Example 3.24. *Let* $X = [0, \infty)$ *and* $B(H)$ *be the set of all bounded linear operators on a Hilbert space* H. *Define* $d : X \times X \to B(H)$ *by* $d(x, y) = | x - y |^3 I$ *for all* $x, y \in X$, *where* I *is the identity operator on* H. *Then* $(X, B(H), d)$ *is a complete* C^**-algebra valued* b*-metric space with the coefficient*

$A = 4I$. *Let* G *be a digraph such that* $V(G) = X$ *and* $E(G) = \Delta \cup \{(x, y) : (x, y) \in (0, 1] \times (0, 1],\ x \geq y\}$.

Let $f,\ g : X \to X$ *be defined by*

$$\begin{aligned} fx &= \frac{x}{6}, \text{ if } x \neq 0 \\ &= 1, \text{ if } x = 0 \end{aligned}$$

and $gx = \frac{x}{2}$ *for all* $x \in X$. *Obviously,* $f(X) \subseteq g(X) = X$.

For $x, y \in X$ *with* $(gx, gy) \in E(\tilde{G})$, *we have*

$$\begin{aligned} d(fx, fy) &= \frac{1}{27} d(gx, gy) \\ &\preceq \frac{1}{9} d(gx, gy) \\ &= B^* \, d(gx, gy) \, B, \end{aligned}$$

where $B = \frac{1}{3} I \in B(H)$ *with* $\| B \|^2 < \frac{1}{\|A\|}$.

We see that f *and* g *have no point of coincidence in* X. *We now verify that the property* $(*)$ *does not hold. In fact,* (gx_n) *is a sequence in* X *with* $gx_n \to 0$ *and* $(gx_n, gx_{n+1}) \in E(\tilde{G})$ *for all* $n \in \mathbb{N}$ *where* $x_n = \frac{2}{n}$. *But there exists no subsequence* (gx_{n_i}) *of* (gx_n) *such that* $(gx_{n_i}, 0) \in E(\tilde{G})$.

Example 3.25. *Let* $X = \mathbb{R}$ *and* $B(H)$ *be the set of all bounded linear operators on a Hilbert space* H. *Choose a positive operator* $T \in B(H)$. *Define* $d : X \times X \to B(H)$ *by* $d(x, y) = | x - y |^5 \, T$ *for all* $x,\ y \in X$. *Then* $(X, B(H), d)$ *is a complete* C^**-algebra valued* b*-metric space with the coefficient* $A = 16I$. *Let* $f,\ g : X \to X$ *be defined by*

$$\begin{aligned} fx &= 2, \text{ if } x \neq 5 \\ &= 3, \text{ if } x = 5 \end{aligned}$$

and $gx = 3x - 4$ *for all* $x \in X$. *Obviously,* $f(X) \subseteq g(X) = X$.
Let G *be a digraph such that* $V(G) = X$ *and* $E(G) = \Delta \cup \{(2, 3),\ (3, 5)\}$. *If* $x = 2,\ y = \frac{7}{3}$, *then* $gx = 2,\ gy = 3$ *and so* $(gx, gy) \in E(\tilde{G})$.
Again, if $x = \frac{7}{3},\ y = 3$, *then* $gx = 3,\ gy = 5$ *and so* $(gx, gy) \in E(\tilde{G})$.
It is easy to verify that condition (3.5) of Theorem 3.9 holds for all $x, y \in X$ *with* $(gx, gy) \in E(\tilde{G})$. *Furthermore,* $2 \in C_{gf}$ *i.e.,* $C_{gf} \neq \emptyset$, f *and* g *are weakly compatible, and* $(X, B(H), d, G)$ *has the property* $(*)$. *Thus, all the conditions of Theorem 3.9 are satisfied and* 2 *is the unique common fixed point of* f *and* g *in* X.

Remark 3.26. *It is observed that in Example 3.25,* f *is not a Fisher* G*-contraction. In fact,* *for* $x = 3,\ y = 5$, *we have*

$$\begin{aligned} B\left[d(fx,y)+d(fy,x)\right] &= B\left[d(2,5)+d(3,3)\right] \\ &= 243BT \\ &= \frac{243}{16}BAT \\ &= \frac{243}{16\times 17}17BAT \\ &\prec T \\ &= d(fx,fy), \end{aligned}$$

for any $B \in B(H)'_+$ *with* $\| BA \| < \frac{1}{\|A\|+1}$. *This implies that* f *is not a Fisher* G*-contraction.*

The following example supports our Theorem 3.15.

Example 3.27. *Let* $X = [0,\infty)$ *and* $B(H)$ *be the set of all bounded linear operators on a Hilbert space* H. *Choose a positive operator* $T \in B(H)$. *Define* $d : X \times X \to B(H)$ *by* $d(x,y) = | x-y |^2\, T$ *for all* $x,\ y \in X$. *Then* $(X, B(H), d)$ *is a complete* C^**-algebra valued* b*-metric space with the coefficient* $A = 2I$. *Let* G *be a digraph such that* $V(G) = X$ *and* $E(G) = \Delta \cup \{(4^t x, 4^t(x+1)) : x \in X \text{ with } x \geq 2,\ t = 0,\ 1,\ 2,\ \cdots\}$.

Let $f,\ g : X \to X$ *be defined by* $fx = 4x$ *and* $gx = 16x$ *for all* $x \in X$. *Clearly,* $f(X) = g(X) = X$.

If $x = 4^{t-2}z,\ y = 4^{t-2}(z+1)$, *then* $gx = 4^t z,\ gy = 4^t(z+1)$ *and so* $(gx, gy) \in E(\tilde{G})$ *for all* $z \geq 2$.

For $x = 4^{t-2}z,\ y = 4^{t-2}(z+1),\ z \geq 2$ *with* $B = \frac{1}{117}I$, *we have*

$$\begin{aligned} d(fx,fy) &= d\left(4^{t-1}z, 4^{t-1}(z+1)\right) \\ &= 4^{2t-2}T \\ &\preceq \frac{1}{117}\,4^{2t-2}(18z^2+18z+9)T \\ &= \frac{1}{117}\,\left[d\left(4^{t-1}z, 4^t z\right) + d\left(4^{t-1}(z+1), 4^t(z+1)\right)\right] \\ &= B\left[d(fx,gx)+d(fy,gy)\right]. \end{aligned}$$

Thus, condition (3.10) is satisfied for all $x, y \in X$ *with* $(gx, gy) \in E(\tilde{G})$. *It is easy to verify that* $0 \in C_{gf}$. *Also, any sequence* (gx_n) *with* $gx_n \to x$ *and* $(gx_n, gx_{n+1}) \in E(\tilde{G})$ *must be a constant sequence and hence property* $(*)$ *holds. Furthermore,* f *and* g *are weakly compatible. Thus, we have all the conditions of Theorem 3.15 and* 0 *is the unique common fixed point of* f *and* g *in* X.

Remark 3.28. *It is easy to observe that in Example 3.27,* f *is a* C^**-algebra valued* G*-Kannan operator with* $B = \frac{16}{117}I$. *But* f *is not a* C^**-algebra valued Kannan operator because, if* $x = 4,\ y = 0$, *then for any arbitrary* $B \in B(H)_+^{'}$ *with* $\| B \| < \frac{1}{\|A\|+1} = \frac{1}{3}$*(which implies* $3B \prec I$*), we have*

$$\begin{aligned} B\left[d(fx,x)+d(fy,y)\right] &= B\left[d(f4,4)+d(f0,0)\right] \\ &= 144BT \\ &= \frac{144}{3\times 256}(3B)(256T) \\ &\prec 256T \\ &= d(fx,fy). \end{aligned}$$

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On rigid Hermitean lattices, II

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ABSTRACT

We study the indexed Hermitean lattice of type 0 generated by a single element $\mathfrak{a}$ subjected to the relation $\mathfrak{a} \leq \mathfrak{b}^{\perp} \wedge \mathfrak{b}\mathfrak{b}^{\perp} = \mathfrak{0}$. We prove that it is finite, provided that two crucial indices are finite. We show that index relations imply algebraic relations and describe the lattice by means of its subdirectly irreducible factors. We finally use the results to confirm a conjecture appeared in 2000.

RESUMEN

Estudiamos el reticulado Hermitiano finito indexado de tipo 0 generado por un solo elemento $\mathfrak{a}$ sujeto a la relación $\mathfrak{a} \leq \mathfrak{b}^{\perp} \wedge \mathfrak{b}\mathfrak{b}^{\perp} = \mathfrak{0}$. Probamos que es finito, suponiendo que dos índices cruciales son finitos. Mostramos que las relaciones de índices implican relaciones algebraicas y describimos el reticulado a travs de sus factores subdirectamente irreductibles. Finalmente, usamos nuestros resultados para confirmar una conjetura aparecida el ao 2000.

Keywords and Phrases: Lattices, semilattices, modular lattices, Hermitean lattices, orthogonal geometry.

