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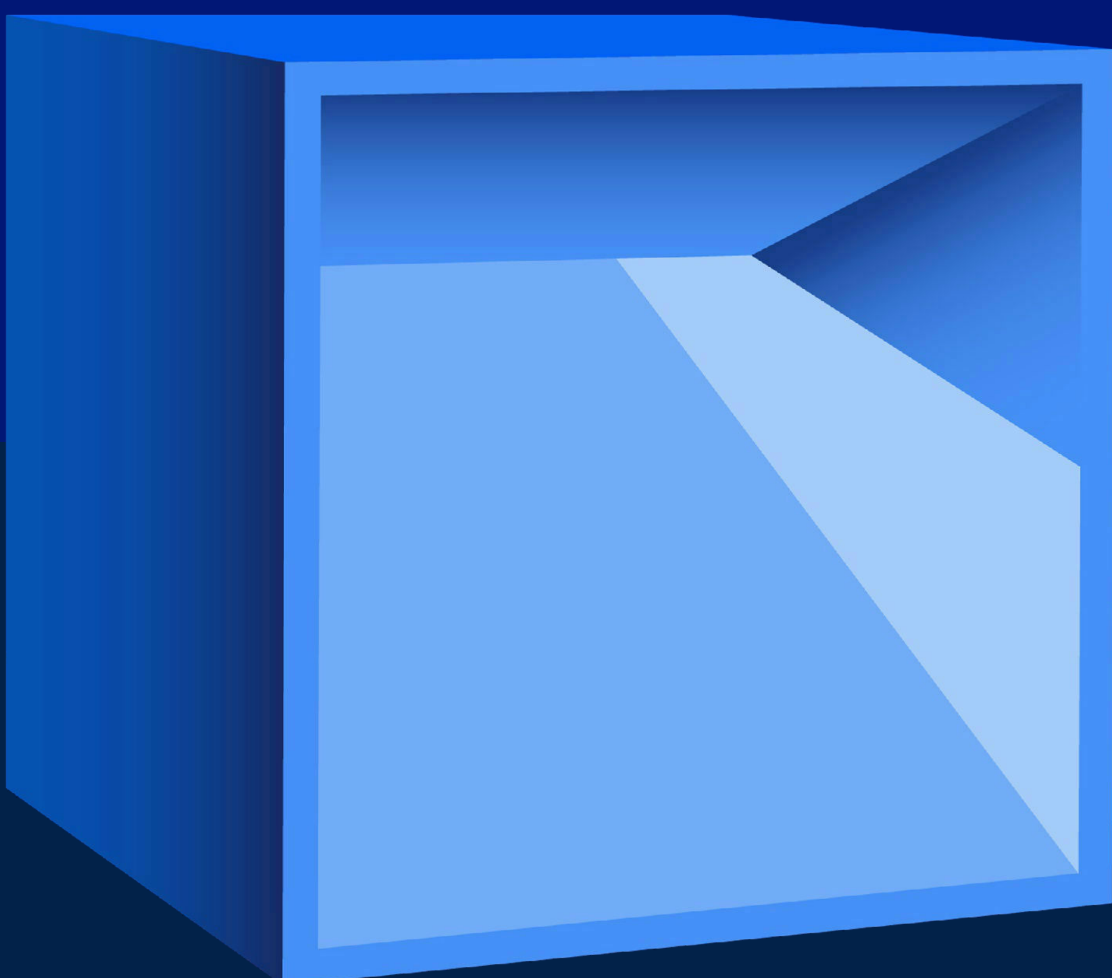

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SUMMARY

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A new iterative method based on the modified proximal-point algorithm for finding a common null point of an infinite family of accretive operators in Banach spaces

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ABSTRACT

In this paper, we introduce and study a new iterative method for finding a common null point of an infinite family of accretive operators with a strongly accretive and Lipschitzian operator, by using the proximal-point algorithm. And also we prove that the common null point is a unique solution of variational inequality without imposing any compactness-type condition on either the operators or the space considered. Finally, some applications of the main results to equilibrium problems and fixed point problems with an infinite family of pseudocontractive mappings are given. The main result is a generalization and improvement of numerous well-known results in the available literature.

RESUMEN

En este artículo, introducimos y estudiamos un nuevo método iterativo para encontrar un cero común de una familia infinita de operadores acretivos con un operador Lischitziano fuertemente acretivo, usando el algoritmo punto-proximal. También demostramos que el cero común es la única solución de una desigualdad variacional sin imponer ninguna condición de tipo compacidad en ninguno de los operadores o los espacios considerados. Finalmente, se entregan algunas aplicaciones de los resultados principales a problemas de equilibrio y problemas de punto fijo con una familia infinita de aplicaciones pseudo-contractivas. El resultado principal es una generalización y mejora de numerosos resultados bien conocidos en la literatura disponible.

Keywords and Phrases: Proximal-point algorithm; Accretive operators; Variational inequality; Common zeros.

2020 AMS Mathematics Subject Classification: 46T05; 47H06; 47H09; 47H10.

1 Introduction

Let H be a real Hilbert space and K be a nonempty subset of H. For a set-valued map $A : H \to 2^H$, the domain of A, $D(A)$, the image of a subset S of H, $A(S)$ the range of A, $R(A)$ and the graph of A, $G(A)$ are defined as follows:

$$D(A) := \{x \in H : Ax \neq \emptyset\}, \ A(S) := \cup\{Ax : x \in S\},$$

$$R(A) := A(H), \ G(A) := \{(x, u) : \ x \in D(A), \ u \in Ax\}.$$

A multi-valued map $A : D(A) \subset H \to 2^H$ is called monotone if the inequality

$$\langle u - v, x - y \rangle \geq 0$$

holds for each $x, y \in D(A)$, $u \in Ax$, $v \in Ay$. A single-valued operator $A : K \to H$ is said to be *strongly positive bounded linear* if there exists a constant $k > 0$ such that

$$\langle Ax, x \rangle \geq k\|x\|^2, \ \ \forall \ x, y \in K.$$

Remark 1. *It is immediate that if A is k-strongly positive bounded linear, then A is k-strongly monotone and $\|A\|$-Lipschitz continuous.*

A monotone operator A is called maximal monotone if its graph $G(A)$ is not properly contained in the graph of any other monotone operator. It is well known that A is maximal monotone if and only if A is monotone and $R(I + rA) = H$ for all $r > 0$ and A is said to satisfy the range condition if $\overline{D(A)} \subset R(I + rA)$. Many problems arising in different areas of mathematics, such as optimization, variational analysis and differential equations, can be modeled by the equation

$$0 \in Ax, \tag{1.1}$$

where A is a monotone mapping. The solution set of this equation coincide to a null points set of A. Such operators have been studied extensively (see, e.g., Bruck Jr [5], Chidume [9], Rockafellar [29], Xu [30] and the references therein).

Consider, for example, the following: let $f : H \to \mathbb{R} \cup \{\infty\}$ be a proper lower semi continuous and convex function. The *subdifferential*, $\partial f : H \to 2^H$ of f at $x \in H$ is defined by

$$\partial f(x) = \big\{x^* \in H : f(y) - f(x) \geq \langle y - x, x^* \rangle \ \ \forall \ y \in H\big\}.$$

It is easy to check that $\partial f : H \to 2^H$ is a *monotone operator* on H, and that $0 \in \partial f(x)$ if and only if x is a minimizer of f. Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of f.

In order to find a solution of problem (1.1), Rockafellar [29] introduced a powerful and successful algorithm which is recognized as Rockafellar proximal- point algorithm: for any initial point $x_0 \in H$, a sequence $\{x_n\}$ is generated by:

$$x_{n+1} = J_{r_n}(x_n + e_n),\ \forall\ n \geq 0,$$

where $J_r = (I + rA)^{-1}$ for all $r > 0$, is the resolvent of A and $\{e_n\}$ is an error sequence in a Hilbert space. In the recent years, the problem of finding a common element of the set of solutions of convex minimization, variational inequality and the set of fixed point problems in real Hilbert spaces, Banach spaces and complete CAT(0) (Hadamard) spaces have been intensively studied by many authors; see, for example, [20, 21, 19, 29, 30] and the references therein.

Very recently, Eslamian and Vahidi [10] introduced a new iterative method base on proximal point algorithm with strongly positive bounded linear operator for solving a system of inclusion problem. They established a strong convergence theorem which extends the corresponding results in [30, 2, 32, 28, 13, 14, 15, 16, 16, 17, 18].

Theorem 2 (Eslamian and Vahidi [10])**.** *Let H be a real Hilbert space and K be a nonempty, closed and convex subset of H. Let $\{B_i\}$, $i \in \mathbb{N}^* := \{1, 2, 3, ...\}$ be an infinite family of operators of H such that $\bigcap_{i=1}^{\infty} {B_i}^{-1}(0) \neq \emptyset$ and $\bigcap_{i=1}^{\infty} \overline{D(B_i)} \subset K \subset \bigcap_{i=1}^{\infty} R(I + rB_i)$, for all $r > 0$. Let $A : H \to H$ be a k-strongly bounded linear operator with a coefficient $\bar{\gamma}$ and f be a $b-$ contraction mapping of K into itself with a constant $b \geq 0$.*
Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:

$$\begin{cases} y_n = \beta_{n,0}x_n + \sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} x_n \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n. \end{cases} \tag{1.2}$$

Let $\{r_n\} \subset]0, \infty[$, $\{\beta_{n,i}\}$ and $\{\alpha_n\}$ be real sequences in $(0, 1)$ satisfying:
(i) $\lim_{n\to\infty} \alpha_n = 0$; *(ii)* $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{\infty} \beta_{n,i} = 1$,
(iii) $\lim_{n\to\infty} \inf r_n > 0$, *and* $\lim_{n\to\infty} \inf \beta_{n,0}\beta_{n,i} > 0$, *for all* $i \in \mathbb{N}$.
Assume that $0 < \gamma < \frac{\bar{\gamma}}{b}$. Then, the sequence $\{x_n\}$ generated by (1.2) converges strongly to $x^* \in \bigcap_{i=1}^{\infty} {B_i}^{-1}(0)$.

Above discussion yields the following questions.

Question 1:Can results of Eslamian and Vahidi [10], and so on be extended from Hilbert spaces to Banach spaces?

Question 2: We know that Lipschitzian mapping is more general than contraction. What happens if the contraction is replaced by Lipschitzian mapping ?
Question 3: We know that k-strongly accretive operators and L-Lipchizian operators is more general than the strong positive bounded linear operators. What happens if the strongly positive bounded linear operators is replaced by k- strongly accretive operators and L-Lipchizian operators ?

The purpose of this paper is to give affirmative answers to these questions mentioned above. Applications are also included to valide our new findings.

2 Preliminairies

Let E be a real Banach space and C be a nonempty, closed and convex subset of E. We denote by J the normalized duality map from E to 2^{E^*} (E^* is the dual space of E) defined by:

$$J(x) := \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}, \forall\, x \in E.$$

Let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S$.
Let E be a normed space with dimE $\geq$ 2. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}; \quad \tau > 0.$$

It is known that a normed linear space E is *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

If there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or\ l_p)\ or\ W_p^m \text{ is}$$

$$\begin{cases} 2 - \text{uniformly smooth and } p - \text{uniformly convex} & \text{if} \quad 2 \leq p < \infty; \\ 2 - \text{uniformly convex and } p - \text{uniformly smooth} & \text{if} \quad 1 < p < 2. \end{cases} \tag{2.1}$$

It is known that a normed linear space E is *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

If there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or \ l_p) \ or \ W_p^m \text{ is } \begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if } \ 2 \leq p < \infty; \\ 2-\text{uniformly convex and } p-\text{uniformly smooth} & \text{if } \ 1 < p < 2. \end{cases}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{ f \in E^* : \langle x, f \rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1} \right\}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. Notice that for $x \neq 0$,

$$J_q(x) = \|x\|^{q-2} J_2(x), \ q > 1.$$

Following Browder [3], we say that a Banach space has a weakly continuous normalized duality map if J is a single-valued and is weak-to-weak* sequentially continous, i.e., if $\{x_n\} \subset E$, $x_n \rightharpoonup x$, then $J(x_n) \rightharpoonup J(x)$ in E^*. Weak continuity of duality map J plays an important role in the fixed point theory for nonlinear operators. Finally recall that a Banach space E satisfies Opial property (see, e.g., [24]) if $\limsup\limits_{n\to+\infty}\|x_n - x\| < \limsup\limits_{n\to+\infty}\|x_n - y\|$ whenever $x_n \rightharpoonup x$, $x \neq y$.
A Banach space E that has a weakly continuous normalized duality map satisfies Opial's property.

Remark 3. *Note also that a duality mapping exists in each Banach space. We recall from [1] some of the examples of this mapping in $l_p, L_p, W^{m,p}$-spaces, $1 < p < \infty$.*

(i) $l_p:\ Jx = \|x\|_{l_p}^{2-p} y \in l_q,\ \ x = (x_1, x_2, \cdots, x_n, \cdots),\ \ y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \cdots, x_n|x_n|^{p-2}, \cdots),$

(ii) $L_p:\ Ju = \|u\|_{L_p}^{2-p} |u|^{p-2} u \in L_q,$

(iii) $W^{m,p}:\ Ju = \|u\|_{W^{m,p}}^{2-p} \sum_{|\alpha \leq m|} (-1)^{|\alpha|} D^\alpha \Big(|D^\alpha u|^{p-2} D^\alpha u \Big) \in W^{-m,q},$

where $1 < q < \infty$ *is such that* $1/p + 1/q = 1$.

Finally recall that a Banach space E satisfies Opial's property (see, e.g., [24]) if $\limsup\limits_{n\to+\infty}\|x_n - x\| < \limsup\limits_{n\to+\infty}\|x_n - y\|$ whenever $x_n \xrightarrow{w} x$, $x \neq y$. Recall that an operator $A : K \to E$ is said to be accretive if there exists $j \in J_q(x - y)$ such that

$$\langle Ax - Ay, j \rangle \geq 0, \ \ \forall x, y \in K.$$

It is said to be strongly accretive if there exists a positive constant $k \in (0,1)$ and such that for all $x, y \in K$, such that

$$\langle Ax - Ay, j\rangle \geq k\|x-y\|^q, \ \ \forall x, y \in K.$$

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, *monotonicity* and *accretivity* coincide. A multi-valued map A defined on a real Banach space E is called m-accretive if it is accretive and $R(I + rA) = E$ for some $r > 0$ and it is said to satisfy the range condition $R(I + rA) = E$ for all $r > 0$.
The operator A in the following example satisfies range condition.

Example 4. *Let* $A : \mathbb{R} \to 2^{\mathbb{R}}$ *defined by*

$$Ax = \begin{cases} sgn(x), & x \neq 0, \\ [-1,1], & x = 0, \end{cases} \tag{2.2}$$

where A is the subdifferential of the absolute value function, $\partial|.|$, then A is m-accretive. It can be shown that if $R(I + rA) = E$ for some $r > 0$, then this holds for all $r > 0$. Hence, m-accretive condition implies range condition.

The demiclosedness of a nonlinear operator T usually plays an important role in dealing with the convergence of fixed point iterative algorithms.

Definition 1. *Let E be a real Banach space and $T : D(T) \subset E \to E$ be a mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $\|x_n - Tx_n\|$ converges to zero, then $p \in F(T)$, where $F(T)$ denote the set of fixed points of the mapping T.*

Lemma 5 (Demiclosedness principle, [3])**.** *Let E be a real Banach space satisfying Opial's property, K be a closed convex subset of E, and $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then $I - T$ is demiclosed; that is,*

$$\{x_n\} \subset K, \ x_n \rightharpoonup x \in K \ \text{ and } \ (I-T)x_n \to y \ \text{ implies that } (I-T)x = y.$$

Lemma 6 ([22])**.** *Let E be a smooth real Banach space. Then, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle \ \forall x, y \in E.$$

Lemma 7 ([31])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*
(a) $\displaystyle\sum_{n=0}^{\infty} \alpha_n = \infty$, *(b)* $\displaystyle\limsup_{n\to\infty} \frac{\sigma_n}{\alpha_n} \leq 0$ *or* $\displaystyle\sum_{n=0}^{\infty} |\sigma_n| < \infty$. *Then* $\displaystyle\lim_{n\to\infty} a_n = 0$.

Theorem 8. *[9] Let $q > 1$ be a fixed real number and E be a smooth Banach space. Then the following statements are equivalent:*
(i) E is q-uniformly smooth.
(ii) There is a constant $d_q > 0$ such that for all $x, y \in E$

$$\|x + y\|^q \leq \|x\|^q + q\langle y \ , \ J_q(x)\rangle + d_q\|y\|^q.$$

(iii) There is a constant $c_1 > 0$ such that

$$\langle x - y \ , \ J_q(x) - J_q(y)\rangle \ \leq c_1\|x - y\|^q \quad \forall \ \ x, y \in E.$$

Lemma 9 ([8])**.** *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B(0)_r := \{x \in E : ||x|| \leq r\}$, a closed ball with center 0 and radius $r > 0$. For any given sequence $\{u_1, u_2, ..., u_n, ...\} \subset B(0)_r$ and any positive real numbers $\{\lambda_1, \lambda_2, ..., \lambda_n, ...\}$ with $\sum\limits_{k=1}^{\infty} \lambda_k = 1$, there exists a continuous, strictly increasing and convex function*

$$g : [0, 2r] \to \mathbb{R}^+, \ g(0) = 0,$$

such that for any integer i, j with $i < j$,

$$\|\sum_{k=1}^{\infty} \lambda_k u_k\|^2 \leq \sum_{k=1}^{\infty} \lambda_k\|u_k\|^2 - \lambda_i\lambda_j g(\|u_i - u_j\|).$$

Lemma 10. *[33] Let H be a real Hilbert space and K a nonempty, closed convex subset of H. Let $A : K \to H$ be a k-strongly monotone and L-Lipschitzian operator with $k > 0, \ \ L > 0$. Assume that $0 < \eta < \dfrac{2k}{L^2}$ and $\tau = \eta\Big(k - \dfrac{L^2\eta}{2}\Big)$. Then for each $t \in \Big(0, min\{1, \ \dfrac{1}{\tau}\}\Big)$, we have*

$$\|(I - t\eta A)x - (I - t\eta A)y\| \leq (1 - t\tau)\|x - y\| \ \forall x, y \in K.$$

Let C be a nonempty subsets of a real Banach space E. A mapping $Q_C : E \to C$ is said to be sunny if

$$Q_C(Q_C x + t(x - Q_C x)) = Q_C x$$

for each $x \in E$ and $t \geq 0$. A mapping $Q_C : E \to C$ is said to be a retraction if $Q_C x = x$ for each $x \in C$.

Lemma 11. *[26] Let C and D be nonempty subsets of a smooth real Banach space E with $D \subset C$ and $Q_D : C \to D$ a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_D z, J(y - Q_D z)\rangle \leq 0 \tag{2.3}$$

for all $z \in C$ and $y \in D$.

Remark 12. *If K is a nonempty closed convex subset of a Hilbert space H, then the nearest point projection P_K from H to K is the sunny nonexpansive retraction.*

The resolvent operator has the following properties:

Lemma 13. *[12] For any $r > 0$.*
(i) A is accretive if and only if the resolvent J_r^A of A is single-valued and nonexpansive;
(ii) A is m-accretive if and only if J_r^A of A is single-valued and nonexpansive and its domain is the entire E;
(iii) $0 \in A(x^*)$ if and only if $x^* \in F(J_r^A)$, where $F(J_r^A)$ denotes the fixed-point set of J_r^A.

Lemma 14. *([23]) For any $r > 0$ and $\mu > 0$, the following holds:*

$$\frac{\mu}{r}x + (1 - \frac{\mu}{r})J_r^A x \in D(J_r^A)$$

and

$$J_r^A x = J_\mu^A(\frac{\mu}{r}x + (1 - \frac{\mu}{r})J_r^A x).$$

Lemma 15. *[7] Let A be a continuous accretive operator defined on a real Banach space E with $D(A) = E$. Then A is m-accretive.*

3 Main results

For our main theorem, we shall need the following lemma.

Lemma 16. *Let $q > 1$ be a fixed real number and E be a q-uniformly smooth real Banach space with constant d_q. Let $A : E \to E$ be a k-strongly accretive and L-Lipschitzian operator with $k > 0$, $L > 0$. Assume that $\eta \in \Big(0, min\Big\{1,\ \Big(\frac{kq}{d_q L^q}\Big)^{\frac{1}{q-1}}\Big\}\Big)$ and $\tau = \eta\Big(k - \frac{d_q L^q \eta^{q-1}}{q}\Big)$. Then for each $t \in \Big(0, min\{1,\ \frac{1}{\tau}\}\Big)$, we have*

$$\|(I - t\eta A)x - (I - t\eta A)y\| \le (1 - t\tau)\|x - y\|,\ \forall x, y \in E. \tag{3.1}$$

Proof. Without loss of generality, assume $k < \frac{1}{q}$. Then, as $\eta < \Big(\frac{kq}{d_q L^q}\Big)^{\frac{1}{q-1}}$, we have $0 < qk - d_q L^q \eta^{q-1}$. Furthermore, from $k < \frac{1}{q}$, we have $qk - d_q L^q \eta^{q-1} < 1$ so that $0 < qk - d_q L^q \eta^{q-1} < 1$. By using (ii) of Theorem 8 and properties of A, it follows that

$$\begin{aligned}
\|(I - t\eta A)x - (I - t\eta A)y\|^q &\le \|x - y\|^q + q\langle t\eta Ay - t\eta Ax\ ,\ J_q(x - y)\rangle + d_q\|t\eta Ax - t\eta Ay\|^q \\
&\le \|x - y\|^q - qt\eta\langle Ax - Ay\ ,\ J_q(x - y)\rangle + d_q(t\eta)^q\|Ax - Ay\|^q \\
&\le \|x - y\|^q - qtk\eta\|x - y\|^q + d_q(Lt\eta)^q\|x - y\|^q \\
&\le \Big(1 - qtk\eta + d_q L^q t^q \eta^q\Big)\|x - y\|^q.
\end{aligned}$$

Therefore

$$\|(I-t\eta A)x-(I-t\eta A)y\| \leq \Big(1-qtk\eta+d_qL^qt\eta^q\Big)^{\frac{1}{q}}\|x-y\|. \tag{3.2}$$

Using definition of τ, inequality (3.2) and inequality $(1+x)^s \leq 1+sx$, for $x>-1$ and $0<s<1$, we have

$$\begin{aligned}\|(I-t\eta A)x-(I-t\eta A)y\| &\leq \Big(1-tk\eta+\frac{d_qL^qt\eta^q}{q}\Big)\|x-y\| \\ &\leq \Big(1-t\eta(k-\frac{d_qL^q\eta^{q-1}}{q})\Big)\|x-y\| \\ &\leq (1-t\tau)\|x-y\|,\end{aligned}$$

which gives us the required result (3.1). This completes the proof. □

Remark 17. *Lemma 16 is one generalization of Lemma 10 for a Banach space.*

We are now in a position to state and prove our main result.

Theorem 18. *Let $q>1$ be a fixed real number and E be a q-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map. Let K be a nonempty, closed and convex subset of E which is a nonexpansive retract of E with Q_K as the nonexpansive retraction. Let $\{B_i\}$, $i\in\mathbb{N}^*$ be an infinite family of accretive operators of E such that $F:=\bigcap_{i=1}^{\infty}B_i^{-1}(0)\neq\emptyset$ and $\bigcap_{i=1}^{\infty}\overline{D(B_i)}\subset K\subset\bigcap_{i=1}^{\infty}R(I+rB_i)$, for all $r>0$. Let $A:K\to E$ be a k-strongly accretive and L-Lipschitzian operator and $f:K\to E$ be a b-Lipschitzian mapping with a constant $b\geq 0$. Let $\{x_n\}$ and $\{y_n\}$ be sequences defined iteratively from arbitrary $x_0\in K$ by:*

$$\begin{cases} y_n=\beta_{n,0}x_n+\displaystyle\sum_{i=1}^{\infty}\beta_{n,i}J_{r_n}^{B_i}x_n, \\ x_{n+1}=Q_K\Big(\alpha_n\gamma f(x_n)+(I-\eta\alpha_nA)y_n\Big).\end{cases} \tag{3.3}$$

Let $\{r_n\}\subset]0,\infty[$, $\{\beta_{n,i}\}$ and $\{\alpha_n\}$ be real sequences in $(0,1)$ satisfying:
(i) $\lim\limits_{n\to\infty}\alpha_n=0$; *(ii)* $\sum_{n=0}^{\infty}\alpha_n=\infty$, $\sum_{i=0}^{\infty}\beta_{n,i}=1$,
(iii) $\lim\limits_{n\to\infty}\inf r_n>0$, *and* $\lim\limits_{n\to\infty}\inf\beta_{n,0}\beta_{n,i}>0$, *for all* $i\in\mathbb{N}$.
Assume that $0<\eta<\Big(\frac{kq}{d_qL^q}\Big)^{\frac{1}{q-1}}$ and $0<b\gamma<\tau$, where $\tau=\eta\Big(k-\frac{d_qL^q\eta^{q-1}}{q}\Big)$. Then the sequence $\{x_n\}$ generated by (3.3) converges strongly to $x^*\in F$, which is a unique solution of variational inequality

$$\langle\eta Ax^*-\gamma f(x^*),J(x^*-p)\rangle\leq 0,\quad\forall p\in F. \tag{3.4}$$

Proof. First of all, we show that the uniqueness of a solution of the variational inequality (3.4). Suppose both $x^* \in F$ and $x^{**} \in F$ are solutions to (3.4). Then

$$\langle \eta Ax^* - \gamma f(x^*), J(x^* - x^{**})\rangle \leq 0 \tag{3.5}$$

and

$$\langle \eta Ax^{**} - \gamma f(x^{**}), J(x^{**} - x^*)\rangle \leq 0. \tag{3.6}$$

Adding up (3.5) and (4.3) yields

$$\langle \eta Ax^{**} - \eta Ax^* + \gamma f(x^*) - \gamma f(x^{**}), J(x^{**} - x^*)\rangle \leq 0. \tag{3.7}$$

$$\begin{aligned}
\frac{d_q L^q \eta^{q-1}}{q} > 0 \quad &\Longleftrightarrow \quad k - \frac{d_q L^q \eta^{q-1}}{q} < k \\
&\Longleftrightarrow \quad \eta\Big(k - \frac{d_q L^q \eta^{q-1}}{q}\Big) < k\eta \\
&\Longleftrightarrow \quad \tau < k\eta.
\end{aligned}$$

It follows that

$$0 < b\gamma < \tau < k\eta.$$

Noticing that

$$\langle \eta Ax^{**} - \eta Ax^* + \gamma f(x^*) - \gamma f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \geq (k\eta - b\gamma)\|x^* - x^{**}\|^2,$$

which implies that $x^* = x^{**}$ and the uniqueness is proved. Below we use x^* to denote the unique solution of (3.4). Without loss of generality, we can assume $\alpha_n \in \Big(0, min\{1\ , \dfrac{1}{\tau}\}\Big)$.
Now, we prove that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Let $p \in F$. Using (3.3) and the fact that $J_{r_n}^{B_i}$ are nonexpansive, we have

$$\begin{aligned}
\|y_n - p\| \quad &= \quad \|\beta_{n,0}x_n + \sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} x_n - p\| \\
&\leq \quad \beta_{n,0}\|x_n - p\| + \sum_{i=1}^{\infty} \beta_{n,i} \|J_{r_n}^{B_i} x_n - p\| \\
&\leq \quad \beta_{n,0}\|x_n - p\| + \sum_{i=1}^{\infty} \beta_{n,i} \|x_n - p\| \\
&\leq \quad \|x_n - p\|.
\end{aligned}$$

Using Lemma 16, we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|Q_K\Big(\alpha_n\gamma f(x_n)+(I-\eta\alpha_n A)y_n\Big)-p\| \\
&\leq \|\alpha_n\gamma f(x_n)+(I-\eta\alpha_n A)y_n-p\| \\
&\leq \alpha_n\gamma\|f(x_n)-f(p)\|+(1-\tau\alpha_n)\|y_n-p\|+\alpha_n\|\gamma f(p)-\eta Ap\| \\
&\leq (1-\alpha_n(\tau-b\gamma))\|x_n-p\|+\alpha_n\|\gamma f(p)-\eta Ap\| \\
&\leq \max\{\|x_n-p\|,\frac{\|\gamma f(p)-\eta Ap\|}{\tau-b\gamma}\}.
\end{aligned}$$

By induction, it is easy to see that

$$\|x_n-p\|\leq \max\{\|x_0-p\|,\frac{\|\gamma f(p)-\eta Ap\|}{\tau-b\gamma}\},\quad n\geq 1.$$

Hence $\{x_n\}$ is bounded also are $\{f(x_n)\}$, and $\{Ax_n\}$.
Let $k\in\mathbb{N}^*$, from Lemma 9 and (3.3), we have

$$\begin{aligned}
\|y_n-p\|^2 &= \|\beta_{n,0}x_n+\sum_{i=1}^{\infty}\beta_{n,i}J_{r_n}^{B_i}x_n-p\|^2 \\
&\leq \beta_{n,0}\|x_n-p\|^2+\sum_{i=1}^{\infty}\beta_{n,i}\|J_{r_n}^{B_i}x_n-p\|^2-\beta_{n,0}\beta_{n,k}g(\|J_{r_n}^{B_k}x_n-x_n\|) \\
&\leq \|x_n-p\|^2-\beta_{n,0}\beta_{n,k}g(\|J_{r_n}^{B_k}x_n-x_n\|).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|Q_K\Big(\alpha_n\gamma f(x_n)+(I-\eta\alpha_n A)y_n\Big)-p\|^2 \\
&\leq \|\alpha_n(\gamma f(x_n)-\eta Ap)+(I-\eta\alpha_n A)(y_n-p)\|^2 \\
&\leq \alpha_n^2\|\gamma f(x_n)-\eta Ap\|^2+(1-\tau\alpha_n)^2\|y_n-p\|^2+2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n) \\
&\quad -\eta Ap\|\|y_n-p\| \\
&\leq \alpha_n^2\|\gamma f(x_n)-\eta Ap\|^2+(1-\tau\alpha_n)^2\|x_n-p\|^2-(1-\tau\alpha_n)^2\beta_{n,0}\beta_{n,k}g(\|J_{r_n}^{B_k}x_n-x_n\|) \\
&\quad +2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta Ap\|\|x_n-p\|
\end{aligned}$$

Thus, for every $k\in\mathbb{N}^*$, we get

$$\begin{aligned}
(1-\tau\alpha_n)^2\beta_{n,0}\beta_{n,k}g(\|J_{r_n}^{B_k}x_n-x_n\|) &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n^2\|\gamma f(x_n)-\eta Ap\|^2 \\
&\quad +2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta Ap\|\|x_n-p\|. \qquad (3.8)
\end{aligned}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded, there exists a constant $C>0$ such that

$$(1-\tau\alpha_n)^2\beta_{n,0}\beta_{n,k}g(\|J_{r_n}^{B_k}x_n-x_n\|)\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n C. \qquad (3.9)$$

Let $VI(A,F)$ the solutions set of variational inequality (3.4). Now, we prove $VI(A,F)$ is nonempty. Let t_0 be a fixed real number such that $t_0\in\Big(0,min\{1\ ,\frac{1}{\tau}\}\Big)$. We observe that $Q_F(I+(t_0\gamma f-t_0\eta A))$

is a contraction, where Q_F is the sunny nonexpansive retraction from E to F. Indeed, for all $x, y \in K$, by Lemma 16, we have

$$\begin{aligned}
\|Q_F(I+(t_0\gamma f - t_0\eta A))x - Q_F(I+(t_0\gamma f - t_0\eta A))x\| &\leq \|(I+(t_0\gamma f - t_0\eta A))x \\
&\quad -(I+(t_0\gamma f - t_0\eta A))x\| \\
&\leq t_0\gamma\|f(x)-f(y)\| \\
&\quad +\|(I-t_0\eta A)x-(I-t_0\eta A)y\| \\
&\leq (1-t_0(\tau-\gamma))\|x-y\|.
\end{aligned}$$

Banach's Contraction Mapping Principle guarantees that $Q_F(I+(t_0\gamma f - t_0\eta A))$ has a unique fixed point, say $x_1 \in E$. That is, $x_1 = Q_F(I+(t_0\gamma f - t_0\eta A))x_1$. Thus, in view of Lemma 11, it is equivalent to the following variational inequality problem

$$\langle \eta A x_1 - \gamma f(x_1), J(x_1 - p)\rangle \leq 0, \quad \forall \ \ p \in F.$$

Hence, $x_1 \in VI(A, F)$. By the uniqueness of the solution of (3.4), we have $x_1 = x^*$.
Next, we prove that $\{x_n\}$ converges strongly to x^*. We divide the proof into two cases.
Case 1. Assume that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0.$$

It then implies from (3.9) that

$$\lim_{n\to\infty} \beta_{n,0}\beta_{n,k} g(\|J_{r_n}^{B_k} x_n - x_n\|) = 0. \tag{3.10}$$

Since $\lim_{n\to\infty} \inf \beta_{n,0}\beta_{n,k} > 0$ and property of g, we have

$$\lim_{n\to\infty} \|x_n - J_{r_n}^{B_k} x_n\| = 0. \tag{3.11}$$

By using the resolvent identity (Lemma 14), for any $r > 0$, we conclude that

$$\begin{aligned}
\|x_n - J_r^{B_k} x_n\| &\leq \|x_n - J_{r_n}^{B_k} x_n\| + \|J_{r_n}^{B_k} x_n - J_r^{B_k} x_n\| \\
&\leq \|x_n - J_{r_n}^{B_k} x_n\| + \|J_r^{B_k} x_n\left(\frac{r}{r_n}x_n + (1-\frac{r}{r_n})J_{r_n}^{B_k} x_n\right) - J_r^{B_k} x_n\| \\
&\leq \|x_n - J_{r_n}^{B_k} x_n\| + \|\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}^{B_k} x_n - x_n\| \\
&\leq \|x_n - J_{r_n}^{B_k} x_n\| + |1-\frac{r}{r_n}|\,\|J_{r_n}^{B_k} x_n - x_n\| \to 0, \ \ n\to\infty, \ \ \forall k \in \mathbb{N}^*.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|x_n - J_r^{B_k} x_n\| = 0. \tag{3.12}$$

We show that $\limsup_{n\to+\infty} \langle \eta A x^* - \gamma f(x^*), J(x^* - x_n)\rangle \leq 0$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to a in K and

$$\limsup_{n\to+\infty} \langle \eta A x^* - \gamma f(x^*), J(x^* - x_n)\rangle = \lim_{j\to+\infty} \langle \eta A x^* - \gamma f(x^*), J(x^* - x_{n_j})\rangle.$$

From (3.12), the fact that $J_r^{B_k}$, $k \in \mathbb{N}^*$ are nonexpansive and Lemma 5, we obtain $a \in F$. On the other hand, the assumption that the duality mapping is weakly continuous and the fact that $x^* \in VI(A, F)$, we then have

$$\begin{aligned}\limsup_{n\to+\infty}\langle \eta Ax^* - \gamma f(x^*), J(x^* - x_n)\rangle &= \lim_{j\to+\infty}\langle \eta Ax^* - \gamma f(x^*), J(x^* - x_{n_j})\rangle \\ &= \langle \eta Ax^* - \gamma f(x^*), J(x^* - a)\rangle \leq 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$. Applying Lemma 6, we get that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|Q_K(\alpha_n\gamma f(x_n) + (I - \eta\alpha_n A)y_n) - x^*\|^2 \\ &\leq \langle \alpha_n\gamma f(x_n) + (I - \eta\alpha_n A)y_n - x^*, J(x_{n+1} - x^*)\rangle \\ &= \langle \alpha_n\gamma f(x_n) + (I - \eta\alpha_n A)y_n - x^* - \alpha_n\gamma f(x^*) + \alpha_n\gamma f(x^*) - \alpha_n\eta Ax^* \\ &\quad +\alpha_n\eta Ax^*, J(x_{n+1} - x^*)\rangle \\ &\leq \Big(\alpha_n\gamma\|f(x_n) - f(x^*)\| + \|(I - \alpha_n\eta A)(y_n - x^*)\|\Big)\|x_{n+1} - x^*\| \\ &\quad +\alpha_n\langle \eta Ax^* - \gamma f(x^*), J(x^* - x_{n+1})\rangle \\ &\leq (1 - \alpha_n(\tau - b\gamma))\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle \eta Ax^* - \gamma f(x^*), J(x^* - x_{n+1})\rangle \\ &\leq (1 - \alpha_n(\tau - b\gamma))\|x_n - x^*\|^2 + 2\alpha_n\langle \eta Ax^* - \gamma f(x^*), J(x^* - x_{n+1})\rangle.\end{aligned}$$

From Lemma 7, its follows that $x_n \to x^*$.

