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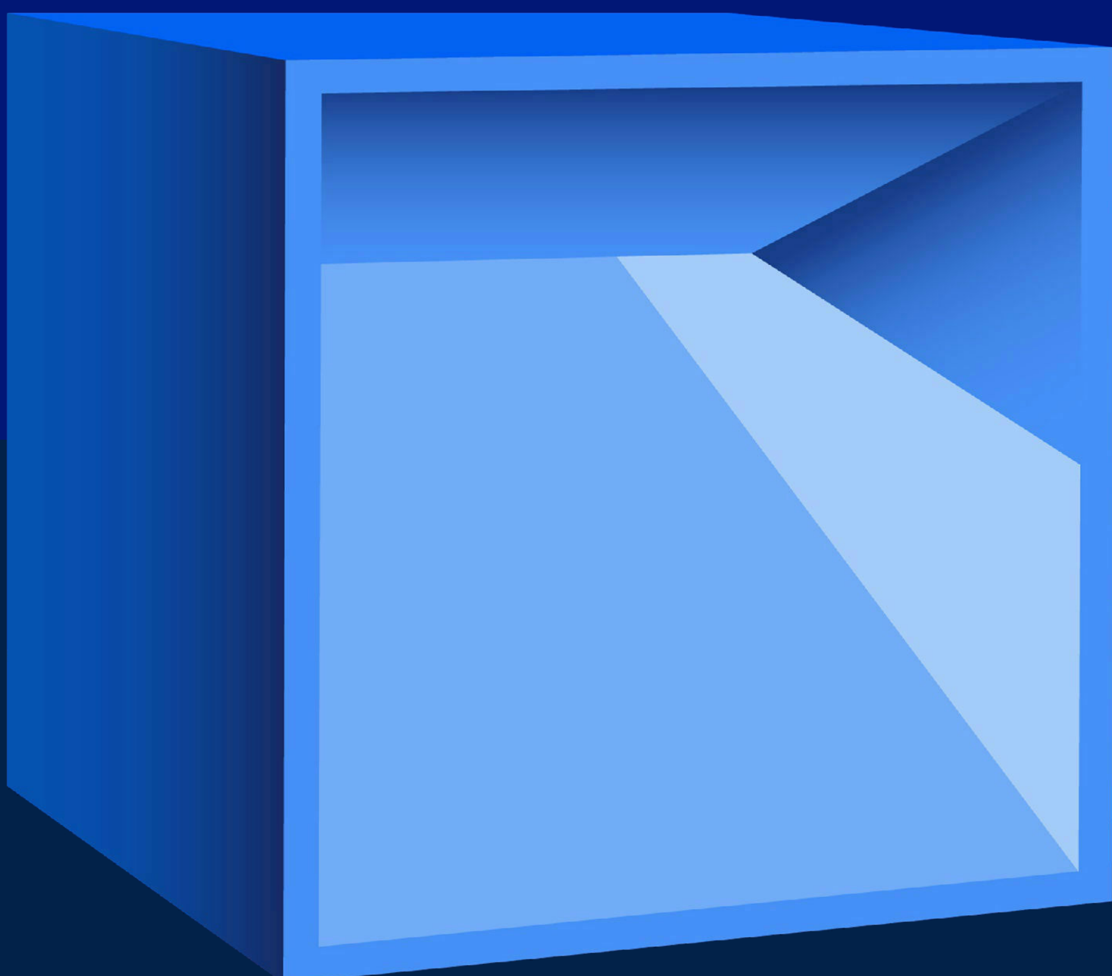

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Bounds for the Generalized (Φ, f)-Mean Difference

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ABSTRACT

In this paper we establish some bounds for the (Φ, f)-*mean difference* introduced in the general settings of measurable spaces and Lebesgue integral, which is a two functions generalization of *Gini mean difference* that has been widely used by economists and sociologists to measure economic inequality.

RESUMEN

En este artículo establecemos algunas cotas para la (Φ, f)-*diferencia media* introducida en el contexto general de espacios medibles e integral de Lebesgue, que es una generalización a dos funciones de la *diferencia media de Gini* que ha sido ampliamente utilizada por economistas y sociólogos para medir desigualdad económica.

Keywords and Phrases: Gini mean difference, Mean deviation, Lebesgue integral, Expectation, Jensen's integral inequality.

2010 AMS Mathematics Subject Classification: 26D15; 26D10; 94A17.

1. Introduction

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure ν on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a ν-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for ν-a.e. (almost every) $x \in \Omega$ and $\int_\Omega w(x)\, d\nu(x) = 1$, consider the *Lebesgue space*

$$L_w(\Omega, \nu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w(x)\,|f(x)|\, d\nu(x) < \infty\}.$$

Let I be an interval of real numbers and $\Phi : I \to \mathbb{R}$ a Lebesgue measurable function on I. For $f : \Omega \to I$ a ν-measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$ we define the *generalized* (Φ, f)-*mean difference* $R_G(\Phi, f; w)$ by

$$R_G(\Phi, f; w) := \frac{1}{2}\int_\Omega \int_\Omega w(x)\, w(y)\, |(\Phi \circ f)(x) - (\Phi \circ f)(y)|\, d\nu(x)\, d\nu(y) \tag{1.1}$$

and the *generalized* (Φ, f)-*mean deviation* $M_D(\Phi, f; w)$ by

$$M_D(\Phi, f; w) := \int_\Omega w(x)\, |(\Phi \circ f)(x) - E(\Phi, f; w)|\, d\nu(x), \tag{1.2}$$

where

$$E(\Phi, f; w) := \int_\Omega (\Phi \circ f)(y)\, w(y)\, d\nu(y)$$

the *generalized* (Φ, f)-expectation.

If $\Phi = e$, where $e(t) = t$, $t \in \mathbb{R}$ is the *identity mapping*, then we can consider the particular cases of interest, the *generalized* f-*mean difference*

$$R_G(f; w) := R_G(e, f; w) = \frac{1}{2}\int_\Omega \int_\Omega w(x)\, w(y)\, |f(x) - f(y)|\, d\nu(x)\, d\nu(y) \tag{1.3}$$

and the *generalized* f-*mean deviation*

$$M_D(f; w) := M_D(e, f; w) = \int_\Omega w(x)\, |f(x) - E(f; w)|\, d\nu(x), \tag{1.4}$$

where $E(f; w) := \int_\Omega f(y)\, w(y)\, d\nu(y)$ is the *generalized* f-expectation.

If $\Omega = [-\infty, \infty]$ and $f = e$ then we have the usual *mean difference*

$$R_G(w) := R_G(f; w) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x)\, w(y)\, |x - y|\, dx dy \tag{1.5}$$

and the *mean deviation*

$$M_D(w) := M_D(f; w) = \int_\Omega w(x)\, |x - E(w)|\, dx, \tag{1.6}$$

where $w : \mathbb{R} \to [0, \infty)$ is a *density function*, this means that w is integrable on $\mathbb{R}$ and $\int_{-\infty}^{\infty} w(t)\,dt = 1$, and

$$E(w) := \int_{-\infty}^{\infty} xw(x)\,dx \tag{1.7}$$

denote the *expectation of* w provided that the integral exists and is finite.

The mean difference $R_G(w)$ was proposed by Gini in 1912 [21], after whom it is usually named, but was discussed by Helmert and other German writers in the 1870's (cf. H. A. David [13], see also [26, p. 48]). It has a certain theoretical attraction, being dependent on the spread of the variate-values among themselves and not on the deviations from some central value ([26, p. 48]). Further, its defining integral (1.5) may converge when that of the *variance* $\sigma(w)$,

$$\sigma(w) := \int_{-\infty}^{\infty} (x - E(w))^2\, w(x)\,dx, \tag{1.8}$$

does not. It is, however, more difficult to compute than the standard deviation.

For some recent results concerning integral representations and bounds for $R_G(w)$ see [5], [6], [8] and [9].

For instance, if $w : \mathbb{R} \to [0, \infty)$ is a density function we define by

$$W(x) := \int_{-\infty}^{x} w(t)\,dt, \quad x \in \mathbb{R}$$

its *cumulative function.* Then we have [5], [6]:

$$\begin{aligned} R_G(w) &= 2\,\mathrm{Cov}(e, W) = \int_{-\infty}^{\infty} (1 - W(y))\, W(y)\,dy \\ &= 2\int_{-\infty}^{\infty} xw(x)\, W(x)\,dx - E(w) \\ &= 2\int_{-\infty}^{\infty} (x - E(w))\,(W(x) - \gamma)\, w(x)\,dx \\ &= 2\int_{-\infty}^{\infty} (x - \delta)\left(W(x) - \frac{1}{2}\right) w(x)\,dx \end{aligned} \tag{1.9}$$

for any γ, $\delta \in \mathbb{R}$ and [6]:

$$R_G(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - y)\,(W(x) - W(y))\, w(x)\, w(y)\,dxdy. \tag{1.10}$$

With the above assumptions, we have the bounds [5]:

$$\frac{1}{2}M_D(w) \le R_G(w) \le 2\sup_{x\in\mathbb{R}} |W(x) - \gamma|\, M_D(w) \le M_D(w), \tag{1.11}$$

for any $\gamma \in [0,1]$, where $W(\cdot)$ is the cumulative distribution of w and $M_D(w)$ is the mean deviation.

Consider the n-tuple of real numbers $a = (a_1, ..., a_n)$ and $p = (p_1, ..., p_n)$ a probability distribution, i.e. $p_i \geq 0$ for each $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then by taking $\Omega = \{1, ..., n\}$ and the discrete measure, we can consider from (1.1) and (1.2) that (see [7])

$$R_G(a;p) := \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j |\Phi(a_i) - \Phi(a_j)|, \tag{1.12}$$

and

$$M_D(a;p) := \frac{1}{2}\sum_{i=1}^{n} p_i \left|\Phi(a_i) - \sum_{j=1}^{n} p_j \Phi(a_j)\right| \tag{1.13}$$

where $a \in I^n := I \times ... \times I$ and $\Phi : I \to \mathbb{R}$.

The quantity $R_G(a;p)$ has been defined in [7] and some results were obtained.

In the case when $\Phi = e$, then we get the special case of Gini mean difference and mean deviation of an empirical distribution that is particularly important for applications,

$$R_G(a;p) := \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j |a_i - a_j|, \tag{1.14}$$

and

$$M_D(a;p) := \frac{1}{2}\sum_{i=1}^{n} p_i \left|a_i - \sum_{j=1}^{n} p_j a_j\right|. \tag{1.15}$$

The following result incorporates an upper bound for the weighted Gini mean difference [7]:

For any $a \in \mathbb{R}^n$ and any p a probability distribution, we have the inequality:

$$\frac{1}{2}M_D(a;p) \leq R_G(a;p) \leq \inf_{\gamma\in\mathbb{R}}\left[\sum_{i=1}^{n} p_i |a_i - \gamma|\right] \leq M_D(a;p). \tag{1.16}$$

The constant $\frac{1}{2}$ in the first inequality in (1.16) is sharp.

For some recent results for discrete Gini mean difference and mean deviation, see [7], [11], [14] and [15].

2. General Bounds

We have:

Theorem 1. *Let* I *be an interval of real numbers and* $\Phi : I \to \mathbb{R}$ *a Lebesgue measurable function on* I. *If* $w : \Omega \to \mathbb{R}$ *is a* ν*-measurable function with* $w(x) \geq 0$ *for* ν*-a.e. (almost every)* $x \in \Omega$ *and* $\int_\Omega w(x)\, d\nu(x) = 1$ *and if* $f : \Omega \to I$ *is a* ν*-measurable function with* $\Phi \circ f \in L_w(\Omega, \nu)$, *then*

$$\frac{1}{2} M_D(\Phi, f; w) \leq R_G(\Phi, f; w) \leq I(\Phi, f; w) \leq M_D(\Phi, f; w), \tag{2.1}$$

where

$$I(\Phi, f; w) := \inf_{\gamma \in \mathbb{R}} \int_\Omega w(x) \left|(\Phi \circ f)(x) - \gamma\right| d\nu(x). \tag{2.2}$$

Demostración. Using the properties of the integral, we have

$$\begin{aligned}
& R_G(\Phi, f; w) \\
&= \frac{1}{2} \int_\Omega \int_\Omega w(x) w(y) \left|(\Phi \circ f)(x) - (\Phi \circ f)(y)\right| d\nu(x)\, d\nu(y) \\
&\geq \frac{1}{2} \int_\Omega w(x) \left|(\Phi \circ f)(x) \int_\Omega w(y)\, d\nu(y) - \int_\Omega w(y) (\Phi \circ f)(y)\, d\nu(y)\right| d\nu(x) \\
&= \frac{1}{2} \int_\Omega w(x) \left|(\Phi \circ f)(x) - \int_\Omega w(y) (\Phi \circ f)(y)\, d\nu(y)\right| d\nu(x) \\
&= \frac{1}{2} M_D(\Phi, f; w)
\end{aligned}$$

and the first inequality in (2.1) is proved.

By the triangle inequality for modulus we have

$$\begin{aligned}
\left|(\Phi \circ f)(x) - (\Phi \circ f)(y)\right| &= \left|(\Phi \circ f)(x) - \gamma + \gamma - (\Phi \circ f)(y)\right| \\
&\leq \left|(\Phi \circ f)(x) - \gamma\right| + \left|(\Phi \circ f)(y) - \gamma\right|
\end{aligned} \tag{2.3}$$

for any $x, y \in \Omega$ and $\gamma \in \mathbb{R}$.

Now, if we multiply (2.3) by $\frac{1}{2}w(x)\,w(y)$ and integrate, we get

$$\begin{aligned}
&R_G(\Phi, f; w)\\
&= \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(\Phi\circ f)(x) - (\Phi\circ f)(y)|\,d\nu(x)\,d\nu(y)\\
&\leq \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,[|(\Phi\circ f)(x) - \gamma| + |(\Phi\circ f)(y) - \gamma|]\,d\nu(x)\,d\nu(y)\\
&= \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(\Phi\circ f)(x) - \gamma|\,d\nu(x)\,d\nu(y)\\
&+ \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(\Phi\circ f)(y) - \gamma|\,d\nu(x)\,d\nu(y)\\
&= \frac{1}{2}\int_\Omega w(x)\,|(\Phi\circ f)(x) - \gamma|\,d\nu(x) + \frac{1}{2}\int_\Omega w(y)\,|(\Phi\circ f)(y) - \gamma|\,d\nu(y)\\
&= \int_\Omega w(x)\,|(\Phi\circ f)(x) - \gamma|\,d\nu(x) \qquad (2.4)
\end{aligned}$$

for any $\gamma \in \mathbb{R}$.

Taking the infimum over $\gamma \in \mathbb{R}$ in (2.4) we get the second part of (2.1).

Since, obviously

$$\begin{aligned}
I(\Phi, f; w) &= \inf_{\gamma\in\mathbb{R}}\int_\Omega w(x)\,|(\Phi\circ f)(x) - \gamma|\,d\nu(x)\\
&\leq \int_\Omega w(x)\left|(\Phi\circ f)(x) - \int_\Omega w(y)\,(\Phi\circ f)(y)\,d\nu(y)\right|d\nu(x)\\
&= M_D(\Phi, f; w),
\end{aligned}$$

the last part of (2.1) is thus proved. ∎

By the Cauchy-Bunyakowsky-Schwarz (CBS) inequality, if $(\Phi\circ f)^2 \in L_w(\Omega, \nu)$, then we have

$$\begin{aligned}
&\left[\int_\Omega w(x)\left|(\Phi\circ f)(x) - \int_\Omega w(y)\,(\Phi\circ f)(y)\,d\nu(y)\right|d\nu(x)\right]^2\\
&\leq \int_\Omega w(x)\left[(\Phi\circ f)(x) - \int_\Omega w(y)\,(\Phi\circ f)(y)\,d\nu(y)\right]^2 d\nu(x)\\
&= \int_\Omega w(x)\,(\Phi\circ f)^2(x)\,d\nu(x)\\
&- 2\int_\Omega w(y)\,(\Phi\circ f)(y)\,d\nu(y)\int_\Omega w(x)\,(\Phi\circ f)(x)\,d\nu(x)\\
&+ \left[\int_\Omega w(y)\,(\Phi\circ f)(y)\,d\nu(y)\right]^2\int_\Omega w(x)\,d\nu(x)\\
&= \int_\Omega w(x)\,(\Phi\circ f)^2(x)\,d\nu(x) - \left[\int_\Omega w(x)\,(\Phi\circ f)(x)\,d\nu(x)\right]^2.
\end{aligned}$$

By considering the *generalized* (Φ, f)*-dispersion*

$$\sigma(\Phi, f; w) := \left(\int_{\Omega} w(x)(\Phi \circ f)^2(x)\,d\nu(x) - \left[\int_{\Omega} w(x)(\Phi \circ f)(x)\,d\nu(x)\right]^2\right)^{1/2},$$

then we have

$$M_D(\Phi, f; w) \leq \sigma(\Phi, f; w) \tag{2.5}$$

provided $(\Phi \circ f)^2 \in L_w(\Omega, \nu)$.

If there exists the constants m, M so that

$$-\infty < m \leq \Phi(t) \leq M < \infty \text{ for almost any } t \in I \tag{2.6}$$

then by the reverse CBS inequality

$$\sigma(\Phi, f; w) \leq \frac{1}{2}(M - m), \tag{2.7}$$

by (2.1) and by (2.5) we can state the following result:

Corollary 1. *Let* I *be an interval of real numbers and* $\Phi : I \to \mathbb{R}$ *a Lebesgue measurable function on* I *satisfying the condition (2.6) for some constants* m, M. *If* $w : \Omega \to \mathbb{R}$ *is a* ν*-measurable function with* $w(x) \geq 0$ *for* ν *-a.e.* $x \in \Omega$ *and* $\int_{\Omega} w(x)\,d\nu(x) = 1$ *and if* $f : \Omega \to I$ *is a* ν*-measurable function with* $(\Phi \circ f)^2 \in L_w(\Omega, \nu)$, *then we have the chain of inequalities*

$$\begin{aligned}\frac{1}{2}M_D(\Phi, f; w) &\leq R_G(\Phi, f; w) \leq I(\Phi, f; w) \leq M_D(\Phi, f; w)\\ &\leq \sigma(\Phi, f; w) \leq \frac{1}{2}(M - m).\end{aligned} \tag{2.8}$$

We observe that, in the discrete case we obtain from (2.1) the inequality (1.16) while for the univariate case with $\int_{-\infty}^{\infty} w(t)\,dt = 1$ we have

$$\frac{1}{2}M_D(w) \leq R_G(w) \leq I(w) \leq M_D(w) \leq \sigma(\Phi, f; w) \tag{2.9}$$

where

$$I(w) := \inf_{\gamma \in \mathbb{R}} \int_{-\infty}^{\infty} w(x)|x - \gamma|\,dx. \tag{2.10}$$

If w is supported on the finite interval $[a, b]$, namely $\int_a^b w(x)\,dx = 1$, then we have the chain of inequalities

$$\frac{1}{2}M_D(w) \leq R_G(w) \leq I(w) \leq M_D(w) \leq \sigma(\Phi, f; w) \leq \frac{1}{2}(M - m). \tag{2.11}$$

3. Bounds for Various Classes of Functions

In the case of functions of bounded variation we have:

Theorem 2. *Let* $\Phi : [a,b] \to \mathbb{R}$ *be a function of bounded variation on the closed interval* $[a,b]$. *If* $w : \Omega \to \mathbb{R}$ *is a* ν*-measurable function with* $w(x) \geq 0$ *for* ν *-a.e.* $x \in \Omega$ *and* $\int_\Omega w(x)\, d\nu(x) = 1$ *and if* $f : \Omega \to [a,b]$ *is a* ν*-measurable function with* $\Phi \circ f \in L_w(\Omega, \nu)$, *then*

$$R_G(\Phi, f; w) \leq \frac{1}{2} \bigvee_a^b (\Phi), \tag{3.1}$$

where $\bigvee_a^b (\Phi)$ *is the total variation of* Φ *on* $[a,b]$.

Demostración. Using the inequality (2.4) we have

$$R_G(\Phi, f; w) \leq \int_\Omega w(x) \left|(\Phi \circ f)(x) - \gamma\right| d\nu(x) \tag{3.2}$$

for any $\gamma \in \mathbb{R}$.

By the triangle inequality, we have

$$\begin{aligned} &\left|(\Phi \circ f)(x) - \frac{1}{2}[\Phi(a) + \Phi(b)]\right| \\ &\leq \frac{1}{2}\left|\Phi(a) - \Phi(f(x))\right| + \frac{1}{2}\left|\Phi(b) - \Phi(f(x))\right| \end{aligned} \tag{3.3}$$

for any $x \in \Omega$.

Since $\Phi : [a,b] \to \mathbb{R}$ is of bounded variation and d is a division of $[a,b]$, namely

$$d \in \mathcal{D}([a,b]) := \{d := \{a = t_0 < t_1 < \ldots < t_n = b\}\},$$

then

$$\bigvee_a^b (\Phi) = \sup_{d \in \mathcal{D}([a,b])} \sum_{i=0}^{n-1} \left|\Phi(t_{i+1}) - \Phi(t_i)\right| < \infty.$$

Taking the division $d_0 := \{a = t_0 < t < t_2 = b\}$ we then have

$$\left|\Phi(t) - \Phi(a)\right| + \left|\Phi(b) - \Phi(t)\right| \leq \bigvee_a^b (\Phi)$$

for any $t \in [a,b]$ and then

$$\left|\Phi(f(x)) - \Phi(a)\right| + \left|\Phi(b) - \Phi(f(x))\right| \leq \bigvee_a^b (\Phi) \tag{3.4}$$

for any $x \in \Omega$.

On making use of (3.3) and (3.4) we get

$$\left|(\Phi \circ f)(x) - \frac{1}{2}[\Phi(a) + \Phi(b)]\right| \leq \frac{1}{2}\bigvee_{a}^{b}(\Phi) \tag{3.5}$$

for any $x \in \Omega$.

If we multiply (3.5) by $w(x)$ and integrate, then we obtain

$$\int_{\Omega} w(x)\left|(\Phi \circ f)(x) - \frac{1}{2}[\Phi(a) + \Phi(b)]\right| \leq \frac{1}{2}\bigvee_{a}^{b}(\Phi). \tag{3.6}$$

Finally, by choosing $\gamma = \frac{1}{2}[\Phi(a) + \Phi(b)]$ in (3.2) and making use of (3.6) we deduce the desired result (3.1). ∎

In the case of absolutely continuous functions we have:

Theorem 3. *Let $\Phi : [a, b] \to \mathbb{R}$ be an absolutely continuous function on the closed interval $[a, b]$. If $w : \Omega \to \mathbb{R}$ is a ν-measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_{\Omega} w(x)\, d\nu(x) = 1$ and if $f : \Omega \to [a, b]$ is a ν-measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then*

$$R_G(\Phi, f; w) \leq \begin{cases} \|\Phi'\|_{[a,b],\infty} R_G(f; w) & \text{if } \Phi' \in L_\infty([\alpha, \beta]), \\ \\ \frac{1}{2^{1/p}}\|\Phi'\|_{[a,b],p} R_G^{1/q}(f; w) & \text{if } \Phi' \in L_p([\alpha, \beta]), \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \tag{3.7}$$

where the Lebesgue norms are defined by

$$\|g\|_{[\alpha,\beta],p} := \begin{cases} \operatorname{essup}_{t\in[\alpha,\beta]} |g(t)| & \text{if } p = \infty, \\ \\ \left(\int_{\alpha}^{\beta} |g(t)|^p\, dt\right)^{1/p} & \text{if } p \geq 1 \end{cases}$$

and $L_p([\alpha, \beta]) := \left\{ g \mid g \text{ measurable and } \|g\|_{[\alpha,\beta],p} < \infty \right\},\ p \in [1, \infty]$.

Demostración. Since f is absolutely continuous, then we have

$$\Phi(t) - \Phi(s) = \int_{s}^{t} \Phi'(u)\, du$$

for any $t, s \in [a, b]$.

Using the Hölder integral inequality we have

$$\begin{aligned} |\Phi(t) - \Phi(s)| &= \left|\int_{s}^{t} \Phi'(u)\, du\right| \\ &\leq \begin{cases} \|\Phi'\|_{[a,b],\infty} |t - s| & \text{if } p = \infty, \\ \\ \|\Phi'\|_{[a,b],p} |t - s|^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \end{aligned} \tag{3.8}$$

for any $t,\ s \in [a,b]$.

Using (3.8) we then have

$$
\begin{aligned}
& |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \\
& \leq \begin{cases} \|\Phi'\|_{[a,b],\infty} |f(x) - f(y)| & \text{if } p = \infty, \\ \\ \|\Phi'\|_{[a,b],p} |f(x) - f(y)|^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases}
\end{aligned} \tag{3.9}
$$

for any $x,\ y \in \Omega$.

If we multiply (3.9) by $\frac{1}{2} w(x) w(y)$ and integrate, then we get

$$
\begin{aligned}
& \frac{1}{2} \int_\Omega \int_\Omega w(x) w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)| \, d\nu(x) \, d\nu(y) \\
& \leq \begin{cases} \frac{1}{2} \|\Phi'\|_{[a,b],\infty} \int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \ \text{ if } p = \infty, \\ \\ \frac{1}{2} \|\Phi'\|_{[a,b],p} \int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)|^{1/q} \, d\nu(x) \, d\nu(y) \\ \\ \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases}
\end{aligned} \tag{3.10}
$$

This proves the first branch of (3.7).

Using Jensen's integral inequality for concave function $\Psi(t) = t^s$, $s \in (0,1)$ we have for $s = \frac{1}{q} < 1$ that

$$
\begin{aligned}
& \int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)|^{1/q} \, d\nu(x) \, d\nu(y) \\
& \leq \left(\int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \right)^{1/q},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \frac{1}{2} \|\Phi'\|_{[a,b],p} \int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)|^{1/q} \, d\nu(x) \, d\nu(y) \\
& \leq \frac{1}{2} \|\Phi'\|_{[a,b],p} \left(\int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \right)^{1/q} \\
& = \|\Phi'\|_{[a,b],p} \left(\frac{1}{2^q} \int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \right)^{1/q} \\
& = \|\Phi'\|_{[a,b],p} \left(\frac{1}{2^{q-1}} \frac{1}{2} \int_\Omega \int_\Omega w(x) w(y) |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \right)^{1/q} \\
& = \frac{1}{2^{\frac{q-1}{q}}} \|\Phi'\|_{[a,b],p} (R_G(f; w))^{1/q} = \frac{1}{2^{1/p}} \|\Phi'\|_{[a,b],p} R_G^{1/q}(f; w)
\end{aligned}
$$

and the second part of (3.7) is proved. ∎

The function $\Phi : [a,b] \to \mathbb{R}$ is called of r-H-*Hölder type* with the given constants $r \in (0,1]$ and $H > 0$ if

$$
|\Phi(t) - \Phi(s)| \leq H |t - s|^r
$$

for any $t,\ s \in [a, b]$.

In the case when $r = 1$, namely, there is the constant $L > 0$ such that

$$|\Phi(t) - \Phi(s)| \leq L|t - s|$$

for any $t, s \in [a, b]$, the function Φ is called L-Lipschitzian on $[a, b]$.

We have:

Theorem 4. *Let $\Phi : [a, b] \to \mathbb{R}$ be a function of r-H-Hölder type on the closed interval $[a, b]$. If $w : \Omega \to \mathbb{R}$ is a ν-measurable function with $w(x) \geq 0$ for ν-a.e. $x \in \Omega$ and $\int_\Omega w(x)\, d\nu(x) = 1$ and if $f : \Omega \to [a, b]$ is a ν-measurable function with $\Phi \circ f \in L_w(\Omega, \nu)$, then*

$$R_G(\Phi, f; w) \leq \frac{1}{2^{1-r}} H R_G^r(f; w). \tag{3.11}$$

In particular, if Φ is L-Lipschitzian on $[a, b]$, then

$$R_G(\Phi, f; w) \leq L R_G(f; w). \tag{3.12}$$

Demostración. We have

$$|(\Phi \circ f)(x) - (\Phi \circ f)(y)| \leq H|f(x) - f(y)|^r \tag{3.13}$$

for any $x,\ y \in \Omega$.

If we multiply (3.13) by $\frac{1}{2} w(x) w(y)$ and integrate, then we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega\int_\Omega w(x) w(y) |(\Phi \circ f)(x) - (\Phi \circ f)(y)|\, d\nu(x)\, d\nu(y) \\ &\leq \frac{1}{2} H \int_\Omega\int_\Omega w(x) w(y) |f(x) - f(y)|^r\, d\nu(x)\, d\nu(y). \end{aligned} \tag{3.14}$$

By Jensen's integral inequality for concave functions we also have

$$\begin{aligned} &\int_\Omega\int_\Omega w(x) w(y) |f(x) - f(y)|^r\, d\nu(x)\, d\nu(y) \\ &\leq \left(\int_\Omega\int_\Omega w(x) w(y) |f(x) - f(y)|\, d\nu(x)\, d\nu(y)\right)^r. \end{aligned} \tag{3.15}$$

Therefore, by (3.14) and (3.15) we get

$$\begin{aligned} R_G(\Phi, f; w) &\leq \frac{1}{2} H \left(\int_\Omega\int_\Omega w(x) w(y) |f(x) - f(y)|\, d\nu(x)\, d\nu(y)\right)^r \\ &= \frac{1}{2^{1-r}} H \left(\frac{1}{2}\int_\Omega\int_\Omega w(x) w(y) |f(x) - f(y)|\, d\nu(x)\, d\nu(y)\right)^r \\ &= \frac{1}{2^{1-r}} H R_G^r(f; w) \end{aligned}$$

and the inequality (3.11) is proved. ∎

We have:

Theorem 5. *Let* $\Phi, \Psi : [a,b] \to \mathbb{R}$ *be continuos functions on* $[a,b]$ *and differentiable on* (a,b) *with* $\Psi'(t) \neq 0$ *for* $t \in (a,b)$. *If* $w : \Omega \to \mathbb{R}$ *is a* ν*-measurable function with* $w(x) \geq 0$ *for* ν *-a.e.* $x \in \Omega$ *and* $\int_\Omega w(x)\, d\nu(x) = 1$ *and if* $f : \Omega \to [a,b]$ *is a* ν*-measurable function with* $\Phi \circ f \in L_w(\Omega, \nu)$, *then*

$$\inf_{t\in(a,b)} \left|\frac{\Phi'(t)}{\Psi'(t)}\right| R_G(\Psi, f; w) \leq R_G(\Phi, f; w) \leq \sup_{t\in(a,b)} \left|\frac{\Phi'(t)}{\Psi'(t)}\right| R_G(\Psi, f; w). \tag{3.16}$$

Demostración. By the Cauchy's mean value theorem, for any t, $s \in [a,b]$ with $t \neq s$ there exists a ξ between t and s such that

$$\frac{\Phi(t) - \Phi(s)}{\Psi(t) - \Psi(s)} = \frac{\Phi'(\xi)}{\Psi'(\xi)}.$$

This implies that

$$\begin{aligned} \inf_{\tau\in(a,b)} \left|\frac{\Phi'(\tau)}{\Psi'(\tau)}\right| |\Psi(t) - \Psi(s)| &\leq |\Phi(t) - \Phi(s)| \\ &\leq \sup_{\tau\in(a,b)} \left|\frac{\Phi'(\tau)}{\Psi'(\tau)}\right| |\Psi(t) - \Psi(s)| \end{aligned} \tag{3.17}$$

for any t, $s \in [a,b]$.

Therefore, we have

$$\begin{aligned} \inf\nolimits_{\tau\in(a,b)} \left|\frac{\Phi'(\tau)}{\Psi'(\tau)}\right| |\Psi(f(x)) - \Psi(f(y))| \quad &\leq |\Phi(f(x)) - \Phi(f(y))| \\ &\leq \sup\nolimits_{t\in(a,b)} \left|\frac{\Phi'(\tau)}{\Psi'(\tau)}\right| |\Psi(f(x)) - \Psi(f(y))| \end{aligned} \tag{3.18}$$

for any $x, y \in \Omega$.

If we multiply (3.18) by $\frac{1}{2} w(x) w(y)$ and integrate, we get the desired result (3.16). ∎

Corollary 2. *Let* $\Phi : [a,b] \to \mathbb{R}$ *be a continuos function on* $[a,b]$ *and differentiable on* (a,b). *If* w *is as in Theorem 5, then we have*

$$\inf_{t\in(a,b)} |\Phi'(t)| R_G(f; w) \leq R_G(\Phi, f; w) \leq \sup_{t\in(a,b)} |\Phi'(t)| R_G(f; w). \tag{3.19}$$

We also have:

Theorem 6. *Let* $\Phi : [a,b] \to \mathbb{R}$ *be an absolutely continuous function on the closed interval* $[a,b]$. *If* $w : \Omega \to \mathbb{R}$ *is a* ν*-measurable function with* $w(x) \geq 0$ *for* ν *-a.e.* $x \in \Omega$ *and* $\int_\Omega w(x)\, d\nu(x) = 1$

and if $f : \Omega \to [a, b]$ *is a* ν*-measurable function with* $\Phi \circ f \in L_w(\Omega, \nu)$, *then*

$$
\begin{aligned}
&R_G(\Phi, f; w) \\
&\leq \begin{cases} \|\Phi'\|_{[a,b],\infty} M(f; w) & \textit{if } p = \infty, \\ \\ \|\Phi'\|_{[a,b],p} M^{1/q}(f; w) & \textit{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \\
&\leq \begin{cases} \frac{1}{2}(b-a)\|\Phi'\|_{[a,b],\infty} & \textit{if } p = \infty, \\ \\ \frac{1}{2^{1/q}}(b-a)^{1/q}\|\Phi'\|_{[a,b],p} & \textit{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases}
\end{aligned} \tag{3.20}
$$

where $M(f; w)$ *is defined by*

$$
M(f; w) := \int_\Omega w(x) \left| f(x) - \frac{a+b}{2} \right| d\nu(x). \tag{3.21}
$$

Demostración. From the inequality (3.8) we have

$$
\begin{aligned}
&\left| (\Phi \circ f)(x) - \Phi\left(\tfrac{a+b}{2}\right) \right| \\
&\leq \begin{cases} \|\Phi'\|_{[a,b],\infty} \left| f(x) - \frac{a+b}{2} \right| & \text{if } p = \infty, \\ \\ \|\Phi'\|_{[a,b],p} \left| f(x) - \frac{a+b}{2} \right|^{1/q} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases}
\end{aligned} \tag{3.22}
$$

for any $x \in \Omega$.

Now, if we multiply (3.22) by $w(x)$ and integrate, then we get

$$
\begin{aligned}
&\int_\Omega w(x) \left| (\Phi \circ f)(x) - \Phi\left(\frac{a+b}{2}\right) \right| d\nu(x) \\
&\leq \begin{cases} \|\Phi'\|_{[a,b],\infty} \int_\Omega w(x) \left| f(x) - \frac{a+b}{2} \right| d\nu(x) & \text{if } p = \infty, \\ \\ \|\Phi'\|_{[a,b],p} \int_\Omega w(x) \left| f(x) - \frac{a+b}{2} \right|^{1/q} d\nu(x) & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases}
\end{aligned} \tag{3.23}
$$

By Jensen's integral inequality for concave functions we have

$$
\int_\Omega w(x) \left| f(x) - \frac{a+b}{2} \right|^{1/q} d\nu(x) \leq \left(\int_\Omega w(x) \left| f(x) - \frac{a+b}{2} \right| d\nu(x) \right)^{1/q}. \tag{3.24}
$$

On making use of (3.2), (3.23) and (3.24) we get the first inequality in (3.20).

The last part of (3.20) follows by the fact that

$$
\left| f(x) - \frac{a+b}{2} \right| \leq \frac{1}{2}(b-a)
$$

for any $x \in \Omega$. ∎

4. Bounds for Special Convexity

When some convexity properties for the function Φ are assumed, then other bounds can be derived as follows.

Theorem 7. *Let* $w : \Omega \to \mathbb{R}$ *be a* ν*-measurable function with* $w(x) \geq 0$ *for* ν *-a.e.* $x \in \Omega$ *and* $\int_\Omega w(x)\, d\nu(x) = 1$ *and* $f : \Omega \to [a,b]$ *be a* ν*-measurable function with* $\Phi \circ f \in L_w(\Omega, \nu)$. *Assume also that* $\Phi : [a,b] \to \mathbb{R}$ *is a continuous function on* $[a,b]$.

(i) If $|\Phi|$ *is concave on* $[a,b]$, *then*

$$R_G(\Phi, f; w) \leq |\Phi(E(f;w))|, \tag{4.1}$$

(ii) If $|\Phi|$ *is convex on* $[a,b]$, *then*

$$R_G(\Phi, f; w) \leq \frac{1}{b-a}\left[(b - E(f;w))\,|\Phi(a)| + (E(f;w) - a)\,\Phi\,|(b)|\right]. \tag{4.2}$$

Demostración. (i) If $|\Phi|$ is concave on $[a,b]$, then by Jensen's inequality we have

$$\int_\Omega w(x)\,|(\Phi \circ f)(x)|\, d\nu(x) \leq \left|\Phi\left(\int_\Omega w(x)\, f(x)\, d\nu(x)\right)\right|. \tag{4.3}$$

From (3.2) for $\gamma = 0$ we also have

$$R_G(\Phi, f; w) \leq \int_\Omega w(x)\,|(\Phi \circ f)(x)|\, d\nu(x). \tag{4.4}$$

This is an inequality of interest in itself.

On utilizing (4.3) and (4.4) we get (4.1).

(ii) Since $|\Phi|$ is convex on $[a,b]$, then for any $t \in [a,b]$ we have

$$|\Phi(t)| = \left|\Phi\left(\frac{(b-t)\,a + b\,(t-a)}{b-a}\right)\right| \leq \frac{(b-t)\,|\Phi(a)| + (t-a)\,\Phi\,|(b)|}{b-a}.$$

This implies that

$$|(\Phi \circ f)(x)| \leq \frac{(b - f(x))\,|\Phi(a)| + (f(x) - a)\,\Phi\,|(b)|}{b-a} \tag{4.5}$$

for any $x \in \Omega$.

If we multiply (4.5) by $w(x)$ and integrate, then we get

$$\begin{aligned}
&\int_\Omega w(x)\,|(\Phi \circ f)(x)|\, d\nu(x) \\
&\leq \frac{1}{b-a}\left[\left(b\int_\Omega w(x)\, d\nu(x) - \int_\Omega w(x)\, f(x)\, d\nu(x)\right)|\Phi(a)|\right. \\
&\left. + \left(\int_\Omega w(x)\, f(x)\, d\nu(x) - a\int_\Omega w(x)\, d\nu(x)\right)\Phi\,|(b)|\right],
\end{aligned}$$

which, together with (4.4), produces the desired result (4.2). ∎

In order to state other results we need the following definitions:

Definition 1 ([19]). *We say that a function* $f : I \to \mathbb{R}$ *belongs to the class* $P(I)$ *if it is nonnegative and for all* $x, y \in I$ *and* $t \in [0,1]$ *we have*

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

It is important to note that $P(I)$ contains all nonnegative monotone, convex and *quasi convex functions,* i.e. functions satisfying

$$f(tx + (1-t)y) \leq \text{máx}\{f(x), f(y)\}$$

for all x, $y \in I$ and $t \in [0,1]$.

For some results on P-functions see [19] and [28] while for quasi convex functions, the reader can consult [18].

Definition 2 ([3]). *Let* s *be a real number,* $s \in (0,1]$. *A function* $f : [0,\infty) \to [0,\infty)$ *is said to be* s*-convex (in the second sense) or Breckner* s*-convex if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ *and* $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [3], [4], [16], [17], [25], [27] and [29].

Theorem 8. *Let* $w : \Omega \to \mathbb{R}$ *be a* ν*-measurable function with* $w(x) \geq 0$ *for* ν *-a.e.* $x \in \Omega$ *and* $\int_\Omega w(x)\, d\nu(x) = 1$ *and* $f : \Omega \to [a,b]$ *be a* ν*-measurable function with* $\Phi \circ f \in L_w(\Omega, \nu)$. *Assume also that* $\Phi : [a,b] \to \mathbb{R}$ *is a continuous function on* $[a,b]$.

(i) If $|\Phi|$ *belongs to the class* P *on* $[a,b]$, *then*

$$R_G(\Phi, f; w) \leq |\Phi(a)| + \Phi|(b)|; \tag{4.6}$$

(ii) If $|\Phi|$ *is quasi convex on* $[a,b]$, *then*

$$R_G(\Phi, f; w) \leq \text{máx}\{|\Phi(a)|, \Phi|(b)|\}; \tag{4.7}$$

(iii) If $|\Phi|$ *is Breckner* s*-convex on* $[a,b]$, *then*

$$\begin{aligned} R_G(\Phi, f; w) &\leq \frac{1}{(b-a)^s}\left[|\Phi(a)| \int_\Omega w(x)(b - f(x))^s\, d\nu(x)\right. \\ &\left. +\Phi|(b)| \int_\Omega w(x)(f(x) - a)^s\, d\nu(x)\right] \\ &\leq \frac{1}{(b-a)^s}\left[|\Phi(a)|(b - E(f;w))^s\, d\nu(x)\right. \\ &\left. +\Phi|(b)|(E(f;w) - a)^s\, d\nu(x)\right]. \end{aligned} \tag{4.8}$$

Demostración. (i) Since $|\Phi|$ belongs to the class P on $[a,b]$, then for any $t \in [a,b]$ we have

$$|\Phi(t)| = \left|\Phi\left(\frac{(b-t)a + b(t-a)}{b-a}\right)\right| \leq |\Phi(a)| + \Phi|(b)|.$$

This implies that

$$|(\Phi \circ f)(x)| \leq |\Phi(a)| + \Phi|(b)| \tag{4.9}$$

for any $x \in \Omega$.

If we multiply (4.9) by $w(x)$ and integrate, then we get

$$\int_\Omega w(x) |(\Phi \circ f)(x)| \, d\nu(x) \leq |\Phi(a)| + \Phi|(b)|, \tag{4.10}$$

which, together with (4.4), produces the desired result (4.6).

(ii) Goes in a similar way.

(iii) By Breckner s-convexity we have

$$|\Phi(t)| = \left|\Phi\left(\frac{(b-t)a + b(t-a)}{b-a}\right)\right| \leq \left(\frac{b-t}{b-a}\right)^s |\Phi(a)| + \left(\frac{t-a}{b-a}\right)^s \Phi|(b)|$$

for any $t \in [a,b]$.

This implies that

$$|(\Phi \circ f)(x)| \leq \frac{1}{(b-a)^s}\left[(b-f(x))^s |\Phi(a)| + (f(x)-a)^s \Phi|(b)|\right] \tag{4.11}$$

for any $x \in \Omega$.

If we multiply (4.11) by $w(x)$ and integrate, then we get

$$\begin{aligned}\int_\Omega w(x)|(\Phi \circ f)(x)| \, d\nu(x) \leq \frac{1}{(b-a)^s}\Bigg[&|\Phi(a)| \int_\Omega w(x)(b-f(x))^s \, d\nu(x) \\ &+\Phi|(b)| \int_\Omega w(x)(f(x)-a)^s \, d\nu(x)\Bigg],\end{aligned} \tag{4.12}$$

which, together with (4.4), produces the first part of (4.8).

The last part follows by Jensen's integral inequality for concave functions, namely

$$\int_\Omega w(x)(b-f(x))^s \, d\nu(x) \leq \left(b - \int_\Omega w(x) f(x) \, d\nu(x)\right)^s$$

and

$$\int_\Omega w(x)(f(x)-a)^s \, d\nu(x) \leq \left(\int_\Omega w(x) f(x) \, d\nu(x) - a\right)^s,$$

where $s \in (0,1)$. ∎

5. Some Examples

Let $f : \Omega \to [0, \infty)$ be a ν-measurable function and $w : \Omega \to \mathbb{R}$ a ν-measurable function with $w(x) \geq 0$ for ν -a.e. $x \in \Omega$ and $\int_\Omega w(x)\, d\nu(x) = 1$. We define, for the function $\Phi(t) = t^p$, $p > 0$, the *generalized* (p, f)-*mean difference* $R_G(p, f; w)$ by

$$R_G(p, f; w) := \frac{1}{2}\int_\Omega \int_\Omega w(x)\, w(y) \left|f^p(x) - f^p(y)\right| d\nu(x)\, d\nu(y) \tag{5.1}$$

and the *generalized* (p, f)-*mean deviation* $M_D(p, f; w)$ by

$$M_D(p, f; w) := \int_\Omega w(x) \left|f^p(x) - E(p, f; w)\right| d\nu(x), \tag{5.2}$$

where

$$E(p, f; w) := \int_\Omega f^p(y)\, w(y)\, d\nu(y) \tag{5.3}$$

is the *generalized* (p, f)-expectation.

If $f : \Omega \to [a, b] \subset [0, \infty)$ is a ν-measurable function, then by (3.1) we have

$$R_G(p, f; w) \leq \frac{1}{2}\left(b^p - a^p\right). \tag{5.4}$$

By (3.7) we have

$$R_G(p, f; w) \leq p\delta_p(a, b)\, R_G(f; w), \tag{5.5}$$

where

$$\delta_p(a, b) := \begin{cases} b^{p-1} \text{ if } p \geq 1, \\ \\ a^{p-1} \text{ if } p \in (0, 1) \end{cases}$$

and

$$R_G(p, f; w) \leq \frac{p}{2^{1/\alpha}}\left[\frac{b^{\alpha(p-1)+1} - a^{\alpha(p-1)+1}}{\alpha(p-1)+1}\right]^{1/\alpha} R_G^{1/\beta}(f; w), \tag{5.6}$$

where $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

From (3.20) we also have

$$\begin{aligned} & R_G(p, f; w) \\ & \leq \begin{cases} \delta_p(a, b)\, M(f; w), \\ \\ p\left(\frac{b^{\alpha(p-1)+1} - a^{\alpha(p-1)+1}}{\alpha(p-1)+1}\right)^{1/\alpha} M^{1/\beta}(f; w) \text{ if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1 \end{cases} \\ & \leq \begin{cases} \frac{1}{2}(b - a)\, \delta_p(a, b), \\ \\ \frac{1}{2^{1/\beta}}(b - a)^{1/\beta}\, p\left(\frac{b^{\alpha(p-1)+1} - a^{\alpha(p-1)+1}}{\alpha(p-1)+1}\right)^{1/\alpha} \text{ if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \end{cases} \end{aligned} \tag{5.7}$$

where $M(f;w)$ is defined by (3.21).

If $p\in(0,1)$, then the function $|\Phi(t)|=t^p$ is concave on $[a,b]\subset[0,\infty)$ and by (4.1) we have

$$R_G(p,f;w)\le E^p(f;w). \tag{5.8}$$

For $p\ge 1$ the function $|\Phi(t)|=t^p$ is convex on $[a,b]\subset[0,\infty)$ and by (4.2) we have

$$R_G(p,f;w)\le \frac{1}{b-a}\left[(b-E(f;w))\,a^p+(E(f;w)-a)\,b^p\right]. \tag{5.9}$$

Let $f:\Omega\to[0,\infty)$ be a ν-measurable function and $w:\Omega\to\mathbb{R}$ a ν-measurable function with $w(x)\ge 0$ for ν -a.e. $x\in\Omega$ and $\int_\Omega w(x)\,d\nu(x)=1$. We define, for the function $\Phi(t)=\ln t$, the *generalized* $(\ln,f)$*-mean difference* $R_G(\ln,f;w)$ by

$$R_G(\ln,f;w):=\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|\ln f(x)-\ln f(y)|\,d\nu(x)\,d\nu(y) \tag{5.10}$$

and the *generalized* (p,f)*-mean deviation* $M_D(\ln,f;w)$ by

$$M_D(\ln,f;w):=\int_\Omega w(x)\,|\ln f(x)-E(\ln,f;w)|\,d\nu(x), \tag{5.11}$$

where

$$E(\ln,f;w):=\int_\Omega w(y)\ln f(y)\,d\nu(y) \tag{5.12}$$

is the *generalized* $(\ln,f)$*-expectation.*

If $f:\Omega\to[a,b]\subset[0,\infty)$ is a ν-measurable function, then by (3.1) we have

$$R_G(\ln,f;w)\le\frac{1}{2}(\ln b-\ln a). \tag{5.13}$$

By (3.7) we have

$$R_G(\ln,f;w)\le\begin{cases}\frac{1}{a}R_G(f;w),\\[2ex] \frac{1}{2^{1/p}}\left(\frac{b^{p-1}-a^{p-1}}{(p-1)b^{p-1}a^{p-1}}\right)^{1/p}R_G^{1/q}(f;w)\ \text{ if } p>1,\ \frac{1}{p}+\frac{1}{q}=1.\end{cases} \tag{5.14}$$

By (3.20) we have

$$\begin{aligned}R_G(\ln,f;w)&\le\begin{cases}\frac{1}{a}M(f;w),\\[2ex] \left(\frac{b^{p-1}-a^{p-1}}{(p-1)b^{p-1}a^{p-1}}\right)^{1/p}M^{1/q}(f;w)\ \text{ if } p>1,\ \frac{1}{p}+\frac{1}{q}=1\end{cases}\\ &\le\begin{cases}\frac{1}{2}\left(\frac{b}{a}-1\right),\\[2ex] \frac{1}{2^{1/q}}(b-a)^{1/q}\left(\frac{b^{p-1}-a^{p-1}}{(p-1)b^{p-1}a^{p-1}}\right)^{1/p}\ \text{ if } p>1,\ \frac{1}{p}+\frac{1}{q}=1.\end{cases}\end{aligned} \tag{5.15}$$

Now, observe that the function $|\Phi(t)| = |\ln t|$ is convex on $(0, 1)$ and concave on $[1, \infty)$. If $f : \Omega \to [a, b] \subset (0, 1)$ is a ν-measurable function, then by (4.2) we have

$$R_G(\ln, f; w) \leq \frac{1}{b-a}\left[(b - E(f; w))|\ln a| + (E(f; w) - a)|\ln b|\right] \tag{5.16}$$

and if $f : \Omega \to [a, b] \subset [1, \infty)$, then by (4.1) we have

$$R_G(\ln, f; w) \leq \ln(E(f; w)). \tag{5.17}$$

The interested reader may state similar bounds for functions Φ such as $\Phi(t) = \exp t$, $t \in \mathbb{R}$ or $\Phi(t) = t \ln t$, $t > 0$. We omit the details.