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1 Introduction

The importance of lattices in infinite-dimensional orthogonal geometry was brought to attention by the pioneering work of Herbert Gross (1936-1989): see in particular [G1] and [G2]. All examples treated in origin are sublattices of some $\mathcal{L}(E)$, the subspace-lattice of an $\aleph_0$-dimensional vector space E over an appropriate division ring k, together with the orthogonal operation induced by a Hermitean form ϕ (i.e. $X \mapsto X^\perp := \{y \in E \mid \phi(y,x) = 0 \ \forall x \in X\}$) and were used to study geometric invariants, for instance dimension of quotient spaces or intersections with the subspace E^* of trace-valued vectors in E. The fact that $E^* \neq E$ only if char(k)= 2 was also playing some role. After some time of concrete investigations with subspace lattices (see e.g. [M1]), the natural idea to insert all considerations into an abstract setting gave rise to the following definitions (cf. [KKW], Ch. IV):

A *Hermitean lattice* (HL for short) is an algebra $(L,\ 0,\ 1,\ \cdot\ ,\ +\ ,\ ^\perp,\ b)$ of type $\langle 0,0,2,2,1,0\rangle$ such that

i) $(L, 0, 1, \cdot\ , +)$ is a modular lattice with universal bounds 0 , 1;

ii) $^\perp : L \to L$ is a unary operation with $1^\perp = 0$ and

$$x \leq (x^\perp y)^\perp \qquad \forall\ x,\ y \in L \tag{1.1}$$

iii) $b \in L$ is a nullary operation with

$$xx^\perp \leq b \qquad \forall\ x \in L.$$

In case b is explicitly not trivial (i.e. $b \neq 1$), the modular law in i) is sometimes replaced by the stronger Fano identity

$$(w+v)(y+z) \leq (w+y)(v+z) + (w+z)(v+y).$$

If we drop the operation "+", then we obtain the notion of *Hermitean semilattice* (HSL for short). In the present paper we will endow HL L with a so-called *index function of type* 0 (IF for short), i.e a function δ from the set of quotients of L into the set of cardinals $\leq \aleph_o$, with the following properties:

$$\delta(x/y) \geq \delta(xz/yz), \tag{1.2}$$

$$\delta(x/y) \geq \delta(x+z/y+z), \tag{1.3}$$

$$\delta(x/y) \geq \delta(y^\perp/x^\perp), \tag{1.4}$$

$$\delta(x/y) + \delta(y/z) = \delta(x/z), \tag{1.5}$$

$$\delta(x/y) = 0 \Longleftrightarrow x = y. \tag{1.6}$$

We will speak about *indexed Hermitean lattices* (IHL). By dropping (1.3), we obtain the notion of *indexed Hermitean semilattices* (IHSL).

A major task of the theory of H(S)L consists in describing the free objects $S[a]$ and $F[a]$, generated by a single element a in the varieties of HSL and HL, respectively. Since such objects are infinite, a more realistic project consists in studying appropriate presentations under (index) relations suggested by geometrical choice (see [G1], [M1], [M2] and also the bibliography in [KKW] for many known examples). One of these options is given by the relation $a \leq b^{\perp}$, which was introduced in [DM3] and gave rise to the concept of *rigid* $H(S)L$. Here we continue such investigation and consider rigid HL with the (somewhat complementary) property $bb^{\perp} = 0$. In the above work the HSL $S := S[a; a \leq b^{\perp} \;\wedge\; bb^{\perp} = 0]$ was already computed, but here we briefly reproduce its description, without proofs, to make this paper more self-contained. Since the corresponding HL is most probably infinite, we work with an IF δ and start our research with the following hypothesis:

$$\delta(a^{\perp}/d_1^{\perp}) < \aleph_0 \quad \wedge \quad \delta(b^{\perp}/c_1^{\perp}) < \aleph_0, \tag{1.7}$$

where

$$c_1 := d^{\perp}e^{\perp},\ d_1 := c^{\perp}e^{\perp},\ \text{ and } c := a^{\perp}e^{\perp},\ d := b^{\perp}e^{\perp},\ e := a^{\perp}b^{\perp}. \tag{1.8}$$

The algebraic relations forced by the index condition (1.7) are given below in (4.4), Theorem 4.1, and have the following important consequence:

$F := F[a; (1) \wedge (4.4)]$ is finite and has 23 subdirectly irreducible factors.

The factors are listed in Tables II, III and IV, Section 7, together with the associated critical quotients.

We will finally use these results to confirm conjecture 2 in [M2] and to suggest an application in orthogonal geometry.

We conclude this introduction with two more remarks:

- Without (1.7), F would be most probably infinite (cf. also the arguments given in [M2]). Thus we can recognize the importance of the intervals $[d_1^{\perp}, a^{\perp}]$ and $[c_1^{\perp}, b^{\perp}]$ in the above HL. Moreover, it is easy to prove that (1.7) is a weakening of the condition $\delta(1/b) < \aleph_0$, which has a natural interpretation in orthogonal geometry (cf. Section 6) and was used as hypothesis in many precedent investigations.

- S appeared naturally as substructure in other works (see [M2] and [DM2]). This important fact was an additional motivation for the present study.

2 Preliminaries

Lemma 2.1. *Any countable HL is indexable.*

Proof. Each HL admits the trivial IF, defined to have value $\aleph_0$ on each nontrivial quotient. □

Lemma 2.2. *The class of IHL is closed with respect to subalgebras, homomorphic images and countable products.*

Proof. This is just a slight generalization of Proposition 21 in [KKW], Ch. IV. □

Clearly, the existence of a nontrivial IF on some HL is controlled by prime quotients. Our lattices do not present difficulties such as described in [S] because the subdirectly irreducible factors are finite and known.

The next result represents the key to obtain algebraic relations from index relations (cf. proof of Theorem 4.1):

Lemma 2.3. *Let* u/v *be any finite quotient of an IHL. If* $v = v^{\perp\perp}$ *then* $u = u^{\perp\perp}$.

Proof. $\delta(u/v) \geq \delta(v^{\perp}/u^{\perp}) \geq \delta(u^{\perp\perp}/v^{\perp\perp}) = \delta(u^{\perp\perp}/v) \geq \delta(u/v)$. □

For the sake of precision we give also the following

Definition 2.4. $S[a : a \leq b^{\perp}]$ *is the initial object of the class of rigid HSL. Similarly,* $F[a : a \leq b^{\perp}]$ *is the initial object of the class of rigid HL.*

Thus any *rigid H(S)L* is a homomorphic image of the initial object.

We could have been even more precise by saying that this is in fact the concept of a *1-generated rigid H(S)L*, a special case of n-*generated rigid H(S)L*, but of course, for the moment, all this is not necessary.

We conclude this section by remarking that the axiom (1.1) is equivalent with the following conditions:

(i) $x \leq x^{\perp\perp}$; (ii) $x \leq y \Rightarrow y^{\perp} \leq x^{\perp}$.

This may facilitate some computations.

3 Description of S

Theorem 3.1. *The HSL* S *has 18 elements and its structure is given by the diagram depicted in Figure 1 (see Section 7).*

Proof. See [DM3]. □

Since we are interested in indices, we consider an IF δ on S and put

$$\begin{array}{llll}\beta_1 := \delta(a/0), & \beta_2 := \delta(b/0), & \beta_3 := \delta(e/0), & \beta_4 := \delta(c/c_1),\\ \beta_5 := \delta(a^{\perp\perp}/a), & \beta_6 := \delta(b^{\perp\perp}/b), & \beta_7 := \delta(c_1/b^{\perp\perp}), & \beta_8 := \delta(d_1/a^{\perp\perp}).\end{array} \tag{3.1}$$

Theorem 3.2. *(Relations among indices in S)*

(i) All other indices of S are determined by $\beta_1, \cdots, \beta_8$ as is shown in Figure 2, Section 7.

(ii) In particular, the following relations hold:

a) $\beta_4 \neq 0$ implies $\beta_1 = \beta_2 = \aleph_0$;

b) $\beta_5 \neq 0$ implies $\beta_1 = \aleph_0$;

c) $\beta_6 \neq 0$ implies $\beta_2 = \aleph_0$;

d) $\beta_7 \neq 0$ or $\beta_8 \neq 0$ implies $\beta_1 = \beta_2 = \beta_3 = \aleph_0$.

Proof. See [DM3]. □

Remark 3.3. *Using the above Theorem, we find 8 subdirectly irreducible factors of S. They are reproduced in Tables I and II, Section 7.*

4 Description of F

Remembering (1.8), let us consider the two descending chains
$\{a_1, a_2, a_3\} := \{a^{\perp}, d_1^{\perp}, d^{\perp}\}$ and $\{b_1, b_2, b_3\} := \{b^{\perp}, c_1^{\perp}, c^{\perp}\}$.

For $1 \leq i, j \leq 3$ we define

$$a_{ij} := a_i(b_j + e^{\perp}), \quad b_{ij} := b_j(a_i + e^{\perp}), \quad e_{ij} := e^{\perp}(a_i + b_j). \tag{4.1}$$

Let I_1, I_2 and I_3 be the modular sublattices of F generated by
$\{a^{\perp}, d_1^{\perp}, d^{\perp}, c, b_{31}^{\perp}, b_{21}^{\perp}, b_{11}^{\perp}, b, e\} \cup \{a_{ij}\}$, $\{b^{\perp}, c_1^{\perp}, c^{\perp}, d, a_{13}^{\perp}, a_{12}^{\perp}, a_{11}^{\perp}, a, e\} \cup \{b_{ij}\}$ and $\{e^{\perp}, c, c_1, b, b_{31}^{\perp}, b_{21}^{\perp}, b_{11}^{\perp}, d, d_1, a_{13}^{\perp},$
$\{e_{ij}\}$, respectively.

By the main result in [DM1], I_1, I_2 and I_3 coincide with the principal ideals of $F_0 :=< I_1 \cup I_2 \cup I_3 >$ generated by $a^{\perp}$, $b^{\perp}$ and $e^{\perp}$ respectively. Moreover, they are distributive and additively generate F_0. We want to show that $F_0 = F$.