Case 2. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing. Set $B_n = \|x_n - x^*\|$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ \ B_k \leq B_{k+1}\}$.

We have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \leq B_{\tau(n)+1}$ for $n \geq n_0$. Let $i \in \mathbb{N}^*$, from (3.9), we have

$$(1 - \tau\alpha_{\tau(n)})^2\beta_{\tau(n),0}\beta_{\tau(n),i}\,g(\|J^{B_i}_{r_{\tau(n)}}x_{\tau(n)} - x_{\tau(n)}\|) \leq \alpha_{\tau(n)}C \to 0 \text{ as } n \to \infty.$$

Furthermore, we have

$$\beta_{\tau(n),0}\beta_{\tau(n),i}\,g(\|J^{B_i}_{r_{\tau(n)}}x_{\tau(n)} - x_{\tau(n)}\|) \to 0 \text{ as } n \to \infty.$$

Hence,

$$\lim_{n\to\infty}\|J^{B_i}_{r_{\tau(n)}}x_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{3.13}$$

By same argument as in Case 1, we can show that $x_{\tau(n)}$ and $y_{\tau(n)}$ are bounded in K and $\limsup\limits_{\tau(n)\to+\infty}\langle \eta Ax^* - \gamma f(x^*), J(x^* - x_{\tau(n)})\rangle \leq 0$. We have for all $n \geq n_0$,

$$0 \leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \leq \alpha_{\tau(n)}[-(\tau - b\gamma)\|x_{\tau(n)} - x^*\|^2 + 2\langle \eta Ax^* - \gamma f(x^*), J(x^* - x_{\tau(n)+1})\rangle],$$

which implies that

$$\|x_{\tau(n)} - x^*\|^2 \leq \frac{2}{\tau - b\gamma}\langle \eta Ax^* - \gamma f(x^*), J(x^* - x_{\tau(n)+1})\rangle.$$

Then, we have

$$\lim_{n\to\infty}\|x_{\tau(n)} - x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Furthermore, for all $n \geq n_0$, we have $B_{\tau(n)} \leq B_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $n > \tau(n)$); because $B_j > B_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence, we have for all $n \geq n_0$,

$$0 \leq B_n \leq \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim\limits_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

As a consequence of Theorem 18, we have the following theorem.

Theorem 19. *Let $q > 1$ be a fixed real number and E be a q-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map. Let $\{B_i\},\ i \in \mathbb{N}^*$ be an infinite family of m-accretive operators of E such that $F := \bigcap\limits_{i=1}^{\infty} {B_i}^{-1}(0) \neq \emptyset$. Let $A : E \to E$ be a k-strongly accretive and L-Lipschitzian operator and and $f : K \to E$ be a b-Lipschitzian mapping with a constant $b \geq 0$. Let $\{x_n\}$ and $\{y_n\}$ be sequences defined iteratively from arbitrary $x_0 \in E$ by:*

$$\begin{cases} y_n = \beta_{n,0}x_n + \displaystyle\sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \eta\alpha_n A)y_n. \end{cases} \tag{3.14}$$

Let $\{r_n\} \subset]0,\infty[,\ \ \{\beta_{n,i}\}$ and $\{\alpha_n\}$ be real sequences in $(0,1)$ satisfying:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$; (ii) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, $\sum\limits_{i=0}^{\infty} \beta_{n,i} = 1$,

(iii) $\lim\limits_{n\to\infty} \inf r_n > 0$, and $\lim\limits_{n\to\infty} \inf \beta_{n,0}\beta_{n,i} > 0$, for all $i \in \mathbb{N}$.

Assume that $0 < \eta < \Big(\dfrac{kq}{d_q L^q}\Big)^{\frac{1}{q-1}}$ and $0 < b\gamma < \tau$, where $\tau = \eta\Big(k - \dfrac{d_q L^q \eta^{q-1}}{q}\Big)$. Then the sequence $\{x_n\}$ generated by (3.14) converges strongly to $x^* \in F$, which is a unique solution of variational inequality (3.4) .

Proof. Since B_i are m-accretive operators, we conclude that B_i are accretive and satisfy the condition $R(I + rB_i) = E$ for all $r > 0$. Setting $K = E$ in Theorem 18, we obtain the desired result. □

Corollary 1. *Let H be a real Hilbert space. Let K be a nonempty, closed and convex subset of H. Let $\{B_i\},\ i \in \mathbb{N}^*$ be an infinite family of monotone operators of H such that $F := \bigcap\limits_{i=1}^{\infty} {B_i}^{-1}(0) \neq \emptyset$ and $\bigcap\limits_{i=1}^{\infty} \overline{D(B_i)} \subset K \subset \bigcap\limits_{i=1}^{\infty} R(I + rB_i)$, for all $r > 0$. Let $A : K \to H$ be a strongly bounded linear* *operator and and $f : K \to E$ be a b-Lipschitzian mapping with a constant $b \geq 0$. Let $\{x_n\}$ and $\{y_n\}$ be sequences defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} y_n = \beta_{n,0}x_n + \sum_{i=1}^{\infty} \beta_{n,i} J_{r_n}^{B_i} x_n, \\ x_{n+1} = P_K\Big(\alpha_n \gamma f(x_n) + (I - \eta\alpha_n A)y_n\Big). \end{cases} \tag{3.15}$$

Let $\{r_n\} \subset]0, \infty[$, $\{\beta_{n,i}\}$ and $\{\alpha_n\}$ be real sequences in $(0, 1)$ satisfying:
(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{\infty} \beta_{n,i} = 1$,
(iii) $\lim_{n\to\infty} \inf r_n > 0$, *and* $\lim_{n\to\infty} \inf \beta_{n,0}\beta_{n,i} > 0$, *for all* $i \in \mathbb{N}$.
Assume that $0 < \eta < \frac{2k}{\|A\|^2}$ and $0 < b\gamma < \tau$, where $\tau = \eta\Big(k - \frac{\|A\|^2\eta}{2}\Big)$. Then the sequence $\{x_n\}$ generated by (3.15) converges strongly to $x^* \in F$, which is the optimality condition for the minimization problem

$$\min_{x\in F} \frac{\eta}{2}\langle Ax, x\rangle - h(x), \tag{3.16}$$

where h is a potential function for γf (i.e. $h^{'}(x) = \gamma f(x)$ on K).

Proof. From Remark 1, we have A is strongly monotone and $\|A\|$-Lipschitz. the proof follows Theorem 18. □

4 Applications

In this section, as applications, we will utilize Theorem 18 to deduced several results. As a direct consequence of Theorem 18, we have the following results:

4.1 Application to equilibrium problems

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the real numbers. The equilibrium problem for F is to find $x \in C$ such that

$$F(x, y) \geq 0, \ \ \forall y \in C. \tag{4.1}$$

The set of solutions is denoted by $EP(F)$. Equilibrium problems which were introduced by Fan [11] and Blum and Oettli [4] have had a great impact and influence on the development of several branches of pure and applied sciences. For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:
$(A1)$ $F(x, x) = 0$ for all $x \in C$;
$(A2)$ F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

$(A3)$ for each $x, y, z \in C$,

$$\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y)$$

$(A4)$ for each $x \in C$, $\quad y \to F(x, y)$ is convex and lower semicontinuous.

Lemma 20. *[6] Assume that $F : C \times C \to \mathbb{R}$ satisfying $(A1)$-$(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows*

$$T_r(x) = \{z \in C,\ F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\},$$

for all $x \in H$. Then, the following hold:
1.T_r is single-valued;
2.T_r is firmly nonexpansive, i.e., $\|T_r(x) - T_r(y)\|^2 \leq \langle T_r x - T_r y, x - y\rangle$ for any $x, y \in H$;
3.$F(T_r) = EP(F)$;
4.$EP(F)$ is closed and convex.

The following lemma appears implicitly in [29].

Lemma 21. *[29] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ satisfy $(A1) - (A4)$. Let A_F be a set-valued mapping of H into itself defined by:*

$$A_F x = \begin{cases} \{z \in H,\ F(x, y) \geq \langle y - x, z\rangle,\ \forall y \in C,\}\ \ \forall x \in C \\ \emptyset,\ \ x \notin C. \end{cases} \tag{4.2}$$

Then $EP(F) = {A_F}^{-1}(0)$ and A_F is a maximal monotone operator with $D(A_F) \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the map T_r defined as Lemma 20 coincides with the resolvent of A_F , i.e,

$$T_r x = \big(I + rA_F\big)^{-1} x.$$

Using Theorem 18 , we prove a strong convergence theorem for an equilibrium problem in a Hilbert space.

Theorem 22. *Let H be a real Hilbert space and $F : H \times H \to \mathbb{R}$ satisfying $(A1)$-$(A4)$ such that $EP(F) \neq \emptyset$. Let $A : H \to H$ be a k-strongly monotone and L-Lipschitzian operator and $f : K \to E$ be a b-Lipschitzian mapping with a constant $b \geq 0$. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be a sequences defined iteratively from arbitrary $x_0 \in H$ by:*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0,\ \forall y \in H \\ y_n = \beta_n x_n + (1 - \beta_n)u_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \eta\alpha_n A)y_n. \end{cases} \tag{4.3}$$

Let $\{r_n\} \subset]0, \infty[$, $\{\beta_n\}$ and $\{\alpha_n\}$ be real sequences in $(0, 1)$ satisfying:
(i) $\lim_{n\to\infty} \alpha_n = 0$; $\quad (ii)$ $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\ \beta_n \in [a, b] \subset (0, 1)$.

(iii) $\lim\limits_{n\to\infty} \inf r_n > 0$.

Assume that $0 < \eta < \dfrac{2k}{L^2}$ *and* $0 < b\gamma < \tau$, *where* $\tau = \eta\Big(k - \dfrac{L^2\eta}{2}\Big)$. *Then the sequence* $\{x_n\}$ *generated by* (4.3) *converge strongly to* $x^* \in EP(f)$, *which is a unique solution of variational inequality*

$$\langle \eta Ax^* - \gamma f(x^*), x^* - p\rangle \le 0, \quad \forall p \in EP(F). \tag{4.4}$$

Proof. Since $F : H \times H \to \mathbb{R}$ satisfying $(A1)$-$(A4)$, we have that the mapping A_F defined by Lemma 21 is a maximal and monotone operator. Put $B = A_F$ in Theorem 19 (with i=1). Then, we obtain that $u_n = T_{r_n}x_n = J^B_{r_n}x_n$. Therefore, we arrive at the desired results. □

4.2 Application to an infinite family of continuous pseudocontractive mappings.

Let K be a nonempty, closed convex subset of a real Banach space$E$. A mapping $T : K \to K$ is said to be pseudocontractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2, \forall x, y \in K.$$

It is well known that the class of pseudocontractive mapping is more general than the class of non-expansive mapping. Moreover, there exists a relationship between the class of accretive mappings and the class of pseudocontractive mappings. A mapping $A : K \to E$ is said to be pseudocontractive if $T := I - A$ is accretive. We can observe that x^* is a zero of the accretive mapping A if and only if it is a fixed point of the pseudocontractive mapping $T := I - A$.
Hence, one has the following result.

Theorem 23. *Let* $q > 1$ *be a fixed real number and* E *be a* q*-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map. Let* K *be a nonempty, closed and convex subset of* E *which is a nonexpansive retract of* E *with* Q_K *as the nonexpansive retraction. Let* $T_i : K \to E$, $i \in \mathbb{N}^*$ *be an infinite family of continuous pseudo-contractive mappings of such that* $\bigcap\limits_{i=1}^{\infty} F(T_i) \neq \emptyset$. *For each* $r > 0$, *let* $J^i_r := (I + r(I - T_i))^{-1}$, $i \in \mathbb{N}^*$. *Let* $A : K \to E$ *be a* k*-strongly accretive and* L*-Lipschitzian operator and* $f : K \to E$ *be an* b*-Lipschitzian mapping with a constant* $b \ge 0$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences defined iteratively from arbitrary* $x_0 \in K$ *by:*

$$\begin{cases} y_n = \beta_{n,0}x_n + \displaystyle\sum_{i=1}^{\infty} \beta_{n,i}J^i_{r_n}x_n \\ x_{n+1} = Q_K\Big(\alpha_n\gamma f(x_n) + (I - \eta\alpha_n A)y_n\Big). \end{cases} \tag{4.5}$$

Let $\{r_n\} \subset]0,\infty[$, $\{\beta_{n,i}\}$ *and* $\{\alpha_n\}$ *be real sequences in* $(0,1)$ *satisfying:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$; $\quad (ii)$ $\displaystyle\sum_{n=0}^{\infty} \alpha_n = \infty$, $\displaystyle\sum_{i=0}^{\infty} \beta_{n,i} = 1$, *(iii)* $\lim_{n\to\infty}\inf r_n > 0$, *and* $\lim_{n\to\infty}\inf \beta_{n,0}\beta_{n,i} > 0$, *for all* $i \in \mathbb{N}^*$.

Assume that $0 < \eta < \left(\frac{kq}{d_q L^q}\right)^{\frac{1}{q-1}}$ *and* $0 < b\gamma < \tau$, *where* $\tau = \eta\Big(k - \frac{d_q L^q \eta^{q-1}}{q}\Big)$. *Then the sequence* $\{x_n\}$ *generated by* (4.5) *converges strongly to* $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$, *which is a unique solution of variational inequality*

$$\langle \eta A x^* - \gamma f(x^*), J(x^* - p)\rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{\infty} F(T_i). \tag{4.6}$$

Proof. For each $i \in \mathbb{N}^*$, we set $B_i = I - T_i$ into Theorem 18. Then $F(T_i) = {B_i}^{-1}(0)$, for all $i \in \mathbb{N}^*$ and hence $\bigcap_{i=1}^{\infty} F(T_i) = \bigcap_{i=1}^{\infty} {B_i}^{-1}(0)$. Furthermore, each B_i is m-accretive. Therefore, the proof is complete from Theorem 18. □

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Mathematical Modeling of Chikungunya Dynamics: Stability and Simulation

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ABSTRACT

Infection due to Chikungunya virus (CHIKV) has a substantially prolonged recuperation period that is a long period between the stage of infection and recovery. However, so far in the existing models (SIR and SEIR), this period has not been given due attention. Hence for this disease, we have modified the existing SEIR model by introducing a new section of human population which is in the recuperation stage or in other words the human population that is no more showing acute symptoms but is yet to attain complete recovery. A mathematical model is formulated and studied by means of existence and stability of its disease free equilibrium (DFE) and endemic equilibrium (EE) points in terms of the associated basic reproduction number (R_0).

RESUMEN

La infección debida al virus Chikungunya (CHIKV) tiene un período de recuperación sustancialmente prolongado, que es un período largo entre la etapa de infección y recuperación. Sin embargo, hasta ahora en los modelos existentes (SIR y SEIR), este período no ha recibido suficiente atención. Por tanto, para esta enfermedad, hemos modificado el modelo SEIR existente introduciendo una nueva sección de población humana que está en la etapa de recuperación o, en otras palabras, la población humana que ya no muestra síntomas agudos pero todavía no se recupera completamente. Se formula y estudia un modelo matemático a través de la existencia y estabilidad de su equilibrio libre de enfermedad (DFE) y puntos de equilibrio endémico (EE) en términos del número de reproducción básico asociado (R_0).

Keywords and Phrases: Equilibrium point, disease free equilibrium, endemic equilibrium, reproduction number, local stability, global stability.

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1 Introduction

In recent past, the study of vector borne diseases has gained considerable attention and mathematics have become a useful tool for such studies. Several temporal deterministic models have been proposed for diseases like dengue, malaria, chikungunya etc. Chikungunya is a disease caused by the chikungunya virus, an RNA genome which is a member of the Alphavirus genus in the family of Togaviridae. It is a mosquito borne viral disease which is transmitted to humans through Aedes aegypti mosquito bite [1]. In 1952, chikungunya was first confirmed as the cause of an epidemic of dengue like illness on the Comoros islands located on the eastern coast of northern Mozambique [2]. Since its discovery, numerous CHIKV outbreaks with irregular intervals of 2-20 years have affected Asian, African, European and American countries. In Thailand, the first report of chikungunya infection occurred in Bangkok in 1958 [3]. In India, the virus emerged in parts of Vellore, Calcutta and Maharashtra in the early 1960's [4]. The virus continued to spread in Sri Lanka in 1969 and many countries of Southeast Asia such as Myanmar, Indonesia and Vietnam [4]. Later, some irregular cases of chikungunya fever were also seen in many provinces of Thailand in the period from 1976 to 1995 [3]. From 1999 to 2000, the reemergence of chikungunya occurred in Democratic Republic of Congo [2], 13,500 cases were reported in Lamu, Kenya in 2004 [5]. In the years 2005-2007, there occurred an outbreak in Reunion islands in the Indian Ocean. In 2007, 197 cases were reported in Europe due to chikungunya [1]. The outbreak mutated to facilitate the disease transmission by Aedes albopictus from the tiger mosquito family. It was a mutation in one of the viral envelope genes which allowed the virus to be present in the mosquito saliva only two days after the infection and seven days in Aedes aegypti mosquitoes. The results indicated that the areas where the tiger mosquitoes are present could have a greater risk of outbreak.

After an effective bite from a mosquito infected with CHIKV, the incubation period (i.e., the time elapsed between exposure to pathogenic organism and when symptoms and signs are first apparent) usually lasts for 3-7 days with fever as the most prominent symptom. The symptoms of chikungunya fever differ from the normal fever as they are accompanied with acute joint pains. Other common symptoms are nausea, rashes, headache and fatigue. Some cases may result in neurological, retinal and carpological complications as well, which makes it difficult for older people to recover as against young people. In some instances, people live with joint pains for years which indicates that the recuperation period can last for a long time. The symptoms of chikungunya are generally mild and the disease may sometimes be misdiagnosed with Zika and Dengue due to similarity in symptoms. There have been very few cases where chikungunya resulted in death and mostly infected individuals are expected to make full recovery with lifelong immunity. As such, there is no preventive vaccine or cure for chikungunya. One can only manage the symptoms by taking medications for temporary relief. To prevent the spread of disease, breeding sites for the mosquitoes should be checked. Using mosquito repellents and wearing long sleeve clothes and full pants can help in preventing mosquito bite. For more such information one may refer to [1].

Increasing globalization and factors contributing to climate change brought about a sudden expansion of mosquito breeding sites. This makes it necessary to improve the vector control techniques and to identify the indexes that monitor thresholds for such programs. Through the 20^{th} century, mathematical modeling has been extensively used to study epidemic diseases. Futhermore, this branch of mathematics is also being used to devise optimal control strategies for various infectious diseases. Like M. Barro et al. [6] introduced an optimal control for a SIR model governed by an ODE system with time delay. And, O. K. Oare [7] considered and analyzed a deterministic multipatch hepatitis C virus model for it.

In context of infection due to chikungunya virus, Y. Dumont et al. [8] proposed a model associated with the time course of the first epidemic of chikungunya in several cities of Reunion Island. A model describing the mosquito population dynamics and the virus transmission to human population was discussed by D. Moulay et al. [9]. Although simplistic, L. Yacob et al. [10] gave a model which provided a close approximation of the peak incidence of the outbreak and the final epidemic size. S. Naowarat and I. M. Tang [11] studied the model taking into consideration the presence of two species of Aedes mosquito (Aedes aegypti and Aedes albopictus). D. H. Palacio and J. Ospina [12] derived measures of disease control, by means of three scenarios, namely a single vector, two vectors, and two vectors and human and non-human reservoirs. It also showed the need to periodically evaluate the effectiveness of vector control measures. F. B. Agusto et al. [13] described the chikungunya model of three age structured transmission dynamics by considering juvenile, adult and senior population, where the dynamics of shift in individuals from one stage to another was studied.

In this paper, we introduce a deterministic model to study the dynamics and transmission of chikungunya virus by considering a very significant section from the class of infected individuals. Usually, the existing models focus on the SIR or the SEIR human population model and SEI mosquito population model. Since the period from the infected stage to the complete recovery stage is quite long for this disease, so it becomes significant to study that particular class of human population which has recovered from acute symptoms of the disease but is yet to attain full recovery. Though the class no longer shows the immediate symptoms like fever, rashes, nausea etc. but at the same time they are bearing the latent and the passive effects of the disease like joint pains, fatigue, headache etc. Generally such ailments continue for a prolonged period which may vary from individual to individual. But as long as the patient is suffering from these ailments, he or she cannot be declared as fully recovered [14]. Focussing on this category of patients, we introduce a new compartment between compartments of the infected and the recovered human population within the existing SEIR model. We refer to it as the recuperation compartment and denote it by R'. So, in this paper our aim is to study, analyse and investigate in detail the model showing the interaction between the human population divided into five compartments resulting into a SEIR$'$R model and the mosquito population into the traditional three compartments which we denote by XYZ model.

The paper is divided as follows: Section 2 deals with the formulation of the model, section 3 analyses its feasibility, section 4 determines the disease free equilibrium (DFE) and establishes its local and global stability , section 5 deals with the existence of endemic equilibrium (EE) and its local stability. Also by means of simulation of the formulated model, we provide a visualization to the dynamics of this disease, in section 6. Finally related to our model, some conclusions are stated.

2 Model Formulation

In this section, an epidemic model is formulated for chikungunya disease. Let N_H represent the total human population which is further subdivided into five categories; susceptibles (S), humans exposed to infection (E), infected humans (I), population in recuperation phase (R$'$) and finally the population that has attained complete recovery (R). So, the traditional SEIR epidemic model has been modified to a more relevant and practically applicable SEIR$'$R model. Hence in this case, at any time t

$$N_H(t) = S(t) + E(t) + I(t) + R'(t) + R(t). \tag{2.1}$$

Let N_M represent the total mosquito population which is further subdivided into 3 parts; susceptible mosquitoes (X), mosquitoes exposed to infection (Y) and infectious mosquitoes (Z). So the total mosquito population is $N_M(t) = X(t) + Y(t) + Z(t)$.

For human population, let μ be the constant birth rate and ζ be the natural death rate. Then the rate of change of susceptible human population is given by

$$\frac{dS}{dt} = \mu - \lambda_H S - \zeta S, \tag{2.2}$$

where $\lambda_H = \dfrac{\beta B_H Z}{N_H}$. B_H is the transmission probability per contact for susceptible humans (S) and β is the mosquito biting rate for transfer of infection from infectious mosquito class (Z) to susceptible human population (S). As only the susceptible human population out of the whole population is prone to get infection, thereby we divide the expression by N_H. The rate of change of exposed human population is given by

$$\frac{dE}{dt} = \lambda_H S - \alpha E - \zeta E, \tag{2.3}$$

where α is the rate of progression from exposed (E) to infected (I) human population. Here the inflow rate is λ_H and outflow rate is $\alpha + \zeta$. Similarly, the rate of change of infected human population is

$$\frac{dI}{dt} = \alpha E - \gamma I - (\zeta + \zeta_1)I, \tag{2.4}$$

where ζ_1 is death rate due to infection and γ is progression rate of infected (I) to recuperated (R$'$) human population. Now, rate of change of human population in recuperation phase is

$$\frac{dR'}{dt} = \gamma I - \lambda R' - (\zeta + \zeta_2)R', \tag{2.5}$$

where ζ_2 is the death rate of humans in recuperated phase due to virus and λ is the rate of progression from recuperation (R$'$) to the recovery phase (R). Finally, rate of change of recovered human population is,

$$\frac{dR}{dt} = \lambda R' - \zeta R. \tag{2.6}$$

Again for the mosquito population, let ρ be the constant birth rate and κ be the natural death rate, then the rate of susceptible mosquito population is given by

$$\frac{dX}{dt} = \rho - \lambda_M X - \kappa X, \tag{2.7}$$

where $\lambda_M = \dfrac{\nu B_M(I + R')}{N_H}$. B_M is the transmission probability per contact for susceptible mosquito population (X) and ν is the mosquito biting rate for transfer of infection from infected (I) or recuperated (R') human population to susceptible mosquito population (X). Again there occurs division by N_H because infection can be transfered to mosquitoes only by a certain fraction of human population. Now, the rate of change of exposed mosquito population is given by

$$\frac{dY}{dt} = \lambda_M X - \psi Y - \kappa Y, \tag{2.8}$$

where ψ is the progression rate from exposed (Y) to infectious (Z) mosquito population. Here the inflow rate is λ_M and outflow rate is $\psi + \kappa$. Similarly, the rate of change of mosquito population carrying infection is

$$\frac{dZ}{dt} = \psi Y - \kappa Z. \tag{2.9}$$

Compiling the above discussion, we get the eight dimensional system of nonlinear ordinary differential equations that forms our Chikungunya Model (CM). The parameters and the variables used in the model (CM) are described in Table 1. To get a clear view of the inter relationships between various compartments in discussion, one may refer to Figure 1 which shows the schematic flow diagram of the model. The model (CM) is as follows:

$$\text{(CM)} \qquad \begin{aligned}
\frac{dS}{dt} &= \mu - \frac{\beta B_H Z S}{N_H} - \zeta S,\\
\frac{dE}{dt} &= \frac{\beta B_H Z S}{N_H} - \alpha E - \zeta E,\\
\frac{dI}{dt} &= \alpha E - \gamma I - (\zeta + \zeta_1) I,\\
\frac{dR'}{dt} &= \gamma I - \lambda R' - (\zeta + \zeta_2) R',\\
\frac{dR}{dt} &= \lambda R' - \zeta R,\\
\frac{dX}{dt} &= \rho - \frac{\nu B_M (I + R') X}{N_H} - \kappa X,\\
\frac{dY}{dt} &= \frac{\nu B_M (I + R') X}{N_H} - \psi Y - \kappa Y,\\
\frac{dZ}{dt} &= \psi Y - \kappa Z.
\end{aligned}$$

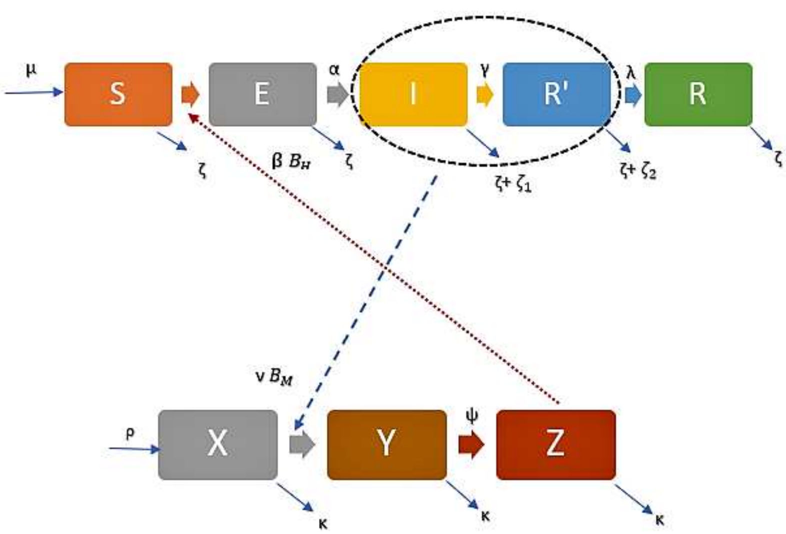


Figure 1: *Schematic diagram of Chikungunya Model (CM)*

Table 1: Description of variables and parameters used in model (CM)

Variables	Description
S	Susceptible human population.
E	Exposed human population. (Population that is infected but yet to show symptoms).
I	Infected Human population showing symptoms.
R'	Human population in recuperation phase.
R	Fully recovered human population.
X	Susceptible mosquito population.
Y	Exposed mosquito population. (carrying infection but not yet capable to spread it).
Z	Infectious mosquito population spreading the disease.
Parameters	**Description**
μ	Human birth rate.
β	Mosquito biting rate for transfer of infection from infectious mosquito class (Z) to susceptible human population (S).
α	Progression rate of exposed to infected human population.
γ	Progression rate of infected to recuperated human population.
λ	Progression rate of recuperated to fully recovered human population.
ρ	Mosquito birth rate.
ν	Mosquito biting rate for transfer of infection from infected human population(I) or population under recuperation phase (R') to susceptible mosquito population (X).
ψ	Progression rate from exposed to infectious mosquito population.
ζ	Natural death rate for human population.
ζ_1	Human death rate in infected stage due to viral infection.
ζ_2	Human death rate due to infection under recovery phase.
κ	Natural death rate for mosquito population.
B_H	Transmission probability per contact in susceptible humans.
B_M	Transmission probability per contact in susceptible mosquitoes.
N_H	Total human population, i.e. $S+E+I+R'+R$.

Table 2: Range of Parameters for the model (CM)

Parameters	Range	References
μ	$400 \times \frac{1}{15 \times 365}$ - $400 \times \frac{1}{12 \times 365}$	[15, 16]
β	0.19 - 0.39	[15, 17]
α	$\frac{1}{4}$ - $\frac{1}{2}$	[4, 15, 18, 19, 20, 21]
γ	$\frac{1}{4}$ - $\frac{1}{2}$	Estimated [14]
λ	$\frac{1}{8}$ - $\frac{1}{4}$	Estimated [14]
ρ	500×0.015 - 500×0.33	[15, 16, 22, 23]
ν	0.19 - 0.39	[15, 17]
ψ	$\frac{1}{6}$ - $\frac{1}{2}$	[9, 18, 20, 24]
ζ	$\frac{1}{60 \times 365}$ - $\frac{1}{18 \times 365}$	[13]
ζ_1	$\frac{1}{10^5}$ - $\frac{1}{10^4}$	[25]
ζ_2	$\frac{1}{10^6}$ - $\frac{1}{10^5}$	[25]
κ	$\frac{1}{42}$ - $\frac{1}{14}$	[9, 18, 19, 20, 21]
B_H	0.001 - 0.54	[8, 15, 26, 18, 27]
B_M	0.005 - 0.35	[8, 26, 27, 28, 29]

Table 3: Values of Parameters for Simulation

Parameters	$R_0 < 1$	$R_0 > 1$
μ	$400 \times \frac{1}{15 \times 365}$	$400 \times \frac{1}{15 \times 365}$
β	0.25	0.30
α	$\frac{1}{3}$	$\frac{1}{4}$
γ	$\frac{1}{3}$	$\frac{1}{4}$
λ	$\frac{1}{7}$	$\frac{1}{8}$
ρ	500×0.1675	500×0.2
ν	0.25	0.30
ψ	$\frac{1}{3.5}$	$\frac{1}{4}$
ζ	$\frac{1}{40 \times 365}$	$\frac{1}{30 \times 365}$
ζ_1	$\frac{1}{10^4}$	$\frac{1}{10^5}$
ζ_2	$\frac{1}{10^5}$	$\frac{1}{10^6}$
κ	$\frac{1}{14}$	$\frac{1}{30}$
B_H	0.24	0.30
B_M	0.24	0.30

3 Preliminary Results

3.1 Positivity of Solutions

In order to establish the epidemiological meaningfullness [13], we prove the non negativity of the state variables for the formulated model at all $t > 0$.

Theorem 3.1: The solution $M(t) = (S, E, I, R', R, X, Y, Z)$ of model (CM) with $M(0) \geq 0$, is non negative for all $t > 0$. Moreover,

$$\lim_{t\to\infty} \sup N_H(t) = \frac{\mu}{\zeta} \text{ and } \lim_{t\to\infty} \sup N_M(t) = \frac{\rho}{\kappa}$$

where $N_H(t) = S(t) + E(t) + I(t) + R'(t) + R(t)$ and $N_M(t) = X(t) + Y(t) + Z(t)$.