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η-Ricci Solitons on 3-dimensional Trans-Sasakian Manifolds

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ABSTRACT

In this paper, we study η-Ricci solitons on 3-dimensional trans-Sasakian manifolds. Firstly we give conditions for the existence of these geometric structures and then observe that they provide examples of η-Einstein manifolds. In the case of ϕ-Ricci symmetric trans-Sasakian manifolds, the η-Ricci soliton condition turns them to Einstein manifolds. Afterward, we study the implications in this geometric context of the important tensorial conditions $R \cdot S = 0$, $S \cdot R = 0$, $W_2 \cdot S = 0$ and $S \cdot W_2 = 0$.

RESUMEN

En este artículo estudiamos solitones η-Ricci en variedades trans-Sasakianas tridimensionales. En primer lugar damos condiciones para la existencia de estas estructuras geométricas y luego observamos que ellas dan ejemplos de variedades η-Einstein. En el caso de variedades trans-Sasakianas ϕ-Ricci simétricas, la condición de solitón η-Ricci las convierte en variedades Einstein. A continuación estudiamos las implicancias en este contexto geométrico de las importantes condiciones tensoriales $R \cdot S = 0$, $S \cdot R = 0$, $W_2 \cdot S = 0$ y $S \cdot W_2 = 0$.

Keywords and Phrases: Trans-Sasakian manifold, η-Ricci solitons.

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1 Introduction

In 1982, the notion of the Ricci flow was introduced by Hamilton [10] to find a canonical metric on a smooth manifold.The Ricci flow is an evolution equation for Riemannian metric $g(t)$ on a smooth manifold M given by

$$\frac{\partial}{\partial t}g(t) = -2S.$$

A solution to this equation (or a Ricci flow) is a one-parameter family of metrics $g(t)$, parameterized by t in a non-degenerate interval I, on a smooth manifold M satisfying the Ricci flow equation. If I has an initial point t_0, then $(M, g(t_0))$ is called the initial condition of or the initial metric for the Ricci flow (or of the solution) [14].

Ricci solitons and η-Ricci solitons are natural generalizations of Einstein metrics. A Ricci soliton on a Riemannian manifold (M, g) is defined by

$$S + \frac{1}{2}\mathcal{L}_X g = \lambda g$$

where $\mathcal{L}_X g$ is the Lie derivative along the vector field X, S is the Ricci tensor of the metric and λ is a real constant. If $X = \nabla f$ for some function f on M, the Ricci soliton becomes gradient Ricci soliton. Ricci solitons appear as self-similar solutions to Hamiltons's Ricci flow and often arise as limits of dilations of singularities in the Ricci flow [11]. A soliton is called shrinking, steady and expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

In 2009, the notion of η-Ricci soliton was introduced by J.C. Cho and M. Kimura [6]. J.C. Cho and M. Kimura proved that a real hypersurface admitting an η-Ricci soliton in a non-flat complex space form is a Hopf-hypersurface [6]. An η-Ricci soliton on a Riemannian manifold (M, g) is defined by the following equation

$$2S + \mathcal{L}_\xi g + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.1}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor of the metric and λ, μ are real constants. If $\mu = 0$, then η-Ricci soliton becomes Ricci soliton.

In the last few years, many authors have worked on Ricci solitons and their generalizations in different Contact metric manfolds in [1], [7], [8], [9], [12] etc. In 2014, B. Y. Chen and S. Deshmukh have established the characterizations of compact shrinking trivial Ricci solitons in [5]. Also, in [2], A. Bhattacharyya, T. Dutta, and S. Pahan studied the torqued vector field and established some applications of torqued vector field on Ricci soliton and conformal Ricci soliton. A.M. Blaga [3], D. G. Prakasha and B. S. Hadimani [17] observed η-Ricci solitons on different contact metric manifolds satisfying some certain curvature conditions.

In this paper we study the existence of η-Ricci soliton on 3-dimensional trans-Sasakian manifold. Next we show that η-Ricci soliton on 3-dimensional trans-Sasakian manifolds becomes η-Einstein Manifold under some conditions. Next we prove that ϕ-Ricci symmetric trans-Sasakian manifold (M, g) manifold satisfying an η-Ricci soliton becomes an Einstein manifold. Next we give an example of an η-Ricci soliton on 3-dimensional trans-Sasaian manifold with $\lambda = -2$ and $\mu = 6$. Later we obtain some different types of curvature tensors and their properties under certain conditions.

2 Preliminaries

The product $\bar{M} = M \times R$ has a natural almost complex structure J with the product metric G being Hermitian metric. The geometry of the almost Hermitian manifold $(\bar{M}, J, G)$ gives the geometry of the almost contact metric manifold (M, ϕ, ξ, η, g). Sixteen different types of structures on M like Sasakian manifold, Kenmotsu manifold etc are given by the almost Hermitian manifold $(\bar{M}, J, G)$. The notion of trans-Sasakian manifolds was introduced by Oubina [15] in 1985. Then J. C. Marrero [13] have studied the local structure of trans-Sasakian manifolds. In general a trans-Sasakian manifold $(M, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type (α, β). An n $(= 2m + 1)$ dimensional Riemannian manifold (M, g) is called an almost contact manifold if there exists a (1,1) tensor field ϕ, a vector field ξ and a 1-form η on M such that

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \eta(\phi X) = 0, \tag{2.2}$$

$$\phi\xi = 0, \tag{2.3}$$

$$\eta(X) = g(X, \xi), \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.5}$$

$$g(X, \phi Y) + g(Y, \phi X) = 0, \tag{2.6}$$

for any vector fields X, Y on M. A 3-dimensional almost contact metric manifold M is called a trans-Sasakian manifold if it satisfies the following condition

$$(\nabla_X\phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.7}$$

for some smooth functions α, β on M and we say that the trans-Sasakian structure is of type (α, β). For 3-dimensional trans-Sasakian manifold, from (2.7) we have,

$$\nabla_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.8}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.9}$$

In a 3-dimensional trans-Sasakian manifold, we have

$$\begin{aligned} R(X,Y)Z &= [\tfrac{r}{2} - 2(\alpha^2 - \beta^2 - \xi\beta)][g(Y,Z)X - g(X,Z)Y] \\ &- [\tfrac{r}{2} - 3(\alpha^2 - \beta^2) + \xi\beta][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ &+ [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)][\phi \ grad \ \alpha - \ grad \ \beta] \\ &- [\tfrac{r}{2} - 3(\alpha^2 - \beta^2) + \xi\beta]\eta(Z)[\eta(Y)X - \eta(X)Y] \\ &- [Z\beta + (\phi Z)\alpha]\eta(Z)[\eta(Y)X - \eta(X)Y] \\ &- [X\beta + (\phi X)\alpha][g(Y,Z)\xi - \eta(Z)Y] \\ &- [Y\beta + (\phi Y)\alpha][g(X,Z)\xi - \eta(Z)X], \end{aligned}$$

$$\begin{aligned} S(X,Y) &= [\tfrac{r}{2} - (\alpha^2 - \beta^2 - \xi\beta)]g(X,Y) \\ &- [\tfrac{r}{2} - 3(\alpha^2 - \beta^2) + \xi\beta]\eta(X)\eta(Y) \\ &- [Y\beta + (\phi Y)\alpha]\eta(X) - [X\beta + (\phi X)\alpha]\eta(Y). \end{aligned}$$

When α and β are constants the above equations reduce to,

$$R(\xi, X)\xi = (\alpha^2 - \beta^2)(\eta(X)\xi - X), \tag{2.10}$$

$$S(X, \xi) = 2(\alpha^2 - \beta^2)\eta(X), \tag{2.11}$$

$$R(\xi, X)Y = (\alpha^2 - \beta^2)(g(X,Y)\xi - \eta(Y)X). \tag{2.12}$$

$$R(X,Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y). \tag{2.13}$$

Definition 2.1. A trans-Sasakian manifold M^3 is said to be η-Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = a g(X,Y) + b\eta(X)\eta(Y),$$

where a, b are smooth functions.

3 η-Ricci solitons on trans-Sasakian manifolds

To study the existence conditions of η-Ricci solitons on 3-dimensional trans-Sasakian manifolds, we prove the following theorem.

Theorem 3.1: Let $(M, g, \phi, \eta, \xi, \alpha, \beta)$ be a 3-dimensional trans-Sasakian manifold with α, β constants $(\beta \neq 0)$. If the symmetric $(0, 2)$ tensor field h satisfying the condition $\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)] = \mathcal{L}_\xi g(X, Y) + 2S(X, Y) + 2\mu\eta(X)\eta(Y)$ is parallel with respect to the Levi-Civita connection associated to g. Then (g, ξ, μ) becomes an η-Ricci soliton.

Proof: We consider a symmetric $(0, 2)$-tensor field h which is parallel with respect to the Levi-Civita connection $(\nabla h = 0)$. Then it follows that

$$h(R(X, Y)Z, W) + h(R(X, Y)Z, W) = 0, \tag{3.1}$$

for an arbitary vector field W, X, Y, Z on M. Put $X = Z = W = \xi$ we get

$$h(R(X, Y)\xi, \xi) = 0, \tag{3.2}$$

for any X, $Y \in \chi(M)$ By using the equation (2.13)

$$h(Y, \xi) = g(Y, \xi)h(\xi, \xi), \tag{3.3}$$

for any $Y \in \chi(M)$. Differentiating the equation (3.3) covariantly with respect to the vector field $X \in \chi(M)$ we have

$$h(\nabla_X Y, \xi) + h(Y, \nabla_X \xi) = g(\nabla_X Y, \xi)h(\xi, \xi) + g(Y, \nabla_X \xi)h(\xi, \xi), \tag{3.4}$$

Using the equation (2.8) we have

$$\beta h(X, Y) - \alpha h(\phi X, Y) = -\alpha g(\phi X, Y)h(\xi, \xi) + \beta h(\xi, \xi)g(X, Y). \tag{3.5}$$

Interchanging X by Y we have

$$\beta h(X, Y) - \alpha h(X, \phi Y) = -\alpha g(X, \phi Y)h(\xi, \xi) + \beta h(\xi, \xi)g(X, Y). \tag{3.6}$$

Then adding the above two equations we get

$$\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)] = \beta h(\xi, \xi)g(X, Y). \tag{3.7}$$

We see that $\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)]$ is a symmetric tensor of type $(0, 2)$. Since $\mathcal{L}_\xi g(X, Y)$, $S(X, Y)$, $\eta(X) = g(X, \xi)$ and $\eta(Y) = g(Y, \xi)$ are symmetric tensors of type $(0, 2)$ and λ, μ are real constants, the sum $\mathcal{L}_\xi g(X, Y) + 2S(X, Y) + 2\mu\eta(X)\eta(Y)$ is a symmetric tensor of type $(0, 2)$.

Therefore, we can take the sum as an another symmetric tensor field of type $(0, 2)$. Hence for we can assume that $\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)] = \mathcal{L}_\xi g(X, Y) + 2S(X, Y) + 2\mu\eta(X)\eta(Y)$.
Then we compute

$$\beta h(\xi, \xi)g(X, Y) = \mathcal{L}_\xi g(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y).$$

As h is parallel so, $h(\xi, \xi)$ is constant. Hence, we can write $h(\xi, \xi) = -\frac{2}{\beta}\lambda$ where β is constant and $\beta \neq 0$.
So, from the equation (3.7) we have

$$\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)] = -2\lambda g(X, Y), \tag{3.8}$$

for any $X, Y \in \chi(M)$. Therefore $\mathcal{L}_\xi g(X, Y) + 2S(X, Y) + 2\mu\eta(X)\eta(Y) = -2\lambda g(X, Y)$ and so (g, ξ, μ) becomes an η-Ricci soliton.

Corollary 3.2: Let $(M, g, \phi, \eta, \xi, \alpha, \beta)$ be a 3-dimensional trans-Sasakian manifold with α, β constants $(\beta \neq 0)$. If the symmetric $(0, 2)$ tensor field h admitting the condition $\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)] = \mathcal{L}_\xi g(X, Y) + 2S(X, Y)$ is parallel with respect to the Levi-Civita connection associated to g with $\lambda = 2n$. Then (g, ξ) becomes a Ricci soliton.

Next theorem shows the necessary condition for the existence of η-Ricci soliton on 3-dimensional trans-Sasakian manifolds.
Theorem 3.3: If 3-dimensional trans-Sasakian manifold satisfies an η-Ricci soliton then the manifold becomes η-Einstein manifold with α and β constants.

Proof: From the equation (1.1) we get

$$2S(X, Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X, Y) - 2\mu\eta(X)\eta(Y). \tag{3.9}$$

By using the equation (2.8) we get

$$S(X, Y) = -(\beta + \lambda)g(X, Y) + (\beta - \mu)\eta(X)\eta(Y) \tag{3.10}$$

and

$$S(X, \xi) = -(\lambda + \mu)\eta(X). \tag{3.11}$$

Also from (2.11) we have

$$\lambda + \mu = 2(\beta^2 - \alpha^2). \tag{3.12}$$

The Ricci operator Q is defined by $g(QX, Y) = S(X, Y)$. Then we get

$$QX = (\mu - \beta + 2(\alpha^2 - \beta^2))X + (\beta - \mu)\eta(X)\xi. \tag{3.13}$$

Then we can easily see that the manifold is an η-Einstein manifold.

We know a manifold is ϕ-Ricci symmetric if $\phi^2 \circ \nabla Q = 0$. Now we prove the next theorem.

Theorem 3.4: If a ϕ-Ricci symmetric trans-Sasakian manifold (M, g) satisfies an η-Ricci soliton then $\mu = \beta$, $\lambda = 2(\beta^2 - \alpha^2) - \beta$ and (M, g) is an Einstein manifold.

Proof: From the equation (3.13) we have

$$(\nabla_X Q)Y = \nabla_X QY - Q(\nabla_X Y)$$

$$= -\alpha(\beta - \mu)\eta(Y)\phi X + \beta(\beta - \mu)\eta(Y)X - (\beta - \mu)\eta(Y)\eta(X)\xi$$

$$+(\beta - \mu)[-\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y)]\xi.$$

Now applying ϕ^2 both sides we have $\mu = \beta$, $\lambda = 2(\beta^2 - \alpha^2) - \beta$ and (M, g) is an Einstein manifold.

We construct an example of η-Ricci soliton on 3-dimensional trans-Sasakian manifolds in the The next section.

4 Example of η-Ricci solitons on 3-dimensional trans-Sasakian manifolds

We consider the three dimensional manifold $M = \{(x, y, z) \in R^3 : y \neq 0\}$ where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = e^{2z}\frac{\partial}{\partial x}, e_2 = e^{2z}\frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M^3)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi(e_1) = e_2, \phi(e_2) = -e_1, \phi(e_3) = 0$. Then using the linearity property of ϕ and g we have

$$\eta(e_2) = 1,\ \phi^2(Z) = -Z + \eta(Z)e_2,\ g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M^3)$. Thus for $e_2 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M. Now, after some calculation we have,

$$[e_1, e_3] = -2e_1, [e_2, e_3] = -2e_2, [e_1, e_2] = 0.$$

The Riemannian connection ∇ of the metric is given by the Koszul's formula which is

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

By Koszul's formula we get,

$$\nabla_{e_1} e_1 = 2e_3, \nabla_{e_2} e_1 = 0, \nabla_{e_3} e_1 = 0, \nabla_{e_1} e_2 = 0, \nabla_{e_2} e_2 = 2e_3,$$

$$\nabla_{e_3} e_2 = 0, \nabla_{e_1} e_3 = -2e_1, \nabla_{e_2} e_3 = -2e_2, \nabla_{e_3} e_3 = 0.$$

From the above it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold of type $(0, -2)$.
Here

$$R(e_1, e_2)e_2 = -4e_1, R(e_3, e_2)e_2 = 4e_2, R(e_1, e_3)e_3 = -4e_1, R(e_2, e_3)e_3 = -4e_2,$$

$$R(e_3, e_1)e_1 = -4e_2, R(e_2, e_1)e_1 = 4e_3.$$

So, we have

$$S(e_1, e_1) = 0, S(e_2, e_2) = 0, , S(e_3, e_3) = -8. \tag{4.1}$$

From the equation (1.1) we get $\lambda = -2$ and $\mu = 6$. Therefore, (g, ξ, λ, μ) is an η-Ricci soliton on $M^3(\phi, \xi, \eta, g)$.

In the next sections we consider η-Ricci Solitons on 3-dimensional trans-Sasakian manifolds satisfying some curvature conditions.

5 η-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $R(\xi, X) \cdot S = 0$

First we suppose that 3-dimensional trans-Sasakian manifolds with η-Ricci solitons satisfy the condition

$$R(\xi, X) \cdot S = 0.$$

Then we have

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0$$

for any $X, Y, Z \in \chi(M)$.

Using the equations (2.12), (3.10), (3.11) we get

$$(\beta - \mu)g(X, Y)\eta(Z) + (\beta - \mu)g(X, Z)\eta(Y) - 2(\beta - \mu)\eta(X)\eta(Y)\eta(Z) = 0.$$

Put $Z = \xi$ we have

$$(\beta - \mu)g(X, Y) - (\beta - \mu)\eta(X)\eta(Y) = 0.$$

Setting $X = \phi X$ and $Y = \phi Y$ in the above equation we get

$$(\beta - \mu)g(\phi X, \phi Y) = 0.$$

Again using the equation (3.12) we have

$$\mu = \beta, \quad \lambda = 2(\beta^2 - \alpha^2) - \beta.$$

Also we can easily see that M is an Einstein manifold. So we have the following theorem.

Theorem 5.1: If a 3-dimensional trans-Sasakian manifold $(M, g, \phi, \eta, \xi, \alpha, \beta)$ with α, β constants admitting an η-Ricci soliton satisfies the condition $R(\xi, X) \cdot S = 0$ then $\mu = \beta$, $\lambda = 2(\beta^2 - \alpha^2) - \beta$ and M is an Einstein manifold.

Corollary 5.2: A 3-dimensional trans-Sasakian manifold with α, β constants satisfies the condition $R(\xi, X) \cdot S = 0$, there is no Ricci soliton with the potential vector field ξ.

6 η-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $S(\xi, X) \cdot R = 0$

We consider 3-dimensional trans-Sasakian manifolds with η-Ricci solitons satisfying the condition

$$S(\xi, X) \cdot R = 0.$$

So we have

$$S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W - S(\xi, Y)R(X, Z)W$$
$$+S(X, Z)R(Y, \xi)W - S(\xi, Z)R(Y, X)W + S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X = 0.$$

Taking inner product with ξ then the above equation becomes

$$\begin{aligned} S(X, R(Y, Z)W) - S(\xi, R(Y, Z)W)\eta(X) + S(X, Y)\eta(R(\xi, Z)W) \\ -S(\xi, Y)\eta(R(X, Z)W) + S(X, Z)\eta(R(Y, \xi)W) - S(\xi, Z)\eta(R(Y, X)W) \\ + S(X, W)\eta(R(Y, Z)\xi) - S(\xi, W)\eta(R(Y, Z)X) = 0. \end{aligned} \tag{6.1}$$

Put $W = \xi$ and using the equations (2.10), (2.12), (3.10), (3.11) we get

$$-(\beta + \lambda)g(X, R(Y, Z)\xi) + (\lambda + \mu)\eta(R(Y, Z)X) = 0. \tag{6.2}$$

Also we have

$$\eta(R(Y, Z)X) = -g(X, R(Y, Z)\xi).$$

So from the equation (6.2) we get

$$(\beta + 2\lambda + \mu)g(X, R(Y, Z)\xi) = 0.$$

Again using the equation (3.12) we have

$$\mu = \beta + 4(\beta^2 - \alpha^2), \quad \lambda = -[2(\beta^2 - \alpha^2) + \beta].$$

So we have the following theorem.

Theorem 6.1: If a 3-dimensional trans-Sasakian manifold $(M, g, \phi, \eta, \xi, \alpha, \beta)$ with $\alpha,\ \beta$ constants admitting an η-Ricci soliton satisfies the condition $S(\xi, X) \cdot R = 0$ then $\mu = \beta + 4(\beta^2 - \alpha^2), \quad \lambda = -[2(\beta^2 - \alpha^2) + \beta]$.

Corollary 6.2: A 3-dimensional trans-Sasakian manifold with α, β constants satisfies the condition $S(\xi, X) \cdot R = 0$, there is no Ricci soliton with the potential vector field ξ.

7 η-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $W_2(\xi, X) \cdot S = 0$

Definition 7.1. Let M be 3-dimensional trans-Sasakian manifold with respect to semi-Symmetric metric connection. The W_2-curvature tensor of M is defined by [16]

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{2}(g(X, Z)QY - g(Y, Z)QX). \tag{7.1}$$

We assume 3-dimensional trans-Sasakian manifolds with η-Ricci solitons satisfying the condition

$$W_2(\xi, X) \cdot S = 0.$$

Then we have

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0$$

for any $X, Y, Z \in \chi(M)$.
Using the equations (2.12), (3.10), (3.11), (7.1) we get

$$\begin{aligned}
&[-\frac{(\beta+\lambda)}{2}(\lambda+\mu) + \frac{(\beta+\lambda)^2}{2} + (\beta-\mu)(\alpha^2-\beta^2) + (\lambda+\mu)\frac{(\beta-\mu)}{2}]g(X, Y)\eta(Z) \\
&+[\frac{(\beta+\lambda)^2}{2} - \frac{(\beta+\lambda)}{2}(\lambda+\mu) + (\beta-\mu)(\alpha^2-\beta^2) + (\lambda+\mu)\frac{(\beta-\mu)}{2}]g(X, Z)\eta(Y) \\
&+[-(\beta+\lambda)(\beta-\mu) - 2(\beta-\mu)(\alpha^2-\beta^2) - (\beta-\mu)(\lambda+\mu)]\eta(X)\eta(Y)\eta(Z) = 0.
\end{aligned}$$

Put $Z = \xi$ in the above equation we get

$$\begin{aligned}
&[-\frac{(\beta+\lambda)}{2}(\lambda+\mu) + \frac{(\beta+\lambda)^2}{2} + (\beta-\mu)(\alpha^2-\beta^2) + (\lambda+\mu)\frac{(\beta-\mu)}{2}]g(X, Y) \\
&+[\frac{(\beta+\lambda)^2}{2} - \frac{(\beta+\lambda)}{2}(\lambda+\mu) + (\beta-\mu)(\alpha^2-\beta^2) + (\lambda+\mu)\frac{(\beta-\mu)}{2}
\end{aligned}$$

$$-(\beta+\lambda)(\beta-\mu)-2(\beta-\mu)(\alpha^2-\beta^2)-(\beta-\mu)(\lambda+\mu)]\eta(X)\eta(Y)=0.$$

Setting $X=\phi X$ and $Y=\phi Y$ in the above equation we get

$$(\beta-\mu)(\frac{(\beta+2\lambda+\mu+2(\alpha^2-\beta^2))}{2})g(\phi X,\phi Y)=0.$$

Again using the equation (3.12) we have

$$\mu=\beta,\quad \lambda=2(\beta^2-\alpha^2)-\beta$$

or

$$\mu=2(\beta^2-\alpha^2)+\beta,\quad \lambda=-\beta.$$

So we have the following theorem.

Theorem 7.1: If a 3-dimensional trans-Sasakian manifold $(M,g,\phi,\eta,\xi,\alpha,\beta)$ with $\alpha,\ \beta$ constants admitting an η-Ricci soliton satisfies the condition $W_2(\xi,X)\cdot S=0$ then $\mu=\beta,\ \ \lambda=2(\beta^2-\alpha^2)-\beta$ or $\mu=2(\beta^2-\alpha^2)+\beta,\ \ \lambda=-\beta$.

Corollary 7.2: A 3-dimensional trans-Sasakian manifold with $\alpha,\ \beta$ constants satisfies the condition $W_2(\xi,X)\cdot S=0$, there is no Ricci soliton with the potential vector field ξ.

8 η-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $S(\xi,X)\cdot W_2=0$

Suppose that 3-dimensional trans-Sasakian manifolds with η-Ricci solitons satisfy the condition

$$S(\xi,X)\cdot W_2=0.$$

So we have

$$S(X,W_2(Y,Z)V)\xi-S(\xi,W_2(Y,Z)V)X+S(X,Y)W_2(\xi,Z)V-S(\xi,Y)W_2(X,Z)V$$

$$+S(X,Z)W_2(Y,\xi)V-S(\xi,Z)W_2(Y,X)V+S(X,V)W_2(Y,Z)\xi-S(\xi,V)W_2(Y,Z)X=0.$$

Taking inner product with ξ then the above equation becomes

$$\begin{aligned} S(X, W_2(Y,Z)V) - S(\xi, W_2(Y,Z)V)\eta(X) + S(X,Y)\eta(W_2(\xi,Z)V) \\ -S(\xi,Y)\eta(W_2(X,Z)V) + S(X,Z)\eta(W_2(Y,\xi)V) - S(\xi,Z)\eta(W_2(Y,X)V) \\ + S(X,V)\eta(W_2(Y,Z)\xi) - S(\xi,V)\eta(W_2(Y,Z)X) = 0. \end{aligned} \tag{8.1}$$

Put $V = \xi$ and using the equations (2.10), (2.12), (3.10), (3.11), (7.1) we get

$$-(\beta+\lambda)g(X, W_2(Y,Z)\xi) + (\lambda+\mu)\eta(W_2(Y,Z)X) = 0. \tag{8.2}$$

Using the equations (3.10), (3.11), (7.1) then the equation (8.2) becomes

$$[(\beta+\lambda)^2 + (\lambda+\mu)^2 + 2(\alpha^2-\beta^2)(\beta+2\lambda+\mu)]g(X, R(Y,Z)\xi) = 0.$$

Using the equation (3.12) we have

$$\mu = \beta, \quad \lambda = 2(\beta^2-\alpha^2) - \beta$$

or

$$\mu = 2(\beta^2-\alpha^2) + \beta, \quad \lambda = -\beta.$$

So we have the following theorem.

Theorem 8.1: If Let a 3-dimensional trans-Sasakian manifold $(M, g, \phi, \eta, \xi, \alpha, \beta)$ with α, β constants admitting an η-Ricci soliton satisfies the condition $S(\xi, X)\cdot W_2 = 0$ then $\mu = \beta$, $\lambda = 2(\beta^2-\alpha^2)-\beta$ or $\mu = 2(\beta^2-\alpha^2)+\beta$, $\lambda = -\beta$.

Corollary 8.2: A 3-dimensional trans-Sasakian manifold with α, β constants satisfies the condition $S(\xi, X)\cdot W_2 = 0$, there is no Ricci soliton with the potential vector field ξ.

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A sufficiently complicated noded Schottky group of rank three

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ABSTRACT

In 1974, Marden proved the existence of non-classical Schottky groups by a theoretical and non-constructive argument. Explicit examples are only known in rank two; the first one by Yamamoto in 1991 and later by Williams in 2009. In 2006, Maskit and the author provided a theoretical method to construct non-classical Schottky groups in any rank. The method assumes the knowledge of certain algebraic limits of Schottky groups, called sufficiently complicated noded Schottky groups. The aim of this paper is to provide explicitly a sufficiently complicated noded Schottky group of rank three and explain how to use it to construct explicit non-classical Schottky groups.

RESUMEN

En 1974, Marden demostró la existencia de grupos de Schottky no-clásicos con un argumento teórico y no-constructivo. Se conocen ejemplos explícitos solo en rango dos; el primero por Yamamoto en 1991 y después por Williams en 2009. En 2006, Maskit y el autor entregaron un método teórico para construir grupos de Schottky no-clásicos en cualquier rango. El método asume el conocimiento de ciertos límites algebraicos de grupos de Schottky, llamados grupos de Schottky anodados suficientemente complicados. El objetivo de este paper es dar un grupo de Schottky anodado suficientemente complicado explícitamente de rango tres y explicar cómo usarlo para construir grupos de Schottky no-clásicos explícitos.

Keywords and Phrases: Riemann surfaces; Schottky groups.

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1 Introduction

A Kleinian group G is called a *Schottky group of rank* $g \geq 2$ if it is generated by loxodromic transformations $A_1, \ldots, A_g \in \mathrm{PSL}_2(\mathbb{C})$ such that there is a collection of $2g$ pairwise disjoint simple loops $\alpha_1, \alpha_1', \alpha_2, \alpha_2', \ldots, \alpha_g, \alpha_g'$ on the Riemann sphere $\widehat{\mathbb{C}}$, all of them bounding a common domain $\mathcal{D}$ of connectivity $2g$, with $A_j(\alpha_j) = \alpha_j'$ and $A_j(\mathcal{D}) \cap \mathcal{D} = \emptyset$. The above set of generators is called *geometrical* and the above loops a *fundamental set of loops* for G. It is well known that G is a free group of rank g and that $\mathcal{D}$ is a fundamental domain for it. In [3] Chukrow proved that every set of g generators of G is geometrical. We say that G is a *classical Schottky group* if it has a set of geometrical generators, called a *classical set of generators*, for which we may find a fundamental set of loops being circles. Classical Schottky groups were firstly considered by Schottky around 1882. In general, a classical Schottky group may have non-classical set of generators.

Examples of classical Schottky groups are given by the finitely generated purely hyperbolic Fuchsian groups representing a closed surface with holes [2]. Moreover, if such a Fuchsian group is a two generator group representing a torus with one hole, then every pair of generators is a classical set of generators [21].

In 1974, Marden [14] provided the existence of non-classical Schottky groups (his proof is non-constructive). In 1975, Zarrow [27] claimed to have constructed an explicit example of a non-classical Schottky group of rank two, but it was lately noted by Sato in [22] to be incorrect. The first explicit (correct) construction was provided by Yamamoto [26] in 1991 and in 2009 another example was provided by Williams in his Ph.D. Thesis [24], both for rank two. It seems that, for rank at least three, there is not explicit example in the literature.

In [7], Maskit and the author described a theoretical method to construct non-classical Schottky groups in any rank $g \geq 2$. The idea is to consider certain Kleinian groups, obtained as geometrically finite algebraic limits of Schottky groups of rank g, called *sufficiently complicated noded Schottky groups of rank* g (see Section 2). In this paper (see Section 3) we provide an explicit construction of a sufficiently complicated noded Schottky group of rank three and we used it to describe how to obtain a infinite family (one-dimensional) non-classical Schottky group of rank three.

To finish this introduction, and as a matter of completeness, let us mention another conjecture related to classical Schottky groups. If Ω is the region of discontinuity of a Schottky group G of rank g, then it is a connected set and Ω/G is a closed Riemann surface of genus g. Conversely, if S is a closed Riemann surface, then there is a Schottky group G such that S and Ω/G are isomorphic (Koebe's uniformization theorem). As we have the existence of non-classical Schottky group, it might be that G is non-classical. A conjecture (due to Bers) asserts that we may chose G to be classical. Some positive answers were obtained by Bobenko [1], Koebe [11], Maskit [17], Seppälä [23] (for the case in which the surface admits antiholomorphic involutions with fixed points) and

McMullen [13] (for the case in which the surface has sufficiently many shorts geodesics). Recently, Hou [8, 9] have announced a proof of this conjecture (by using Haussdorf dimension of the limit set of Kleinian groups) and another approach in [6] (by using Belyi curves).

2 Sufficiently complicated Noded Schottky groups

2.1 Noded Schottky groups

A *noded Schottky group of rank* $g \geq 2$ is geometrically defined as follows. Consider a collection of pairwise disjoint open topological discs D_1, $D_1', \ldots, D_g$ and D_g' on the Riemann sphere $\widehat{\mathbb{C}}$ so that the corresponding boundaries $\widehat{\alpha}_1 = \partial D_1, \widehat{\alpha}_1' = \partial D_1', \ldots, \widehat{\alpha}_g = \partial D_g, \widehat{\alpha}_g' = \partial D_g'$ are simple loops and they only intersect in at most finitely many points. Let $\widehat{A}_1, \ldots, \widehat{A}_g$ be Möbius transformations such that $\widehat{A}_j(\widehat{\alpha}_j) = \widehat{\alpha}_j'$ and $\widehat{A}_j(D_j) \cap D_j' = \emptyset$, for each $j = 1, \ldots, g$. Observe that the transformation $\widehat{A}_j$ may only be loxodromic or parabolic. The group $\widehat{G}$, generated by these transformations, is a Kleinian group isomorphic to a free group of rank g. If p is a point of intersection of two of the above loops, then either it is a fixed point of a parabolic transformation of $\widehat{G}$ or it has trivial $\widehat{G}$-stabilizer. In the second situation, one may deform in a suitable manner these loops in order to avoid the intersection at p and not adding extra intersections. In this way, two possibilities appear (up to performing the above deformation), either: (i) G is a Schottky group of rank g or (ii) there are intersection points, each of them being a fixed point of a parabolic transformation of $\widehat{G}$. In case (ii) we say that $\widehat{G}$ is a *noded Schottky group of rank* g; we call the above set of loops *a fundamental set of loops* and the generators a set of *geometrical generators*.

Remark 1. In [18], as an application of the Klein-Maskit's combination theorems, it was noted that a noded Schottky group $\widehat{G}$ is geometrically finite, that each of its parabolic elements is a conjugate of a power of one of the transformations fixing a common point of two of the fundamental loops, and that the complement $\widehat{\mathcal{D}}$ of the union of the closures of $D_1, D_1', \ldots, D_g$, D_g' is a fundamental domain for $\widehat{G}$. Different as for the case of Schottky groups, not every set of free generators of a noded Schottky group is necessarily geometrical.

2.2 The extended region of discontinuity

If Ω is the region of discontinuity of a noded Schottky group $\widehat{G}$ of rank g, then by adding to it the parabolic fixed points of $\widehat{G}$, and with the appropriate cusped topology (see [5, 12]), we obtain its *extended region of discontinuity* Ω^+; it happens that $S^+ = \Omega^+/\widehat{G}$ is a stable Riemann surface of genus g. In the case that the number of nodes of S^+ is $3g-3$, we say that $\widehat{G}$ is a maximal noded Schottky group (in this case, there are exactly $2g-2$ connected components of the complement of the nodes of S^+, each one being a triple-punctured sphere). In [4] it was observed that every stable Riemann surface of genus g is obtained as above; so every point of the Deligne-Mumford

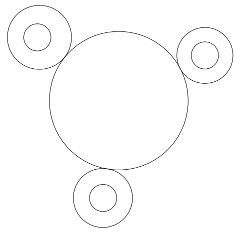

Figure 1—A stable Riemann surface of genus 3 with 3 nodes

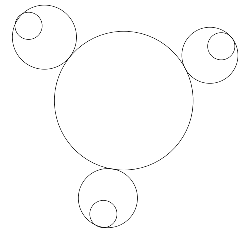

Figure 2—A stable Riemann surface of genus 3 with 6 nodes

compactification of the moduli space of genus g can be realized by a suitable noded Schottky group of rank g.

2.3 Neoclassical noded Schottky groups

A noded Schottky group for which there is a set of geometrical generators admitting a fundamental set of loops all of which are Euclidean circles is called *neoclassical*; the corresponding set of generators is called a *neoclassical set of generators*.

In [7] it was proved that if G is a noded Schottky group such that Ω^+/G is a stable Riemann surface as in Figure 1, then it cannot be neoclassical (this should be still true for every noded Schottky group of rank $g \geq 4$ whose corresponding stable Riemann surface has $g+1$ components, one being of genus zero and the others being of genus one).

2.4 Sufficiently complicated noded Schottky groups

The space of deformations of a Schottky group of rank g, denoted by $\mathcal{S}_{alg}$, is a subset of the representation space of the free group of rank g in $\mathrm{PSL}_2(\mathbb{C})$, modulo conjugation. Regard $\mathbb{H}^3$ as being the set $\{(z,t) : z \in \mathbb{C}, t > 0 \in \mathbb{R}\}$. We likewise identify $\mathbb{C}$ with the boundary of $\mathbb{H}^3$, except for the point at infinity; that is, we identify $\mathbb{C}$ with $\{(z,t) : t = 0\}$.

2.4.1 The relative conical neighbourhood of a noded Schottky group

Let $\widehat{G}$ be a noded Schottky group of rank $g \geq 2$, with a set of geometrical generators $\widehat{A}_1, \ldots, \widehat{A}_g$, and corresponding fundamental set of loops $\widehat{\alpha}_1, \ldots, \widehat{\alpha}'_g$, these being the corresponding boundary loops of a collection of pairwise disjoint open discs $D_1, \ldots, D'_g$. The complement of the closures of these discs is a fundamental domain $\widehat{D}$ for $\widehat{G}$. Let $\widehat{P}_1, \ldots, \widehat{P}_q$ be a maximal set of primitive parabolic elements of $\widehat{G}$ generating non-conjugate cyclic subgroups, where $q \geq 1$ (we may assume the fix point of these parabolic transformations to be contained in the intersection of two fundamental loops). We denote by $\Omega(\widehat{G})$ its region of discontinuity and by $\Omega^+(\widehat{G})$ its extended region of discontinuity.

Next, we proceed to recall a construction done in [7] of a one-real family of Schottky groups G^τ whose geometric limit is $\widehat{G}$.

(I) The infinite shoebox construction

For each $i = 1, \dots, q$, choose a particular Möbius transformation H_i conjugating $\widehat{P}_i$ to the transformation $P(z) = z + 1$ and consider the renormalized group $H_i \widehat{G} H_i^{-1}$. For this group, there is a number $\tau_0 > 1$ so that the set $\{|\operatorname{Im}(z)| \geq \tau_0\}$ is precisely invariant under the stabilizer $\operatorname{Stab}(\infty)$ of ∞ in the group $H_i \widehat{G} H_i^{-1}$. In this normalization, for each parameter τ, with $\tau > \tau_0$, we define the *infinite shoebox* to be the set $B_{0,\tau} = \{(z, t) : |\operatorname{Im}(z)| \leq \tau, t \leq \tau\}$. Since $\tau_0 > 1$, we easily observe that for every $\tau > \tau_0$, the complement of $B_{0,\tau}$ in $\mathbb{H}^3 \cup \mathbb{C}$ is precisely invariant under $\operatorname{Stab}(\infty) \subset H_i \widehat{G} H_i^{-1}$, where we are now regarding Möbius transformations as hyperbolic isometries, which act on the closure of $\mathbb{H}^3$. Then for $\widehat{G}$, the infinite shoebox with parameter τ at z_i, the fixed point of $\widehat{P}_i$, is $B_{i,\tau} = H_i^{-1}(B_{0,\tau})$. If $\widehat{P}$ is any parabolic element of $\widehat{G}$, conjugate to some power of $\widehat{P}_i$, then the corresponding infinite shoebox at the fixed point of $\widehat{P}$, is given by $T(B_{i,\tau})$, where $\widehat{P} = T\widehat{P}_i T^{-1}$. It was observed in [19] that, for each fixed $\tau > \tau_0$, $\widehat{G}$ acts as a group of conformal homeomorphisms on the *expanded regular set* $B^\tau = \bigcap \widehat{A}(B_{i,\tau})$, where the intersection is taken over all $\widehat{A} \in \widehat{G}$ and all $i = 1, \dots, q$. Further, $\widehat{G}$ acts as a (topological) Schottky group (in the sense of our geometrical definition) on the boundary of B^τ. Each parabolic $\widehat{P} \in \widehat{G}$ appears to have two fixed points on the boundary of B^τ; that is, $\widehat{P}$, as it acts on the boundary of B^τ, appears to be loxodromic. The *flat part* of B^τ is the intersection of B^τ with the extended complex plane $\widehat{\mathbb{C}}$. The complement of the flat part (on the boundary of B^τ) is the disjoint union of 3-sided *boxes*, where each box has two *vertical sides* (translates of the sets $\{\operatorname{Im}(z) = \pm\tau, 0 < t < \tau\}$) and one *horoball side* (a translate of the set $\{|\operatorname{Im}(z)| \leq \tau, t = \tau\}$). For each $i = 1 \dots, q$ and for each integer $n \geq 1$, we set $B_{i,\tau,n} = H_i^{-1}(B_{0,\tau} \cap \{|\operatorname{Re}(z)| \leq n\})$ and $B^{\tau,n} = \bigcap_{\widehat{A} \in \widehat{G}} \widehat{A}(B_{i,\tau,n})$. The *truncated flat part* of $B^{\tau,n}$ is the intersection $B^{\tau,n} \cap \widehat{\mathbb{C}}$. The boundary of the truncated flat part near a parabolic fixed point, renormalized so as to lie at ∞, is a Euclidean rectangle. Let us renormalize $\widehat{G}$ so that $\infty \in \Omega(\widehat{G})$. Then, for each $\tau > \tau_0$, there is a conformal map f^τ, mapping the boundary of B^τ to $\widehat{\mathbb{C}}$, and conjugating $\widehat{G}$ onto a Schottky group G^τ, where f^τ is defined by the requirement that, near ∞, $f^\tau(z) = z + O(|z|^{-1})$. The group G^τ depends on the choice of the Möbius transformations $H_1, \dots, H_q$ as well as on the choice of τ. The elements $A_1^\tau = f^\tau \widehat{A}_1 (f^\tau)^{-1}, \dots, A_g^\tau = f^\tau \widehat{A}_g (f^\tau)^{-1}$ provide a set of free generators for the Schottky group G^τ.

(II) Vertical projection loops

Next, we proceed to construct a fundamental set of loops for G^τ for the above generators in terms of the fundamental set of loops for $\widehat{G}$. In [19] it was shown that, with the above normalization, f^τ converges to the identity I, uniformly on compact subsets of $\Omega(\widehat{G})$, and, for each $j \in \{1, \dots, g\}$, A_j^τ converges to $\widehat{A}_j$, as $\tau \to \infty$. In particular, if we fix τ_0, and fix n, then $f^\tau \to$ I uniformly on compact subsets of the truncated flat part of $B^{\tau_0,n}$. The boundary of $B^{\tau_0,n}$ consists of a disjoint union of quadrilaterals with circular sides. After renormalization, the part of the boundary of $B^{\tau_0,n}$ corresponding to $\{|\operatorname{Im}(z)| = \tau_0\}$ is the *horizontal* part of the boundary, while the part of the boundary corresponding to $\{|\operatorname{Re}(z)| = n\}$ is the *vertical* part of the boundary. We make a fixed choice of the conjugating maps, H_i, $i = 1,\dots,q$, and we fix a choice of the parameter $\tau > \tau_0$ in the above construction. We may deform all the loops $\widehat{\alpha}_i$ and $\widehat{\alpha}'_i$, within $\Omega^+(\widehat{G})$ to an equivalent fundamental set of loops, with the same geometric generators $\widehat{A}_1,\dots,\widehat{A}_g$, so that, after appropriate renormalization, each connected component of each of the deformed loops appears, in each component of the complement of the flat part of B^τ, as a pair of half-infinite Euclidean vertical lines, one in $\{\operatorname{Im}(z) \geq \tau\}$, the other in $\{\operatorname{Im}(z) \leq -\tau\}$, both with the same real part (the technical details of such a deformation can be found in [7]). We call such a deformed loops the *vertical projection loops*. These vertical projection loops, which we still denoting as $\widehat{\alpha}_1,\dots,\widehat{\alpha}'_g$, yields a fundamental set of loops, $\alpha_1^\tau,\dots,{\alpha'_g}^\tau$ for the generators $A_1^\tau,\dots,A_g^\tau$ of the Schottky group G^τ.

(III) The relative conical neighborhood

The *relative conical neighborhood* of $\widehat{G}$ is to be defined as the set of all marked Schottky groups $G^\tau = \langle A_1^\tau,\dots,A_g^\tau\rangle$, with the fundamental set of loops $\alpha_1^\tau,\dots,{\alpha'_g}^\tau$, as constructed above.

Remark 2. Recall that we are assuming that ∞ is an interior point of the flat part corresponding to τ_0, and $f^\tau(z) = z+O(|z|^{-1})$ near ∞. As, with these normalizations, $f^\tau \to I$ uniformly on compact subsets of $\Omega(\widehat{G})$, we obtain that $G^\tau \to \widehat{G}$ algebraically. It now follows, from the Jørgensen-Marden criterion [10], that $G^\tau \to \widehat{G}$ geometrically and that each relative conical neighborhood contains infinitely many distinct marked Schottky groups. It is also easy to see, as in [19], that, for each primitive parabolic element $\widehat{P} \in \widehat{G}$, as $\tau \to \infty$, there is a corresponding geodesic on $S^\tau = \Omega(G^\tau)/G^\tau$ whose length tends to zero. It follows that each relative conical neighborhood of a noded Schottky group contains Schottky groups representing infinitely many distinct Riemann surfaces.

2.4.2 Pinchable loops of Schottky groups

Let G be a Schottky group of rank $g \geq 2$, with generators $A_1,\dots,A_g$, and let $\pi : \Omega(G) \to S$ be a regular covering with deck group G.

(IV) Pinchable loops

Let $\gamma_1,\dots,\gamma_q$ be a set of simple disjoint geodesics loops on S. Each γ_j corresponds to a conjugacy class of a cyclic subgroup of G (including the trivial subgroup) by the lifting under π; let $\langle W_j\rangle$ be a representative of such a class. If these q cyclic subgroups are non-trivial, they are pairwise non-conjugated in G and the generators W_j are non-trivial powers in G (i.e., there is no $T_j \in G$ so that $W_j = T_j^{m_j}$ for some $m_j \geq 2$), then we say that this set of geodesics is *pinchable* in G.

Remark 3. (1) It was shown in [20] (see also Yamamoto [25]) that if $\gamma_1, \dots, \gamma_q$ is a set of pinchable simple disjoint geodesics loops on S in G, defined by the words $W_1, \dots, W_q$, as above, then there is a noded Schottky group $\widehat{G}$, and there is an isomorphism $\psi : G \to \widehat{G}$, where $\psi(W_1), \dots, \psi(W_q)$, and their powers and conjugates, are exactly the parabolic elements of $\widehat{G}$. More precisely, it was shown in [20] that there is a path in Schottky space, $\mathcal{S}_{alg}$, which converges to a set of generators for $\widehat{G}$, along which the lengths of the geodesics, $\gamma_1, \dots, \gamma_q$, all tend to zero. (2) On the other direction, let us consider a noded Schottky group $\widehat{G}$ of rank $g \geq 2$, with a set of geometrical generators $\widehat{A}_1, \dots, \widehat{A}_g$ and corresponding fundamental set of loops $\widehat{\alpha}_1, \dots, \widehat{\alpha}'_g$. Let G^τ be a Schottky group of rank g with fundamental set of loops $\alpha_1^\tau, \dots, {\alpha'_g}^\tau$ and generators $A_1^\tau, \dots, A_g^\tau$, in a relative conical neighborhoodof $\widehat{G}$ as previously described in Section 2.4.1. Let $S = \Omega(G^\tau)/G^\tau$ be the closed Riemann surface of genus g represented by G^τ, and let $V_i \subset S$ be the projection of α_i^τ, $i = 1, \dots, g$. Then $V_1, \dots, V_g$ is a set of g homologically independent simple disjoint loops on S. Let $\psi : G^\tau \to \widehat{G}$ be the isomorphism defined by $A_i^\tau \mapsto \widehat{A}_i$, $i = 1, \dots, g$. The elements of G^τ which are sent to parabolic elements of $\widehat{G}$ are called the *pinched elements* of G^τ. There are simple disjoint geodesics $\gamma_1, \dots, \gamma_q$ on S, defined by pinched elements of G^τ, given by the words $W_1, \dots, W_q$ in the generators $A_1^\tau, \dots, A_g^\tau$, so that every parabolic element of $\widehat{G}$ is a power of a conjugate of one of their ψ-image. It happens that this collection of loops $\gamma_1, \dots, \gamma_q$ is a set of pinchable geodesics of G^τ. The construction in [19] shows that we can choose the above parameter τ so that the γ_i are all arbitrarily short.

Proposition 1 ([7]). *Every non-empty set of* $k < 3g - 3$ *pinchable geodesics is contained in a maximal set of* $3g - 3$ *pinchable geodesics.*

(V) Valid sets of fundamental loops and their complexity

Let $\gamma_1, \dots, \gamma_q \subset S$ be a pinchable set of geodesics in G. Set $\widehat{S}^+$ the stable Riemann surface obtained from S by pinching these q geodesics; it consists of a finite number of compact Riemann surfaces, called *parts*, which are joined together at a finite number of nodes. Also, let $\widehat{G}$ be the noded Schottky group obtained from G by pinching these q geodesics.