To this end it will be useful to define the following indices:

$\alpha_i := \beta_i$ for $i = 1, 2, 3, 4, 5, 6$ and further

$$
\begin{array}{lll}
\alpha_7 := \delta(e^\perp/e_{11}), & \alpha_8 := \delta(c^\perp/b_{13}), & \alpha_9 := \delta(d^\perp/a_{31}),\\
\alpha_{10} := \delta(1/a^\perp + b^\perp + e^\perp), & \alpha_{11} := \delta(b_{11}^\perp/b^{\perp\perp}), & \alpha_{12} := \delta(a_{11}^\perp/a^{\perp\perp}),\\
\alpha_{13} := \delta(d_1^\perp/a_{21} + d^\perp), & \alpha_{14} := \delta(c_1^\perp/b_{12} + c^\perp), & \alpha_{15} := \delta(b_{33}/d_1 + e),\\
\alpha_{16} := \delta(d_1/a_{13}^\perp), & \alpha_{17} := \delta(c_1/b_{31}^\perp), & \alpha_{18} := \delta(a_{22}/a_{23} + a_{32}),\\
\alpha_{19} := \delta(a_{12}^\perp/a_{11}^\perp), & \alpha_{20} := \delta(a_{13}^\perp/a_{12}^\perp), & \alpha_{21} := \delta(b_{31}^\perp/b_{21}^\perp),\\
\alpha_{22} := \delta(b_{23}/b_{33}), & \alpha_{23} := \delta(a_{32}/a_{33}). &
\end{array}
\tag{4.2}
$$

Theorem 4.1. *(Description of* I_1*,* I_2 *and* I_3 *in* F*):*

1) The plain structure of I_1*,* I_2 *and* I_3 *is represented by the diagrams depicted in Fig 3, Fig 4 and Fig 5 of Section 7.*

2) The ideals are connected by the following relations between indices:

$$
\begin{array}{rcl}
\alpha_4 &=& \delta(d/d_1) = \delta(c/c_1),\\
\alpha_{11} &=& \delta(b_{11}^\perp/b^{\perp\perp}) = \delta(b^\perp/b_{11}^{\perp\perp}),\\
\alpha_{12} &=& \delta(a_{11}^\perp/a^{\perp\perp}) = \delta(a^\perp/a_{11}^{\perp\perp}),\\
\alpha_{15} &=& \delta(b_{33}/d_1 + e) = \delta(a_{33}/c_1 + e) = \delta(e_{33}/c_1 + d_1),\\
\alpha_{16} &=& \delta(d_1/a_{13}^\perp) = \delta(a_{13}/a_{23}) = \delta(b_{13}/b_{23}) = \delta(e_{13}/e_{23}),\\
\alpha_{17} &=& \delta(c_1/b_{31}^\perp) = \delta(a_{31}/a_{32}) = \delta(b_{31}/b_{32}) = \delta(e_{31}/e_{32}),\\
\alpha_{18} &=& \delta(a_{22}/a_{23} + a_{32}) = \delta(e_{22}/e_{23} + e_{32}),\\
\alpha_{19} &=& \delta(a_{12}^\perp/a_{11}^\perp) = \delta(b_{21}^\perp/b_{11}^\perp) = \delta(e_{11}/e_{12} + e_{21}),\\
\alpha_{20} &=& \delta(a_{13}^\perp/a_{12}^\perp) = \delta(a_{12}/a_{13} + a_{22}) = \delta(b_{12}/b_{13} + b_{22}) = \delta(e_{12}/e_{13} + e_{22}),\\
\alpha_{21} &=& \delta(b_{31}^\perp/b_{21}^\perp) = \delta(a_{21}/a_{22} + a_{31}) = \delta(b_{21}/b_{22} + b_{31}) = \delta(e_{21}/e_{22} + e_{31}),\\
\alpha_{22} &=& \delta(b_{23}/b_{33}) = \delta(a_{23}/c + a_{33}),\\
\alpha_{23} &=& \delta(a_{32}/a_{33}) = \delta(b_{32}/b_{33} + d) = \delta(e_{32}/d + e_{33}).
\end{array}
\tag{4.3}
$$

3) $I_1 \cup I_2 \cup I_3$ *is orthogonally closed in force of the following relations:*

$$
\begin{array}{lll}
a_{11} + d_1^\perp = a_{11}^{\perp\perp}, & a_{12} + d_1^\perp = a_{12}^{\perp\perp}, & a_{13} + d_1^\perp = a_{13}^{\perp\perp},\\
b_{11} + c_1^\perp = b_{11}^{\perp\perp}, & b_{21} + c_1^\perp = b_{21}^{\perp\perp}, & b_{31} + c_1^\perp = b_{31}^{\perp\perp}.
\end{array}
\tag{4.4}
$$

Proof. 1) This is routine verification.

2) $\delta(d/d_1) \geq \delta(d_1^\perp/d^\perp) \geq \delta(d_1^\perp e^\perp/d^\perp e^\perp) \geq \delta(c/c_1) \geq \delta(c_1^\perp/c^\perp) \geq$ $\delta(c_1^\perp e^\perp/c^\perp e^\perp) \geq \delta(d/d_1)$. This shows the first equality. The second and third ones are evident. As to the fourth, just consider the free modular lattice generated by the triple $(d^\perp, c^\perp, e^\perp)$. The other equalities are proved analogously.

3) We just show the first equality (the others follow in the same manner):

$$\delta(a_{11} + d_1^{\perp}/d_1^{\perp}) = \delta(a_{11}/a_{11}d_1^{\perp}) = \delta(a_{11}/a_{21}) \leq \delta(a^{\perp}/d_1^{\perp}) < \aleph_0 \text{ (by (1.7))}.$$

Thus $a_{11} + d_1^{\perp} = (a_{11} + d_1^{\perp})^{\perp\perp}$ by Lemma 2.3, because $d_1^{\perp} = (d_1^{\perp})^{\perp\perp}$.

Since $d_1^{\perp} \leq a_{11}^{\perp\perp}$ (because $c + e \leq a_{11}$), we obtain the desired equality.

The rest is easy and it follows $F = F_0$. ☐

Theorem 4.2. *(Forced relations among indices):*

i) *If* $\alpha_7 \neq 0$ *then* $\alpha_1 = \alpha_2 = \aleph_0$*;*

ii) *If* $\alpha_8 \neq 0$ *then* $\alpha_1 = \alpha_3 = \aleph_0$*;*

iii) *If* $\alpha_9 \neq 0$ *then* $\alpha_2 = \alpha_3 = \aleph_0$*;*

iv) *For* $i \in \{10, 11, 12, 15, 16, 17, 19\}$*, if* $\alpha_i \neq 0$ *then* $\alpha_1 = \alpha_2 = \alpha_3 = \aleph_0$*;*

v) *For* $i \in \{13, 14, 18, 20, 21, 22, 23\}$*, if* $\alpha_i \neq 0$ *then* $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \aleph_0$*;*

vi) $\alpha_{11} + \alpha_{12} + \alpha_{16} + \alpha_{17} + \alpha_{19} + \alpha_{20} + \alpha_{21} < \aleph_0$

Proof. Each implication follows in a way as was shown in the proof of Theorem 3.2, possibly in conjunction with Lemma 2.3.

The well known rule $(x+y)^{\perp} = x^{\perp}y^{\perp}$ may also be useful for computations. The last assertion is just the translation of (1.7) in terms of the indices α_i. ☐

5 The subdirectly irreducible factors of F.

In order to discover the factors of F it is sufficient to work out I_1, I_2 and I_3 at the same time, using the relations given in Theorem 4.1 and Theorem 4.2.

The essence of the procedure consists in collecting all prime quotients that are connected with a given one via the algebraic operations: this will produce automatically the corresponding subdirectly irreducible factor, together with the associated relation.

Observe how useful are indices in this procedure: on the one hand they are associated in natural way to congruences, on the other hand the forced relations among them give directly the non minimal congruences in the subdirectly irreducible factors.

A little final caution is needed: there is a quotient which does not appear in the ideals, namely $1/(a^{\perp}+b^{\perp}+e^{\perp})$ (see the factor corresponding to α_9 in Table III). Since $(a^{\perp}+b^{\perp})^{\perp\perp} = (a^{\perp}+e^{\perp})^{\perp\perp} = (e^{\perp} + b^{\perp})^{\perp\perp} = 1$ this is the only exception.

The factors are labelled from 1 to 23 in Tables II, III and IV. The last table contains all non distributive members.

Remark 5.1. *From all the above results we deduce in particular that Conjecture 2 in [M2] is true: in fact, the finite codimensions indicated in the conjecture correspond to the ones given by (1.7).*

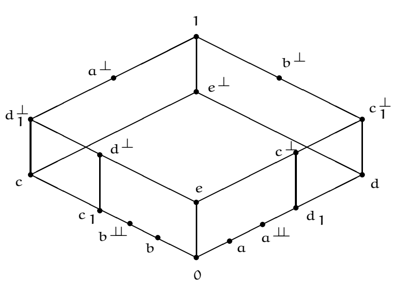
Remark 5.2. *There are plain lattice isomorphisms between different factors. Nevertheless we chose to give explicitly all diagrams, in order to facilitate visualization. It is worth noticing that the majority of this plain isomorphisms are induced by the map* $\mathsf{a} \mapsto \mathsf{b}$ *and* $\mathsf{b} \mapsto \mathsf{a}$*, which defines an involution of* S *that extends naturally to* F*. More precisely, there are eight pairs of symmetric factors, namely (1,2), (5,6), (8,9), (11,12), (13,14), (16,17), (20,21) and (2,23), all other factors being self symmetric.*

6 Remarks concerning applications to Hermitean spaces

It is possible to prove that all factors of F are implemented by Hermitean models. Hence they can be used to describe the congruence class of a subspace A in a Hermitean space (E, ϕ) of denumerable dimension under the starting assumptions, where $\mathsf{A}, \mathsf{E}, \mathsf{E}^*$ correspond to $\mathsf{a}, 1, \mathsf{b}$, respectively.

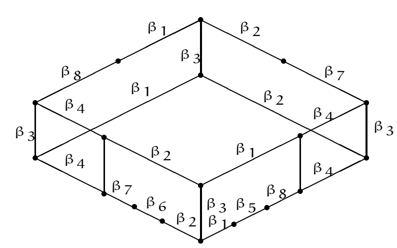
In general, these IHL will not suffice to build a complete set of geometric invariants, but they constitute a very important part. Details on these aspects cannot be discussed in the present work.