Proof: Let $t_1 = \sup\,\{t > 0 : M(t) > 0\}$. Clearly $t_1 > 0$. Consider the first equation of the model (CM),

$$\frac{dS}{dt} = \mu - \frac{\beta B_H S Z}{N_H} - \zeta S.$$

Solving the differential equation we have,

$$\frac{d}{dt}\left\{S(t)\exp\left[\left(\int_0^{t_1} \frac{\beta B_H Z(\tau)}{N_H(\tau)} d\tau + \zeta t\right)\right]\right\} = \mu \exp\left[\left(\int_0^{t_1} \frac{\beta B_H Z(\tau)}{N_H(\tau)} d\tau + \zeta t\right)\right]$$

$$\implies S(t_1)\exp\left[\left(\int_0^{t_1} \frac{\beta B_H Z(\tau)}{N_H(\tau)} d\tau + \zeta t_1\right)\right] - S(0) = \int_0^{t_1} \mu \exp\left[\left(\int_0^{u} \frac{\beta B_H Z(\tau)}{N_H(\tau)} d\tau + \zeta u\right)\right] du.$$

Furthermore,

$$\begin{aligned} S(t_1) =& S(0)\exp\left[\left(-\int_0^{t_1} \frac{\beta B_H Z(\tau)}{N_H(\tau)} d\tau + \zeta t_1\right)\right] \\ &+ \exp\left[\left(-\int_0^{t_1} \frac{\beta B_H Z(\tau)}{N_H(\tau)} d\tau + \zeta t_1\right)\right]\int_0^{t_1} \mu \exp\left[\left(\int_0^{u} \frac{\beta B_H Z(\tau)}{N_H(\tau)} d\tau + \zeta u\right)\right] du > 0. \end{aligned}$$

Similarly, the non negativity can be shown for all the state variables, i.e., $M(t_1) > 0$ and therefore $M(t) > 0$ for all $t > 0$. In fact, we now have, $0 < S(t) \leq N_H(t)$, $0 < E(t) \leq N_H(t)$, $0 < I(t) \leq N_H(t)$, $0 < R'(t) \leq N_H(t)$, $0 < R(t) \leq N_H(t); 0 < X(t) \leq N_M(t)$, $0 < Y(t) \leq N_M(t)$, $0 < Z(t) \leq N_M(t)$. As the total human population is given by $N_H(t) = S(t) + E(t) + I(t) + R'(t) + R(t)$, the rate of change of human population with respect to time is given by

$$\begin{aligned} \frac{dN_H}{dt} &= \mu - \zeta(S + E + I + R' + R) - \zeta_1 I - \zeta_2 R' \\ &= \mu - \zeta N_H - \zeta_1 I - \zeta_2 R' \\ &\leq \mu - \zeta N_H. \end{aligned} \tag{3.1}$$

Now for $N_M(t) = X(t) + Y(t) + Z(t)$,

$$\frac{dN_M}{dt} \leq \rho - \kappa N_M.$$

Let $N = \dfrac{\mu}{\zeta}$. As $t \to \infty$, the disease will disappear. Therefore, $\lim\limits_{t\to\infty} \sup I(t) = 0$ and $\lim\limits_{t\to\infty} \sup R'(t) = 0$. Now, $\dfrac{dN_H}{dt} = \mu - \zeta N_H$ this implies $N_H(t) = \dfrac{\mu}{\zeta} + \left(N_H(0) - \dfrac{\mu}{\zeta}\right) e^{-\zeta t}$, which further implies $\lim\limits_{t\to\infty} N_H(t) = \dfrac{\mu}{\zeta} = N$. This follows that $0 < \lim\limits_{t\to\infty} \sup N_H(t) \leq N = \dfrac{\mu}{\zeta}$ if $\lim\limits_{t\to\infty} \sup I(t) = 0$ and $\lim\limits_{t\to\infty} \sup R'(t) = 0$. And if $N_H > N = \dfrac{\mu}{\zeta}$ then from (3.1), $\dfrac{dN_H}{dt} < 0$. Similarly, it can be seen that $0 < \lim\limits_{t\to\infty} \sup N_M(t) \leq \dfrac{\rho}{\kappa}$.

3.2 Invariant Region

Consider $\Re = \Re_H \times \Re_M \subset \mathbb{R}^5_+ \times \mathbb{R}^3_+$, where

$$\Re_H = \left\{S, E, I, R', R : N_H(t) \leq \frac{\mu}{\zeta}\right\},$$
$$\Re_M = \left\{X, Y, Z : N_M(t) \leq \frac{\rho}{\kappa}\right\}.$$

Now, we establish the positive invariance [13], of the region $\Re$ associated to the model (CM). That is, we show that solutions in $\Re$ remain in $\Re$ for all $t > 0$.

Theorem 3.2: The region $\Re \subset \mathbb{R}^8_+$ is positively invariant for the model (CM), with non-negative initial conditions in $\mathbb{R}^8_+$.

Proof : As seen in Theorem 3.1, $\dfrac{dN_H}{dt} \leq \mu - \zeta N_H$ and $\dfrac{dN_M}{dt} \leq \rho - \kappa N_M$. By using standard comparison theorem [30], it can be seen that, $N_H(t) \leq \dfrac{\mu}{\zeta} = N$. So, clearly every solution in $\Re_H$ remains in $\Re_H$ for all $t > 0$. Similar is the case for every solution of $\Re_M$. Hence, the region $\Re$ is positively invariant and contains all solutions of $\mathbb{R}^8_+$ for model (CM).

In the following sections, we show the existence and stability of the disease free equilibrium (DFE) and endemic equilibrium (EE) for the model (CM).

4 Disease Free Equilibrium (DFE)

In this section, we find a unique disease free equilibrium (DFE) for the model (CM) and then analyse its stability.

4.1 Existence of Equilibrium

To determine the disease free equilibrium (DFE) of the model, we consider the sections of populations that are free from disease and put their time derivatives equal to zero. Let DFE be denoted by $E_d = (S^*, E^*, I^*, R^{'*}, R^*, X^*, Y^*, Z^*)$. As sections of susceptible and recovered humans as well as susceptible mosquitoes are the only sections free from disease therefore $E_d = (S^*, 0, 0, 0, R^*, X^*, 0, 0)$. Solving the differential equations of the model (CM), DFE is obtained as $E_d = \left(\frac{\mu}{\zeta}, 0, 0, 0, 0, \frac{\rho}{\kappa}, 0, 0\right)$.

4.2 Reproduction Number

Let the basic reproduction number be denoted by R_0, which is defined as the expected number of secondary cases produced by a single (typical) infection in a population that is completely disease free. To find the threshold quantity R_0 [31, 32], we consider the next generation matrix G, which comprises of two matrices F and V^{-1}, where $F = \dfrac{d\mathcal{F}_i(x_0)}{dx_j}$ and $V = \dfrac{d\mathcal{V}_i(x_0)}{dx_j}$ for $1 \le i, j \le 5$. Here, $\mathcal{F}_i$ represents the new infection, whereas $\mathcal{V}_i$ corresponds to the transfers of infection from one compartment to another. Let x_0 be the disease free equilibrium state. Hence, the reproduction number is the largest eigen value of the next generation matrix G (defined as the product of matrices F and V^{-1}), that is the largest eigen value of the matrix, $G = FV^{-1}$. Corresponding to the model (CM),

$$\mathcal{F} = \begin{bmatrix} \frac{\beta B_H S Z}{N_H} \\ 0 \\ 0 \\ \frac{\nu B_M (I+R')X}{N_H} \\ 0 \end{bmatrix} \quad \text{and} \quad \mathcal{V} = \begin{bmatrix} \alpha E + \zeta E \\ -\alpha E + \gamma I + (\zeta + \zeta_1) I \\ -\gamma I + \lambda R' + (\zeta + \zeta_2) R' \\ \psi Y + \kappa Y \\ -\psi Y + \kappa Z \end{bmatrix}.$$

Next, we find the Jacobian F and V of the matrices $\mathcal{F}$ and $\mathcal{V}$ respectively and the eigen values of the matrix $G = FV^{-1}$, gives the reproduction number as

$$R_0 = \frac{\sqrt{\rho\nu\psi\zeta\alpha\beta B_H B_M(\lambda + \gamma + \zeta + \zeta_2)}}{\kappa\sqrt{\mu(\psi + \kappa)(\zeta + \alpha)(\zeta + \gamma + \zeta_1)(\zeta + \lambda + \zeta_2)}}.$$

4.3 Local Stability

Theorem 4.1 : The DFE of the chikungunya model (CM) is locally asymptotically stable, if $R_0 < 1$ and unstable if $R_0 > 1$, where R_0 is the associated reproduction number.

Proof : We consider the system of non linear differential equations, corresponding to the model (CM) to evaluate its Jacobian matrix. Let J_D denote the Jacobian of the system at DFE that is,

$$J_D = \begin{bmatrix} -\zeta & 0 & 0 & 0 & 0 & 0 & 0 & -B_H\beta \\ 0 & -\alpha-\zeta & 0 & 0 & 0 & 0 & 0 & B_H\beta \\ 0 & \alpha & -\gamma-\zeta-\zeta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & -\lambda-\zeta-\zeta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & -\zeta & 0 & 0 & 0 \\ 0 & 0 & -\frac{\nu B_M\rho\zeta}{\kappa\mu} & -\frac{\nu B_M\rho\zeta}{\kappa\mu} & 0 & -\kappa & 0 & 0 \\ 0 & 0 & \frac{\nu B_M\rho\zeta}{\kappa\mu} & \frac{\nu B_M\rho\zeta}{\kappa\mu} & 0 & 0 & -\psi-\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \psi & -\kappa \end{bmatrix}$$

Clearly, the trace of the matrix J_D is negative and determinant of matrix J_D [33, 34], is given by

$$\det(J_D) = \frac{-\zeta^2[\kappa^2\mu(\psi+\kappa)(\zeta(\zeta+\alpha+\gamma)+\alpha\gamma+\zeta\zeta_1+\zeta_1\alpha)(-\zeta-\lambda-\zeta_2)] + \rho\nu\zeta\psi\alpha\beta B_H B_M(\zeta+\lambda+\gamma+\zeta_2)}{\mu}.$$

For $R_0 < 1$, we have

$$\sqrt{\rho\nu\psi\zeta\alpha\beta B_H B_M(\zeta+\gamma+\lambda+\zeta_2)} < \kappa\sqrt{\mu(\psi+\kappa)(\zeta+\alpha)(\zeta+\lambda+\zeta_2)(\zeta+\gamma+\zeta_1)}.$$

Therefore,

$$\kappa^2\mu(\psi+\kappa)(\zeta+\lambda+\zeta_2)(\zeta(\zeta+\alpha+\gamma)+\alpha\gamma+\zeta\zeta_1+\zeta_1\alpha) - \psi[\rho\nu\zeta\alpha\beta B_H B_M(\zeta+\lambda+\gamma+\zeta_2)] > 0$$

or $\det(J_D) > 0$. Hence, DFE is locally asymptotically stable if $R_0 < 1$.

4.4 Global Stability

Consider the feasible region $\Re_1 = \{D \in \Re : S \leq S^*, X \leq X^*\}$ where $D = (S, E, I, R', R, X, Y, Z)$, S^* and X^* are the components of DFE (E_d).

Lemma 4.1: The region $\Re_1$ is positively invariant for the model (CM).

Proof: From the first equation of the model (CM),

$$\begin{aligned}\frac{dS}{dt} &= \mu - \frac{\beta B_H Z S}{N_H} - \zeta S\\ &\leq \mu - \zeta S\\ &\leq \zeta\left(\frac{\mu}{\zeta} - S\right)\\ &\leq \zeta(S^* - S)\\ S &\leq S^* + (S(0) - S^*)e^{-\zeta t}\end{aligned}$$

Thus, if $S^* = \frac{\mu}{\zeta}$ for all $t \geq 0$ and $S(0) \leq S^*$, then $S \leq S^*$ for all $t \geq 0$. Similarly, for

$$\begin{aligned}\frac{dX}{dt} &= \rho - \frac{\nu B_M (I + R')X}{N_H} - \kappa X\\ &\leq \rho - \kappa X\\ &\leq \kappa(X^* - X)\\ X &\leq X^* + (X(0) - X^*)e^{-\kappa t}\end{aligned}$$

Thus, if $X^* = \frac{\rho}{\kappa}$ for all $t \geq 0$ and $X(0) \leq X^*$, then $X \leq X^*$ for all $t \geq 0$. Hence, it has been shown that the region $\Re_1$ is positively invariant and attracts all solutions in $\Re^8_+$ for the model (CM). Now in order to establish the global asymptotic stability of DFE [35], we rewrite the model (CM) as

$$\left[\begin{array}{c}\frac{dT_U}{dt} = F(T_U, T_I)\\ \frac{dT_I}{dt} = G(T_U, T_I), \qquad G(T_U, 0) = 0\end{array}\right] \qquad \text{(RM)}$$

where $T_U = (S, R, X) \in \mathbb{R}^3$ and $T_I = (E, I, R', Y, Z) \in \mathbb{R}^5$.
Let $E^*_D = (T^*_U, 0)$ be DFE of (RM) where $T^*_U = \left(\frac{\mu}{\zeta}, 0, \frac{\rho}{\kappa}\right)$. We now state the following two conditions which must be satisfied to guarantee global asymptotic stability:
(H1) For $\dfrac{dT_U}{dt} = F(T_U, 0)$, T^*_U is globally asymptotically stable.
(H2) $G(T_U, T_I) = AT_I - \hat{G}(T_U, T_I), \hat{G}(T_U, T_I) \geq 0$, $(T_U, T_I) \in \Re$ where $A = \dfrac{\partial G(T^*_U, 0)}{\partial T_I}$ is an M-matrix which by definition has the off diagonal elements non-negative.
Theorem 4.2: The fixed point $E^*_D = (T^*_U, 0)$ is globally asymptotic stable (g.a.s) equilibrium of (RM) provided that $R_0 < 1$ and that assumptions (H1) and (H2) are satisfied.

Proof: For the system (RM),

$$\frac{dT_U}{dt} = F(T_U, 0) = \begin{bmatrix}\mu - \zeta S\\ 0\\ \rho - \kappa X\end{bmatrix}$$

We solve the above linear differential system to get the $S(t) = \dfrac{\mu}{\zeta} + S^*(0)e^{-\mu t}$, $R(t) = 0$ and $X(t) = \dfrac{\rho}{\kappa} + X^*(0)e^{-\kappa t}$ which implies $S(t) \to \dfrac{\mu}{\zeta}$, $R(t) \to 0$ and $X(t) \to \dfrac{\rho}{\kappa}$ as $t \to \infty$.

Therefore, disease free point T_U^* is a globally asymptotic stable (g.a.s) equilibrium of $\dfrac{dT_U}{dt} = F(T_U, 0)$. Hence (H1) holds. Clearly it can be seen that

$$G(T_U, T_I) = \begin{bmatrix} \frac{\beta B_H Z S}{N_H} - \alpha E - \zeta E \\ \alpha E - \gamma I - (\zeta + \zeta_1) I \\ \gamma I - \lambda R' - (\zeta + \zeta_2) R' \\ \frac{\nu B_M (I + R') X}{N_H} - \psi Y - \kappa Y \\ \psi Y - \kappa Z \end{bmatrix}$$

Also from (H2) $G(T_U, T_I) = AT_I - \hat{G}(T_U, T_I)$, where

$$A = \frac{\partial G(T_U^*, 0)}{\partial T_I} = \begin{bmatrix} -\alpha - \zeta & 0 & 0 & 0 & \beta B_H \\ \alpha & -\gamma - \zeta - \zeta_1 & 0 & 0 & 0 \\ 0 & \gamma & -\lambda - \zeta - \zeta_2 & 0 & 0 \\ 0 & \frac{\nu B_M \rho \zeta}{\kappa \mu} & \frac{\nu B_M \rho \zeta}{\kappa \mu} & -\psi - \kappa & 0 \\ 0 & 0 & 0 & \psi & -\kappa \end{bmatrix}.$$

Therefore,

$$\frac{\partial G(T_U^*, 0)}{\partial T_I} T_I = \begin{bmatrix} -\alpha E - \zeta E - \beta B_H Z \\ \alpha E - \gamma I - (\zeta + \zeta_1) I \\ \gamma I - \lambda R' - (\zeta + \zeta_2) R' \\ \frac{\nu B_M a \zeta}{\kappa \mu} (I + R') - (\psi + \kappa) Y \\ \psi Y - \kappa Z \end{bmatrix}.$$

In view of (H2), $\hat{G}(T_U, T_I) = \frac{\partial G(T_U^*, 0)}{\partial T_I} T_I - G(T_U, T_I)$ which gives

$$\hat{G}(T_U, T_I) = \begin{bmatrix} \beta B_H Z (1 - \frac{S}{N_H}) \\ 0 \\ 0 \\ \nu B_M (I + R') \left[\frac{\rho \zeta}{\mu \kappa} - \frac{X}{N_H} \right] \\ 0 \end{bmatrix}.$$

Clearly, $\beta B_H Z (1 - \dfrac{S}{N_H}) \geq 0$ as $\dfrac{S}{N_H} < 1$. Also, $\dfrac{\kappa X}{\rho} \leq \dfrac{\zeta N_H}{\mu}$ or $\dfrac{\rho \zeta}{\mu \kappa} \geq \dfrac{X}{N_H} \Rightarrow \dfrac{X^*}{S^*} \geq \dfrac{X}{N_H}$ and from Lemma 4.1 we know $X^* \geq X$ and $N_H^* = S^* \geq N_H$, which implies $\nu B_M (I + R') \left[\dfrac{\rho \zeta}{\mu \kappa} - \dfrac{X}{N_H} \right] \geq 0$. Therefore, (H2) holds true. Hence, $E_D^* = (T_U^*, 0)$ is globally asymptotically stable in the region $\Re$ whenever $R_0 \leq 1$.

5 Endemic Equilibrium

In this section, we first determine the endemic equilibrium points for the model (CM), establish its existence and then analyse its stability.

5.1 Endemic Equilibrium Points

Let endemic equilibrium points be denoted by $E_e = (S^{**}, E^{**}, I^{**}, R'^{**}, R^{**}, X^{**}, Y^{**}, Z^{**})$. The components of E_e are obtained by imposing constant solutions in the model (CM) and solving the algebraic equations. By computations, we have

$$
\begin{aligned}
S^{**} &= \frac{\mu N_H}{\zeta N_H + Z^{**}\beta B_H},\\
E^{**} &= \frac{\beta B_H Z^{**}\mu}{(\alpha+\zeta)(\beta B_H Z^{**} + \zeta N_H)},\\
I^{**} &= \frac{\alpha\beta B_H Z^{**}\mu}{(\gamma+\zeta+\zeta_1)(\alpha+\zeta)(\beta B_H Z^{**} + \zeta N_H)},\\
R'^{**} &= \frac{\alpha\gamma\beta B_H Z^{**}\mu}{(\lambda+\zeta+\zeta_2)(\gamma+\zeta+\zeta_1)(\alpha+\zeta)(\beta B_H Z^{**} + \zeta N_H)},\\
R^{**} &= \frac{\lambda\alpha\beta B_H Z^{**}\mu\gamma}{\zeta(\lambda+\zeta+\zeta_2)(\gamma+\zeta+\zeta_1)(\alpha+\zeta)(\beta B_H Z^{**} + \zeta N_H)},\\
X^{**} &= \frac{\rho}{\lambda_M + \kappa},\\
Y^{**} &= \frac{\rho\lambda_M}{(\lambda_M+\kappa)(\psi+\kappa)},\\
Z^{**} &= \frac{\rho\psi\lambda_M}{\kappa(\lambda_M+\kappa)(\psi+\kappa)}.
\end{aligned}
$$

5.2 Existence and Uniqueness of Endemic Equilibrium(E_e)

Theorem 5.1 : Chikungunya Model (CM) has a unique endemic equilibrium if $R_0 > 1$.
As seen in section 2,

$$
\begin{aligned}
\lambda_M &= \frac{\nu B_M (I^{**} + R^{'**})}{N_H}\\
&= \frac{\nu B_M \zeta\alpha\beta\mu B_H Z^{**}(\zeta+\zeta_2+\lambda+\gamma)}{\mu(\beta B_H Z^{**} + \mu)(\alpha+\zeta)(\zeta+\zeta_1+\gamma)(\zeta+\zeta_2+\lambda)}\\
&= \frac{R_0^2 \mu Z^{**}\kappa^2(\psi+\kappa)}{\rho\psi(\beta B_H Z^{**} + \mu)}
\end{aligned}
$$

Also, $\lambda_H = \dfrac{\beta B_H Z^{**}}{N_H} = \dfrac{\beta B_H \rho\psi\lambda_M}{\kappa N_H(\lambda_M+\kappa)(\psi+\kappa)}$, or equivalently $\lambda_M = \dfrac{\lambda_H \mu\kappa^2(\psi+\kappa)}{\beta B_H\rho\psi\zeta - \mu\kappa(\psi+\kappa)\lambda_H}$.

Equating both values of λ_M, we get the following linear equation in terms of λ_H:

$$\lambda_H(\rho\psi\beta B_H + R_0^2\mu\kappa(\psi+\kappa)) = (R_0^2-1)\beta B_H\rho\psi\zeta.$$

The unique solution to this equation exists and is given by

$$\lambda_H = \frac{(R_0^2-1)\beta B_H\rho\psi\zeta}{\rho\psi\beta B_H + R_0^2\mu\kappa(\psi+\kappa)},$$

which is positive if $R_0^2 > 1$. This implies $Z^{**} > 0$, for $R_0 > 1$. Hence, unique endemic equilibrium exists for $R_0 > 1$.

5.3 Local Stability

Theorem 5.2: The endemic equilibrium of the chikungunya model (CM) is locally asymptotically stable if $R_0 > 1$.

Proof: We evaluate the Jacobian matrix for the system of nonlinear differential equations corresponding to the model (CM). Let J_e denote the Jacobian of the system at E_e (which exists for $R_0 > 1$). Clearly, $J_E = (J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8)^{\mathrm{T}}$ where

$J_1 = \left(\frac{-\beta B_H Z^{**}}{N_H} + \frac{\beta B_H Z^{**}S^{**}}{(N_H)^2} - \zeta, \frac{\beta B_H Z^{**}S^{**}}{(N_H)^2}, \frac{\beta B_H Z^{**}S^{**}}{(N_H)^2}, \frac{\beta B_H Z^{**}S^{**}}{(N_H)^2}, \frac{\beta B_H Z^{**}S^{**}}{(N_H)^2}, 0, 0, \frac{-\beta B_H S^{**}}{N_H}\right),$

$J_2 = \left(\frac{\beta B_H Z^{**}}{N_H} - \frac{\beta B_H Z^{**}S^{**}}{(N_H)^2}, \frac{-\beta B_H Z^{**}S^{**}}{(N_H)^2} - \alpha - \zeta, \frac{-\beta B_H Z^{**}S^{**}}{(N_H)^2}, \frac{-\beta B_H Z^{**}S^{**}}{(N_H)^2}, \frac{-\beta B_H Z^{**}S^{**}}{(N_H)^2}, 0, 0, \frac{\beta B_H S^{**}}{N_H}\right),$

$J_3 = (0, \alpha, -\gamma-\zeta-\zeta_1, 0, 0, 0, 0, 0)$, $J_4 = (0, 0, \gamma, -\lambda-\zeta-\zeta_2, 0, 0, 0, 0)$,

$J_5 = (0, 0, 0, \lambda, -\zeta, 0, 0, 0)$,

$J_6 = \left(\frac{\nu B_M(I^{**}+R'^{**})X^{**}}{(N_H)^2}, \frac{\nu B_M(I^{**}+R'^{**})X^{**}}{(N_H)^2}, \frac{\nu B_M(I^{**}+R'^{**})X^{**}}{(N_H)^2} - \frac{\nu B_M X^{**}}{N_H}, \frac{\nu B_M(I^{**}+R'^{**})X^{**}}{(N_H)^2} - \frac{\nu B_M X^{**}}{N_H},\right.$
$\left. 0, -\frac{\nu B_M(I^{**}+R'^{**})}{N_H} - \kappa, 0, 0\right),$

$J_7 = \left(\frac{-\nu B_M(I^{**}+R'^{**})X^{**}}{(N_H)^2}, \frac{-\nu B_M(I^{**}+R'^{**})X^{**}}{(N_H)^2}, \frac{-\nu B_M(I^{**}+R'^{**})X^{**}}{(N_H)^2} + \frac{\nu B_M X^{**}}{N_H}, \frac{-\nu B_M(I^{**}+R'^{**})X^{**}}{(N_H)^2} + \frac{\nu B_M X^{**}}{N_H},\right.$
$\left. 0, \frac{\nu B_M(I^{**}+R'^{**})}{N_H}, -\kappa-\psi, 0\right)$, $J_8 = (0, 0, 0, 0, 0, 0, \psi, -\kappa)$

Further, we reduce J_E to the following upper triangular matrix (U_E). $U_E = (U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8)^{\mathrm{T}}$ where

$U_1 = \left(\frac{-\beta B_H Z^{**}}{N_H} + \frac{\beta B_H Z^{**}S^{**}}{(N_H)^2} - \zeta, \frac{\beta B_H Z^{**}S^{**}}{(N_H)^2}, \frac{\beta B_H Z^{**}S^{**}}{(N_H)^2}, \frac{\beta B_H Z^{**}S^{**}}{(N_H)^2}, \frac{\beta B_H Z^{**}S^{**}}{(N_H)^2}, 0, 0, \frac{-\beta B_H S^{**}}{N_H}\right),$

$U_2 = \left(0, \frac{-\beta B_H Z^{**}S^{**}}{(N_H)^2} - \alpha - \zeta, \frac{-\beta B_H Z^{**}S^{**}}{(N_H)^2}, \frac{-\beta B_H Z^{**}S^{**}}{(N_H)^2}, \frac{-\beta B_H Z^{**}S^{**}}{(N_H)^2}, 0, 0, \frac{\beta B_H S^{**}}{N_H}\right),$

$U_3 = (0, 0, -\gamma-\zeta-\zeta_1, 0, 0, 0, 0, 0)$, $U_4 = (0, 0, 0, -\lambda-\zeta-\zeta_2, 0, 0, 0, 0)$,

$U_5 = (0, 0, 0, 0, -\zeta, 0, 0, 0)$, $U_6 = \left(0, 0, 0, 0, 0, -\frac{\nu B_M(I^{**}+R'^{**})}{N_H} - \kappa, 0, 0\right),$

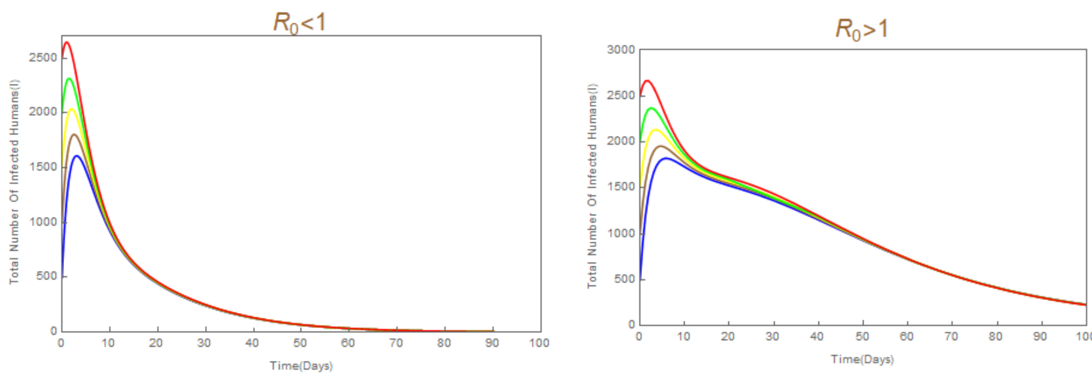
$U_7 = (0, 0, 0, 0, 0, 0, -\kappa-\psi, 0)$, $U_8 = (0, 0, 0, 0, 0, 0, 0, -\kappa)$

Attached are the eigen values of U_E:
$$\left(-\zeta, -\kappa, -\psi-\kappa, -\gamma-\zeta-\zeta_1, -\lambda-\zeta-\zeta_2, -\frac{\nu B_M(I^{**}+R^{'**})}{N_H}-\kappa, \frac{Z^{**}\beta B_H(S^{**}-N_H)}{(N_H)^2}-\zeta, \frac{-\beta B_H Z^{**}S^{**}}{(N_H)^2}-\alpha-\zeta\right)$$
each of which is negative and by the criterion given in [36], the endemic equilibrium point (E_e) is locally asymptotically stable if $R_0 > 1$.

6 Numerical Simulation

The values of parameters that would be used for simulation of the model (CM) are listed in Table 3. The values used for simulation are taken with reference to their ranges, as stated in Table 2.

Fig. 2a and Fig. 2b are visualizations of the existence and stability of equilibria for the cases, $R_0 < 1$ and $R_0 > 1$, respectively. Also, it illustrates that for $R_0 < 1$, the infection dies out over a period of time as it is the case of DFE. However, in the same time period, it can been seen that the infection continues to persist in the population when $R_0 > 1$ as it is the case of EE.



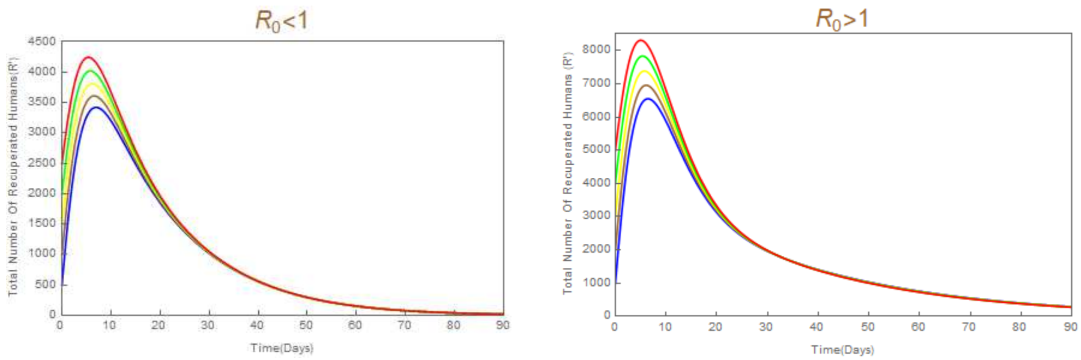
(a) (b)

Figure 2: *Total number of Infected Humans (I) with respect to time.*

In Fig. 3a, it is clear that the recuperated population ultimately falls down to zero for the case when $R_0 < 1$, where finally the disease dies out and ultimately the entire population will shift to the recovered section with no more inflow into the recuperated part. In contrast, for the same time period, if $R_0 > 1$ (Fig. 3b), the disease persists in the population. Therefore, we can see a substantial proportion of population which is still in the recuperated phase.

Fig. 4 and Fig. 5, both show the time duration around which the number of infected population comes to a fall which is actually the same for recuperated population to reach the peak.

In Fig. 6a, again for $R_0 < 1$, as the disease dies out so it is evidently a situation when the population of the infectious mosquitoes dies out. In contrast to it, for $R_0 > 1$ (Fig. 6b), the number



(a) (b)

Figure 3: *Total number of Recuperated Humans* (R') *with respect to time.*

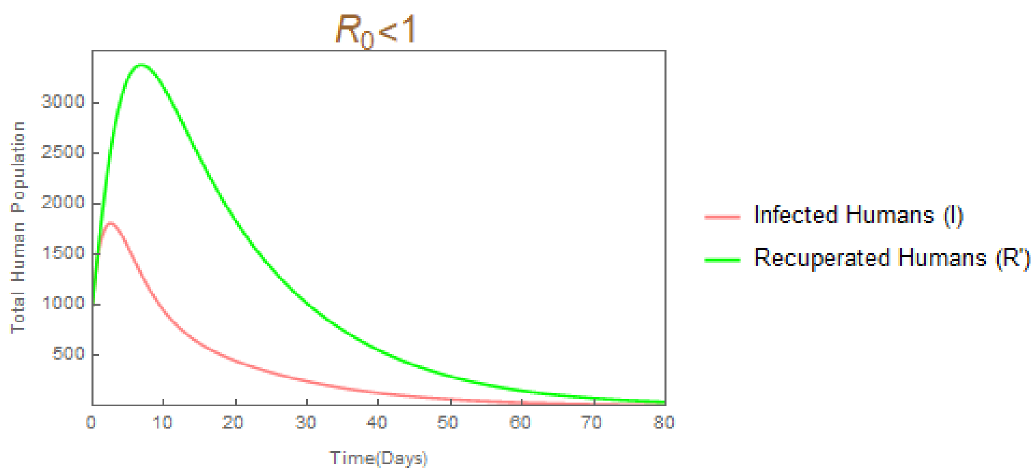


Figure 4: *Total number of Infected (I) and Recuperated* (R') *Humans when* $R_0 < 1$.

of infectious mosquitoes continue to persist in population as it is the case of endemic equilibrium (EE).

Fig. 7 shows the change in the number of infected, recuperated and recovered population with respect to time in accordance with model (CM) whereas Fig. 8 is a simulation of the model (CM) without recuperated section of population.

The curve representing the recovered population in Fig. 8, is an increasing curve showing a rapid increase in the number of people attaining full recovery. But this does not fit in accordance to the case of Chikungunya infection. However, in Fig. 7, we can see the convexity of the curve representing recovered population for a substantial period of time and this is because of the presence of recuperation factor which has been considered in our model. During this period, the recuperation curve is rising higher which is practically more relevant and well in consensus with the nature of this particular disease.