Let $V_1, \dots, V_g$, be a fundamental set of loops for G on S (that is, the components of the lifting of these loops under π are simple loops and such a lifting set of loops contains a fundamental set of loops for G) and let $\widehat{V}_1, \dots, \widehat{V}_g$ be the corresponding loops on $\widehat{S}^+$ obtained by pinching $\gamma_1, \dots, \gamma_q$. We observe that the lifts of the $\widehat{V}_i$ to $\Omega^+(\widehat{G})$ are all loops, but they are generally not disjoint and they need not to be simple. There are certainly some number of these lifts passing through each parabolic fixed point, and some of them might pass more than once through the same parabolic fixed point. The set of loops, $V_1, \dots, V_g$, is called a *valid set of fundamental loops* for $\gamma_1, \dots, \gamma_q$, if every lift of every $\widehat{V}_i$ to $\Omega^+(\widehat{G})$ is a simple loop; that is, it passes at most once through each parabolic fixed point (i.e., the set of loops, $\widehat{V}_1, \dots, \widehat{V}_g$, forms a fundamental set of loops for $\widehat{G}$ on

$\widehat{S}^+$). We note that there are exactly q equivalence classes of parabolic fixed points in $\widehat{G}$, one for each of the loops γ_i.

Proposition 2 ([4, 7]). *There is at least one valid set of fundamental loops* $V_1, \ldots, V_g$*, for every set of pinchable geodesics,* $\gamma_1, \ldots, \gamma_q$.

(VI) The complexity

Let us now consider a valid set of fundamental loops, $V_1, \ldots, V_g$, for a set of pinchable geodesics $\gamma_1, \ldots, \gamma_q$. We can deform the V_i on S so that they are all geodesics. Then the geometric intersection number, $V_i \bullet \gamma_j$, of V_i with γ_j is well defined; it is the number of points of intersection of these two geodesics. Looking on the corresponding noded surface $\widehat{S}^+$, $V_i \bullet \gamma_j$ is the number of times the curve $\widehat{V}_i$ obtained from V_i by contracting γ_j to a point, passes through that point (node). The *complexity* of $V_1, \ldots, V_g$, with respect to $\gamma_1, \ldots, \gamma_q$, is given by

$$\Xi(\gamma_1, \ldots, \gamma_q; V_1, \ldots, V_g) = \max_{1 \leq j \leq q} \sum_{i=1}^{g} V_i \bullet \gamma_j,$$

and the *complexity* $\Xi(\gamma_1, \ldots, \gamma_q)$ is the minimum of $\Xi(\gamma_1, \ldots, \gamma_q; V_1, \ldots, V_g)$, where the minimum is taken over all valid sets of fundamental loops. If $\Xi(\gamma_1, \ldots, \gamma_q) \geq n$, then, for every valid fundamental set $V_1, \ldots, V_g$, there is a node P on S^+ so that the total number of crossings of P by $\widehat{V}_1, \ldots, \widehat{V}_g$ is at least n.

Proposition 3 ([7]). *Let* $g \geq 2$ *and* G *be a Schottky group of rank* g*. For each positive integers* n *there are only finitely many topologically distinct maximal (i.e.* $q = 3g-3$*) pinchable set of geodesics in* G *and complexity* n*. In particular, there are infinitely many topologically distinct maximal noded Schottky groups of rank* g *and there are only finitely many topologically distinct maximal neoclassical noded Schottky groups in each rank* g.

(VII) Sufficiently complicated pinchable sets of geodesics

Now, we consider a maximal set of pinchable geodesics in G, say $\gamma_1, \ldots, \gamma_{3g-3}$; so $\widehat{G}$ is a maximal noded Schottky group. Observe that $\widehat{G}$ is rigid, and that every part of S^+ is a sphere with three distinct nodes. Also, every connected component $\Delta \subset \Omega(\widehat{G})$ is a Euclidean disc Δ, where $\Delta/\operatorname{Stab}(\Delta)$ is a sphere with three punctures; the three punctures correspond to the three nodes of the corresponding part of $\widehat{S}^+$.

Let $V_1, \ldots, V_g$ be a valid set of fundamental loops on S for the given set of pinchable geodesics (the existence is given by Proposition 2), and let $\widehat{V}_1, \ldots, \widehat{V}_g$ be the corresponding loops on $\widehat{S}^+$. For each $i = 1, \ldots, g$, the intersection of a lifting of $\widehat{V}_i$ with a component of $\widehat{G}$ (i.e., a connected component of $\Omega(\widehat{G})$) is called a *strand* of that lifting $\widehat{V}_i$. Similarly, the loops $\widehat{V}_1, \ldots, \widehat{V}_g$ appear on the corresponding parts of $\widehat{S}^+$ as collections of *strands* connecting the nodes on the boundary of each part. There are two possibilities for these strands; either a strand connects two distinct nodes on some part, or it starts and ends at the same node. Since the loops $V_1, \dots, V_g$ are simple and disjoint, there are at most three sets of *parallel* strands of the $\widehat{V}_i$ in each part; that is, there are at most three sets of strands, where any two strands in the same set are homotopic arcs with fixed endpoints at the nodes. We regard each of these sets of strands on a single part as being a *superstrand*, so that there are at most 3 superstrands on any one part. We next look in some component Δ of $\Omega(\widehat{G})$, and look at a parabolic fixed point x on its boundary, where x corresponds to the node N on the part S_i of $\widehat{S}^+$. In general, there will be infinitely many liftings of superstrands emanating from x in Δ, but, modulo $\mathrm{Stab}(\Delta)$ there are only finitely many. In fact, there are at most 4 such liftings of superstrands emanating from x. If there is exactly one superstrand on S_i with one endpoint at N, and the other endpoint at a different node, then modulo $\mathrm{Stab}(\Delta)$ there will be exactly the one lifting of this superstrand emanating from x. If there is only one superstrand on S_i with both endpoints at the same node N, then this superstrand has two liftings starting at x, one in each direction; so, in this case, we see two lifts of superstrands modulo $\mathrm{Stab}(\Delta)$ emanating from x. It follows that, modulo $\mathrm{Stab}(\Delta)$, we can have 0, 1, 2, 3 or 4 liftings of superstrands starting at x. We note that these liftings of superstrands all end at distinct parabolic fixed points on the boundary of Δ. We say that the fundamental set of loops, $\widehat{V}_1, \dots, \widehat{V}_p$ is *sufficiently complicated* if there are two (different) lifts $\widehat{\alpha}_i$ and $\widehat{\alpha}_j$, of some $\widehat{V}_i$ and some not necessarily distinct $\widehat{V}_j$, respectively, so that $\widehat{\alpha}_i$ and $\widehat{\alpha}_j$ both pass through the parabolic fixed point z_1, into a component Δ_1 of $\widehat{G}$, then both travel through Δ_1 to the same parabolic fixed point on its boundary, z_2, and into another component Δ_2, which they again traverse together to the same boundary point, z_3, necessarily a parabolic fixed point, where they enter Δ_3, and they leave Δ_3 at different parabolic fixed points.

2.4.3 Sufficiently complicated noded Schottky groups

A maximal noded Schottky group $\widehat{G}$ is *sufficiently complicated* if every set of valid fundamental loops on $\widehat{S}^+$ is sufficiently complicated. We note that (keeping the notation of last section), inside Δ_1, $\widehat{\alpha}_i$ and $\widehat{\alpha}_j$ are disjoint; they both enter Δ_1 at the same point, and they both leave Δ_1 at the same point; hence they cannot both be circles. In [7] the following result, stating a sufficient condition in terms of the complexity for a maximal noded Schottky group to be sufficiently complicated, was obtained.

Theorem 1 ([7]). *If a maximal noded Schottky group $\widehat{G}$ has complexity at least 11, then it is sufficiently complicated.*

The previous theorem, together Proposition 3, asserts the existence of infinitely many topologically different sufficiently complicated maximal noded Schottky groups in every rank $g \geq 2$. The following result states sufficient conditions for a Schottky group to be non-classical.

Theorem 2 ([7]). *Let $\widehat{G}$ be a maximal noded Schottky group.*

(1) *If $\widehat{G}$ is sufficiently complicated, then, for τ sufficiently large, the Schottky group G^{τ} in the relative conical neighborhood of $\widehat{G}$ is non-classical.*

(2) *If $S^{+} = \Omega^{+}(\widehat{G})/\widehat{G}$ is the stable Riemann surface as shown in figure 2, then $\widehat{G}$ is sufficiently complicated.*

3 Explicit construction of a sufficiently complicated noded Schottky group

In this section we construct explicitly a noded Schottky group as in part (2) of Theorem 2, so part (1) of the same theorem asserts that any Schottky group sufficiently near to $\widehat{G}$ is necessarily non-classical.

3.1 A family Schottky groups of rank three

Let L_0 be the unit circle, L_1 be the real line, L_2 be the line through the points 0 and $w_0 = e^{\pi i/3}$, and set (see Figure 7)

$$\mathcal{F} = \left\{(p,r) : 1/2 < p < 1,\ 0 < r < r^{*}(p) := \frac{\sqrt{1+p^2+p^4}+p^2-1}{\sqrt{3}p}\right\}.$$

For each $(p,r) \in \mathcal{F}$ we set L_3 to be the circle with center at 0 and radius r and L_4 to be the circle orthogonal to L_0, intersecting L_1 at the points p and $1/p$ with angle $\pi/3$ (see Figure 3). The circle L_4 has its center at $c = \left(\frac{1+p^2}{2p}\right) + \frac{i}{\sqrt{3}}\left(\frac{1-p^2}{2p}\right)$ and it has radius $R = \frac{2}{\sqrt{3}}\left(\frac{1-p^2}{2p}\right)$. The condition $p > 1/2$ asserts that L_2 and L_4 are disjoint (tangency occurs when $p = 1/2$) and the condition $r < r^{*}(p)$ asserts that L_3 and L_4 are disjoint (tangency occurs when $r = r^{*}(p)$). Let τ_j be the reflection on L_j, for $j = 0,1,2,3,4$, so

$$\tau_0(z) = 1/\overline{z}, \quad \tau_1(z) = \overline{z}, \quad \tau_2(z) = w^2\overline{z}, \quad \tau_3(z) = r^2/\overline{z}, \quad \tau_4(z) = \frac{c\overline{z}-1}{\overline{z}-\overline{c}},$$

and let $K_{r,p} = \langle \tau_0, \tau_1, \tau_2, \tau_3, \tau_4 \rangle$. It turns out that $K_{r,p}$ is an extended Kleinian group with connected region of discontinuity $\Omega_{r,p}$ and so that $\Omega_{r,p}/K_{r,p}$ is a closed disc with 5 branch values, of orders 2, 2, 2, 2 and 3, on its border. As a consequence of the Klein-Maskit combination theorems [18], the group $K_{r,p}$ has no parabolic transformations, its limit set is a Cantor set and it is geometrically finite. If we set $W = \tau_2\tau_1$, $J = \tau_0\tau_1$ and $L = \tau_1\tau_4$, then $W^3 = L^3 = J^2 = (WJ)^2 = (LJ)^2 = 1$.

Now, if $A_1 = L^{-1}W^{-1} = \tau_4\tau_2$, $A_2 = \tau_1 A_1 \tau_1$, and $A_3 = \tau_0\tau_3$, so

$$A_1(z) = \frac{cw_0z-1}{w_0z-\overline{c}}, \quad A_2(z) = \frac{\overline{c}w_0^2z-1}{w_0^2z-c}, \quad A_3(z) = r^2z,$$

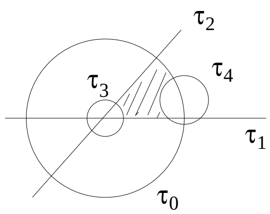

Figure 3—A set of lines and circles

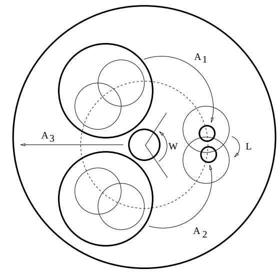

Figure 4—The Schottky group $G_{r,p}$ of rank 3: the six darkest loops are a fundamental set of loops

then we set $G_{r,p} = \langle A_1, A_2, A_3 \rangle$. In Figure 4 we show the situation for values of p near to 1 and r near to 0; in which case $G_{r,p}$ turns out to be a classical Schottky group of rank three.

Lemma 1. *If* $(p, r) \in \mathcal{F}$, *then* $G_{r,p}$ *is a Schottky group of rank three.*

Proof. It can be seen that $G_{r,p}$ is a finite index normal subgroup of $K_{r,p}$ and

$$K_{r,p}/G_{r,p} = \langle \tau_0 : \tau_0^2 = 1 \rangle \times \langle \tau_1, \tau_2 : \tau_1^2 = \tau_2^2 = (\tau_2\tau_1)^3 = 1 \rangle \cong \mathbb{Z}_2 \oplus D_3.$$

In particular, $G_{r,p}$ has the same region of discontinuity as for $K_{r,p}$ (so a function group), it is geometrically finite and does not have parabolic transformations. As any of the elliptic transformations of $K_{r,p}$ goes into an element of the same order in the quotient $K_{r,p}/G_{r,p}$, we also have that $G_{r,p}$ is torsion free. Now, as a consequence of the classification of function groups [15, 16], the group $G_{r,p}$ is a Schottky group of rank three (in Figure 4 there is shown a fundamental set of loops). □

The closed Riemann surface $S_{r,p} = \Omega_{r,p}/G_{r,p}$ of genus 3 admits the group $\mathbb{Z}_2 \oplus D_3$ as a group of conformal/anticonformal automorphisms. On $S_{r,p}$ we have simple closed curves γ_1,..., γ_6 which are pinchable (see Figures 5 and 6) with respect to the Schottky group $G_{r,p}$; these pinchable curves correspond to the conjugacy classes of cyclic groups of $G_{r,p}$ as follows:

γ_1 corresponds to A_2^{-1}; γ_2 corresponds to A_1; γ_3 corresponds to $A_1^{-1}A_2$;
γ_4 corresponds to $A_1^{-1}A_2A_3^{-1}A_2^{-1}A_1A_3$; γ_5 corresponds to $A_2^{-1}A_3^{-1}A_2A_3$;
γ_6 corresponds to $A_1A_3^{-1}A_1^{-1}A_3$.

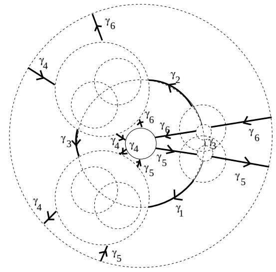

Figure 5—A set of pinchable curves seen at the Schottky uniformization

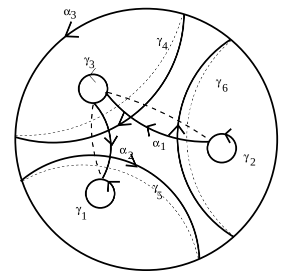

Figure 6—A set of pinchable curves seen on the Riemann surface $S_{r,p}$

3.2 A sufficiently complicated noded Schottky groups of rank three

To obtain the desired noded Schottky group, we need to move the pair $(p, r) \in \mathcal{F}$ to some point in the boundary in order to have that the loxodromic transformations A_2^{-1}, A_1, $A_1^{-1}A_2$, $A_1^{-1}A_2A_3^{-1}A_2^{-1}A_1A_3$, $A_2^{-1}A_3^{-1}A_2A_3$ and $A_1A_3^{-1}A_1^{-1}A_3$ are transformed into parabolic transformations. As the order three automorphism W permutes cyclically $\gamma_1, \gamma_2, \gamma_3$ and also $\gamma_4, \gamma_5, \gamma_6$, we only need to take care of A_1 and $A_1A_3^{-1}A_1^{-1}A_3$. First, in order to transform A_1 into a parabolic transformation we only need to have $\tau_4(w_0) = w_0$, equivalently, that the circle L_4 is tangent to the line L_2 at w_0. This happens exactly when $p = 1/2$. Now, assuming $p = 1/2$, in order for $A_1A_3^{-1}A_1^{-1}A_3$ to be a parabolic transformation we only need to have tangency of the circle L_3 with L_4, that is, $r = r^*(1/2) = \frac{\sqrt{7}-\sqrt{3}}{2}$. The group $G_{r^*(1/2),1/2}$ turns out to be a noded Schottky group that uniformizes a stable Riemann surface as shown in Figure 2 and, by (2) in Theorem 2, it is a sufficiently complicated noded Schottky group.

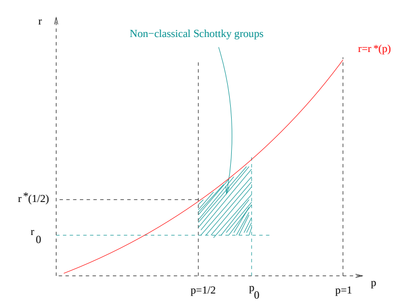

Figure 7—The region $\mathcal{F}$ and the filled part for the non-classical Schottky groups

3.3 Non-classical Schottky groups of rank three

By (1) in Theorem 2, there exist $(p_0, r_0) \in \mathcal{F}$ with the property that if $(p, r) \in \mathcal{F}$, $1/2 < p < p_0$ and $r_0 < r < r^*(1/2)$, then $G_{r,p}$ is a non-classical Schottky group of rank three (see filled part region in Figure 7). Moreover, each of these Schottky groups is contained in a Kleinian group $K_{r,p}$

as a finite index normal subgroup with $K_{r,p}/G_{r,p} \cong \mathbb{Z}_2 \oplus D_3$, in other words, the closed Riemann surfaces $S_{r,p} = \Omega(G_{r,p})/G_{r,p}$ admit a group of conformal automorphisms isomorphic to $\mathbb{Z}_2 \oplus D_3$. The family of these Riemann surfaces degenerates to a stable Riemann surface $S_{r^*(1/2),1/2}$ as Figure 2 keeping the above group of automorphisms invariant.

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Super-Halley method under majorant conditions in Banach spaces

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ABSTRACT

In this paper, we have studied local convergence of Super-Halley method in Banach spaces under the assumption of second order majorant conditions. This approach allows us to obtain generalization of earlier convergence analysis under majorizing sequences. Two important special cases of the convergence analysis based on the premises of Kantorovich and Smale type conditions have also been concluded. To show efficacy of our approach we have given three numerical examples.

RESUMEN

En este artículo, hemos estudiado la convergencia local del método Super-Halley en espacios de Banach, asumiendo condiciones mayorantes de segundo orden. Este punto de vista nos permite obtener generalizaciones de análisis de convergencia bajo sucesiones mayorantes obtenidos anteriormente. También se han concluido dos casos especiales del análisis de convergencia basados en las premisas de condiciones tipo Kantorovich y Smale. Para mostrar la eficacia de nuestro enfoque, damos tres ejemplos numéricos.

Keywords and Phrases: Nonlinear equations; Super-Halley method; Majorant conditions; Local Convergence; Semilocal Convergnce; Smale-type conditions; Kantorovich-type conditions.

2010 AMS Mathematics Subject Classification: 65D10, 65G99, 65K10, 47H17, 49M15, 47H99.

1 Introduction

Let f be a given operator that maps from some nonempty open convex subset Ω of a Banach space $\mathbb{X}$ to another Banach space $\mathbb{Y}$. Approximating a locally unique solution $\bar{x}$ of a nonlinear equation

$$f(x) = 0 \tag{1.1}$$

is widely studied in both theoretical and applied areas of mathematics. Generally, this is done by using some iterative processes. An iterative process is a mathematical procedure that, from one or several initial approximations of a solution of (1.1), a sequence of iterates $\{x_n\}_{n\in\mathbb{N}}$ is constructed so that each subsequent iterate of the sequence is a better approximation to the previous approximation to the solution of (1.1); that is, the sequence $\{\|x_n - \bar{x}\|\}_{n\in\mathbb{N}}$ is convergent to zero. Usually, in order to study convergence analysis of the method, we could consider the study of local and semilocal convergence analysis. If the convergence analysis seeks assumptions around a solution $\bar{x}$, then it is called local convergence and this type of analysis estimates the radii of convergence balls, where as we could also consider assumptions around an initial point x_0 to study convergence analysis of the method. In that case the convergence analysis is called semilocal one and it gives criteria ensuring the convergence. It is also very important to give convergence ball of an iterative method, because that shows the extent to which we can choose an initial guesses for that method.

One of the most important iterative methods to solve this problem is Newton's method given by

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n), \quad k = 0, 1, 2, \dots \tag{1.2}$$

where $x_0 \in \Omega$ is an initial point. As anybody can recall that one of the most famous results to study convergence of Newton's method (1.2) is the well known Kantorovich method[16], which guarantees convergence of the method to a solution, using semilocal conditions. It does not require a priori existence of a solution, proving instead the existence of the solution and its uniqueness on some region. Many researches have also done works related to Kantorovich-like method (for details see [4, 8, 10, 26, 27, 29] and references there in). Also, Smale's point theory [21] assumes that the nonlinear operator is analytic at the initial point, which is an important result concerning Newton's method. Wan and Han[25, 22] has discussed the generalization and the particular cases of Smale's point estimate theory.

For a positive number α and $x \in \mathbb{X}$, we consider $B(x, \alpha)$ to stand for the open ball with radius α and center x and $\bar{B}(x, \alpha)$ is the corresponding close ball. In [7, 6], Ferreira and Svaiter had studied the local convergence of Newton's method (1.2) under the following majorant conditions:

$$\|f'(\bar{x})^{-1}[f'(y) - f'(x)]\| \leq h'(\|y - x\| + \|x - \bar{x}\|) - h'(\|x - \bar{x}\|), \tag{1.3}$$

for $x, y \in B(\bar{x}, R)$, $R > 0$, where $\|y - x\| + \|x - \bar{x}\| < R$ and $h : (0, R) \to \mathbb{R}$ is a continuously differentiable, convex and strictly increasing function that satisfies $h(0) > 0$, $h'(0) = -1$ and has a zero in

$(0, R)$. Note that by this study the assumptions for guaranteeing Q-quadratic convergence of the respective iterative methods has been relaxed and a new estimate of the Q-quadratic convergence has been obtained.

Recently, inspired by these ideas, Ling and Xu[17] have presented a new convergence analysis of Halley's method which makes a relationship of the majorizing function h and the nonlinear operator f under majorant conditions similar to given in (1.3). Argyros and Ren[1] also presented a local convergence of Halley's method which gives a ball convergence of the method under assumptions similar to (1.3).

On the other hand, one of the famous third order iterative method to solve nonlinear equation (1.1) in Banach space is the Super-Halley method denoted by

$$x_{n+1} = x_n - \left[I + \frac{1}{2}L_f(x_n)[I - L_f(x_n)]^{-1}\right]f'(x_n)^{-1}f(x_n), \tag{1.4}$$

where for $x \in \mathbb{X}$, $L_f(x)$ is the linear operator defined as

$$L_f(x) = f'(x)^{-1}f''(x)f'(x)^{-1}f(x).$$

The results concerning the convergence of this method have been studied in [2, 9, 20] under different types of assumptions by using recurrence relations. On the other hand Ezquerro and Hernández[5] and Gutiérrez and Hernández[11] have studied semilocal convergence of this method by using majorizing sequences. Now, if the nonlinear operator f is analytic at the initial point then motivated by the ideas of Argyros and Ren[1] and Ling and Xu[17], we have studied local convergence of Super-Halley method (1.4) using second order majorant condition. This majorant condition generalizes the earlier results on Super-Halley method[5, 11] using majorizing sequences. Two particular cases namely results based of affine invariant Lipschitz-type condition and Smale-type condition have also been derived. Numerical efficacy of the method has also been derived by way of a number of numerical examples.

Rest of the paper is organized as follows. Some preliminaries results are contained in section 2. In section 2.1, we studied local convergence analysis of Super-Halley method. Two special cases of main result are presented in section 3. In section 4, we have shown a number of numerical examples to show efficacy of our study. Section 5 forms the conclusion part of the paper.

2 Preliminaries

In this section we provide some basic results which is required for our convergence analysis of the method.

Assume $a > 0$ and $\phi : (0, a) \to \mathbb{R}$ be a twice continuously differentiable function. Let $x, y \in B(\bar{x}, a) \subset \Omega$, with $\|y - x\| + \|x - \bar{x}\| < a$. We say that the operator f satisfy a second order majorizing function ϕ at $\bar{x}$ if the following conditions hold on f:

$$\|f'(\bar{x})^{-1}[f''(y) - f''(x)]\| \leq \phi''(\|y - x\| + \|x - \bar{x}\|) - \phi''(\|x - \bar{x}\|), \tag{2.1}$$

with the assumptions:

$$\left.\begin{array}{l} (M1)\ \phi(0) > 0, \phi''(0) > 0, \phi'(0) = -1, \\ (M2)\ \phi'' \text{ is convex and strictly increasing in } (0, a), \\ (M3)\ \phi \text{ has atleast one zero in } (0, a) \text{ with } t^* \text{ as the smallest zero and } \phi'(t^*) < 0. \end{array}\right\} \tag{2.2}$$

and

$$\|f'(\bar{x})^{-1}f(\bar{x})\| \leq \phi(0), \quad \|f'(\bar{x})^{-1}f''(\bar{x})\| \leq \phi''(0). \tag{2.3}$$

In this paper we assume that ϕ is the majorizing function of f. Note that if we define

$$\Theta_f(x) := x - \left[I + \frac{1}{2}L_f(x)[I - L_f(x)]^{-1}\right] f'(x)^{-1}f(x) \tag{2.4}$$

where $L_f(x) = f'(x)^{-1}f''(x)f'(x)^{-1}f(x)$, then $\Theta_f(x)$ can be taken as the iterative function of Super-Halley method as it can be written as $x_{n+1} = \Theta_f(x_n)$. Also the scalar sequence $\{t_n\}$ can be generated by applying the method to $\phi(t)$. In this case we can write $t_{n+1} = \Theta_\phi(t_n)$ with

$$\Theta_\phi(t) := t - \left[1 + \frac{L_\phi(t)}{2(1 - L_\phi(t))}\right] \frac{\phi(t)}{\phi'(t)}, \quad L_\phi(t) = \frac{\phi(t)\phi''(t)}{\phi'(t)^2}, \quad t \in (0, a). \tag{2.5}$$

Now we can easily establish some basic properties of the majorizing function ϕ, the iterative functions $\Theta_f(x)$ and $\Theta_\phi(t)$ which are described in the following lemmas.

Lemma 2.1. *Let ϕ satisfies assumptions $(M1) - (M3)$. Then*

(i) ϕ' *is strictly convex and strictly increasing on* $(0, a)$.

(ii) ϕ *is strictly convex on* $(0, a)$, $\phi(t) > 0$ *for* $t \in (0, t^*)$ *and equation* $\phi(t) = 0$ *has at most one root in* (t^*, a).

(iii) $-1 < \phi'(t) < 0$ *for* $t \in (0, t^*)$.

(iv) $0 \leq L_\phi(t) \leq \frac{1}{2}$ *for* $t \in [0, t^*]$.

Proof. The proof is similar to one given in [17], so omitted.

Lemma 2.2. *Let ϕ satisfies assumptions $(M1)-(M3)$. Then for all $t \in (0, t^*)$, $t < \Theta_\phi(t) < t^*$. Moreover, $\phi'(t^*) < 0$ if and only if there exist $t \in (t^*, a)$ such that $\phi(t) < 0$.*

Proof. It is not to be mentioned that by using Lemma 2.1, one can have $\phi(t) > 0$, $-1 < \phi'(t) < 0$ and $0 \leq L_\phi(t) \leq \frac{1}{2}$ for $t \in (0, t^*)$. This gives $\left[1 + \frac{L_\phi(t)}{2(1 - L_\phi(t))}\right] \frac{\phi(t)}{\phi'(t)} < 0$ and hence $t < \Theta_\phi(t)$.

Also, for any $t \in (0, t^*)$, from the definition of directional derivative and assumption (M2) it follows that since $\phi''(t)$ is increasing in $(0, a)$ and $t < t^* < a$, we have $\phi''(t) < \phi''(t^*)$ and $\phi''(t) > 0$ which implies that left directional derivative $D^-\phi''(t) > 0$.

As $\phi''(t)\phi(t) - 2\phi'(t)^2 \leq -4\phi''(t)\phi(t)$, we obtain $D^-\Theta_\phi(t) = 1 + \frac{2\phi'(t)\phi(t)D^-\phi''(t)}{(\phi''(t)\phi(t) - 2\phi'(t)^2)\phi''(t)} > 0$ for $t \in (0, t^*)$.

And this implies that $\Theta_\phi(t) < \Theta_\phi(t^*) = t^*$ for any $t \in (0, t^*)$. So the first part of this lemma is complete. Now, if $\phi'(t^*) < 0$, then it is obvious that there exists $t \in (t^*, a)$ such that $\phi(t) < 0$. Conversely, noting that $\phi'(t^*) = 0$, then we have $\phi(t) > \phi(t^*) + \phi'(t)(t^* - t)$ for $t \in (t^*, a)$, which implies that $\phi'(t^*) < 0$. This completes the proof.

Remark Following properties are implied by the condition $\phi'(t^*) < 0$ in (M3):

- $\phi(t^{**}) = 0$ for some $t^{**} \in (t^*, a)$.
- $\phi(t) < 0$ for some $t \in (t^*, a)$.

Lemma 2.3. *Let* ϕ *satisfies assumptions* $(M1) - (M3)$. *Then*

$$t^* - \Theta_\phi(t) \leq \left[\frac{1}{2}\frac{\phi''(t^*)^2}{\phi'(t^*)^2} + \frac{1}{3}\frac{D^-\phi''(t^*)}{-\phi'(t^*)}\right](t^* - t)^3, t \in (0, t^*). \tag{2.6}$$

Proof. We can derive the following relation, by using the definition of Θ_ϕ in (2.5)

$$\begin{aligned} t^* - \Theta_\phi(t) &= \frac{1}{1 - L_\phi(t)}\left[(1 - L_\phi(t))(t^* - t) + \frac{\phi(t)}{2\phi'(t)}(1 - L_\phi(t)) + \frac{\phi(t)}{2\phi'(t)}\right] \\ &= -\frac{1}{\phi'(t)(1 - L_\phi(t)}\int_0^1 [\phi''(t + \sigma(t^* - t)) - \phi''(t)](t^* - t)^2(1 - \sigma)d\sigma \\ &\quad + \frac{(t^* - t)\phi''(t)}{2(1 - L_\phi(t))\phi'(t)^2}\int_0^1 \phi''(t + \sigma(t^* - t))(t^* - t)^2(1 - \sigma)d\sigma \\ &\quad - \frac{(t^* - t)\phi''(t)}{2(1 - L_\phi(t))\phi'(t)}\left(\phi(t) + \frac{\phi(t)^2}{\phi'(t)(t^* - t)}\right) \end{aligned}$$

Since ϕ'' is convex and $t < t^*$, it follows from Lemma 2.1 that

$$\phi''(t + \sigma(t^* - t)) - \phi''(t) \leq [\phi''(t^*) - \phi''(t)]\frac{\sigma(t^* - t)}{(t^* - t)}$$

So by noting that ϕ'' is strictly increasing, we have

$$\begin{aligned} t^* - \Theta_\phi(t) \quad &\leq \quad -\frac{\phi''(t^*) - \phi''(t)}{6\phi'(t)(1-L_\phi(t))}(t^*-t)^2 + \frac{\phi''(t^*)\phi''(t)}{4\phi'(t)^2(1-L_\phi(t))}(t^*-t)^3 \\ &\quad -\frac{\phi(t)\phi''(t)}{2(1-L_\phi(t))\phi'(t)}(t^*-t) - \frac{\phi(t)^2\phi''(t)}{2(1-L_\phi(t))\phi'(t)^2} \end{aligned}$$

Since $\phi'(t) < 0$, $\phi''(0) > 0$ and ϕ', ϕ'' are strictly increasing on $(0, t^*)$ and $0 \leq L_\phi(t) \leq \frac{1}{2}$ for $t \in [0, t^*]$ by Lemma 2.1, so we have

$$t^* - \Theta_\phi(t) \leq \Big[\frac{1}{2}\frac{\phi''(t^*)^2}{\phi'(t^*)^2} + \frac{1}{3}\frac{D^-\phi''(t^*)}{-\phi'(t^*)}\Big](t^*-t)^3. \tag{2.7}$$

As ϕ' is increasing, $\phi'(t^*) < 0$ and $\phi'(t) < 0$ t in $(0, t^*)$, we have

$$\frac{\phi''(t^*)) - \phi''(t)}{-\phi'(t)} \leq \frac{\phi''(t^*) - \phi''(t)}{-\phi'(t^*)} = \frac{1}{-\phi'(t^*)}\frac{\phi''(t^*)) - \phi''(t)}{t^* - t}(t^*-t) \leq \frac{D^-\phi''(t^*)}{-\phi'(t^*)}(t^*-t)$$

where the last inequality follows from definitions of directional derivative. Combining the above inequality with (2.7), we conclude that (2.6) holds. This completes the proof. □

Let $\{t_k\}$ denote the majorizing sequence generated by

$$t_0 = 0, t_{k+1} = \Theta_\phi(t_k) = t_k - \Big[I + \frac{L_\phi(t_k)}{2(1-L_\phi(t_k))}\Big]\frac{\phi(t_k)}{\phi'(t_k)}, \quad k = 0, 1, 2, \ldots \tag{2.8}$$

We arrive at the following theorem using Lemma 2.3 that

Theorem 2.4. *Let the sequence* $\{t_k\}$ *be defined by* (2.8)*. Then* $\{t_k\}$ *is well defined, strictly increasing and is contained in* $(0, t^*)$*. Moreover,* $\{t_k\}$ *satisfies (2.6) and converges to* t^* *with* $Q-\mathrm{cubic}$*, i.e.:*

$$t^* - t_{k+1} \leq \Big[\frac{1}{2}\frac{\phi''(t^*)^2}{\phi'(t^*)^2} + \frac{1}{3}\frac{D^-\phi''(t^*)}{-\phi'(t^*)}\Big](t^*-t_k)^3, \quad t_k \in (0, t^*).$$

2.1 Local convergence results for Super-Halley method

This section is devoted to giving the local convergence analysis of (1.4). For that the following lemmas will play important role.

Lemma 2.5. *Assume* $\|x - \bar{x}\| \leq t < t^*$*. If* $\phi : (0, t^*) \to \mathbb{R}$ *is a twice continuously differentiable function which majorizes* f *at* $\bar{x}$*, then*

(i) $f'(x)$ *is nonsingular and*

$$\|f'(x)^{-1}f'(\bar{x})\| \leq -\frac{1}{\phi'(\|x-\bar{x}\|)} \leq -\frac{1}{\phi'(t)}. \tag{2.9}$$

(ii) $\|f'(\bar{x})^{-1}f''(x)\| \leq \phi''(\|x-\bar{x}\|) \leq \phi''(t)$.

Proof. Let us take $x \in B(\bar{x}, t)$, $0 \leq t < t^*$. Since

$$f'(x) = f'(\bar{x}) + \int_0^1 [f''(\bar{x} + \sigma(x-\bar{x})) - f''(\bar{x})](x-\bar{x})d\sigma + f''(\bar{x})(x-\bar{x}),$$

we get

$$\begin{aligned}
\|I - f'(\bar{x})^{-1}f'(x)\| &\leq \int_0^1 \|f'(\bar{x})^{-1}[f''(\bar{x}+\sigma(x-\bar{x})) - f''(\bar{x})]\|\|x-\bar{x}\|d\sigma \\
&\quad + \|f'(\bar{x})^{-1}f''(\bar{x})\|\|x-\bar{x}\| \\
&\leq \int_0^1 [\phi''(\sigma(\|(x-\bar{x})\|)) - \phi''(0)]\|x-\bar{x}\|d\sigma + \phi''(0)\|x-\bar{x}\| \\
&= \phi'(\|(x-\bar{x})\|) - \phi'(0).
\end{aligned}$$

So, we conclude that

$$\|I - f'(\bar{x})^{-1}f'(x)\| \leq \phi'(t) - \phi'(0) < 1$$

as $\phi'(0) = -1$ and $-1 < \phi'(t) < 0$ for $(0, t^*)$ using Lemma 2.1. Therefore, it follows from Banach lemma that $f'(\bar{x})^{-1}f'(x)$ is nonsingular and (2.9) holds as

$$\|f'(x)^{-1}f'(\bar{x})\| \leq \frac{1}{1-\phi'(\|x-\bar{x}\|)+\phi'(0)} = -\frac{1}{\phi'(\|x-\bar{x}\|)} \leq -\frac{1}{\phi'(t)}.$$

Thus we conclude that, f' is nonsingular in $B(\bar{x}, t^*)$. By using majorant conditions, we have

$$\begin{aligned}
\|f'(\bar{x})^{-1}f''(x)\| &\leq \|f'(\bar{x})^{-1}[f''(x) - f''(\bar{x})]\| + \|f'(\bar{x})^{-1}f''(\bar{x})\| \\
&\leq \phi''(\|x-\bar{x}\|) - \phi''(0) + \phi''(0) = \phi''(\|x-\bar{x}\|) \leq \phi''(t).
\end{aligned}$$

The last inequality holds true because of ϕ'' is strictly increasing. This completes the proof. □

Now the main local convergence result for the Super-Halley method (1.4) is presented as follows.

Theorem 2.6. *Let* f *satisfies the second order majorant conditions* (2.1)-(2.3). *Then, the sequence of iterates* $\{x_n\}$ *generated by Super-Halley method (1.4) is well defined, contained in* $B(\bar{x}, t^*)$ *and converges to the unique solution* $\bar{x}$ *of (1.1). Moreover, the following error estimate hold*

$$\|\bar{x} - x_{k+1}\| \leq (t^* - t_{k+1})\left(\frac{\|\bar{x}-x_k\|}{t^*-t_k}\right)^3, \quad k = 0, 1, 2, \ldots \tag{2.10}$$

Thus the sequence $\{x_n\}$ *generated by Super-Halley method (1.4) converges* $Q-$*cubic as follows*

$$\|\bar{x} - x_{k+1}\| \leq \left[\frac{1}{2}\frac{h''(t^*)^2}{h'(t^*)^2} + \frac{1}{3}\frac{D^-h''(t^*)}{-h'(t^*)}\right]\|\bar{x}-x_k\|^3, \quad k = 0, 1, 2, \ldots \tag{2.11}$$

Proof. By using $f(\bar{x}) = 0$ and some standard analytic techniques, one can have

$$\begin{aligned}
\bar{x} - x_{k+1} &= -\Gamma_f(x_k)f'(x_k)^{-1}[-f'(x_k)(\bar{x}-x_k) - f(x_k)] - \Gamma_f(x_k)L_f(x_k)(\bar{x}-x_k) \\
&\quad + \Big[\frac{1}{2}f'(x_k)^{-1}f(x_k) - \frac{1}{2}\Gamma_f(x_k)f'(x_k)^{-1}f(x_k)\Big] \\
&= -\Gamma_f(x_k)f'(x_k)^{-1}\int_0^1 (1-\sigma)[f''(x_k+\sigma(\bar{x}-x_k)) - f''(x_k)](\bar{x}-x_k)^2 d\sigma \\
&\quad + f''(x_k)\Gamma_f(x_k)f'(x_k)^{-1}\Big[f'(x_k)^{-1}\int_0^1 (1-\sigma)f''(x_k+\sigma(\bar{x}-x_k)) \\
&\quad \times(\bar{x}-x_k)^2 d\sigma\Big](\bar{x}-x_k) \\
&\quad - \frac{1}{2}f''(x_k)\Gamma_f(x_k)f'(x_k)^{-1}\Big[f'(x_k)^{-1}\int_0^1 (1-\sigma)f''(x_k+\sigma(\bar{x}-x_k)) \\
&\quad \times(\bar{x}-x_k)^2 d\sigma\Big]^2,
\end{aligned}$$

where $\Gamma_f(x) = (I - L_f(x))^{-1}$. Using majorant condition, we can get

$$\begin{aligned}
\int_0^1 \|f'(\bar{x})^{-1}[f''(x_k+\sigma(\bar{x}-x_k)) - f''(x_k)]\|(1-\sigma)d\sigma &\le \int_0^1 \|[\phi''(\sigma\|\bar{x}-x_k\| + \|x_k-\bar{x}\|) \\
&\quad - \phi''(\|x_k-\bar{x}\|)]\|(1-\sigma)d\sigma.
\end{aligned}$$

Also, we know that, if $u, v, w \in (0, a)$ and $u < v < w$, then because of convexity[7] of $\phi(x)$ in $(0, a)$, we have

$$\phi(v) - \phi(u) \le [\phi(w) - \phi(u)]\frac{v-u}{w-u}.$$

Therefore,

$$\begin{aligned}
\phi''(\sigma\|\bar{x}-x_k\| + \|x_k-\bar{x}\|) - \phi''(\|x_k-\bar{x}\|) &\le \phi''(\sigma\|\bar{x}-x_k\| + t_k) - \phi''(t_k) \\
&\le [\phi''(\sigma(t^*-t_k)+t_k) - \phi''(t_k)]\frac{\|\bar{x}-x_k\|}{t^*-t_k}.
\end{aligned}$$

Thus Lemma 2.5 and above inequality implies

$$\begin{aligned}
\|\bar{x}-x_{k+1}\| &\le -\frac{1}{(1-L_\phi(t_k))\phi'(t_k)}\Big[\int_0^1 [\phi''(\sigma(t^*-t_k)+t_k) - \phi''(t_k)](1-\sigma)d\sigma\Big]\frac{\|\bar{x}-x_k\|^3}{t^*-t_k} \\
&\quad + \frac{\phi''(t_k)}{(1-L_\phi(t_k))\phi'(t_k)^2}\Big[\int_0^1 [\phi''(\sigma(t^*-t_k)+t_k)(1-\sigma)d\sigma\Big]\|\bar{x}-x_k\|^3 \\
&\quad + \frac{1}{2}\frac{\phi''(t_k)}{(1-L_\phi(t_k))\phi'(t_k)^2}\Big[\int_0^1 [\phi''(\sigma(t^*-t_k)+t_k)(1-\sigma)d\sigma\Big]^2\|\bar{x}-x_k\|^4 \\
&\le \frac{\phi(t_k)}{(1-L_\phi(t_k))\phi'(t_k)}\Big(\frac{\|\bar{x}-x_k\|}{t^*-t_k}\Big)^3 = (t^*-t_{k+1})\Big(\frac{\|\bar{x}-x_k\|}{t^*-t_k}\Big)^3.
\end{aligned}$$

Finally, we want to show that the solution $\bar{x}$ of (1.1) is unique in $\bar{B}(\bar{x}, t^*)$. For that assume $\bar{y}$ be another solution in $\bar{B}(\bar{x}, t^*)$. Then proceeding similarly as above we get

$$\|\bar{y}-x_{k+1}\| \le (t^*-t_{k+1})\Big(\frac{\|\bar{y}-x_k\|}{t^*-t_k}\Big)^3.$$

Since the sequence $\{x_k\}$ converges to $\bar{x}$ and $\{t_k\}$ converges to t^*, we conclude that $\bar{y} = \bar{x}$. Therefore, $\bar{x}$ is the unique zero of (1.1) in $\bar{B}(\bar{x}, t^*)$. □

Remark 2.7. *It is to be noted that if we replace* $\bar{x}$ *with the initial approximation* x_0 *in* (2.1), *then after some manipulations we can obtain a semilocal convergence analysis of our iteration method. This analysis approach enables us to drop out the assumption of existence of a second root for the majorizing function, still guarantee* $Q-$*cubic convergence rate. Thus the semilocal convergence theorem for the iteration method (1.4) is as follows:*

Theorem 2.8. *Suppose* $f : \Omega \subseteq \mathbb{X} \to \mathbb{Y}$ *be a twice continuously differentiable nonlinear operator, and* Ω *is open and convex. Consider that for a given initial guess* $x_0 \in \Omega$, $f'(x_0)$ *is nonsingular that is* $f'(x_0)^{-1}$ *exists and that* ϕ *is the majorizing function to* f *at* x_0 *and* ϕ *satisfies the assumptions* $(M1)--(M3)$. *Then sequence* $\{x_k\}$ *generated by the method (1.4) for solving equation (1.1) with a starting point* x_0 *is well defined, contained in* $B(x_0, t^*)$ *and converges to a solution* $\bar{x} \in \bar{B}(x_0, t^*)$ *of the Eq.(1.1). The solution is unique in* $B(x_0, \sigma)$, *where* σ *is defined as* $\sigma := \sup\{t \in (t^*, R) : \phi(t) \leq 0\}$. *For* $k = 0, 1, 2, \ldots,$ *a priori error estimate and a posteriori error estimate are given respectively as*

$$\|\bar{x} - x_{k+1}\| \leq (t^* - t_{k+1})\Big(\frac{\|\bar{x} - x_k\|}{t^* - t_k}\Big)^3,$$

and

$$\|\bar{x} - x_{k+1}\| \leq (t^* - t_{k+1})\Big(\frac{\|x_{k+1} - x_k\|}{t_{k+1} - t_k}\Big)^3.$$

Also the method converges $Q-$*cubically as*

$$\|\bar{x} - x_{k+1}\| \leq \Big[\frac{1}{2}\frac{\phi''(t^*)^2}{\phi'(t^*)^2} + \frac{1}{3}\frac{D^-\phi''(t^*)}{-\phi'(t^*)}\Big](\|\bar{x} - x_k\|)^3.$$

3 Special cases and applications

This section consists of two special cases of the local convergence results obtained in previous section. Namely, convergence results under affine covariant Kantorovich-type condition and the Smale-type γ-condition.

3.1 Kantorovich-type

Suppose that f satisfies the affine covariant Lipschitz condition (see Han and Wang[12]) as given by:

$$\|f'(\bar{x})^{-1}[f''(y) - f''(x)]\| \leq \lambda_1 \|y - x\|, \quad x, y \in \Omega. \tag{3.1}$$

and the following initial conditions

$$\|f'(\bar{x})^{-1} f(\bar{x})\| \leq \beta, \tag{3.2}$$

$$\|f'(\bar{x})^{-1}f''(\bar{x})\| \leq \lambda_2. \tag{3.3}$$

Consider the scalar valued function

$$\phi(t) = \frac{\lambda_1}{6}t^3 + \frac{\lambda_2}{2}t^2 - t + \beta. \tag{3.4}$$

This function was considered as majorizing function in [28, 5, 11] for establishing convergence of super-Halley method. If we choose the above cubic polynomial as the majorizing function ϕ in (2.1), then the majorant condition (2.1) reduced to the Lipschitz condition (3.1) and in this way the results based on Lipschitz condition have been generalized by our assumptions of majorant conditions. The assumptions (M1) and (M2) are satisfied for f if the following criterion holds

$$\beta \leq \frac{2(\lambda_2 + 2(\lambda_2^2 + 2\lambda_1)^{1/2})}{3(\lambda_2 + 2(\lambda_2^2 + 2\lambda_1)^{1/2})^2}. \tag{3.5}$$

Therefore, Theorem 2.6 reduces to the following form:

Theorem 3.1. *Suppose that f satisfies the conditions (3.1)-(3.3) with the assumptions given in (3.5). Then, the sequence $\{x_k\}$ generated by Super-Halley method (1.4) for solving equation (1.1) with a starting point x_0 is well defined, contained in $B(\bar{x}, t^*)$ and converges to a solution $\bar{x} \in \bar{B}(\bar{x}, t^*)$ of the Eq.(1.1). Note that t^* is the smallest positive root of ϕ defined by (3.4) in $[0, r_1]$ where $r_1 = (-\lambda_2 + (\lambda_2^2 + 2\lambda_1)^{1/2})/\lambda_1$ is the positive root of ϕ'. The limit $\bar{x}$ of the sequence $\{x_k\}$ is the unique zero of Eq.(1.1) in $B(\bar{x}, t^{**})$, where t^{**} is the root of ϕ in the interval $(r_1, +\infty)$. Moreover, the following error estimates holds*

$$\|\bar{x} - x_{k+1}\| \leq (t^* - t_{k+1})\Big(\frac{\|\bar{x} - x_k\|}{t^* - t_k}\Big)^3, \ k = 0, 1, 2, \ldots$$

and the sequence generated by Super-Halley method (1.4) converges Q-cubic as follows

$$\|\bar{x} - x_{k+1}\| \leq \Big[\frac{3(\lambda_1 + \lambda_2 t^*)^2 + 2\lambda_2(1 - \lambda_1 t^* - \lambda_2 t^{*2}/2)}{6(1 - \lambda_1 t^* - \lambda_2 t^{*2}/2)^2}\Big](\|\bar{x} - x_k\|)^3, \ k = 0, 1, 2, \ldots$$

3.2 Smale-type

This subsection contains the local convergence results for the Super-Halley method (1.4) under the γ-Condition.

In [21], Smale has studied the convergence and error estimation of Newton's method under the hypotheses that f is analytic and satisfies

$$\|f'(\bar{x})^{-1}f^{(n)}(\bar{x})\| \leq n!\gamma^{n-1}, \qquad n > 2$$

where

$$\gamma := \sup_{k>1} \|\frac{f'(\bar{x})^{-1} f^{(n)}(\bar{x})}{k!}\|^{\frac{1}{k-1}}$$

Smale's result is completely improved by Wang and Han[24, 25] by introducing a majorizing function

$$\phi(t) = \beta - t + \frac{\gamma t^2}{1-\gamma t}, \qquad t \in \Big[0, \frac{1}{\gamma}\Big) \tag{3.6}$$

where

$$\|f'(\bar{x})^{-1} f(\bar{x})\| \leq \beta \tag{3.7}$$

and

$$\|f'(\bar{x})^{-1} f''(\bar{x})\| \leq 2\gamma \tag{3.8}$$

If we choose this function as the majorizing function ϕ in (2.1), then it reduces to the following condition:

$$\begin{aligned} \|f'(\bar{x})^{-1}[f''(y) - f''(x)]\| \quad \leq \quad & \frac{2\gamma}{(1-\gamma\|y-x\| - \gamma\|x-\bar{x}\|)^3} \\ & -\frac{2\gamma}{(1-\gamma\|x-\bar{x}\|)^3}, \qquad \gamma > 0 \end{aligned} \tag{3.9}$$

where $\|y - x\| + \|x - \bar{x}\| < \frac{1}{\gamma}$, and the assumptions $(M1)$ and $(M2)$ are satisfied for ϕ. Also, if $\alpha := \beta\gamma < 3 - 2\sqrt{2}$, then assumption $(M3)$ is satisfied for ϕ. Therefore, the concrete form of Theorem 2.6 is given as follows.