7 Diagrams



$\mathsf{S} = \mathsf{S}[\mathsf{a}; \mathsf{a} \leq \mathsf{b}^\perp \wedge \mathsf{b}\mathsf{b}^\perp = 0]$

Figure 1



$\mathsf{S} = \mathsf{S}[\mathsf{a}; \mathsf{a} \leq \mathsf{b}^\perp \wedge \mathsf{b}\mathsf{b}^\perp = 0]$

Figure 2

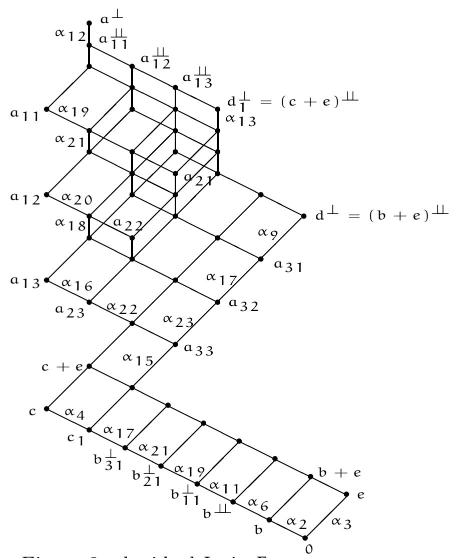


Figure 3: the ideal I_1 in F

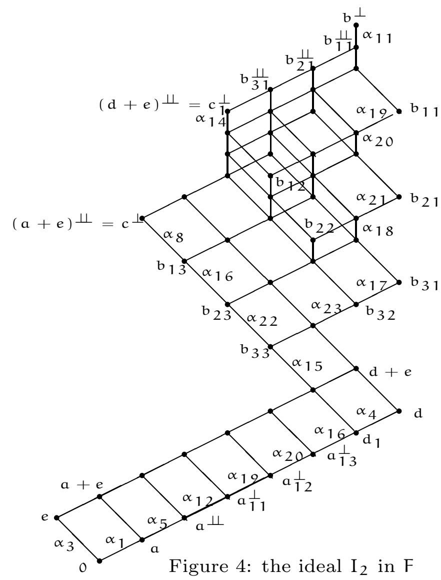


Figure 4: the ideal I_2 in F

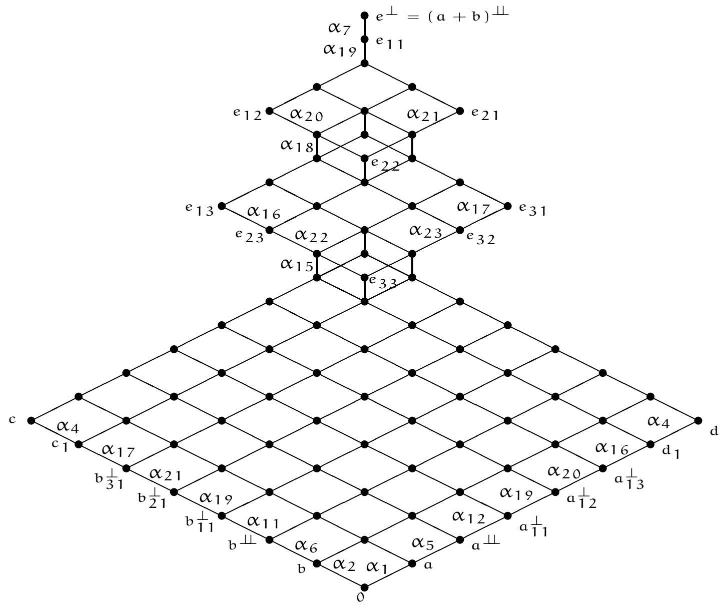


Figure 5: the ideal I_3 in F

Table I

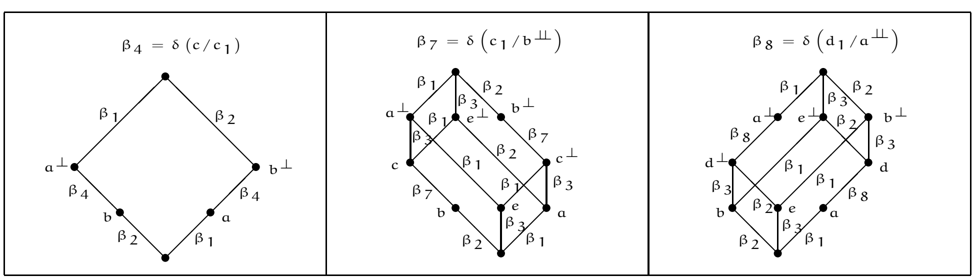


Table II

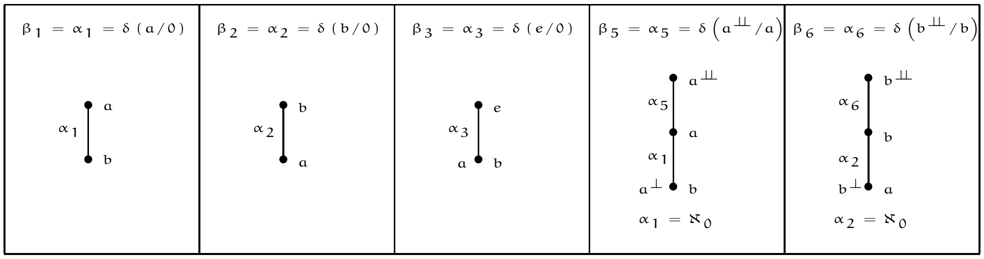

β1 = α1 = δ(a/0)
β2 = α2 = δ(b/0)
β3 = α3 = δ(e/0)
β5 = α5 = δ(a⊥⊥/a)
β6 = α6 = δ(b⊥⊥/b)
a
α1
b
b
α2
a
e
α3
a b
a⊥⊥
α5
a
α1
a⊥ b
α1 = ℵ0
b⊥⊥
α6
b
α2
b⊥ a
α2 = ℵ0

Table III

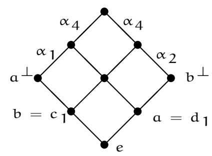
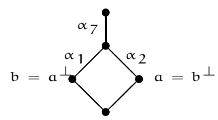
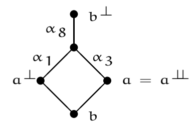
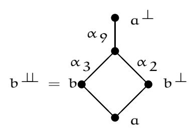
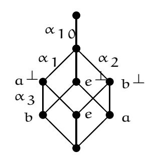
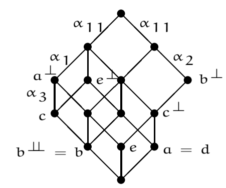
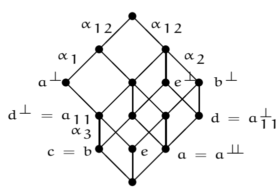
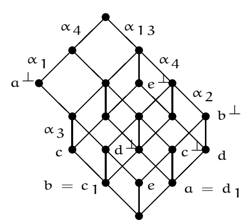
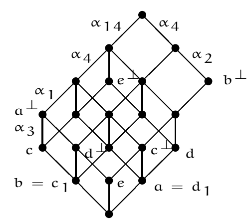
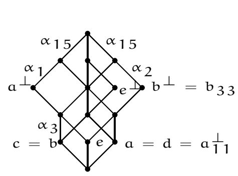
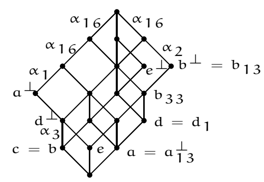
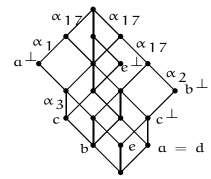
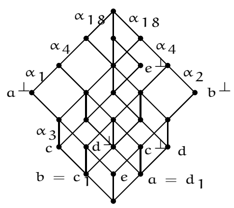
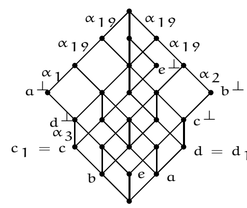
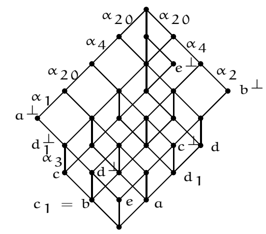
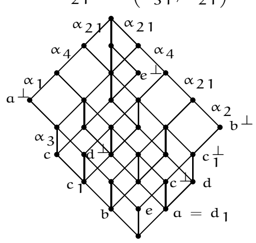
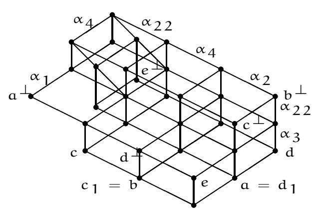
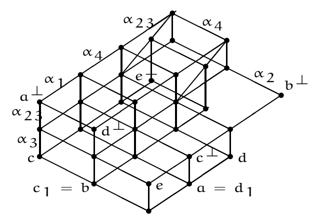
$\alpha_4 = \delta\left(c/c_1\right)$  $\alpha_1 = \alpha_2 = \aleph_0$	$\alpha_7 = \delta\left(e^{\perp}/e_{11}\right)$  $\alpha_1 = \alpha_2 = \aleph_0$	$\alpha_8 = \delta\left(c^{\perp}/b_{13}\right)$  $\alpha_1 = \alpha_3 = \aleph_0$
$\alpha_9 = \delta\left(d^{\perp}/a_{31}\right)$  $\alpha_2 = \alpha_3 = \aleph_0$	$\alpha_{10} = \delta\left(\frac{1}{a^{\perp} + b^{\perp} + e^{\perp}}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \aleph_0$	$\alpha_{11} = \delta\left(b^{\perp}_{11}/b^{\perp\perp}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \aleph_0$
$\alpha_{12} = \delta\left(a^{\perp}_{11}/a\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \aleph_0$	$\alpha_{13} = \delta\left(\frac{d^{\perp}_1}{a_{21} + d^{\perp}}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \aleph_0$	$\alpha_{14} = \delta\left(\frac{c^{\perp}_1}{b_{12} + c^{\perp}}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \aleph_0$

Table IV

$\alpha_{15} = \delta\left(\frac{b_{33}}{d_1 + e}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \aleph_0$	$\alpha_{16} = \delta\left(d_1 / a_{13}^{\perp}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \aleph_0$	$\alpha_{17} = \delta\left(c_1 / b_{31}^{\perp}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \aleph_0$
$\alpha_{18} = \delta\left(\frac{a_{22}}{a_{23} + a_{32}}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \aleph_0$	$\alpha_{19} = \delta\left(a_{11}^{\perp\perp} / a_{12}^{\perp\perp}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \aleph_0$	$\alpha_{20} = \delta\left(a_{13}^{\perp} / a_{12}^{\perp}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \aleph_0$
$\alpha_{21} = \delta\left(b_{31}^{\perp} / b_{21}^{\perp}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \aleph_0$	$\alpha_{22} = \delta\left(b_{23} / b_{33}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \aleph_0$	$\alpha_{23} = \delta\left(a_{32} / a_{33}\right)$  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \aleph_0$

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Anti-invariant $\xi^\perp$-Riemannian Submersions From Hyperbolic β-Kenmotsu Manifolds

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ABSTRACT

In this paper, we introduce anti-invariant $\xi^\perp$-Riemannian submersions from Hyperbolic β-Kenmotsu Manifolds onto Riemannian manifolds. Necessary and sufficient conditions for a special anti-invariant $\xi^\perp$-Riemannian submersion to be totally geodesic are studied. Moreover, we obtain decomposition theorems for the total manifold of such submersions.