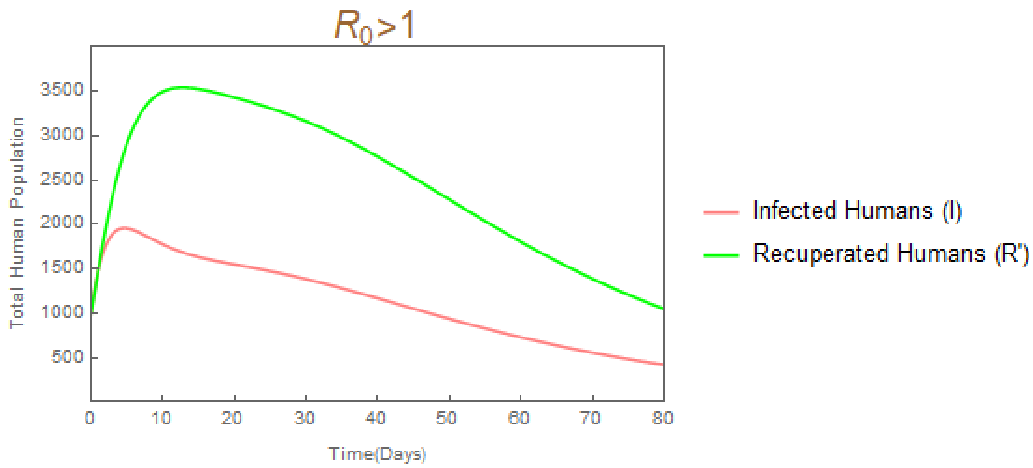
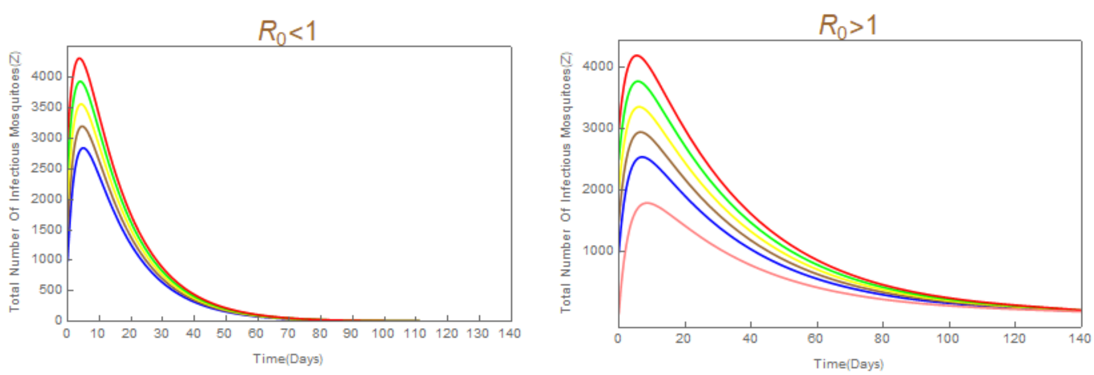


Figure 5: *Total number of Infected (I) and Recuperated* (R') *Humans when* $R_0 > 1$.



(a) (b)

Figure 6: *Total number of Infectious Mosquitoes (Z) with respect to time.*

7 Conclusion

In this paper, a new deterministic model is formulated to study the transmission dynamics of Chikungunya virus (CHIKV). Making a considerable refinement to the existing models present in the literature, a so far neglected section of human population is introduced, namely the population in the recuperation phase. The study shows that the disease free equilibrium (DFE) of the model is locally as well as globally asymptotically stable whenever existence of an associated reproduction number R_0, is less than 1 and unstable otherwise. Also, an endemic equilibrium (EE) exists whenever R_0 is greater than 1 and is locally asymptotically stable too. Simulations of the model make it evident that introduction of the said compartment is well justified, as this model provides a more realistic illustration for Chikungunya infection wherein the quantitative behaviour of disease has given a better visualisation. Moreover, the qualitative behaviour of the disease as studied by

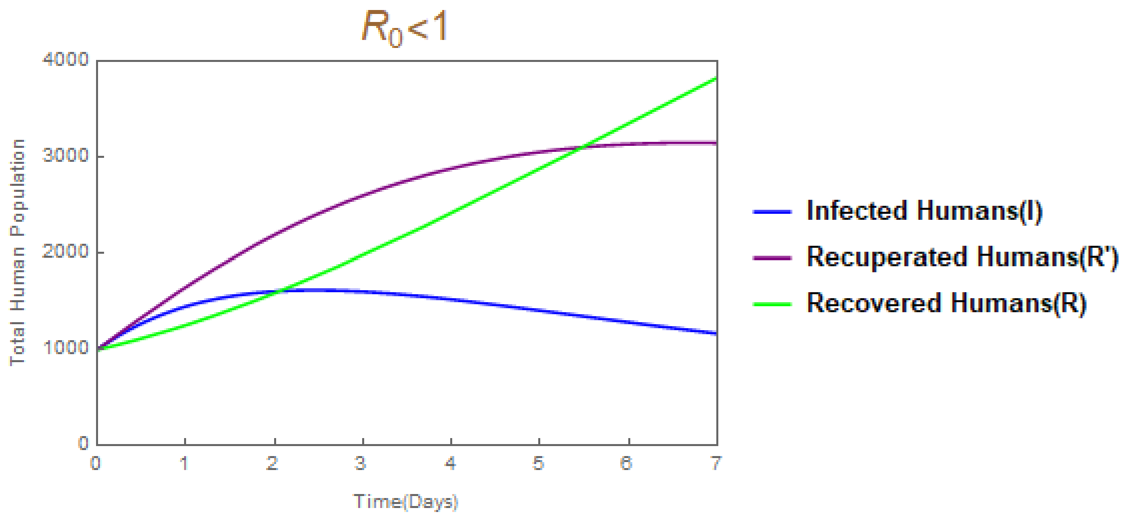


Figure 7: *Variation of Infected, Recuperated and Recovered Human Population with time for model (CM).*

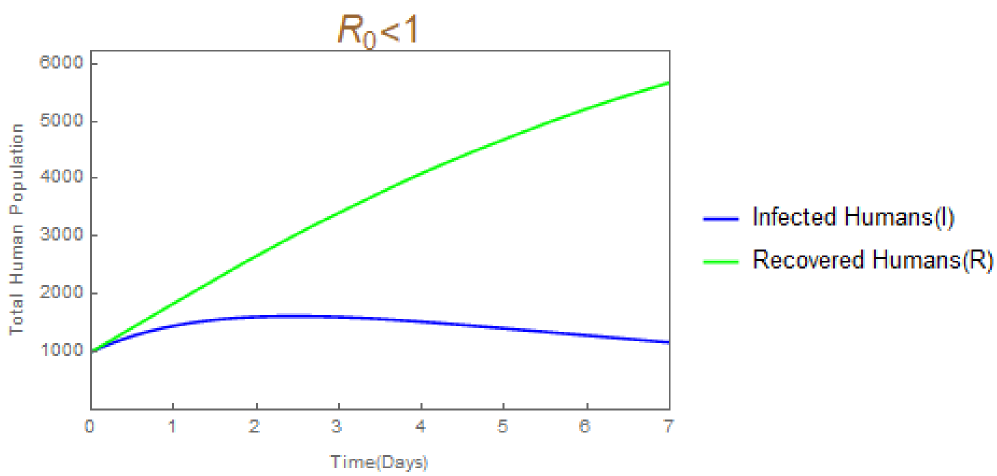


Figure 8: *Variation of Infected and Recovered Human Population with time for model (CM) without recuperation section.*

various researchers in [14] is very well taken into consideration through our model.

If we do not consider the recuperation section in model (CM), then the following model becomes a special case of our model. It is clearly seen that our model (CM) gives a better illustration to the dynamics of the Chikungunya virus and hence, the proposed model is indeed more realistic and practical.

$$
\begin{aligned}
\frac{dS}{dt} &= \mu - \frac{\beta B_H Z S}{N_H} - \zeta S,\\
\frac{dE}{dt} &= \frac{\beta B_H Z S}{N_H} - \alpha E - \zeta E,\\
\frac{dI}{dt} &= \alpha E - \gamma I - (\zeta + \zeta_1) I,\\
\frac{dR}{dt} &= \gamma I - \zeta R,\\
\frac{dX}{dt} &= \rho - \frac{\nu B_M I X}{N_H} - \kappa X,\\
\frac{dY}{dt} &= \frac{\nu B_M I X}{N_H} - \psi Y - \kappa Y,\\
\frac{dZ}{dt} &= \psi Y - \kappa Z, \qquad \text{where} \quad N_H(t) = S(t) + E(t) + I(t) + R(t).
\end{aligned}
$$

Comparison of the above model with our model (CM) is done in section 6 with the help of the graphs shown in Fig. 7 and Fig. 8.

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Contractive mapping theorems in Partially ordered metric spaces

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ABSTRACT

The purpose of this paper is to establish some coincidence, common fixed point theorems for monotone f-non decreasing self mappings satisfying certain rational type contraction in the context of a metric spaces endowed with partial order. Also, the results involving an integral type of such classes of mappings are discussed in application point of view. These results generalize and extend well known existing results in the literature.

RESUMEN

El propósito de este artículo es establecer teoremas de coincidencia y de punto fijo común para auto mapeos monótonos f-no decrecientes satisfaciendo ciertas contracciones de tipo racional en el contexto de espacios métricos dotados de un orden parcial. Adicionalmente, resultados que involucran clases de mapeos de tipo integral son discutidos desde un punto de vista de las aplicaciones. Estos resultados generalizan y extienden resultados bien conocidos, existentes en la literatura.

Keywords and Phrases: Partially ordered metric spaces; Rational contractions; Compatible mappings; Weakly compatible mappings.

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1 Introduction

Ever since in Fixed point theory and Approximation theory, the classical Banach contraction principle plays a vital role to obtain an unique solution of the results. Of course, it is very important and popular tool in different fields of mathematics to solve the existing problems in nonlinear analysis. Since then a lot of variety of generalizations of this Banach contraction principle [1] have been taken place in a metric fixed point theory by improving the underlying contraction condition [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Thereafter vigorous research work has been obtained by weakening its hypotheses on various spaces such as rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi-metric spaces, probabilistic metric spaces, D-metric spaces, G-metric spaces, F-metric spaces, cone metric spaces, and so on to prove the existing results. Prominent work on the existence and uniqueness of a fixed point and common fixed point theorems involving monotone mappings on cone metric spaces, partially ordered metric spaces and others spaces [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30] generate natural interest to establish usable fixed point results.

The aim of this paper is to prove some coincidence, common fixed point results in the frame work of partially ordered metric spaces for a pair of self-mappings satisfying a generalized contractive condition of rational type. These results generalize and extend the results of Harjani et al.[19] and Chandok [28] in ordered metric space. Also the applications of these results are presented on taking integral type contractions in the same space.

2 Preliminaries

The following definitions are frequently used in results given in upcoming sections.

Definition 1. *The triple $(X, d, \preceq)$ is called a partially ordered metric space, if $(X, \preceq)$ is a partially ordered set together with (X, d) is a metric space.*

Definition 2. *If (X, d) is a complete metric space, then the triple $(X, d, \preceq)$ is called a partially ordered complete metric space.*

Definition 3. *Let $(X, \preceq)$ be a partially ordered set. A self-mapping $f : X \to X$ is said to be strictly increasing, if $f(x) \prec f(y)$, for all $x, y \in X$ with $x \prec y$ and is also said to be strictly decreasing, if $f(x) \succ f(y)$, for all $x, y \in X$ with $x \prec y$.*

Definition 4. *A point $x \in A$, where A is a non-empty subset of a metric space (X, d) is called a common fixed (coincidence) point of two self-mappings f and T if $fx = Tx = x(fx = Tx)$.*

Definition 5. *The two self-mappings f and T defined over a subset A of a metric space (X, d) are called commuting if $fTx = Tfx$ for all $x \in A$.*

Definition 6. *Two self-mappings f and T defined over $A \subset X$ are compatible, if for any sequence $\{x_n\}$ with $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} Tx_n = \mu$, for some $\mu \in A$ then $\lim_{n\to+\infty} d(Tfx_n, fTx_n) = 0$.*

Definition 7. *Two self-mappings f and T defined over $A \subset X$ are said to be weakly compatible, if they commute at their coincidence points. i.e., if $fx = Tx$ then $fTx = Tfx$.*

Definition 8. *Let f and T be two self-mappings defined over a partially ordered set $(X, \preceq)$. A mapping T is called a monotone f non-decreasing if*

$$fx \preceq fy \text{ implies } Tx \preceq Ty, \text{for all } x, y \in X.$$

Definition 9. *Let A be a non-empty subset of a partially ordered set $(X, \preceq)$. If very two elements of A are comparable then it is called well ordered set.*

Definition 10. *A partially ordered metric space $(X, d, \preceq)$ is called an ordered complete, if for each convergent sequence $\{x_n\}_{n=0}^{+\infty} \subset X$, one of the following condition holds*

- *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ implies $x_n \preceq x$, for all $n \in \mathbb{N}$ that is, $x = \sup\{x_n\}$ or*
- *if $\{x_n\}$ is a nonincreasing sequence in X such that $x_n \to x$ implies $x \preceq x_n$, for all $n \in \mathbb{N}$ that is, $x = \inf\{x_n\}$.*

3 Main Results

In this section, we prove some coincidence, common fixed point theorems in the context of ordered metric space.

Theorem 1. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the self-mappings f and T on X are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following condition*

$$d(Tx, Ty) \leq \alpha \frac{d(fx, Tx)\ d(fy, Ty)}{d(fx, fy)} + \beta\left[d(fx, Tx) + d(fy, Ty)\right] + \gamma d(fx, fy) \tag{3.1}$$

for all x, y in X with $f(x) \neq f(y)$ are comparable, where $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq T(x_0)$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Proof. Let $x_0 \in X$ such that $f(x_0) \preceq T(x_0)$. Since from hypotheses, we have $T(X) \subseteq f(X)$ then, we can choose a point $x_1 \in X$ such that $fx_1 = Tx_0$. But $Tx_1 \in f(X)$ then, again there exists another point $x_2 \in X$ such that $fx_2 = Tx_1$. By continuing the same way, we can construct a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$, for all n.

Again, by hypotheses, we have $f(x_0) \preceq T(x_0) = f(x_1)$ and T is a monotone f-nondecreasing mapping then, we get $T(x_0) \preceq T(x_1)$. Similarly, we obtain $T(x_1) \preceq T(x_2)$, since $f(x_1) \preceq f(x_2)$ and then by continuing the same procedure, we obtain that

$$T(x_0) \preceq T(x_1) \preceq T(x_2) \preceq \text{.........} T(x_n) \preceq T(x_{n+1}) \preceq \text{........}$$

The equality $T(x_{n+1}) = T(x_n)$ is impossible because $f(x_{n+2}) \neq f(x_{n+1})$ for all $n \in \mathbb{N}$. Thus $d(T(x_n), T(x_{n+1})) > 0$ for all $n \geq 0$ therefore, from contraction condition (3.1), we have

$$\begin{aligned} d(Tx_{n+1}, Tx_n) \leq \alpha \frac{d(fx_{n+1}, Tx_{n+1})\, d(fx_n, Tx_n)}{d(fx_{n+1}, fx_n)} + \beta \left[d(fx_{n+1}, Tx_{n+1}) + d(fx_n, Tx_n)\right] \\ + \gamma d(fx_{n+1}, fx_n) \end{aligned}$$

which intern implies that

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &\leq \alpha d(Tx_n, Tx_{n+1}) + \beta \left[d(Tx_n, Tx_{n+1}) + d(Tx_{n-1}, Tx_n)\right] \\ &\quad + \gamma d(Tx_n, Tx_{n-1}) \end{aligned}$$

Finally, we arrive at

$$d(Tx_{n+1}, Tx_n) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right)\, d(Tx_n, Tx_{n-1})$$

Continuing the same process up to $(n-1)$ times, we get

$$d(Tx_{n+1}, Tx_n) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right)^n d(Tx_1, Tx_0)$$

Let $k = \frac{\beta+\gamma}{1-\alpha-\beta} \in [0, 1)$, then from triangular inequality for $m \geq n$, we have

$$\begin{aligned} d(Tx_m, Tx_n) &\leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + \text{........} + d(Tx_{n+1}, Tx_n) \\ &\leq \left(k^{m-1} + k^{m-2} + \text{..........} + k^n\right)\, d(Tx_1, Tx_0) \\ &\leq \frac{k^n}{1-k}\, d(Tx_1, Tx_0) \end{aligned}$$

as $m, n \to +\infty$, $d(Tx_m, Tx_n) \to 0$, which shows that the sequence $\{Tx_n\}$ is a Cauchy sequence in X. So, by the completeness of X, there exists a point $\mu \in X$ such that $Tx_n \to \mu$ as $n \to +\infty$. Again, by the continuity of T, we have

$$\lim_{n \to +\infty} T(Tx_n) = T\left(\lim_{n \to +\infty} Tx_n\right) = T\mu.$$

But $fx_{n+1} = Tx_n$, then $fx_{n+1} \to \mu$ as $n \to +\infty$ and from the compatibility for T and f, we have

$$\lim_{n \to +\infty} d(T(fx_n), f(Tx_n)) = 0.$$

Further by triangular inequality, we have

$$d(T\mu, f\mu) = d(T\mu, T(fx_n)) + d(T(fx_n), f(Tx_n)) + d(f(Tx_n), f\mu)$$

On taking limit as $n \to +\infty$ in both sides of the above equation and using the fact that T and f are continuous then, we get $d(T\mu, f\mu) = 0$. Thus, $T\mu = f\mu$. Hence, μ is a coincidence point of T and f in X. ∎

Corollary 1. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the self-mappings f and T on X are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following condition*

$$d(Tx, Ty) \leq \alpha \frac{d(fx, Tx)\ d(fy, Ty)}{d(fx, fy)} + \beta\left[d(fx, Tx) + d(fy, Ty)\right]$$

for all x, y in X with $f(x) \neq f(y)$ are comparable and for some $\alpha, \beta \in [0, 1)$ with $0 \leq \alpha + 2\beta < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq T(x_0)$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Proof. Set $\gamma = 0$ in Theorem 1. ∎

Corollary 2. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the self-mappings f and T on X are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following condition*

$$d(Tx, Ty) \leq \beta\left[d(fx, Tx) + d(fy, Ty)\right] + \gamma d(fx, fy)$$

for all x, y in X with $f(x) \neq f(y)$ are comparable and for some $\beta, \gamma \in [0, 1)$ with $0 \leq 2\beta + \gamma < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq T(x_0)$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Proof. The proof can be obtained by setting $\alpha = 0$ in Theorem 1. ∎

We may remove the continuity criteria of T in Theorem 1 is still valid by assuming the following hypothesis in X:

If $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Theorem 2. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that f and T are self-mappings on X, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying*

$$d(Tx, Ty) \leq \alpha \frac{d(fx, Tx)\ d(fy, Ty)}{d(fx, fy)} + \beta\left[d(fx, Tx) + d(fy, Ty)\right] + \gamma d(fx, fy) \tag{3.2}$$

for all x, y in X with $f(x) \neq f(y)$ are comparable and for some $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq T(x_0)$ and $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

If $f(X)$ is a complete subset of X, then T and f have a coincidence point in X. Further, if T and f are weakly compatible, then T and f have a common fixed point in X. Moreover, the set *of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point in X.*

Proof. Suppose $f(X)$ is a complete subset of X. As we know from the proof of Theorem 1, the sequence $\{Tx_n\}$ is a Cauchy sequence and hence $\{fx_n\}$ is also a Cauchy sequence in $(f(X), d)$ as $fx_{n+1} = Tx_n$ and $T(X) \subseteq f(X)$. Since $f(X)$ is complete then there exists some $fu \in f(X)$ such that

$$\lim_{n\to+\infty} T(x_n) = \lim_{n\to+\infty} f(x_n) = f(u).$$

Also note that the sequences $\{Tx_n\}$ and $\{fx_n\}$ are nondecreasing and from hypotheses, we have $T(x_n) \preceq f(u)$ and $f(x_n) \preceq f(u)$ for all $n \in \mathbb{N}$. But T is a monotone f-nondecreasing then, we get $T(x_n) \preceq T(\mu)$ for all n. Letting $n \to +\infty$, we obtain that $f(u) \preceq T(u)$.

Suppose that $f(u) \prec T(u)$ then define a sequence $\{u_n\}$ by $u_0 = u$ and $fu_{n+1} = Tu_n$ for all $n \in \mathbb{N}$. An argument similar to that in the proof of Theorem 1 yields that $\{fu_n\}$ is a nondecreasing sequence and $\lim\limits_{n\to+\infty} f(u_n) = \lim\limits_{n\to+\infty} T(u_n) = f(v)$ for some $v \in X$. So from hypotheses, it is clear that $\sup f(u_n) \preceq f(v)$ and $\sup T(u_n) \preceq f(v)$, for all $n \in \mathbb{N}$. Notice that

$$f(x_n) \preceq f(u) \preceq f(u_1) \preceq .... \preceq f(u_n) \preceq .... \preceq f(v).$$

Case:1 Suppose if there exists some $n_0 \geq 1$ such that $f(x_{n_0}) = f(u_{n_0})$ then, we have

$$f(x_{n_0}) = f(u) = f(u_{n_0}) = f(u_1) = T(u).$$

Hence, u is a coincidence point of T and f in X.

Case:2 Suppose that $f(x_{n_0}) \neq f(u_{n_0})$ for all n then, from (3.2), we have

$$\begin{aligned} d(fx_{n+1}, fu_{n+1}) =& d(Tx_n, Tu_n) \\ \leq& \alpha \frac{d(fx_n, Tx_n)\ d(fu_n, Tu_n)}{d(fx_n, fu_n)} + \beta\left[d(fx_n, Tx_n) + d(fu_n, Tu_n)\right] \\ &+ \gamma d(fx_n, fu_n) \end{aligned}$$

Taking limit as $n \to +\infty$ on both sides of the above inequality, we get

$$\begin{aligned} d(fu, fv) \leq& \gamma\ d(fu, fv) \\ &< d(fu, fv), \text{ since } \gamma < 1. \end{aligned}$$

Thus, we have

$$f(u) = f(v) = f(u_1) = T(u).$$

Hence, we conclude that u is a coincidence point of T and f in X.

Now, suppose that T and f are weakly compatible. Let w be a coincidence point then,

$$T(w) = T(f(z)) = f(T(z)) = f(w), \text{ since } w = T(z) = f(z), \text{ for some } z \in X.$$

Now by contraction condition, we have

$$\begin{aligned} d(T(z), T(w)) \leq & \alpha \frac{d(fz, Tz)\ d(fw, Tw)}{d(fz, fw)} + \beta\left[d(fz, Tz) + d(fw, Tw)\right] + \gamma d(fz, fw) \\ \leq & \gamma\ d(T(z), T(w)) \end{aligned}$$

as $\gamma < 1$, then $d(T(z), T(w)) = 0$. Therefore, $T(z) = T(w) = f(w) = w$. Hence, w is a common fixed point of T and f in X.

Now suppose that the set of common fixed points of T and f is well ordered, we have to show that the common fixed point of T and f is unique. Let u and v be two common fixed points of T and f such that $u \neq v$ then from (3.2), we have

$$\begin{aligned} d(u, v) \leq & \alpha \frac{d(fu, Tu)\ d(fv, Tv)}{d(fu, fv)} + \beta\left[d(fu, Tu) + d(fv, Tv)\right] + \gamma d(fu, fv) \\ \leq & \gamma\ d(u, v) \\ < & \ d(u, v), \text{ since } \gamma < 1, \end{aligned}$$

which is a contradiction. Thus, $u = v$. Conversely, suppose T and f have only one common fixed point then the set of common fixed points of T and f being a singleton is well ordered. This completes the proof. ∎

Corollary 3. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that f and T are self-mappings on X, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying*

$$d(Tx, Ty) \leq \alpha \frac{d(fx, Tx)\ d(fy, Ty)}{d(fx, fy)} + \beta\left[d(fx, Tx) + d(fy, Ty)\right]$$

for all x, y in X with $f(x) \neq f(y)$ are comparable and for some $\alpha, \beta \in [0, 1)$ with $0 \leq \alpha + 2\beta < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq T(x_0)$ and $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

If $f(X)$ is a complete subset of X, then T and f have a coincidence point in X. Further, if T and f are weakly compatible, then T and f have a common fixed point in X. Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point in X.

Proof. Set $\gamma = 0$ in Theorem 2. ∎

Corollary 4. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that f and T are self-mappings on X, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying*

$$d(Tx, Ty) \leq \beta\left[d(fx, Tx) + d(fy, Ty)\right] + \gamma d(fx, fy)$$

for all x, y in X for which $f(x) \neq f(y)$ are comparable and for some $\beta, \gamma \in [0, 1)$ with $0 \leq 2\beta + \gamma < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq T(x_0)$ and $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

If $f(X)$ is a complete subset of X, then T and f have a coincidence point in X. Further, if T and f are weakly compatible, then T and f have a common fixed point in X. Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point in X.

Proof. Set $\alpha = 0$ in Theorem 2. ∎

Remark 1. *(i). If $\beta = 0$, in Theorem 1 and Theorem 2, we obtain Theorem 2.1 and Theorem 2.3 of Chandok [28].*

(ii). If $f = I$ and $\beta = 0$, in Theorem 1 and Theorem 2, then we get Theorem 2.1 and Theorem 2.3 of Harjani et al. [19].

4 Applications

In this section, we state some applications of the main result for a self mapping involving the integral type contractions.

Let us denote τ, a set of all functions φ defined on $[0, +\infty)$ satisfying the following conditions:

(1) each φ is Lebesgue integrable mapping on each compact subset of $[0, +\infty)$ and

(2) for any $\epsilon > 0$, we have $\int_0^{\epsilon} \varphi(t)dt > 0$.

Theorem 3. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the self-mappings f and T on X are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following condition*

$$\begin{aligned} \int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt + \beta \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt \\ + \gamma \int_0^{d(fx,fy)} \varphi(t)dt \end{aligned} \tag{4.1}$$

for all x, y in X with $f(x) \neq f(y)$ are comparable, $\varphi(t) \in \tau$ and for some $\alpha, \beta, \gamma \in [0, 1)$ such that $0 \leq \alpha + 2\beta + \gamma < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq T(x_0)$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Similarly, we can obtain the following results in complete partially ordered metric space, by putting $\gamma = 0$ and $\alpha = 0$ in an integral contraction of Theorem 3.

Theorem 4. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the self-mappings f and T on X are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following condition*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt + \beta \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt \tag{4.2}$$

for all x, y in X with $f(x) \neq f(y)$ are comparable, $\varphi(t) \in \tau$ and where $\alpha, \beta \in [0, 1)$ such that $0 \leq \alpha + 2\beta < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq T(x_0)$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Theorem 5. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the self-mappings f and T on X are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following condition*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \beta \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt + \gamma \int_0^{d(fx,fy)} \varphi(t)dt \tag{4.3}$$

for all x, y in X with $f(x) \neq f(y)$ are comparable, $\varphi(t) \in \tau$ and for some $\beta, \gamma \in [0, 1)$ such that $0 \leq 2\beta + \gamma < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq T(x_0)$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Corollary 5. *By replacing $\beta = 0$ in Theorem 3, we obtain the Corollary 2.5 of Chandok [28].*

We illustrate the usefulness of the obtained results for the existence of the coincidence point in the space.

Example 1. *Define a metric $d : X \times X \to [0, +\infty)$ by $d(x, y) = |x - y|$, where $X = [0, 1]$ with usual order $\leq$. Suppose that T and f be two self mappings on X such that $Tx = \frac{x^2}{2}$ and $fx = \frac{2x^2}{1+x}$, then T and f have a coincidence in point X.*

Proof. By definition of a metric d, it is clear that (X, d) is a complete metric space. Obviously, $(X, d, \leq)$ is a partially ordered complete metric space with usual order. Let $x_0 = 0 \in X$ then $f(x_0) \leq T(x_0)$ and also by definition; T, f are continuous, T is a monotone f- nondecreasing and $T(X) \subseteq f(X)$.
Now for any distinct x, y in X, we have

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{2}|x^2 - y^2| = \frac{1}{2}(x + y)|x - y| \\ &< \frac{\alpha}{4} \frac{x^2y^2}{(x + y + xy)} \frac{|x - 3||y - 3|}{|x - y|} + \frac{\beta}{2} \frac{x^2(1 + y)|x - 3| + y^2(1 + x)|y - 3|}{(1 + x)(1 + y)} \\ &\quad + \gamma \frac{2(x + y + xy)}{(1 + x)(1 + y)}|x - y| \end{aligned}$$

$$
\begin{aligned}
&< \alpha \frac{\frac{x^2|x-3|}{2(1+x)} \cdot \frac{y^2|y-3|}{2(1+y)}}{2|x-y| \frac{x+y+xy}{(1+x)(1+y)}} + \beta \left[\frac{x^2|x-3|}{2(1+x)} + \frac{y^2|y-3|}{2(1+y)} \right] + \gamma \frac{2(x+y+xy)}{(1+x)(1+y)} |x-y| \\
&< \alpha \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} + \beta \left[d(fx,Tx) + d(fy,Ty) \right] + \gamma d(fx,fy)
\end{aligned}
$$

Then, the contraction condition in Theorem 1 holds by selecting proper values of α, β, γ in $[0,1)$ such that $0 \leq \alpha + 2\beta + \gamma < 1$. Therefore T, f have a coincidence point $0 \in X$. ∎

Similarly the following is one more example of main Theorem 1.

Example 2. *A distance function* $d : X \times X \to [0, +\infty)$ *by* $d(x,y) = |x-y|$, *where* $X = [0,1]$ *with usual order* $\leq$. *Define two self mappings* T *and* f *on* X *by* $Tx = x^2$ *and* $fx = x^3$, *then* T *and* f *have two coincidence points* 0, 1 *in* X *with* $x_0 = \frac{1}{2}$.

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D-metric Spaces and Composition Operators Between Hyperbolic Weighted Family of Function Spaces

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ABSTRACT

The aim of this paper is to introduce new hyperbolic classes of functions, which will be called $\mathcal{B}^*_{\alpha,\,\log}$ and $F^*_{\log}(p,q,s)$ classes. Furthermore, we introduce D-metrics space in the hyperbolic type classes $\mathcal{B}^*_{\alpha,\,\log}$ and $F^*_{\log}(p,q,s)$. These classes are shown to be complete metric spaces with respect to the corresponding metrics. Moreover, necessary and sufficient conditions are given for the composition operator C_ϕ to be bounded and compact from $\mathcal{B}^*_{\alpha,\,\log}$ to $F^*_{\log}(p,q,s)$ spaces.

RESUMEN

El objetivo de este artículo es introducir nuevas clases hiperbólicas de funciones, que serán llamadas clases $\mathcal{B}^*_{\alpha,\,\log}$ y $F^*_{\log}(p,q,s)$. A continuación, introducimos D-espacios métricos en las clases de tipo hiperbólicas $\mathcal{B}^*_{\alpha,\,\log}$ y $F^*_{\log}(p,q,s)$. Mostramos que estas clases son espacios métricos completos con respecto a las métricas correspondientes. Más aún, damos condiciones necesarias y suficientes para que el operador composición C_ϕ sea acotado y compacto desde el espacio $\mathcal{B}^*_{\alpha,\,\log}$ a $F^*_{\log}(p,q,s)$.

Keywords and Phrases: D-metric spaces, Logarithmic hyperbolic classes, Composition operators.

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1 Introduction

Let ϕ be an analytic self-map of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and let $\partial\mathbb{D}$ be its boundary. Let $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$ and let $B(\mathbb{D})$ be the subset of $H(\mathbb{D})$ consisting of those $f \in H(\mathbb{D})$ for which $|f(z)| < 1$ for all $z \in \mathbb{D}$.

Let the Green's function of $\mathbb{D}$ be defined as $g(z,a) = \log\frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation related to the point $a \in \mathbb{D}$.

A linear composition operator C_ϕ is defined by $C_\phi(f) = (f \circ \phi)$ for f in the set $H(\mathbb{D})$ of analyticfunctions on $\mathbb{D}$ (see [9]). A function $f \in B(\mathbb{D})$ belongs to α-Bloch space $\mathcal{B}_\alpha, 0 < \alpha < \infty$, if

$$||f||_{\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}} (1-|z|)^\alpha |f'(z)| < \infty.$$

The little α-Bloch space $\mathcal{B}_{\alpha,\,0}$ consisting of all $f \in \mathcal{B}_\alpha$ so that

$$\lim_{|z|\to 1^-} (1-|z|^2)|f'(z)| = 0.$$

Definition 1. *[15] For an analytic function f on $\mathbb{D}$ and $0 < \alpha < \infty$, if*

$$||f||_{\mathcal{B}^\alpha_{\log}} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| \left(\log \frac{2}{1-|z|^2}\right) < \infty,$$

then, f belongs to the weighted α-Bloch spaces $\mathcal{B}^\alpha_{\log}$.

If $\alpha = 1$, the weighted Bloch space $\mathcal{B}_{\log}$ is the set for all analytic functions f in $\mathbb{D}$ for which $||f||_{\mathcal{B}_{\log}} < \infty$.

The expression $||f||_{\mathcal{B}_{\log}}$ defines a seminorm while the norm is defined by

$$||f||_{\mathcal{B}_{\log}} = |f(0)| + ||f||_{\mathcal{B}_{\log}}.$$

Definition 2. *[14] For $0 < p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$, a function $f \in H(\mathbb{D})$ is in $F(p,q,s)$, if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < \infty.$$

Moreover, if

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) = 0,$$

then $f \in F_0(p,q,s)$.

El-Sayed and Bakhit [5] gave the following definition:

Definition 3. *For $0 < p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$, a function $f \in H(\mathbb{D})$ is said to belong to $F_{\log}(p, q, s)$, if*

$$\sup_{I \subset \partial\mathbb{D}} \frac{\left(\log \frac{2}{|I|}\right)^p}{|I|^s} \int_{S(I)} |f'(z)|^p (1 - |z|^2)^q \left(\log \frac{1}{|z|}\right)^s dA(z) < \infty.$$

Where $|I|$ denotes the arc length of $I \subset \partial\mathbb{D}$ and $S(I)$ is the Carleson box defined by (see [8, 6])

$$S(I) = \{z \in \mathbb{D} : 1 - |I| < |z| < 1, \frac{z}{|z|} \in |I|\}.$$

The interest in the $F_{\log}(p, q, s)$-spaces rises from the fact that they cover some well known function spaces. It is immediate that $F_{\log}(2, 0, 1) = BMOA_{\log}$ and $F_{\log}(2, 0, p) = Q^p_{\log}$, where $0 < p < \infty$.