Theorem 3.2. *Suppose* f *satisfies* (3.7)-(3.9). *If* $\alpha := \beta\gamma < 3 - 2\sqrt{2}$, *then the sequence* $\{x_k\}$ *generated by the super-Halley method (1.4) for solving the equation (1.1) with a starting point* x_0 *is well defined, is contained in* $B(\bar{x}, t^*)$ *and converges to a solution* $\bar{x} \in \bar{B}(\bar{x}, t^*)$ *of the Eq.(1.1). The limit* $\bar{x}$ *of the sequence* $\{x_k\}$ *is unique in* $B(\bar{x}, t^{**})$, *where* t^* *and* t^{**} *are given as*

$$t^* = \frac{\alpha + 1 - \sqrt{(\alpha+1)^2 - 8\alpha}}{4\gamma} \quad \text{and} \quad t^{**} = \frac{\alpha + 1 + \sqrt{(\alpha+1)^2 - 8\alpha}}{4\gamma}$$

respectively. Moreover, the following error bound holds: for all $k \geq 0$, *we have*

$$\|\bar{x} - x_{k+1}\| \leq (t^* - t_{k+1})\Big(\frac{\|\bar{x} - x_k\|}{t^* - t_k}\Big)^3, \ k = 0, 1, 2, \ldots$$

and the sequence $\{x_k\}$ *converges* Q-*cubic as follows*

$$\|\bar{x} - x_{k+1}\| \leq \frac{2\gamma^2}{[2(1-\gamma t^*)^2 - 1]^2}(\|\bar{x} - x_k\|)^3, \ k = 0, 1, 2, \ldots$$

4 Numerical Examples

This section is devoted to illustrate the above theoretical results by a number of numerical examples.

Example 4.1. *Let* $X = Y = \mathbb{R}$ *with* $\Omega = B(0,1)$ *and the function* f *on* Ω *is*

$$f(x) = e^x - 1 \tag{4.1}$$

and for $\bar{x} = 0$,

$$f'(\bar{x}) = 1, f''(\bar{x}) = 1.$$

Also, we obtain that

$$\lambda_1 = e, \lambda_2 = 1, \beta = 0$$

Therefore, the convergence criterion (3.5) *holds and the Theorem 3.1 is applicable to conclude that the sequence generated by super-Halley method* (1.4) *with initial point* $x_0 = 0.25$ *converges to a root of* (4.1). *In this case, we have* $t^* = 0$ *and* $t^{**} = 1.03304078$*, that is the existence and uniqueness ball are* $B(0.25, 0)$ *and* $\bar{B}(0.25, 1.03304078)$ *respectively and the error bound is* 1.525807581.

Example 4.2. *Let* $X = C[0,1]$ *the space of continuous functions defined on interval* $[0,1]$ *equipped with max norm and let* $\Omega = U[0,1]$ *and the function* f *on* Ω *is .*

$$f(x)(s) = x(s) - 2\lambda \int_0^1 G(s,t)x(t)^3 dt \tag{4.2}$$

Therefore we have

$$f'(x)u(s) = u(s) - 6\lambda \int_0^1 G(s,t)x(t)^2 u(t) dt, \qquad u \in \Omega,$$

and

$$f''(x)[uv](s) = -\lambda \int_0^1 G(s,t)x(t)(uv)(t) dt, \qquad u, v \in \Omega,$$

Now, let $M = \max_{s \in [0,1]} \int_0^1 |G(s,t)| dt$. *Then* $M = \frac{1}{8}$.. *Also, for any* $x, y \in \Omega$, *we have*

$$\|f'(\bar{x})^{-1}[f''(x) - f''(\bar{x})]\| \leq \frac{3|\lambda|}{2} \|x - \bar{x}\|.$$

So, we obtain the values of β, λ_2 *and* λ_1 *in as follows*

$$\beta = 0, \qquad \lambda_2 = 0, \qquad \lambda_1 = \frac{3|\lambda|}{2}.$$

Therefore, the convergence criterion (3.5) holds and the Theorem 3.1 is applicable to conclude that the sequence generated by Super-Halley method (1.4) with initial point x_0 *converges to a zero of* f *defined by (4.2).*

For the different values of λ *i.e. for* $\lambda = 0.0625, 0.125, 0.25, 0.5, 1$ *and the initial point* $x_0 = 0.25$ *the corresponding domain of existence and uniqueness of solution, are given in Table-4.2.*

Table-4.2: *Domains of existence and uniqueness of solution for Super-Halley's method*

λ	*convergence ball in our work*	
	existence	*uniqueness*
0.0625	$\bar{B}(0.25, 0)$	$B(0.25, 8)$
0.125	$\bar{B}(0.25, 0)$	$B(0.25, 5.656854249)$
0.25	$\bar{B}(0.25, 0)$	$B(0.25, 4)$
0.5	$\bar{B}(0.25, 0)$	$B(0.25, 2.828427125)$
1	$\bar{B}(0.25, 0)$	$B(0.25, 2)$

Now, we present a numerical example to illustrate the Smale-type conditions.

Example 4.3. *Let* $X = Y = \mathbb{R}$ *with* $\Omega = U[0, 1]$ *and the function* f *on* Ω *is*

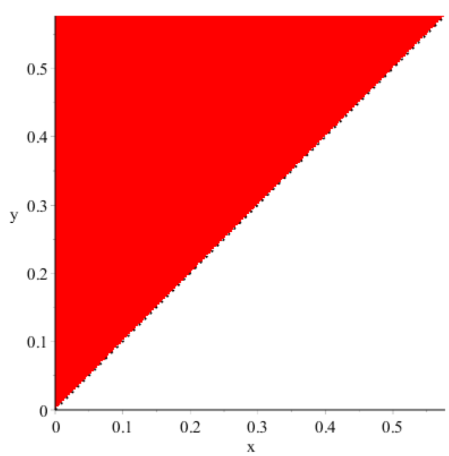
$$f(x) = e^x + 2x^2 - 1 \tag{4.3}$$

For $\bar{x} = 0$,

$$f'(\bar{x}) = 1, f''(\bar{x}) = 5$$

and we obtain that

$$\beta = 0, \gamma = 2.5$$



Therefore, the convergence criterion (3.9) holds (which can be seen from the above graph in case $y > x$*) and the Theorem 3.2 is applicable to conclude that the sequence generated by Super-Halley method (1.4) converges to a zero of* f *defined by (4.3) with* $t^* = 0$ *and* $t^{**} = 0.2$.

5 Conclusions

In this paper, the local convergence of Super-Halley method has been studied under majorant conditions on second derivative of f. Convergence ball of the method has been included. Two special cases: one Kantorovich-type conditions and another Smale-type conditions have also been studied. A number of numerical examples also given to illustrate our study.

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Certain results on the conharmonic curvature tensor of (κ, μ)-contact metric manifolds

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ABSTRACT

The paper presents a study of (κ, μ)-contact metric manifolds satisfying certain conditions on the conharmonic curvature tensor.

RESUMEN

El artículo presenta un estudio de variedades (κ, μ)-contacto métricas satisfaciendo ciertas condiciones sobre el tensor de curvatura conharmónico.

Keywords and Phrases: (κ, μ)-contact metric manifold, conharmonically flat, conharmonically locally ϕ-symmetric, ϕ-conharmonically semisymmetric, h-conharmonically semisymmetric.

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1 Introduction

In 1995, Blair et al.[3] introduced the idea of a class of contact metric manifolds for which the characteristic vector field ξ belongs to the (κ,μ)-nullity distribution for some real numbers κ and μ and such type of manifolds are called (κ,μ)-contact metric manifold. The non-Sasakian (κ,μ)-contact metric manifolds have two classes, namely, the class consists of the unit tangent sphere bundles of spaces of constant curvature, equipped with the natural contact metric structure and the class contains all the three-dimensional unimodular Lie groups, except the commutative one admitting the structure of a left invariant (κ,μ)-contact metric manifold [3, 4, 9]. Boeckx [4] given a full classification of (κ,μ)-contact metric manifolds. (κ,μ)-contact metric manifolds have been studied by several authors in [5, 6, 13, 11] and others.

A rank-four tensor N that remains invariant under conharmonic transformation for a $(2n+1)$-dimensional Riemannian manifold M is given by

$$\begin{aligned} N(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{1.1}$$

which is also of the form

$$\begin{aligned} N(X,Y,Z,T) &= R(X,Y,Z,T) - \frac{1}{2n-1}[S(Y,Z)g(X,T) - S(X,Z)g(Y,T) \\ &\quad + g(Y,Z)g(QX,T) - g(X,Z)g(QY,T)], \end{aligned} \tag{1.2}$$

where R, S and Q represents the Riemannian curvature tensor, Ricci tensor and Ricci operator respectively.

A manifold whose conharmonic curvature vanishes at every point of the manifold is called conharmonically flat manifold. Such a curvature tensor have been extensively studied by Siddiqui and Ahsan [12], Ozgur [8], Avijit Sarkar et al. [10], Asghari and Taleshian [7] and many others.

Our present work is organised in the following way: After introduction, section 2 includes basics related to (κ,μ)-contact metric manifold which will be used later. Section 3 deals with conharmonically flat (κ,μ)-contact metric manifolds. We proved that conharmonically locally ϕ-symmetric (κ,μ)-contact metric manifold is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ in section 4. Section 5 and 6 are devoted to the study of h-Conharmonically semisymmetric and ϕ-Conharmonically semisymmetric non-Sasakian (κ,μ)-contact metric manifolds respectively. Finally, we have shown that if the conharmonic curvature tensor on a (κ,μ)-contact metric manifold is divergent free then the Ricci tensor S is a Codazzi tensor.

2 Preliminaries

A $(2n+1)$-dimensional differentiable manifold M^{2n+1} is called a contact manifold [1] if it carries a global 1-form η such that $\eta \wedge (d\eta)^{2n+1} \neq 0$ everywhere on M^{2n+1}. It is well known that a contact metric manifold admits an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$-tensor field, ξ is the characteristic vector field, and a Riemannian metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \quad g(X, \xi) = \eta(X), \tag{2.1}$$

$$\eta(\xi) = 1, \quad g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y). \tag{2.2}$$

$$d\eta(X, Y) = g(X, \phi Y), \quad g(X, \phi Y) = -g(Y, \phi X), \tag{2.3}$$

for all vector fields $X, Y \in TM^{2n+1}$ and then we call a structure as contact metric structure. A manifold M^{2n+1} with such a structure is said to be contact metric manifold and it is denoted by (ϕ, ξ, η, g).

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad d\eta(\xi, X) = 0. \tag{2.4}$$

We define a $(1,1)$-tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds_\xi$ is the Lie differentiation in the direction of ξ. Since the tensor field h is self-adjoint and anticommutes with ϕ, we have

$$h\xi = 0, \quad \phi h + h\phi = 0, \quad trh = tr\phi h = 0, \tag{2.5}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.6}$$

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.7}$$

where ∇ is the Levi-Civita connection and if $X \neq 0$ is an eigenvector of h corresponding to the eigenvalue λ, then ϕX is an eigenvector of h corresponding to the eigenvalue $-\lambda$. Blair et al. [3] studied the (κ, μ)-nullity condition and the (κ, μ)-nullity distribution $N(\kappa, \mu)$ of a contact metric manifold M is defined by [3]

$$\begin{aligned} & N(\kappa, \mu) : p \longrightarrow N_p(\kappa, \mu) \\ & = [Z \in T_pM : R(X, Y)Z = (\kappa I + \mu h)\{g(Y, Z)X - g(X, Z)Y\}], \end{aligned} \tag{2.8}$$

for all $X, Y \in TM^{2n+1}$. A contact metric manifold M^{2n+1} with $\xi \in N(\kappa, \mu)$ is called a (κ, μ)-contact metric manifold. In a (κ, μ)-contact metric manifold, we have

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{2.9}$$

for all $X, Y \in TM^{2n+1}$.

In a (κ, μ)-contact metric manifold, the following relations hold [3, 11]:

$$\begin{aligned}
h^2 &= (\kappa-1)\phi^2, &(2.10)\\
(\nabla_X\phi)Y &= g(X+hX,Y)\xi-\eta(Y)(X+hX), &(2.11)\\
R(\xi,X)Y &= \kappa[g(X,Y)\xi-\eta(Y)X]+\mu[g(hX,Y)\xi-\eta(Y)hX], &(2.12)\\
S(X,\xi) &= 2n\kappa\eta(X), &(2.13)\\
S(X,Y) &= [2(n-1)-n\mu]g(X,Y)+[2(n-1)+\mu]g(hX,Y) &(2.14)\\
&\quad +[2(1-n)+n(2\kappa+\mu)]\eta(X)\eta(Y), &\\
QX &= [2(n-1)-n\mu]X+[2(n-1)+\mu]hX &(2.15)\\
&\quad +[2(n-1)+n(2\kappa+\mu)], &\\
S(\phi X,\phi Y) &= S(X,Y)-2nk\eta(X)\eta(Y)-2(2n-2+\mu)g(hX,Y), &(2.16)\\
g(QX,Y) &= S(X,Y). &(2.17)
\end{aligned}$$

From (2.6), we have

$$\begin{aligned}
(\nabla_X\eta)Y &= g(X+hX,\phi Y), &(2.18)\\
(\nabla_X h)Y &= \{(1-\kappa)g(X,\phi Y)+g(X,h\phi Y)\}\xi+\eta(Y)\{h(\phi X+\phi hX)\} &(2.19)\\
&\quad -\mu\eta(X)\phi hY, &
\end{aligned}$$

where S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator and r is the scalar curvature of the manifold. It is well known that in a Sasakian manifold, the Ricci operator Q commutes with ϕ. But in a (κ,μ)-contact metric manifold Q does not commute with ϕ. In general, in a (κ,μ)-contact metric manifold Blair et al.[3] proved the following:

Proposition 1. *Let M^n be a (κ,μ)-contact metric manifold, then the relation*

$$Q\phi-\phi Q=2[2(n-1)+\mu]h\phi,$$

holds.

From the definition of η-Einstein manifold, it follows easily that $Q\phi=\phi Q$. Hence from Proposition 2.1 we obtain either $\mu=-2(n-1)$, or the manifold is Sasakian. Using $\mu=-2(n-1)$, from (2.14) we obtain that the manifold is an η-Einstein manifold. Therefore Yildiz and De [13] proved the following:

Proposition 2. *In a non-Sasakian (κ,μ)-contact metric manifold, the following conditions are equivalent:*
(i) η*-Einstein manifold,*
(ii) $Q\phi=\phi Q$.

For $n = 1$, from Proposition 2.1 and Proposition 2.2, Yildiz and De [13] obtained the following:

Corolary 1. *A 3-dimensional non-Sasakian (κ,μ)-contact η-Einstein manifold is an $N(k)$-contact metric manifold.*

Lemma 2.1. *[2]:Let M^{2n+1} (ϕ,ξ,η,g) be a contact metric manifold with $R(X,Y)\xi = 0$ for all vector fields X, Y tangent to M^{2n+1}. Then M^{2n+1} is locally isometric to the Riemannian product* $E^{n+1}(0)\times S^n(4)$.

3 Conharmonically flat (κ,μ)-contact metric manifolds

From (1.2), for a $(2n+1)$-dimensional conharmonically flat (κ,μ)-contact metric manifold, we have

$$\begin{aligned} R(X,Y,Z,T) &= \frac{1}{2n-1}[S(Y,Z)g(X,T) - S(X,Z)g(Y,T) + g(Y,Z)g(QX,T) \\ &\quad -g(X,Z)g(QY,T)]. \end{aligned} \tag{3.1}$$

Substituting $Z = \xi$ in (3.1) and using (2.1), (2.9) and (2.13), we obtain

$$\begin{aligned} &\kappa[\eta(Y)g(X,T) - \eta(X)g(Y,T)] + \mu[\eta(Y)g(hX,T) - \eta(X)g(hY,T)] \\ &= \frac{1}{2n-1}[2n\kappa\eta(Y)g(X,T) - 2n\kappa\eta(X)g(Y,T) + \eta(Y)g(QX,T) - \\ &\eta(X)g(QY,T)]. \end{aligned} \tag{3.2}$$

Again, by taking $Y = \xi$ and using (2.1), (2.2), (2.5) and (2.13), (3.2) becomes

$$S(X,T) = -\kappa g(X,T) + (2n+1)\kappa\eta(X)\eta(T) + (2n-1)\mu g(hX,T). \tag{3.3}$$

From the equation (3.3), it follows that if $\mu = 0$, then the manifold is an η-Einstein manifold. Conversely, if the manifold is η-Einstein, then we can write

$$S(X,T) = a_1 g(X,T) + b_1\eta(X)\eta(T). \tag{3.4}$$

On equating (3.3) and (3.4), we find

$$a_1 g(X,T) + b_1\eta(X)\eta(T) = -\kappa g(X,T) + (2n+1)\kappa\eta(X)\eta(T) + (2n-1)\mu g(hX,T) \tag{3.5}$$

Now, in (3.5) replacing T by ϕX and using (2.3), we get

$$(2n-1)\mu g(hX,\phi X) = 0, \tag{3.6}$$

for all X. Consequently, $\mu = 0$.
Hence, an n-dimensional conharmonically flat (κ,μ)-contact metric manifold is an η-Einstein manifold if and only if $\mu = 0$. But from (2.14), it follows that a (κ,μ)-contact metric manifold is

η-Einstein if and only if $\{2(n-1)+\mu\}=0$. If we consider a $(2n+1)$-dimensional $(n>1)$ conharmonically flat η-Einstein (κ,μ)-contact metric manifold, then $n=1$, which contradicts the fact that $n>1$.
Hence, the theorem can be stated as follows:

Theorem 3.1. *An $(2n+1)$-dimensional $(n>1)$ conharmonically flat (κ,μ)-contact metric manifold cannot be an η-Einstein manifold.*

4 Conharmonically locally ϕ-symmetric (κ,μ)-contact metric manifolds

Definition 4.1. *An $(2n+1)$-dimensional $(n>1)$ (κ,μ)-contact metric manifold M^{2n+1} is said to be conharmonically locally ϕ-symmetric if it satisfies*

$$\phi^2((\nabla_W N)(X,Y)Z)=0, \tag{4.1}$$

for all X,Y,Z,W orthogonal to ξ.

Taking covariant differentiation of (1.1), we have

$$\begin{aligned}(\nabla_W N)(X,Y)Z &= (\nabla_W R)(X,Y)Z-\frac{1}{2n-1}[(\nabla_W S)(Y,Z)X-(\nabla_W S)(X,Z)Y \\ &\quad +g(Y,Z)(\nabla_W Q)(X)-g(X,Z)(\nabla_W Q)(Y)],\end{aligned} \tag{4.2}$$

where ∇ denotes the Levi-Civita connection on the manifold.
Differentiating equations (2.8), (2.14) and (2.15) covariantly with respect to W, we obtain

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= W\kappa\{g(Y,Z)X-g(X,Z)Y\}+W\mu\{g(Y,Z)hX-g(X,Z)hY\} \\ &\quad +\mu[g(Y,Z)(\{(1-\kappa)g(W,\phi X)+g(W,h\phi X)\}\xi \\ &\quad +\eta(X)\{h(\phi W+\phi hW)\}-\mu\eta(W)\phi hX) \\ &\quad -g(X,Z)(\{(1-\kappa)g(W,\phi Y)+g(W,h\phi Y)\}\xi \\ &\quad +\eta(Y)\{h(\phi W+\phi hW)\}-\mu\eta(W)\phi hY)],\end{aligned} \tag{4.3}$$

$$\begin{aligned}(\nabla_W S)(Y,Z)X &= \{2(1-n)+n(2\kappa+\mu)\}[g(W,\phi Y)\eta(Z)X \\ &\quad +g(hW,\phi Y)\eta(Z)X+g(W,\phi Z)\eta(Y)X+g(hW,\phi Z)\eta(Y)X] \\ &\quad +(2(n-1)+\mu)[\{(1-\kappa)g(W,\phi Y)\eta(Z)X+g(W,h\phi Y)\eta(Z)X \\ &\quad +g(h(\phi W+\phi hW),Z)\eta(Y)X\}-\mu g(\phi hY,Z)\eta(W)X]\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}(\nabla_W Q)(X) &= \{2(n-1)+\mu\}[\{(1-\kappa)g(W,\phi X)+g(W,h\phi X)\}\xi \\ &+\eta(X)\{h(\phi W+\phi hX)\}-\mu\eta(W)\phi hX] \\ &+\{2(n-1)+n(2\kappa+\mu)\}g(W,\phi X)\xi \\ &+\{2(n-1)+n(2\kappa+\mu)\}g(hW,\phi X)\xi \\ &-\{2(n-1)+n(2\kappa+\mu)\}\eta(X)\phi W \\ &-\{2(n-1)+n(2\kappa+\mu)\}\eta(X)\phi hW.\end{aligned} \tag{4.5}$$

Now, considering equations (4.3), (4.4) and (4.5) in (4.2) and also taking X, Y, Z, W orthogonal to ξ, we get

$$\begin{aligned}(\nabla_W N)(X,Y)Z &= W\kappa[g(Y,Z)X-g(X,Z)Y]+W\kappa[g(Y,Z)hX-g(X,Z)hY] \\ &+\mu[(1-\kappa)g(Y,Z)g(W,\phi X)\xi+(1-\kappa)g(Y,Z)g(W,h\phi X)\xi \\ &-(1-\kappa)g(X,Z)g(W,\phi Y)\xi-(1-\kappa)g(X,Z)g(W,h\phi Y)\xi] \\ &-\frac{1}{2n-1}[\{2(n-1)+\mu\}\{(1-\kappa)[g(Y,Z)g(W,\phi X)\xi \\ &-g(X,Z)g(W,\phi Y)\xi]+g(Y,Z)g(W,h\phi X)\xi \\ &-g(X,Z)g(W,h\phi Y)\xi\} \\ &+\{2(n-1)+n(2\kappa+\mu)\}[g(Y,Z)g(W,\phi X)\xi \\ &+g(Y,Z)g(hW,\phi X)\xi-g(X,Z)g(W,\phi Y)\xi \\ &-g(X,Z)g(hW,\phi Y)\xi]].\end{aligned} \tag{4.6}$$

Applying ϕ^2 on both sides of (4.6), one can obtain

$$\begin{aligned}\phi^2((\nabla_W N)(X,Y)Z) &= \phi^2\{W\kappa[g(Y,Z)X-g(X,Z)Y]+W\kappa[g(Y,Z)hX \\ &-g(X,Z)hY]+\mu[(1-\kappa)g(Y,Z)g(W,\phi X)\xi \\ &+(1-\kappa)g(Y,Z)g(W,h\phi X)\xi-(1-\kappa)g(X,Z)g(W,\phi Y)\xi \\ &-(1-\kappa)g(X,Z)g(W,h\phi Y)\xi] \\ &-\frac{1}{2n-1}[\{2(n-1)+\mu\}\{(1-\kappa)[g(Y,Z)g(W,\phi X)\xi \\ &-g(X,Z)g(W,\phi Y)\xi]+g(Y,Z)g(W,h\phi X)\xi \\ &-g(X,Z)g(W,h\phi Y)\xi\} \\ &+\{2(n-1)+n(2\kappa+\mu)\}\{g(Y,Z)g(W,\phi X)\xi \\ &+g(Y,Z)g(hW,\phi X)\xi-g(X,Z)g(W,\phi Y)\xi \\ &-g(X,Z)g(hW,\phi Y)\xi\}]\}.\end{aligned} \tag{4.7}$$

From (4.1) and using (2.1), (4.7) becomes

$$(W\kappa)[g(X,Z)Y - g(Y,Z)X] + (W\kappa)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ +(W\mu)[g(X,Z)hY - g(Y,Z)hX] = 0. \tag{4.8}$$

Again, considering X, Y orthogonal to ξ, one can get

$$(W\kappa)[g(X,Z)Y - g(Y,Z)X] + (W\mu)[g(X,Z)hY - g(Y,Z)hX] = 0. \tag{4.9}$$

By taking inner product of (4.9) with V, we have

$$(W\kappa)[g(X,Z)g(Y,V) - g(Y,Z)g(X,V)] + (W\mu)[g(X,Z)g(hY,V) \\ -g(Y,Z)g(hX,V)] = 0. \tag{4.10}$$

On contraction, the above equation yields

$$-2n(W\kappa)g(Y,Z) + (W\mu)g(Z,hY) = 0. \tag{4.11}$$

Setting $Y = \xi$ in (4.11) and using (2.5), we get

$$2n(W\kappa)\eta(Z) = 0. \tag{4.12}$$

If we assume that $\kappa = 0$ in (4.11) then either $\mu = 0$ or $g(Z, hY) = 0$. Further, if $\kappa = 0 = \mu$ in (2.9), then we get $R(X,Y)\xi = 0$ for all X, Y and in the light of Lemma 2.1, the manifold under consideration is locally isometric to the Riemannian product $E^{n+1} \times S^n(4)$.
So from Lemma 2.1, we can state the theorem as follows:

Theorem 4.2. *Let M^{2n+1} (ϕ, ξ, η, g) be a conharmonically locally ϕ-symmetric (κ, μ)-contact metric manifold. Then the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$.*

5 h-Conharmonically semisymmetric non-Sasakian (κ, μ)-contact metric manifolds

Definition 5.1. *A Riemannian manifold (M^{2n+1}, g) is said to be h-conharmonically semisymmetric if it satisfies*

$$N(X,Y) \cdot h = 0. \tag{5.1}$$

The following lemma which was proved in [3] is helpful to state our theorem.

Lemma 5.1. *[3]: Let* $M^{2n+1}(\phi, \xi, \eta, g)$ *be a contact metric manifold with* ξ *belonging to the* (κ, μ)*-nullity distribution. Then for any vector fields* X, Y, Z*,*

$$\begin{aligned} R(X,Y)hZ - hR(X,Y)Z &= \{\kappa[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)] \\ &+\mu(\kappa-1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\}\xi \\ &+\kappa\{g(Y,\phi Z)\phi hX - g(X,\phi Z)\phi hY + g(Z,\phi hY)\phi X \\ &-g(Z,\phi hX)\phi Y + \eta(Z)[\eta(X)hY - \eta(Y)hX]\} \\ &-\mu\{\eta(Y)[(1-\kappa)\eta(Z)X + \mu\eta(X)hZ] \\ &-\eta(X)[(1-\kappa)\eta(Z)Y + \mu\eta(Y)hZ] + 2g(X,\phi Y)\phi hZ\}. \end{aligned} \tag{5.2}$$

Let M^{2n+1} be h-conharmonically semisymmetric non-Sasakian (κ, μ)-contact metric manifold. The condition $N(X,Y)\cdot h = 0$ can be expressed as follows,

$$(N(X,Y)\cdot h)Z = N(X,Y)hZ - hN(X,Y)Z = 0, \tag{5.3}$$

for any vector fields X, Y, Z.
With the help of (1.1) and (5.2), (5.3) can be written as

$$\begin{aligned} &[\kappa\{g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)\} + \mu(\kappa-1)\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}]\xi \\ &+\kappa\{g(Y,\phi Z)\phi hX - g(X,\phi Z)\phi hY + g(Z,\phi hY)\phi X - g(Z,\phi hX)\phi Y + \eta(Z)[\eta(X)hY \\ &-\eta(Y)hX]\} - \mu\{\eta(Y)[(1-\kappa)\eta(Z)X + \mu\eta(X)hZ] - \eta(X)[(1-\kappa)\eta(Z)Y + \mu\eta(Y)hZ] \\ &+2g(X,\phi Y)\phi hZ\} - \frac{1}{2n-1}[S(Y,hZ)X - S(X,hZ)Y + g(Y,hZ)QX - g(X,hZ)QY \\ &-S(Y,Z)hX + S(X,Z)hY - g(Y,Z)QhX + g(X,Z)QhY] = 0. \end{aligned} \tag{5.4}$$

By taking inner product of (5.4) with T, we get

$$\begin{aligned} &[\kappa\{g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)\} + \mu(\kappa-1)\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}]\eta(T) \\ &+\kappa\{g(Y,\phi Z)g(\phi hX,T) - g(X,\phi Z)g(\phi hY,W) + g(Z,\phi hY)g(\phi X,T) \\ &-g(Z,\phi hX)g(\phi Y,W) + \eta(Z)[\eta(X)g(hY,W) - \eta(Y)g(hX,T)]\} \\ &-\mu\{\eta(Y)[(1-\kappa)\eta(Z)g(X,T) + \mu\eta(X)g(hZ,T)] - \eta(X)[(1-\kappa)\eta(Z)g(Y,T) \\ &+\mu\eta(Y)g(hZ,T)] + 2g(X,\phi Y)g(\phi hZ,T)\} - \frac{1}{2n-1}[S(Y,hZ)g(X,T) \\ &-S(X,hZ)g(Y,T) + g(Y,hZ)S(X,T) - g(X,hZ)S(Y,T) - S(Y,Z)g(hX,T) \\ &+S(X,Z)g(hY,T) - g(Y,Z)S(hX,T) + g(X,Z)S(hY,T)] = 0. \end{aligned} \tag{5.5}$$

Setting $Y = T = \xi$ in (5.5) and using (2.2) and (2.5), we get

$$\begin{aligned} \frac{1}{2n-1}S(X,hZ) &= -\mu(1-\kappa)g(X,Z) + [2(1-\mu) + (1-\kappa)]\eta(X)\eta(Z) \\ &+[\kappa - \frac{2(2n+1)\kappa}{2(n-1)}g(X,hZ)]. \end{aligned} \tag{5.6}$$

Replacing X by hX in the above equation and using (2.10), we have

$$S(X, Z) = -\kappa g(X, Z) + \kappa\eta(X)\eta(Z) - 2\mu(n-1)g(hX, Z). \tag{5.7}$$

If we consider $\mu = 0$ in (5.7) then it is an η-Einstein manifold.
Using (2.14) in (5.7) and simplifying, we finally obtain

$$S(X, Z) = n_1 g(X, Z) + n_2\eta(X)\eta(Z), \tag{5.8}$$

where $n_1 = \frac{-\kappa[2(n-1)+\mu]+\mu(2n-1)[2(n-1)+n\mu]}{[2(n-1)+\mu]+\mu(2n-1)}$
and
$n_2 = \frac{\kappa[2(n-1)+\mu]+\mu(2n-1)[2(1-n)+n(2\kappa+\mu)]}{[2(n-1)+\mu]+\mu(2n-1)}$.
Thus from (5.8), we can conclude the following theorem:

Theorem 5.2. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a non-Sasakian (κ, μ)-contact metric manifold. If M is h-conharmonically semisymmetric, then the manifold is an η-Einstein manifold with constant coefficients.*

From Proposition 2.2 and Theorem 5.5 we can state the following:

Corolary 2. *If M^{2n+1} is a h-conharmonically semisymmetric (κ, μ)-contact metric manifold then the Ricci operator Q commutes with ϕ i.e., $Q\phi = \phi Q$.*

6 ϕ-Conharmonically semisymmetric non-Sasakian (κ, μ)-contact metric manifolds

Definition 6.1. *A Riemannian manifold (M^{2n+1}, g) is said to be ϕ-conharmonically semisymmetric if*

$$N(X, Y) \cdot \phi = 0. \tag{6.1}$$

Now we need the following lemma:

Lemma 6.1. *[3]: Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a contact metric manifold with ξ belonging to the (κ, μ)-nullity distribution. Then for any vector fields X, Y, Z,*

$$\begin{aligned} R(X,Y)\phi Z - \phi R(X,Y)Z \quad = \quad & \{(1-\kappa)[g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y)] \\ & +(1-\mu)[g(\phi hY, Z)\eta(X) - g(\phi hX, Z)\eta(Y)]\}\xi \\ & -g(Y + hY, Z)(\phi X + \phi hX) + g(X + hX, Z)(\phi Y \\ & +\phi hY) - g(\phi Y + \phi hY, Z)(X + hX) + g(\phi X \\ & +\phi hX, Z)(Y + hY) - \eta(Z)\{(1-\kappa)[\eta(X)\phi Y \\ & -\eta(Y)\phi X] + (1-\mu)[\eta(X)\phi hY - \eta(Y)\phi hX]\}. \end{aligned} \tag{6.2}$$

Let M^{2n+1} be a $(2n+1)$-dimensional ϕ-conharmonically semisymmetric non-Sasakian (κ,μ)-contact metric manifold. The condition $N(X,Y)\cdot\phi=0$ turns into,

$$(N(X,Y)\cdot\phi)Z = N(X,Y)\phi Z - \phi N(X,Y)Z = 0, \tag{6.3}$$

for any vector fields X, Y, Z.
In view of (1.1) and (6.2), (6.3) becomes

$$\begin{aligned}
&\{(1-\kappa)[g(\phi Y,Z)\eta(X)-g(\phi X,Z)\eta(Y)]+(1-\mu)[g(\phi hY,Z)\eta(X)-g(\phi hX,Z)\eta(Y)]\}\xi \\
&-g(Y+hY,Z)(\phi X+\phi hX)+g(X+hX,Z)(\phi Y+\phi hY)-g(\phi Y+\phi hY,Z)(X+hX)\\
&+g(\phi X+\phi hX,Z)(Y+hY)-\eta(Z)\{(1-\kappa)[\eta(X)\phi Y-\eta(Y)\phi X]+(1-\mu)[\eta(X)\phi hY\\
&-\eta(Y)\phi hX)]\}-\frac{1}{2n-1}[S(Y,\phi Z)X-S(X,\phi Z)Y+g(Y,\phi Z)QX-g(X,\phi Z)QY\\
&-S(Y,Z)\phi X+S(X,Z)\phi Y-g(Y,Z)Q\phi X+g(X,Z)Q\phi Y]=0.
\end{aligned}\tag{6.4}$$

Taking inner product of (6.4) with T, we get

$$\begin{aligned}
&\{(1-\kappa)[g(\phi Y,Z)\eta(X)-g(\phi X,Z)\eta(Y)]+(1-\mu)[g(\phi hY,Z)\eta(X)\\
&-g(\phi hX,Z)\eta(Y)]\}\eta(T)-g(Y,Z)g(\phi X,T)-g(hY,Z)g(\phi X,T)-g(Y,Z)g(\phi hX,T)\\
&-g(hY,Z)g(\phi hX,T)+g(X,Z)g(\phi Y,T)+g(hX,Z)g(\phi Y,T)+g(X,Z)g(\phi hY,T)\\
&+g(hX,Z)g(\phi hY,T)-g(\phi Y,Z)g(X,T)-g(\phi Y,Z)g(hX,T)-g(\phi hY,Z)g(X,T)\\
&-g(\phi hY,Z)g(hX,T)+g(\phi X,Z)g(Y,T)+g(\phi hX,Z)g(Y,T)+g(\phi X,Z)g(hY,T)\\
&+g(\phi hX,Z)g(hY,T)-\eta(Z)\{(1-\kappa)[\eta(X)g(\phi Y,T)-\eta(Y)g(\phi X,T)]\\
&+(1-\mu)[\eta(X)g(\phi hY,T)-\eta(Y)g(\phi hX,T)]\}-\frac{1}{2n-1}[S(Y,\phi Z)g(X,T)\\
&-S(X,\phi Z)g(Y,T)+g(Y,\phi Z)g(QX,T)-g(X,\phi Z)g(QY,T)-S(Y,Z)g(\phi X,T)\\
&+S(X,Z)g(\phi Y,T)-g(Y,Z)g(Q\phi X,T)+g(X,Z)g(Q\phi Y,T)]=0.
\end{aligned}\tag{6.5}$$

Treating $Y=T=\xi$ in (6.5) and using (2.1), (2.2), (2.4), (2.5) and (2.13), we have

$$\frac{1}{2n-1}S(X,\phi Z)=\{(\kappa-2)+\frac{2(2n+1)\kappa}{2n-1}\}g(X,\phi Z)-\mu g(\phi X,hZ). \tag{6.6}$$

Substituting X by ϕX in (6.6) and using (2.1), (2.2) and (2.16), one can get

$$\begin{aligned}
S(X,Z) \;=\; &[(\kappa-2)(2n-1)+2n\kappa]g(X,Z)-[(\kappa-2)(2n-1)]\eta(X)\eta(Z)\\
&+[\mu(\kappa-1)(2n-1)+2\{2(n-1)+\mu\}]g(hX,Z).
\end{aligned}\tag{6.7}$$

Making use of (2.14), (6.7) yields

$$S(X,Z)=n_3 g(X,Z)+n_4\eta(X)\eta(Z), \tag{6.8}$$

where $n_3 = \frac{\{(\kappa-2)(2n-1)+2n\kappa\}\{2(n-1)+\mu\}-\{\mu(\kappa-1)(2n-1)+2[2(n-1)+\mu]\}\{2(n-1)-n\mu\}}{[2(n-1)+\mu]-[\mu(\kappa-1)(2n-1)+2\{2(n-1)+\mu\}]}$ and

$n_4 = \frac{[(2-\kappa)(2n-1)][2(n-1)+\mu]-\{\mu(\kappa-1)(2n-1)+2[2(n-1)+\mu]\}[2(1-n)+2n(2\kappa+\mu)]}{[2(n-1)+\mu]-[\mu(\kappa-1)(2n-1)+2\{2(n-1)+\mu\}]}$.
Hence from (6.8), the theorem can be stated as follows:

Theorem 6.2. *If a $(2n+1)$-dimensional non-Sasakian (κ, μ)-contact metric manifold M^{2n+1} is ϕ-conharmonically semisymmetric then the manifold is an η-Einstein manifold with constant coefficients.*

Similarly, from Proposition 2.2 and Theorem 6.6, we get the following statement:

Corolary 3. *If M^{2n+1} is a ϕ-conharmonically semisymmetric (κ, μ)-contact metric manifold then the Ricci operator Q commutes with ϕ i.e., $Q\phi = \phi Q$.*

7 (κ, μ)-contact metric manifold with divergent free conharmonic curvature tensor

In this section, we study divergent free conharmonic curvature tensor on (κ, μ)-contact metric manifold.
Let $M^{2n+1}(\phi, \xi, \eta, g)$ $(n > 1)$ be a (κ, μ)-contact metric manifold satisfying the following condition

$$(DivN)(X, Y)Z = 0. \tag{7.1}$$

In view of (7.1), (1.1) leads to

$$\begin{aligned}(DivR)(X, Y)Z \;=\;& \frac{1}{2n-1}[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) + g(Y, Z)dr(X) \\ &-g(X, Z)dr(Y)].\end{aligned} \tag{7.2}$$

The above equation simplifies to,

$$\begin{aligned}&\frac{2(n-1)}{(2n-1)}[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)] - \frac{1}{(2n-1)}[g(Y, Z)dr(X) \\ &-g(X, Z)dr(Y)] = 0.\end{aligned} \tag{7.3}$$

On contracting and taking summation over i, $1 \le i \le n$ in (7.3), we get

$$2(3n-1)dr(Y) = 0, \tag{7.4}$$

which implies

$$dr(Y) = 0, \tag{7.5}$$

since $2(3n-1) \neq 0$.
Further, considering (7.5) in (7.3), we obtain

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = 0, \tag{7.6}$$

which gives

$$(\nabla_X Q)Y = (\nabla_Y Q)X. \tag{7.7}$$

Thus, we can state:

Theorem 7.1. *Let* $M^{2n+1}(\phi, \xi, \eta, g)$ $(n > 1)$ *be a* (κ, μ)*-contact metric manifold. If the manifold has divergent free conharmonic curvature tensor then the Ricci tensor* S *is a Codazzi tensor.*

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Nonlinear elliptic $p(u)-$ Laplacian problem with Fourier boundary condition

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ABSTRACT

We study a nonlinear elliptic $p(u)-$ Laplacian problem with Fourier boundary conditions and L^1- data. The existence and uniqueness results of entropy solutions are established.

RESUMEN

Estudiamos un problema $p(u)-$Laplaciano elíptico nolineal con condiciones de borde Fourier y datos L^1. Se establecen resultados de existencia y unicidad de soluciones de entropía.

Keywords and Phrases: variable exponent, $p(u)-$Laplacian, Young measure, Fourier boundary condition, entropy solution.

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1 Introduction

In this paper, we consider the following nonlinear Fourier boundary value problem

$$\begin{cases} b(u) - \mathrm{div} a(x,u,\nabla u) = f & \text{in } \Omega \\ a(x,u,\nabla u).\eta + \lambda u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ is a bounded open domain with Lipschitz boundary $\partial\Omega$, η is the outer unit normal vector on $\partial\Omega$ and $\lambda > 0$.

The operator $\mathrm{div} a(x,u,\nabla u)$ is called $p(u)$-Laplacian. It is more complicated than $p(x)$-Laplacian in the term of nonlinearity. A prototype of this operator is $\mathrm{div}\big(|\nabla u|^{p(u)-2}.\nabla u\big)$. The variable exponent p depend both on the space variable x and on the unknown solution u. The problem (1.1) is a generalization of the following nonlinear problem

$$\begin{cases} b(u) - \mathrm{div} a(x,\nabla u) = f & \text{in } \Omega \\ a(x,\nabla u).\eta + \lambda u = g & \text{on } \partial\Omega, \end{cases}$$

studied in [15] by Nyanquini and Ouaro. The authors used an auxiliary result due to Le (see [16], Theorem 3.1) to prove the existence of the weak solution when $f \in L^\infty(\Omega)$, $g \in L^\infty(\partial\Omega)$ and by approximation methods they obtained the entropy solution when $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$.

In the present paper, as the function $(x,z,\eta) \mapsto a(x,z,\eta)$ is more general than $(x,\eta) \mapsto a(x,\eta)$, it is difficult to use the sub-supersolution method, as in [16], to get the existence of the weak solution. Therefore, we use the technic of pseudo-monotone operators in Banach spaces, some a priori estimates and the convergence in term of Young measure to obtain the existence of entropy solutions of problem (1.1). Indeed, we define an approximation problem, and we prove that this problem has a solution u_n which converges to u, an entropy solution of problem (1.1).

In this paper, we consider the following basic assumptions on the data for the study of the problem (1.1).

(A_1) f and g are some functions such as $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$ and $g \not\equiv 0$.

(A_2) b is nondecreasing surjective and continuous function defined on $\mathbb{R}$ such that $b(0) = 0$. Problem (1.1) is adapted into a generalized Leray-Lions framework under the assumption that $a : \Omega \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}^N$ is a Carathéodory function with:

(A_3) $a(x,z,0) = 0$ for all $z \in \mathbb{R}$, and a.e. $x \in \Omega$;

(A_4) $\big(a(x,z,\xi) - a(x,z,\eta)\big).(\xi - \eta) > 0$ for all $\xi, \eta \in \mathbb{R}^N, \xi \neq \eta$, as well as the growth and the coercivity assumptions with variable exponent

(A_5) $\big|a(x,z,\xi)\big|^{p'(x,z)} \leq C_1\big(|\xi|^{p(x,z)} + \mathcal{M}(x)\big)$
and

(A_6) $a(x,z,\xi).\xi \geq \dfrac{1}{C_2}|\xi|^{p(x,z)}$.

Here, C_1 and C_2 are positive constants and $\mathcal{M}$ is a positive function such that $\mathcal{M} \in L^1(\Omega)$.

$p : \Omega\times\mathbb{R} \to [p_-, p_+]$ is a Carathéodory function, $1 < p_- \leq p_+ < +\infty$ and $p'(x,z) = \dfrac{p(x,z)}{p(x,z)-1}$ is the conjugate exponent of $p(x,z)$, with

$$p_- := ess \inf_{(x,z)\in\Omega\times\mathbb{R}} p(x,z) \ \text{ and } \ p_+ := ess \sup_{(x,z)\in\Omega\times\mathbb{R}} p(x,z).$$

The study of $p(u)$-Laplacian problem was recently developped by Andreianov et *al.* (see [2]). These authors established the partial existence and uniqueness result to the weak solution in the cases of homogeneous Dirichlet boundary condition.
The interest of the study of this kind of problem is due to the fact that they can model various phenomena which arise in the study of elastic mechanic (see [6]), electrorheological fluids (see [20]) or image restoration (see [9]).
In this paper, we study the existence of the weak solution for approximation problem and we also establish the existence and uniqueness results of the entropy solution when the data are in L^1.
In this work, we use the Sobolev embedding results and the convergence in term of Young measure (see [10, 12]).
Here, we consider a Fourier boundary condition which bring some difficulties to treat the term at the boundary.
We were inspired by the work of Ouaro and Tchousso (see [15]), where the authors defined for the first time a new space by taking into account the boundary.
For the next part of the paper (section 2), we introduce some preliminary results. In section 3, we study the existence and uniqueness of entropy solution when the data are in L^1.

2 Preliminary

• We will use the so-called truncation function

$$T_k(s) := \begin{cases} s & \text{if } |s| \leq k \\ k sign_0(s) & \text{if } |s| > k \end{cases}, \quad \text{where } sign_0(s) := \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0. \end{cases}$$

The truncation function possesses the following properties.

$$T_k(-s) = -T_k(s), |T_k(s)| = \min\{|s|, k\},$$
$$\lim_{k\to+\infty} T_k(s) = s \ \text{ and } \ \lim_{k\to 0} \frac{1}{k}T_k(s) = sign_0(s).$$

We also need to truncate vector valued-function with the help of the mapping

$$h_m : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad h_m(\lambda) = \begin{cases} \lambda, & \text{if } \ |\lambda| \leq m \\ m\dfrac{\lambda}{|\lambda|} & \text{if } \ |\lambda| > m, \end{cases} \quad \text{where } \ m > 0.$$

For a Lebesgue measurable set $A \subset \Omega$, χ_A denotes its characteristic function and $meas(A)$ denotes its Lebesgue measure. Let $u : \Omega \to \mathbb{R}$ be a function and $k \in \mathbb{R}$, we write $\{|u| \leq k\}$ or $[|u| \leq k]$ for the set $\{x \in \Omega : \ \ |u(x)| \leq k\}$, (respectively, $\geq, =, <, >$).

The function $a(.,.,.)$ appearing in (1.1) satisfies a generalized Leray-Lions assumptions given in Introduction. View that, $a(.,.,.)$ satisfies (A_5) and (A_6), we must work in Lebesgue and Sobolev spaces with variable exponent, that depend on x and on $u(x)$. For the study of problem (1.1), we need the Sobolev spaces $W^{1,\pi(\cdot)}(\Omega)$, where $\pi(.) = p(., u(.))$.

Definition 1. *Let $\pi : \Omega \longrightarrow [1, +\infty)$ be a measurable function for $\pi(.) = p(., u(.))$.*
•$L^{\pi(\cdot)}(\Omega)$ is the space of all measurable function $f : \Omega \longrightarrow \mathbb{R}$ such that the modular

$$\rho_{\pi(.)}(f) := \int_\Omega |f|^{\pi(x)} dx < +\infty.$$

If p_+ is finite, this space is equipped with the Luxembourg norm

$$||f||_{L^{\pi(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0; \quad \rho_{\pi(.)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

In the sequel, we will use the same notation $L^{\pi(\cdot)}(\Omega)$ for the space $(L^{\pi(\cdot)}(\Omega))^N$ of vector-valued functions.
•$W^{1,\pi(\cdot)}(\Omega)$ is the space of all functions $f \in L^{\pi(\cdot)}(\Omega)$ such that the gradient of f (taken in the sense of distributions) belongs to $L^{\pi(\cdot)}(\Omega)$. The space $W^{1,\pi(\cdot)}(\Omega)$ is equipped with the norm

$$||u||_{W^{1,\pi(\cdot)}(\Omega)} := ||u||_{L^{\pi(\cdot)}(\Omega)} + ||\nabla u||_{L^{\pi(\cdot)}(\Omega)}.$$

When $1 < p_- \leq \pi(.) \leq p_+ < +\infty$, all the above spaces are separable and reflexive Banach spaces.

We denote $\pi_n(x) := p(x, u_n(x))$.

Proposition 1. *(See [1], Proposition 2.3)*
For all measurable function $\pi : \Omega \to [p_-, p_+]$, the following properties hold.

i) $L^{\pi(\cdot)}(\Omega)$ and $W^{1,\pi(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.

ii) $L^{\pi'(\cdot)}(\Omega)$ can be identified with the dual space of $L^{\pi(\cdot)}(\Omega)$, and the following Hölder type inequality holds:

$$\forall f \in L^{\pi(\cdot)}(\Omega), g \in L^{\pi'(\cdot)}(\Omega), \qquad \left| \int_\Omega fg dx \right| \leq 2||f||_{L^{\pi(\cdot)}(\Omega)}||g||_{L^{\pi'(\cdot)}(\Omega)}.$$

iii) One has $\rho_{\pi(.)}(f) = 1$ *if and only if* $||f||_{L^{\pi(.)}(\Omega)} = 1$*; further,*

$$\text{if } \rho_{\pi(.)}(f) \leq 1, \text{ then } ||f||^{p_+}_{L^{\pi(.)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq ||f||^{p_-}_{L^{\pi(.)}(\Omega)};$$

$$\text{if } \rho_{\pi(.)}(f) \geq 1, \text{ then } ||f||^{p_-}_{L^{\pi(.)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq ||f||^{p_+}_{L^{\pi(.)}(\Omega)}.$$

In particular, if $(f_n)_{n\in\mathbb{N}}$ *is a sequence in* $L^{\pi(.)}(\Omega)$*, then* $||f_n||_{L^{\pi(.)}(\Omega)}$ *tends to zero (resp., to infinity) if and only if* $\rho_{\pi(.)}(f_n)$ *tends to zero (resp., to infinity), as* $n \to +\infty$.