RESUMEN

En este artículo se introducen las submersiones $\xi^\perp$-Riemannianas anti-invariantes desde variedades hiperbólicas β-Kenmotsu sobre variedades Riemannianas. Se estudian condiciones necesarias y suficientes para que ciertas submersiones $\xi^\perp$-Riemannianas anti-invariantes especiales sean totalmente geodésicas. Más aún, se obtienen teoremas de descomposión para la variedad total de dichas submersiones.

Keywords and Phrases: Riemannian submersion Anti-invariant $\xi^\perp$-Riemannian submersions, Hyperbolic β-Kenmotsu Manifolds, Integrability Conditions. geometry.

2010 AMS Mathematics Subject Classification: 53C25, 53C20, 53C50, 53C40.

1 Introduction

The geometry of Riemannian submersions between Riemannian manifolds has been intensively studied and sevral results has been pulished (see O'Neill [7] and Gray [4]). In [11] Waston defined almost Hermitian submersion between almost Hermitian manifolds and in most cases he show that the base manifold and each fiber has the same kind of structure as the total space. He also show that the vertical and horizontal distributions are invariant. On the other hand, the geometry of anti-invariant Riemannian submersions is different from the geometry of almost Hermitian submersions. For example, since every holomorphic map between Kahler manifolds is harmonic [2], it follows that any holomorphic submersion between Kahler manifolds is harmonic. However, this result is not valid for anti-invariant Riemannian submersions, which was first studied by Sahin in [8]. Similarly, Ianus and Pastore [5] shows ϕ-holomorphic maps between contact manifolds are harmonic. This implies that any contact submersion is harmonic. However, this result is not valid for anti-invariant Riemannian submersions. In [1], Chinea defined almost contact Riemannian submersion between almost contact metric manifolds. In [6], Lee studied the vertical and horizontal distribution are ϕ-invariant. Moreover, the characteristic vector field ξ is horizontal. We note that only ϕ-holomorphic submersions have been consider on an almost contact manifolds [3]. It was 1976, Upadhyay and Dube [10] introduced the notion of almost hyperbolic contact (f, g, η, ξ)-structure. Some properties of CR-submanifolds of trans hyperbolic Sasakian manifold were studied in [9]. In this paper, we consider a Riemannian submersion from a Hyperbolic β-Kenmotsu Manifolds under the assumption that the fibers are anti-invariant with respect to the tensor field of type $(1, 1)$ of almost hyperbolic contact manifold. This assumption implies that the horizontal distribution is not invariant under the action of tensor field of the total manifold of such submersions. In other words, almost hyperbolic contact are useful for describing the geometry of base manifolds, anti-invariant submersion are however served to determine the geometry of total manifold.

The paper is organized as follows: In Section 2, we present the basic information needed for this paper. In Section 3, we give the definition of anti-invariant $\xi^{\perp}$-Riemannian submersions. We also introduce a special anti-invariant $\xi^{\perp}$-Riemannian submersions and obtain necessary and sufficient conditions for such submersions to be totally geodesic or harmonic. In Section 4, we give decomposition theorems by using the existence of anti-invariant $\xi^{\perp}$-Riemannian submersions and observe that such submersions put some restrictions on the geometry of the total manifold.

2 Preliminaries

In this section, we define almost hyperbolic contact manifolds, recall the notion of Riemannian submersion between Riemannian manifolds and give a brife review of basic facts if Riemannian submersion.

Let M be an almost hyperbolic contact metric manifold with an almost hyperbolic contact metric structure (ϕ, ξ, η, g_M), where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and

g_M is a compatible Riemannian metric on M such that

$$\phi^2 = I - \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = -1, \tag{2.1}$$

$$g_M(\phi X, \phi Y) = -g_M(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

$$g_M(X, \phi Y) = -g_M(\phi X, Y), \quad g_M(X, \xi) = \eta(X) \tag{2.3}$$

An almost hyperbolic contact metric structure (ϕ, ξ, η, g_M) on M is called trans-hyperbolic Sasakian [9] if and only if

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)\phi X) + \beta(g(\phi X, Y) - \eta(Y)\phi X) \tag{2.4}$$

for all X, Y tangent to M, α and β are smooth functions on M and we say that the trans-hyperbolic Sasakian structure of type (α, β). From the above condition it follows that

$$\nabla_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi), \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \tag{2.6}$$

where ∇ is the Riemannian connection of Levi-Civita covariant differentiation.

More generally one has the notion of a hyperbolic β-Kenmotsu structure which be defined by

$$(\nabla_X \phi)Y = \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.7}$$

where β is non-zero smooth function. Also we have

$$\nabla_X \xi = \beta[X - \eta(X)\xi]. \tag{2.8}$$

Thus $\alpha = 0$ and therefore a trans-hyperbolic Sasakian structure of type $(0, \beta)$ with a non-zero constant is always hyperbolic β-Kenmotsu manifold.

Let (M^m, g_M) and (N^n, g_N) be Riemannian manifolds, where $dimM = m$, $dimN = N$ and $m > n$. A Riemannian submersion $F : M \to N$ is a map from M onto N satisfying the following axioms:

(1) (S1) F has maximal rank

(2) (S2) The differential F_* preserves the lengths of horizontal vectors.

For each $q \in N$, $F^{-1}(q)$ is an $(m - n)$-dimensional submanifold of M. The submanifold $F^{-1}(q)$ are called fibers. A vector field on M is called vertical if it is always tangent to fibers. A vector field on M is called horizontal if it is always orthogonal to fibers. A vector field X on M is called basic if X is horizontal and F-related to a vector field X_* on N, i.e., $F_* X_p = X_* F(p)$ for all $p \in M$. Note that we denote the projection morphisms on the distributions $kerF_*$ and $(kerF_*)$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

We recall the following lemma from O'Neill [7].

Lemma 2.1. *Let* $F : M \to N$ *be a Riemannian submersion between Riemannian manifolds and* X, Y *be basic vector fields of* M. *Then*

(1) *(1)* $g_M(X, Y) = g_N(X_*, Y_*) \circ F$.

(2) *(2)* the horizontal part $[X, Y]^H$ *of* $[X, Y]$ *is a basic vector field and corresponds to* $[X_*, Y_*]$, *i.e.,* $F_*([X, Y]) = [X_*, Y_*]$.

(3) *(3)* $[V, X]$ *is vertical for any vector field* V *of* $kerF_*$.

(4) *(4)* $((\nabla)^M_X Y)^H$ *is the basic vector field corresponding to* $\nabla^N_{X_*} Y_*$.

The geometry of Riemannian submersion is characterized by O'Neill's tensor T and A defined for vector fields E, F on M by

$$A_E F = H\nabla_{HE} VF + V\nabla_{HE} HF \tag{2.9}$$

$$T_E F = H\nabla_{VE} VF + V\nabla_{VE} HF \tag{2.10}$$

where ∇ is the Levi-Civita connection of g_M. It is easy to see that a Riemannian submersion $F : M \to N$ has totally geodesic fibers if and only if T vanishes identically. For any $E \in (TM)$, $T_C = T_{VC}$ and A is horizontal, $A = A_{HE}$. We note that the tensor T and A satisfy

$$T_U W = T_W U, \qquad U, W \in (kerF_*) \tag{2.11}$$

$$A_X Y = -A_Y X = \frac{1}{2} V[X, Y], \qquad X, Y \in (kerF_*)^\perp \tag{2.12}$$

On the other hand, from (2.6) and (2,7), we have

$$\nabla_V W = T_V W + \bar{\nabla}_V W \tag{2.13}$$

$$\nabla_V X = H\nabla_V X + T_V X \tag{2.14}$$

$$\nabla_X V = A_X V + V\nabla_X V \tag{2.15}$$

$$\nabla_X Y = H\nabla_X Y + A_X V \tag{2.16}$$

for $X, Y \in (kerF_*)^\perp$ and $V, W \in (kerF_*)$, where $\bar{\nabla}_V W = V\nabla_V W$. If X is basic then $H\nabla_V X = A_X V$.

Finally, we recall the notion of harmonic maps between Riemannian manifolds. Let (M, g_M) and (N, g_N) be Riemannian manifolds and supposed that $\phi : M \to N$ is a smooth map. Then the differential ϕ_* of ϕ can be viewed a section of the bundle $Hom(TM, \phi^{-1}TN) \to M$, where $\phi^{-1}TN$ is the pullback bundle which has fibers $(\phi^{-1}TN)_p = T_{\phi(p)}N$, $p \in M$. $Hom(TM, \phi^{-1}TN)$

has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection ∇^{ϕ}. Then the second fundamental form of ϕ is given by

$$(\nabla\phi*)(X,Y) = \nabla^{\phi}_X \phi * (Y) - \phi * (\nabla^M_X Y) \tag{2.17}$$

for $X, Y \in TM$. It is known that the second fundamental form is symmetric. A smooth map $\phi : (M, g_M) \to (N, g_N)$ is said to be harmonic if $trace(\nabla\phi*) = 0$. On the other hand, the tensor field of ϕ is the section $\tau(\phi)$ of $(\phi^{-1}TN)$ defined by

$$\tau(\phi) = div\phi* = \sum_{i=1}^{m} (\nabla\phi*)(e_i, e_i), \tag{2.18}$$

where $\{e_1, ...e_m\}$ is the orthogonal frame on M. Then it follows that ϕ is harmonic if and only if $\tau(\phi) = 0$ (see [7]).

3 Anti-invariant $\xi^{\perp}$- Riemannian Submersions

In this section, we define anti-invariant $\xi^{\perp}$- Riemannian submersion from hyperbolic β-Kenmotsu manifold onto a Riemannian manifold and investigate the integrability of distributions and obtain a necessary and sufficient condition for such submersions to be totally geodesic map. We also investigate the harmonicity of a special Riemannian submersion.