2 Preliminaries

Definition 4. *[11] The hyperbolic Bloch space $\mathcal{B}^*_\alpha$ is defined as*

$$\mathcal{B}^*_\alpha = \{f : f \in B(\mathbb{D}) \text{ and } \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha f^*(z) < \infty\}.$$

Denoting $f^*(z) = \frac{|f'(z)|}{1 - |f(z)|^2}$, the hyperbolic derivative of $f \in B(\mathbb{D})$. [7]

The little hyperbolic Bloch space $\mathcal{B}^*_{\alpha,\,0}$ is a subspace of $\mathcal{B}^*_\alpha$ consisting of all $f \in \mathcal{B}^*_\alpha$ so that

$$\lim_{|z| \to 1^-} (1 - |z|^2)^\alpha f^*(z) = 0.$$

The space $\mathcal{B}^*_\alpha$ is Banach space with the norm defined as

$$||f||_{\mathcal{B}^*_\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|)^\alpha |f^*(z)|.$$

Definition 5. *For $0 < p, s < \infty$, $-2 < q < \infty$, $\alpha = \frac{q+2}{p}$ and $q + s > -1$, a function $f \in H(\mathbb{D})$ is said to belong to $F^*(p, q, s)$, if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (f^*(z))^p (1 - |z|^2)^{\alpha p - 2} g^s(z, a) dA(z) < \infty.$$

Definition 6. *For $f \in B(\mathbb{D})$ and $0 < \alpha < \infty$, if*

$$||f||_{\mathcal{B}^*_{\alpha,\,\log}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha (f^*(z)) \left(\log \frac{2}{1 - |z|^2}\right) < \infty,$$

then f belongs to the $\mathcal{B}^*_{\alpha,\,\log}$.

We must consider the following lemmas in our study:

Lemma 2.1. *[12] Let $0 < r \leq t \leq 1$, then*

$$\log \frac{1}{t} \leq \frac{1}{r}(1-t^2)$$

Lemma 2.2. *[12] Let $0 \leq k_1 < \infty,\ 0 \leq k_2 < \infty$, and $k_1 - k_2 > -1$, then*

$$C(k_1, k_2) = \int_{\mathbb{D}} \left(\log \frac{1}{|z|}\right)^{k_1} (1-|z|^2)^{-k_2} dA(z) < \infty.$$

To study composition operators on $\mathcal{B}^*_{\alpha,\log}$ and $F^*_{\log}(p,q,s)$ spaces, we need to prove the following result:

Theorem 1. *If $0 < p < \infty$, $1 < s < \infty$ and $\alpha = \frac{q+2}{p}$ with $q + s > -1$. Then the following are equivalent:*
(A) $f \in \mathcal{B}^*_{\alpha,\log}$.

(B) $f \in F^*_{\log}(p,q,s)$.

(C) $\sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} (f^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi(z)|^2)^s dA(z) < \infty$,
(D) $\sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} (f^*(z))^p (1-|z|^2|^{\alpha p-2} g^s(z,a) dA(z) < \infty$.

Proof. Let $0 < p < \infty,\ -2 < q < \infty,\ 1 < s < \infty$ and $0 < r < 1$. By subharmonicity we have for an analytic function $g \in \mathbb{D}$ that

$$|g(0)|^p \leq \frac{1}{\pi r^2} \int_{\mathbb{D}(0,r)} |g(w)|^p dA(w).$$

For $a \in \mathbb{D}$, the substitution $z = \varphi_a(z)$ results in Jacobian change in measure given by

$$dA(w) = |\varphi_a'(z)|^2\ dA(z).$$

For a Lebesgue integrable or a non-negative Lebesgue measurable function f on $\mathbb{D}$, we thus have the following change of variable formula:

$$\int_{\mathbb{D}(0,r)} f(\varphi_a(w)) dA(w) = \int_{\mathbb{D}(a,r)} f(z)|\varphi_a'(z)|^2 dA(z).$$

Let $g = \frac{f' \circ \varphi_a}{1-|f\circ\varphi_a|^2}$ then we have

$$\left(\frac{|f'(a)|}{1-|f(a)|^2}\right)^p = (f^*(a))^p \leq \frac{1}{\pi r^2} \int_{\mathbb{D}(0,r)} \left(\frac{|f'(\varphi_a(w))|}{1-|f(\varphi_a(w))|^2}\right)^p dA(w)$$
$$= \frac{1}{\pi r^2} \int_{\mathbb{D}(a,r)} (f^*(z))^p |\varphi_a'(z)|^2 dA(z).$$

Since

$$|\varphi_a'(z)| = \frac{1-|\varphi_a(z)|^2}{1-|z|^2},$$

and

$$\frac{1-|\varphi_a(z)|^2}{1-|z|^2} \le \frac{4}{1-|a|^2} \qquad a, z \in \mathbb{D}.$$

So we obtain that

$$(f^*(a))^p \le \frac{16}{\pi r^2(1-|a|^2)^2}\int_{\mathbb{D}(a,r)} (f^*(z))^p dA(z).$$

Again $f \in \mathcal{B}^*_{\alpha,\,\log}$, and $(1-|z|^2)^2 \approx (1-|a|^2)^2 \approx \mathbb{D}(a,r)$, for $z \in \mathbb{D}(a,r)$. Thus, we have

$$\left(\log\frac{2}{1-|a|^2}\right)^p (f^*(a))^p(1-|a|^2)^{\alpha p}$$

$$\begin{aligned}
&\le \frac{16}{\pi r^2(1-|a|^2)^{2-\alpha p}} \times \left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p dA(z)\\
&\le \frac{16}{\pi r^2} \times \left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p (1-|z|^2)^{\alpha p-2} dA(z)\\
&\le \frac{16}{\pi r^2} \times \left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p (1-|z|^2)^{\alpha p-2} \times \left(\frac{1-|\varphi_a(z)|^2}{1-|\varphi_a(z)|^2}\right)^s dA(z)\\
&\le \frac{16}{\pi r^2(1-r^2)^s} \times \left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\
&\le M(r) \times \left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a'(z)|^2)^s dA(z).
\end{aligned}$$

Where $M(r)$ is a constant depending on r. Thus, the quantity (A) is less than or equal to constant times the quantity (C).

From the fact

$$(1-|\varphi_a(z)|^2) \le 2\log\frac{1}{|\varphi_a(z)|} = 2g(z,a) \qquad for\ a,\ z \in \mathbb{D},$$

we have

$$\begin{aligned}
&\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\
\le\ &\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p (1-|z|^2)^{\alpha p-2} g^s(z,a) dA(z).
\end{aligned}$$

Hence, the quantity (C) is less than or equal to a constant times (D). By taking $\alpha = \frac{q+2}{p}$, it follows $f \in F^*_{\log}(p,q,s)$. Thus, the quantity (C) is less than or equal to a constant times the quantity (B).

Finally, from the following inequality, let $z = \varphi_a(w)$ then $w = \varphi_a(z)$. Hence,

$$\begin{aligned}
&\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} (f^*(\varphi_a(w)))^p (1-|\varphi_a(w)|^2)^{\alpha p-2} \left(\log\frac{1}{|w|}\right)^s |\varphi_a'(w)|^2 dA(w) \\
= &\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} (f^*(\varphi_a(w)))^p (1-|\varphi_a(w)|^2)^{\alpha p} \left(\log\frac{1}{|w|}\right)^s \frac{|\varphi_a'(w)|^2}{(1-|\varphi_a(w)|^2)^2} dA(w) \\
= &\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} (f^*(\varphi_a(w)))^p (1-|\varphi_a(w)|^2)^{\alpha p} \left(\log\frac{1}{|w|}\right)^s \frac{1}{(1-|w|^2)^2} dA(w) \\
\leq &\ ||f||^p_{\mathcal{B}^*_{\alpha,\log}} \left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} \left(\log\frac{1}{|w|}\right)^s (1-|w|^2)^{-2} dA(w) \\
= &\ C(s,2)||f||^p_{\mathcal{B}^*_{\alpha,\log}}.
\end{aligned}$$

By lemma 2.2, $C(s,2) = \int_{\mathbb{D}} \left(\log \frac{1}{|w|}\right)^s (1-|w|^2)^{-2} dA(w) < \infty, \quad for\ 1 < s < \infty.$

Thus, the quantity (D) is less than or equal to a constant times the quantity (A). Hence, it is proved.

Let us we give the following equivalent definition for $F^*_{\log}(p,q,s)$.

Definition 7. *For $0 < p, s < \infty$, $-2 < q < \infty$, $\alpha = \frac{q+2}{p}$ and $q + s > -1$, a function $f \in H(\mathbb{D})$ is said to belong to $F^*_{\log}(p,q,s)$, if*

$$\sup_{a\in\mathbb{D}} \left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} (f^*(z))^p (1-|z|^2)^{\alpha p-2} (1-|\varphi_a(z)|^2)^s dA(z) < \infty.$$

Definition 8. *A composition operator $C_\phi : \mathcal{B}^*_{\alpha,\log} \to F^*_{\log}(p,q,s)$ is said to be bounded if there is a positive constant C so that $||C_\phi f||_{F^*_{\log}}(p,q,s) \leq C||f||_{\mathcal{B}^*_{\alpha,\log}}$ for all $f \in \mathcal{B}^*_{p,\alpha}$.*

Definition 9. *A composition operator $C_\phi : \mathcal{B}^*_{\alpha,\log} \to F^*_{\log}(p,q,s)$ is said to be compact if it maps any ball in $\mathcal{B}^*_{p,\alpha}$ onto a precompact set in $F^*(p,q,s)$.*

The following lemma follows by standard arguments similar to those outline in [13]. Hence, we omit the proof.

Lemma 2.3. *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself. Let $0 < p, s, \alpha < \infty, -2 < q < \infty$, then $C_\phi : \mathcal{B}^*_{\alpha,\log} \to F^*_{\log}(p,q,s)$ is compact if and only if for any bounded sequence $\{f_n\}_{n\in N} \in \mathcal{B}^*_{\alpha,\log}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ we have $\lim_{n\to\infty} ||C_\phi f_n||_{F^*_{\log}(p,q,s)} = 0$.*

3 D-metric space

Topological properties of generalized metric space called D- metric space was introduced in [1], see for example, ([2] and [3]). This structure of D-metric space is quite different from a 2-metric space and natural generalization of an ordinary metric space in some sense.

Definition 10. *[4] Let X denote a nonempty set and $\mathbb{R}$ the set of real numbers. A function $D : X\times X\times X \to \mathbb{R}$ is said to be a D-metric on X if it satisfies the following properties:*

(i) $D(x, y, z) \geq 0$ for all $x, y, z \in X$ and equality holds if and only if $x = y = z$ (nonnegativity),

(ii) $D(x, y, z) = D(x, z, y) = \cdots$ (symmetry),

*(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$
(tetrahedral inequality).*

A nonempty set X together with a D-metric D is called a D-metric space and is represented by (X, D). The generalization of a D-metric space with D-metric as a function of n variables is provided in Dhage [2].

Example1.1: [4] Let (X, d) be an ordinary metric space and define a function D_1 on X^3 by

$$D_1(x, y, z) = max\{d(x, y), d(y, z), d(z, x)\},$$

for all $x, y, z \in X$. Then, the function D_1 is a D-metric on X and (X, D_1) is a D-metric space.

Example1.2: [4] Let (X, d) be an ordinary metric space and define a function D_2 on X^3 by

$$D_2(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

for $x, y, z \in X$. Then, D_2 is a metric on X and (X, D_2) is a D-metric space.

Remark 1. *Geometrically, the D-metric D_1 represents the diameter of a set consisting of three points x, y and z in X and the D-metric $D_2(x, y, z)$ represents the perimeter of a triangle formed by three points x, y, z in X as its vertices.*

Definition 11. *(Cauchy sequence , completeness)[10] For every $m, n > N$. A sequence (x_n) in a metric space $X = (X, d)$ is said to be-Cauchy if for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that*

$$d(x_m, x_n) < \varepsilon.$$

The space X is said to be complete if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

The following theorem can be found in [4]:

Theorem 2. *[4] Let d be an ordinary metric on X and let D_1 and D_2 be corresponding associated D-metrics on X. Then, (X, D_1) and (X, D_2) are complete if and only if (X, d) is complete.*

4 D-metrics in $\mathcal{B}^*_{\alpha,\log}$ and $F^*_{\log}(p,q,s)$

In this section, we introduce a D-metric on $\mathcal{B}^*_{\alpha,\log}$ and $F^*_{\log}(p,q,s)$.
Let $0 < p, s < \infty, -2 < q < \infty$, and $0 < \alpha < 1$. First, we can find a D-metric in $\mathcal{B}^*_{\alpha,\log}$, for $f, g, h \in \mathcal{B}^*_{\alpha,\log}$ by defining

$$D(f,g,h;\mathcal{B}^*_{\alpha,\log}) := D_{\mathcal{B}^*_{\alpha,\log}}(f,g,h) + ||f-g||_{\mathcal{B}_{\alpha,\log}} + ||g-h||_{\mathcal{B}_{\alpha,\log}} + ||h-f||_{\mathcal{B}_{\alpha,\log}}$$
$$+|f(0)-g(0)| + |g(0)-h(0)| + |h(0)-f(0)|,$$

where

$$D_{\mathcal{B}^*_{\alpha,\log}}(f,g,h) := d_{\mathcal{B}^*_{\alpha,\log}}(f,g) + d_{\mathcal{B}^*_{\alpha,\log}}(g,h) + d_{\mathcal{B}^*_{\alpha,\log}}(h,f)$$

and

$$D_{\mathcal{B}^*_{\alpha,\log}}(f,g,h) := \left(\sup_{z\in\mathbb{D}} |f^*(z)-g^*(z)| + \sup_{z\in\mathbb{D}} |g^*(z)-h^*(z)| + \sup_{z\in\mathbb{D}} |h^*(z)-f^*(z)|\right)$$
$$\times\left((1-|z|^2)^{\alpha}\left(\log\frac{2}{1-|z|^2}\right)\right).$$

Also, for $f, g, h \in F^*_{\log}(p,q,s)$ we introduce a D-metric on $F^*_{\log}(p,q,s)$ by defining

$$D(f,g,h;F^*_{\log}(p,q,s)) := D_{F^*_{\log}(p,q,s)}(f,g,h) + ||f-g||_{F_{\log(p,q,s)}} + ||g-h||_{F_{\log(p,q,s)}} +$$
$$||h-f||_{F_{\log(p,q,s)}} + |f(0)-g(0)| + |g(0)-h(0)| + |h(0)-f(0)|,$$

where

$$D_{F^*_{\log}(p,q,s)}(f,g,h) := d_{F^*_{\log}(p,q,s)}(f,g) + d_{F^*_{\log}(p,q,s)}(g,h) + d_{F^*_{\log}(p,q,s)}(h,f)$$

and

$$d_{F^*_{\log}(p,q,s)}(f,g) := \left(\sup_{z\in\mathbb{D}} \ell^p(a)\int_{\mathbb{D}} |f^*(z)-g^*(z)|^p(1-|z|^2)^q(1-|\varphi(z)|^2)^s dA(z)\right)^{\frac{1}{p}}.$$

Proposition 1. *The class $\mathcal{B}^*_{\alpha,\log}$ equipped with the D-metric $D(.,.;\mathcal{B}^*_{\alpha,\log})$ is a complete metric space. Moreover, $\mathcal{B}^*_{\alpha,\log,0}$ is a closed (and therefore complete) subspace of $\mathcal{B}^*_{\alpha,\log}$.*

Proof. Let $f, g, h, a \in \mathcal{B}^*_{\alpha,\log}$. Then, clearly

(i) $D(f,g,h;\mathcal{B}^*_{\alpha,\log}) \geq 0$, for all $f, g, h \in \mathcal{B}^*_{\alpha,\log}$.

(ii)$D(f,g,h;\mathcal{B}^*_{\alpha,\,\log}) = D(f,h,g;\mathcal{B}^*_{\alpha,\,\log}) = D(g,h,f;\mathcal{B}^*_{\alpha,\,\log})$.

(iii)$D(f,g,h;\mathcal{B}^*_{\alpha,\,\log}) \leq D(f,g,a;\mathcal{B}^*_{\alpha,\,\log}) + D(f,a,h;\mathcal{B}^*_{\alpha,\,\log}) + D(a,g,h;\mathcal{B}^*_{\alpha,\,\log})$

for all $f,g,h,a \in \mathcal{B}^*_{\alpha,\,\log}$.

(iv)$D(f,g,h;\mathcal{B}^*_{\alpha,\,\log}) = 0$ implies $f = g = h$.

Hence, D is a D-metric on $\mathcal{B}^*_{\alpha,\,\log}$, and $(\mathcal{B}^*_{\alpha,\,\log}, D)$ is D-metric space.

To prove the completeness, we use Theorem 2, let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in the metric space $(\mathcal{B}^*_{\alpha,\,\log}, d)$, that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ such that $d(f_n, f_m; \mathcal{B}^*_{\alpha,\,\log}) < \varepsilon$, for all $n, m > N$. Since $(f_n) \subset B(\mathbb{D})$, the family (f_n) is uniformly bounded and hence normal in $\mathbb{D}$. Therefore, there exists $f \in B(\mathbb{D})$ and a subsequence $(f_{n_j})_{j=1}^{\infty}$ such that f_{n_j} converges to f uniformly on compact subsets of $\mathbb{D}$. It follows that f_n also converges to f uniformly on compact subsets, and by the Cauchy formula, the same also holds for the derivatives. Now let $m > N$. Then, the uniform convergence yields

$$\begin{aligned}
& \left|f^*(z) - f^*_m(z)\right|(1-|z|^2)^{\alpha}\left(\log\frac{2}{1-|z|^2}\right) \\
= \quad & \lim_{n\to\infty}\left|f^*_n(z) - f^*_m(z)\right|(1-|z|^2)^{\alpha}\left(\log\frac{2}{1-|z|^2}\right) \\
\leq \quad & \lim_{n\to\infty} d(f_n, f_m; \mathcal{B}^*_{\alpha,\,\log}) \leq \varepsilon
\end{aligned}$$

for all $z \in \mathbb{D}$, and it follows that $||f||_{\mathcal{B}^*_{\alpha,\,\log}} \leq ||f_m||_{\mathcal{B}^*_{\alpha,\,\log}} + \varepsilon$. Thus $f \in \mathcal{B}^*_{\alpha,\,\log}$ as desired. Moreover, the above inequality and the compactness of the usual $\mathcal{B}^*_{\alpha,\,\log}$ space imply that $(f_n)_{n=1}^{\infty}$ converges to f with respect to the metric d, and $(\mathcal{B}^*_{\alpha,\,\log}, D)$ is complete D-metric space.
Since $\lim\limits_{n\to\infty} d(f_n, f_m; \mathcal{B}^*_{\alpha,\,\log}) \leq \varepsilon$, the second part of the assertion follows.

Next we give characterization of the complete D-metric space $D(.,.; F^*_{\log}(p,q,s))$.

Proposition 2. *The class $F^*_{\log}(p,q,s)$ equipped with the D-metric $D(.,.; F^*_{\log}(p,q,s))$ is a complete metric space. Moreover, $F^*_{\log,\,0}(p,q,s)$ is a closed (and therefore complete) subspace of $F^*_{\log}(p,q,s)$.*

Proof. Let $f,g,h,a \in F^*_{\log}(p,q,s)$. Then clearly

(i) $D(f,g,h; F^*_{\log}(p,q,s)) \geq 0$, for all $f,g,h \in F^*_{\log}(p,q,s)$.

(ii)$D(f,g,h; F^*_{\log}(p,q,s)) = D(f,h,g; F^*_{\log}(p,q,s)) = D(g,h,f; F^*_{\log}(p,q,s))$.

(iii)$D(f,g,h;F^*_{\log}(p,q,s)) \leq D(f,g,a;F^*_{\log}(p,q,s)) + D(f,a,h;F^*_{\log}(p,q,s))$

$$+D(a,g,h;F^*_{\log}(p,q,s))$$

for all $f,g,h,a \in F^*_{\log}(p,q,s)$.

(iv)$D(f,g,h;F^*_{\log}(p,q,s)) = 0$ implies $f = g = h$.

Hence, D is a D-metric on $F^*_{\log}(p,q,s)$, and $(F^*_{\log}(p,q,s), D)$ is D-metric space.

For the complete proof, by using Theorem 2, let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in the metric space $(F^*_{\log}(p,q,s), d)$, that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ so that $d(f_n, f_m; F^*_{\log}(p,q,s)) < \varepsilon$, for all $n, m > N$. Since $(f_n) \subset B(\mathbb{D})$, such that f_{n_j} converges to f uniformly on compact subsets of $\mathbb{D}$. It follows that f_n also converges to f uniformly on compact subsets, now let $m > N$, and $0 < r < 1$. Then, the Fatou's yields

$$\begin{aligned} &\int_{\mathbb{D}(0,r)} \left| f^*(z) - f^*_m(z) \right|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) \\ = &\int_{\mathbb{D}(0,r)} \lim_{n\to\infty} \left| f^*_n(z) - f^*_m(z) \right|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) \\ \leq &\lim_{n\to\infty} \int_{\mathbb{D}(0,r)} \left| f^*(z) - f^*_m(z) \right|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) \leq \varepsilon^p, \end{aligned}$$

and by taking $r \to 1^-$, it follows that,

$$\begin{aligned} &\int_{\mathbb{D}} (f^*(z))^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) \\ &\leq 2^p \varepsilon^p + 2^p \int_{\mathbb{D}} (f^*_m(z))^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z). \end{aligned}$$

This yields

$$||f||^p_{F^*_{\log}(p,q,s)} \leq 2^p ||f_m||^p_{F^*_{\log}(p,q,s)} + 2^p \varepsilon^p.$$

And thus $f \in F^*_{\log}(p,q,s)$. We also find that $f_n \to f$ with respect to the metric of $(F^*_{\log}(p,q,s), D)$ and $(F^*_{\log}(p,q,s), D)$ is complete D-metric space. The second part of the assertion follows.

5 Composition operators of $C_\phi : \mathcal{B}^*_{\alpha,\,\log} \to F^*_{\log}(p,q,s)$

In this section, we study boundedness and compactness of composition operators on $\mathcal{B}^*_{\alpha,\,\log}$ and $F^*_{\log}(p,q,s)$ spaces. We need the following notation:

$$\Phi_\phi(\alpha,p,s;a) = \ell^p(a) \int_{\mathbb{D}} |\phi'(z)|^p \frac{(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s}{(1-|\phi(z)|^2)^{\alpha p} \left(\log \frac{2}{(1-|\phi(z)|^2)} \right)^p} \, dA(z),$$

where $\ell^p(a) = \left(\log \frac{2}{1-|a|^2}\right)^p$.

For $0 < \alpha < 1$, we suppose there exist two functions $f, g \in \mathcal{B}^*_{\alpha,\, \log}$ such that for some constant C,

$$(|f^*(z)| + |g^*(z)|) \geq \frac{C}{(1-|z|^2)^\alpha \left(\log \frac{2}{1-|a|^2}\right)^p} > 0, \qquad for\ each\ z\ \in\ \mathbb{D}.$$

Now, we provide the following theorem:

Theorem 3. *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself and let $0 < p, 1 < s < \infty, 0 < \alpha \leq 1$. Then the induced composition operator C_ϕ maps $\mathcal{B}^*_{\alpha,\, \log}$ into $F^*_{\log}(p, \alpha p - 2, s)$ is bounded if and only if,*

$$\sup_{z\in\mathbb{D}} \Phi_\phi(\alpha, p, s; a) < \infty. \tag{5.1}$$

Proof. First assume that $\sup_{z\in\mathbb{D}} \Phi_\phi(\alpha, p, s; a) < \infty$ is held, and $f \in \mathcal{B}^*_{\alpha,\, \log}$ with $||f||_{\mathcal{B}_{\alpha,\, \log}} \leq 1$, we can see that

$$\begin{aligned}
& ||C_\phi f||^p_{F^*_{\log}(p,\alpha p-2,s)} \\
= \quad & \sup_{a\in\mathbb{D}} \ell^p(a) \int_{\mathbb{D}} ((f\circ\phi)^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
= \quad & \sup_{a\in\mathbb{D}} \ell^p(a) \int_{\mathbb{D}} (f^*(\phi(z)))^p |\phi'(z)|^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
\leq \quad & ||f||^p_{\mathcal{B}^*_{\alpha,\, \log}} \sup_{a\in\mathbb{D}} \ell^p(a) \int_{\mathbb{D}} \frac{|\phi'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s}{(1-|\phi(z)^2|)^{p\alpha}(\log \frac{2}{1-|z|^2})} dA(z) \\
= \quad & ||f||^p_{\mathcal{B}^*_{\alpha,\, \log}} \Phi_\phi(\alpha, p, s; a) < \infty.
\end{aligned}$$

For the other direction, we use the fact that for each function $f \in \mathcal{B}^*_{\alpha,\, \log}$, the analytic function

$C_\phi(f) \in F^*_{\log}(p, \alpha p - 2, s)$. Then, using the functions of lemma 1.2

$$\begin{aligned}
& 2^p\Big\{||C_\phi f_1||^p_{F^*_{\log}(p,\alpha p-2,s)} + ||C_\phi f_2||^p_{F^*_{\log}(p,\alpha p-2,s)}\Big\} \\
= \ & 2^p\Big\{\sup_{a\in\mathbb{D}} \ell^p(a)\int_{\mathbb{D}}\Big[((f_1\circ\phi)^*(z))^p + ((f_2\circ\phi)^*(z))^p\Big] \\
& \times(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\Big\} \\
\geq \ & \Big\{\sup_{a\in\mathbb{D}} \ell^p(a)\int_{\mathbb{D}}\Big[(f_1\circ\phi)^*(z) + (f_2\circ\phi)^*(z)\Big]^p \\
& \times(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\Big\} \\
\geq \ & \Big\{\sup_{a\in\mathbb{D}} \ell^p(a)\int_{\mathbb{D}}\Big[(f_1^*(\phi))(z) + (f_2^*(\phi))(z)\Big]^p \\
& \times|\phi'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\Big\} \\
\geq \ & C\Big\{\sup_{a\in\mathbb{D}} \ell^p(a)\int_{\mathbb{D}}|\phi'(z)|^p\frac{(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s}{(1-|\phi(z)|^2)^{\alpha p}\left(\log\frac{2}{(1-|\phi(z)|^2)}\right)^p}dA(z)\Big\} \\
\geq \ & C\sup_{a\in\mathbb{D}}\Phi_\phi(\alpha,p,s;a).
\end{aligned}$$

Hence C_ϕ is bounded, the proof is completed.

The composition operator $C_\phi : \mathcal{B}^*_{\alpha,\,\log} \to F^*_{\log}(p, \alpha p - 2, s)$ is compact if and only if for every sequence $f_n \in \mathbb{N} \subset F^*_{\log}(p, \alpha p - 2, s)$ is bounded in $F^*_{\log}(p, \alpha p - 2, s)$ norm and $f_n \to 0$, $n \to \infty$, uniformly on compact subset of the unit disk (where $\mathbb{N}$ be the set of all natural numbers), hence,

$$||C_\phi(f_n)||_{F^*_{\log}(p,\alpha p-2,s)} \to 0, n \to \infty.$$

Now, we describe compactness in the following result:

Theorem 4. *Let $0 < p, 1 < s < \infty, \alpha < \infty$. If ϕ is an analytic self-map of the unit disk, then the induced composition operator $C_\phi : \mathcal{B}^*_{\alpha,\,\log} \to F^*_{\log}(p, \alpha p - 2, s)$ is compact if and only if $\phi \in F^*_{\log}(p, \alpha p - 2, s)$, and*

$$\lim_{r\to 1}\sup_{a\in\mathbb{D}}\Phi_\phi(\alpha,p,s;a) \to 0. \tag{5.2}$$

Proof. Let $C_\phi : \mathcal{B}^*_{\alpha,\,\log} \to F^*_{\log}(p, \alpha p - 2, s)$ be compact. This means that
$\phi \in F^*_{\log}(p, \alpha p - 2, s)$.

Let

$$U^1_r = \{z : |\phi(z)| > r, r \in (0, 1)\},$$

and

$$U_r^2 = \{z : |\phi(z)| \leq r, r \in (0,1)\}.$$

Let $f_n(z) = \frac{z^n}{n}$ if $\alpha \in [0, \infty)$ or $f_n(z) = \frac{z^n}{n^{1-\alpha}}$ if $\alpha \in (0,1)$. Without loss of generality, we only consider $\alpha \in (0,1)$. Since $||f_n||_{\mathcal{B}^*_{\alpha,\,\log}} \leq M$ and $f_n(z) \to 0$ as $n \to \infty$, locally uniformly on the unit disk, then $||C_\phi(f_n)||_{F^*_{\log}(p,\alpha p-2,s)}$, $n \to \infty$. This means that for each $r \in (0,1)$ and for all $\varepsilon > 0$, there exist $N \in \mathbb{N}$ so that if $n \geq N$, then

$$\frac{N^{\alpha p}}{r^{p(1-N)}} \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} |\phi'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon.$$

If we choose r so that $\frac{N^{\alpha p}}{r^{p(1-N)}} = 1$, then

$$\sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} |\phi'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon. \tag{5.3}$$

Let now f be with $||f||_{\mathcal{B}^*_{\alpha,\,\log}} \leq 1$. We consider the functions $f_t(z) = f(tz)$, $t \in (0,1)$. $f_t \to f$ uniformly on compact subset of the unit disk as $t \to 1$ and the family (f_t) is bounded on $\mathcal{B}^*_{\alpha,\,\log}$, thus

$$||(f_t \circ \phi\,) - (f \circ \phi\,)|| \to 0.$$

Due to compactness of C_ϕ, we get that for $\varepsilon > 0$ there is $t \in (0,1)$ so that

$$\sup_{a\in\mathbb{D}} \ell^p(a) \int_{\mathbb{D}} |F_t(\phi(z))|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon,$$

where

$$F_t(\phi(z)) = \big[(f \circ \phi\,)^* - (f_t \circ \phi\,)^*\big].$$

Thus, if we fix t, then

$$\begin{aligned}
& \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} ((f\circ\phi)^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
\leq\ & 2^p \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} |F_t(\phi(z))|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
& +2^p \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} ((f_t\circ\phi)^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
\leq\ & 2^p\varepsilon + ||f_t^*||^p_{H^\infty} \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} |\phi'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
\leq\ & 2^p\varepsilon + 2^p\varepsilon||f_t^*||^p_{H^\infty}.
\end{aligned}$$

i.e,

$$\begin{aligned} & \sup_{a\in\mathbb{D}} \ell^p(a)\int_{U_r^1}((f\circ\phi)^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\ \leq \; & 2^p\varepsilon(1+||f_t^*||^p_{H^\infty}), \end{aligned} \tag{5.4}$$

where we have used (4). On the other hand, for each $||f||_{\mathcal{B}^*_{\alpha,\,\log}} \leq 1$ and $\varepsilon > 0$, there exists a δ depending on f and ε, so that for $r \in [\delta, 1)$,

$$\sup_{a\in\mathbb{D}} \ell^p(a)\int_{U_r^1}((f\circ\phi)^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon. \tag{5.5}$$

Since C_ϕ is compact, then it maps the unit ball of $\mathcal{B}^*_{\alpha,\,\log}$ to a relatively compact subset of $F^*_{\log}(p,q,s)$. Thus, for each $\varepsilon > 0$, there exists a finite collection of functions $f_1, f_2, ..., f_n$ in the unit ball of $\mathcal{B}^*_{\alpha,\,\log}$ so that for each $||f||_{\mathcal{B}^*_{\alpha,\,\log}}$, there is $k \in \{1,2,3,...,n\}$ so that

$$\sup_{a\in\mathbb{D}} \ell^p(a)\int_{U_r^1}|F_k(\phi(z))|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon,$$

where

$$F_k(\phi(z)) = \big[(f\circ\phi\,)^* - (f_k\circ\phi\,)^*\big].$$

Also, using (5), we get for $\delta = max_{1\leq k\leq n}\delta(f_k,\varepsilon)$ and $r \in [\delta, 1)$, that

$$\sup_{a\in\mathbb{D}} \ell^p(a)\int_{U_r^1}((f_k\circ\phi)^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon.$$

Hence, for any $f, ||f||_{\mathcal{B}^*_{\alpha,\,\log}} \leq 1$, combining the two relations as above, we get the following

$$\sup_{a\in\mathbb{D}} \ell^p(a)\int_{U_r^1}((f\circ\phi)^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \leq 2^p\varepsilon.$$

Therefore, we get that (2) holds. For the sufficiency, we use that $\phi \in F^*_{\log}(p, \alpha p-2, s)$ and (2) holds.

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in the unit ball of $\mathcal{B}^*_{\alpha,\,\log}$ so that $f_n \to 0$ as $n \to \infty$, uniformly on the compact subsets of the unit disk. Let also $r \in (0,1)$. Then,

$$\begin{aligned} & ||f_n\circ\phi||^p_{F^*_{\log}(p,\alpha p-2,s)} \leq 2^p|f_n(\phi(0))| \\ & +2^p\sup_{a\in\mathbb{D}} \ell^p(a)\int_{U_r^2}((f_n\circ\phi)^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\ & +2^p\sup_{a\in\mathbb{D}} \ell^p(a)\int_{U_r^1}((f_n\circ\phi)^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\ = \; & 2^p(I_1+I_2+I_3). \end{aligned}$$

Since $f_n \to 0$ as $n \to \infty$, locally uniformly on the unit disk, then $I_1 = |f_n(\phi(0))|$ goes to zero as $n \to \infty$ and for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ so that for each $n > N$,

$$\begin{aligned} I_2 &= \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^2} ((f_n \circ \phi)^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\ &\leq \varepsilon ||\phi||^p_{F^*_{\log}(p,\alpha p-2,s)}. \end{aligned}$$

We also observe that

$$\begin{aligned} I_3 &= \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} ((f_n \circ \phi)^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\ &\leq ||f||^p_{\mathcal{B}^*_{\alpha,\,\log}} \\ &\quad \times \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} |\phi'(z)|^p \frac{(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s}{(1-|\phi(z)|^2)^{\alpha p}\left(\log \frac{2}{(1-|\phi(z)|^2)}\right)^p} dA(z). \end{aligned}$$

Under the assumption that (2) holds, then for every $n > N$ and for every $\varepsilon > 0$, there exists r_1 so that for every $r > r_1$, $I_3 < \varepsilon$.
Thus, if $\phi(z) \in F^*_{\log}(p, \alpha p - 2, s)$, we get

$$||f_n \circ \phi||^p_{F^*_{\log}(p,\alpha p-2,s)} \leq 2^p \left\{ 0 + \varepsilon ||\phi||^p_{F^*_{\log}(p,\alpha p-2,s)} + \varepsilon \right\} \leq C\varepsilon.$$

Combining the above, we get $||C_\phi(f_n)||^p_{F^*_{\log}(p,\alpha p-2,s)} \to 0$ as $n \to \infty$ which proves compactness. Thus, the theorem we presented is proved.