For a measurable function $f \in W^{1,\pi(.)}(\Omega)$ we introduce the following notation:

$$\rho_{1,\pi(.)}(f) = \int_\Omega |f|^{\pi(.)}dx + \int_\Omega |\nabla f|^{\pi(.)}dx.$$

Replacing $p(x)$ by $\pi(x)$ in [8]-Proposition 2.2, we obtain the following result that is fundamental in this paper.

Proposition 2. *(See [23, 24]) If* $f \in W^{1,\pi(.)}(\Omega)$*, the following properties hold:*

i) $||f||_{W^{1,\pi(.)}(\Omega)} > 1 \Rightarrow ||f||^{p_-}_{W^{1,\pi(.)}(\Omega)} < \rho_{1,\pi(.)}(f) < ||f||^{p_+}_{W^{1,\pi(.)}(\Omega)}$;

ii) $||f||_{W^{1,\pi(.)}(\Omega)} < 1 \Rightarrow ||f||^{p_+}_{W^{1,\pi(.)}(\Omega)} < \rho_{1,\pi(.)}(f) < ||f||^{p_-}_{W^{1,\pi(.)}(\Omega)}$;

iii) $||f||_{W^{1,\pi(.)}(\Omega)} < 1 \ (\textit{respectively} = 1; > 1) \Leftrightarrow \rho_{1,\pi(.)}(f) < 1 \ (\textit{respectively} \ = 1; > 1)$.

The following lemma prove that the space $W^{1,\pi(.)}(\Omega)$ is stable by truncation.

Lemma 2.1. *If* $u \in W^{1,\pi(.)}(\Omega)$ *then* $T_k(u) \in W^{1,\pi(.)}(\Omega)$.

Now, we give the embedding results.

Proposition 3. *(See [1], Proposition 2.4) Assume that* $\pi : \Omega \to [p_-, p_+]$ *has a representative which can be extended into a continuous function up to the boundary* $\partial\Omega$ *and satisfying the log-Hölder continuity assumption:*

$$\exists L > 0, \ \ \forall x, y \in \overline{\Omega}, x \neq y, \quad -\big(\log|x-y|\big)|\pi(x) - \pi(y)| \leq L. \tag{2.1}$$

i) Then, $\mathcal{D}(\Omega)$ *is dense in* $W^{1,\pi(.)}(\Omega)$.

ii) $W^{1,\pi(.)}(\Omega)$ *is embedded into* $L^{\pi^*(.)}(\Omega)$*, where* $\pi^*(.)$ *is the Sobolev embedding exponent defined as in (2.2) below. If* q *is a measurable variable exponent such that ess* $\inf_{x\in\Omega}(\pi^*(.) - q(.)) > 0$*, then the embedding of* $W^{1,\pi(.)}(\Omega)$ *into* $L^{q(.)}(\Omega)$ *is compact.*

For a given $\pi(.)$, a function taking values in $[p_-, p_+]$, $\pi^*(.)$ denotes the optimal Sobolev embedding defined for any $x \in \Omega$ by

$$\pi^*(x) = \begin{cases} \dfrac{N\pi(x)}{N-\pi(x)} & \text{if} \quad \pi(x) < N \\ \text{any real value} & \text{if} \quad \pi(x) = N \\ +\infty & \text{if} \quad \pi(x) > N. \end{cases} \tag{2.2}$$

Put

$$\pi^\partial(x) := \big(\pi(x)\big)^\partial := \begin{cases} \dfrac{(N-1)\pi(x)}{N-\pi(x)} & \text{if} \quad \pi(x) < N \\ +\infty & \text{if} \quad \pi(x) \geq N. \end{cases} \tag{2.3}$$

Proposition 4. *(See [18], Proposition 2.3)*
Let $\pi(.) \in C(\overline{\Omega})$ and $p_- > 1$. If $q(x) \in C(\partial\Omega)$ satisfies the condition:

$$1 \leq q(x) < \pi^\partial(x), \quad \forall x \in \partial\Omega,$$

then, there is a compact embedding

$$W^{1,\pi(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega).$$

In particular there is compact embedding

$$W^{1,\pi(.)}(\Omega) \hookrightarrow L^{\pi(.)}(\partial\Omega).$$

T**Young measures and nonlinear weak-* convergence.**
Throughout the paper, we denote by δ_c the Dirac measure on $\mathbb{R}^d$ ($d \in \mathbb{N}$), concentrated at the point $c \in \mathbb{R}^d$.
In the following theorem, we gather the results of Ball [7], Pedregal [19] and Hungerbühler [13] which will be needed for our purposes (we limit the statement to the case of a bounded domain Ω). Let us underline that the results of (ii),(iii), expressed in terms of the convergence in measure, are very convenient for the applications that we have in mind.

Theorem 2.1. *(i) Let $\Omega \subset \mathbb{R}^N, N \in \mathbb{N}$, and a sequence $(v_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^d$ -valued functions, $d \in \mathbb{N}$, such that $(v_n)_{n\in\mathbb{N}}$ is equi-integrable on Ω. Then, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a parametrized family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^d$ ($d \in \mathbb{N}$) , weakly measurable in x with respect to the Lebesgue measure in Ω, such that for all Carathéodory function $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^t$, $t \in \mathbb{N}$, we have*

$$\lim_{k\to+\infty} \int_\Omega F(x, v_{n_k})dx = \int_\Omega \int_{\mathbb{R}^d} F(x,\lambda)d\nu_x(\lambda)dx, \tag{2.4}$$

whenever the sequence $(F(.,v_n(.)))_{n\in\mathbb{N}}$ *is equi-integrable on* Ω. *In particular,*

$$v(x) := \int_{\mathbb{R}^d} \lambda d\nu_x(\lambda) \tag{2.5}$$

is the weak limit of the sequence $(v_{n_k})_{k\in\mathbb{N}}$ *in* $L^1(\Omega)$.
The family $(\nu_x)_{x\in\Omega}$ *is called the Young measure generated by the subsequence* $(v_{n_k})_{k\in\mathbb{N}}$.

(ii) If Ω *is of finite measure, and* $(\nu_x)_{x\in\Omega}$ *is the Young measure generated by a sequence* $(v_n)_{n\in\mathbb{N}}$, *then* $\nu_x = \delta_{v(x)}$ *for a.e.* $x\in\Omega \Leftrightarrow v_n$ *converges in measure in* Ω *to* v *as* $n\to+\infty$.

(iii) If Ω *is of finite measure,* $(u_n)_{n\in\mathbb{N}}$ *generates a Dirac Young measure* $(\delta_{u(x)})_{x\in\Omega}$ *on* $\mathbb{R}^{d_1}$, *and* $(v_n)_{n\in\mathbb{N}}$ *generates a Young measure* $(\nu_x)_{x\in\Omega}$ *on* $\mathbb{R}^{d_2}$, *then the sequence* $(u_n,v_n)_{n\in\mathbb{N}}$ *generates the Young measure* $(\delta_{u(x)}\otimes\nu_x)_{x\in\Omega}$ *on* $\mathbb{R}^{d_1+d_2}$. *Whenever a sequence* $(v_n)_{n\in\mathbb{N}}$ *generates a Young measure* $(\nu_x)_{x\in\Omega}$, *following the terminology of [11] we will say that* $(v_n)_{n\in\mathbb{N}}$ *nonlinear weak-* converges, and* $(\nu_x)_{x\in\Omega}$ *is the nonlinear weak-* limit of the sequence* $(v_n)_{n\in\mathbb{N}}$. *In the case where* $(v_n)_{n\in\mathbb{N}}$ *possesses a nonlinear weak-* convergent subsequence, we will say that it is nonlinear weak-* compact. ([1], Theorem 2.10(i)) It means that any equi-integrable sequence of measurable functions is nonlinear weak-* compact on* Ω.

Lemma 2.2. *(See [1], Theorem 3.11 and [2] Step 2 of proof of Theorem 2.6). Assume that* $(u_n)_{n\in\mathbb{N}}$ *converges a.e. on* Ω *to some function* u, *then*

$$\begin{aligned}&|p(x,u_n(x))-p(x,u(x))| \ \text{ converges in measure to } 0 \text{ on } \Omega,\\ &\text{and for all bounded subset } K \text{ of } \mathbb{R}^N,\\ &\sup_{\xi\in K}|a(x,u_n(x),\xi)-a(x,u(x),\xi)| \quad \text{converges in measure to } 0 \text{ on } \Omega.\end{aligned} \tag{2.6}$$

For the sequel, we assume that $p(.,.)$ is log Hölder continuous uniformly on $\overline{\Omega}\times[-M,M]$ and $p_- > N$. We recall some notations.
For any $u\in W^{1,\pi(.)}(\Omega)$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense.
We will identify at boundary u and $\tau(u)$.
Set

$$\mathcal{T}^{1,\pi(.)}(\Omega) = \big\{u:\Omega\to\mathbb{R}, \ \text{ measurable such that } \ T_k(u)\in W^{1,\pi(.)}(\Omega), \ \text{ for any } \ k>0\big\}.$$

3 Entropy solution

In this part, we study the existence and uniqueness of the entropy solution to the problem (1.1). We give some notations.

We define $\mathcal{T}_{tr}^{1,\pi(.)}(\Omega)$ as the set of the functions $u \in \mathcal{T}^{1,\pi(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p_+}(\Omega)$ satisfying the following conditions:

(C_1) $u_n \to u$ a.e. in Ω.

(C_2) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^1(\Omega)$.

(C_3) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense as introduced in [4, 5]. In the sequel the trace of $u \in \mathcal{T}_{tr}^{1,\pi(.)}(\Omega)$ on $\partial\Omega$ will be denoted $tr(u)$. If $u \in W^{1,\pi(.)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover, for $u \in \mathcal{T}_{tr}^{1,\pi(.)}(\Omega)$ and for all $k > 0$, $tr(T_k(u)) = T_k(tr(u))$ and if $\varphi \in W^{1,\pi(.)}(\Omega)$ then $u - \varphi \in \mathcal{T}_{tr}^{1,\pi(.)}(\Omega)$ and $tr(u - \varphi) = tr(u) - tr(\varphi)$.
As in [1]-Proposition 3.5, we give the following result.

Proposition 5. *Let $u \in \mathcal{T}^{1,\pi(.)}(\Omega)$. There exists a unique measurable function $w : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = w\chi_{\{|u|<k\}}$ for $k > 0$. The function w is denoted by ∇u. Moreover, if $u \in W^{1,\pi(.)}(\Omega)$ then $w \in L^{\pi(.)}(\Omega)$ and $w = \nabla u$ in the usual sense.*

Remark 3.1. *The space $\mathcal{T}_{tr}^{1,\pi(.)}(\Omega)$ in our context will be a subset of $\mathcal{T}^{1,\pi(.)}(\Omega)$ consisting to the function can be approximated by function of $W^{1,p_+}(\Omega)$. Since the weak solution of approximated problem (3.2) belongs to $W^{1,p_+}(\Omega)$.*

Now, we introduce the notion of entropy solution due to Ouaro and al. [14, Definition 3.1].

Definition 2. *A measurable function $u : \Omega \to \mathbb{R}$ for $\pi(.) = p(., u(.))$ is called entropy solution of the problem (1.1) if*

$$u \in \mathcal{T}_{tr}^{1,\pi(.)}(\Omega), \quad b(u) \in L^1(\Omega), \quad u \in L^1(\partial\Omega)$$

and for all $k > 0$,

$$\begin{aligned}\int_\Omega b(u)T_k(u-\varphi)dx + \int_\Omega a(x,u,\nabla u).\nabla T_k(u-\varphi)dx \quad &+ \quad \lambda\int_{\partial\Omega} uT_k(u-\varphi)d\sigma \\ &\leq \quad \int_\Omega fT_k(u-\varphi)dx + \int_{\partial\Omega} gT_k(u-\varphi)d\sigma,\end{aligned} \tag{3.1}$$

where $\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$.

The following theorem gives existence result of entropy solution.

Theorem 3.2. *Assume that $(A_3) - (A_6)$ hold and $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$. Then, there exists at least one entropy solution to the problem (1.1).*

The proof of Theorem 3.2 is done in two parts.

Part 1: The approximate problem.

Let $f_n = T_n(f)$ and $g_n = T_n(g)$. Then, $f_n \in L^\infty(\Omega)$ and $g_n \in L^\infty(\partial\Omega)$. Moreover, $(f_n)_{n\in\mathbb{N}}$ strongly converges to f in $L^1(\Omega)$ and $(g_n)_{n\in\mathbb{N}}$ strongly converges to g in $L^1(\partial\Omega)$ such that $||f_n||_{L^1(\Omega)} \leq ||f||_{L^1(\Omega)}$ and $||g_n||_{L^1(\partial\Omega)} \leq ||g||_{L^1(\partial\Omega)}$.

We consider the following problem

$$\begin{cases} T_n(b(u_n)) - \mathrm{div} a(x,u_n,\nabla u_n) - \varepsilon\triangle_{p_+}u_n + \varepsilon|u_n|^{p_+-2}u_n = f_n & \text{in} \quad \Omega \\ \big(a(x,u_n,\nabla u_n) + \varepsilon|\nabla u_n|^{p_+-2}\nabla u_n\big).\eta + \lambda T_n(u_n) = g_n & \text{on} \quad \partial\Omega, \end{cases} \tag{3.2}$$

where

$$-\triangle_{p_+}u_n := -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_+-2}\frac{\partial u_n}{\partial x_i}\Big).$$

In this part, we show that the problem (3.2) admits at least one weak solution u_n, for all $\varepsilon > 0$. We define the following reflexive space

$$E = W^{1,p_+}(\Omega) \times L^{p_+}(\partial\Omega).$$

Let

$$X_0 = \{(u,v) \in E : \quad v = \tau(u)\}.$$

In the sequel, we will identify an element $(u,v) \in X_0$ with its representative $u \in W^{1,p_+}(\Omega)$ (since $W^{1,p_+}(\Omega) \hookrightarrow\hookrightarrow L^{p_+}(\partial\Omega)$).

Theorem 3.3. *There exists at least one weak solution u_n for the problem* (3.2) *in the sense that $u_n \in X_0$ and for all $v \in X_0$,*

$$\begin{aligned} \int_\Omega T_n(b(u_n))v dx + \int_\Omega a(x,u_n,\nabla u_n)\nabla v dx \quad &+ \quad \int_{\partial\Omega}\lambda T_n(u_n)v d\sigma \\ &+ \quad \varepsilon\int_\Omega \big[|u_n|^{p_+-2}u_n v + |\nabla u_n|^{p_+-2}\nabla u_n\nabla v\big]dx \\ &= \quad \int_\Omega f_n v dx + \int_{\partial\Omega} g_n v d\sigma. \end{aligned} \tag{3.3}$$

To prove the Theorem 3.3, we need the following result.

Lemma 3.1. *(See [22], Corollary 2.2). If an operator $\mathcal{A}$ is of type (M), bounded and coercive on a separable Banach space to its dual, then $\mathcal{A}$ is surjective.*

We define the operator A_n by

$$A_n u = Au + B_n u,$$

where

$$< Au, v >= \int_\Omega a(x, u, \nabla u)\nabla v dx$$

and

$$< B_n u, v >= \int_\Omega T_n(b(u))v dx + \lambda \int_{\partial\Omega} T_n(u)v d\sigma + \varepsilon \int_\Omega \left[|u|^{p_+-2}uv + |\nabla u|^{p_+-2}\nabla u \nabla v\right] dx,$$

with $u, v \in X_0$.

Proof of the Theorem 3.3. The proof is organized in three Steps.

Step 1: A_n is bounded.

By using Hölder type inequality and (A_5) with constant exponent p_+, we deduce that A is bounded. Moreover, B_n is bounded. Indeed, let $u \in F$, where F is a bounded subset of X_0.
As b is onto, we have

$$\begin{aligned} < B_n u, v > &= \int_\Omega T_n(b(u))v dx + \lambda \int_{\partial\Omega} T_n(u)v d\sigma + \varepsilon \int_\Omega \left[|u|^{p_+-2}uv + |\nabla u|^{p_+-2}\nabla u \nabla v\right] dx \\ &\leq \int_\Omega |b(u)||v| dx + \lambda \int_{\partial\Omega} |u||v| d\sigma + \varepsilon \int_\Omega \left[|u|^{p_+-1}|v| + |\nabla u|^{p_+-1}|\nabla v|\right] dx \\ &\leq C(\lambda)\big(||v||_{L^1(\Omega)} + ||v||_{L^1(\partial\Omega)}\big) + \varepsilon\big[||u||^{\frac{p_+}{(p_+)'}}_{L^{p_+}(\Omega)}||v||_{L^{p_+}(\Omega)} + ||\nabla u||^{\frac{p_+}{(p_+)'}}_{L^{p_+}(\Omega)}||\nabla v||_{L^{p_+}(\Omega)}\big] \\ &\leq C(\lambda)\big(||v||_{L^1(\Omega)} + ||v||_{L^1(\partial\Omega)}\big) + C(\varepsilon)||v||_{W^{1,p_+}(\Omega)}. \end{aligned}$$

Therefore, A_n is bounded.

We recall the following notion:

Definition 3. *An operator $A : V \to V'$ is type of (M) if:*

$$\left.\begin{array}{l} u_n \rightharpoonup u \ \ in \ \ V \\ A(u_n) \rightharpoonup \chi \ \ in \ \ V' \\ \limsup\limits_{n\to\infty} < A(u_n), u_n > \leq < \chi, u > \end{array}\right\} \Rightarrow \chi = A(u).$$

Step 2: A_n is pseudo-monotone.

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in X_0 such that

$$\begin{cases} u_k \rightharpoonup u \ \text{ in } \ X_0 \\ A_n u_k \rightharpoonup \chi \ \text{ in } \ X_0' \\ \limsup\limits_{k\to\infty} < A_n u_k, u_k >=< \chi, u > . \end{cases}$$

We will prove that $\chi = A_n u$.
As $T_n(b(u_k))u_k \geq 0$ and $\lambda T_n(u_k)u_k \geq 0$, by Fatou's Lemma, we deduce that

$$\liminf_{k\to\infty} \left(\int_\Omega T_n(b(u_k))u_k dx + \lambda \int_{\partial\Omega} T_n(u_k)u_k d\sigma \right) \geq \int_\Omega T_n(b(u))u dx + \lambda \int_{\partial\Omega} T_n(u)u d\sigma.$$

One the other hand, thanks to the Lebesgue dominated convergence Theorem, we have

$$\lim_{k\to\infty}\left(\int_\Omega T_n(b(u_k))vdx+\lambda\int_{\partial\Omega}T_n(u_k)vd\sigma+\varepsilon\int_\Omega\left[|u_k|^{p_+-2}u_kv+|\nabla u_k|^{p_+-2}\nabla u_k\nabla v\right]dx\right)$$
$$=\int_\Omega T_n(b(u))vdx+\lambda\int_{\partial\Omega}T_n(u)vd\sigma+\varepsilon\int_\Omega\left[|u|^{p_+-2}uv+|\nabla u|^{p_+-2}\nabla u\nabla v\right]dx,$$

for any $v\in X_0$. Therefore, for k large enough,

$$T_n(b(u_k))+\lambda T_n(u_k)+\varepsilon\left[|u_k|^{p_+-2}u_k+|\nabla u_k|^{p_+-2}\nabla u_k\right]\rightharpoonup T_n(b(u))+\lambda T_n(u)+\varepsilon\left[|u|^{p_+-2}u+|\nabla u|^{p_+-2}\nabla u\right]\ \text{ in }\ X_0'.$$

Thus,

$$Au_k\rightharpoonup\chi-\Big(T_n(b(u))+\lambda T_n(u)+\varepsilon[|u|^{p_+-2}u+|\nabla u|^{p_+-2}\nabla u]\Big)\ \text{ in }\ X_0',\ \text{ as }\ k\to+\infty.$$

Now, we are going to prove that A is of type (M).

Let us set

$$a_1(u,v,w)=\int_\Omega a(x,u,\nabla v)\nabla wdx.$$

Then, $w\mapsto a_1(u,v,w)$ is continuous on $W^{1,p_+}(\Omega)$, thus

$$a_1(u,v,w)=\langle A(u,v),w\rangle,\ \ A(u,v)\in(W^{1,p_+}(\Omega))',$$

and verify

$$A(u,u)=Au,\ \text{ where }\ Au:=-\text{div}a(x,u,\nabla u).$$

Let us prove that A is of type of Calculus of variation.

• As $A(u,.)$ is bounded, we prove that $v\mapsto A(u,v)$ is hemi-continuous from $W^{1,p_+}(\Omega)\to(W^{1,p_+}(\Omega))'$.
Since $a(x,u,\nabla(v_1+tv_2))\rightharpoonup a(x,u,\nabla v_1)$ in $L^{p'_+}(\Omega)$ as $t\to 0$ and $u,v_1,v_2\in W^{1,p_+}(\Omega)$ then, $a_1(u,v_1+tv_2,w)\to a_1(u,v_1,w)$ as $t\to 0$.
In the same manner we prove that $u\mapsto A(u,v)$ is hemi-continuous from $W^{1,p_+}(\Omega)\to(W^{1,p_+}(\Omega))'$.

Moreover, for all $u,v\in W^{1,p_+}(\Omega)$, we have

$$\begin{aligned}<A(u,u)-A(u,v),u-v> &= <A(u,u),u-v>-<A(u,v),u-v>\\ &= a_1(u,u,u-v)-a_1(u,v,u-v)\\ &= \int_\Omega a(x,u,\nabla u)\nabla(u-v)dx-\int_\Omega a(x,u,\nabla v)\nabla(u-v)dx\\ &= \int_\Omega\big(a(x,u,\nabla u)-a(x,u,\nabla v)\big)\nabla(u-v)dx\geq 0.\end{aligned}$$

• Let us suppose that $u_k \rightharpoonup u$ in $W^{1,p_+}(\Omega)$ and $< A(u_k,u_k) - A(u_k,u), u_k - u > \to 0$. We prove that

$$\forall v \in \ W^{1,p_+}(\Omega), \ \ A(u_k,v) \rightharpoonup A(u,v) \ \text{ in } \ (W^{1,p_+}(\Omega))'.$$

Let's set

$$\int_\Omega F_k dx = \big\langle A(u_k,u_k) - A(u_k,u), u_k - u\big\rangle, \ \text{ then } \ F_k \to 0.$$

As $u_k \rightharpoonup u$, we have

$$a(x,u_k,\nabla v) \rightharpoonup a(x,u,\nabla v) \ \text{ in } \ L^{p'_+}(\Omega)$$

(see [17], Lemma 2.2 with $m=1$). Therefore, $A(u_k,v) \rightharpoonup A(u,v)$ in $(W^{1,p_+}(\Omega))'$.

• Now, we suppose that $u_k \rightharpoonup u$ in $W^{1,p_+}(\Omega)$ and $A(u_k,v) \rightharpoonup \Theta$ in $(W^{1,p_+}(\Omega))'$. We prove that

$$\big\langle A(u_k,v), u_k\big\rangle \to \big\langle \Theta, u\big\rangle.$$

Then, by using ([17], Lemma 2.1), we obtain that $a(x,u_k,\nabla v) \to a(x,u,\nabla v)$ in $L^{p'_+}(\Omega)$ and thus, $a_1(u_k,v,u_k) \to a_1(u,v,u)$.
Therefore,

$$< A(u_k,v), u_k >= a_1(u_k,v,u_k) \to < A(u,v), u > \ \text{ and } \ \Theta = A(u,v).$$

Hence, A is of type of Calculus of variation. Finally, by using ([17], Proposition 2.6 and Proposition 2.5), we prove that A is of type (M).

As the operator A is of type (M), so we have immediately

$$Au = \chi - \bigg(T_n(b(u)) + \lambda T_n(u) + \varepsilon\big[|u|^{p_+-2}u + |\nabla u|^{p_+-2}\nabla u\big]\bigg).$$

Therefore, we deduce that $A_n u = \chi$.

Step 3: A_n is coercive.

$$\begin{aligned}
< A_n u, u > &= \int_\Omega a(x,u,\nabla u).\nabla u dx + \int_\Omega T_n(b(u)) u dx \\
&+ \lambda \int_{\partial\Omega} T_n(u) u dx + \varepsilon \int_\Omega \big[|u|^{p_+} + |\nabla u|^{p_+}\big] dx \\
&\geq \varepsilon \int_\Omega \big[|u|^{p_+} + |\nabla u|^{p_+}\big] dx \\
&\geq \varepsilon ||u||^{p_+}_{W^{1,p_+}(\Omega)}.
\end{aligned}$$

We deduce that

$$\frac{< A_n u, u >}{||u||_{W^{1,p_+}(\Omega)}} \to +\infty \ \text{ as } \ ||u||_{W^{1,p_+}(\Omega)} \to +\infty.$$

Hence, A_n is coercive.

Then, according to Lemma 3.1, A_n is surjective.
Thus, for any $F_n =< T_n(f), T_n(g) >\subset E' \subset X_0'$, there exists at least one solution $u_n \in X_0$ of the problem

$$< A_n u_n, v >=< F_n, v > \quad \text{for all} \;\; v \in X_0.$$

Therefore, u_n is a weak solution of the problem (3.2). This ends the proof of Theorem 3.3.

Remark 3.4. *If u_n is a weak solution of the problem* (3.2)*, then $u_n \in W^{1,\pi_n(.)}(\Omega)$, since $W^{1,p_+}(\Omega) \hookrightarrow W^{1,\pi_n(.)}(\Omega)$ continuously. Moreover, $a(x, u_n, \nabla u_n)$ satisfies $(A_3) - (A_6)$ with variable exponent $\pi_n(x) := p(x, u_n(x))$.*

Part 2: A priori estimates and convergence results.
This part is done in three steps, we make a priori estimates, some convergence results and other based on the Young measure and nonlinear weak$-^*$ convergence.

Step 1: A priori estimates

Lemma 3.2. *Suppose that $(A_3) - (A_6)$ hold with variable exponent $\pi_n(.)$ and $f_n \in L^\infty(\Omega)$, $g_n \in L^\infty(\partial\Omega)$. Let u_n be a weak solution of* (3.2)*. Then, for all $k > 0$,*

$$\int_\Omega \big|\nabla T_k(u_n)\big|^{\pi_n(.)} dx \leq C_2 k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big), \tag{3.4}$$

$$\int_\Omega \big|T_n(b(u_n))\big| dx \leq ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}, \tag{3.5}$$

$$\int_{\partial\Omega} \big|T_n(u_n)\big| dx \leq \frac{1}{\lambda}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big). \tag{3.6}$$

Proof of Lemma 3.2. By taking $v = T_k(u_n)$ in the weak formulation (3.3), we obtain

$$\begin{aligned} \int_\Omega T_n(b(u_n))T_k(u_n)dx \quad &+ \quad \int_\Omega a(x, u_n, \nabla u_n).\nabla T_k(u_n)dx + \int_{\partial\Omega} \lambda T_n(u_n)T_k(u_n)d\sigma \\ &+ \quad \varepsilon \int_\Omega \big[|u_n|^{p_+-2}u_n T_k(u_n) + |\nabla u_n|^{p_+-2}\nabla u_n \nabla T_k(u_n)\big] dx \\ &= \quad \int_\Omega f_n T_k(u_n)dx + \int_{\partial\Omega} g_n T_k(u_n)d\sigma. \end{aligned} \tag{3.7}$$

Since all the terms of the left hand side of (3.7) are nonnegative, we deduce that

$$\int_\Omega a(x, u_n, \nabla u_n).\nabla T_k(u_n)dx \leq \int_\Omega f_n T_k(u_n)dx + \int_{\partial\Omega} g_n T_k(u_n)d\sigma. \tag{3.8}$$

By using (A_6) and (3.8), we get

$$\begin{aligned}\int_\Omega \left|\nabla T_k(u_n)\right|^{\pi_n(.)}dx &\leq C_2\left(\int_\Omega f_n T_k(u_n)dx + \int_{\partial\Omega} g_n T_k(u_n)d\sigma\right)\\ &\leq C_2 k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).\end{aligned}$$

From (3.7), we deduce that

$$\begin{aligned}\int_\Omega T_n(b(u_n))T_k(u_n)dx &\leq \int_\Omega f_n T_k(u_n)dx + \int_{\partial\Omega} g_n T_k(u_n)d\sigma\\ &\leq k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big)\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\lambda\int_{\partial\Omega} T_n(u_n)T_k(u_n)dx &\leq \int_\Omega f_n T_k(u_n)dx + \int_{\partial\Omega} g_n T_k(u_n)d\sigma\\ &\leq k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).\end{aligned} \tag{3.10}$$

Dividing (3.9) and (3.10) by k and letting k goes to 0, we obtain

$$\int_\Omega T_n(b(u_n))sign_0(u_n)dx \leq ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}$$

and

$$\lambda\int_{\partial\Omega} T_n(u_n)sign_0(u_n)dx \leq ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}.$$

Hence,

$$\int_\Omega \left|T_n(b(u_n))\right|dx \leq ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}$$

and

$$\int_{\partial\Omega} \left|T_n(u_n)\right|dx \leq \frac{1}{\lambda}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).$$

Lemma 3.3. *Assume that (A_3)-(A_6) hold. If u_n is a weak solution of the problem (3.2), $f_n \in L^\infty(\Omega)$ and $g_n \in L^\infty(\partial\Omega)$, then for all $k > 0$*

$$\int_\Omega \left|\nabla T_k(u_n)\right|^{p_-}dx \leq C\big(||f||_{L^1(\Omega)}, ||g||_{L^1(\partial\Omega)}, meas(\Omega)\big)(k+1) \tag{3.11}$$

and

$$\int_{\partial\Omega} \left|T_k(u_n)\right|d\sigma \leq \frac{1}{\lambda}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big), \tag{3.12}$$

for all $n \geq k > 0$.

Proof of Lemma 3.3. Firstly, we prove (3.11). We know that

$$\int_\Omega \big|\nabla T_k(u_n)\big|^{\pi_n(.)}dx \leq C_2 k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big). \tag{3.13}$$

Let us note that

$$\begin{aligned}\int_\Omega \big|\nabla T_k(u_n)\big|^{p_-}dx &= \int_{\{|\nabla T_k(u_n)|>1\}} \big|\nabla T_k(u_n)\big|^{p_-}dx + \int_{\{|\nabla T_k(u_n)|\leq 1\}} \big|\nabla T_k(u_n)\big|^{p_-}dx \\ &\leq \int_{\{|\nabla T_k(u_n)|>1\}} \big|\nabla T_k(u_n)\big|^{p_-}dx + meas(\Omega) \\ &\leq \int_\Omega \big|\nabla T_k(u_n)\big|^{\pi_n(.)}dx + meas(\Omega). \end{aligned} \tag{3.14}$$

By using (3.13) and (3.14), we get

$$\begin{aligned}\int_\Omega \big|\nabla T_k(u_n)\big|^{p_-}dx &\leq \max\big(C_2\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big), meas(\Omega)\big)(k+1) \\ &:= C\big(||f||_{L^1(\Omega)}, ||g||_{L^1(\partial\Omega)}, meas(\Omega)\big)(k+1). \end{aligned} \tag{3.15}$$

Now, from the formula (3.6), we obtain $||T_n(u_n)||_{L^1(\partial\Omega)} \leq \dfrac{1}{\lambda}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big)$ and as $|T_k(u_n)| \leq |T_n(u_n)|$ for all $n \geq k > 0$, one deduces that

$$\int_{\partial\Omega} |T_k(u_n)|d\sigma \leq \frac{1}{\lambda}\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).$$

Lemma 3.4. *For any $k > 0$, we have*

$$||T_k(u_n)||_{W^{1,\pi_n(.)}(\Omega)} \leq 1 + C\big(k, f, g, p_-, p_+, meas(\Omega)\big)$$

and for all $k \geq 1$,

$$meas\big(\{|u_n| > k\}\big) \leq \frac{C}{\min\big(b(k), |b(-k)|\big)}.$$

Proof of Lemma 3.4. By using (3.4), we have

$$\int_\Omega \big|\nabla T_k(u_n)\big|^{\pi_n(.)}dx \leq C_2 k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big). \tag{3.16}$$

We also have

$$\int_\Omega |T_k(u_n)|^{\pi_n(.)}dx = \int_{\{|u_n|\leq k\}} |T_k(u_n)|^{\pi_n(.)}dx + \int_{\{|u_n|>k\}} |T_k(u_n)|^{\pi_n(.)}dx.$$

Furthermore,

$$\begin{aligned}\int_{\{|u_n|>k\}} |T_k(u_n)|^{\pi_n(.)}dx &= \int_{\{|u_n|>k\}} k^{\pi_n(.)}dx \\ &\leq \begin{cases} k^{p_+}meas(\Omega) & \text{if} \quad k \geq 1 \\ meas(\Omega) & \text{if} \quad k < 1 \end{cases}\end{aligned}$$

and

$$\begin{aligned}\int_{\{|u_n|\leq k\}}|T_k(u_n)|^{\pi_n(.)}dx &\leq \int_{\{|u_n|\leq k\}}k^{\pi_n(.)}dx\\ &\leq \begin{cases}k^{p_+}meas(\Omega) & \text{if} \quad k\geq 1\\ meas(\Omega) & \text{if} \quad k<1.\end{cases}\end{aligned}$$

This allow us to write

$$\int_\Omega |T_k(u_n)|^{\pi_n(.)}dx \leq 2(1+k^{p_+})meas(\Omega). \tag{3.17}$$

Hence, adding (3.16) and (3.17) one gets

$$\rho_{1,\pi_n(.)}(T_k(u_n)) \leq C_2k\big(||f||_{L^1(\Omega)}+||g||_{L^1(\partial\Omega)}\big)+2(1+k^{p_+})meas(\Omega).$$

For $||T_k(u_n)||_{W^{1,\pi_n(.)}(\Omega)} \geq 1$, we have according to Proposition 2 that

$$||T_k(u_n)||^{p_-}_{W^{1,\pi_n(.)}(\Omega)} \leq \rho_{1,\pi_n(.)}(T_k(u_n)) \leq \big[C_2k\big(||f||_{L^1(\Omega)}+||g||_{L^1(\partial\Omega)}\big)+2(1+k^{p_+})meas(\Omega)\big],$$

which implies that

$$\begin{aligned}||T_k(u_n)||_{W^{1,\pi_n(.)}(\Omega)} &\leq \big[C_2k\big(||f||_{L^1(\Omega)}+||g||_{L^1(\partial\Omega)}\big)+2(1+k^{p_+})meas(\Omega)\big]^{\frac{1}{p_-}}\\ &:= C(k,f,g,p_+,p_-,meas(\Omega)).\end{aligned}$$

Thus,

$$||T_k(u_n)||_{W^{1,\pi_n(.)}(\Omega)} < 1+C(k,f,g,p_+,p_-,meas(\Omega)).$$

Moreover, from (3.5), we have

$$\int_{\partial\Omega}\big|T_n(b(u_n))\big|dx \leq ||f||_{L^1(\Omega)}+||g||_{L^1(\partial\Omega)}.$$

We deduce that the sequence $(T_n(b(u_n)))_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^1(\Omega)$. Thus, $(b(u_n))_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^1(\Omega)$. So, there exists a positive constant C such that

$$\int_\Omega |b(u_n)|dx \leq C.$$

Furthermore, for all $k\geq 1$, we have

$$\int_{\{|u_n|>k\}}|b(u_n)|dx \leq \int_\Omega |b(u_n)|dx \leq C.$$

As b is continuous, nondecreasing and surjective, we infer

$$\int_{\{|u_n|>k\}}\min\big(b(k),|b(-k)|\big)dx \leq \int_{\{|u_n|>k\}}|b(u_n)|dx \leq C.$$

Therefore,

$$meas\big(\{|u_n|>k\}\big) \leq \frac{C}{\min\big(b(k),|b(-k)|\big)}, \quad \forall k\geq 1.$$

Then, the proof of Lemma 3.4 is complete.

From the Lemma 3.4, we deduce that for any $k > 0$, the sequence $\big(T_k(u_n)\big)_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,\pi_n(.)}(\Omega)$ and also in $W^{1,p_-}(\Omega)$.
Then, up to a subsequence still denoted $T_k(u_n)$, we can assume that for any $k > 0$, $T_k(u_n)$ weakly converges to s_k in $W^{1,p_-}(\Omega)$ and also $T_k(u_n)$ strongly converges to s_k in $L^{p_-}(\Omega)$.
By using the above a priori estimates, we obtain the following convergence results .

Step 2: The convergence results
The proof of the following proposition use the Lemma 3.4.

Proposition 6. *Assume that $(A_3)-(A_6)$ hold and let u_n be a weak solution of the problem (3.2), then the sequence $(u_n)_{n\in\mathbb{N}}$ is Cauchy in measure.*
In particular, there exists a measurable function u and a subsequence still denoted u_n such that $u_n \to u$ in measure, as $n \to +\infty$.

As $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, so (up to a subsequence) it converges almost everywhere to some measurable function u.

As for any $k > 0$, T_k is continuous; then $T_k(u_n) \to T_k(u)$ a.e. $x \in \Omega$, so $s_k = T_k(u)$.
Therefore,

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W^{1,p_-}(\Omega)$$

and by compact embedding Theorem, we have

$$T_k(u_n) \to T_k(u) \text{ in } L^{p_-}(\Omega) \text{ (respectively in } L^{p_-}(\partial\Omega)) \text{ and a.e. in } \Omega \text{ (respectively a.e. on } \partial\Omega).$$

Lemma 3.5. *u_n converges a.e. on $\partial\Omega$ to some function v.*

Proof of Lemma 3.5
Since $T_k(u_n) \rightharpoonup T_k(u)$ in $W^{1,p_-}(\Omega)$ and $W^{1,p_-}(\Omega) \hookrightarrow L^{p_-}(\partial\Omega)$ (compact embedding), then $T_k(u_n) \to T_k(u)$ in $L^{p_-}(\partial\Omega)$ and a.e. on $\partial\Omega$. Therefore, $T_k(u_n) \to T_k(u)$ in $L^1(\partial\Omega)$ and a.e. in $\partial\Omega$. We deduce that there exists $E \subset \partial\Omega$ such that $T_k(u_n) \to T_k(u)$ on $\partial\Omega \setminus E$ with $\mu(E) = 0$, where μ is area measure on $\partial\Omega$.
For every $k > 0$, let $E_k = \{x \in \partial\Omega \text{ such that } |T_k(u)| < k\}$ and $F = \partial\Omega \setminus \bigcup_{k>0} E_k$. By using Fatou's Lemma, we have

$$\begin{aligned} \int_{\partial\Omega} \big|T_k(u)\big| d\sigma &\leq \liminf_{n\to+\infty} \int_{\partial\Omega} \big|T_k(u_n)\big| d\sigma \\ &\leq \frac{||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}}{\lambda}. \end{aligned} \tag{3.18}$$

Now, we use (3.18) to get

$$\mu(F)=\frac{1}{k}\int_F \big|T_k(u)\big|d\sigma \quad\leq\quad \frac{1}{k}\int_{\partial\Omega}\big|T_k(u)\big|d\sigma \leq \frac{||f||_{L^1(\Omega)}+||g||_{L^1(\partial\Omega)}}{k\lambda}.$$

We obtain $\mu(F)=0$, as k goes to ∞. Let' s now define on $\partial\Omega$ the function v by

$$v(x)=T_k(u(x)),\ \ x\in E_k.$$

We take $x\in\partial\Omega\setminus(E\cup F)$, then there exists $k>0$ such that $x\in E_k$ and we have

$$u_n(x)-v(x)=\big(u_n(x)-T_k(u_n(x))\big)+\big(T_k(u_n(x))-T_k(u(x))\big).$$

Since $x\in E_k$, we have $|T_k(u(x))|<k$ and so $|T_k(u_n(x))|<k$, from which we deduce that $|u_n(x)|<k$. Therefore,

$$u_n(x)-v(x)=T_k(u_n(x))-T_k(u(x))\to 0,\ \ \text{as}\ \ n\to+\infty.$$

This means that u_n converges to v a.e. on $\partial\Omega$, but for all $x\in E_k$, $T_k(u(x))=u(x)$. Thus, $v=u$ a.e. on $\partial\Omega$. Therefore,

$$u_n\to u\ \ \text{a.e. on}\ \ \partial\Omega.$$

The following assertions are based on the Young measure and nonlinear weak $-^*$ convergence results (see [7, 19, 13]).

Step 3: The convergence in term of Young measure

Assertion 1

The sequence $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ converges to a Young measure $\nu_x^k(\lambda)$ on $\mathbb{R}^N$ in the sense of the nonlinear weak-* convergence and

$$\nabla T_k(u)=\int_{\mathbb{R}^N}\lambda d\nu_x^k(\lambda). \tag{3.19}$$

Proof. Using Lemma 3.3, $\nabla T_k(u_n)$ is uniformly bounded in $L^{p_-}(\Omega)$, so, equi-integrable on Ω. Moreover, $\nabla T_k(u_n)$ weakly converges to $\nabla T_k(u)$ in $L^{p_-}(\Omega)$. Therefore, using the representation of weakly convergence sequences in $L^1(\Omega)$ in terms of Young measures (see Theorem 2.1 and formula (2.5)), we can write

$$\nabla T_k(u)=\int_{\mathbb{R}^N}\lambda d\nu_x^k(\lambda)\ \ \square$$

Assertion 2. $|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x^k(\lambda)dx$ on $\mathbb{R}^N\times\Omega$, moreover, $T_k(u)\in W^{1,\pi(.)}(\Omega)$.

Proof. We know that $p(.,u_n(.))\to p(.,u(.))$ in measure on Ω. Now, using Theorem 2.1 (ii),

(iii) $(p(.,u_n(.)), \nabla T_k(u_n))_{n\in\mathbb{N}}$ converges on $\mathbb{R}\times\mathbb{R}^N$ to Young measure $\mu_x^k = \delta_{\pi(x)} \otimes \nu_x^k$.
Thus, we can apply the weak convergence properties (2.4) to the Carathéodory function $F_m(x,\lambda_0,\lambda) \in \Omega\times(\mathbb{R}\times\mathbb{R}^N) \mapsto |h_m(\lambda)|^{\lambda_0}$ with $m\in\mathbb{N}$, where h_m is defined in the preliminaries. Then, we obtain

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{\pi(x)} d\nu_x^k(\lambda)dx &= \int_{\Omega\times(\mathbb{R}\times\mathbb{R}^N)} |h_m(\lambda)|^{\lambda_0} d\mu_x^k(\lambda_0,\lambda)dx\\
&= \int_\Omega\int_{\mathbb{R}\times\mathbb{R}^N} F_m(x,\lambda_0,\lambda)d\mu_x^k(\lambda_0,\lambda)dx\\
&= \lim_{n\to+\infty}\int_\Omega F_m(x,p(x,u_n(x)),\nabla T_k(u_n(x)))dx\\
&= \lim_{n\to+\infty}\int_\Omega |h_m(\nabla T_k(u_n))|^{p(.,u_n(.))}dx\\
&\le \lim_{n\to+\infty}\int_\Omega |\nabla T_k(u_n)|^{p(.,u_n(.))}dx\\
&\le C_2k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big) \quad \text{(using (3.4))}.
\end{aligned}$$

$h_m(\lambda)\to\lambda$, as $m\to+\infty$ and $m\mapsto h_m(\lambda)$ is increasing. Then, using Lebesgue convergence Theorem , we deduce from last inequality that

$$\int_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(x)} d\nu_x^k(\lambda)dx \le C_2k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).$$

Hence, $|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x^k(\lambda)dx$ on $\mathbb{R}^N\times\Omega$.
From (3.19), the last inequality and Jensen inequality, we get

$$\int_\Omega |\nabla T_k(u)|^{\pi(x)}dx = \int_\Omega \left|\int_{\mathbb{R}^N} \lambda d\nu_x^k(\lambda)\right|^{\pi(x)} dx \le \int_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(x)} d\nu_x^k dx < \infty.$$

Thus, $\nabla T_k(u)\in L^{\pi(.)}(\Omega)$. Moreover, $\int_\Omega |T_k(u)|^{\pi(.)}dx \le \max\big(k^{p_+},k^{p_-}\big)meas(\Omega)$. Hence, $T_k(u)\in L^{\pi(.)}(\Omega)$ and we conclude that $T_k(u)\in W^{1,\pi(.)}(\Omega)$. $\square$

Assertion 3.
i) The sequence $\big(\Phi_n^k\big)_{n\in\mathbb{N}}$ defined by $\Phi_n^k := a(x,u_n,\nabla T_k(u_n))$ is equi-integrable on Ω.
ii) The sequence $\big(\Phi_n^k\big)_{n\in\mathbb{N}}$ weakly converges to Φ^k in $L^1(\Omega)$ and we have

$$\Phi^k(x) = \int_{\mathbb{R}^N} a(x,u,\lambda)d\nu_x^k(\lambda). \tag{3.20}$$

Proof. i) Using the growth assumption (A_5) with variable exponent $p(.,u_n(.))$ and relation (3.4), we deduce that (Φ_n^k) is bounded in $L^{\pi_n'(.)}(\Omega)$, so, $L^{\pi_n'(.)}-$ equi-integrable on Ω.
Moreover, as $\pi_n'(.)>1$, we obtain

$$|a(x,u_n,\nabla T_k(u_n))| \le 1 + |a(x,u_n,\nabla T_k(u_n))|^{\pi_n'(.)}.$$

Thus, for all subset $E \subset \Omega$, we have

$$\int_E |a(x,u_n,\nabla T_k(u_n))|dx \leq meas(E) + \int_E |a(x,u_n,\nabla T_k(u_n))|^{\pi'_n(.)}dx.$$

Therefore, for $meas(E)$ small enough, (Φ_n^k) is equi-integrable on Ω.
ii) Set $\tilde{\Phi}_n^k = a(x,u(x),\nabla v_n)$ with $\nabla v_n = \nabla T_k(u_n).\chi_{S_n}$ where $S_n = \big\{x \in \Omega,\ \ |\pi(x) - \pi_n(x)| < \frac{1}{2}\big\}$.
Applying (A_5) with variable exponent $\pi(.)$ on $a(x,u(x),\nabla v_n)$, we have for all subset $E \subset \Omega$,

$$\begin{aligned}\int_E |a(x,u(x),\nabla v_n)|dx &\leq C\int_E \big(1 + \mathcal{M}(x) + |\nabla v_n|^{\pi(.)-1}\big)dx \\ &\leq C\int_E \big(1 + \mathcal{M}(x))dx + \int_{E\cap S_n} |\nabla T_k(u_n)|^{\pi(.)-1}dx.\end{aligned}$$

The first term of the right hand side of the last inequality is small for $meas(E)$ small enough. For $x \in S_n$, $\pi(x) < \pi_n(x) + \frac{1}{2}$, thus

$$\int_{E\cap S_n} |\nabla T_k(u_n)|^{\pi(.)-1}dx \leq \int_{E\cap S_n} \big(1 + |\nabla T_k(u_n)|^{\pi_n(.)-\frac{1}{2}}\big)dx$$

and

$$\int_\Omega |\nabla T_k(u_n)|^{(\pi_n(.)-\frac{1}{2})(2\pi_n(.))'}dx = \int_\Omega |\nabla T_k(u_n)|^{\pi_n(.)}dx < \infty,$$

which is equivalent to saying $|\nabla T_k(u_n)|^{\pi_n(.)-\frac{1}{2}} \in L^{(2\pi_n(.))'}(\Omega)$. Now, using Hölder type inequality,

$$\begin{aligned}\int_{E\cap S_n} |\nabla T_k(u_n)|^{\pi(.)-1}dx &\leq \int_E \left(1 + |\nabla T_k(u_n)|^{\pi_n(.)-\frac{1}{2}}\right)dx \\ &\leq meas(E) + 2\big|\big|\nabla T_k(u_n)\big|\big|_{L^{\pi_n(.)}(\Omega)} ||\chi_E||_{L^{2\pi_n(.)}(\Omega)}.\end{aligned} \tag{3.21}$$

According to Proposition 1,

$$\begin{aligned}||\chi_E||_{L^{2\pi_n(.)}(\Omega)} &\leq \max\left\{\big(\rho_{2\pi_n(.)}\big(\chi_E\big)\big)^{\frac{1}{2p_-}}, \big(\rho_{2\pi_n(.)}(\chi_E)\big)^{\frac{1}{2p_+}}\right\} \\ &= \max\left\{\big(meas(E)\big)^{\frac{1}{2p_-}}, \big(meas(E)\big)^{\frac{1}{2p_+}}\right\}.\end{aligned}$$

The right-hand side of (3.21) is uniformly small for $meas(E)$ small, and the equi-integrability of $\tilde{\Phi}_n^k$ follows. Therefore, (up to a subsequence) $\tilde{\Phi}_n^k$ weakly converges in $L^1(\Omega)$ to $\tilde{\Phi}^k$, as $n \to +\infty$.
Now, we prove that $\tilde{\Phi}^k = \Phi^k$; more precisely, we show that $\tilde{\Phi}_n^k - \Phi_n^k$ strongly converges in $L^1(\Omega)$ to 0.
Let $\beta > 0$, by (3.4), $\displaystyle\int_\Omega |\nabla T_k(u_n)|^{\pi_n(.)}dx$ is uniformly bounded, which implies that $\displaystyle\int_\Omega |\nabla T_k(u_n)|dx$ is finite, since

$$\int_\Omega |\nabla T_k(u_n)|dx \leq \int_\Omega (1 + |\nabla T_k(u_n)|^{\pi_n(x)})dx.$$

By Chebyschev Inequality, we have

$$meas(\{|\nabla T_k(u_n)| > L\}) \leq \frac{\int_\Omega |\nabla T_k(u_n)|dx}{L}.$$

Therefore, $\sup_{n\in\mathbb{N}} meas(\{|\nabla T_k(u_n)| > L\})$ tends to 0 for L large enough. Since $\tilde{\Phi}_n^k - \Phi_n^k$ is equi-integrable, there exists $\delta = \delta(\beta)$ such that for all $A \subset \Omega$, $meas(A) < \delta$ and $\int_A |\tilde{\Phi}_n^k - \Phi_n^k| dx < \frac{\beta}{4}$.
Therefore, if we choose L large enough, we get $\frac{\int_\Omega |\nabla T_k(u_n)| dx}{L} < \delta$, so $meas(\{|\nabla T_k(u_n)| > L\}) < \delta$.
Hence,

$$\int_{\{|\nabla T_k(u_n)|>L\}} |\tilde{\Phi}_n^k - \Phi_n^k| dx < \frac{\beta}{4}.$$

By Lemma 2.2, we also have

$$meas\left(\left\{x \in \Omega;\ \sup_{\lambda\in K} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| \geq \sigma\right\}\right) \longrightarrow 0,$$

as $n \to +\infty$.
Thus, by the above equi-integrability, for all $\sigma > 0$, there exists $n_0 = n_0(\sigma, L) \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\int_{\left\{x\in\Omega;\ \sup_{|\lambda|\leq L} |a(x,u_n(x),\lambda)-a(x,u(x),\lambda)|\geq\sigma\right\}} |\tilde{\Phi}_n^k - \Phi_n^k| dx < \frac{\beta}{4}.$$

Using the definition of Φ_n^k and $\tilde{\Phi}_n^k$, we have

$$\Phi_n^k - \tilde{\Phi}_n^k = a(x, u_n(x), \nabla T_k(u_n)) - a(x, u(x), \nabla T_k(u_n)) \ \text{ on } \ S_n.$$

Now, we reason on

$$S_{n,L,\sigma} := \left\{x \in \Omega;\ \sup_{|\lambda|\leq L} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| < \sigma, |\nabla T_k(u_n)| \leq L\right\}.$$

We get

$$\begin{aligned}\int_{S_{n,L,\sigma}} |\tilde{\Phi}_n^k - \Phi_n^k| dx &\leq \int_{S_{n,L,\sigma}} \sup_{|\lambda|\leq L} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| dx \\ &\leq \sigma meas(\Omega).\end{aligned}$$

We observe that

$$\int_{S_n} |\tilde{\Phi}_n^k - \Phi_n^k| dx = \int_{S_n\cap S_{n,L,\sigma}} |\tilde{\Phi}_n^k - \Phi_n^k| dx + \int_{S_n\setminus S_{n,L,\sigma}} |\tilde{\Phi}_n^k - \Phi_n^k| dx$$

and

$$S_n \setminus S_{n,L,\sigma} \subset \left\{x \in \Omega;\ \sup_{|\lambda|\leq L} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| \geq \sigma\right\} \cup \left\{|\nabla T_k(u_n)| > L\right\}.$$

Consequently, by choosing $\sigma = \sigma(\beta) < \frac{\beta}{4meas(\Omega)}$, we get

$$\int_{S_n} |\tilde{\Phi}_n^k - \Phi_n| dx < \frac{\beta}{4} + \frac{\beta}{4} + \frac{\beta}{4} = \frac{3\beta}{4},$$

for all $n \geq n_0(\sigma, L)$. By Lemma 2.2, we also have $meas(\{x \in \Omega, |\pi(x) - \pi_n(x)| \geq \frac{1}{2}\}) \to 0$ for n large enough; which means that $meas(\Omega \setminus S_n)$ converges to 0 for n large enough. Thus,

$$\int_{\Omega \setminus S_n} |\tilde{\Phi}_n^k - \Phi_n^k| dx = \int_{\Omega \setminus S_n} |\Phi_n^k| dx \leq \frac{\beta}{4}.$$

Therefore, for all $\beta > 0$ there exists $n_0 = n_0(\beta)$ such that for all $n \geq n_0$, $\displaystyle\int_\Omega |\tilde{\Phi}_n^k - \Phi_n^k| dx \leq \beta$. Hence, $\tilde{\Phi}_n^k - \Phi_n^k$ strongly converges to 0 in $L^1(\Omega)$. We prove that

$$\Phi^k(x) = \int_{\mathbb{R}^N} a(x, u(x), \lambda) d\nu_x^k(\lambda) \quad a.e. \ x \in \Omega \ \text{ and } \ \Phi^k \in L^{\pi'(.)}(\Omega).$$

Notice that

$$\lim_{n \to +\infty} \int_\Omega |\nabla T_k(u_n)|(1 - \chi_{S_n}) dx = \lim_{n \to +\infty} \int_{\Omega \setminus S_n} |\nabla T_k(u_n)| dx = 0,$$

since $(\nabla T_k(u_n))_{n \in \mathbb{N}}$ is equi-integrable and $meas(\Omega \setminus S_n)$ converges to 0 for n large enough. Therefore, $(\nabla T_k(u_n))_{n \in \mathbb{N}}$ and $\nabla T_k(u_n)\chi_{S_n}$ converge to the same Young measure $\nu_x^k(\lambda)$. Moreover, by applying Theorem 2.1 i) to the Carathéodory function $F(x, (\lambda_0, \lambda)) := a(x, \lambda_0, \lambda)$, we infer that

$$\tilde{\Phi}(x) = \Phi(x) = \int_{\mathbb{R}^N} a(x, u(x), \lambda) d\nu_x^k(\lambda) \ \text{ a.e. } \ x \in \Omega.$$

Using (A_5), it follows that $|a(x, u(x), \lambda)|^{\pi'(.)} \leq C(\mathcal{M}(x) + |\lambda|^{\pi(.)})$. Thus, with Jensen Inequality, it follows that

$$\begin{aligned}
\int_\Omega |\Phi^k(x)|^{\pi'(.)} dx &= \int_\Omega \left| \int_{\mathbb{R}^N} a(x, u(x), \lambda) d\nu_x^k(\lambda) \right|^{\pi'(.)} dx \\
&\leq \int_{\Omega \times \mathbb{R}^N} |a(x, u(x), \lambda)|^{\pi'(.)} d\nu_x^k(\lambda) dx \\
&\leq C \int_{\Omega \times \mathbb{R}^N} \left(\mathcal{M}(x) + |\lambda|^{\pi(.)} \right) d\nu_x^k(\lambda) dx < \infty.
\end{aligned}$$

Hence, $\Phi^k \in L^{\pi'(.)}(\Omega)$. □

Assertion 4

(a) For all $k' > k > 0$, we have $\Phi^k = \Phi^{k'} \chi_{\{|u| < k\}}$.