Definition 3.1. *Let* $(M, g_M, \phi, \xi, \eta)$ *be a hyperbolic* β*-Kenmotsu manifold and* (N, g_N) *a Riemannian manifold. Suppose that there exists a Riemannian submersion* $F : M \to N$ *such that* ξ *is normal to* $kerF_*$ *and* $kerF_*$ *is anti-invariant with respect to* ϕ*, ie.,* $\phi(kerF_*) \subset (kerF_*)^{\perp}$*. Then we say that* F *is an anti-invariant* $\xi^{\perp}$*-Riemannian submersion.*

Now, we assume that $F : (M, g_M, \phi, \xi, \eta) \to (N, g_N)$ is an anti-invariant $\xi^{\perp}$-Riemannian submersion. First of all, from Definition 3.1, we have $(kerF_*)^{\perp} \cap (kerF_*) \neq 0$. We denote the complementary orthogonal distribution to $\phi(kerF_*)$ in $(kerF_*)^{\perp}$ by μ. Then we have

$$(kerF_*)^{\perp} = \phi(kerF_*) \oplus \mu, \tag{3.1}$$

where $\phi(\mu) \subset \mu$. Hence μ contains ξ. Thus, for $X \in (kerF_*)^{\perp}$, we have

$$\phi X = BX + CX, \tag{3.2}$$

where $BX \in (kerF_*)$ and $CX \in (\mu)$. On the other hand, since $F_*(kerF_*)^{\perp} = TN$ and F is a Riemannian submersion, using (3.2), we have

$$g_N (F_*\phi V, F_*\phi CX) = 0$$

for any $X \in (kerF_*)^{\perp}$ and $V \in (kerF_*)$, which implies

$$TN = F_*(\phi((kerF_*)) \oplus F_*(\mu).$$

Example 3.2. *Let us consider a 5-dimensional manifold* $\bar{M} = \{(x_1, x_2, x_3, x_4, z) \in \mathbb{R}^5 : z \neq 0\}$, *where* (x_1, x_2, x_3, x_4, z) *are standard coordinates in* $\mathbb{R}^5$.

We choose the vector fields
$E_1 = e^{-z}\frac{\partial}{\partial x_1}$, $E_2 = e^{-z}\frac{\partial}{\partial x_2}$, $E_3 = e^{-z}\frac{\partial}{\partial x_3}$, $E_4 = e^{-z}\frac{\partial}{\partial x_4}$, $E_5 = e^{-z}\frac{\partial}{\partial x_1}$,
which are linearly independent at each point of $\bar{M}$. We define g by

$$g = e^{2z}G,$$

where G is the Euclidean metric on R^5. Hence $\{E_1, E_2, E_3, E_4, E_5\}$ is an orthonormal basis of $\bar{M}$.

We consider an 1-form η defined by

$$\eta = e^z dz, \quad \eta(X) = g(X, E_5), \qquad \forall X \in T\bar{M}.$$

We defined the $(1, 1)$ tensor field ϕ by

$$\phi\left\{\sum_{i=2}^{2}\left(x_i\frac{\partial}{\partial x_i} + x_{i+2}\frac{\partial}{\partial x_{i+2}} + z\frac{\partial}{\partial z}\right)\right\} = \sum_{i=2}^{2}\left(x_i\frac{\partial}{\partial x_{i+2}} - x_{i+2}\frac{\partial}{\partial x_i}\right).$$

Thus, we have

$$\phi(E_1) = E_3,\ \phi(E_2) = E_4,\ \phi(E_3) = -E_1,\ \phi(E_4) = -E_2,\ \phi(E_5) = 0.$$

The linear property of g and ϕ yields that

$$\eta(E_5) = -1, \quad \phi^2(X) = X - \eta(X)E_5$$

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on $\bar{M}$. Thus, $\bar{M}$ (ϕ, ξ, η, g) defines an almost hyperbolic contact metric manifold with $\xi = E_5$. Moreover, let $\bar{\nabla}$ be the Levi-Civita connection with respect to metric g. Then we have $[E_1, E_2] = 0$. Similarly $[E_1, \xi] = e^{-z}E_1$, $[E_2, \xi] = e^{-z}E_2$, $[E_3, \xi] = e^{-z}E_3$, $[E_4, \xi] = e^{-z}E_4$, $[E_i, E_j] = 0$, $1 \leq i \neq\leq 4$.
The Riemannian connection $\bar{\nabla}$ of the metric g is given by

$$2g(\bar{\nabla}_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

By Koszul's formula, we obtain the following equations

$$\bar{\nabla}_{E_1}E_1 = -e^{-z}\xi,\ \bar{\nabla}_{E_2}E_2 = -e^{-z}\xi,\ \bar{\nabla}_{E_3}E_3 = -e^{-z}\xi,\ \bar{\nabla}_{E_4}E_4 = -e^{-z}\xi,$$

$$\bar{\nabla}_\xi\xi = 0,\ \bar{\nabla}_\xi E_i = 0,\ \bar{\nabla}_{E_i}\xi = e^{-z}E_i,\ 1 \leq i \leq 4$$

and $\bar{\nabla}_{E_i}E_i = 0$ for all $1 \leq i$, $j \leq 4$. Thus, we see that M is a trans-hyperbolic Sasakian manifold of type $(0, e^{-z})$, which is hyperbolic β-Kenmotsu manifold. Here $\alpha = 0$ and $\beta = e^{-z}$.

Now, we define $(1, 1)$ tensor field as follows

$$\phi(x_1, x_2, x_3, x_4, z) = (-x_3, -x_4, x_1, x_3, z).$$

Now, we can give the following example.

Example 3.3. *Let* $(M_1, g_1 = e^{2z}G, \phi, \xi, \eta)$ *be an almost Hyperbolic contact manifolds and* M_2 *be* $\mathbb{R}^3$. *The Riemannian metric tensor field* g_2 *is defined by* $g_2 = e^{2z}(dy_1 \otimes dy_1 + dy_2 \otimes dy_2 + dy_3 \otimes dy_3)$ *on* M_2.

Let ϕ be a submersion defined by

$$\begin{array}{cccc} \phi: & \mathbb{R}^5 & \longrightarrow & \mathbb{R}^3 \\ & (x_1, x_2, x_3, x_4, z) & & (\frac{x_1 + x_3}{\sqrt{2}}, z, \frac{x_1 + x_2}{\sqrt{2}}) \end{array}$$

Then it follows that

$$\ker\phi_* = span\{V_1 = \partial x_1 - \partial x_3, \quad V_2 = \partial x_2 - \partial x_2\}$$

and

$$(\ker\phi_*)^{\perp} = span\{X_1 = \partial x_1 + \partial x_3, \quad X_2 = \partial x_2 + \partial x_2, \quad X_3 = z = \xi\}$$

Hence we have $\phi V_1 = X_1$ and $\phi V_2 = X_2$. It means that $\phi(\ker\phi) \subset (\ker\phi)^{\perp}$. A straight computations, we get $\phi_* X_1 = \partial y_1$, $\phi_* X_2 = \partial y_3$ and $\phi_* X_3 = \partial y_2$. Hence, we have

$$g_1(X_i, X_i) = g_2(\phi_* X_i, \phi_* X_i), \quad \text{for} \quad i = 1, 2, 3.$$

Thus ϕ is a anti-invariant $\xi^{\perp}$ Riemannian submersion.

Lemma 3.4. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N). *Then we have*

$$g_M(CY, \phi V) = 0, \tag{3.3}$$

$$g_M(\nabla_X CY, \phi V) = -g_M(CY, \phi A_X V) \tag{3.4}$$

for $X, Y \in ((\ker F_*)^{\perp})$ and $V \in (\ker F_*)$.

Proof. For $Y \in ((\ker F_*)^{\perp})$ and $V \in (\ker F_*)$, using (2.2), we have

$$g_M(CY, \phi V) = g_M(\phi Y - BY, \phi V) = g_M(\phi Y, \phi V) = -g_M(Y, V) - \eta(Y)\eta(V) = -g_M(Y, V) = 0$$

since $BY \in (\ker F_*)$ and $\phi V, \xi \in ((\ker F_*)^{\perp})$. Differentiating (3.3) with respect to X, we get

$$\begin{aligned} g_M(\nabla_X CY, \phi V) &= - g_M(CY, \nabla_X \phi V) \\ &= g_M(CY, (\nabla_X \phi)V) - g_M(CY, \phi(\nabla_X V)) \\ &= - g_M(CY, \phi(\nabla_X V)) \\ &= - g_M(CY, \phi A_X V) - g_M(CY, \phi \nu \nabla_X V) \\ &= - g_M(CY, \phi A_X V) \end{aligned}$$

due to $\phi\nu\nabla_X V \in (\ker F_*))$. Our assertion is complete. ∎

We study the integrability of the distribution $(\ker F_*)^\perp$ and then we investigate the geometry of leaves of $\ker F_*$ and $(\ker F_*)^\perp$. We note it is known that the distribution $(\ker F_*)$ is integrable.

Theorem 3.5. *Let F be an anti-invaraint $\xi^\perp$-Riemannian submersion from a hyperbolic β-Kenmotsu manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). The followings are equivalent.*

(1) $(\ker F_*)^\perp$ is integrable,

(2)

$$\begin{aligned} g_N((\nabla F_*)(Y, BX), F_*\phi V) &= g_N((\nabla F_*)(X, BY), F_*\phi V) \\ &+ g_M(CY, \phi A_X V) - g_M(CX, \phi A_Y V) \\ &+ \beta\eta(Y) g_M(X, V) - \beta\eta(X) g_M(Y, V), \end{aligned}$$

(3)

$$\begin{aligned} g_M(A_X BY - A_Y BY, \phi V) &= g_M(CY, \phi A_X V) - g_M(CX, \phi A_Y V) \\ &+ \beta\eta(Y) g_M(X, V) - \beta\eta(X) g_M(Y, V). \end{aligned}$$

for $X, Y \in (\ker F_*)^\perp$ and $V \in (\ker F_*)$.