6 Conclusions

We have obtained some essential and important D-metric spaces. Moreover, the important properties for D-metric on $\mathcal{B}^*_{\alpha,\,\log}$ and $F^*_{\log}(p,q,s)$ are investigated in Section 4. Finally, we introduced composition operators in hyperbolic weighted family of function spaces.

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Hyers-Ulam stability of an additive-quadratic functional equation

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ABSTRACT

In this paper, we introduce the following (a, b, c)-mixed type functional equation of the form

$$g(ax_1 + bx_2 + cx_3) - g(-ax_1 + bx_2 + cx_3) + g(ax_1 - bx_2 + cx_3) - g(ax_1 + bx_2 - cx_3) + 2a^2[g(x_1) + g(-x_1)] + 2b^2[g(x_2) + g(-x_2)] + 2c^2[g(x_3) + g(-x_3)] + a[g(x_1) - g(-x_1)] + b[g(x_2) - g(-x_2)] + c[g(x_3) - g(-x_3)] = 4g(ax_1 + cx_3) + 2g(-bx_2) + 2g(bx_2)$$

where a, b, c are positive integers with $a > 1$, and investigate the solution and the Hyers-Ulam stability of the above functional equation in Banach spaces by using two different methods.

RESUMEN

En este artículo introducimos la siguiente ecuación funcional de tipo (a, b, c)-mixta de la forma

$$g(ax_1 + bx_2 + cx_3) - g(-ax_1 + bx_2 + cx_3) + g(ax_1 - bx_2 + cx_3) - g(ax_1 + bx_2 - cx_3) + 2a^2[g(x_1) + g(-x_1)] + 2b^2[g(x_2) + g(-x_2)] + 2c^2[g(x_3) + g(-x_3)] + a[g(x_1) - g(-x_1)] + b[g(x_2) - g(-x_2)] + c[g(x_3) - g(-x_3)] = 4g(ax_1 + cx_3) + 2g(-bx_2) + 2g(bx_2)$$

donde a, b, c son enteros positivos con $a > 1$, e investigamos la solución y la estabilidad de Hyers-Ulam de la ecuación funcional anterior en espacios de Banach usando dos métodos diferentes.

Keywords and Phrases: Hyers-Ulam stability, mixed type functional equation, Banach space, fixed point.

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1 Introduction

The stability problem of functional equations originated form a question of Ulam [28] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam [28] for Banach spaces. Hyers theorem was generalized by Aoki [3] for additive mappings and Rassias [12] for quadratic mappings. During the last three decades the stability theorem of Rassias [26] provided a lot of influence for the development of stability theory of a large variety of functional equations (see [1, 2, 4, 7, 9, 11, 14, 17, 18, 21, 22, 23, 27]). One of the most famous functional equations is the following additive functional equation

$$g(x+y) = g(x) + g(y) \tag{1.1}$$

In 1821, it was first solved by Cauchy in the class of continuous real-valued functions. It is often called Cauchy additive functional equation in honour of Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social science ([6, 24, 26]). Every solution of the additive functional equation (1.1) is called an additive mapping.

The function $g(x) = x^2$ satisfies the functional equation

$$g(x+y) + g(x-y) = 2g(x) + 2g(y) \tag{1.2}$$

and therefore, the functional equation (1.2) is called quadratic functional equation. The Hyers-Ulam stability theorem for the quadratic functional equation (1.2) was proved by Skof [25] for the mapping $g : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space.

Moslehian and Rassias [20] studied the Hyers-Ulam stability problem in non-Archimedean normed spaces. Mirzavaziri and Moslehian [19] studied the Hyers-Ulam stability of a quadratic functional equation in Banach spaces by using the fixed point method and Ciepliński [5] surveyed the Hyers-Ulam stability of functional equations by using the fixed point method. Ebadian, Ghobadipour and Eshaghi Gordji [8] proved the Hyers-Ulam stability of bimultipliers and Jordan bimultipliers in C^*-ternary algebras by using the fixed point method for a three variable additive functional equation.

Motivated by Ebadian *et al.* [8], we introduce the following three variable generalized additive-quadratic functional equation of the form $Dg(x_1, x_2, x_3) :=$

$$\begin{aligned} g(ax_1 + bx_2 + cx_3) - g(-ax_1 + bx_2 + cx_3) + g(ax_1 - bx_2 + cx_3) - g(ax_1 + bx_2 - cx_3) \\ + 2a^2[g(x_1) + g(-x_1)] + 2b^2[g(x_2) + g(-x_2)] + 2c^2[g(x_3) + g(-x_3)] \\ + a[g(x_1) - g(-x_1)] + b[g(x_2) - g(-x_2)] + c[g(x_3) - g(-x_3)] \end{aligned}$$

$$- [4g(ax_1 + cx_3) + 2g(-bx_2) + 2g(bx_2)] = 0 \quad (1.3)$$

where a, b, c are positive integers with $a > 1$, and investigate the solution and the Hyers-Ulam stability of the three variable generalized additive-quadratic functional equation (1.3) in Banach spaces by using the direct method and the fixed point method.

2 Solution of the functional equation (1.3): when g is odd

In this section, we investigate the solution of the functional equation (1.3) for an odd mapping case. Throughout this section, let X and Y be real vector spaces.

Theorem 1. *If an odd mapping $g : X \to Y$ satisfies the functional equation (1.1) if and only if $g : X \to Y$ satisfies the functional equation (1.3).*

Proof. Assume that $g : X \to Y$ satisfies the functional equation (1.1).
Since g is odd, $g(0) = 0$.
Replacing (x, y) by (x, x) and by $(x, 2x)$ respectively in (1.1), we obtain

$$g(2x) = 2g(x) \text{ and } g(3x) = 3g(x) \quad (2.1)$$

for all $x \in X$. In general for any positive integer d, we have

$$g(dx) = dg(x) \quad (2.2)$$

for all $x \in X$. It is easy to verify from (1.1) that

$$g(d^2x) = d^2g(x) \text{ and } g(d^3x) = d^3g(x) \quad (2.3)$$

for all $x \in X$. Replacing (x, y) by $(ax_1 + bx_2, cx_3)$ in (1.1), we get

$$g(ax_1 + bx_2 + cx_3) = g(ax_1 + bx_2) + g(cx_3) \quad (2.4)$$

for $x_1, x_2, x_3 \in X$. Replacing x_1 by $-x_1$ in (2.4), we get

$$g(-ax_1 + bx_2 + cx_3) = g(-ax_1 + bx_2) + g(cx_3) \quad (2.5)$$

for $x_1, x_2, x_3 \in X$. Replacing x_2 by $-x_2$ in (2.4), we have

$$g(ax_1 - bx_2 + cx_3) = g(ax_1 - bx_2) + g(cx_3) \quad (2.6)$$

for $x_1, x_2, x_3 \in X$. Replacing x_3 by $-x_3$ in (2.4), we obtain

$$g(ax_1 + bx_2 - cx_3) = g(ax_1 + bx_2) + g(-cx_3) \quad (2.7)$$

for $x_1, x_2, x_3 \in X$. By (2.4), (2.5), (2.6), (2.7), (1.1) and (2.3), we get

$$g(ax_1 + bx_2 + cx_3) - g(-ax_1 + bx_2 + cx_3) + g(ax_1 - bx_2 + cx_3) - g(ax_1 + bx_2 - cx_3) = 2ag(x_1) - 2bg(x_2) + 2cg(x_3) \quad (2.8)$$

for $x_1, x_2, x_3 \in X$. Adding $2ag(x_1) - 2bg(x_2) + 2cg(x_3) + 2a^2g(x_1) + 2b^2g(x_2) + 2c^2g(x_3)$ to both sides and using the oddness of g, we get (1.3).

Conversely, assume that g satisfies (1.3). Letting $x_3 = 0$ in (1.3), we have

$$\begin{aligned} &g(ax_1 + bx_2 + cx_3) - g(-ax_1 + bx_2 + cx_3) + g(ax_1 - bx_2 + cx_3) - g(ax_1 + bx_2 - cx_3) \\ &\quad + 2a^2[g(x_1) + g(-x_1)] + 2b^2[g(x_2) + g(-x_2)] + 2c^2[g(x_3) + g(-x_3)] + a[g(x_1) - g(-x_1)] \\ &\quad + b[g(x_2) - g(-x_2)] + c[g(x_3) - g(-x_3)] \\ &= 2g(ax_1 - bx_2) + 2ag(x_1) + 2bg(x_2) \end{aligned}$$

for all $x_1, x_2 \in X$, since g is odd. So

$$2g(ax_1 - bx_2) + 2ag(x_1) + 2bg(x_2) = 4g(ax_1) \quad (2.9)$$

for all $x_1, x_2 \in X$. Letting $x_2 = 0$ in (2.9), we have $2g(ax_1) + 2ag(x_1) = 4g(ax_1)$ and so $g(ax_1) = ag(x_1)$ for all $x_1 \in X$. Letting $x_1 = 0$ in (2.9), we have $-2g(bx_2) + 2bg(x_2) = 0$ and so $g(bx_2) = bg(x_2)$ for all $x_2 \in X$. It follows from (2.9) that

$$2g(ax_1 - bx_2) + 2g(ax_1) + 2g(bx_2) = 4g(ax_1)$$

for all $x_1, x_2 \in X$ and so

$$g(x - y) + g(y) = g(x)$$

for all $x, y \in X$. Letting $z = x - y$ in the above equation, we get $g(z) + g(y) = g(z + y)$ for all $z, y \in X$. ∎

3 Solution of the functional equation (1.3): when g is even

In this section, we investigate the solution of the functional equation (1.3) for an even mapping case. Throughout this section, let X and Y to be real vector spaces.

Theorem 2. *If an even mapping $g : X \to Y$ satisfies the functional equation (1.2) if and only if $g : X \to Y$ satisfies the functional equation (1.3).*

Proof. Assume that $g : X \to Y$ satisfies the functional equation (1.2).
Setting $x = y = 0$ in (1.2), we get $g(0) = 0$.

Replacing (x, y) by (x, x) and by $(x, 2x)$, respectively, in (1.2), we obtain

$$g(2x) = 4g(x) \text{ and } g(3x) = 9g(x) \tag{3.1}$$

for all $x \in X$. In general for any positive integer d, we have

$$g(dx) = d^2 g(x) \tag{3.2}$$

for all $x \in X$. It is easy to verify from (1.2) that

$$g(d^2 x) = d^4 g(x) \text{ and } g(d^3 x) = d^6 g(x) \tag{3.3}$$

for all $x \in X$. Replacing (x, y) by (ax_1, cx_3) in (1.2), we get

$$g(ax_1 + cx_3) + g(ax_1 - cx_3) = 2g(ax_1) + 2g(cx_3) \tag{3.4}$$

for $x_1, x_2, x_3 \in X$.
Multiplying 2 on both sides and using (3.3), we get

$$2g(ax_1 + cx_3) + 2g(ax_1 - cx_3) = 4a^2 g(x_1) + 4c^2 g(x_3) \tag{3.5}$$

for $x_1, x_2, x_3 \in X$.
Adding $2g(ax_1 + cx_3)$ to (3.5) on both sides and using (3.3), we obtain

$$2g(ax_1 + cx_3) + 2g(ax_1 - cx_3) + 2g(ax_1 + cx_3) = 4a^2 g(x_1) + 4c^2 g(x_3) + 2g(ax_1 + cx_3) \tag{3.6}$$

for $x_1, x_2, x_3 \in X$. So

$$4g(ax_1 + cx_3) = 4a^2 g(x_1) + 4c^2 g(x_3) + 2g(ax_1 + cx_3) - 2g(ax_1 - cx_3). \tag{3.7}$$

Adding and subtracting $2g(bx_2)$ to (3.7), we get

$$\begin{aligned} 4g(ax_1 + cx_3) = 4a^2 g(x_1) + 4c^2 g(x_3) + g(ax_1 + cx_3 + bx_2) + g(ax_1 + cx_3 - bx_2) \\ - g(ax_1 - cx_3 + bx_2) - g(ax_1 - cx_3 - bx_2) \end{aligned} \tag{3.8}$$

for $x_1, x_2, x_3 \in X$.
Adding $4g(bx_2)$ to (3.8) on both sides, we obtain

$$\begin{aligned} 4g(ax_1 + cx_3) + 4g(bx_2) = 4a^2 g(x_1) + 4c^2 g(x_3) + g(ax_1 + cx_3 + bx_2) + g(ax_1 + cx_3 - bx_2) \\ - g(ax_1 - cx_3 + bx_2) - g(ax_1 - cx_3 - bx_2) + 4g(bx_2) \end{aligned} \tag{3.9}$$

for $x_1, x_2, x_3 \in X$. By (3.9) and (3.3), we get

$$\begin{aligned} 4g(ax_1 + cx_3) + 4g(bx_2) = 4a^2 g(x_1) + 4c^2 g(x_3) + 4b^2 g(x_2) + g(ax_1 + cx_3 + bx_2) \\ + g(ax_1 + cx_3 - bx_2) - g(ax_1 - cx_3 + bx_2) - g(-ax_1 + cx_3 + bx_2) \end{aligned} \tag{3.10}$$

for $x_1, x_2, x_3 \in X$. Using (3.10), (3.3) and the evenness of g, we get

$$g(ax_1 + bx_2 + cx_3) + g(ax_1 - bx_2 + cx_3) - g(ax_1 + bx_2 - cx_3) - g(-ax_1 + bx_2 + cx_3) + 4a^2 g(x_1) + 4b^2 g(x_2) + 4c^2 g(x_3) = 4g(ax_1 + cx_3) + 4g(bx_2) \quad (3.11)$$

for all $x_1, x_2, x_3 \in X$.

Conversely, assume that $g : X \to Y$ satisfies the functional equation (1.3). Replacing (x_1, x_2, x_3) by $\left(\frac{x}{a}, 0, \frac{y}{c}\right)$ in (1.3), we get

$$g(x-y) - g(-x+y) + g(x+y) - g(x-y) + 4g(x) + 4g(y) = 4g(x+y) \quad (3.12)$$

for all $x, y \in X$. Using (1.3) and the evenness of g, we get

$$g(x+y) + g(x-y) = 2g(x) + 2g(y),$$

which is quadratic. ∎

4 Stability results for (1.3): Odd case and direct method

In this section, we present the Hyers-Ulam stability of the functional equation (1.3) for an odd mapping case.

Theorem 3. *Let $j \in \{-1, 1\}$ and $\alpha : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(a^{kj}x_1, a^{kj}x_2, a^{kj}x_3)}{a^{kj}} < \infty$$

for all $x_1, x_2, x_3 \in X$. Let $g : X \to Y$ be an odd mapping satisfying the inequality

$$\|Dg(x_1, x_2, x_3)\| \le \alpha(x_1, x_2, x_3) \quad (4.1)$$

for all $x_1, x_2, x_3 \in X$. There exists a unique additive mapping $A : X \to Y$ which satisfies the functional equation (1.3) and

$$\|g(x) - A(x)\| \le \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(a^{kj}x_1, 0, 0)}{a^{kj}} \quad (4.2)$$

for all $x_1 \in X$. The mapping $A(x)$ is defined by,

$$A(x) = \lim_{k\to\infty} \frac{g(a^{kj}x_1)}{a^{kj}} \text{ for all } x \in X$$

Proof. Assume that $j = 1$. Replacing (x_1, x_2, x_3) by $(x, 0, 0)$ in (4.2) and using the oddness of g, we get

$$\|2g(ax) - 2ag(x)\| \le \alpha(x, 0, 0) \quad (4.3)$$

for all $x \in X$. It follows from (4.3) that

$$\left\| \frac{g(ax)}{a} - g(x) \right\| \leq \frac{1}{2a}\alpha(x,0,0) \tag{4.4}$$

for all $x \in X$. Replacing x by ax in (4.4) and dividing by a, we obtain

$$\left\| \frac{g(a^2x)}{a^2} - \frac{g(ax)}{a} \right\| \leq \frac{1}{2a^2}\alpha(ax,0,0) \tag{4.5}$$

for all $x \in X$. It follows from (4.4) and (4.5) that

$$\left\| \frac{g(a^2x)}{a^2} - g(x) \right\| \leq \frac{1}{2a}\left[\alpha(x,0,0) + \frac{\alpha(ax,0,0)}{a}\right] \tag{4.6}$$

for all $x \in X$. Similarly, for any positive integer n, we have

$$\left\| g(x) - \frac{g(a^nx)}{a^n} \right\| \leq \frac{1}{2a}\sum_{k=0}^{n-1}\frac{\alpha(a^kx,0,0)}{a^k} \leq \frac{1}{2a}\sum_{k=0}^{\infty}\frac{\alpha(a^kx,0,0)}{a^k} \tag{4.7}$$

for all $x \in X$. In order to prove convergence of the sequence $\left\{\frac{g(a^kx)}{a^k}\right\}$, replacing x by a^mx and dividing a^m in (4.7) for any $m, n > 0$, we get

$$\begin{aligned} \left\| \frac{g(a^mx)}{a^m} - \frac{g(a^{m+n}x)}{a^{m+n}} \right\| &= \frac{1}{2a^m}\left\| g(a^mx) - \frac{g(a^ma^nx)}{a^n} \right\| \\ &\leq \frac{1}{2a}\sum_{m=0}^{n-1}\frac{\alpha(a^{m+n}x,0,0)}{a^{m+n}} \\ &\leq \frac{1}{2a}\sum_{m=0}^{n-1}\frac{\alpha(a^{m+n}x,0,0)}{a^{m+n}} \\ &\to 0 \text{ as } m \to \infty. \end{aligned}$$

Hence the sequence $\left\{\frac{g(a^nx)}{a^n}\right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ such that

$$A(x) = \lim_{n\to\infty}\frac{g(a^nx)}{a^n}, \ \forall x \in X. \tag{4.8}$$

Letting $n \to \infty$ in (4.8), we see that (4.8) holds for $x \in X$.
To prove that A satisfies (1.3), replacing (x_1, x_2, x_3) by (a^nx, a^nx, a^nx) and dividing a^n in (4.1), we obtain

$$\frac{1}{a^n}\left\| Dg(a^nx, a^nx, a^nx)\right\| \leq \frac{1}{a^n}\alpha(a^nx, a^nx, a^nx)$$

for all $x_1, x_2, x_3 \in X$. Letting $m \to \infty$ in the above inequality and using the definition of $A(x)$, we see that $DA(x_1, x_2, x_3) = 0$. Hence A satisfies (1.3) for all $x_1, x_2, x_3 \in X$.
To show that A is unique, let $B(x)$ be another additive mapping satisfying (4.2). Then

$$\begin{aligned} \|A(x) - B(x)\| &= \frac{1}{a^n}\|A(a^nx) - B(a^nx)\| \\ &\leq \frac{1}{a^n}\{\|A(a^nx) - g(a^nx)\| + \|g(a^nx) - B(a^nx)\|\} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence A is unique.

Assume that $j = -1$. Replacing x by $\frac{x}{a}$ in (4.3), we get

$$\left\| ag(x) - a^2 g\left(\frac{x}{a}\right)\right\| \leq \alpha\left(\frac{x}{a}, 0, 0\right) \tag{4.9}$$

for all $x \in X$. The rest of the proof is similar to the proof of the case $j = 1$. This completes the proof of the theorem. ∎

The following corollary is an immediate consequence of Theorem 3 concerning the stability of (1.3).

Corollary 1. *Let ϵ and p be nonnegative real numbers. Let $g : X \to Y$ be an odd mapping satisfiying the inequality*

$$\begin{aligned} &\|Dg(x_1, x_2, x_3)\| \\ &\leq \begin{cases} \epsilon; & \\ \epsilon\left(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p\right); & p > 1 \quad or \quad p < 1 \\ \epsilon\left(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_1\|^{3p}\|x_2\|^{3p}\|x_3\|^{3p}\right); & p > \frac{1}{3} \quad or \quad p < \frac{1}{3} \end{cases} \end{aligned} \tag{4.10}$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|g(x) - A(x)\| \leq \begin{cases} \frac{\epsilon}{2|a-1|}; & \\ \frac{\epsilon\|x\|^p}{2|a-a^p|}; & p > 1 \quad or \quad p < 1 \\ \frac{\epsilon\|x\|^{3p}}{2|a-a^{3p}|}; & p > \frac{1}{3} \quad or \quad p < \frac{1}{3} \end{cases} \tag{4.11}$$

for all $x \in X$.

Proof. Letting

$$\alpha(x_1, x_2, x_3) = \begin{cases} \epsilon; \\ \epsilon\left(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p\right); \\ \epsilon\left(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p + \|x_1\|^{3p}\|x_2\|^{3p}\|x_3\|^{3p}\right) \end{cases}$$

for all $x_1, x_2, x_3 \in X$, we can get the result. ∎

5 Stability results for (1.3): Even case and direct method

In this section, we discuss the Hyers-Ulam stability of the functional equation (1.3) for an even mapping case by using the direct method.

Theorem 4. *Let $j \in \{-1, 1\}$ and $\alpha : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(a^{kj}x_1, a^{kj}x_2, a^{kj}x_3)}{a^{kj}} < \infty \tag{5.1}$$

for all $x_1, x_2, x_3 \in X$. Let $g : X \to Y$ be an even mapping satisfying $g(0) = 0$ and the inequality

$$\|Dg(x_1, x_2, x_3)\| \leq \alpha(x_1, x_2, x_3) \tag{5.2}$$

for all $x_1, x_2, x_3 \in X$. There exists a unique additive mapping $Q : X \to Y$ which satisfies the functional equation (1.3) and

$$\|g(x) - Q(x)\| \leq \frac{1}{4a^2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(a^{kj}x, 0, 0)}{a^{2kj}} \tag{5.3}$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{n\to\infty} \frac{g(a^{kj}x)}{a^{2kj}} \tag{5.4}$$

for all $x \in X$.

Proof. Assume that $j = 1$. Replacing (x_1, x_2, x_3) by $(x, 0, 0)$ in (5.2), we get

$$\|4g(ax) - 4a^2 g(x)\| \leq \alpha(x, 0, 0) \tag{5.5}$$

for all $x \in X$. It follows from (5.5) that

$$\left\| \frac{g(ax)}{a^2} - g(x) \right\| \leq \frac{1}{4a^2} \alpha(x, 0, 0) \tag{5.6}$$

for all $x \in X$. Replacing x by ax in (5.6) and dividing by a^2, we obtain

$$\left\| \frac{g(a^2x)}{a^4} - \frac{g(ax)}{a^2} \right\| \leq \frac{1}{4a^4} \alpha(ax, 0, 0) \tag{5.7}$$

for all $x \in X$. It follows from (5.6) and (5.7) that

$$\left\| \frac{g(a^2x)}{a^4} - g(x) \right\| \leq \frac{1}{4a^2} \left[\alpha(x, 0, 0) + \frac{\alpha(ax, 0, 0)}{a^2} \right] \tag{5.8}$$

for all $x \in X$. Inductively, we have

$$\left\| g(x) - \frac{g(a^n x)}{a^{2n}} \right\| \leq \frac{1}{4a^2} \sum_{k=0}^{n-1} \frac{\alpha(a^k x, 0, 0)}{a^{2k}} \leq \frac{1}{a^3} \sum_{k=0}^{\infty} \frac{\alpha(a^k x, 0, 0)}{a^{2k}} \tag{5.9}$$

for all $x \in X$. In order to prove convergence of the sequence $\left\{\frac{g(a^k x)}{a^{2k}}\right\}$, replacing x by $a^m x$ and dividing a^m in (5.9) for any $m, n > 0$, we get

$$\begin{aligned}\left\|\frac{g(a^m x)}{a^{2m}} - \frac{g(a^{m+n}x)}{a^{2(m+n)}}\right\| &= \frac{1}{a^{2m}}\left\|g(a^m x) - \frac{g(a^m a^n x)}{a^{2n}}\right\| \\ &\leq \frac{1}{a^3}\sum_{m=0}^{n-1}\frac{\alpha(a^{m+n}x,0,0)}{a^{2(m+n)}} \\ &\leq \frac{1}{a^3}\sum_{m=0}^{n-1}\frac{\alpha(a^{m+n}x,0,0)}{a^{2(m+n)}} \\ &\to 0 \text{ as } m \to \infty.\end{aligned}$$

Hence the sequence $\left\{\frac{g(a^n x)}{a^{2n}}\right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $Q : X \to Y$ such that

$$Q(x) = \lim_{n\to\infty}\frac{g(a^n x)}{a^{2n}},\ \forall x \in X. \tag{5.10}$$

Letting $n \to \infty$ in (5.10) we see that (5.10) holds for $x \in X$.
To prove that Q satisfies (1.3), replacing (x_1, x_2, x_3) by $(a^n x, a^n x, a^n x)$ and dividing a^{2n} in (5.2), we obtain

$$\frac{1}{a^{2n}}\|Dg(a^n x, a^n x, a^n x)\| \leq \frac{1}{a^{2n}}\alpha(a^n x, a^n x, a^n x)$$

for all $x_1, x_2, x_3 \in X$. Letting $n \to \infty$ in the above inequality and using the definition of $Q(x)$, we see that $DQ(x_1, x_2, x_3) = 0$. Hence Q satisfies (1.3) for all $x_1, x_2, x_3 \in X$.
To show that Q is unique, let $B(x)$ be another quadratic mapping satisfying (5.4). Then

$$\begin{aligned}\|Q(x) - B(x)\| &= \frac{1}{a^{2n}}\|Q(a^n x) - B(a^n x)\| \\ &\leq \frac{1}{a^{2n}}\{\|Q(a^n x) - g(a^n x)\| + \|g(a^n x) - B(a^n x)\|\} \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence Q is unique.

Assume that $j = -1$. Replacing x by $\frac{x}{a}$ in (5.5), we get

$$\left\|ag(x) - a^2 g\left(\frac{x}{a}\right)\right\| \leq \frac{1}{4}\alpha\left(\frac{x}{a}, 0, 0\right) \tag{5.11}$$

for all $x \in X$. The rest of the proof is similar to the proof of the case $j = 1$. This completes the proof of the theorem. ∎

The following corollary is an immediate consequence of Theorem 4 concerning the stability of (1.3).

Corollary 2. *Let ϵ and p be nonnegative real numbers. Let $g_q : X \to Y$ be an even mapping satisfiying $g(0) = 0$ and the inequality*

$$\|Dg(x_1, x_2, x_3)\| \tag{5.12}$$
$$\leq \begin{cases} \epsilon; & \\ \epsilon\left(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p\right); & p > 2 \quad or \quad p < 2 \\ \epsilon\left(\|x_1\|^p\|x_2\|^p\|x_3\|^p + \{\|x_1\|^{3p}\|x_2\|^{3p}\|x_3\|^{3p}\}\right); & p > \frac{2}{3} \quad or \quad p < \frac{2}{3} \end{cases}$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|g(x) - Q(x)\| \leq \begin{cases} \frac{\epsilon}{4|a^2-1|} \\ \frac{\epsilon\|x\|^p}{4|a^2-a^p|} \\ \frac{\epsilon\|x\|^{3p}}{4|a^2-a^{3p}|} \end{cases} \tag{5.13}$$

for all $x \in X$.

Proof. Letting

$$\alpha(x_1, x_2, x_3) = \begin{cases} \epsilon; \\ \epsilon\left(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p\right); \\ \epsilon\left(\|x_1\|^p\|x_2\|^p\|x_3\|^p + \{\|x_1\|^{3p}\|x_2\|^{3p}\|x_3\|^{3p}\}\right); \end{cases}$$

for all $x_1, x_2, x_3 \in X$, we get the result. ∎

6 Stability results of (1.3): Mixed case

In this section, we establish the Hyers-Ulam stability of the functional equation(1.3) for a mixed mapping case.

Theorem 5. *Let $j \in \{-1, 1\}$ and $\alpha : X^3 \to [0, \infty)$ be a function satisfying (1.3) for all $x_1, x_2, x_3 \in X$. Let $g : X \to Y$ be a mapping satisfying the inequality*

$$\|Dg(x_1, x_2, x_3)\| \leq \alpha(x_1, x_2, x_3) \tag{6.1}$$

for all $x_1, x_2, x_3 \in X$. There exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (1.3) and

$$\begin{aligned} \|f(x) - A(x) - Q(x)\| \leq \frac{1}{2}\Big\{\Big[\frac{1}{2a}\sum_{k=\frac{1-j}{2}}^{\infty}\Big[\frac{\alpha(a^{kj}x, 0, 0)}{a^{kj}} + \frac{\alpha(-a^{kj}x, 0, 0)}{a^{kj}}\Big]\Big] \\ + \frac{1}{4n^2}\Big[\sum_{k=\frac{1-j}{2}}^{\infty}\Big[\frac{\alpha(a^{kj}x, 0, 0)}{a^{2kj}} + \frac{\alpha(-a^{kj}x, 0, 0)}{a^{2kj}}\Big]\Big]\Big\} \end{aligned}$$

for all $x \in X$. The mapping $A(x)$ and $Q(x)$ are defined in (4.2) and (5.10), respectively.

Proof. Let $g_o(x) = \frac{g_a(x)-g_a(-x)}{2}$ for all $x \in X$. Then $g_o(0) = 0$ and $g_o(-x) = -g_o(x)$ for all $x \in X$.
Hence

$$\begin{aligned}\|Dg_o(x_1,x_2,x_3)\| &\le \frac{1}{2}\big\{\|Dg_a(x_1,x_2,x_3)\| + \|Dg_a(-x_1,-x_2,-x_3)\|\big\}\\ &\le \frac{\alpha(x_1,x_2,x_3)}{2} + \frac{\alpha(-x_1,-x_2,-x_3)}{2}\end{aligned}$$

for all $x_1, x_2, x_3 \in X$. By Theorem 3, we have

$$\|g_o(x) - A(x)\| \le \frac{1}{4a}\sum_{k=\frac{1-j}{2}}^{\infty}\Big[\frac{\alpha(a^{kj}x,0,0)}{a^{kj}} + \frac{\alpha(-a^{kj}x,0,0)}{a^{kj}}\Big] \tag{6.2}$$

for all $x \in X$.

Let $g_e(x) = \frac{g_q(x)+g_q(-x)}{2}$ for all $x \in X$. Then $g_e(0) = 0$ and $g_e(-x) = g_e(x)$ for all $x \in X$.
Hence,

$$\begin{aligned}\|Dg_e(x_1,x_2,x_3)\| &\le \frac{1}{2}\big\{\|Dg_q(x_1,x_2,x_3)\| + \|Dg_q(-x_1,-x_2,-x_3)\|\big\}\\ &\le \frac{\alpha(x_1,x_2,x_3)}{2} + \frac{\alpha(-x_1,-x_2,-x_3)}{2}\end{aligned}$$

for all $x_1, x_2, x_3 \in X$.
By Theorem 4, we have

$$\|g_e(x) - Q(x)\| \le \frac{1}{8a^2}\sum_{k=\frac{1-j}{2}}^{\infty}\Big[\frac{\alpha(a^{kj}x,0,0)}{a^{2kj}} + \frac{\alpha(-a^{kj}x,0,0)}{a^{2kj}}\Big] \tag{6.3}$$

for all $x \in X$. Then

$$g(x) = g_e(x) + g_o(-x) \tag{6.4}$$

for all $x \in X$. It follows from (6.2), (6.3) and (6.4) that

$$\begin{aligned}\|g(x) - A(x) - Q(x)\| &= \|g_e(x) + g_o(-x) - A(x) - Q(x)\|\\ &\le \|g_o(-x) - A(x)\| + \|g_e(x) - Q(x)\|\\ &\le \frac{1}{4a}\sum_{k=\frac{1-j}{2}}^{\infty}\Big[\frac{\alpha(a^{kj}x,0,0)}{a^{kj}} + \frac{\alpha(-a^{kj}x,0,0)}{a^{kj}}\Big]\\ &\quad + \frac{1}{8a^2}\sum_{k=\frac{1-j}{2}}^{\infty}\Big[\frac{\alpha(a^{kj}x,0,0)}{a^{2kj}} + \frac{\alpha(-a^{kj}x,0,0)}{a^{2kj}}\Big]\end{aligned}$$

for all $x \in X$. Hence the theorem is proved. ∎

Using Corollaries 1 and 2, we have the following corollary concerning the stability of (1.3).

Corollary 3. *Let λ and s be a nonnegative real numbers. Let $g_q : X \to Y$ be a mapping satisfiying the inequality*

$$\|D_g(x_1,x_2,x_3)\| \leq \begin{cases} \lambda; & \\ \lambda(\|x_1\|^s + \|x_2\|^s + \|x_3\|^s); & s \neq 1,2 \\ \lambda(\|x_1\|^s + \|x_2\|^s + \|x_3\|^s) + \{\|x_1\|^{3s} + \|x_2\|^{3s} + \|x_3\|^{3s}\}; & s \neq \frac{1}{3}, \frac{2}{3} \end{cases} \tag{6.5}$$

for all $x_1, x_2, x_3 \in X$. Then there exist a unique additive function $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|g(x) - A(x) - Q(x)\| \leq \begin{cases} \frac{\lambda}{2}\left[\frac{1}{|a-1|} + \frac{1}{2|a^2-1|}\right] \\ \frac{\lambda\|x\|^S}{2}\left[\frac{1}{|a-a^S|} + \frac{1}{2|a^2-a^S|}\right] \\ \frac{\lambda\|x\|^{3S}}{2}\left[\frac{1}{|a-a^{3S}|} + \frac{1}{2|a^2-a^{3S}|}\right] \end{cases} \tag{6.6}$$

for all $x \in X$.

7 Fixed point stability of (1.3): Odd mapping case

The following theorems are useful to prove our fixed point stability results.