(b) For all $k > 0$,

$$\int_\Omega \Phi^k . \nabla T_k(u) dx \geq \int_{\Omega \times \mathbb{R}^N} a(x, u(x), \lambda) . \lambda d\nu_x^k(\lambda) dx. \tag{3.22}$$

(c) The "div-curl" inequality holds:

$$\int_{\Omega \times \mathbb{R}^N} \big(a(x, u(x), \lambda) - a(x, u(x), \nabla T_k(u(x))) \big) (\lambda - \nabla T_k(u(x))) d\nu_x^k(\lambda) dx \leq 0. \tag{3.23}$$

(d) For all $k > 0$,

$$\Phi^k = a(x, u(x), \nabla T_k(u)) \text{ for a.e. } x \in \Omega$$

and $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ in measure on Ω, as $n \to +\infty$.

Proof.

(a) Let $k' > k > 0$ and $g_n^k := a(x, u_n, \nabla T_{k'}(u_n))\chi_{[|u|<k]}$. By Assertion 3-ii), $(g_n^k)_{n\in\mathbb{N}}$ weakly converges to $\Phi^{k'}\chi_{[|u|<k]}$ in $L^1(\Omega)$. If we prove that $(g_n^k)_{n\in\mathbb{N}}$ weakly converges to Φ^k in $L^1(\Omega)$, then the wished result will come of the uniqueness of the limit. Let us put

$$h_n^k := a(x, u_n, \nabla T_{k'}(u_n))\chi_{[|u_n|<k]}.$$

As $\nabla T_k(u_n) \equiv \nabla T_{k'}(u_n)\chi_{[|u_n|<k]}$, for all $k' > k > 0$, then, we get

$$h_n^k := a(x, u_n, \nabla T_{k'}(u_n))\chi_{[|u_n|<k]} \equiv a(x, u_n, \nabla T_k(u_n)),$$

so, $(h_n^k)_{n\in\mathbb{N}}$ weakly converges to Φ^k in $L^1(\Omega)$ by Assertion 3-ii). Set

$$d_n^k := g_n^k - h_n^k = a(x, u_n, \nabla T_{k'}(u_n))(\chi_{[|u|<k]} - \chi_{[|u_n|<k]}).$$

On the one hand, thanks to Assertion 3-i), $(d_n^k)_{n\in\mathbb{N}}$ is equi-integrable. On the other hand $d_n^k \to 0$ a.e. on Ω. Indeed, $\chi_{[|u_n|<k]} = \chi_{(-k,k)}(u_n)$ and if $|u_n| \neq k$ a.e. on Ω, $\chi_{(-k,k)}(.)$ is continuous on $\mathbb{R}$. In other words $\chi_{(-k,k)}(.)$ is continuous on the image of Ω by u a.e. $k > 0$. Moreover, $u_n \to u$ a.e. on Ω, then $\chi_{[|u_n|<k]} \to \chi_{[|u|<k]}$ a.e. in Ω. Now, using Vitali's Theorem $(d_n^k)_{n\in\mathbb{N}}$ strongly converges to 0 in $L^1(\Omega)$, so it weakly converges in $L^1(\Omega)$. Hence, $(g_n^k)_{n\in\mathbb{N}}$ and $(h_n^k)_{n\in\mathbb{N}}$ weakly converge to the same limit Φ^k in $L^1(\Omega)$.

(b) Let $\mathcal{S}$ be a set of $W^{2,\infty}$ functions $S : \mathbb{R} \to \mathbb{R}$ such that $S'(.)$ has a compact support.
We construct a sequence $(S_M)_{M\in\mathbb{N}} \subset \mathcal{S}$ such that
- S'_M and S''_M are uniformly bounded;
- for all $M \in \mathbb{N}$, $S'_M = 1$ on $[-M+1, M-1]$, supp$S' \subset [-M, M]$;
- the sequence $(b(z)S'_M(z))_{M\in\mathbb{N}}$ is non-decreasing for all $z \in \mathbb{R}$.

For all $\varphi \in C^\infty(\overline{\Omega})$, $v = \varphi S'_M(u_n)$ is an admissible test function in the weak formulation (3.3). We have

$$\begin{aligned}
\int_\Omega T_n(b(u_n))S'_M(u_n)\varphi dx \quad &+ \quad \int_\Omega S'_M(u_n)a(x, u_n, \nabla T_M(u_n)).\nabla\varphi dx \\
&+ \quad \int_\Omega S''_M(u_n)a(x, u_n, \nabla T_M(u_n)).\nabla T_M(u_n)\varphi dx + \int_{\partial\Omega} \lambda T_n(u_n)S'_M(u_n)\varphi d\sigma \\
&+ \quad \varepsilon\int_\Omega \big[|\nabla u_n|^{p_+-2}\nabla u_n\nabla(\varphi S'_M(u_n)) + |u_n|^{p_+-2}u_nS'_M(u_n)\varphi\big]dx \\
&= \quad \int_\Omega f_nS'_M(u_n)\varphi dx + \int_{\partial\Omega} g_nS'_M(u_n)\varphi d\sigma. \qquad (3.24)
\end{aligned}$$

Since u_n converges to u a.e. in Ω and a.e. on $\partial\Omega$, by continuity of b, S'_M and the compacteness of suppS'_M, we obtain

$$\int_\Omega T_n(b(u_n))S'_M(u_n)\varphi dx \to \int_\Omega b(u)S'_M(u)\varphi dx, \text{ as } n\to+\infty \tag{3.25}$$

and

$$\lambda\int_\Omega T_n(u_n)S'_M(u_n)\varphi d\sigma \to \lambda\int_\Omega uS'_M(u)\varphi d\sigma, \text{ as } n\to+\infty. \tag{3.26}$$

Moreover, we have $|f_nS'_M(u_n)\varphi|\leq ||S'_M||_{L^\infty(\mathbb{R}}|f||\varphi|\in L^1(\Omega)$, $f_nS'_M(u_n)\varphi\to fS'_M(u)\varphi$ a.e. in Ω. and $|g_nS'_M(u_n)\varphi|\leq ||S'_M||_{L^\infty(\mathbb{R})}|g||\varphi|\in L^1(\mathbb{R})$, $g_nS'_M(u_n)\varphi\to gS'_M(u)\varphi$ a.e. on $\partial\Omega$. Thus, by Lebesgue dominated convergence Theorem

$$\int_\Omega f_nS'_M(u_n)\varphi dx \to \int_\Omega fS'_M(u)\varphi dx, \text{ as } n\to+\infty \tag{3.27}$$

and

$$\int_{\partial\Omega} g_nS'_M(u_n)\varphi d\sigma \to \int_{\partial\Omega} gS'_M(u)\varphi d\sigma, \text{ as } n\to+\infty. \tag{3.28}$$

Let us prove now, that

$$\int_\Omega S'_M(u_n)a(x,u_n,\nabla T_M(u_n)).\nabla\varphi dx \to \int_\Omega S'_M(u)\Phi^M.\nabla\varphi dx, \text{ as } n\to+\infty. \tag{3.29}$$

For all $L>0$, we have

$$\begin{aligned}\int_\Omega S'_M(u_n)a(x,u_n,\nabla T_M(u_n)).\nabla\varphi dx &= \int_{\{|\nabla\varphi|\leq L\}} S'_M(u_n)\Phi_n^M.\nabla\varphi dx\\ &+ \int_{\{|\nabla\varphi|>L\}} S'_M(u_n)\Phi_n^M.\nabla\varphi dx.\end{aligned} \tag{3.30}$$

For the first term of the right-hand side of (3.30), we have

$$\int_{\{|\nabla\varphi|\leq L\}} S'_M(u_n)\Phi_n^M.\nabla\varphi dx \to \int_{\{|\nabla\varphi|\leq L\}} S'_M(u)\Phi^M.\nabla\varphi dx, \text{ as } n\to+\infty. \tag{3.31}$$

Thanks $\Phi_n^M \rightharpoonup \Phi^M$ in $L^1(\Omega)$ and $\nabla\varphi S'_M(u_n)\chi_{\{|\nabla\varphi|\leq L\}} \to^* \nabla\varphi S'_M(u)\chi_{\{|\nabla\varphi|\leq L\}}$ in $L^\infty(\Omega)$. Furthermore, the second term of the right hand-side of (3.30) converges to zero for L large enough, uniformly in n. Indeed, using Hölder type inequality and the fact that $L^{p_+}(\Omega)\hookrightarrow L^{\pi_n(\cdot)}(\Omega)$, we get

$$\begin{aligned}&\left|\int_{\{|\nabla\varphi|>L\}}\Phi_n^M\nabla\varphi S'_M(u_n)dx\right|\\ \leq\ & C||S'_M||_{L^\infty(\mathbb{R})}||\Phi_n^M||_{L^{\pi'_n(\cdot)}(\Omega)}||\nabla\varphi\chi_{\{|\nabla\varphi|>L\}}||_{L^{\pi_n(\cdot)}(\Omega)}\\ \leq\ & C\big(p_-,||S'_M||_{L^\infty(\mathbb{R})},meas(\Omega)\big)||\Phi_n^M||_{L^{\pi'_n(\cdot)}(\Omega)}||\nabla\varphi||_{L^{p_+}(\Omega)}meas\big(\{|\nabla\varphi|>L\}\big).\end{aligned}$$

From (A_5), (3.4) and Proposition 2, we obtain

$$||\Phi_M^n||_{L^{\pi'_n(\cdot)}(\Omega)}<C.$$

Moreover, $\varphi \in C^\infty(\overline{\Omega})$ and $C^\infty(\overline{\Omega})$ is dense in the space $W^{1,p_+}(\Omega)$. Then, by Proposition 2 and the fact that $\lim_{L\to+\infty} meas(\{|\nabla\varphi| > L\}) = 0$, we get

$$meas\big(\{|\nabla\varphi| > L\}\big)||\Phi_n^M||_{L^{\pi'_n(\cdot)}(\Omega)}||\nabla\varphi||_{L^{p_+}(\Omega)} \to 0, \quad \text{as } L \to +\infty.$$

Hence, the second term of the right hand-side of (3.30) converges to zero, as L tends to infinity.
Thus, as $n \to +\infty$ and $L \to +\infty$ in (3.30), we deduce (3.29).
Let us consider the third term of left hand-side of (3.24), we obtain

$$\begin{aligned}
\int_\Omega |S''_M(u_n)|a(x,u_n,\nabla T_M(u_n)).\nabla T_M(u_n)\varphi dx &\leq C\int_{\{|u_n|<M\}} |S''_M(u_n)|a(x,u_n,\nabla T_M(u_n)).\nabla T_M(u_n)dx \\
&\leq C'\int_{\{M-1<|u_n|<M\}} a(x,u_n,\nabla T_M(u_n))\nabla T_M(u_n)dx \\
&+ C\int_{\{|u_n|\leq M-1\}} \underbrace{|S''_M(u_n)}_{=0}|a(x,u_n,\nabla T_M(u_n)).\nabla T_M(u_n)dx,
\end{aligned} \tag{3.32}$$

where $C = C\big(||\varphi||_{L^\infty(\Omega)}\big)$, $C' = C\big(||S''_M||_{L^\infty(\mathbb{R})}, ||\varphi||_{L^\infty(\Omega)}\big)$ and $|a(x,u_n,\nabla T_M(u_n)).\nabla T_M(u_n)$ is finite. Otherwise,

$$\int_{\{M-1<|u_n|<M\}} a(x,u_n,\nabla T_M(u_n))\nabla T_M(u_n)dx \to 0, \quad \text{as } M \to +\infty.$$

Since, thanks to Lemma 3.4, $\lim_{M\to+\infty} meas(\{M-1 < |u_n| < M\}) = 0$ and $a(x,u_n,\nabla T_M(u_n))\nabla T_M(u_n)$ is equi-integrable.
Finally, using (3.25), (3.26) (3.27), (3.28), (3.29), (3.32) and passing to the limit in (3.24), as n tends to infinty and as ε goes to 0, we obtain

$$\int_\Omega b(u)S'_M(u)\varphi dx + \int_\Omega S'_M(u)\Phi^M.\nabla\varphi dx + \lambda\int_{\partial\Omega} uS'_M(u)\varphi d\sigma = \int_\Omega fS'_M(u)\varphi dx + \int_{\partial\Omega} gS'_M(u)\varphi d\sigma. \tag{3.33}$$

For $k > 0$ fixed, $T_k(u) \in W^{1,\pi(.)}(\Omega)$ and the exponent $\pi(.)$ verify (2.1). Therefore, $C^\infty(\overline{\Omega})$ is dense in $W^{1,\pi(.)}(\Omega)$, so, we replace φ by $T_k(u)$. Now, for $M > k$, thanks to (a), we replace $\Phi^M.\nabla T_k(u)$ by $\Phi^k.\nabla T_k(u)$ in (3.33).
S'_M converges a.e. to 1 on $\mathbb{R}$, as $M \to +\infty$, then using the monotone convergence theorem in the first term of left hand-side of (3.33) and dominated convergence theorem in the other term of (3.33), we get

$$\int_\Omega \big[b(u)T_k(u) + \Phi^k.\nabla T_k(u)\big]dx + \lambda\int_{\partial\Omega} uT_k(u)d\sigma = \int_\Omega fT_k(u)dx + \int_{\partial\Omega} gT_k(u)d\sigma. \tag{3.34}$$

The relation (3.7) is equivalent to

$$\begin{aligned}\int_\Omega T_n(b(u_n))T_k(u_n)dx \quad &+ \quad \int_\Omega a(x,u_n,\nabla T_k(u_n)).\nabla T_k(u_n)dx + \int_{\partial\Omega}\lambda T_n(u_n)T_k(u_n)d\sigma\\ &+ \quad \varepsilon\int_\Omega \left[|u_n|^{p_+-2}u_nT_k(u_n) + |\nabla u_n|^{p_+-2}\nabla u_n\nabla T_k(u_n)\right]dx\\ &= \quad \int_\Omega f_nT_k(u_n)dx + \int_{\partial\Omega} g_nT_k(u_n)d\sigma.\end{aligned} \tag{3.35}$$

The sequences $\big(T_n(b(u_n))T_k(u_n)\big)_{n\in\mathbb{N}}, \big(T_n(u_n)T_k(u_n)\big)_{n\in\mathbb{N}}$ are nonnegative and converge a.e. in Ω to $b(u)T_k(u)$ and a.e. on $\partial\Omega$ to $uT_k(u)$. By Fatou's Lemma, we have

$$\liminf_{n\to+\infty}\int_\Omega T_n(b(u_n))T_k(u_n)dx \geq \int_\Omega b(u)T_k(u)dx \tag{3.36}$$

and

$$\lambda\liminf_{n\to+\infty}\int_{\partial\Omega} T_n(u_n)T_k(u_n)dx \geq \lambda\int_{\partial\Omega} uT_k(u)d\sigma. \tag{3.37}$$

Now, we consider the right hand side of (3.35). We have
$|f_nT_k(u_n)| \leq k|f| \in L^1(\Omega)$, $f_nT_k(u_n) \to fT_k(u)$ a.e. in Ω and $|g_nT_k(u_n)| \leq k|g| \in L^1(\partial\Omega)$, $g_nT_k(u_n) \to gT_k(u)$ a.e. on $\partial\Omega$. Thus, by Lebesgue dominated convergence Theorem

$$\int_\Omega f_nT_k(u_n)dx \to \int_\Omega fT_k(u)dx, \text{ as } n\to+\infty \tag{3.38}$$

and

$$\int_{\partial\Omega} g_nT_k(u_n)d\sigma \to \int_{\partial\Omega} gT_k(u)d\sigma, \text{ as } n\to+\infty. \tag{3.39}$$

Combining (3.36),(3.37), (3.38), (3.39) and using (3.35), we get

$$\begin{aligned}&\liminf_{n\to+\infty}\left(\int_\Omega f_nT_k(u_n)dx + \int_{\partial\Omega} g_nT_k(u_n)d\sigma\right) - \left(\int_\Omega b(u)T_k(u)dx + \lambda\int_{\partial\Omega} uT_k(u)d\sigma\right)\\ \geq\ &\liminf_{n\to+\infty}\left(\int_\Omega f_nT_k(u_n)dx + \int_{\partial\Omega} g_nT_k(u_n)d\sigma - \int_\Omega T_n(b(u_n))T_k(u_n)dx - \lambda\int_{\partial\Omega} T_n(u_n)T_k(u_n)d\sigma\right),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\int_\Omega fT_k(u)dx + \int_{\partial\Omega} gT_k(u)d\sigma - \left(\int_\Omega b(u)T_k(u)dx + \lambda\int_{\partial\Omega} uT_k(u)d\sigma\right)\\ \geq\ &\liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)dx + \varepsilon\int_\Omega \left[|u_n|^{p_+-2}u_nT_k(u_n) + |\nabla u_n|^{p_+-2}\nabla u_n\nabla T_k(u_n)\right]dx\\ \geq\ &\liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla T_k(u_n)\nabla T_k(u_n)dx.\end{aligned}$$

Thus, by using the relation (3.34), we obtain

$$\int_\Omega \Phi^k\nabla T_k(u)dx \geq \liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)dx. \tag{3.40}$$

(c) From [1]-Lemma 2.1, $m \mapsto a(x, u_n, h_m(\nabla T_k(u_n))).h_m(\nabla T_k(u_n))$ is increasing and converges to $a(x, u_n, \nabla T_k(u_n)).\nabla T_k(u_n)$ for m large enough. Thus, we deduce that

$$a(x, u_n, h_m(\nabla T_k(u_n))).h_m(\nabla T_k(u_n)) \leq a(x, u_n, \nabla T_k(u_n)).\nabla T_k(u_n) = \Phi_n^k.\nabla T_k(u_n).$$

Therefore, using (b) and Theorem 2.1, we get

$$\begin{aligned}
\int_\Omega \Phi^k.\nabla T_k(u)dx &\geq \liminf_{n\to+\infty} \int_\Omega \Phi_n^k.\nabla T_k(u_n)dx \\
&\geq \lim_{n\to+\infty} \int_\Omega a(x, u_n, h_m(\nabla T_k(u_n))).h_m(\nabla T_k(u_n))dx \\
&= \int_{\Omega\times\mathbb{R}^N} a(x, u, h_m(\lambda)).h_m(\lambda)d\nu_x^k(\lambda)dx. \qquad (3.41)
\end{aligned}$$

Using Lebesgue convergence Theorem in (3.41), we get for m large enough

$$\int_\Omega \Phi^k.\nabla T_k(u)dx \geq \int_{\Omega\times\mathbb{R}^N} a(x, u, \lambda).\lambda d\nu_x^k(\lambda)dx. \qquad (3.42)$$

We have

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^N} (a(x, u(x), \lambda) - a(x, u(x), \nabla T_k(u(x)))(\lambda - \nabla T_k(u(x)))d\nu_x^k(\lambda)dx \\
= &\int_{\Omega\times\mathbb{R}^N} a(x, u(x), \lambda).\lambda d\nu_x^k(\lambda)dx - \int_{\Omega\times\mathbb{R}^N} a(x, u(x), \lambda).\nabla T_k(u(x))d\nu_x^k(\lambda)dx \\
- &\int_{\Omega\times\mathbb{R}^N} a(x, u(x), \nabla T_k(u(x))).\lambda d\nu_x^k(\lambda)dx + \int_{\Omega\times\mathbb{R}^N} a(x, u(x), \nabla T_k(u(x))).\nabla T_k(u(x))d\nu_x^k(\lambda)dx \\
= &\int_{\Omega\times\mathbb{R}^N} a(x, u(x), \lambda).\lambda d\nu_x^k(\lambda)dx - \int_\Omega \left(\int_{\mathbb{R}^N} a(x, u(x), \lambda)d\nu_x^k(\lambda)\right)\nabla T_k(u(x))dx \\
- &\int_\Omega a(x, u(x), \nabla T_k(u(x))).\left(\int_{\mathbb{R}^N} \lambda d\nu_x^k\right)dx + \int_\Omega a(x, u(x), \nabla T_k(u(x))).\nabla T_k(u(x))\left(\int_{\mathbb{R}^N} d\nu_x^k\right)dx \\
= &\int_{\Omega\times\mathbb{R}^N} a(x, u(x), \lambda).\lambda d\nu_x^k(\lambda)dx - \int_\Omega \Phi^k.\nabla T_k(u(x))dx \leq 0.
\end{aligned}$$

We pass from the first equality to the second equality by using Fubini-Tonelli Theorem and from the second inequality to the third one by using (3.19), (3.20) and the fact that ν_x is probability measures on $\mathbb{R}^N$. Finally (3.42) give us the desired inequality.

(d) Using (3.23) and the strict monotonicity assumption (A_4), we deduce that

$$\big(a(x, u(x), \lambda) - a(x, u(x), \nabla T_k(u(x))\big)\big(\lambda - \nabla T_k(u(x))\big) = 0 \text{ a.e. } x \in \Omega, \quad \lambda \in \mathbb{R}^N.$$

Thus, $\lambda = \nabla T_k(u(x))$ a.e. $x \in \Omega$ with respect to the measure ν_x^k on $\mathbb{R}^N$. Therefore, the measure ν_x^k reduces to the Dirac measure $\delta_{\nabla T_k(u(x))}$. Using (3.20), we obtain

$$\Phi^k = \int_{\mathbb{R}^N} a(x, u(x), \lambda)d\nu_x^k(\lambda) = a(x, u(x), \nabla T_k(u(x))) \text{ a.e. } x \in \Omega.$$

Now, by using Theorem 2.1 (ii), we deduce that $\nabla T_k(u_n)$ converges in measure to $\nabla T_k(u)$. $\square$

Lemma 3.6. *u is an entropy solution of (1.1).*

Proof of the Lemma 3.6.
Let u_n be a weak solution of the problem (3.2). Then, by Assertion $4-(d)$, $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ converges to $\nabla T_k(u)$ in measure, thus (up to a subsequence still denoted $(\nabla T_k(u_n))_{n\in\mathbb{N}}$), $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ converges to $\nabla T_k(u)$ a.e. Ω. Moreover, we deduce from Lemma 3.4 that $\nabla T_k(u_n)$ is uniformly bounded in $L^{p_-}(\Omega)$, so, p_-−equi-integrable on Ω. Then, by using Vitali's Theorem

$$\nabla T_k(u_n)\to\nabla T_k(u) \text{ in } L^{p_-}(\Omega), \text{ which implies that } \nabla T_k(u_n)\to\nabla T_k(u) \text{ in } L^1(\Omega).$$

Furthermore, thanks to Assertion 2, $u\in\mathcal{T}^{1,\pi(.)}(\Omega)$ and it follows from Lemma 3.5 that

$$u_n\to u \text{ a.e on } \partial\Omega.$$

Therefore, $u\in\mathcal{T}_{tr}^{1,\pi(.)}(\Omega)$. Now, using Lemma 3.2, the fact that $T_n(b(u_n))\to b(u)$ a.e. in Ω and $u_n\to u$ a.e. on $\partial\Omega$, it follows from Fatou's Lemma that

$$\int_\Omega |b(u)|\leq \liminf_{n\to+\infty}\int_\Omega \big|T_n(b(u_n))\big|dx\leq ||f||_{L^1(\Omega)}+||g||_{L^1(\partial\Omega)}$$

and

$$\int_{\partial\Omega}|u|\leq\liminf_{n\to+\infty}\int_{\partial\Omega}\big|T_n(u_n)\big|dx\leq\frac{1}{\lambda}\big(||f||_{L^1(\Omega)}+||g||_{L^1(\partial\Omega)}\big).$$

Hence, $b(u)\in L^1(\Omega)$ and $u\in L^1(\partial\Omega)$.
Let $\varphi\in\mathcal{C}^\infty(\overline{\Omega})$, then we can choose $T_k(u_n-\varphi)$ as a test function in (3.3) ($C^\infty(\overline{\Omega})$ is dense in the space $W^{1,p_+}(\Omega)$ and $T_k(u_n-\varphi)\in L^\infty(\partial\Omega)$) to get

$$\begin{aligned}\int_\Omega T_n(b(u_n))T_k(u_n-\varphi)dx \quad &+ \quad \int_\Omega a(x,u_n,\nabla u_n).\nabla T_k(u_n-\varphi)dx+\int_{\partial\Omega}\lambda T_n(u_n)T_k(u_n-\varphi)d\sigma\\ &+ \quad \varepsilon\int_\Omega\big[|\nabla u_n|^{p_+-2}\nabla u_n\nabla T_k(u_n-\varphi)+|u_n|^{p_+-2}u_nT_k(u_n-\varphi)\big]dx\\ &= \quad \int_\Omega f_nT_k(u_n-\varphi)dx+\int_{\partial\Omega}g_nT_k(u_n-\varphi)d\sigma. \end{aligned}\tag{3.43}$$

For the first term of the left hand side of (3.43), we have

$$\begin{aligned}\int_\Omega T_n(b(u_n))T_k(u_n-\varphi)dx \quad &= \quad \int_\Omega\big[T_n(b(u_n))-T_n(b(\varphi))\big]T_k(u_n-\varphi)dx\\ &+ \quad \int_\Omega T_n(b(\varphi))T_k(u_n-\varphi)dx.\end{aligned}$$

By Fatou's Lemma, we infer

$$\liminf_{n\to+\infty}\int_\Omega T_n(b(u_n))T_k(u_n-\varphi)dx\geq\int_\Omega b(u)T_k(u-\varphi)dx, \tag{3.44}$$

since,

$$\big[T_n(b(u_n)) - T_n(b(\varphi))\big]T_k(u_n-\varphi) \to \big(b(u - b(\varphi)\big)T_k(u-\varphi) \ \text{ a.e.}$$

with

$$\big[T_n(b(u_n)) - T_n(b(\varphi))\big]T_k(u_n-\varphi) \geq 0$$

and

$$T_n(b(\varphi))T_k(u_n-\varphi) \to b(\varphi)T_k(u-\varphi) \ \text{ in } \ L^1(\Omega).$$

In the same manner

$$\liminf_{n\to+\infty} \lambda \int_{\partial\Omega} T_n(u_n)T_k(u_n-\varphi)d\sigma \geq \lambda \int_{\partial\Omega} uT_k(u-\varphi)d\sigma. \tag{3.45}$$

For the fourth term of the left hand side of (3.43), we prove that

$$\lim_{n\to+\infty} \varepsilon \int_\Omega \big[|\nabla u_n|^{p_+-2}\nabla u_n \nabla T_k(u_n-\varphi) + |u_n|^{p_+-2}u_n T_k(u_n-\varphi)\big]dx \geq 0 \ \text{ as } \ \varepsilon \to 0. \tag{3.46}$$

Setting $l = k + ||\varphi||_{L^\infty(\Omega)}$ we have,

$$\begin{aligned}
&\varepsilon \int_\Omega |\nabla u_n|^{p_+-2}\nabla u_n \nabla T_k(u_n-\varphi)dx = \varepsilon \int_{\{|u_n-\varphi|<k\}} |\nabla T_l(u_n)|^{p_+-2}\nabla T_l(u_n)\nabla\big(T_l(u_n)-\varphi\big)dx \\
= \ & \varepsilon \int_{\{|u_n-\varphi|<k\}} |\nabla T_l(u_n)|^{p_+}dx - \varepsilon \int_{\{|u_n-\varphi|<k\}} |\nabla T_l(u_n)|^{p_+-2}\nabla T_l(u_n)\nabla\varphi dx \\
\geq \ & -\varepsilon \int_{\{|u_n-\varphi|<k\}} |\nabla T_l(u_n)|^{p_+-2}\nabla T_l(u_n)\nabla\varphi dx.
\end{aligned} \tag{3.47}$$

Moreover, by taking $v = T_l(u_n)$ in (3.3), we deduce that

$$\varepsilon \int_\Omega \big[|\nabla u_n|^{p_+-2}\nabla u_n \nabla T_l(u_n) + |u_n|^{p_+-2}u_n T_l(u_n)\big]dx \leq l\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big),$$

which implies that

$$\varepsilon \int_\Omega |\nabla T_l(u_n)|^{p_+}dx \leq l\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).$$

Therefore, $\varepsilon\nabla T_l(u_n)$ is uniformly bounded in $L^{p_+}(\Omega)$. From, Assertion $4-(d)$, $\nabla T_l(u_n)$ converges a.e. in Ω (up to a subsequence) to $\nabla T_l(u)$. So, by Vitali's Theorem, $\varepsilon\nabla T_l(u_n)$ converges to $\varepsilon\nabla T_l(u)$ in $L^{p_+}(\Omega)$. Thus, $\varepsilon|\nabla T_l(u_n)|^{p_+-2}\nabla T_l(u_n)\chi_{\{|u_n-\varphi|<k\}}$ converges to $\varepsilon|\nabla T_l(u)|^{p_+-2}\nabla T_l(u)\chi_{\{|u-\varphi|<k\}}$ in $L^{p'_+}(\Omega)$. Using (3.47), we obtain

$$\lim_{n\to+\infty} \varepsilon \int_\Omega |\nabla u_n|^{p_+-2}\nabla u_n \nabla T_k(u_n-\varphi)dx \geq -\varepsilon \int_{\{|u-\varphi|<k\}} |\nabla T_l(u)|^{p_+-2}\nabla T_l(u)\nabla\varphi dx.$$

Therefore,

$$\lim_{n\to+\infty} \varepsilon \int_\Omega |\nabla u_n|^{p_+-2}\nabla u_n \nabla T_k(u_n-\varphi)dx \geq 0, \ \text{ as } \ \varepsilon \to 0. \tag{3.48}$$

Now, we prove that

$$\lim_{n\to+\infty} \varepsilon\int_\Omega |u_n|^{p_+-2}u_nT_k(u_n-\varphi)dx \geq 0, \text{ as } \varepsilon\to 0.$$

We have

$$\begin{aligned}\int_\Omega |u_n|^{p_+-2}u_nT_k(u_n-\varphi)dx &= \int_\Omega \Big(|u_n|^{p_+-2}u_n-|\varphi|^{p_+-2}\varphi\Big)T_k(u_n-\varphi)dx\\ &+ \int_\Omega |\varphi|^{p_+-2}\varphi T_k(u_n-\varphi)dx\\ &\geq \int_\Omega |\varphi|^{p_+-2}\varphi T_k(u_n-\varphi)dx, \end{aligned}\tag{3.49}$$

since $\big(|u_n|^{p_+-2}u_n-|\varphi|^{p_+-2}\varphi\big)T_k(u_n-\varphi)$ is nonnegative.
Furthermore, $T_k(u_n-\varphi)$ converges weakly* to $T_k(u-\varphi)$ in $L^\infty(\Omega)$ and $|\varphi|^{p_+-2}\varphi\in L^{p'_+}(\Omega)$, so

$$\lim_{n\to+\infty}\int_\Omega |\varphi|^{p_+-2}\varphi T_k(u_n-\varphi)dx = \int_\Omega |\varphi|^{p_+-2}\varphi T_k(u-\varphi)dx.\tag{3.50}$$

Combining (3.49) and (3.50), we obtain

$$\lim_{n\to+\infty} \varepsilon\int_\Omega |u_n|^{p_+-2}u_nT_k(u_n-\varphi)dx \geq 0, \text{ as } \varepsilon\to 0.\tag{3.51}$$

The relations (3.48) and (3.51) give us (3.46).
For the second term of the left hand side of (3.43), we recall that $l=k+||\varphi||_{L^\infty(\Omega)}$ and we get

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n).\nabla T_k(u_n-\varphi)dx = \int_\Omega a(x,u_n,\nabla T_l(u_n)).\nabla(T_l(u_n)-\varphi)\chi_{\{|u_n-\varphi|<k\}}dx\\ &= \int_\Omega a(x,u_n,\nabla T_l(u_n)).\nabla T_l(u_n)\chi_{\{|u_n-\varphi|<k\}}dx - \int_\Omega a(x,u_n,\nabla T_l(u_n)).\nabla\varphi\chi_{\{|u_n-\varphi|<k\}}dx.\end{aligned}\tag{3.52}$$

Moreover, $a(x,u_n,\nabla T_l(u_n)).\nabla T_l(u_n)\chi_{\{|u_n-\varphi|<k\}}$ is nonnegative and converges a.e. in Ω to $a(x,u,\nabla T_l(u))\nabla T_l(u)\chi_{\{|u-\varphi|<k\}}$. Thanks to Fatou's Lemma, we get

$$\liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla T_l(u_n)).\nabla T_l(u_n)\chi_{\{|u_n-\varphi|<k\}}dx \geq \int_\Omega a(x,u,\nabla T_l(u)).\nabla T_l(u)\chi_{\{|u-\varphi|<k\}}dx.\tag{3.53}$$

We now focus our attention on $\displaystyle\int_\Omega a(x,u_n,\nabla T_l(u_n))\nabla\varphi\chi_{\{|u_n-\varphi|<k\}}$.
Let us prove that $a(x,u_n,\nabla T_l(u_n)).\nabla\varphi\chi_{\{|u_n-\varphi|<k\}}$ is equi-integrable. Let E be a subset of Ω.

$$\begin{aligned}\int_E a(x,u_n,\nabla T_l(u_n)).\nabla\varphi\chi_{\{|u_n-\varphi|<k\}}dx &\leq \int_E |a(x,u_n,\nabla T_l(u_n))||\nabla\varphi|dx\\ &\leq \int_E \frac{1}{\pi'_n(.)}|a(x,u_n,\nabla T_l(u_n))|^{\pi'_n(.)}dx + \int_E\frac{1}{\pi_n(.)}|\nabla\varphi|^{\pi_n(.)}dx\\ &\leq C_1\int_E\big(\mathcal{M}(x)+|\nabla T_l(u_n)|^{\pi_n(.)}\big)dx + \int_E|\nabla\varphi|^{\pi_n(.)}dx.\end{aligned}$$

Moreover,

$$\begin{aligned}\int_E |\nabla\varphi|^{\pi_n(\cdot)}dx &= \int_{E\cap\{|\nabla\varphi|\le 1\}} |\nabla\varphi|^{\pi_n(\cdot)}dx + \int_{E\cap\{|\nabla\varphi|>1\}} |\nabla\varphi|^{\pi_n(\cdot)}dx \\ &\le meas(E) + \int_E |\nabla\varphi|^{p_+}dx,\end{aligned}$$

since $|\nabla\varphi|^{p_+}, \mathcal{M} \in L^1(\Omega)$ and $|\nabla T_l(u_n)|^{\pi_n(\cdot)}$ is equi-integrable $\big($using density argument for $C^\infty(\overline{\Omega})$ and (3.4)$\big)$. Then, we obtain

$$\lim_{meas(E)\to 0}\int_E a(x,u_n,\nabla T_l(u_n))\nabla\varphi\chi_{\{|u_n-\varphi|<k\}}dx = 0.$$

Furthermore,

$$a(x,u_n,\nabla T_l(u_n))\nabla\varphi\chi_{\{|u_n-\varphi|<k\}} \to a(x,u,\nabla T_l(u)).\nabla\varphi\chi_{\{|u-\varphi|<k\}} \quad \text{a.e. in} \quad \Omega.$$

By applying Vitali's Theorem, we obtain

$$\lim_{n\to+\infty}\int_\Omega a(x,u_n,\nabla T_l(u_n))\nabla\varphi\chi_{\{|u_n-\varphi|<k\}}dx = \int_\Omega a(x,u,\nabla T_l(u)).\nabla\varphi\chi_{\{|u-\varphi|<k\}}dx. \tag{3.54}$$

Using (3.52), (3.53) and (3.54) we get

$$\begin{aligned}\liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla u_n).\nabla T_k(u_n-\varphi)dx &\ge \int_\Omega a(x,u,\nabla T_l(u))\nabla(T_l(u)-\varphi)\chi_{\{|u-\varphi|<k\}}dx \\ &= \int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)dx. \qquad (3.55)\end{aligned}$$

Now, we consider the right hand side of (3.43). For the first term of the right hand side of (3.43), since $f_n \to f$ in $L^1(\Omega)$ and $T_k(u_n-\varphi) \rightharpoonup^* T_k(u-\varphi)$ in $L^\infty(\Omega)$, we have

$$\lim_{n\to+\infty}\int_\Omega f_n T_k(u_n-\varphi)dx = \int_\Omega f T_k(u-\varphi)dx. \tag{3.56}$$

For the second term of the right hand side of (3.43), by using the fact that g_n strongly converges to g in $L^1(\partial\Omega)$, we obtain

$$\lim_{n\to+\infty}\int_{\partial\Omega} g_n T_k(u_n-\varphi)d\sigma = \int_{\partial\Omega} g T_k(u-\varphi)d\sigma, \tag{3.57}$$

since

$$T_k(u_n-\varphi) \rightharpoonup^* T_k(u-\varphi) \text{ in } L^\infty(\partial\Omega), \tag{3.58}$$

because $u_n \to u$ a.e. on $\partial\Omega$.

Using (3.44), (3.45), (3.46), (3.55), (3.56), (3.57) and (3.43), we get

$$\begin{aligned}
& \int_\Omega b(u)T_k(u-\varphi)dx + \int_\Omega a(x,u,\nabla u).\nabla T_k(u-\varphi)dx + \int_{\partial\Omega} \lambda u T_k(u-\varphi)d\sigma \\
\leq\ & \liminf_{n\to+\infty} \Bigg(\int_\Omega T_n(b(u_n))T_k(u_n-\varphi)dx + \int_\Omega a(x,u_n,\nabla u_n).\nabla T_k(u_n-\varphi)dx \\
+\ & \int_{\partial\Omega} \lambda T_n(u_n)T_k(u_n-\varphi)d\sigma + \varepsilon \int_\Omega \left[|\nabla u_n|^{p_+-2}\nabla u_n \nabla T_k(u_n-\varphi) + |u_n|^{p_+-2}u_n T_k(u_n-\varphi) \right] dx \Bigg) \\
=\ & \int_\Omega f T_k(u-\varphi)dx + \int_{\partial\Omega} g T_k(u-\varphi)d\sigma, \text{ as } \varepsilon \to 0,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\int_\Omega b(u)T_k(u-\varphi)dx + \int_\Omega a(x,u,\nabla u).\nabla T_k(u-\varphi)dx \quad + & \int_{\partial\Omega} \lambda u T_k(u-\varphi)d\sigma \\
\leq & \int_\Omega f T_k(u-\varphi)dx + \int_{\partial\Omega} g T_k(u-\varphi)d\sigma,
\end{aligned} \tag{3.59}$$

for $\varphi \in C^\infty(\overline{\Omega})$.

As $\pi(.)$ verifies the log-Hölder condition (2.1), $C^\infty(\overline{\Omega})$ is dense in the space $W^{1,\pi(.)}(\Omega)$. Moreover, $W^{1,\pi(.)}(\Omega) \hookrightarrow W^{1,p_-}(\Omega) \hookrightarrow L^\infty(\Omega)$, since $\pi(.) \geq p_- > N$ and Ω is a bounded open domain with Lipschitz boundary $\partial\Omega$. Therefore, the inequality (3.59) holds true for $\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$. Hence, u is an entropy solution of (1.1). $\square$

Now, we state the uniqueness result of entropy solution. This result uses the same arguments as [2]-Theorem 2.8.

Theorem 3.5. *Assume that b is strictly increasing. Assume that $a = a(x,z,\eta)$ satisfies $(A_3)-(A_6)$ and $\mathcal{M}$ constant. Moreover, a satisfies:*
for all bounded subset K of $\mathbb{R}\times\mathbb{R}^N$, there exists a constant $C(K)$ such that

$$\begin{aligned}
& a.e.\ x \in \Omega, \ \text{ for all } (z,\eta),(\tilde{z},\eta) \in K, \\
& |a(x,z,\eta) - a(x,\tilde{z},\eta)| \leq C(K)|z-\tilde{z}|.
\end{aligned} \tag{3.60}$$

Finally, suppose the following regularity property:

$$\begin{aligned}
& \text{for all } \ f \in L^\infty(\Omega) \ \text{ and } \ g \in L^\infty(\partial\Omega) \\
& \text{there exists an entropy solution of (1.1)}, \\
& \text{which is Lipchitz continuous on } \overline{\Omega}.
\end{aligned} \tag{3.61}$$

Then, for all $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$ the problem (1.1) admits a unique entropy solution.

Remark 3.6. *As in [2, Theorem* 2.8*], the condition (3.61) goes back to idea of [3]. Moreover, in the Theorem 3.5 the relation* (3.60) *is used to obtain the inequality* (3.69) *below.*

Proof. The proof of this theorem is done in two steps.

Step 1. A priori estimates.

Lemma 3.7. *If v is an entropy solution of (1.1), there exists a positive constant C such that*

$$\rho_{p(.,v(.))}\big(|\nabla v|\chi_F\big) \leq Ck,$$

where $F = \{h-k < |v| < h\}$, $h > k > 0$.

Proof. Let $\varphi = T_{h-k}(v)$ as test function in the entropy inequality (3.1), we get

$$\begin{aligned}\int_\Omega a(x,v,\nabla v).\nabla T_k(v - T_{h-k}(v))dx \quad &+ \quad \int_\Omega b(v)T_k(v-T_{h-k}(v))dx + \lambda\int_{\partial\Omega} vT_k(v-T_{h-k}(v))d\sigma \\ &\leq \quad \int_\Omega fT_k(v-T_{h-k}(v))dx + \int_{\partial\Omega} gT_k(v-T_{h-k}(v))d\sigma.\end{aligned}$$

Thus,

$$\int_{\{h-k<|v|<h\}} a(x,v,\nabla v).\nabla T_k(v-T_{h-k}(v))dx \leq k\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big)$$

and using (A_6), we have

$$\int_F |\nabla v|^{p(x,v(x))}dx \leq kC_2\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).$$

Consequently,

$$\rho_{p(.,v(.))}\big(|\nabla v|\chi_F\big) \leq Ck, \quad \text{where} \quad C = C_2\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big). \qquad \square$$

We give the following lemma.