Proof. For $Y \in (\ker F_*)^\perp$ and $V \in (\ker F_*)$, from Definition 3.1, $\phi V \in (\ker F_*)^\perp$ and $\phi Y \in (\ker F_*) \oplus \mu$. Using (2.2) and (2.4), we note that for $X \in (\ker F_*)^\perp$,

$$\begin{aligned} g_M(\nabla_X Y, V) = g_M(\nabla_X \phi Y, \phi V) - \beta\eta(Y) g_M(X, V) \\ -(\alpha + \beta)\eta(X)\eta(Y)\eta(V). \end{aligned} \tag{3.5}$$

Therefore, from (3.5), we get

$$\begin{aligned} g_M([X, Y], V) &= g_M(\nabla_X \phi Y, \phi V) - g_M(\nabla_Y \phi X, \phi V) \\ &= \beta\eta(X) g_M(Y, V) - \beta\eta(Y) g_M(X, V) \\ &= g_M(\nabla_X BY, \phi V) + g_M(\nabla_X CY, \phi V) \\ &- g_M(\nabla_Y BX, \phi V) - g_M(\nabla_Y CX, \phi V) \\ &- \beta\eta(Y) g_M(X, V) + \beta\eta(X) g_M(Y, V). \end{aligned}$$

Since F is a Riemannian submersion, we obtain

$$\begin{aligned} g_M([X, Y], V) &= g_N(F_*\nabla_X BY, F_*\phi V) + g_M(\nabla_X CY, \phi V) \\ &- g_N(F_*\nabla_Y BX, F_*\phi V) - g_M(\nabla_Y CX, \phi V) \\ &- \beta\eta(Y) g_M(X, V) + \beta\eta(X) g_M(Y, V). \end{aligned}$$

Thus, from (2.15) and (3.4), we have

$$g_M([X,Y],V) = g_N(-(\nabla F_*(X,BY) + (\nabla F_*)(Y,BX), F_*\phi V)$$
$$-g_M(CY, \phi A_X V + g_M(CX, \phi A_Y V)$$
$$-\beta\eta(Y)g_M(X,V) + \beta\eta(X)g_M(Y,V).$$

which proves (1) $\Longleftrightarrow$ (2).
On the other hand, using (2.14), we obtain

$$(\nabla F_*)(Y,BX) - (\nabla F_*)(X,BY) = -F_*(\nabla_Y BX - \nabla_X BY) = -F_*(A_Y BX - A_X BY),$$

which shows that (2) $\Longleftrightarrow$ (3) □

Corollary 3.6. *Let* F *be an anti-invaraint* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N) *with* $(\ker F_*)^{\perp} = \phi(\ker F_*) \oplus <\xi>$. *Then the following are equivalent:*

(1) $(\ker F_*)^{\perp}$ is integrable

(2) $(\nabla F_*)(X, \phi Y) + \beta\eta(X)F_*Y = (\nabla F_*)(Y, \phi X) + \beta\eta(Y)F_*X$

(3) $A_X\phi Y + \beta\eta(X)Y = A_Y\phi X + \beta\eta(Y)X$, for $X, Y \in (\ker F_*)^{\perp}$.

Theorem 3.7. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N). *The following are equivalent:*

(1) $(\ker F_*)^{\perp}$ defines a totally geodesic foliation on M.

(2) $g_M(A_X BY, \phi V) = g_M(CY, \phi A_X Y) - \beta\eta(X)g_M(X,V) - \beta\eta(X)g_M(Y,V)$,

(3) $g_N((\nabla F_*)(Y, \phi X), F_*\phi V) = g_M(CY, \phi A_X V) - \beta\eta(X)g_M(X,V) - \beta\eta(X)g_M(Y,V)$, for $X, Y \in (\ker F_*)^{\perp}$ and $V \in (\ker F_*)$.

Proof. For $X, Y \in (\ker F_*)^{\perp}$ and $V \in (\ker F_*)$, from (3.5), we have

$$g_M(\nabla_X Y, V) = g_M(A_X BY, \phi V) + g_M(\nabla_X CY, \phi V) - \beta\eta(Y)g_M(X,V) - \beta\eta(X)\eta(Y)\eta(V)$$

Then from (3.4), we have

$$g_M(\nabla_X Y, V) = g_M(A_X BY, \phi V) + g_M(CY, \phi A_X V) - \beta\eta(Y)g_M(X,V) - \beta\eta(X)\eta(Y)\eta(V)$$

which shows (1) $\Longleftrightarrow$ (2). On the other hand, from (2.12) and (2.14), we have

$$g_M(A_X BY, \phi V) = g_N(-(\nabla F_*)(X, BY), F_*\phi V),$$

which proves (2) $\Longleftrightarrow$ (3). □

Corollary 3.8. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N) *with* $(\ker F_*)^{\perp} = \phi(\ker F_*) \oplus <\xi>$*. Then the following are equivalent:*

(1) $(\ker F_*)^{\perp}$ defines a totally geodesic folition on M

(2) $A_X\phi Y = \beta\eta(Y)X - (\alpha + \beta)\eta(X)Y$

(3) $(\nabla F_*)(Y, \phi X) = \beta\eta(Y)F_*X - \beta)\eta(X)F_*Y$

for $X, Y \in (\ker F_*)^{\perp}$.

Theorem 3.9. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N)*. The following are equivalent:*

(1) $\ker F_*$ defines a totally geodesic folition on M

(2) $-g_N(\nabla F_*)(V, \phi X, F_*\phi W) = 0$

(3) $T_V BX + A_{CX}V \in (\mu)$,

for $X, \in (\ker F_*)^{\perp}$ and $V, W \in (\ker F_*)$

Proof. For $X, \in (\ker F_*)^{\perp}$ and $V, W \in (\ker F_*)$, $g_M(W, \xi) = 0$ implies that from (2.4)

$$g_M(\nabla_V W, \xi) = -g_M(W, \nabla_V \xi) = g_M(W, \beta(V - \eta(V)\xi)) = 0.$$

Thus we have

$$\begin{aligned} g_M(\nabla_V W, X) &= -g_M(\phi\nabla_V W, \phi X) - \eta((\nabla_V W)\eta(X) \\ &= -g_M(\phi\nabla_V W, \phi X) \\ &= -g_M(\nabla_V \phi W, \phi X) + g_M((\nabla_V \phi)W, \phi X) \\ &= g_M(\phi W, \nabla_V \phi X). \end{aligned}$$

Since F is Riemannian submersion, we have

$$g_M(\nabla_V W, X) = g_N(F_*\phi W, F_*\nabla_V \phi X) = -g_N(F_*\phi W, (\nabla F_*)(V\phi X)),$$

which proves $(1) \Longleftrightarrow (2)$.
By direct calculation, we derive

$$-g_N(F_*\phi W, (\nabla F_*)(V\phi X)) = g_M(\phi W, \nabla_V \phi X)$$

$$= g_M(\phi W, \nabla_V BX + \nabla_V CX)$$

$$= g_M(\phi W, \nabla_V BX + [V, CX] + \nabla_{CX} V).$$

□

Since $[V, CX] \in (\ker F_*)$, from (2.10) and (2.12), we obtain

$$-g_N(F_*\phi W, (\nabla F_*)(V\phi X)) = g_M(\phi W, T_V BX + A_{CX} V),$$

which proves (2) $\Longleftrightarrow$ (3).
As an analouge of a Lagrangian Riemannian submersion in [11], we have a similar result;

Corollary 3.10. *Let* F *be an anti-invaraint* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N) *with* $(\ker F_*)^{\perp} = \phi(\ker F_*) \oplus <\xi, >$. *Then the following are equivalent:*

(1) $(\ker F_*)^{\perp}$ defines a totally geodesic folition on M

(2) $-(\nabla F_*)(V, \phi X) = 0$

(3) $T_V \phi W = 0$,

$X, \in (\ker F_*)^{\perp}$ and $V, W \in (\ker F_*)$.

Proof. From Theorem 3.6, it is enough to show (2) $\Longleftrightarrow$ (3). Using (2.14) and (2.11), we have

$$-g_N(F_*\phi W, (\nabla F_*)(V\phi X)) = g_M(\nabla_V \phi W, \phi X)$$
$$= g_M(T_V \phi W, \phi X).$$

Since $T_V \phi W \in (\ker F_*)$, the proof is complete. □

We note that a differentiable map F between two Riemannian manifolds is called totally geodesic if $\nabla F_* = 0$. For the special Riemannian submersion, we have the following characterization.

Theorem 3.11. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N) *with* $(\ker F_*)^{\perp} = \phi(\ker F_*) \oplus <\xi, >$. *Then* F *is a totally geodesic map if and only if*

$$T_V \phi W = 0, \qquad V, W \in (\ker F_*) \tag{3.6}$$

and

$$A_X \phi W = 0, \qquad X \in (\ker F_*^{\perp}). \tag{3.7}$$

Proof. First of all, we recall that the second fundamental form of a Riemannian submersion satisfies

$$(\nabla F_*)(X, Y) = 0 \qquad \forall \; X, Y \in (\ker F_*^{\perp}). \tag{3.8}$$

For $V, W \in (\ker F_*)$, we get

$$(\nabla F_*)(X, Y) = F_*(\phi T_V \phi W). \tag{3.9}$$

On the other hand, from (2.1), (2.2) and (2.14), we get

$$(\nabla F_*)(X, W) = F_*(\phi A_X \phi W), X \in (\ker F_*^{\perp}). \tag{3.10}$$

Therefore, F is totally geodesic if and only if

$$\phi(T_V \phi W) = 0 \quad \forall \ V, W \in (\ker F_*^{\perp}). \tag{3.11}$$

and

$$\phi(A_X \phi W) = 0 \quad \forall \quad X \in (\ker F_*^{\perp}). \tag{3.12}$$

From (2.2), (2.6) and (2.7), we have

$$T_V \phi W = 0 \quad \forall \quad V, W \in (\ker F_*). \tag{3.13}$$

and

$$A_X \phi W = 0 \quad \forall \quad X \in (\ker F_*^{\perp}).$$

From (2.4), F is totally geodesic if and only the equation (3.6) and (3.7) hold □

Finally, in this section, we give a necessary and sufficient condition for a special Riemannian submersion to be harmonic as an analouge of Lagrangian Riemannian submersion in [11].