Theorem 6. *[12]* (**Banach Contraction Principle**) *Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping.*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ *for some (Lipschitz constant) $L < 1$.*

(i) The mapping T has one and only fixed point $x^* = T(x^*)$;

(ii) The fixed point for each given element x^* is globally contractive, that is,

(A2) $\lim_{n\to\infty} T^n x = x^*$ *for any starting point $x \in X$;*

(iii) One has the following estimation inequalities

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L} d(T^n x, T^{n+1}x), \forall n \geq 0, \forall x \in X$;

(A4) $d(x, x^*) = \frac{1}{1-L} d(x, x^*), \forall x \in X$.

Theorem 7. *[12]* (**Alternative Fixed Point Theorem**) *Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then for each given element $x \in X$,*

(B1) $d(T^n x, T^{n+1}x) = \infty, \forall n \geq 0$;

(B2) there exists a natural number n_0 such that

(i) $d(T^n x, T^{n+1}x) < \infty, \forall n \geq 0$;

(ii) *The sequence* $\{T^n x\}$ *is convergent to a fixed point* y^* *of* T;

(iii) y^* *is the unique fixed point of* T *in the set* $Y = \{y \in Y : d(T^{n_0}, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in Y$.

In this method, we investigate the Hyers-Ulam stability of the functional equation (1.3) for an odd mapping case by using fixed point method.

Theorem 8. *Let* $g : W \to B$ *be an odd mapping for which there exists a function* $\alpha : W^3 \to [0, \infty)$ *with the condition*

$$\lim_{n\to\infty} \frac{\alpha(a_i^k x_1, a_i^k x_2, a_i^k x_3)}{a_i^k} = 0, \tag{7.1}$$

for $a_i = \begin{cases} a & i = 0 \\ \frac{1}{a} & i = 1, \end{cases}$ *such that the functional inequality*

$$\|Dg(x_1, x_2, x_3)\| \leq \alpha(x_1, x_2, x_3) \tag{7.2}$$

for all $x_1, x_2, x_3 \in W$. *If there exists* $L = L(i)$ *such that the function* $x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{a}, 0, 0\right)$ *has the property*

$$\frac{1}{a_i}\beta(a_i x) = L\left(\beta(x)\right) \tag{7.3}$$

for all $x \in W$. *Then there exists a unique additive function* $A : W \to B$ *satisfying the functional equation (1.3) and*

$$\|g(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x) \tag{7.4}$$

for all $x \in W$.

Proof. Consider the set $X = \{P | P : W \to B, P(0) = 0\}$ and introduce the generalized metric on X.

$$d(p, q) = \inf\{k \in (0, \infty) : \|p(x) - q(x)\| \leq \beta(x), x \in W\}$$

It is easy to see that (X, d) is complete.
Define $T : X \to X$ by $T_p(x) = \frac{1}{a_i} p(a_i x)$ for all $x \in W$. Now $p, q \in X$,

$$\begin{aligned}
& d(p, q) \leq k \\
& \Rightarrow \|p(x) - q(x)\| \leq k\beta(x), x \in W. \\
& \Rightarrow \left\| \frac{1}{a_i} p(a_i x) - \frac{1}{a_i} q(a_i x) \right\| \leq \frac{1}{a_i} k\beta(a_i x), \forall x \in W \\
& \Rightarrow \|T_p(x) - T_q(x)\| \leq Lk\beta(x), \forall x \in W \\
& \Rightarrow d(T_p, T_q) \leq Lk.
\end{aligned}$$

This implies $d(T_p, T_q) \leq Ld(p, q)$ for all $p, q \in X$. That is, T is a strictly contractive mapping on X with Lipschitz constant L. It follows from (4.3) that

$$\|2g(ax) - 2ag(x)\| \leq \alpha(x, 0, 0) \tag{7.5}$$

for all $x \in W$. It follows from (7.5) that,

$$\left\| g(x) - \frac{g(ax)}{a} \right\| \leq \frac{1}{2a}\alpha(x, 0, 0) \tag{7.6}$$

for all $x \in W$. Using (6.2), for this case $i = 0$, it reduces to

$$\left\| g(x) - \frac{g(ax)}{a} \right\| \leq \frac{1}{a}\beta(x) \tag{7.7}$$

for all $x \in W$. Thus

$$d(g_a, Tg_a) \leq \frac{1}{a} = L = L^1 < \infty.$$

Again replacing x by $\frac{x}{a}$ in (7.5), we get

$$\left\| g(x) - ag\left(\frac{x}{a}\right) \right\| \leq \frac{1}{2}\alpha\left(\frac{x}{a}, 0, 0\right) \tag{7.8}$$

for all $x \in W$.
By using (7.3) for the case $i = 1$, it reduces to

$$\left\| g(x) - ag\left(\frac{x}{a}\right) \right\| \leq \beta(x). \tag{7.9}$$

That is, $d(g, Tg) \leq 1 \Rightarrow d(g, Tg) \leq 1 = L^0 < \infty$. In the above case, we have $d(g, Tg) \leq L^{1-i}$. Therefore $(B_2(i))$ holds. From $(B_2(ii))$, it follows that there exists a fixed point A of T in X such that

$$A(x) = \lim_{i \to \infty} \frac{g_a(a_i^k x)}{a_i^k}, \forall x \in W. \tag{7.10}$$

In order to prove $A : W \to B$ is additive, replacing (x_1, x_2, x_3) by $(a_i^k x_1, a_i^k x_2, a_i^k x_3)$ in (7.2) and dividing a_i^k, it follows from (7.3) and (7.10) that A satisfies (1.3) for all $x_1, x_2, x_3 \in W$. By $(B_2(iii))$, A is the unique fixed point of T in the set, $Y = \{g \in X : d(Tg, A) < \infty\}$.
Using the fixed point alternative result, A is the unique function such that

$$\|g(x) - A(x)\| \leq k\beta(x)$$

for all $x \in W$ and $k > 0$. Finally, by $(B_2(iv))$, we obtain

$$d(g, A) \leq \frac{1}{1-L}d(g, Tg).$$

That is, $d(g, A) \leq \frac{L^{1-i}}{1-L}$. Hence we conclude that

$$\|g(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x)$$

for all $x \in W$. This completes the proof of the theorem. ∎

Corollary 4. *Let $g : W \to B$ be an odd mapping and assume that there exist real numbers λ and s such that*

$$\|D_{g_a}(x_1, x_2, x_3)\| \leq \begin{cases} \lambda; \\ \lambda(\|x_1\|^s + \|x_2\|^s + \|x_3\|^s); \\ \lambda(\|x_1\|^s + \|x_2\|^s + \|x_3\|^s) + \{\|x_1\|^{3s} + \|x_2\|^{3s} + \|x_3\|^{3s}\} \end{cases} \tag{7.11}$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique additive mapping $A : W \to B$ such that

$$\|g(x) - A(x)\| \leq \begin{cases} \frac{\lambda}{2|a-1|}; \\ \frac{\lambda\|x\|^s}{2|a-a^s|}; & s \neq 1 \\ \frac{\lambda\|x\|^{3s}}{2|a-a^{3s}|}; & s \neq \frac{1}{3} \end{cases} \tag{7.12}$$

for all $x \in X$.

Proof. Let

$$\alpha(x_1, x_2, x_3) = \begin{cases} \lambda; \\ \lambda(\|x_1\|^S + \|x_2\|^S + \|x_3\|^S); \\ \lambda(\|x_1\|^S + \|x_2\|^S + \|x_3\|^S) + \{\lambda(\|x_1\|^{3S} + \|x_2\|^{3S} + \|x_3\|^{3S})\}; \end{cases}$$

for all $x_1, x_2, x_3 \in W$.
Now,

$$\frac{\alpha(a_i^k x_1, a_i^k x_2, a_i^k x_3)}{a_i^k}$$

$$= \begin{cases} \dfrac{\lambda}{a_i^k}; \\ \dfrac{\lambda}{a_i^k}(\|a_i^k x_1\|^S + \|a_i^k x_2\|^S + \|a_i^k x_3\|^S); \\ \dfrac{\lambda}{a_i^k}(\|a_i^k x_1\|^S + \|a_i^k x_2\|^S + \|a_i^k x_3\|^S) + \{\|a_i^k x_1\|^{3S} + \|a_i^k x_2\|^{3S} + \|a_i^k x_3\|^{3S}\} \end{cases}$$

$$= \begin{cases} \to 0 \quad \text{as} \quad k \to \infty \\ \to 0 \quad \text{as} \quad k \to \infty \\ \to 0 \quad \text{as} \quad k \to \infty. \end{cases} \tag{7.13}$$

That is, (7.1) holds. But we have $\beta(x) = \frac{1}{2}\alpha\left(\frac{x}{a}, 0, 0\right)$. Hence

$$\beta(x) = \frac{1}{2}\alpha\left(\frac{x}{a}, 0, 0\right) = \begin{cases} \frac{\lambda}{2} \\ \frac{\lambda}{2a^S}(\|x\|^S) \\ \frac{\lambda}{2a^S}(\|x\|^S). \end{cases}$$

Also

$$\frac{1}{a_i}\beta(a_i,x) = \begin{cases} \frac{\lambda}{2a_i} \\ \frac{\lambda}{2a_i}(\|a_i x\|^S) \\ \frac{\lambda}{2a_i}(\|a_i x\|^S) \end{cases} = \begin{cases} a_i^{-1}\beta(x) \\ a_i^{S-1}\beta(x) \\ a_i^{3S-1}\beta(x). \end{cases}$$

Hence the inequality (7.7) holds. Either $L = a^{-1}$ for $s = 0$ if $i = 0$ and $L = \frac{1}{a^{-1}}$ for $s = 0$ if $i = 1$.

Either $L = a^{s-1}$ for $s < 1$ if $i = 0$ and $L = \frac{1}{a^{s-1}}$ for $s > 1$ if $i = 1$.

Either $L = a^{3s-1}$ for $s < 1$ if $i = 0$ and $L = \frac{1}{a^{3s-1}}$ for $s > 1$ if $i = 1$.

Now from (7.2), we prove the following cases:

Case: 1 $L = a^{-1}, i = 0$

$$\|g_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{(a^{-1})^{1-0}}{1-a^{-1}}\frac{\lambda}{2} = \frac{\lambda}{2(a-1)}. \tag{7.14}$$

Case: 2 $L = \left(\frac{1}{a}\right)^{-1}, i = 1$

$$\|g_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{(a)^{1-1}}{1-a}\frac{\lambda}{2} = \frac{\lambda}{2(1-a)}. \tag{7.15}$$

Case: 3 $L = a^{s-1}, s < 1, i = 0$

$$\|g_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{(a^{s-1})^{1-0}}{1-a^{S-1}}\frac{\lambda}{2a^S}\|x\|^S = \frac{\lambda\|x\|^S}{2|a-a^S|}. \tag{7.16}$$

Case: 4 $L = \left(\frac{1}{a}\right)^{S-1}, S > 1, i = 1$

$$\|g_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{(a^{1-s})^{1-1}}{1-a^{1-S}}\frac{\lambda}{2a^S}\|x\|^S = \frac{\lambda\|x\|^S}{2(a^S-a)}. \tag{7.17}$$

Case: 5 $L = a^{3s-1}, S < \frac{1}{3}, i = 0$

$$\|g_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{(a^{3S-1})^{1-0}}{1-a^{3S-1}}\frac{\lambda}{2a^{3S}}\|x\|^S = \frac{\lambda\|x\|^S}{2(a-a^{3S})}. \tag{7.18}$$

Case: 6 $L = \left(\frac{1}{a}\right)^{-1}, i = 1$

$$\|g_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{(a^{1-3S})^{1-1}}{1-a^{1-3S}}\frac{\lambda}{2a^{3S}}\|x\|^S = \frac{\lambda\|x\|^S}{2(a^{3S}-a)}. \tag{7.19}$$

Hence the proof of the corollary is completed.

∎

8 Fixed point stability of (1.3): Even mapping case

In this method, we investigate the Hyers-Ulam stability of the functional equation (1.3) for an even case mapping by using fixed point method.

Theorem 9. *Let $g : W \to B$ be an even mapping for which there exists a function $\alpha : W^3 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{\alpha(a_i^k x_1, a_i^k x_2, a_i^k x_3)}{a_i^{2k}} = 0 \tag{8.1}$$

for $a_i = \begin{cases} a & i = 0 \\ \frac{1}{a} & i = 1, \end{cases}$ *such that the functional inequality*

$$\|Dg(x_1, x_2, x_3)\| \le \alpha(x_1, x_2, x_3) \tag{8.2}$$

for all $x_1, x_2, x_3 \in W$. If there exists $L = L(i)$ such that the function

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{a}, 0, 0\right) \tag{8.3}$$

has the property

$$\frac{1}{a_i^2}\beta(a_i x) = L\left(\beta(x)\right) \tag{8.4}$$

for all $x \in W$, then there exists a unique quadratic mapping $Q : W \to B$ satisfying the functional equation (1.3) and

$$\|g(x) - Q(x)\| \le \frac{L^{1-i}}{1-L}\beta(x) \tag{8.5}$$

for all $x \in W$.

Proof. Consider the set $X = \{P | P : W \to B, P(0) = 0\}$ and introduce the generalized metric on X.

$$d(p, q) = \inf\{k \in (0, \infty) : \|p(x) - q(x)\| \le \beta(x), x \in W\}$$

It is easy to see that (X, d) is complete.
Define $T : X \to X$ by $T_p(x) = \frac{1}{a_i^2} p(a_i x)$ for all $x \in W$. Now $p, q \in X$,

$$\begin{aligned}
& d(p, q) \le k \\
& \Rightarrow \|p(x) - q(x)\| \le k\beta(x), x \in W. \\
& \Rightarrow \left\| \frac{1}{a_i^2} p(a_i x) - \frac{1}{a_i^2} q(a_i x) \right\| \le \frac{1}{a_i^2} k\beta(a_i x), \forall x \in W \\
& \Rightarrow \|T_p(x) - T_q(x)\| \le Lk\beta(x), \forall x \in W \\
& \Rightarrow d(T_p, T_q) \le Lk.
\end{aligned}$$

This implies $d(T_p, T_q) \le Ld(p, q)$ for all $p, q \in X$. That is, T is a strictly contractive mapping on X with Lipschitz constant L.
Replacing (x_1, x_2, x_3) by $(x, 0, 0)$ in (9.1) and using the evenness of g, we get

$$\|4g(ax) - 4a^2 g(x)\| \le \alpha(x, 0, 0), \tag{8.6}$$

$$\left\| g(x) - \frac{g(ax)}{n^2} \right\| \le \tfrac{1}{4a^2}\alpha(x, 0, 0) \tag{8.7}$$

for all $x \in W$. By using (8.4), for this case $i = 0$, it reduces to

$$\left\| g(x) - \frac{g(ax)}{a^2} \right\| \leq \frac{1}{2a^2}\beta(x) \tag{8.8}$$

for all $x \in W$. That is,

$$d(g, Tg) \leq \frac{1}{a^2} \Rightarrow d(g, Tg) \leq \frac{1}{a^2} = L = L^1 < \infty.$$

Again replacing x by $\frac{x}{a}$ in (8.6), we get

$$\left\| g(x) - a^2 g\left(\frac{x}{a}\right) \right\| \leq \frac{1}{4}\alpha\left(\frac{x}{a}, 0, 0\right) \tag{8.9}$$

for all $x \in W$. That is,

$$d(g, Tg) \leq \frac{1}{2} < 1 \Rightarrow d(g, Tg) \leq 1 = L^0 < \infty.$$

In above case, we get $d(g, Tg) \leq L^{1-i}$.

The rest of the proof is similar to that of the previous theorem. This completes the proof of the theorem. ∎

Corollary 5. *Let $g : W \to B$ be an even mapping and assume that there exist real numbers λ and s such that*

$$\|D_g(x_1, x_2, x_3)\| \leq \begin{cases} \lambda; & \\ \lambda(\|x_1\|^s + \|x_2\|^s + \|x_3\|^s); & s \neq 2 \\ \lambda(\|x_1\|^s + \|x_2\|^s + \|x_3\|^s) + \{\|x_1\|^{3s} + \|x_2\|^{3s} + \|x_3\|^{3s}\}; & s \neq \frac{1}{3} \end{cases} \tag{8.10}$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique quadratic mapping $Q : W \to B$ such that

$$\|g_q(x) - Q(x)\| \leq \begin{cases} \frac{\lambda}{4|a^2-1|}; \\ \frac{\lambda\|x\|^s}{4|a^2-a^s|} \\ \frac{\lambda\|x\|^{3s}}{4|a^2-a^{3s}|} \end{cases} \tag{8.11}$$

for all $x \in X$.

9 Fixed point stability of (1.3): Mixed mapping case

In this method, we present the Hyers-Ulam stability of the functional equation (1.3) for a mixed mapping case by using fixed point method.

Theorem 10. *Let $g : W \to B$ be a mapping for which there exists a function $\alpha : W^3 \to [0, \infty)$ with the condition (7.1) and (8.1) for $a_i = \begin{cases} a & i = 0 \\ \frac{1}{a} & i = 1, \end{cases}$ such that the functional inequality*

$$\|Dg(x_1, x_2, x_3)\| \leq \alpha(x_1, x_2, x_3) \tag{9.1}$$

for all $x_1, x_2, x_3 \in W$. If there exists $L = L(i)$ such that the function

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{a}, 0, 0\right)$$

satisfies (7.3) and (8.3) for all $x \in W$, then there exist a unique additive mapping $A : W \to B$ and a quadratic mapping $Q : W \to B$ satisfying the functional equation (1.3) and

$$\|g(x) - A(x) - Q(x)\| \leq \frac{L^{1-i}}{1-L}[\beta(x) + \beta(-x)]$$

holds for all $x \in W$.

Proof. It follows from (6.2) and Theorem 8 that

$$\|g_o(x) - A(x)\| \leq \frac{1}{2}\frac{L^{1-i}}{1-L}[\beta(x) + \beta(-x)]. \tag{9.2}$$

Similarly, it follows from (7.5) and Theorem 9 that

$$\|g_e(x) - Q(x)\| \leq \frac{1}{2}\frac{L^{1-i}}{1-L}[\beta(x) + \beta(-x)] \tag{9.3}$$

for all $x \in W$. Then $g(x) = g_o(x) + g_e(x)$ for all $x \in W$.
From (8.11), (9.2) and (9.3), we have

$$\begin{aligned}\|g(x) - A(x) - Q(x)\| &= \|g_e(x) + g_o(x) - A(x) - Q(x)\| \\ &\leq \|g_o(x) - A(x)\| + \|g_e(x) - Q(x)\| \\ &= \frac{L^{1-i}}{1-L}[\beta(x) + \beta(-x)]\end{aligned}$$

for all $x \in W$. Hence the theorem is proved. ∎

Corollary 6. *Let $g : W \to B$ be a mapping and assume that there exist real numbers λ and s such that*

$$\|D_g(x_1, x_2, x_3)\| \leq \begin{cases} \lambda; & \\ \lambda(\|x_1\|^s + \|x_2\|^s + \|x_3\|^s); & s \neq 1, 2 \\ \lambda(\|x_1\|^s + \|x_2\|^s + \|x_3\|^s) + \{\|x_1\|^{3s} + \|x_2\|^{3s} + \|x_3\|^{3s}\}; & s \neq \frac{1}{3}, \frac{2}{3} \end{cases}$$

for all $x_1, x_2, x_3 \in X$. Then there exist a unique additive mapping $A : W \to B$ and a unique quadratic mapping $Q : W \to B$ such that

$$\|g(x) - A(x) - Q(x)\| \leq \begin{cases} \frac{\lambda}{2|a-1|} + \frac{\lambda}{4|a^2-1|} \\ \frac{\lambda\|x\|^S}{2|a-a^S|} + \frac{\lambda\|x\|^S}{4|a^2-a^S|} \\ \frac{\lambda\|x\|^{3S}}{2|a-a^{3S}|} + \frac{\lambda\|x\|^{3S}}{4|a^2-a^{3S}|} \end{cases}$$

for all $x \in X$.

Declarations

Availablity of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

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Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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Results on para-Sasakian manifold admitting a quarter symmetric metric connection

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ABSTRACT

In this paper we have studied pseudosymmetric, Ricci-pseudosymmetric and projectively pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection and constructed examples of 3-dimensional and 5-dimensional para-Sasakian manifold admitting a quarter-symmetric metric connection to verify our results.

RESUMEN

En este artículo hemos estudiado variedades para-Sasakianas seudosimétricas, Ricci-seudosimétricas y proyectivamente seudosimétricas que admiten una conexión métrica cuarto-simétrica, y construimos ejemplos de variedades para-Sasakianas 3-dimensional y 5-dimensional que admiten una conexión métrica cuarto-simétrica para verificar nuestros resultados.

Keywords and Phrases: Para-Sasakian manifold, pseudosymmetric, Ricci-pseudosymmetric, projectively pseudosymmetric, quarter-symmetric metric connection.

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1 Introduction

One of the most important geometric property of a space is symmetry. Spaces admitting some sense of symmetry play an important role in differential geometry and general relativity. Cartan [5] introduced locally symmetric spaces, i.e., the Riemannian manifold (M, g) for which $\nabla R = 0$, where ∇ denotes the Levi-Civita connection of the metric. The integrability condition of $\nabla R = 0$ is $R \cdot R = 0$. Thus, every locally symmetric space satisfies $R \cdot R = 0$, whereby the first R stands for the curvature operator of (M, g), i.e., for tangent vector fields X and Y one has $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$, which acts as a derivation on the second R which stands for the Riemann-Christoffel curvature tensor. The converse however does not hold in general. The spaces for which $R \cdot R = 0$ holds at every point were called semi-symmetric spaces and which were classified by Szabo [19].

Semisymmetric manifolds form a subclass of the class of pseudosymmetric manifolds. In some spaces $R \cdot R$ is not identically zero, these turn out to be the pseudo-symmetric spaces of Deszcz [9, 10, 11], which were characterized by the condition $R \cdot R = L\, Q(g, R)$, where L is a real function on M and $Q(g, R)$ is the Tachibana tensor of M.

If at every point of M the curvature tensor satisfies the condition

$$R(X, Y) \cdot \mathcal{J} = L_{\mathcal{J}}[(X \wedge_g Y) \cdot \mathcal{J}], \tag{1.1}$$

then a Riemannian manifold M is called pseudosymmetric (resp., Ricci-pseudosymmetric, projectively pseudosymmetric) when $\mathcal{J} = R(resp., S, P)$. Here $(X \wedge_g Y)$ is an endomorphism and is defined by $(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y$ and $L_{\mathcal{J}}$ is some function on $U_{\mathcal{J}} = \{x \in M : \mathcal{J} \neq 0\}$ at x. A geometric interpretation of the notion of pseudosymmetry is given in [13]. It is also easy to see that every pseudosymmetric manifold is Ricci-pseudosymmetric, but the converse is not true.

An analogue to the almost contact structure, the notion of almost paracontact structure was introduced by Sato [18]. An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be of even dimension as well. Kaneyuki and Williams [14] studied the almost paracontact structure on a pseudo-Riemannian manifold. Recently, almost paracontact geometry in particular, para-Sasakian geometry has taking interest, because of its interplay with the theory of para-Kahler manifolds and its role in pseudo-Riemannian geometry and mathematical physics ([4, 7, 8], etc.,).

As a generalization of semi-symmetric connection, quarter-symmetric connection was introduced. Quarter-symmetric connection on a differentiable manifold with affine connection was defined and studied by Golab [12]. From thereafter many geometers studied this connection on different manifolds.

Para-Sasakian manifold with respect to quarter-symmetric metric connection was studied by

De et.al., [16, 1], Pradeep Kumar et.al., [17] and Bisht and Shanker [15].

Motivated by the above studies in this article we study properties of projective curvature tensor on para-Sasakian manifold admitting a quarter-symmetric metric connection. The organization of the paper is as follows: In Section 2, we present some basic notions of para-Sasakian manifold and quarter-symmetric metric connection on it. Section 3 and 4 are respectively devoted to study the pseudosymmetric and Ricci-pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection. Here we prove that if a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is Pseudosymmetric (resp., Ricci pseudosymmetric) then M^n is an Einstein manifold with respect to quarter-symmetric metric connection or it satisfies $L_{\tilde{R}} = -2$ (resp., $L_{\tilde{S}} = -2$). Section 5 and 6 are concerned with projectively flat and projectively pseudosymmetric para-Sasakian manifold M^n admitting a quarter-symmetric metric connection. Finally, we construct examples of 3-dimensional and 5-dimensional para-Sasakian manifold admitting a quarter-symmetric metric connection and we find some of its geometric characteristics.

2 Preliminaries

A differential manifold M^n is said to admit an almost paracontact Riemannian structure (ϕ, ξ, η, g), where ϕ is a tensor field of type (1, 1), ξ is a vector field, η is a 1-form and g is a Riemannian metric on M^n such that

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields $X, Y \in \chi(M^n)$. If (ϕ, ξ, η, g) on M^n satisfies the following equations

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.3}$$

$$d\eta = 0 \quad and \quad \nabla_X \xi = \phi X, \tag{2.4}$$

then M^n is called para-Sasakian manifold [3].

In a para-Sasakian manifold, the following relations hold [6]:

$$(\nabla_X \eta)Y = -g(X, Y) + \eta(X)\eta(Y), \tag{2.5}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.6}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.7}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.9}$$

for every vector fields X, Y, Z on M^n. Here ∇ denotes the Levi-Civita connection, R denotes the Riemannian curvature tensor and S denotes the Ricci curvature tensor.

Here we consider a quarter-symmetric metric connection $\tilde{\nabla}$ on a para-Sasakian manifold [16] given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{2.10}$$

The relation between curvature tensor $\tilde{R}(X,Y)Z$ of M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ and the curvature tensor $R(X,Y)Z$ with respect to the Levi-Civita connection ∇ is given by

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + 3g(\phi X, Z)\phi Y - 3g(\phi Y, Z)\phi X\\ &+\{\eta(X)Y - \eta(Y)X\}\eta(Z) - [g(Y,Z)\eta(X) - \eta(Y)g(X,Z)]\xi.\end{aligned} \tag{2.11}$$

Also from (2.11) we obtain

$$\tilde{S}(Y,Z) = S(Y,Z) + 2g(Y,Z) - (n+1)\eta(Y)\eta(Z) - 3trace\phi\, g(\phi Y, Z), \tag{2.12}$$

where $\tilde{S}$ and S are Ricci tensors of connections $\tilde{\nabla}$ and ∇ respectively.

3 Pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection

A para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is said to be pseudosymmetric if

$$\tilde{R}(X,Y)\cdot\tilde{R} = L_{\tilde{R}}[(X \wedge_g Y)\cdot\tilde{R}], \tag{3.1}$$

holds on the set $U_{\tilde{R}} = \{x \in M^n : \tilde{R} \neq 0 \ at\ x\}$, where $L_{\tilde{R}}$ is some function on $U_{\tilde{R}}$.

Suppose that M^n be pseudosymmetric, then in view of (3.1) we have

$$\begin{aligned}&\tilde{R}(\xi,Y)\tilde{R}(U,V)W - \tilde{R}(\tilde{R}(\xi,Y)U,V)W - \tilde{R}(U,\tilde{R}(\xi,Y)V)W\\ &-\tilde{R}(U,V)\tilde{R}(\xi,Y)W = L_{\tilde{R}}[(\xi\wedge_g Y)\tilde{R}(U,V)W - \tilde{R}((\xi\wedge_g Y)U,V)W\\ &-\tilde{R}(U,(\xi\wedge_g Y)V)W - \tilde{R}(U,V)(\xi\wedge_g Y)W].\end{aligned} \tag{3.2}$$

By virtue of (2.7) and (2.11), (3.2) takes the form

$$\begin{aligned}&(L_{\tilde{R}}+2)[\eta(\tilde{R}(U,V)W)Y - g(Y,\tilde{R}(U,V)W)\xi - \eta(U)\tilde{R}(Y,V)W + g(Y,U)\tilde{R}(\xi,V)W\\ &-\eta(V)\tilde{R}(U,Y)W + g(Y,V)\tilde{R}(U,\xi)W - \eta(W)\tilde{R}(U,V)Y + g(Y,W)\tilde{R}(U,V)\xi] = 0.\end{aligned} \tag{3.3}$$

Taking inner product of (3.3) with ξ and using (2.6) and (2.11), we get

$$\begin{aligned}&(L_{\tilde{R}}+2)[g(Y,R(U,V)W) + 3g(\phi U,W)g(\phi V,Y) - 3g(\phi V,W)g(\phi U,Y)\\ &+\eta(W)\{\eta(U)g(V,Y) - \eta(V)g(U,Y)\} - \{g(V,W)\eta(U) - \eta(V)g(U,W)\}\eta(Y)\\ &+2\{g(V,W)g(Y,U) - g(V,Y)g(U,W)\}] = 0.\end{aligned} \tag{3.4}$$

Assuming that $L_{\tilde{R}} + 2 \neq 0$, the above equation becomes

$$\begin{aligned} g(Y, R(U,V)W) + 3g(\phi U, W)g(\phi V, Y) - 3g(\phi V, W)g(\phi U, Y) \\ +\eta(W)\{\eta(U)g(V,Y) - \eta(V)g(U,Y)\} - [g(V,W)\eta(U) - \eta(V)g(U,W)]\eta(Y) \\ +2[g(V,W)g(Y,U) - g((V,Y)g(U,W)] = 0. \end{aligned} \tag{3.5}$$

Putting $V = W = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $i = 1, 2, 3, \cdots, n$, we get

$$\tilde{S}(Y,U) = -2(n-1)g(Y,U). \tag{3.6}$$

Hence, we can state the following:

Theorem 1. *If a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is pseudosymmetric then M^n is an Einstein manifold with respect to quarter-symmetric metric connection or it satisfies $L_{\tilde{R}} = -2$.*

4 Ricci-pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection

A para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is said to be Ricci-pseudosymmetric if the following condition is satisfied

$$\tilde{R}(X,Y) \cdot \tilde{S} = L_{\tilde{S}}[(X \wedge_g Y) \cdot \tilde{S}], \tag{4.1}$$

on $U_{\tilde{S}}$.

Let para-Sasakian manifold M^n admitting a quarter-symmetric metric connection be Ricci-pseudosymmetric. Then we have

$$\tilde{S}(\tilde{R}(X,Y)Z,W) + \tilde{S}(Z,\tilde{R}(X,Y)W) = L_{\tilde{S}}[\tilde{S}((X \wedge_g Y)Z,W) + \tilde{S}(Z,(X \wedge_g Y)W)]. \tag{4.2}$$

By taking $Y = W = \xi$ and making use of (2.7), (2.8) and (2.11), the above equation turns into

$$(L_{\tilde{S}} + 2)[\tilde{S}(X,Z) + 2(n-1)g(X,Z)] = 0 \tag{4.3}$$

Thus, we have the following assertion:

Theorem 2. *If a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is Ricci-pseudosymmetric then M^n is an Einstein manifold with respect to quarter-symmetric metric connection or it satisfies $L_{\tilde{S}} = -2$.*

5 Projectively flat para-Sasakian manifold admitting a quarter-symmetric metric connection

The projective curvature tensor on a Riemannian manifold is defined by [2]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y]. \tag{5.1}$$

For an n-dimensional para-Sasakian manifold M^n admitting a quarter-symmetric metric connection, the projective curvature tensor is given by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-1)}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y]. \tag{5.2}$$

Theorem 3. *A projectively flat para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is an Einstein manifold with respect to quarter-symmetric metric connection.*

Proof. Consider a projectively flat para-Sasakian manifold admitting a quarter-symmetric metric connection. Then from (5.2) we have

$$g(\tilde{R}(X,Y)Z,W) = \frac{1}{(n-1)}[\tilde{S}(Y,Z)g(X,W) - \tilde{S}(X,Z)g(Y,W)]. \tag{5.3}$$

Setting $X = W = \xi$ in (5.3) and using (2.7), (2.8), (2.11) and (2.12), we get

$$\tilde{S}(X,Z) = -2(n-1)g(X,Z). \tag{5.4}$$

Hence, the proof is completed. □

6 Projectively pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection

A para-Sasakian manifold admitting a quarter-symmetric metric connection is said to be projectively pseudosymmetric if

$$\tilde{R}(X,Y)\cdot\tilde{P} = L_{\tilde{P}}[(X\wedge_g Y)\cdot\tilde{P}], \tag{6.1}$$

holds on the set $U_{\tilde{P}} = \{x \in M^n : \tilde{P} \neq 0 \text{ at } x\}$, where $L_{\tilde{P}}$ is some function on $U_{\tilde{P}}$.

Let M^n be projectively pseudosymmetric, then we have

$$\begin{aligned}
&\tilde{R}(X,\xi)\tilde{P}(U,V)\xi - \tilde{P}(\tilde{R}(X,\xi)U,V)\xi - \tilde{P}(U,\tilde{R}(X,\xi)V)\xi \\
&-\tilde{P}(U,V)\tilde{R}(X,\xi)\xi = L_{\tilde{P}}[(X\wedge_g\xi)\tilde{P}(U,V)\xi - \tilde{P}((X\wedge_g\xi)U,V)\xi \\
&\qquad -\tilde{P}(U,(X\wedge_g\xi)V)\xi - \tilde{P}(U,V)(X\wedge_g\xi)\xi].
\end{aligned} \tag{6.2}$$

By virtue of (2.11), (2.12) and (5.2), (6.2) becomes

$$(L_{\tilde{P}} + 2)\tilde{P}(U, V)X = 0. \tag{6.3}$$

So, one can state that:

Theorem 4. *If a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is projectively pseudosymmetric then M^n is projectively flat with respect to quarter-symmetric metric connection or $L_{\tilde{P}} = -2$.*

In view of theorem 3, one can state the above theorem as

Theorem 5. *If a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is projectively pseudosymmetric then M^n is an Einstein manifold with respect to quarter-symmetric metric connection or $L_{\tilde{P}} = -2$.*

7 Examples

7.1 Example

We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be a linearly independent global frame field on M given by

$$E_1 = e^z \frac{\partial}{\partial y}, \qquad E_2 = e^z(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}), \qquad E_3 = \frac{\partial}{\partial z},$$

If g is a Riemannian metric defined by

$$g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for $1 \leq i, j \leq 3$, and if η is the 1-form defined by $\eta(Z) = g(Z, E_3)$ for any vector field $Z \in \chi(M)$. We define the (1, 1)-tensor field ϕ as

$$\phi(E_1) = E_1, \quad \phi(E_2) = -E_2, \quad \phi(E_3) = 0.$$

The linearity property of ϕ and g yields that

$$\begin{aligned} &\eta(E_3) = 1, \\ &\phi^2 U = U - \eta(U)E_3, \\ &g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \end{aligned}$$

for any $U, V \in \chi(M)$.