Lemma 3.8. *If u is an entropy solution of (1.1), then*

$$meas\big(\{|u| > h\}\big) \leq \frac{||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}}{\min\big(b(h), |b(-h)|\big)}, \quad \forall h \geq 1.$$

Proof. Let us take $\varphi = 0$ and $k = h$ in entropy inequality (3.1).
Since

$$\int_\Omega a(x,u,\nabla u).\nabla T_h(u)dx + \lambda\int_{\partial\Omega} uT_h(u)d\sigma \geq 0,$$

the relation (3.1) gives

$$\int_\Omega b(u)T_h(u)dx \leq \int_\Omega fT_h(u)dx + \int_{\partial\Omega} gT_h(u)d\sigma.$$

Then,

$$\int_{\{|u|\leq h\}} b(u)T_h(u)dx + \int_{\{|u|>h\}} b(u)T_h(u)dx \leq h\big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big),$$

or

$$\int_{\{|u|>h\}} \frac{b(u)T_h(u)}{h}dx = \int_{\{u>h\}} b(u)dx + \int_{\{u<-h\}} -b(u)dx \leq \big(||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}\big).$$

Therefore,

$$\int_{\{|u|>h\}} |b(u)|dx \leq ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}.$$

Since b is nondecreasing, we deduce

$$\int_{\{|u|>h\}} \min(b(h), |b(-h)|)dx \leq \int_{\{|u|>h\}} |b(u)| \leq ||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}, \;\; \forall h \geq 1.$$

So,

$$meas\big(\{|u| > h\}\big) \leq \frac{||f||_{L^1(\Omega)} + ||g||_{L^1(\partial\Omega)}}{\min\big(b(h), |b(-h)|\big)}, \;\; \forall h \geq 1. \qquad \square$$

Step 2. Uniqueness.
The existence has already been proved. Now, we show the uniqueness. For more details see [2]-Proof of Theorem 2.8.
Let u be a Lipschitz continuous entropy solution of (1.1) with $f \in L^\infty(\Omega), g \in L^\infty(\partial\Omega)$ and v be an entropy solution, with $\hat{f} \in L^1(\Omega), \hat{g} \in L^1(\partial\Omega)$.
Since Ω is open bounded domain with smooth boundary $\partial\Omega$, the space of Lipschitz functions $C^{0,1}(\overline{\Omega})$ and $W^{1,\infty}(\Omega)$ are homeomorphic and they can be identified. Therefore, u belongs to $W^{1,\infty}(\Omega)$. Thus, for all $h > 0$, we can write the entropy inequality corresponding to the solution u, with $T_h(v)$ as a test function and to the solution v, with $T_h(u)$ as a test function. For all $k > 0$, we get

$$\begin{cases} \displaystyle\int_\Omega a(x, u, \nabla u).\nabla T_k(u - T_h(v))dx + \int_\Omega b(u)T_k(u - T_h(v))dx \\ \displaystyle +\lambda \int_{\partial\Omega} uT_k(u - T_h(v))d\sigma \leq \int_\Omega fT_k(u - T_h(v))dx + \int_{\partial\Omega} gT_k(u - T_h(v))d\sigma \end{cases} \tag{3.62}$$

and

$$\begin{cases} \displaystyle\int_\Omega a(x, v, \nabla v).\nabla T_k(v - T_h(u))dx + \int_\Omega b(v)T_k(v - T_h(u))dx \\ \displaystyle +\lambda \int_{\partial\Omega} vT_k(v - T_h(u))d\sigma \leq \int_\Omega \hat{f}T_k(v - T_h(u))dx + \int_{\partial\Omega} \hat{g}T_k(v - T_h(u))d\sigma. \end{cases} \tag{3.63}$$

Adding (3.62) and (3.63) we obtain

$$\begin{cases} \displaystyle\int_\Omega a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx + \int_\Omega a(x,v,\nabla v).\nabla T_k(v-T_h(u))dx \\ \displaystyle + \int_\Omega b(u)T_k(u-T_h(v))dx + \int_\Omega b(v)T_k(v-T_h(u))dx \\ \displaystyle +\lambda\int_{\partial\Omega} uT_k(u-T_h(v))d\sigma + \lambda\int_{\partial\Omega} vT_k(v-T_h(u))d\sigma \\ \displaystyle \leq \int_\Omega \big[fT_k(u-T_h(v)) + \hat{f}T_k(v-T_h(u))\big]dx \\ \displaystyle + \int_{\partial\Omega}\big[gT_k(u-T_h(v)) + \hat{g}T_k(v-T_h(u))\big]d\sigma. \end{cases} \tag{3.64}$$

Set $A=\{0<|u-v|<k, |v|\leq h\}$; $B=A\cap\{|u|\leq h\}$; $C=A\cap\{|u|>h\}$ and $A'=\{0<|v-u|<k, |u|\leq h\}$; $B'=A'\cap\{|v|\leq h\}$; $C'=A'\cap\{|v|>h\}$. We start with the first integral in (3.64). We have

$$\begin{aligned} & \int_{\{0<|u-T_h(v)|<k\}} a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx \\ = & \int_{\{0<|u-T_h(v)|<k\}\cap\{|v|\leq h\}} a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx \\ + & \int_{\{0<|u-T_h(v)|<k\}\cap\{|v|>h\}} a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx \\ = & \int_{\{0<|u-v|<k\}\cap\{|v|\leq h\}} a(x,u,\nabla u).\nabla(u-v)dx \\ + & \int_{\{0<|u-hsign(v)|<k\}\cap\{|v|>h\}} a(x,u,\nabla u).\nabla u dx \\ \geq & \int_A a(x,u,\nabla u)\nabla(u-v)dx \\ = & \int_B a(x,u,\nabla u)\nabla(u-v)dx + \int_C a(x,u,\nabla u)\nabla(u-v)dx. \end{aligned}$$

Then, we get

$$\begin{cases} \displaystyle\int_{\{0<|u-T_h(v)|<k\}} a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx \\ \displaystyle\geq \int_B a(x,u,\nabla u)\nabla(u-v)dx - \int_C a(x,u,\nabla u)\nabla v dx. \end{cases} \tag{3.65}$$

Now we use the fact that ∇u is bounded. By assumption of the theorem ($\mathcal{M}$ is constant), $|a(x,u,\nabla u)|\leq C(|\nabla u|^{p(x,u(x))}+1)\in L^\infty(\Omega)$. Therefore, there exists a constant K such that

$$\begin{aligned} \left|\int_C a(x,u,\nabla u)\nabla v dx\right| & \leq \int_C |a(x,u,\nabla u)||\nabla v|dx \\ & \leq K\int_C |\nabla v|dx \leq K\int_{\{h-k<|v|<h\}} |\nabla v|dx, \end{aligned} \tag{3.66}$$

since $C\subset\{h-k<|v|<h\}$.

Thanks to Lemma 3.8, $\displaystyle\lim_{h\to+\infty} meas(\{h-k<|v|<h\})=0$ and by Lemma 3.7, $|\nabla v|\chi_F\in L^1(\Omega)$.

So, the right hand side of (3.66) converges to zero, as h goes to infinity.
Consequently, the second integral of the right hand side of (3.65) converges to zero, as h goes to infinity. Then, we can write that

$$\begin{aligned}\int_{\{0<|u-T_h(v)|<k\}} a(x,u,\nabla u).\nabla T_k(u-T_h(v))dx &\geq \int_B a(x,u,\nabla u).\nabla(u-v)dx \\ &+ I_h, \text{ with } \lim_{h\to+\infty} I_h = 0.\end{aligned}$$

As $B = B'$, we may adopt the same procedure to treat the second integral of (3.64) to obtain

$$\begin{aligned}\int_{\{0<|v-T_h(u)|<k\}} a(x,v,\nabla v).\nabla T_k(v-T_h(u))dx &\geq -\int_B a(x,v,\nabla v).\nabla(u-v)dx \\ &+ J_h, \text{ with } \lim_{h\to+\infty} J_h = 0.\end{aligned}$$

For the other terms in the left hand side of (3.64), we denote by

$$K_h = \int_\Omega b(u)T_k(u-T_h(v))dx + \int_\Omega b(v)T_k(v-T_h(u))dx$$

and

$$L_h = \lambda\int_{\partial\Omega} uT_k(u-T_h(v))d\sigma + \lambda\int_{\partial\Omega} vT_k(v-T_h(u))d\sigma.$$

We have

$$b(u)T_k(u-T_h(v)) \to b(u)T_k(u-v) \text{ a.e. in } \Omega \text{ as } h\to+\infty$$

and

$$|b(u)T_k(u-T_h(v))| \leq k|b(u)| \in L^1(\Omega).$$

Then, by Lebesgue dominated convergence Theorem, we get

$$\lim_{h\to+\infty}\int_\Omega b(u)T_k(u-T_h(v))dx = \int_\Omega b(u)T_k(u-v)dx$$

and

$$\lim_{h\to+\infty}\int_\Omega b(v)T_k(v-T_h(u))dx = \int_\Omega b(v)T_k(v-u)dx.$$

Then,

$$\lim_{h\to+\infty} K_h = \int_\Omega (b(u)-b(v))T_k(u-v)dx.$$

Similarly, we obtain

$$\lim_{h\to+\infty} L_h = \lambda\int_{\partial\Omega}(u-v)T_k(u-v)d\sigma.$$

Now, we consider the right hand side of (3.64), we have

$$\lim_{h\to+\infty}\left[fT_k(u-T_h(v)) + \hat{f}T_k(v-T_h(u))\right] = (f-\hat{f})T_k(u-v) \text{ a.e. in } \Omega$$

and

$$\left|fT_k(u-T_h(v))+\hat{f}T_k(v-T_h(u))\right|\le k(|f|+|\hat{f}|)\in L^1(\Omega).$$

By Lebesgue dominated convergence Theorem, we get

$$\lim_{h\to+\infty}\int_\Omega f\big[T_k(u-T_h(v))+T_k(v-T_h(u))\big]dx=\int_\Omega \big(f-\hat{f}\big)T_k(u-v)dx.$$

Similarly, we have

$$\lim_{h\to+\infty}\int_{\partial\Omega} g\big[T_k(u-T_h(v))+\hat{g}T_k(v-T_h(u))\big]d\sigma=\int_{\partial\Omega}\big(g-\hat{g}\big)T_k(u-v)d\sigma.$$

After passing to the limit as h goes to $+\infty$ in (3.64), we get

$$\begin{cases}\displaystyle\int_{\{0<|u-v|<k\}}\big(a(x,u,\nabla u)-a(x,v,\nabla v)\big)\nabla(u-v)dx\\ \displaystyle+\int_\Omega(b(u)-b(v))T_k(u-v)dx+\lambda\int_{\partial\Omega}(u-v)T_k(u-v)d\sigma\\ \displaystyle\le\int_\Omega\big(f-\hat{f}\big)T_k(u-v)dx+\int_{\partial\Omega}\big(g-\hat{g}\big)T_k(u-v)d\sigma,\end{cases}\tag{3.67}$$

which is equivalent to

$$\begin{cases}\displaystyle\int_{\{0<|u-v|<k\}}\big(a(x,u,\nabla v)-a(x,v,\nabla v)\big)\nabla(u-v)dx\\ \displaystyle+\int_{\{0<|u-v|<k\}}\big(a(x,u,\nabla u)-a(x,u,\nabla v)\big)\nabla(u-v)dx\\ \displaystyle+\int_\Omega(b(u)-b(v))T_k(u-v)dx+\lambda\int_{\partial\Omega}(u-v)T_k(u-v)d\sigma\\ \displaystyle\le\int_\Omega\big(f-\hat{f}\big)T_k(u-v)dx+\int_{\partial\Omega}\big(g-\hat{g}\big)T_k(u-v)d\sigma.\end{cases}\tag{3.68}$$

Dividing (3.68) by k and letting k goes to 0, we have

$$\begin{cases}\displaystyle\lim_{k\to0}\frac{1}{k}\int_{\{0<|u-v|<k\}}\big(a(x,u,\nabla v)-a(x,v,\nabla v)\big)\nabla(u-v)dx\\ \displaystyle+\lim_{k\to0}\frac{1}{k}\int_{\{0<|u-v|<k\}}\big(a(x,u,\nabla u)-a(x,u,\nabla v)\big)\nabla(u-v)dx\\ \displaystyle+\int_\Omega|b(u)-b(v)|dx+\lambda\int_{\partial\Omega}|u-v|d\sigma\\ \displaystyle\le\int_\Omega\big(f-\hat{f}\big)sign(u-v)dx+\int_{\partial\Omega}\big(g-\hat{g}\big)sign(u-v)d\sigma.\end{cases}\tag{3.69}$$

Thanks to the relation (3.60), the first integral of (3.69) goes to 0 as $k\to0$ (See [2], proof of Theorem 2.8-Step 2). Thus, we obtain

$$\begin{cases}\displaystyle\lim_{k\to0}\frac{1}{k}\int_{\{0<|u-v|<k\}}\big(a(x,u,\nabla u)-a(x,u,\nabla v)\big)\nabla(u-v)dx\\ \displaystyle+\int_\Omega|b(u)-b(v)|dx+\lambda\int_{\partial\Omega}|u-v|d\sigma\\ \displaystyle\le\int_\Omega|f-\hat{f}|dx+\int_{\partial\Omega}\big|g-\hat{g}\big|d\sigma.\end{cases}\tag{3.70}$$

Since, the three integral of left-hand in (3.70) are positive, we deduce that

$$\int_\Omega |b(u)-b(v)|dx + \lambda\int_{\partial\Omega}|u-v|d\sigma \le \int_\Omega |f-\hat{f}|dx + \int_{\partial\Omega}|g-\hat{g}|d\sigma. \tag{3.71}$$

Let us take a sequence $(f_i)_{i\in\mathbb{N}} \subset L^\infty(\Omega)$ and $(g_i)_{i\in\mathbb{N}} \subset L^\infty(\partial\Omega)$ and let $(u_i)_{i\in\mathbb{N}}$ be the corresponding sequence of Lipschitz continuous entropy solutions. By (3.71), we have

$$\begin{aligned}\int_\Omega |b(u)-b(v)|dx &+ \lambda\int_{\partial\Omega}|u-v|d\sigma \le \int_\Omega \big[|b(u)-b(u_i)|+|b(v)-b(u_i)|\big]dx \\ &+ \lambda\int_{\partial\Omega}\big[|u-u_i|+|v-u_i|\big]d\sigma \\ &\le \int_\Omega \big[|f-f_i|+|\hat{f}-f_i|\big]dx + \int_{\partial\Omega}\big[|g-g_i|+|\hat{g}-g_i|\big]d\sigma, \end{aligned} \tag{3.72}$$

so that at the limit as $i\to\infty$ in left hand-side of (3.72) and using the density argument ($L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$ are dense (respectively) in $L^1(\Omega)$ and $L^1(\partial\Omega)$), we infer that

$$b(u)=b(v) \text{ a.e. in } \Omega \text{ and } u=v \text{ a.e. on } \partial\Omega.$$

Hence,

$$u=v \text{ a.e. in } \Omega \text{ and } u=v \text{ a.e. on } \partial\Omega.$$

Since b is assumed strictly increasing. $\square$

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On Katugampola fractional order derivatives and Darboux problem for differential equations

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ABSTRACT

In this paper, we investigate the existence and uniqueness of solutions for the Darboux problem of partial differential equations with Caputo-Katugampola fractional derivative.

RESUMEN

En este artículo investigamos la existencia y unicidad de soluciones para el problema de Darboux de ecuaciones diferenciales parciales con derivada fraccional de Caputo-Katugampola.

Keywords and Phrases: Darboux problem, Fractional differential equations, Caputo-Katugampola derivative.

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1 Introduction

To investigate many different fields of science and engineering, the fractional calculus represents a powerful tool, with many applications in mathematical physics, hydrology, finance, astrophysics, thermodynamics, statistical mechanics, biophysics, control theory, cosmology, bioengineering and so on, [5, 6].
In recent years, there has been an important works in ordinary and partial fractional differential equations. For the Caputo fractional-order ordinary differential equations case, see Kilbas et al. [7], Miller and Ross [8]. In addition, Yunru Bai and Hua Kong have treated the existence of solution for nonlinear Caputo-Hadamard fractional differential equations in [9]. For the Caputo fractional-order partial differential equations case, see the work of Tian Liang Guo and KanJian Zhang in [10]. Furthermore, Xianmin Zhang has investigated the Caputo-Hadamard partial fractional differential equations in [11]. The choice of an appropriate fractional derivative (or integral) depends on the considered system, and for this reason there are a large number of works devoted to different fractional operators.

Recently, U. Katugampola presented new types of fractional operators, which generalize both the Riemann-Liouville and Hadamard fractional operators [4]. Although the Katugampola fractional integral operator is an Erdélyi-Kober type operator [13] author in [14] argued that is not possible to obtain Hadamard equivalence operators from Erdélyi-Kober type operators. In this sense, Almeida, Malinowska and Odzijewicz [2] introduced a new fractional operator, called the Caputo-Katugampola derivative, which generalizes the concept of Caputo and Caputo-Hadamard fractional derivatives. It turns out that, the new operator is the left inverse of the Katugampola fractional integral and keeps some of the fundamental properties of the Caputo and Caputo-Hadamard fractional derivatives. Such derivative is the generalization of the Caputo and Caputo-Hadamard fractional derivative. The existence and uniqueness of the solution of the ordinary Caputo-Katugampola differential equations is given in [3]. A. Cernea in [12] studied a Darboux problem associated to a fractional hyperbolic integro-differential inclusion defined by Caputo-Katugampola fractional derivative and several existence results for this problem are proved.

In this paper, we study the existence and uniqueness of solutions of the following partial differential equation with Caputo-Katugampola fractional derivative

$$ {}^{C}D_{a^+}^{\alpha;\rho} u\left(x,y\right) = f\left(x,y,u\left(x,y\right)\right), \left(x,y\right) \in J = \left[a_1,b_1\right] \times \left[a_2,b_2\right], \tag{1.1} $$

$$ \begin{aligned} &u\left(x,a_2\right) = \varphi\left(x\right), \, x \in \left[a_1,b_1\right], \\ &u\left(a_1,y\right) = \psi\left(y\right), \, \, y \in \left[a_2,b_2\right], \\ &\varphi\left(a_1\right) = \psi\left(a_2\right), \end{aligned} \tag{1.2} $$

where $f : J \times \mathbb{R} \to \mathbb{R}$, $\varphi : [a_1, b_1] \to \mathbb{R}$ and $\psi : [a_2, b_2] \to \mathbb{R}$ are given continuous functions.

The rest of the paper is organized as follows. Some definitions and preliminaries are presented in Sect. 2. Finally, the existence and uniqueness results, is given in Sect. 3.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Definition 1. *[2, 3, 4] Given $\alpha > 0$, $\rho > 0$ and an interval $[a, b]$ of R, where $0 < a < b$. The Katugampola fractional integral of a function $u \in L^1([a, b])$ is defined by*

$$I_{a^+}^{\alpha,\rho} u(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1} u(s)}{(t^\rho - s^\rho)^{1-\alpha}} ds,$$

where Γ is the Gamma function.

Definition 2. *[2, 3, 4] Given $\alpha > 0$, $\rho > 0$ and an interval $[a, b]$ of R, where $0 < a < b$. The Katugampola fractional derivative is defined by*

$$D_{a^+}^{\alpha,\rho} u(t) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_a^t \frac{s^{\rho-1} u(s)}{(t^\rho - s^\rho)^{\alpha}} ds.$$

Definition 3. *[2, 3, 4] Given $0 < \alpha < 1$, $\rho > 0$ and an interval $[a, b]$ of R, where $0 < a < b$. The Caputo-Katugampola fractional derivative is defined by*

$$\begin{aligned} {}^{C}D_{a^+}^{\alpha,\rho} u(t) =& D_{a^+}^{\alpha,\rho} [u(t) - u(a)] \\ =& \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_a^t \frac{s^{\rho-1} [u(s) - u(a)]}{(t^\rho - s^\rho)^{\alpha}} ds. \end{aligned}$$

Definition 4. *Let $0 < a_i < b_i$, $i = 1, 2$ reals numbers, $a = (a_1, a_2)$ and $u : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be an integrable function. The mixed Katugampola fractional integrals of order $\alpha = (\alpha_1, \alpha_2)$, and parameter $\rho = (\rho_1, \rho_2)$ is defined by*

$$I_{a^+}^{\alpha,\rho} u(x, y) = \frac{\rho_1^{1-\alpha_1} \rho_2^{1-\alpha_2}}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_{a_1^+}^{x} \int_{a_2^+}^{y} \frac{s^{\rho_1-1} t^{\rho_2-1}}{(x^{\rho_1} - s^{\rho_1})^{1-\alpha_1} (y^{\rho_2} - t^{\rho_2})^{1-\alpha_2}} u(s, t)\, dt ds.$$

where $\alpha_1, \alpha_2, \rho_1$ and ρ_2 are strictly positives.

Definition 5. *Let $0 < a_i < b_i$, $i = 1, 2$ reals numbers, $a = (a_1, a_2)$ and $u : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be a function. The mixed Katugampola fractional derivative of order $\alpha = (\alpha_1, \alpha_2)$, and parameter*

$\rho = (\rho_1, \rho_2)$ is defined by

$$\begin{aligned} D^{\alpha,\rho}_{a^+} u(x,y) &= x^{1-\rho_1} y^{1-\rho_2} D^2_{x,y} I^{1-\alpha,\rho}_{a^+} u(x,y) \\ &= \frac{x^{1-\rho_1} y^{1-\rho_2} \rho_1^{\alpha_1} \rho_2^{\alpha_2}}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} D^2_{x,y} \int_{a_1^+}^{x} \int_{a_2^+}^{y} \frac{s^{\rho_1 - 1} t^{\rho_2 - 1}}{(x^{\rho_1} - s^{\rho_1})^{\alpha_1} (y^{\rho_2} - t^{\rho_2})^{\alpha_2}} \\ &\quad \times u(s,t)\, dtds. \end{aligned}$$

Where $(\alpha_1, \alpha_2) \in (0,1)^2$, $D^2_{x,y} = \frac{\partial^2}{\partial x \partial y}$ and ρ_1, ρ_2 are strictly positives.

Definition 6. *Let $0 < a_i < b_i$, $i = 1, 2$ reals numbers, $a = (a_1, a_2)$ and $u : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be a function. The mixed Caputo-Katugampola fractional derivative of order $\alpha = (\alpha_1, \alpha_2)$, and parameter $\rho = (\rho_1, \rho_2)$ is defined by*

$$^{C}D^{\alpha,\rho}_{a^+} u(x,y) = D^{\alpha,\rho}_{a^+} \left(u(x,y) - u(x,a_2) - u(a_1,y) + u(a_1,a_2)\right)$$

where $(\alpha_1, \alpha_2) \in (0,1)^2$ and ρ_1, ρ_2 are strictly positives.

Lemma 2.1. *Let $0 < a_i < b_i$, $i = 1, 2$ reals numbers, $a = (a_1, a_2)$ and $u : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ is an absolutely continuous function. The mixed Caputo-Katugampola fractional derivative of order $\alpha = (\alpha_1, \alpha_2)$, and parameter $\rho = (\rho_1, \rho_2)$ is given by*

$$\begin{aligned} ^{C}D^{\alpha,\rho}_{a^+} u(x,y) &= I^{1-\alpha,\rho}_{a^+} \left(x^{1-\rho_1} y^{1-\rho_2} D^2_{x,y} u(x,y)\right) \\ &= \frac{\rho_1^{\alpha_1} \rho_2^{\alpha_2}}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} \int_{a_1^+}^{x} \int_{a_2^+}^{y} \frac{D^2_{s,t} u(s,t)}{(x^{\rho_1} - s^{\rho_1})^{\alpha_1} (y^{\rho_2} - t^{\rho_2})^{\alpha_2}} dtds \end{aligned}$$

almost everywhere, where $(\alpha_1, \alpha_2) \in (0,1)^2$, $D^2_{s,t} = \frac{\partial^2}{\partial s \partial t}$ and ρ_1, ρ_2 are strictly positives.

Lemma 2.2. *Let $0 < a_i < b_i$, $i = 1, 2$ reals numbers, $a = (a_1, a_2)$ and $u : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be an integrable function. Then*

$$I^{\alpha,\rho}_{a^+} I^{\beta,\rho}_{a^+} u(x,y) = I^{\alpha+\beta,\rho}_{a^+} u(x,y) \tag{2.1}$$

almost everywhere, where $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ and parameter $\rho = (\rho_1, \rho_2)$. If additionally u is a continuous function, then the identity (2.1) holds everywhere.

Proof. Using Fubini's Theorem we get

$$
\begin{aligned}
I_{a^+}^{\alpha,\rho} I_{a^+}^{\beta,\rho} u(x,y) &= \frac{\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{a_1^+}^{x}\int_{a_2^+}^{y} \frac{s_1^{\rho_1-1} s_2^{\rho_2-1} I_{a^+}^{\beta,\rho} u(s_1,s_2)}{\left(x^{\rho_1}-s_1^{\rho_1}\right)^{1-\alpha_1}\left(y^{\rho_2}-s_2^{\rho_2}\right)^{1-\alpha_2}} ds_2 ds_1 \\
&= \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \frac{\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{a_1^+}^{x}\int_{a_2^+}^{y} \frac{s_1^{\rho_1-1} s_2^{\rho_2-1}}{\left(x^{\rho_1}-s_1^{\rho_1}\right)^{1-\alpha_1}\left(y^{\rho_2}-s_2^{\rho_2}\right)^{1-\alpha_2}} \times \\
&\quad \int_{a_1^+}^{s_1}\int_{a_2^+}^{s_2} \frac{t_1^{\rho_1-1} t_2^{\rho_2-1}}{\left(s_1^{\rho_1}-t_1^{\rho_1}\right)^{1-\beta_1}\left(s_2^{\rho_2}-t_2^{\rho_2}\right)^{1-\beta_2}} u(t_1,t_2)\, dt_2 dt_1 ds_2 ds_1 \\
&= \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \frac{\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{a_1^+}^{x}\int_{a_2^+}^{y} t_1^{\rho_1-1} t_2^{\rho_2-1} u(t_1,t_2) \times \qquad (2.2) \\
&\quad \int_{t_1}^{x}\int_{t_2}^{y} \frac{s_1^{\rho_1-1} s_2^{\rho_2-1} ds_2 ds_1 dt_2 dt_1}{\left(x^{\rho_1}-s_1^{\rho_1}\right)^{1-\alpha_1}\left(y^{\rho_2}-s_2^{\rho_2}\right)^{1-\alpha_2}\left(s_1^{\rho_1}-t_1^{\rho_1}\right)^{1-\beta_1}\left(s_2^{\rho_2}-t_2^{\rho_2}\right)^{1-\beta_2}}.
\end{aligned}
$$

Using the change of variables

$$
x = \frac{\left(s_1^{\rho_1}-t_1^{\rho_1}\right)^{1-\beta_1}}{\left(x^{\rho_1}-t_1^{\rho_1}\right)^{1-\alpha_1}} \ \texttt{and}\ y = \frac{\left(s_2^{\rho_2}-t_2^{\rho_2}\right)^{1-\beta_2}}{\left(y^{\rho_2}-t_2^{\rho_2}\right)^{1-\alpha_2}},
$$

we get

$$
\begin{aligned}
&\int_{t_1}^{x}\int_{t_2}^{y} \frac{s_1^{\rho_1-1} s_2^{\rho_2-1}}{\left(x^{\rho_1}-s_1^{\rho_1}\right)^{1-\alpha_1}\left(y^{\rho_2}-s_2^{\rho_2}\right)^{1-\alpha_2}} \frac{1}{\left(s_1^{\rho_1}-t_1^{\rho_1}\right)^{1-\beta_1}\left(s_2^{\rho_2}-t_2^{\rho_2}\right)^{1-\beta_2}} ds_2 ds_1 \\
&= \int_{t_1}^{x} \frac{s_1^{\rho_1-1}}{\left(x^{\rho_1}-s_1^{\rho_1}\right)^{1-\alpha_1}\left(s_1^{\rho_1}-t_1^{\rho_1}\right)^{1-\beta_1}} ds_1 \times \int_{t_2}^{y} \frac{s_2^{\rho_2-1}}{\left(y^{\rho_2}-s_2^{\rho_2}\right)^{1-\alpha_2}\left(s_2^{\rho_2}-t_2^{\rho_2}\right)^{1-\beta_2}} ds_2 \\
&= \frac{\left(x^{\rho_1}-t_1^{\rho_1}\right)}{\rho_1}\frac{\left(y^{\rho_2}-t_2^{\rho_2}\right)}{\rho_2} \int_0^1 (1-x)^{\alpha_1-1} x^{\beta_1} dx \int_0^1 (1-y)^{\alpha_1-1} y^{\beta_1} dy \\
&= \frac{\left(x^{\rho_1}-t_1^{\rho_1}\right)}{\rho_1}\frac{\left(y^{\rho_2}-t_2^{\rho_2}\right)}{\rho_2} B(\alpha_1,\beta_1)\, B(\alpha_2,\beta_2) \\
&= \frac{\left(x^{\rho_1}-t_1^{\rho_1}\right)}{\rho_1}\frac{\left(y^{\rho_2}-t_2^{\rho_2}\right)}{\rho_2} \frac{\Gamma(\alpha_1)\Gamma(\beta_1)}{\Gamma(\alpha_1+\beta_1)} \frac{\Gamma(\alpha_2)\Gamma(\beta_2)}{\Gamma(\alpha_2+\beta_2)}. \qquad (2.3)
\end{aligned}
$$

From (2.2) and (2.3) we obtain (2.1). ∎

Lemma 2.3. *Let $0 < a_i < b_i$, $i = 1, 2$ reals numbers, $a = (a_1, a_2)$ and $u : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be an integrable function. Then*

$$
D_{a^+}^{\alpha,\rho} I_{a^+}^{\alpha,\rho} u(x,y) = u(x,y)
$$

almost everywhere, where $\alpha = (\alpha_1, \alpha_2) \in (0,1)^2$ and parameter $\rho = (\rho_1, \rho_2)$.

Proof. From Lemma (2.2), we get

$$
\begin{aligned}
D_{a^+}^{\alpha,\rho} I_{a^+}^{\alpha,\rho} u(x,y) &= x^{1-\rho_1} y^{1-\rho_2} D_{x,y}^2 I_{a^+}^{1-\alpha,\rho} I_{a^+}^{\alpha,\rho} u(x,y) \\
&= x^{1-\rho_1} y^{1-\rho_2} D_{x,y}^2 I_{a^+}^{1,\rho} u(x,y) \\
&= u(x,y).
\end{aligned}
$$

□

3 Existence and uniqueness results

For the existence and uniqueness of solutions for the problem (1.1)-(1.2) we need the following lemma.

Lemma 3.1. *The function $u \in C(J)$ is a solution of fractional order problem (1.1)-(1.2) if and only if*

$$u(x,y) = \varphi(x) + \psi(y) - \varphi(a_1) + I_{a^+}^{\alpha;\rho} f(x,y,u(x,y)). \tag{3.1}$$

Proof. First suppose that u is a solution of the integral equation (3.1). Applied ${}^{C}D_{a^+}^{\alpha;\rho}$ and using Lemma 2.3 we obtain that u solves the the equation (1.1). Since the integral is zero when $x = a_1$, or $y = a_2$, then the initial conditions in (1.2) are satisfied. Hence u solves the problem (1.1)-(1.2). Conversely, if u is a solution of the problem (1.1)-(1.2). Let

$$\begin{aligned} h(x,y) &= f(x,y,u(x,y)) \\ &= D_{a^+}^{\alpha,\rho}(u(x,y) - u(x,a_2) - u(a_1,y) + u(a_1,a_2)) \\ &= x^{1-\rho_1} y^{1-\rho_2} D_{x,y}^2 I_{a^+}^{1-\alpha,\rho}[u(x,y) - u(x,a_2) - u(a_1,y) + u(a_1,a_2)]. \end{aligned} \tag{3.2}$$

Applying the operator $I_{a^+}^{1,\rho}$ to (3.2), we get

$$I_{a^+}^{1,\rho} h(x,y) = I_{a^+}^{1-\alpha,\rho}[u(x,y) - u(x,a_2) - u(a_1,y) + u(a_1,a_2)].$$

Applying the operator $D_{a^+}^{1-\alpha,\rho}$ to this equation we find

$$\begin{aligned} [u(x,y) - u(x,a_2) - u(a_1,y) + u(a_1,a_2)] &= D_{a^+}^{1-\alpha,\rho} I_{a^+}^{1,\rho} h(x,y) \\ &= \left(x^{1-\rho_1} y^{1-\rho_2}\right) D_{x,y}^2 I_{a^+}^{\alpha,\rho} I_{a^+}^{1,\rho} h(x,y) \\ &= I_{a^+}^{\alpha,\rho} h(x,y). \end{aligned}$$

Hence, the proof is complete. □

3.1 Existence of solutions

In this subsection we study the existence of solutions for the problem (1.1)-(1.2).

Theorem 3.1. *Let $k > 0, h_1^* > a_1$ and $h_2^* > a_2$.*
Define

$$G = \{(x,y,u) : (x,y) \in [a_1, h_1^*] \times [a_2, h_2^*], |u - \varphi(x) - \psi(y) + \varphi(a_1)| \le k\},$$

$$M = \sup_{(x,y,u)\in G} |f(x,y,u)|$$

and

$$(h_1,h_2)=\begin{cases}(h_1^*,h_2^*) & \textit{if } M=0,\\ \left(\min\left(h_1^*,\left(\frac{k^{\frac{1}{2}}\rho_1^{\alpha_1}\Gamma(\alpha_1+1)}{M^{\frac{1}{2}}}\right)^{\frac{1}{\alpha_1}}\right),\min\left(h_2^*,\left(\frac{k^{\frac{1}{2}}\rho_2^{\alpha_2}\Gamma(\alpha_2+1)}{M^{\frac{1}{2}}}\right)^{\frac{1}{\alpha_2}}\right)\right) & \textit{otherwise.}\end{cases}$$

Then, there exists a function $u\in C\left[a_1,h_1\right]\times\left[a_2,h_2\right]$ that solves the problem (1.1)-(1.2).

Proof. If $M=0$ then $f(x,y,u)=0$, for all $(x,y,u)\in G$. In this case it is clear that the function $u:[a_1,h_1]\times[a_2,h_2]\to\mathbb{R}$ with $u(x,y)=\varphi(x)+\psi(y)-\varphi(a_1)$ is a solution of the problem (1.1)-(1.2).

For $M\neq 0$, using Lemma 3.1 we obtain that the problem (1.1)-(1.2) is equivalent to the Volterra integral equation (3.1).

Define the function T by

$$T(x,y)=\varphi(x)+\psi(y)-\varphi(a_1). \tag{3.3}$$

and the set U by

$$U=\left\{u\in C\left([a_1,h_1]\times[a_2,h_2]\right),\left\|u-T\right\|_\infty\leq k\right\}. \tag{3.4}$$

The set U is nonempty since $T\in U$. It is clear that U is a closed and convex subset of the Banach space of all continuous functions on $[a_1,h_1]\times[a_2,h_2]$.

We define the operator A on this set U by

$$(Au)(x,y)=T(x,y)+\frac{\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{a_1^+}^{x}\int_{a_2^+}^{y}\frac{s^{\rho_1-1}t^{\rho_2-1}f(s,t,u(s,t))}{\left(x^{\rho_1}-s^{\rho_1}\right)^{1-\alpha_1}\left(y^{\rho_2}-t^{\rho_2}\right)^{1-\alpha_2}}dtds. \tag{3.5}$$

We have to show that A has a fixed point. This is done through the Schauder's Fixed Point Theorem.

It is easy to see that A is continuous. Now we show that A is defined to U into itself, let $u\in U$ and $(x,y)\in[a_1,h_1]\times[a_2,h_2]$ then

$$\begin{aligned}\left|(Au)(x,y)-T(x,y)\right| &= \frac{\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{a_1^+}^{x}\int_{a_2^+}^{y}\frac{s^{\rho_1-1}t^{\rho_2-1}\left|f(s,t,u(s,t))\right|}{\left(x^{\rho_1}-s^{\rho_1}\right)^{1-\alpha_1}\left(y^{\rho_2}-t^{\rho_2}\right)^{1-\alpha_2}}dtds\\
&\leq \frac{M\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{a_1^+}^{x}\int_{a_2^+}^{y}\frac{s^{\rho_1-1}t^{\rho_2-1}}{\left(x^{\rho_1}-s^{\rho_1}\right)^{1-\alpha_1}\left(y^{\rho_2}-t^{\rho_2}\right)^{1-\alpha_2}}dtds\\
&\leq \frac{M}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\left(\frac{x^{\rho_1}-a_1^{\rho_1}}{\rho_1}\right)^{\alpha_1}\left(\frac{y^{\rho_2}-a_2^{\rho_2}}{\rho_2}\right)^{\alpha_2}\\
&\leq \frac{M}{\rho_1^{\alpha_1}\rho_2^{\alpha_2}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}h_1^{\rho_1\alpha_1}h_2^{\rho_2\alpha_2}\\
&\leq \frac{M}{\rho_1^{\alpha_1}\rho_2^{\alpha_2}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}h_1^{\alpha_1}h_2^{\alpha_2}\\
&\leq \frac{M}{\rho_1^{\alpha_1}\rho_2^{\alpha_2}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\frac{k\rho_1^{\alpha_1}\rho_2^{\alpha_2}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{M}\\
&\leq k.\end{aligned}$$

Thus, we have $Au \in U$ if $u \in U$. We will now show that $AU = \{Au : u \in U\}$ is relatively compact. This is done by the using Arzela-Ascoli Theorem.
Firstly, we show that $A(U)$ is uniformly bounded. Indeed, let $u \in U$ and $(x, y) \in [a_1, h_1] \times [a_2, h_2]$ and from the previous step we get

$$\|Au\|_\infty \le \|T\|_\infty + k.$$

Secondly, we show that $A(U)$ is equicontinuous. Indeed, let $(x_1, y_1) \in [a_1, h_1] \times [a_2, h_2], (x_2, y_2) \in [a_1, h_1] \times [a_2, h_2]$ such that $x_1 < x_2$ and $y_1 < y_2$, we have

$$\begin{aligned}
& \left|(Au)(x_1,y_1) - (Au)(x_2,y_2)\right| \\
\le\ & \left|T(x_1,y_1) - T(x_2,y_2)\right| + \frac{M\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{a_1^+}^{x_1}\int_{a_2^+}^{y_1} \frac{s^{\rho_1-1}t^{\rho_2-1}}{\left(x_1^{\rho_1} - s^{\rho_1}\right)^{1-\alpha_1}\left(y_1^{\rho_2} - t^{\rho_2}\right)^{1-\alpha_2}} \\
& - \frac{s^{\rho_1-1}t^{\rho_2-1}}{\left(x_2^{\rho_1} - s^{\rho_1}\right)^{1-\alpha_1}\left(y_2^{\rho_2} - t^{\rho_2}\right)^{1-\alpha_2}} dtds \\
& + \frac{M\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{a_1^+}^{x_1}\int_{y_1}^{y_2} \frac{s^{\rho_1-1}t^{\rho_2-1}}{\left(x_2^{\rho_1} - s^{\rho_1}\right)^{1-\alpha_1}\left(y_2^{\rho_2} - t^{\rho_2}\right)^{1-\alpha_2}} dtds \\
& + \frac{M\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{x_2}\int_{a_2^+}^{y_1} \frac{s^{\rho_1-1}t^{\rho_2-1}}{\left(x_2^{\rho_1} - s^{\rho_1}\right)^{1-\alpha_1}\left(y_2^{\rho_2} - t^{\rho_2}\right)^{1-\alpha_2}} dtds \\
& + \frac{M\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} \frac{s^{\rho_1-1}t^{\rho_2-1}}{\left(x_2^{\rho_1} - s^{\rho_1}\right)^{1-\alpha_1}\left(y_2^{\rho_2} - t^{\rho_2}\right)^{1-\alpha_2}} dtds \\
\le\ & \left|T(x_1,y_1) - T(x_2,y_2)\right| \\
& + \frac{3M}{\rho_1^{\alpha_1}\rho_2^{\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)} \left[\left(x_2^{\rho_1} - a_1^{\rho_1}\right)^{\alpha_1}\left(y_2^{\rho_2} - y_1^{\rho_2}\right)^{\alpha_2} + \left(y_2^{\rho_2} - a_2^{\rho_2}\right)^{\alpha_2}\left(x_2^{\rho_1} - x_1^{\rho_1}\right)^{\alpha_1}\right]
\end{aligned}$$

Hence, $A(U)$ is equicontinous, since T is uniformly continuous in $[a_1, h_1] \times [a_2, h_2]$. As a consequence of the Schauder's Fixed Point Theorem, we deduce that A has a fixed point u in U. This fixed point is the required solution of the problem (1.1)-(1.2). Hence, the proof is complete. ☐

3.2 Uniqueness of solutions

In this subsection we discuss the uniqueness results for the problem (1.1)-(1.2).
Let $u_1, u_2 \in C([a_1, h_1] \times [a_2, h_2])$, and $(x, y) \in [a_1, h_1] \times [a_2, h_2]$.
Suppose there exists a constant $L > 0$ independent of x, y, u_1, and u_2 such that

$$\left|f(x,y,u_1) - f(x,y,u_2)\right| \le L\left|u_1 - u_2\right|, \tag{3.6}$$

then we have

$$\left\|(Au_1) - (Au_2)\right\|_{C([a_1,x]\times[a_2,y])} \le \frac{L\left\|u_1 - u_2\right\|_{C([a_1,x]\times[a_2,y])}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} \left(\frac{x^{\rho_1}}{\rho_1}\right)^{\alpha_1}\left(\frac{y^{\rho_2}}{\rho_2}\right)^{\alpha_2}. \tag{3.7}$$

Indeed, let $u_1, u_2 \in C([a_1,h_1]\times[a_2,h_2])$, $(x,y)\in[a_1,h_1]\times[a_2,h_2]$ and $(v,w)\in[a_1,x]\times[a_2,y]$, we have

$$\begin{aligned}
& \left|(Au_1)(v,w)-(Au_2)(v,w)\right| \\
= & \frac{\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{a_1^+}^{v}\int_{a_2^+}^{w}\frac{s^{\rho_1-1}t^{\rho_2-1}\left|f(s,t,u_1(s,t))-f(s,t,u_2(s,t))\right|}{\left(v^{\rho_1}-s^{\rho_1}\right)^{1-\alpha_1}\left(w^{\rho_2}-t^{\rho_2}\right)^{1-\alpha_2}}dtds \\
\leq & \frac{L\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{a_1^+}^{v}\int_{a_2^+}^{w}\frac{s^{\rho_1-1}t^{\rho_2-1}}{\left(v^{\rho_1}-s^{\rho_1}\right)^{1-\alpha_1}\left(w^{\rho_2}-t^{\rho_2}\right)^{1-\alpha_2}}\left|u_1(s,t)-u_2(s,t)\right|dtds \\
\leq & \frac{L\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\left\|u_1-u_2\right\|_{C([a_1,x]\times[a_2,y])}\int_{a_1^+}^{v}\int_{a_2^+}^{w}\frac{s^{\rho_1-1}t^{\rho_2-1}}{\left(v^{\rho_1}-s^{\rho_1}\right)^{1-\alpha_1}\left(w^{\rho_2}-t^{\rho_2}\right)^{1-\alpha_2}}dtds \\
\leq & \frac{L}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\left\|u_1-u_2\right\|_{C([a_1,x]\times[a_2,y])}\left(\frac{v^{\rho_1}}{\rho_1}\right)^{\alpha_1}\left(\frac{w^{\rho_2}}{\rho_2}\right)^{\alpha_2} \\
\leq & \frac{L}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\left\|u_1-u_2\right\|_{C([a_1,x]\times[a_2,y])}\left(\frac{x^{\rho_1}}{\rho_1}\right)^{\alpha_1}\left(\frac{y^{\rho_2}}{\rho_2}\right)^{\alpha_2}.
\end{aligned}$$

From the above inequality we get (3.7).

$$\left\|(Au_1)-(Au_2)\right\|_{C([a_1,x]\times[a_2,y])}\leq\frac{L\left\|u_1-u_2\right\|_{C([a_1,x]\times[a_2,y])}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\left(\frac{x^{\rho_1}}{\rho_1}\right)^{\alpha_1}\left(\frac{y^{\rho_2}}{\rho_2}\right)^{\alpha_2}.$$

Next, we have the following result

Theorem 3.2. *Suppose that the assumptions of Theorem 3.1 are satisfied. Also let $j\in\mathbb{N},(x,y)\in[a_1,h_1]\times[a_2,h_2]$ and $u_1,u_2\in U$. Suppose f satisfies the Lipschitz condition with respect to the third variable with the Lipschitz constant L. Then*

$$\left\|A^j u_1-A^j u_2\right\|_{C([a_1,x]\times[a_2,y])}\leq\frac{\left(\frac{x^{\rho_1}}{\rho_1}\right)^{\alpha_1 j}\left(\frac{y^{\rho_2}}{\rho_2}\right)^{\alpha_2 j}}{\Gamma(1+\alpha_1 j)\Gamma(1+\alpha_2 j)}\left\|u_1-u_2\right\|_{C([a_1,x]\times[a_2,y])}. \tag{3.8}$$

Proof. We will prove (3.8) by induction. In the case $j=0$, the inequality holds. Assume (3.8) is true for $j-1 \in \mathbb{N}_0$ then for all $(x,y) \in [a_1,h_1] \times [a_2,h_2]$ and $(v,w) \in [a_1,x] \times [a_2,y]$ we have

$$\begin{aligned}
& \left|\left(A^j u_1\right)(v,w) - \left(A^j u_2\right)(v,w)\right| \\
= & \left|\left(AA^{j-1} u_1\right)(v,w) - \left(AA^{j-1} u_2\right)(v,w)\right| \\
= & \frac{\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{a_1^+}^{v}\int_{a_2^+}^{w} \frac{s^{\rho_1-1}t^{\rho_2-1}\left|f\left(s,t,A^{j-1}u_1(s,t)\right) - f\left(s,t,A^{j-1}u_2(s,t)\right)\right|}{\left(v^{\rho_1}-s^{\rho_1}\right)^{1-\alpha_1}\left(w^{\rho_2}-t^{\rho_2}\right)^{1-\alpha_2}} dtds \\
\leq & \frac{L\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{a_1^+}^{v}\int_{a_2^+}^{w} \frac{s^{\rho_1-1}t^{\rho_2-1}\left|A^{j-1}u_1(s,t) - A^{j-1}u_2(s,t)\right|}{\left(v^{\rho_1}-s^{\rho_1}\right)^{1-\alpha_1}\left(w^{\rho_2}-t^{\rho_2}\right)^{1-\alpha_2}} dtds \\
\leq & \frac{L\rho_1^{1-\alpha_1}\rho_2^{1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{a_1^+}^{v}\int_{a_2^+}^{w} \frac{s^{\rho_1-1}t^{\rho_2-1}\left\|A^{j-1}u_1 - A^{j-1}u_2\right\|_{C([a_1,s]\times[a_2,t])}}{\left(v^{\rho_1}-s^{\rho_1}\right)^{1-\alpha_1}\left(w^{\rho_2}-t^{\rho_2}\right)^{1-\alpha_2}} dtds \\
\leq & \frac{L^j\rho_1^{1-\alpha_1 j}\rho_2^{1-\alpha_2 j}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(1+\alpha_1(j-1))\Gamma(1+\alpha_2(j-1))}\left\|u_1-u_2\right\|_{C([a_1,x]\times[a_2,y])} \\
& \int_{a_1^+}^{v}\int_{a_2^+}^{w} \frac{s^{\rho_1+\alpha_1\rho_1(j-1)-1}t^{\rho_2+\alpha_2\rho_2(j-1)-1}}{\left(v^{\rho_1}-s^{\rho_1}\right)^{1-\alpha_1}\left(w^{\rho_2}-t^{\rho_2}\right)^{1-\alpha_2}} dtds \\
\leq & \frac{L^j\rho_1^{1-\alpha_1 j}\rho_2^{1-\alpha_2 j}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(1+\alpha_1(j-1))\Gamma(1+\alpha_2(j-1))}\left\|u_1-u_2\right\|_{C([a_1,x]\times[a_2,y])} \\
& \times\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(1+\alpha_1(j-1))\Gamma(1+\alpha_2(j-1))}{\Gamma(1+\alpha_1 j)\Gamma(1+\alpha_2 j)}\frac{x^{\rho_1\alpha_1 j}}{\rho_1}\frac{y^{\rho_2\alpha_2 j}}{\rho_2} \\
\leq & \frac{\left(\frac{x^{\rho_1}}{\rho_1}\right)^{\alpha_1 j}\left(\frac{y^{\rho_2}}{\rho_2}\right)^{\alpha_2 j}}{\Gamma(1+\alpha_1 j)\Gamma(1+\alpha_2 j)}\left\|u_1-u_2\right\|_{C([a_1,x]\times[a_2,y])}.
\end{aligned}$$

Hence, the proof is complete. □

Theorem 3.3. *Let k, h_1^* and h_1^* are positive numbers, define the set G as in Theorem 3.1 and assume that the function $f : G \to \mathbb{R}$ satisfies a Lipschitz condition with respect to the third variable with the Lipschitz constant L. Then, there exists a unique solution $u \in C([a_1,h_1]\times[a_2,h_2])$ for the problem (1.1)-(1.2). Where h_1, h_2 are the same as in Theorem 3.1.*

Proof. According to Theorem 3.1, the problem (1.1)-(1.2) has a solution. To prove the uniqueness, we adopt Theorem 3.2, we use the operato A as defined in (3.5), the function T as defined in (3.3) and the set U as defined in (3.4). We will apply Weissinger's Fixed Point Theorem to prove that A has a unique fixed point.
Let $j \in \mathbb{N}$ and $u_1, u_2 \in C([a_1,h_1]\times[a_2,h_2])$. From (3.8) and taking the norms on $[a_1,h_1]\times[a_2,h_2]$, we get

$$\left\|A^{j-1}u_1 - A^{j-1}u_2\right\|_{C([a_1,h_1]\times[a_2,h_2])} \leq \frac{\left(\frac{x^{\rho_1}}{\rho_1}\right)^{\alpha_1 j}\left(\frac{y^{\rho_2}}{\rho_2}\right)^{\alpha_2 j}}{\Gamma(1+\alpha_1 j)\Gamma(1+\alpha_2 j)}\left\|u_1-u_2\right\|_{C([a_1,h_1]\times[a_2,h_2])}.$$

Let $\omega_j = \frac{\left(\frac{x^{\rho_1}}{\rho_1}\right)^{\alpha_1 j}\left(\frac{y^{\rho_2}}{\rho_2}\right)^{\alpha_2 j}}{\Gamma(1+\alpha_1 j)\Gamma(1+\alpha_2 j)}$. It is clear that

$$\sum_{j=0}^{\infty}\omega_j = \sum_{j=0}^{\infty}\frac{\left(\left(\frac{x^{\rho_1}}{\rho_1}\right)^{\alpha_1}\left(\frac{y^{\rho_2}}{\rho_2}\right)^{\alpha_2}\right)^j}{\Gamma\left(1+\alpha_1 j\right)\Gamma\left(1+\alpha_2 j\right)} = \mathbb{E}\left((\alpha_i, 1)_{i=1,2}\,;(\left(\frac{x^{\rho_1}}{\rho_1}\right)^{\alpha_1}\left(\frac{y^{\rho_2}}{\rho_2}\right)^{\alpha_2})\right),$$

hence the series converges. This completes the proof. □

4 Conclusion

Here we have studied the existence and uniqueness of the solutions for the Darboux problem of partial differential equations with Caputo-Katugampola fractional derivative.