Theorem 3.12. *Let F be an anti-invaraint $\xi^{\perp}$-Riemannian submersion from a hyperbolic β-Kenmotsu manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N) with $(\ker F_*)^{\perp} = \phi(\ker F_*) \oplus < \xi >$. Then F is harmonic if and only if $\mathrm{Trace}(\phi T_V) = 0$ for $V \in (\ker F_*)$.*

Proof. From [5], we know that F is harmonic if and only if F has minimal fibers. Thus F is harmonic if and only if $\sum_{i=1}^{m_1} T_{e_i} e_i = 0$. On the other hand, from (2.4), (2.11) and (2.10), we have

$$T_V \phi W = \phi T_V W \tag{3.14}$$

due to $\xi \in (\ker F_*^{\perp})$ for any $V, W \in (\ker F_*)$. Using (3.14), we get

$$\sum_{i=1}^{m_1} g_M(T_{e_i} \phi e_i, V) = \sum_{i=1}^{m_1} g_M(\phi T_{e_i} \phi e_i, V) = -\sum_{i=1}^{m_1} g_M(T_{e_i} e_i, \phi V)$$

for any $V \in (\ker F_*)$. Thus skew-symmetric T implies that

$$\sum_{i=1}^{m_1} g_M(\phi T_{e_i} \phi e_i, V) = -\sum_{i=1}^{m_1} g_M(T_{e_i} e_i, \phi V).$$

Using (2.8) and (2.2), we have

$$\sum_{i=1}^{m_1} g_M(e_i, \phi T_V e_i) = -\sum_{i=1}^{m_1} g_M(\phi e_i, T_V e_i) = -\sum_{i=1}^{m_1} g_M(T_{e_i} e_i, \phi V)$$

which shows our assertion. □

4 Decomposition theorems

In this section, we obtain decomposition theorems by using the existence of anti-invariant $\xi^{\perp}$-Riemannian submersions. First, we recall the following.

Theorem 4.1. [10] *Let g be a Riemannian metric on the manifold $B = M \times N$ and assume that the canonical foliations D_M and D_N intersect perpendicular every where. Then g is the metric tensor of*

(1) (i) a twisted product $M \times_f N$ if and only if D_M is totally geodesic foliation and D_N is a totally umbilical foliation.

(2) (ii) a warped product $M \times_f N$ if and only if D_M is totally geodesic foliation and D_N is a spheric foliation, i.e., it is umbilical and its mean curvature vector field is parallel.

(3) (iii) a usual product of Riemannian manifold if and only if D_M and D_N are totally geodesic foliations.

Our first decomposition theorem for anti-invariant $\xi^{\perp}$-Riemannian submersion comes from Theorem 3.4 and 3.6 in terms of the second fundamental forms of such submersions.

Theorem 4.2. *Let F be an anti-invariant $\xi^{\perp}$-Riemannian submersion from a hyperbolic β-Kenmotsu manifold $(M, g_M, \phi, \xi, \eta)$ on to a Riemannian manifold (N, g_N). Then M is locally product manifold if and only if*

$$-g_N((\nabla F_*)(Y, \phi X), F_*\phi V) = g_M(CY, \phi A_X V) - \beta\eta(Y) g_M(X, V)$$

and

$$-g_N((\nabla F_*)(V, \phi X), F_*\phi W) = 0$$

for $X, Y \in (\ker F_*^{\perp})$ and $V, W \in (\ker F_*)$.
From Corollary 3.5 and 3.7, we have the following decomposition theorem:

Theorem 4.3. *Let F be an anti-invariant $\xi^{\perp}$-Riemannian submersion from a hyperbolic β-Kenmotsu manifold $(M, g_M, \phi, \xi, \eta)$ on to a Riemannian manifold (N, g_N) with $(\ker F_*^{\perp}) \oplus < \xi >$. Then M is a locally product manifold if and only if $A_X \phi Y = (\alpha + \beta)\eta(Y)X$ and $T_V \phi W = 0$, for $X, Y \in (\ker F_*^{\perp})$ and $V, W \in (\ker F_*)$.*

Next we obtain a decomposition theorem which is related to the notion of a twisted product manifold.

Theorem 4.4. *Let F be an anti-invariant $\xi^{\perp}$-Riemannian submersion from a hyperbolic β-Kenmotsu manifold $(M, g_M, \phi, \xi, \eta)$ on to a Riemannian manifold (N, g_N) with $(\ker F_*^{\perp}) \oplus < \xi >$. Then M is locally twisted product manifold of the form $M_{\ker F_*^{\perp}} \times_f M_{\ker F_*}$ if and only if*

$$T_V\phi X = -g_M(X, T_V V)\,\|V\|^{-2} - \beta\eta(Y)g_M(\phi X, \phi V).$$

and

$$A_X\phi Y = \beta\eta(Y)X$$

for $X, Y \in (\ker F_*^{\perp})$ and $V \in (\ker F_*)$, where $M_{(\ker F_*^{\perp})}$ and $M_{(\ker F_*)}$ are integrable manifolds of the distributions $(\ker F_*^{\perp})$ and $(\ker F_*)$.

Proof. For $X \in (\ker F_*^{\perp})$ and $V \in (\ker F_*)$, from (2.4) and (2.11), we obtain

$$g_M(\nabla_V W, X) = g_M(T_V\phi W, \phi X) = -g_M(\phi W, T_V\phi X)$$

Since T_V is skew-symmetric. This implies that $\ker F_*$ is totally umbilical if and only if

$$T_V\phi X - \beta\eta(V)g_M(\phi X, \phi V) = -X(\lambda)\phi V,$$

where λ is a function on M. By direct computation,

$$T_V\phi X = -g_M(X, T_V V)\,\|V\|^{-2} - \beta\eta(Y)g_M(\phi X, \phi V).$$

□

Then the proof follows from Corollary 3.5
However, in the sequel, we show that the notion of anti-invariant $\xi^{\perp}$-Riemannian submersion puts some restrictions on the source manifold.

Theorem 4.5. *Let* $(M, g_M, \phi, \xi, \eta)$ *be a hyperbolic* β*-Kenmotsu manifold and* (N, g_N) *be a Riemannian manifold . Then there does not exist an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from* M *to* N *with* $(\ker F_*)^{\perp} = \phi(\ker F_*)^{\perp} \oplus < \xi >$ *such that* M *is a locally proper twisted product manifold of the form* $M_{\ker F_*} \times_f M_{(\ker F_*)^{\perp}}$*.*

Proof. Suppose that $F : (M, g_M, \phi, \xi, \eta) \longrightarrow (N, g_N)$ is an anti-invaraiant $\xi^{\perp}$-Riemannian submersion with $(\ker F_*)^{\perp} = \phi(\ker F_*)^{\perp} \oplus < \xi >$ and M is a locally twisted product of the form $M_{\ker F_*} \times_f M_{(\ker F_*)^{\perp}}$ .Then $M_{\ker F_*}$ is a totally geodesic foliation and $M_{(\ker F_*^{\perp})}$ is a totally umbilical foliation. We denote the second fundamental form of $M_{(\ker F_*^{\perp})}$ by h. Then we have

$$g_M(\nabla_X Y, V) = g_M(h(X, Y), V) \quad X, Y \in ((\ker F_*)^{\perp}, V \in (\ker F_*). \tag{4.1}$$

Since $M(^{\perp}_{\ker F_*})$ is a totally umbilical foliation, we have

$$g_M(\nabla_X Y, V) = g_M(H, V)g_M(X, Y),$$

where H is the mean curvature vector field of $M_{(\ker F_*)^{\perp}}$. On the other hand, from (3.5), we derive

$$g_M(\nabla_X Y, V) = -g_M(\phi Y, \nabla_X\phi V) - \beta\eta(Y)g(X, V) - \beta\eta(X)\eta(Y)\eta(V). \tag{4.2}$$

Using (2.13), we obtain

$$g_M(\nabla_X Y, V) = g_M(\phi Y, A_X\phi V) - \beta\eta(Y)g(X,V) - \beta\eta(X)\eta(Y)\eta(V) \tag{4.3}$$

$$= g_M(Y, A_X\phi V) - \beta g(X,V) - \beta\eta(X)\eta(V)\xi)$$

Therefore, from (4.1), (4.3) and (2.2), we have

$$A_X\phi V = g_M(H,V)\phi X + \eta(A_X\phi V)\xi.$$

Since $A_X\phi V \in (\ker F_*)$,

$$\eta(A_X\phi V) = g_M(A_X\phi V, \xi) = 0.$$

Thus, we have

$$A_X\phi V = g_M(H,V)\phi X.$$

Hence, we derive

$$g_M(A_X\phi V, \phi X) - \beta\eta(X)\eta(V)g(Y,\phi X) = -g_M(H,V)\left\{\|X\|^2 - \eta^2(X)\right\}$$

$$g_M(\nabla_X\phi V, \phi X) = -g_M(H,V)\left\{\|X\|^2 - \eta^2(X)\right\} + \beta\eta(X)\eta(V)g(Y,\phi X)$$

$$g_M(\nabla_X Y, V) + \beta\eta(Y)g(X,V) - \beta\eta(X)\eta(Y)\eta(V)$$

$$= -g_M(H,V)\left\{\|X\|^2 - \eta^2(X)\right\} + \beta\eta(X)\eta(V)g(Y,\phi X).$$

Thus using (2.9), we have $A_X X = 0$, which implies

$$\beta\eta(X)g_M(X,V) = -g_M(H,V)\left\{\|X\|^2 - \eta^2(X)\right\} + \beta\eta(X)\eta(Y)[\eta(V) - g_M(Y,\phi X)]$$

for every $X \in ((\ker F_*^{\perp}), V \in (\ker F_*)$. Choosing X which is orthogonal to ξ, $g_M(H,V)\|X\|^2 = 0$. Since g_M is the Riemannian metric and $H \in (\ker F_*)$, we conclude that $H = 0$, which shows $\ker F_*^{\perp}$ is totally geodesic, so M is usual product of Riemannian manifolds. □

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