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Level sets regularization with application to optimization problems

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ABSTRACT

Given a coupling function c and a non empty subset of $\mathbb{R}$, we define a closure operator. We are interested in extended real-valued functions whose sub-level sets are closed for this operator. Since this class of functions is closed under pointwise suprema, we introduce a regularization for extended real-valued functions. By decomposition of the closure operator using polarity scheme, we recover the regularization by bi-conjugation. We apply our results to derive a strong duality for a minimization problem.

RESUMEN

Dada una función de acoplamiento c y un subconjunto no vacío de $\mathbb{R}$, definimos un operador clausura. Estamos interesados en funciones extendidas a valores reales cuyos conjuntos de sub-nivel son cerrados para este operador. Dado que esta clase de funciones es cerrada bajo supremos puntuales, introducimos una regularización para funciones extendidas a valores reales. Gracias a la descomposición del operador clausura usando el esquema de polaridad, recuperamos la regularización por bi-conjugación. Aplicamos nuestros resultados para derivar una dualidad fuerte para un problema de minimización.

Keywords and Phrases: Duality, regularization, level sets, c-elementary functions, polarity, conjugacy.

2010 AMS Mathematics Subject Classification: 49N15.

1 Introduction

Regularization and conjugation of extended real-valued functions play an important role in optimization theory since it is a base of duality theory. Until the Fenchel's work([4]), many authors have introduced and studied different kinds of regularization and conjugation among which we can cite Moreau ([8]), Crouzeix ([1]), Rockafellar ([13]), Martínez-Legaz ([7]), Singer ([15]), Penot-Volle ([11]), Volle ([16, 17]). In [16], M. Volle used a dual pair of polarities to introduce and study level set regularization and conjugacy.

In this work, we introduce and study level set regularization and conjugacy by means of a coupling function and a nonvoid subset of the real numbers. The outline of the paper is as follows. In Section 2, we recall Moreau conjugation scheme. Section 3 is devoted to the study of the $\Gamma-$regularization of extended real-valued functions and hull of sets. We introduce these notions and give some properties (Proposition 2, 4 and Theorem 3.8). In Section 4, we introduce the level set regularization of extended real-valued functions. By decomposition of a closure operator via a couple of dual polarities, we show that this regularization coincides with the bi-conjugation relative to the polarity couple (Proposition 8 and Theorem 4.6). We derive an analytic expression of level set regularization of extended real-valued functions (Proposition 10). Section 5 is devoted to an application of our theory to a minimization problem. A perturbational dual of this problem is defined and a necessary and sufficient condition is given to ensure a strong duality property for this problem (Theorem 5.1, Corollary 5.2 and Corollary 5.3).

2 Preliminaries

Let us start this section by recalling the Moreau conjugation ([8]). Let U, V two nonvoid sets and $c : U \times V \to \mathbb{R}$, a coupling function. Given an extended real-valued function $h : U \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, we define the $c-$conjugate of h by $h^c(v) := \sup\limits_{u\in U}\{c(u,v) - h(u)\}$, for any $v \in V$. By exchanging the role of U and V, we define the $c-$conjugate of a given function $k : V \to \overline{\mathbb{R}}$ by $k^c(u) := \sup\limits_{v\in V}\{c(u,v) - k(v)\}$ for any $u \in U$. The $c-$bi-conjugate of a given $h : U \to \overline{\mathbb{R}}$ is then defined by $h^{cc}(u) := \sup\limits_{v\in V}\{c(u,v) - h^c(v)\}$ for any $u \in U$.

Example 2.1. *The usual Legendre-Fenchel conjugacy is obtained by taking $U := X$, a topological vector space with topological dual $V := X^*$ and c the standard bilinear coupling function.*

Example 2.2. *Other examples of coupling functions have been considered in the literature among which, we can cite:*

(1) $U = V = \{x \in \mathbb{R}^n | x_1 > 0, \ldots, x_n > 0\}$ *and* $c(u,v) = \min\limits_{1\leq i\leq n} v_i u_i$, *([14]),*

(2) (U,d) *a metric space,* $\alpha > 0$*,* $V = U$ *and* $c(u,v) = -\alpha d(u,v)$*, ([6]),*

(3) (U,d) *a metric space,* $V = U\times]0,+\infty[$ *and* $c(u,(v,\alpha)) = -\alpha d(u,v)$*, ([6]),*

(4) $U = V = \mathbb{R}^n$*,* $0 < \alpha \leq 1$*,* $\beta > 0$ *and* $c(u,v) = -\beta\|u - v\|^\alpha$*, ([7]),*

(5) U *a topological space,* $V = \mathscr{C}(U,\mathbb{R})$*, space of countinuous real functions on* U *and* $c(u,v) = v(u)$*, ([5],[10]).*

Given a function $h : U \to \overline{\mathbb{R}}$, the following notation and definitions will be needed: dom $h = \{u \in U \mid h(u) < +\infty\}$, the effective domain of h, $[h \leq t] := \{u \in U \mid h(u) \leq t\}$, the t-sub-level set of h(or level set of h in short).

Given a subset A of U, we define its indicator function i_A by $i_A(u) = 0$ if $u \in A$ and $i_A(u) = +\infty$ if $u \in U \setminus A$. Following the terminology introduced in [9] we will also use the valley function v_A of A defined by $v_A(u) = -\infty$ if $u \in A$ and $v_A(u) = +\infty$ if $u \in U \setminus A$.

3 $\Gamma-$regularization of functions and hull of sets

3.1 $\Gamma-$regularization of functions

The notion of continuous affine functions can be generalized by those of $c-$elementary functions. In this work, we call $c-$elementary function on U (resp. on V), the function of the form $c(.,v) - r$ (resp. $c(u,.) - r$) with $v \in V$ (resp. $u \in U$) and $r \in \mathbb{R}$. The upper hull (i.e., the supremum) of a family of $c-$elementary functions is called c-regular. We denote by $\Gamma_c(U)$, the set of $c-$regular functions defined on U. We call c-hull or Γ_c-regularization of $h : U \to \overline{\mathbb{R}}$, the greatest $c-$regular minorant of h. This function is denoted by h^{Γ_c}. It is well known ([8]) that

$$h^{cc} = h^{\Gamma_c}, \text{ for each } h : U \to \overline{\mathbb{R}}. \tag{3.1}$$

Remark 3.1. *The equality* (3.1) *is still valid if the coupling function is an extended real-valued function. In this case, one must interpret the conjugate* h^c *as follows*

$$h^c(v) = -\inf_{u\in U}\{h(u) - c(u,v)\},$$

with the usual conventions $(+\infty) - (+\infty) = (-\infty) - (-\infty) = +\infty$.

There exists an equivalent approach to generalized convex duality in terms of Φ -convexity [2], which consists of working with a set U and a class of functions $\Phi \subset \overline{\mathbb{R}}^U$.

3.2 Hull of sets

Let $\mathbb{P}$ be a nonvoid subset of $\mathbb{R}$. The following definition generalizes the notion of half space.

Definition 1. *We call $(c,\mathbb{P})-$elementary subset of U any subset of U of the form $\{u \in U \mid r - c(u,v) \in \mathbb{P}\}$, where $(v,r) \in V \times \mathbb{R}$. We note it by $E^{\mathbb{P}}_{v,r}$.*

Note that, if $\mathbb{P} = \mathbb{R}$, then $E^{\mathbb{P}}_{v,r} = U$ for any $(v,r) \in V \times \mathbb{R}$. In this case, the only $(c,\mathbb{P})-$elementary subset of U is U itself. The $(c,\mathbb{P})-$elementary subsets of U allow us to define a notion of hull of a subset A of U.

Definition 2. *The $(c,\mathbb{P})-$hull of $A \subset U$ is the intersection of all $(c,\mathbb{P})-$elementary subsets of U containing A. The $(c,\mathbb{P})-$hull of A is denoted by $\langle A\rangle_{c,\mathbb{P}}$.*

Remark 3.2. *If there is not $(c,\mathbb{P})-$elementary subset of U containing A, then $\langle A\rangle_{c,\mathbb{P}} = U$ by convention.*

Proposition 1. *If $\mathbb{P} \neq \mathbb{R}$, then $\langle \emptyset\rangle_{c,\mathbb{P}} = \emptyset$.*

Proof Let $s \in \mathbb{R} \setminus \mathbb{P}$. Assume $\langle \emptyset\rangle_{c,\mathbb{P}} \neq \emptyset$. Let $a \in \langle \emptyset\rangle_{c,\mathbb{P}}$. Then $r - c(a,v) \in \mathbb{P}$ for any $(v,r) \in V \times \mathbb{R}$. In particular $s = (s + c(a,v)) - c(a,v) \in \mathbb{P}$, absurd. □

It follows from the definition of $\langle . \rangle_{c,\mathbb{P}}$ that, for each $A \subset U$, and for each $u \in U$, one has

$$u \notin \langle A\rangle_{c,\mathbb{P}} \Longleftrightarrow \exists (v,r) \in V \times \mathbb{R} \ : \ A \subset E^{\mathbb{P}}_{v,r} \ \text{ and } r - c(u,v) \notin \mathbb{P}. \tag{3.2}$$

By definition 1, one has $A \subset \langle A\rangle_{c,\mathbb{P}}$, for any $A \subset U$. Moreover, if $A \subset B$ then $\langle A\rangle_{c,\mathbb{P}} \subset \langle B\rangle_{c,\mathbb{P}}$. Therefore, $\langle\langle A\rangle_{c,\mathbb{P}}\rangle_{c,\mathbb{P}} = \langle A\rangle_{c,\mathbb{P}}$, $\forall A \subset U$. We deduce that $\langle . \rangle_{c,\mathbb{P}}$ is an algebraic closure operator.

Definition 3. *A subset A of U is said to be $(c,\mathbb{P})-$regular if $A = \langle A\rangle_{c,\mathbb{P}}$. We denote $\mathcal{R}_{c,\mathbb{P}}(U)$, the set of all $(c,\mathbb{P})$-regular subsets of U.*

Observe that $(c,\mathbb{P})-$elementary sets are $(c,\mathbb{P})-$regular. More generally, any intersection of $(c,\mathbb{P})-$regular subsets is $(c,\mathbb{P})-$regular and the $(c,\mathbb{P})-$regular hull of $A \subset U$ coincides with the intersection of all $(c,\mathbb{P})-$regular subsets of U containing A.

In what follows, we will use the following values for $\mathbb{P}$: $\mathbb{P}_1 = \mathbb{R}_+ := [0,+\infty[$, $\mathbb{P}_2 = \mathbb{R}^*_+ :=]0,+\infty[$, $\mathbb{P}_3 = \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $\mathbb{P}_4 = \{0\}$. For $i = 1,2,3,4$, $\langle . \rangle_{c,i} := \langle . \rangle_{c,\mathbb{P}_i}$ for short. For $i = 1,2,3$, the set $\langle A\rangle_{c,i}$ can be explained as follows:

Proposition 2. *For any $A \subset U$, one has:*

$$\langle A\rangle_{c,1} = \{u \in U \mid c(u,v) \leq \sup_{a \in A} c(a,v), \ \forall v \in V\}, \tag{3.3}$$

$$\langle A\rangle_{c,2} = \{u \in U \mid \forall v \in V, \exists a \in A \mid c(u,v) \leq c(a,v)\}, \tag{3.4}$$

$$\langle A\rangle_{c,3} = \{u \in U \mid \forall v \in V, \exists a \in A \mid c(u,v) = c(a,v)\}. \tag{3.5}$$

Proof By (3.2), one has:

$$\begin{aligned} a \notin \langle A\rangle_{c,1} &\iff \exists (v,r) \in V\times\mathbb{R} : A \subset [c(.,v) \le r] \text{ and } r < c(a,v) \\ &\iff \exists (v,r) \in V\times\mathbb{R} : \sup_{u\in A} c(u,v) \le r < c(a,v) \\ &\iff \exists v \in V : \sup_{u\in A} c(u,v) < c(a,v). \end{aligned}$$

Thus, $a \in \langle A\rangle_{c,1} \iff \forall v \in V,\ c(a,v) \le \sup_{u\in A} c(u,v)$, and (3.3) holds.

$$\begin{aligned} a \notin \langle A\rangle_{c,2} &\iff \exists (v,r) \in V\times\mathbb{R} : A \subset [c(.,v) < r] \text{ and } r \le c(a,v) \\ &\iff \exists (v,r) \in V\times\mathbb{R} : c(u,v) < r \le c(a,v),\ \forall u \in A \\ &\iff \exists v \in V :\ c(u,v) < c(a,v),\ \forall u \in A. \end{aligned}$$

Thus, $a \in \langle A\rangle_{c,2} \iff \forall v \in V, \exists u \in A :\ c(a,v) \le c(u,v)$, and (3.4) holds.

$$\begin{aligned} a \notin \langle A\rangle_{c,3} &\iff \exists (v,r) \in V\times\mathbb{R} : A \subset [c(.,v) \ne r] \text{ and } r = c(a,v) \\ &\iff \exists v \in V :\ c(u,v) \ne c(a,v),\ \forall u \in A. \end{aligned}$$

Thus $a \in \langle A\rangle_{c,3} \iff \forall v \in V,\ \exists u \in A :\ c(a,v) = c(u,v)$, and (3.5) holds. □

Remark 3.3. *Observe that one cannot remove the real parameter r in the definition of $\langle A\rangle_{c,4}$.*

Example 3.4. *We observe the situation in topological vector case. Assume U is a topological vector space with topological dual V and c the standard coupling function. The $c-$elementary functions are affine continuous functions, and we have:*

1. *$(c,\mathbb{P}_1)-$elementary sets are $\emptyset$, U and closed half spaces. Moreover, if U is locally convex, then by Hahn-Banach separation theorem and (3.3), $\langle A\rangle_{c,1} = \overline{\text{conv}}A$, the closed convex hull of A.*
2. *$(c,\mathbb{P}_2)-$elementary sets are $\emptyset$, U and half open spaces. The $(c,\mathbb{P}_2)-$hull of a subset of U is its evenly convex hull ([4],[7],...).*
3. *$(c,\mathbb{P}_3)-$elementary sets are $\emptyset$, U and complementary set of closed hyperplane. The $(c,\mathbb{P}_3)-$hull of a subset of U is its evenly co-affine hull ([15]). Observe that ([15], corollary 2.2) A is evenly convex if and only if A is evenly co-affine and convex.*
4. *$(c,\mathbb{P}_4)-$elementary sets are $\emptyset$, U and closed hyperplane. Moreover, if U is locally convex, then by the Hahn-Banach separation theorem and (3.2), the $(c,\mathbb{P}_4)-$hull of a non empty subset of U is its closed affine hull.*

Proposition 3. *Let P and Q be two nonvoid subsets of $\mathbb{R}$. Assume that any (c,P)-elementary set is (c,Q)-regular. Then, $\langle A\rangle_{c,Q} \subset \langle A\rangle_{c,P}, \forall A \subset U$.*

Proof Let $a \notin \langle A\rangle_{c,P}$. By definition, there exists an (c, P)-elementary set E such that $A \subset E$ and $a \notin E$. Since E is also (c, Q)-regular, it follows from (3.2) that $a \notin \langle A\rangle_{c,Q}$, and we are done. $\square$

Corollary 3.5. *For any $A \subset U$, one has:*

$$\langle A\rangle_{c,1} \supset \langle A\rangle_{c,2} \supset \langle A\rangle_{c,3} \quad \text{and } \langle A\rangle_{c,4} \supset \langle A\rangle_{c,3}.$$

Proof Let $v \in V$ and $r \in \mathbb{R}$, it is obvious that

$$\{u \in U \mid c(u, v) \leq r\} = \bigcap_{s>r} \{u \in U \mid c(u, v) < s\}.$$

Consequently, any $(c, \mathbb{P}_1)-$elementary subset is $(c, \mathbb{P}_2)-$regular, and by Proposition 3, one has $\langle A\rangle_{c,1} \supset \langle A\rangle_{c,2}$ for any $A \subset U$.

It is easy to verify that:

$$\{u \in U \mid c(u, v) < r\} = \bigcap_{s\geq r} \{u \in U \mid c(u, v) \neq s\},$$
$$\{u \in U \mid c(u, v) = r\} = \bigcap_{s\neq r} \{u \in U \mid c(u, v) \neq s\},$$

therefore, $\langle A\rangle_{c,2} \supset \langle A\rangle_{c,3} \subset \langle A\rangle_{c,4}$. $\square$

We derive from Corollary 3.5, the following comparison between the sets $\mathcal{R}_{c,\mathbb{P}_i}(U)$:

$$\mathcal{R}_{c,\mathbb{P}_1}(U) \subset \mathcal{R}_{c,\mathbb{P}_2}(U) \subset \mathcal{R}_{c,\mathbb{P}_3}(U) \text{ and } \mathcal{R}_{c,P_4}(U) \subset \mathcal{R}_{c,\mathbb{P}_3}(U). \tag{3.6}$$

Remark 3.6. *Observe that*

1. $\mathbb{P}_1 \supset \mathbb{P}_2$ *and* $\mathcal{R}_{c,\mathbb{P}_1}(U) \subset \mathcal{R}_{c,\mathbb{P}_2}(U)$.
2. $\mathbb{P}_3 \supset \mathbb{P}_2$ *and* $\mathcal{R}_{c,\mathbb{P}_3}(U) \supset \mathcal{R}_{c,\mathbb{P}_2}(U)$.
3. $\mathcal{R}_{c,\mathbb{P}_1}(U) \subset \mathcal{R}_{c,\mathbb{P}_3}(U)$ *whereas* $\mathbb{P}_1$ *and* $\mathbb{P}_3$ *are not comparable in the sense of inclusion.*
4. *In particular, in the case of locally convex vector space, we recover the fact that every closed convex subset is evenly convex.*

Proposition 4. *Assume that the coupling function c satisfies the property:*

$$\forall v \in V,\ \exists w \in V \quad | \quad -c(., v) = c(., w). \tag{3.7}$$

Then, one has:

$$\mathcal{R}_{c,\mathbb{P}_4}(U) \subset \mathcal{R}_{c,\mathbb{P}_1}(U) \subset \mathcal{R}_{c,\mathbb{P}_2}(U) \subset \mathcal{R}_{c,\mathbb{P}_3}(U). \tag{3.8}$$

Proof By (3.6), we only need to show that $\mathcal{R}_{c,\mathbb{P}_4}(U) \subset \mathcal{R}_{c,\mathbb{P}_1}(U)$. Let $(v, r) \in V \times \mathbb{R}$. We have

$$[c(.,v) = r] = [c(.,v) \leq r] \bigcap [-c(.,v) \leq -r].$$

By assumption on the coupling function, there exists $w \in V$ such that $-c(.,v) = c(.,w)$. Consequently,

$$[c(.,v) = r] = [c(.,v) \leq r] \bigcap [c(.,w) \leq -r].$$

We conclude with Proposition 3. □

Example 3.7. *Assume that $U = V = \mathbb{R}^n$, and coupling function c is defined by $c(u,v) = \|u - v\|$, where $\|.\|$ is the euclidean norm. The non trivial $(c, \mathbb{P}_4)-$elementary sets are spheres (not convex) whereas the non trivial $(c, \mathbb{P}_1)-$elementary sets are closed balls (closed convex). In this case, $\mathcal{R}_{c,\mathbb{P}_1}(U)$ and $\mathcal{R}_{c,\mathbb{P}_4}(U)$ are not comparable. Observe that in this case assumption (3.7) does not hold.*

Proposition 5. *For any $A \subset U$, one has $\langle A \rangle_{c,1} = [i_A^{\Gamma_c} \leq 0]$.*

Proof By (3.2), one has

$$\begin{aligned} a \notin \langle A \rangle_{c,1} &\Longleftrightarrow \exists v \in V \ : \ i_A^c(v) < c(a,v) \\ &\Longleftrightarrow 0 < \sup_{v \in V} \{c(a,v) - i_A^c(v)\} \\ &\Longleftrightarrow 0 < i_A^{\Gamma_c}(a) \\ &\Longleftrightarrow 0 \notin \left[i_A^{\Gamma_c} \leq 0 \right]. \end{aligned}$$

Thus $\langle A \rangle_{c,1} = \left[i_A^{\Gamma_c} \leq 0 \right]$. □

The following result makes the link between hull of set and Γ-regularization of function by means of indicator function.

Theorem 3.8. *Assume that the coupling function c satisfies the condition:*

$$\forall \ (v, \beta) \in V \times \mathbb{R}_+^*, \quad \exists \bar{v} \in V \ | \ \beta c(.,v) = c(.,\bar{v}). \tag{3.9}$$

Then for each $A \subset U$ such that dom $i_A^c \neq \emptyset$, *one has:* $i_A^{\Gamma_c} = i_{\langle A \rangle_{c,1}}$.

Proof Let $b \in U$.

(1) Assume that $b \notin \langle A \rangle_{c,1}$. By (3.3), there exists $(v, \epsilon) \in V \times \mathbb{R}_+^*$ such that $c(b,v) - \sup_{a \in A} c(a,v) \geq \epsilon$. From (3.9), one has:

$$\forall n \geq 1, \exists v_n \in V \ : \ nc(.,v) = c(.,v_n).$$

Consequently,

$$n\epsilon \leq c(b, v_n) - \sup_{a \in A} c(a, v_n) = c(b, v_n) - i_A^c(v_n) \leq i_A^{\Gamma_c}(b), \quad \forall n \geq 1.$$

Therefore $i_A^{\Gamma_c}(b) = +\infty$.

(2) Assume that $b \in \langle A \rangle_{c,1}$. By (3.3), one has

$$c(b, w) - \sup_{a \in A} c(a, w) \leq 0, \quad \forall v \in V.$$

Thus

$$i_A^{\Gamma_c}(b) = \sup_{v \in V} \left\{ c(b, v) - \sup_{a \in A} c(a, v) \right\} \leq 0.$$

Let $v \in \text{dom } i_A^c$. By (3.9), one gets

$$\forall n \geq 1, \ \exists \ v_n \in V \ : \ \frac{1}{n} c(., v) = c(., v_n).$$

Consequently,

$$\frac{1}{n} \left(c(b, v) - \sup_{a \in A} c(a, v) \right) = c(b, v_n) - \sup_{a \in A} c(a, v_n) \leq i_A^{\Gamma_c}(b), \quad \forall n \geq 1.$$

Therefore,

$$0 = \lim_{n \to +\infty} \frac{1}{n} \left(c(b, v) - \sup_{a \in A} c(a, v) \right) \leq i_A^{\Gamma_c}(b),$$

and finally, $i_A^{\Gamma_c}(b) = 0$.

□

Remark 3.9. *Assumption* (3.9) *is satisfied by coupling functions (1), (3) and (5) of Example 2.2. Coupling functions (2) and (4) of the same example do not satisfy assumption* (3.9).

4 Level set regularization of functions

In this section, we introduce a notion of $(c, \mathbb{P})-$level set regularization of extended real-valued functions. We show that this level set regularization can be interpreted as bi-conjugacy relative to a couple of dual polarities by decomposition of the closure operator. We then give some other expressions of these regularizations.

4.1 Definitions and properties

Definition 4. *A function $h : U \to \overline{\mathbb{R}}$ is said to be $(c, \mathbb{P})-$level regular if all of its sub-level sets are $(c, \mathbb{P})-$regular, i.e $\langle [h \leq r] \rangle_{c,\mathbb{P}} = [h \leq r], \ \forall r \in \mathbb{R}$.*

We denote $\mathcal{N}_{c,\mathbb{P}}(U)$, the set of $(c,\mathbb{P})-$level regular functions defined on U to $\overline{\mathbb{R}}$. Observe that this set contains the constant function $-\infty$.

Proposition 6. *The set $\mathcal{N}_{c,\mathbb{P}}(U)$ is closed under pointwise suprema, i.e given $(h_i)_{i\in I}$ a family of $(c,\mathbb{P})-$level regular functions,then $h := \sup\limits_{i\in I} h_i$ is $(c,\mathbb{P})-$level regular.*

Proof Let $r \in \mathbb{R}$. Since $[h \leq r] = \cap_{i\in I}[h_i \leq r]$, the conclusion follows from the fact that any intersection $(c,\mathbb{P})-$regular sets is $(c,\mathbb{P})-$regular. $\square$

We define the $(c,\mathbb{P})-$level set regularization of an extended real-valued function as follows.

Definition 5. *The $(c,\mathbb{P})-$level set regularization of a function $h : U \to \overline{\mathbb{R}}$ is the greatest $(c,\mathbb{P})-$level regular minorant of h. This function is denoted by $h^{\langle\rangle_{c,\mathbb{P}}}$.*

Example 4.1. *Assume U is topological vector space with topological dual V and c the standard coupling function. $(c,\mathbb{P}_2)-$level regular functions are evenly quasi-convex functions. Moreover, if U is locally convex then $(c,\mathbb{P}_1)-$level regular functions are lower semi-continuous quasi-convex functions.*

Example 4.2. *Assume that U is a metric space, $V = \mathscr{C}(U,\mathbb{R})$ a space of continuous functions from U to $\mathbb{R}$, and $c : U \times V \to \mathbb{R}$ defined by $c(u,v) = v(u)$. A function $h : U \to \mathbb{R} \cup \{+\infty\}$ is $(c,\mathbb{P}_1)-$level regular if and only if h is lower semi-continuous ([3], corollary 11).*

Proposition 7. *Any $c-$elementary function is $(c,\mathbb{P}_i)-$level regular for $i = 1,2,3$. More precisely, one has*

$$\Gamma_c(U) \subset \mathcal{N}_{c,\mathbb{P}_1}(U) \subset \mathcal{N}_{c,\mathbb{P}_2}(U) \subset \mathcal{N}_{c,\mathbb{P}_3}(U) \text{ and } \mathcal{N}_{c,\mathbb{P}_4}(U) \subset \mathcal{N}_{c,\mathbb{P}_3}(U).$$

Proof Let $h := c(.,v) - r$ an $c-$elementary function. For any $t \in \mathbb{R}$, we have

$$[h \leq t] = \{u \in U \mid t + r - c(u,v) \geq 0\},$$

which is obviously $(c,\mathbb{P}_1)-$elementary set. Therefore $\Gamma_c(U) \subset \mathcal{N}_{c,\mathbb{P}_1}(U)$. The other inclusions follow from (3.6). $\square$

Remark 4.3. *$c-$elementary functions are not necessary $(c,\mathbb{P}_4)-$level set regular functions. For example, in the topological case, one cannot write a half space as an intersection of affine hyperplanes.*

Example 4.4. *Let $n \geq 1$, an integer number. Assume that $U = V = \mathbb{R}^n$ and c a standard scalar product of $\mathbb{R}^n$. Let $h_1, h_2 : \mathbb{R}^n \to [\![0;n]\!]$ two functions defined by*

$$h_1(x) = \begin{cases} 0 & \text{if } x = 0 \\ \max\{i \in [\![1,n]\!] \mid x_i \neq 0\} & \text{if } x \neq 0, \end{cases}$$

$$h_2(x) = \begin{cases} 0 & \text{if } x_i \neq 0, \ \forall i \in [\![1,n]\!] \\ \max\{i \in [\![1,n]\!] \mid x_i = 0\} & \text{else .} \end{cases}$$

For any $r \in \mathbb{R}$, we have

$$[h_1 \leq r] = \begin{cases} \emptyset & \text{if } \ r < 0 \\ \{x \in \mathbb{R}^n \mid x_{i+1} = \ldots = x_n = 0\} & \text{if } \ i \leq r < i+1, \quad i = 0,1,\ldots,n-1 \\ \mathbb{R}^n & \text{if } \ n \leq r, \end{cases}$$

$$[h_2 \leq r] = \begin{cases} \emptyset & \text{if } r < 0 \\ \{x \in \mathbb{R}^n \ \mid x_{i+1} \neq 0,\ldots,x_n \neq 0\} & \text{if } i \leq r < i+1, \quad i = 0,1,\ldots,n-1 \\ \mathbb{R}^n & \text{if } \ n \leq r. \end{cases}$$

It is clear that:

(1) h_1 is $(c,\mathbb{P}_4)-$level regular. In particular, $h_1 \in \mathcal{N}_{c,\mathbb{P}_i}(U)$, for $i = 1,2,3,4$.

(2) h_2 is $(c,\mathbb{P}_3)-$level regular but not $(c,\mathbb{P}_2)-$level regular since $[h_2 \leq n-1] = \{x \in \mathbb{R}^n \mid x_n \neq 0\}$ is not convex.

Example 4.5. *Let $U = V = \mathbb{R}$, c the standard product of $\mathbb{R}$. The indicator function of $\mathbb{R}^*$,$i_{\mathbb{R}^*}$ is $(c,\mathbb{P}_3)-$level regular but not quasi-convex.*

4.2 Decomposition of $\langle\rangle_{c,\mathbb{P}}$

Let us consider a map $\Delta_{c,\mathbb{P}} : 2^U \to 2^{V\times\mathbb{R}}$ defined by:

$$\Delta_{c,\mathbb{P}}(A) := \{(v,r) \in V \times \mathbb{R} \mid A \subset E^{\mathbb{P}}_{v,r}\}, \tag{4.1}$$

which, we simply denote Δ in the sequel. Given $(A_i)_{i\in I}$ a family of subsets of U, we have

$$\begin{aligned} \Delta \bigcup_{i\in I} A_i &:= \left\{(v,r) \in V \times \mathbb{R} \mid \bigcup_{i\in I} A_i \subset E^{\mathbb{P}}_{v,r}\right\} \\ &= \big\{(v,r) \in V \times \mathbb{R} \mid A_i \subset E^{\mathbb{P}}_{v,r}, \quad \forall i \in I\big\} \\ &= \bigcap_{i\in I} \big\{(v,r) \in V \times \mathbb{R} \mid A_i \subset E^{\mathbb{P}}_{v,r}\big\} \\ &= \bigcap_{i\in I} \Delta A_i. \end{aligned}$$

Therefore Δ is said to be a polarity ([16]). We associate to Δ, its dual polarity $\Delta^* : 2^{V\times\mathbb{R}} \to 2^U$ defined by

$$\Delta^*(B) = \bigcup\{A \in 2^U \mid B \subset \Delta(A)\}. \tag{4.2}$$

Observe that for each $(v,r) \in V \times \mathbb{R}$ and for each $u \in U$, one has

$$u \in \Delta^*(v,r) \Longleftrightarrow (v,r) \in \Delta(u) \Longleftrightarrow u \in E^{\mathbb{P}}_{v,r}, \tag{4.3}$$

therefore $\Delta^*(v,r) = E^{\mathbb{P}}_{v,r}$. Since Δ^* is a polarity, then we have for each $B \subset V \times \mathbb{R}$,

$$\Delta^*(B) = \Delta^*\big(\bigcup_{(v,r)\in B} \{(v,r)\}\big) = \bigcap_{(v,r)\in B} \Delta^*(v,r) = \bigcap_{(v,r)\in B} E^{\mathbb{P}}_{v,r}. \tag{4.4}$$

The operator $\langle\rangle_{c,\mathbb{P}}$ can be decomposed as follows.

Proposition 8. *For any $A \subset U$, we have $(\Delta^* \circ \Delta)(A) = \langle A\rangle_{c,\mathbb{P}}$.*

Proof Given $A \subset U$, one has

$$(\Delta^* \circ \Delta)(A) = \Delta^*(\{(v,r) \in V \times \mathbb{R} \mid A \subset E^{\mathbb{P}}_{v,r}\}) = \bigcap_{A\subset E^{\mathbb{P}}_{v,r}} E^{\mathbb{P}}_{v,r} = \langle A\rangle_{c,\mathbb{P}}.$$

□

4.3 Conjugacy associated to polarities Δ and Δ^* ([16, 17])

The conjugate of a function $h : U \to \overline{\mathbb{R}}$ relative to the polarity Δ is the function $h^{\Delta} : V \times \mathbb{R} \to \overline{\mathbb{R}}$ given by

$$h^{\Delta}(v,r) := \sup_{u\notin\Delta^*(v,r)} -h(u) = \sup_{u\notin E^{\mathbb{P}}_{v,r}} -h(u). \tag{4.5}$$

Analogously, the conjugate of a function $k : V \times \mathbb{R} \to \overline{\mathbb{R}}$ relative to the polarity Δ^* is defined by

$$k^{\Delta^*}(u) := \sup_{(v,r)\notin\Delta(u)} -k(v,r) = \sup_{u\notin E^{\mathbb{P}}_{v,r}} -k(v,r). \tag{4.6}$$

Thus, the bi-conjugacy relative to polarities Δ, Δ^* of a function $h : U \to \overline{\mathbb{R}}$ is the function $h^{\Delta\Delta^*} : U \to \overline{\mathbb{R}}$ given by

$$h^{\Delta\Delta^*}(a) := \sup_{(v,r)\notin\Delta(a)} \inf_{u\notin\Delta^*(v,r)} h(u) = \sup_{a\notin E^{\mathbb{P}}_{v,r}} \inf_{u\notin E^{\mathbb{P}}_{v,r}} h(u). \tag{4.7}$$

It is well known ([16]) that this conjugacy can be interpreted by means of coupling function $\delta : U \times (V \times \mathbb{R}) \to \overline{\mathbb{R}}$ defined by

$$\delta(u,(v,r)) = \begin{cases} 0 & \text{si} \quad u \notin E^{\mathbb{P}}_{v,r} \\ -\infty & \text{si} \quad u \in E^{\mathbb{P}}_{v,r}. \end{cases}$$

More precisely, given a function $h : U \to \overline{\mathbb{R}}$, we have $h^{\Delta} = h^{\delta}$ and $h^{\Delta\Delta^*} = h^{\delta\delta}$.

Theorem 4.6 ([16])**.** *The $(c,\mathbb{P})-$level regularization of a function $h : U \to \overline{\mathbb{R}}$ coincides with bi-conjugacy relative to polarities Δ, Δ^*: $h^{\langle\rangle_{c,\mathbb{P}}} = h^{\Delta\Delta^*}$.*

Corollary 4.7. *For any subset A of U, we have*

$$i_A^{\langle\rangle_{c,\mathbb{P}}} = i_{\langle A\rangle_{c,\mathbb{P}}} \quad \textit{and} \quad v_A^{\langle\rangle_{c,\mathbb{P}}} = v_{\langle A\rangle_{c,\mathbb{P}}}.$$

Proof Let $A \subset U$. Let $a \in A$. It follows from Theorem 4.6 that

$$i_A^{\langle\rangle_{c,\mathbb{P}}}(a) = \sup_{a \notin E_{v,r}^{\mathbb{P}}} \inf_{u \notin E_{v,r}^{\mathbb{P}}} i_A(u).$$

We distinguish two cases:

- We first assume that $a \notin \langle A \rangle_{c,\mathbb{P}}$. There exists $(v,r) \in V \times \mathbb{R}$ such that $a \notin E_{v,r}^{\mathbb{P}}$ and $A \subset E_{v,r}^{\mathbb{P}}$. Consequently,

$$\inf_{u \notin E_{v,r}^{\mathbb{P}}} i_A(u) = +\infty.$$

- Secondly, assume that $a \in \langle A \rangle_{c,\mathbb{P}}$. For all $(v,r) \in V \times \mathbb{R}$ such that $a \notin E_{v,r}^{\mathbb{P}}$, there exists $u \in A$ such that $u \notin E_{v,r}^{\mathbb{P}}$. Consequently,

$$\begin{aligned} i_A^{\langle\rangle_{c,\mathbb{P}}}(a) &= \sup_{a \notin E_{v,r}^{\mathbb{P}}} \inf_{u \notin E_{v,r}^{\mathbb{P}}} i_A(u) \\ &= \begin{cases} 0 & \text{if } a \in \langle A \rangle_{c,\mathbb{P}} \\ +\infty & \text{if } a \notin \langle A \rangle_{c,\mathbb{P}} \end{cases} \\ &= i_{\langle A \rangle_{c,\mathbb{P}}}(a). \end{aligned}$$

Analogously, we have

$$\begin{aligned} v_A^{\langle\rangle_{c,\mathbb{P}}}(a) &= \sup_{a \notin E_{v,r}^{\mathbb{P}}} \inf_{u \notin E_{v,r}^{\mathbb{P}}} v_A(u) \\ &= \begin{cases} -\infty & \text{if } a \in \langle A \rangle_{c,\mathbb{P}} \\ +\infty & \text{if } a \notin \langle A \rangle_{c,\mathbb{P}} \end{cases} \\ &= v_{\langle A \rangle_{c,\mathbb{P}}}(a). \end{aligned}$$

$\square$

Proposition 9. *For any function $h : U \to \overline{\mathbb{R}}$ and for any real number t, one has*

$$[h^{\langle\rangle_{c,\mathbb{P}}} \leq t] = \bigcap_{s>t} \langle [h \leq s] \rangle_{c,\mathbb{P}}.$$

Proof Let $s > t$ and $a \notin \langle [h \leq s] \rangle_{c,\mathbb{P}}$. There exists $(\bar{v}, \bar{r}) \in V \times \mathbb{R}$ such that $a \notin E_{\bar{v},\bar{r}}^{\mathbb{P}}$ and $[h \leq s] \subset E_{\bar{v},\bar{r}}^{\mathbb{P}}$. We deduce that

$$h^{\langle\rangle_{c,\mathbb{P}}}(a) = \sup_{a \notin E_{v,r}^{\mathbb{P}}} \inf_{u \notin E_{v,r}^{\mathbb{P}}} h(u) \geq \inf_{u \notin E_{\bar{v},\bar{r}}^{\mathbb{P}}} h(u) \geq s > t.$$

Let $a \notin \left\langle [h^{\langle\rangle_{c,\mathbb{P}}} \leq s] \right\rangle_{c,\mathbb{P}}$. There exists $s \in \mathbb{R}$ such that $h^{\langle\rangle_{c,\mathbb{P}}}(a) > s > t$. By Theorem 4.6, there exists $(v,r) \in V \times \mathbb{R}$ such that

$$a \notin E_{v,r}^{\mathbb{P}} \quad \text{and} \quad \inf_{u \notin E_{v,r}^{\mathbb{P}}} h(u) > s,$$

thus $[h \leq s] \subset E_{v,r}^{\mathbb{P}}$, and finally $a \notin \langle [h \leq s] \rangle_{c,\mathbb{P}}$. $\square$

4.4 Other expressions of $(c, \mathbb{P})-$level regularizations

We now give another expression of the $(c, \mathbb{P})-$level regularization of an extended real-valued function h. These expressions give the value of the $(c, \mathbb{P})-$level regularization of h at a given point. Given $h : U \to \overline{\mathbb{R}}$ and $a \in U$, we define sets $\mathcal{I}_h(a)$ and $\mathcal{J}_h(a)$ by:

$$\mathcal{I}_h(a) := \{t \in \mathbb{R} \mid a \notin \langle [h \leq t] \rangle_{c,\mathbb{P}}\} \quad \text{and} \quad \mathcal{J}_h(a) = \{t \in \mathbb{R} \mid a \in \langle [h \leq t] \rangle_{c,\mathbb{P}}\}. \tag{4.8}$$

Sets $\mathcal{I}_h(a)$ and $\mathcal{J}_h(a)$ are two intervals of $\mathbb{R}$ such that $\mathcal{I}_h(a) \cap \mathcal{J}_h(a) = \emptyset$ and $\mathcal{I}_h(a) \cup \mathcal{J}_h(a) = \mathbb{R}$. Moreover, for any $(r, s) \in \mathcal{I}_h(a) \times \mathcal{J}_h(a)$, we have $r < s$. We deduce that $\sup \mathcal{I}_h(a) = \inf \mathcal{J}_h(a)$.

Proposition 10.

$$h^{\langle\rangle_{c,\mathbb{P}}}(a) = \sup \left\{ t \in \mathbb{R} : a \notin \langle [h \leq t] \rangle_{c,\mathbb{P}} \right\} = \inf \left\{ t \in \mathbb{R} : a \in \langle [h \leq t] \rangle_{c,\mathbb{P}} \right\}.$$

Proof Let $t \in \mathcal{I}_h(a)$. There exists $(v, r) \in V \times \mathbb{R}$ such that $a \notin E^{\mathbb{P}}_{v,r}$ and $[h \leq t] \subset E^{\mathbb{P}}_{v,r}$. Therefore

$$\inf_{u \notin E^{\mathbb{P}}_{v,r}} h(u) \geq t \quad \text{and so} \quad \sup_{a \notin E^{\mathbb{P}}_{v,r}} \inf_{u \notin E^{\mathbb{P}}_{v,r}} h(u) \geq t.$$

By Theorem 4.6, one gets $h^{\langle\rangle_{c,\mathbb{P}}}(a) \geq t$. Hence $h^{\langle\rangle_{c,\mathbb{P}}}(a) \geq \sup \mathcal{I}_h(a)$. Conversely, let $t < h^{\langle\rangle_{c,\mathbb{P}}}(a)$, then $a \notin [h^{\langle\rangle_{c,\mathbb{P}}} \leq t]$ and by Proposition 9, there exists $s > t$ such that $a \notin \langle [h \leq s] \rangle_{c,\mathbb{P}}$. Consequently, $\sup \mathcal{I}_h(a) \geq s > t$. Hence $\sup \mathcal{I}_h(a) \geq h^{\langle\rangle_{c,\mathbb{P}}}(a)$. $\square$

5 Applications to an optimization problem: sub-level set duality

Let us consider the following minimization problem:

$$\min_x f(x), \qquad s.t \quad x \in X, \tag{$\mathscr{P}$}$$

where X is a nonempty set and $f : X \to \mathbb{R} \cup \{+\infty\}$ is an extended real-valued function.

5.1 Level set perturbational duality

We consider a perturbation function $F : X \times U \to \overline{\mathbb{R}}$ satisfying

$$\exists a \in U \ : \ F(., a) = f(.). \tag{5.1}$$

We associate to F a valued function $h : U \to \overline{\mathbb{R}}$ defined by

$$h(u) := \inf_{x \in X} F(x, u). \tag{5.2}$$

We denote by α the optimal value of $(\mathscr{P})$. It is obvious that $\alpha = h(a)$.

The perturbational dual of $(\mathscr{P})$ ([16]) is given by

$$\max_{(v,r)} -h^{\Delta}(v,r) \quad s.t \quad a \notin E^{\mathbb{P}}_{v,r}. \tag{$\mathscr{D}$}$$

We denote by β the optimal value of $(\mathscr{D})$. By definition, one has

$$-\infty \leq \beta := \sup(\mathscr{D}) = h^{\Delta\Delta^*}(a) \leq h(a) =: \alpha = \inf(\mathscr{P}) \leq +\infty. \tag{5.3}$$

Thus, the weak duality holds. The following theorem gives a necessary and sufficient condition to assure the strong duality.

Theorem 5.1. *The following statements are equivalent:*

(1) The strong duality holds for $(\mathscr{P})$ *i.e* $\inf(\mathscr{P}) = \max(\mathscr{D})$,

(2) $a \notin \langle [h < \alpha] \rangle_{c,\mathbb{P}}$.

Proof. Assume that (1) holds. There exists $(\bar{v}, \bar{r}) \in V \times \mathbb{R}$ such that

$$a \notin E^{\mathbb{P}}_{\bar{v},\bar{r}} \text{ and } \alpha = h(a) = -h^{\Delta}(\bar{v},\bar{r}) := \inf_{u \notin E^{\mathbb{P}}_{\bar{v},\bar{r}}} h(u).$$

We deduce that $[h < \alpha] \subset E^{\mathbb{P}}_{\bar{v},\bar{r}}$. Thus $a \notin \langle [h < \alpha] \rangle_{c,\mathbb{P}}$. Conversely, assume that (2) holds. There exists $(\bar{v}, \bar{r}) \in V \times \mathbb{R}$ such that $a \notin E^{\mathbb{P}}_{\bar{v},\bar{r}}$ and $[h < \alpha] \subset E^{\mathbb{P}}_{\bar{v},\bar{r}}$. Therefore

$$\inf_{u \notin E^{\mathbb{P}}_{\bar{v},\bar{r}}} h(u) \geq \alpha \geq \beta.$$

Remember that

$$\beta := \sup_{a \notin E^{\mathbb{P}}_{v,r}} -h^{\Delta}(v,r) = \sup_{a \notin E^{\mathbb{P}}_{v,r}} \inf_{u \notin E^{\mathbb{P}}_{v,r}} h(u).$$

Thus

$$\beta \geq \inf_{u \notin E^{\mathbb{P}}_{\bar{v},\bar{r}}} h(u) = -h^{\Delta}(\bar{v},\bar{r}) \geq \alpha \geq \beta.$$

Hence $\beta = -h^{\Delta}(\bar{v},\bar{r}) = \alpha$. □

Theorem 5.1 is interesting in evenly convex case which is used in economic theory.

5.2 Evenly quasi-convex duality ([1],[7],[11],[12],[17])

We assume X and U are topological vector spaces, $V = U^*$ the topological dual of U, $c = \langle , \rangle$ the standard coupling function between U and U^*.

Corollary 5.2. *Assume that function $F : X \times U \to \overline{\mathbb{R}}$ is quasi-convex and for each $x \in X$, $F(x,.) : U \to \overline{\mathbb{R}}$ is upper semi-continuous. One has:*

$$\inf(\mathscr{P}) = \max_{u^* \in U^*} \inf_{\substack{(x,u) \in X \times U \\ \langle u-a, u^* \rangle \geq 0}} F(x,u).$$

Proof Since F is quasi-convex and for each $x \in X$, $F(x,.)$ is upper semi-continuous then h is quasi-convex and upper semi-continuous. Consequently, $[h < \alpha]$ is open convex set and so it is evenly convex. As $a \notin [h < \alpha]$, it results from Theorem 5.1 that

$$\inf(\mathscr{P}) = \max(\mathscr{D}) = \max_{r - \langle a, u^* \rangle \leq 0} \inf_{r - \langle u, u^* \rangle \leq 0} h(u) = \max_{u^* \in U^*} \inf_{\langle u, u^* \rangle \geq \langle a, u^* \rangle} h(u),$$

where the last equality follows from the fact that for each $u^* \in U^*$, function $k_{u^*} : \mathbb{R} \to \overline{\mathbb{R}}$ defined by $k_{u^*}(r) = \inf_{r - \langle u, u^* \rangle \leq 0} h(u)$ is not decreasing. □

Corollary 5.3. *Assume that function $F : X \times U \to \overline{\mathbb{R}}$ is quasi-convex and for each $x \in X$, $F(x,.) : U \to \overline{\mathbb{R}}$ is upper semi-continuous. One has:*

$$\inf(\mathscr{P}) = \max_{u^* \in U^*} \inf_{\substack{(x,u) \in X \times U \\ \langle u-a, u^* \rangle = 0}} F(x,u).$$

Proof We know that under these assumptions on F, $[h < \alpha]$ is convex open set, therefore it is $(\langle , \rangle, \mathbb{R}^*)-$regular. Since $a \notin [h < \alpha]$, it results from Theorem 5.1 that

$$\begin{aligned}
\inf(\mathscr{P}) &= \max(\mathscr{D}) \\
&= \max_{\substack{(u^*, r) \in U^* \times \mathbb{R} \\ \langle a, u^* \rangle \, = \, r}} \inf_{\substack{u \in U \\ \langle u, u^* \rangle = r}} h(u) \\
&= \max_{u^* \in U^*} \max_{\substack{r \in \mathbb{R} \\ \langle a, u^* \rangle = r}} \inf_{\substack{u \in U \\ \langle u, u^* \rangle = r}} h(u) \\
&= \max_{u^* \in U^*} \inf_{\substack{u \in U \\ \langle u-a, u^* \rangle = 0}} h(u) \\
&= \max_{u^* \in U^*} \inf_{\substack{(x,u) \in X \times U \\ \langle u-a, u^* \rangle = 0}} F(x,u) \quad \text{by definition of } h.
\end{aligned}$$

□

6 Conclusion

In this work, we introduced a closure operator by means of coupling function and a subset of $\mathbb{R}$. This operator allowed us to define a hull of sets and level set regularization of extended real-valued functions. By decomposition of closure operator, we showed that a level set regularization of any extended real-valued function coincides with its bi-conjugacy relative to a couple of dual polarities. We derive an analytic expression of a level set regularization of extended real-valued function. Our results are applied to derive a strong duality for a minimization problem.

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