Now we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = E_2.$$

The Riemannian connection ∇ of the metric g known as Koszul's formula and is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ -g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Using Koszul's formula we get the followings in matrix form

$$\begin{pmatrix} \nabla_{E_1} E_1 & \nabla_{E_1} E_2 & \nabla_{E_1} E_3 \\ \nabla_{E_2} E_1 & \nabla_{E_2} E_2 & \nabla_{E_2} E_3 \\ \nabla_{E_3} E_1 & \nabla_{E_3} E_2 & \nabla_{E_3} E_3 \end{pmatrix} = \begin{pmatrix} -E_3 & 0 & E_1 \\ 0 & -E_3 & E_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly (ϕ, ξ, η, g) is a para-Sasakian structure on M. Thus $M(\phi, \xi, \eta, g)$ is a 3-dimensional para-Sasakian manifold.

Using (2.10) and the above equation, one can easily obtain the following:

$$\begin{pmatrix} \tilde{\nabla}_{E_1} E_1 & \tilde{\nabla}_{E_1} E_2 & \tilde{\nabla}_{E_1} E_3 \\ \tilde{\nabla}_{E_2} E_1 & \tilde{\nabla}_{E_2} E_2 & \tilde{\nabla}_{E_2} E_3 \\ \tilde{\nabla}_{E_3} E_1 & \tilde{\nabla}_{E_3} E_2 & \tilde{\nabla}_{E_3} E_3 \end{pmatrix} = \begin{pmatrix} -2E_3 & 0 & 2E_1 \\ 0 & -2E_3 & 2E_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

With the help of the above results it can be easily verified that

$$\begin{aligned} &R(E_1, E_2)E_3 = 0, && R(E_2, E_3)E_3 = -E_2, && R(E_1, E_3)E_3 = -E_1, \\ &R(E_1, E_2)E_2 = -E_1, && R(E_2, E_3)E_2 = E_3, && R(E_1, E_3)E_2 = 0, \\ &R(E_1, E_2)E_1 = E_2, && R(E_2, E_3)E_1 = 0, && R(E_1, E_3)E_1 = E_3. \end{aligned}$$

and

$$\begin{aligned} &\tilde{R}(E_1, E_2)E_3 = 0, && \tilde{R}(E_2, E_3)E_3 = -2E_2, && \tilde{R}(E_1, E_3)E_3 = -2E_1, \\ &\tilde{R}(E_1, E_2)E_2 = -4E_1, && \tilde{R}(E_2, E_3)E_2 = 2E_3, && \tilde{R}(E_1, E_3)E_2 = 0, \\ &\tilde{R}(E_1, E_2)E_1 = 4E_2, && \tilde{R}(E_2, E_3)E_1 = 0, && \tilde{R}(E_1, E_3)E_1 = 2E_3. \end{aligned} \tag{7.1}$$

Since E_1, E_2, E_3 forms a basis, any vector field $X, Y, Z \in \chi(M)$ can be written as $X = a_1E_1 + b_1E_2 + c_1E_3$, $Y = a_2E_1 + b_2E_2 + c_2E_3$, $Z = a_3E_1 + b_3E_2 + c_3E_3$, where $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, 2, 3$. Using the expressions of the curvature tensors, we find values of Riemannian curvature and Ricci curvature with respect to quarter-symmetric metric connection as;

$$\begin{aligned}\tilde{R}(X,Y)Z &= [-4\{a_1b_2 - b_1a_2\}b_3 + 2\{c_1a_2 - a_1c_2\}c_3]E_1 \\ &+ [-4\{b_1a_2 - a_1b_2\}a_3 + 2\{c_1b_2 - b_1c_2\}c_3]E_2 \\ &+ [-2\{c_1a_2 - a_1c_2\}a_3 - 2\{c_1b_2 - b_1c_2\}b_3]E_3, \quad (7.2)\\ \tilde{S}(E_1,E_1) &= \tilde{S}(E_2,E_2) = -6,\ \tilde{S}(E_3,E_3) = -4. \quad (7.3)\end{aligned}$$

Using (7.1), (7.3) and the expression of the endomorphism $(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y$, one can easily verify that

$$\tilde{S}(\tilde{R}(X,E_3)Y,E_3) + \tilde{S}(Y,\tilde{R}(X,E_3)E_3) = -2[\tilde{S}((X \wedge_g E_3)Y,E_3) + \tilde{S}(Y,(X \wedge_g E_3)E_3)], \quad (7.4)$$

here $L_{\tilde{S}} = -2$. Thus, the above equation verify one part of the Theorem 2 of section 4.

Moreover, the manifold under consideration satisfies

$$\tilde{R}(X,Y)Z = -\tilde{R}(Y,X)Z,$$
$$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0.$$

Hence, from the above equations one can say that this example verifies the condition (c) of Theorem 3.1 in [1] and first Bianchi identity.

7.2 Example

We consider a 5-dimensional manifold $M = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5\}$, where (x_1,x_2,x_3,x_4,x_5) are standard coordinates in $\mathbb{R}^5$. We choose the vector fields

$$E_1 = \frac{\partial}{\partial x_1},\quad E_2 = \frac{\partial}{\partial x_2},\quad E_3 = \frac{\partial}{\partial x_3},\quad E_4 = \frac{\partial}{\partial x_4},\quad E_5 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5},$$

which are linearly independent at each point of M.

Let g be a Riemannian metric defined by

$$g(E_i,E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for $1 \leq i,j \leq 5$, and if η is the 1-form defined by $\eta(Z) = g(Z,E_5)$ for any vector field $Z \in \chi(M)$. Let ϕ be the (1, 1)-tensor field defined by

$$\phi(E_1) = E_1,\quad \phi(E_2) = E_2,\quad \phi(E_3) = E_3,\quad \phi(E_4) = E_4,\quad \phi(E_5) = 0.$$

The linearity property of ϕ and g yields that

$$\begin{aligned}
&\eta(E_5) = 1,\\
&\phi^2 U = U - \eta(U)E_5,\\
&g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V),
\end{aligned}$$

for any $U, V \in \chi(M)$.

Now we have

$$\begin{aligned}
&[E_1, E_2] = 0, \quad [E_1, E_3] = 0, \quad [E_1, E_4] = 0, \quad [E_1, E_5] = E_1,\\
&[E_2, E_3] = 0, \quad [E_2, E_4] = 0, \quad [E_2, E_5] = E_2,\\
&[E_3, E_4] = 0, \quad [E_3, E_5] = E3, \quad [E_4, E_5] = E_4.
\end{aligned}$$

By virtue of Koszul's formula we get the followings in matrix form

$$\begin{pmatrix}
\nabla_{E_1}E_1 & \nabla_{E_1}E_2 & \nabla_{E_1}E_3 & \nabla_{E_1}E_4 & \nabla_{E_1}E_5\\
\nabla_{E_2}E_1 & \nabla_{E_2}E_2 & \nabla_{E_2}E_3 & \nabla_{E_2}E_4 & \nabla_{E_2}E_5\\
\nabla_{E_3}E_1 & \nabla_{E_3}E_2 & \nabla_{E_3}E_3 & \nabla_{E_3}E_4 & \nabla_{E_3}E_5\\
\nabla_{E_4}E_1 & \nabla_{E_4}E_2 & \nabla_{E_4}E_3 & \nabla_{E_4}E_4 & \nabla_{E_4}E_5\\
\nabla_{E_5}E_1 & \nabla_{E_5}E_2 & \nabla_{E_5}E_3 & \nabla_{E_5}E_4 & \nabla_{E_5}E_5
\end{pmatrix} = \begin{pmatrix}
-E_5 & 0 & 0 & 0 & E_1\\
0 & -E_5 & 0 & 0 & E_2\\
0 & 0 & -E_5 & 0 & E_3\\
0 & 0 & 0 & -E_5 & E_4\\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Above expressions satisfies all the properties of para-Sasakian manifold. Thus $M(\phi, \xi, \eta, g)$ is a 5-dimensional para-Sasakian manifold.

From the above expressions and the relation of quarter symmetric metric connection and Riemannian connection, one can easily obtain the following:

$$\begin{pmatrix}
\tilde\nabla_{E_1}E_1 & \tilde\nabla_{E_1}E_2 & \tilde\nabla_{E_1}E_3 & \tilde\nabla_{E_1}E_4 & \tilde\nabla_{E_1}E_5\\
\tilde\nabla_{E_2}E_1 & \tilde\nabla_{E_2}E_2 & \tilde\nabla_{E_2}E_3 & \tilde\nabla_{E_2}E_4 & \tilde\nabla_{E_2}E_5\\
\tilde\nabla_{E_3}E_1 & \tilde\nabla_{E_3}E_2 & \tilde\nabla_{E_3}E_3 & \tilde\nabla_{E_3}E_4 & \tilde\nabla_{E_3}E_5\\
\tilde\nabla_{E_4}E_1 & \tilde\nabla_{E_4}E_2 & \tilde\nabla_{E_4}E_3 & \tilde\nabla_{E_4}E_4 & \tilde\nabla_{E_4}E_5\\
\tilde\nabla_{E_5}E_1 & \tilde\nabla_{E_5}E_2 & \tilde\nabla_{E_5}E_3 & \tilde\nabla_{E_5}E_4 & \tilde\nabla_{E_5}E_5
\end{pmatrix} = \begin{pmatrix}
-2E_5 & 0 & 0 & 0 & 2E_1\\
0 & -2E_5 & 0 & 0 & 2E_2\\
0 & 0 & -2E_5 & 0 & 2E_3\\
0 & 0 & 0 & -2E_5 & 2E_4\\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

With the help of the above results it can be easily obtain the non-zero components of curvature tensors as

$$\begin{aligned}
&R(E_1,E_2)E_1 = E_2, \quad R(E_1,E_2)E_2 = -E_1, \quad R(E_1,E_3)E_1 = E_3, \quad R(E_1,E_3)E_3 = -E_1,\\
&R(E_1,E_4)E_1 = E_4, \quad R(E_1,E_4)E_4 = -E_1, \quad R(E_1,E_5)E_1 = E_5, \quad R(E_1,E_5)E_5 = -E_1,\\
&R(E_2,E_3)E_2 = E_3, \quad R(E_2,E_3)E_3 = -E_2, \quad R(E_2,E_4)E_2 = E_4, \quad R(E_2,E_4)E_4 = -E_2,\\
&R(E_2,E_5)E_2 = E_5, \quad R(E_2,E_5)E_5 = -E_2, \quad R(E_3,E_4)E_3 = E_4, \quad R(E_3,E_4)E_4 = -E_3,\\
&R(E_3,E_5)E_3 = E_5, \quad R(E_3,E_5)E_5 = -E_3, \quad R(E_4,E_5)E_4 = E_5, \quad R(E_4,E_5)E_5 = -E_4,
\end{aligned}$$

and

$$\tilde{R}(E_1,E_2)E_1 = 4E_2,\quad \tilde{R}(E_1,E_2)E_2 = -4E_1,\quad \tilde{R}(E_1,E_3)E_1 = 4E_3,\quad \tilde{R}(E_1,E_3)E_3 = -4E_1,$$
$$\tilde{R}(E_1,E_4)E_1 = 4E_4,\quad \tilde{R}(E_1,E_4)E_4 = -4E_1,\quad \tilde{R}(E_1,E_5)E_1 = 2E_5,\quad \tilde{R}(E_1,E_5)E_5 = -2E_1,$$
$$\tilde{R}(E_2,E_3)E_2 = 4E_3,\quad \tilde{R}(E_2,E_3)E_3 = -4E_2,\quad \tilde{R}(E_2,E_4)E_2 = 4E_4,\quad \tilde{R}(E_2,E_4)E_4 = -4E_2,$$
$$\tilde{R}(E_2,E_5)E_2 = 2E_5,\quad \tilde{R}(E_2,E_5)E_5 = -2E_2,\quad \tilde{R}(E_3,E_4)E_3 = 4E_4,\quad \tilde{R}(E_3,E_4)E_4 = -4E_3,$$
$$\tilde{R}(E_3,E_5)E_3 = 2E_5,\ \tilde{R}(E_3,E_5)E_5 = -2E_3,\ \tilde{R}(E_4,E_5)E_4 = 2E_5,\ \tilde{R}(E_4,E_5)E_5 = -2E_4. \quad (7.5)$$

Since E_1, E_2, E_3, E_4, E_5 forms a basis, any vector field $X, Y, Z \in \chi(M)$ can be written as $X = a_1E_1 + b_1E_2 + c_1E_3 + d_1E_4 + f_1E_5$, $Y = a_2E_1 + b_2E_2 + c_2E_3 + d_2E_4 + f_2E_5$, $Z = a_3E_1 + b_3E_2 + c_3E_3 + d_3E_4 + f_3E_5$, where $a_i, b_i, c_i, d_i, f_i \in \mathbb{R}$, $i = 1, 2, 3, 4, 5$. Using the expressions of the curvature tensors, we find values of Riemannian curvature and Ricci curvature with respect to quarter-symmetric metric connection as;

$$\begin{aligned}
\tilde{R}(X,Y)Z &= [-4\{a_1(b_2b_3 + c_2c_3 + d_2d_3) - a_2(b_1b_3 + c_1c_3 + d_1d_3)\} - 2(a_1f_2 - f_1a_2)f_3]E_1 \\
&+ [-4\{b_1(a_2a_3 + c_2c_3 + d_2d_3) - b_2(a_1a_3 + c_1c_3 + d_1d_3)\} - 2(b_1f_2 - f_1b_2)f_3]E_2 \\
&+ [-4\{c_1(a_2a_3 + b_2b_3 + d_2d_3) - c_2(a_1a_3 + b_1b_3 + d_1d_3)\} - 2(c_1f_2 - f_1c_2)f_3]E_3 \\
&+ [-4\{d_1(a_2a_3 + b_2b_3 + c_2c_3) - d_2(a_1a_3 + b_1b_3 + c_1c_3)\} - 2(d_1f_2 - f_1d_2)f_3]E_4 \\
&+ [2\{(a_1f_2 - f_1a_2)a_3 + (b_1f_2 - f_1b_2)b_3 + (c_1f_2 - f_1c_2)c_3 + (d_1f_2 - f_1d_2)d_3\}]E_5, \\
\tilde{S}(E_1,E_1) &= \tilde{S}(E_2,E_2) = \tilde{S}(E_3,E_3) = \tilde{S}(E_4,E_4) = -14,\ \tilde{S}(E_5,E_5) = -8. \qquad (7.6)
\end{aligned}$$

In view of (7.5), (7.6) and the expression of the endomorphism one can easily verify the equation (7.4) and hence the Theorem 2 of section 4 is verified. This example also verifies the condition (c) of Theorem 3.1 in [1] and first Bianchi identity.

Above two examples verifies the one part of the Theorem 2, that is, if a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is Ricci pseudosymmetric then M^n satisfies $L_{\tilde{S}} = -2$ (M^n is not Einstein manifold with respect to quarter-symmetric metric connection). Another part of the theorem is that, if a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is Ricci pseudosymmetric then M^n is an Einstein manifold with respect to quarter-symmetric metric connection ($L_{\tilde{S}} \neq -2$). Now, the second part of the Theorem 2 can be verified by using the proper example.

7.3 Example

We consider a 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are standard coordinates in $\mathbb{R}^5$. Let $\{E_1, E_2, E_3, E_4, E_5\}$ be a linearly independent global frame field on M given by

$$E_1 = \frac{\partial}{\partial x}, \quad E_2 = e^{-x}\frac{\partial}{\partial y}, \quad E_3 = e^{-x}\frac{\partial}{\partial z}, \quad E_4 = e^{-x}\frac{\partial}{\partial u}, \quad E_5 = e^{-x}\frac{\partial}{\partial v}.$$

Let g be a Riemannian metric defined by

$$g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for $1 \leq i, j \leq 5$, and if η is the 1-form defined by $\eta(Z) = g(Z, E_1)$ for any vector field $Z \in \chi(M)$. Let the (1, 1)-tensor field ϕ be defined by

$$\phi(E_1) = 0, \quad \phi(E_2) = E_2, \quad \phi(E_3) = E_3, \quad \phi(E_4) = E_4, \quad \phi(E_5) = E_5.$$

With the help of linearity property of ϕ and g, we have

$$\begin{aligned} &\eta(E_1) = 1, \\ &\phi^2 V = V - \eta(V)E_1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \end{aligned}$$

for any $X, Y \in \chi(M)$.

Now we have

$$\begin{aligned} &[E_1, E_2] = -E_2, \quad [E_1, E_3] = -E_3, \quad [E_1, E_4] = -E_4, \quad [E_1, E_5] = -E_5, \\ &[E_2, E_3] = [E_2, E_4] = [E_2, E_5] = [E_3, E_4] = [E_3, E_5] = E_4, E_5] = 0. \end{aligned}$$

With the help of Koszul's formula we get the followings in matrix form

$$\begin{pmatrix} \nabla_{E_1}E_1 & \nabla_{E_1}E_2 & \nabla_{E_1}E_3 & \nabla_{E_1}E_4 & \nabla_{E_1}E_5 \\ \nabla_{E_2}E_1 & \nabla_{E_2}E_2 & \nabla_{E_2}E_3 & \nabla_{E_2}E_4 & \nabla_{E_2}E_5 \\ \nabla_{E_3}E_1 & \nabla_{E_3}E_2 & \nabla_{E_3}E_3 & \nabla_{E_3}E_4 & \nabla_{E_3}E_5 \\ \nabla_{E_4}E_1 & \nabla_{E_4}E_2 & \nabla_{E_4}E_3 & \nabla_{E_4}E_4 & \nabla_{E_4}E_5 \\ \nabla_{E_5}E_1 & \nabla_{E_5}E_2 & \nabla_{E_5}E_3 & \nabla_{E_5}E_4 & \nabla_{E_5}E_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ E_2 & -E_1 & 0 & 0 & 0 \\ E_3 & 0 & -E_1 & 0 & 0 \\ E_4 & 0 & 0 & -E_1 & 0 \\ E_5 & 0 & 0 & 0 & -E_1 \end{pmatrix}.$$

In this case, (ϕ, ξ, η, g) is a para-Sasakian structure on M and hence $M(\phi, \xi, \eta, g)$ is a 5-dimensional para-Sasakian manifold.

Using (2.10) and the above equation, one can easily obtain the following:

$$\begin{pmatrix} \tilde{\nabla}_{E_1}E_1 & \tilde{\nabla}_{E_1}E_2 & \tilde{\nabla}_{E_1}E_3 & \tilde{\nabla}_{E_1}E_4 & \tilde{\nabla}_{E_1}E_5 \\ \tilde{\nabla}_{E_2}E_1 & \tilde{\nabla}_{E_2}E_2 & \tilde{\nabla}_{E_2}E_3 & \tilde{\nabla}_{E_2}E_4 & \tilde{\nabla}_{E_2}E_5 \\ \tilde{\nabla}_{E_3}E_1 & \tilde{\nabla}_{E_3}E_2 & \tilde{\nabla}_{E_3}E_3 & \tilde{\nabla}_{E_3}E_4 & \tilde{\nabla}_{E_3}E_5 \\ \tilde{\nabla}_{E_4}E_1 & \tilde{\nabla}_{E_4}E_2 & \tilde{\nabla}_{E_4}E_3 & \tilde{\nabla}_{E_4}E_4 & \tilde{\nabla}_{E_4}E_5 \\ \tilde{\nabla}_{E_5}E_1 & \tilde{\nabla}_{E_5}E_2 & \tilde{\nabla}_{E_5}E_3 & \tilde{\nabla}_{E_5}E_4 & \tilde{\nabla}_{E_5}E_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2E_2 & -2E_1 & 0 & 0 & 0 \\ 2E_3 & 0 & -2E_1 & 0 & 0 \\ 2E_4 & 0 & 0 & -2E_1 & 0 \\ 2E_5 & 0 & 0 & 0 & -2E_1 \end{pmatrix}.$$

From above results it can be easily obtain the non-zero components of Riemannian curvature and Ricci curvature tensors as

$$\begin{aligned}
&R(E_1,E_2)E_1 = E_2, \quad R(E_1,E_2)E_2 = -E_1, \quad R(E_1,E_3)E_1 = E_3, \quad R(E_1,E_3)E_3 = -E_1,\\
&R(E_1,E_4)E_1 = E_4, \quad R(E_1,E_4)E_4 = -E_1, \quad R(E_1,E_5)E_1 = E_5, \quad R(E_1,E_5)E_5 = -E_1,\\
&R(E_2,E_3)E_2 = E_3, \quad R(E_2,E_3)E_3 = -E_2, \quad R(E_2,E_4)E_2 = E_4, \quad R(E_2,E_4)E_4 = -E_2,\\
&R(E_2,E_5)E_2 = E_5, \quad R(E_2,E_5)E_5 = -E_2, \quad R(E_3,E_4)E_3 = E_4, \quad R(E_3,E_4)E_4 = -E_3,\\
&R(E_3,E_5)E_3 = E_5, \quad R(E_3,E_5)E_5 = -E_3, \quad R(E_4,E_5)E_4 = E_5, \quad R(E_4,E_5)E_5 = -E_4,
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{R}(E_1,E_2)E_1 = 2E_2, \quad \tilde{R}(E_1,E_2)E_2 = -2E_1, \quad \tilde{R}(E_1,E_3)E_1 = 2E_3, \quad \tilde{R}(E_1,E_3)E_3 = -2E_1,\\
&\tilde{R}(E_1,E_4)E_1 = 2E_4, \quad \tilde{R}(E_1,E_4)E_4 = -2E_1, \quad \tilde{R}(E_1,E_5)E_1 = 2E_5, \quad \tilde{R}(E_1,E_5)E_5 = -2E_1,\\
&\tilde{R}(E_2,E_3)E_2 = 2E_3, \quad \tilde{R}(E_2,E_3)E_3 = -2E_2, \quad \tilde{R}(E_2,E_4)E_2 = 2E_4, \quad \tilde{R}(E_2,E_4)E_4 = -2E_2,\\
&\tilde{R}(E_2,E_5)E_2 = 2E_5, \quad \tilde{R}(E_2,E_5)E_5 = -2E_2, \quad \tilde{R}(E_3,E_4)E_3 = 2E_4, \quad \tilde{R}(E_3,E_4)E_4 = -2E_3,\\
&\tilde{R}(E_3,E_5)E_3 = 2E_5,\ \tilde{R}(E_3,E_5)E_5 = -2E_3,\ \tilde{R}(E_4,E_5)E_4 = 2E_5,\ \tilde{R}(E_4,E_5)E_5 = -2E_4, \quad (7.7)\\
&\tilde{S}(E_1,E_1) = \tilde{S}(E_2,E_2) = \tilde{S}(E_3,E_3) = \tilde{S}(E_4,E_4) = \tilde{S}(E_5,E_5) = -8, \quad (7.8)\\
&\tilde{S}(X,Y) = -2(5-1)g(X,Y) = -8g(X,Y),
\end{aligned}$$

where $X = a_1E_1 + b_1E_2 + c_1E_3 + d_1E_4 + f_1E_5$ and $Y = a_2E_1 + b_2E_2 + c_2E_3 + d_2E_4 + f_2E_5$.

From (7.7), (7.8) and the expression of the endomorphism one can easily substantiate, the equation (7.4) and hence second part of the Theorem 2 (for $L_{\tilde{S}} \neq -2$).

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Fixed point theorems on cone S-metric spaces using implicit relation

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ABSTRACT

In this paper, we establish some fixed point theorems in the framework of cone S-metric spaces using implicit relation. Our results extend, unify and generalize several results from the current existing literature. Especially, they extend the corresponding results of Sedghi and Dung [24] to the setting of complete cone S-metric spaces.

RESUMEN

En este artículo, establecemos algunos teoremas de punto fijo en el marco de espacios S-métricos del cono usando una relación implícita. Nuestros resultados extienden, unifican y generalizan diversos resultados de la literatura actual existente. Especialmente, extienden los resultados correspondientes de Sedghi y Dung [24] en el contexto de espacios S-métricos de cono completo.

Keywords and Phrases: Fixed point, implicit relation, cone S-metric space, cone.

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1 Introduction and Preliminaries

In 2007, Huang and Zhang [8] introduced the concept of cone metric spaces as a generalization of metric spaces by replacing the set of real numbers by a general Banach space E which is partially ordered with respect to a cone $P \subset E$ and establish some fixed point theorems for contractive mappings in normal cone metric spaces.

In 2012, Sedghi et al. [23] introduced the concept of S-metric space which is different from other space and proved fixed point theorems in S-metric space. They also give some examples of S-metric space which shows that S-metric space is different from other spaces.

In 2016, Rahman and Sarwar [20] have discussed the fixed point results of Altman integral type mappings in S-metric spaces and in the same year Ozgur and Tas [14] have studied new contractive conditions of integral type in complete S-spaces.

Recently, Dhamodharan and Krishnakumar [6] introduced the concept of cone S-metric space and proved some fixed point theorems using various contractive conditions in the above said space.

Due to great importance of the fixed point theory, it is immensely interesting to study fixed point theorems on different concepts. Many authors studied the fixed points for mappings satisfying contractive conditions in complete S-metric spaces (see, e.g., [6, 11, 13, 14, 20, 23, 25, 26]) and others).

Popa [15] and [16], on the other hand, considered an implicit contraction type condition instead of the usual explicit condition. This direction of research produced a consistent literature on fixed point and common fixed point theorems in various ambient spaces. For more details see [1, 2, 3, 9, 17, 18, 19, 24].

Motivated and inspired by Popa [15, 16], Sedghi and Dung [24] and others, this paper is aimed to study and establish some fixed point theorems in the setting of complete cone S-metric spaces under implicit contractive condition which is used in [24]. Following the current literature there is ample vicinity to explore and improve this new avenue of research area. Here, we prove an important result of cone S-metric space and then obtain some classical fixed point theorems as corollaries, for example, Banach's contraction mapping principle, Kannan's fixed point theorem, Chatterjae's fixed point theorem, Reich fixed point theorem and Ćirić's fixed point theorem in this setting. Our results extend and generalize several results from the existing literature, especially, the results of Sedghi and Dung [24] from complete S-metric spaces to the setting of complete cone S-metric spaces.

The present work is to encouraged by its possible application, especially in discrete models for numerical analysis, where iterative schemes are extensively used due to their versatility for computer simulation. These models play an important role in applied mathematics.

We need the following definitions and lemmas in the sequel.

Definition 1. *([8]) Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:*

(c_1) *P is closed, nonempty and $P \neq \{0\}$;*

(c_2) *$a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;*

(c_3) *$P \cap (-P) = \{0\}$.*

Given a cone $P \subset E$, we define a partial ordering $\leq$ in E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$, where P^0 stands for the interior of P. If $P^0 \neq \emptyset$ then P is called a solid cone (see [28]).

There exist two kinds of cones- normal (with the normal constant K) and non-normal ones ([7]).

Let E be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering defined by P. Then P is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ imply } \|x\| \leq K\|y\|, \tag{1.1}$$

or equivalently, if $(\forall n)$ $x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{1.2}$$

The least positive number K satisfying (1.1) is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Suppose E is a Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is partial ordering in E with respect to P.

Example 1. *([12]) Let $K > 1$ be given. Consider the real vector space*

$$E = \left\{ ax + b : a, b \in R; x \in \left[1 - \frac{1}{K}, 1\right] \right\}$$

with supremum norm and the cone

$$P = \left\{ ax + b \in E : a \geq 0, b \geq 0 \right\}$$

in E. The cone P is regular and so normal.

Definition 2. *([8, 29]) Let X be a nonempty set. Suppose that the mapping $d\colon X \times X \to E$ satisfies:*

(CM_1) *$0 \leq d(x,y)$ for all $x, y \in X$ with $x \neq y$ and $d(x,y) = 0 \Leftrightarrow x = y$;*

(CM_2) *$d(x,y) = d(y,x)$ for all $x, y \in X$;*

(CM_3) *$d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.*

Then d is called a cone metric [8] on X and (X,d) is called a cone metric space [8] or simply CMS.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Lemma 1. *([22]) Every regular cone is normal.*

Example 2. *([8]) Let $E = \mathbb{R}^2$, $P = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and $d\colon X \times X \to E$ defined by $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space with normal cone P where $K = 1$.*

Clearly, the above example shows that the class of cone metric spaces contains the class of metric spaces.

Definition 3. *([23, 14]) Let X be a nonempty set and $S\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all x, y, z, $t \in X$:*

(SM_1) *$S(x,y,z) \geq 0$;*

(SM_2) *$S(x,y,z) = 0$ if and only if $x = y = z$;*

(SM_3) *$S(x,y,z) \leq S(x,x,t) + S(y,y,t) + S(z,z,t)$.*

Then the function S is called an S-metric on X and the pair (X,S) is called an S-metric space or simply SMS.

Example 3. *([27]) Let X be a nonempty set and d be the ordinary metric on X. Then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.*

Example 4. *([23]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.*

Example 5. *([23]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|x - z\| + \|y - z\|$ is an S-metric on X.*

Example 6. *([24]) Let $X = \mathbb{R}$ be the real line. Then $S(x,y,z) = \|x - z\| + \|y - z\|$ for all $x, y, z \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.*

Definition 4. *([6]) Suppose that E is a real Banach space, P is a cone in E with $int\, P \neq \emptyset$ and $\leq$ is partial ordering with respect to P. Let X be a nonempty set and let the function $S\colon X^3 \to E$ satisfy the following conditions:*

(CSM_1) $S(x, y, z) \geq 0$;

(CSM_2) $S(x, y, z) = 0$ *if and only if* $x = y = z$;

(CSM_3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a), \forall x, y, z, a \in X$.

Then the function S is called a cone S-metric on X and the pair (X, S) is called a cone S-metric space or simply CSMS.

Example 7. *([6]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and d be the ordinary metric on X. Then the function $S\colon X^3 \to E$ defined by $S(x, y, z) = \Big(d(x, z) + d(y, z), \alpha(d(x, z) + d(y, z))\Big)$, where $\alpha > 0$ is a cone S-metric on X.*

Lemma 2. *([6]) Let (X, S) be a cone S-metric space. Then we have $S(x, x, y) = S(y, y, x)$.*

Definition 5. *([6]) Let (X, S) be a cone S-metric space.*

(i) A sequence $\{u_n\}$ in X converges to u if and only if $S(u_n, u_n, u) \to 0$ as $n \to \infty$, that is, there exists $n_0 \in N$ such that for all $n \geq n_0$, $S(u_n, u_n, u) \ll c$ for each $c \in E$, $0 \ll c$. We denote this by $\lim_{n\to\infty} u_n = u$ or $\lim_{n\to\infty} S(u_n, u_n, u) = 0$.

(ii) A sequence $\{u_n\}$ in X is called a Cauchy sequence if $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$, that is, there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $S(u_n, u_n, u_m) \ll c$ for each $c \in E$, $0 \ll c$.

(iii) The cone S-metric space (X, S) is called complete if every Cauchy sequence is convergent.

In the following lemma, we see the relationship between a cone metric and a cone S-metric.

Lemma 3. *([6]) Let (X, d) be a cone metric space. Then, the following properties are satisfied:*

(1) *$S(u, v, z) = d(u, z) + d(v, z)$ for all $u, v, z \in X$, is a cone S-metric on X.*

(2) *$u_n \to u$ in (X, d) if and only if $u_n \to u$ in (X, S_d).*

(3) *$\{u_n\}$ is Cauchy in (X, d) if and only if $\{u_n\}$ is Cauchy in (X, S_d).*

(4) *(X, d) is complete if and only if (X, S_d) is complete.*

Lemma 4. *([24]) Let $f\colon X \to Y$ be a map from an S-metric space X to an S-metric space Y. Then f is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.*

Now, we introduce an implicit relation to investigate some fixed point theorems on cone S-metric spaces. Let ψ be the family of all continuous functions of five variables $\phi\colon \mathbb{R}_+^5 \to \mathbb{R}_+$. For some $k \in [0, 1)$, we consider the following conditions.

(A_1) For all $x, y, z \in \mathbb{R}_+$, if $y \leq \phi(x, x, y, z, 0)$ with $z \leq 2x + y$, then $y \leq kx$.

(A_2) For all $y \in \mathbb{R}_+$, if $y \leq \phi(y, 0, 0, y, y)$, then $y = 0$.

(A_3) If $x_i \leq y_i + z_i$ for all $x_i, y_i, z_i \in \mathbb{R}_+$, $i \leq 5$, then

$$\phi(x_1, \ldots, x_5) \leq \phi(y_1, \ldots, y_5) + \phi(z_1, \ldots, z_5).$$

Moreover, for all $y \in X$, $\phi(0, 0, 2y, y, 0) \leq ky$.

Remark 1. *Note that the coefficient k in conditions (A_1) and (A_3) may be different, for example, k_1 and k_3 respectively. But we may assume that they are equal by taking $k = \max\{k_1, k_3\}$.*

2 Main Results

In this section, we shall prove some fixed point theorems using implicit relation in the setting of cone S-metric spaces.

Theorem 1. *Let T be a self-map on a complete cone S-metric space (X, S), P be a normal cone with normal constant K and*

$$\begin{aligned} S(Tx, Tx, Ty) \quad \leq \quad & \phi\Big(S(x, x, y), S(x, x, Tx), S(y, y, Ty), \\ & S(x, x, Ty), S(y, y, Tx)\Big) \end{aligned} \tag{2.1}$$

for all $x, y \in X$ and some $\phi \in \psi$. Then we have

(1) If ϕ satisfies the condition (A_1), then T has a fixed point. Moreover, for any $x_0 \in X$ and the fixed point x, we have

$$S(Tx_n, Tx_n, x) \leq \Big(\frac{2k^n}{1-k}\Big) S(x_0, x_0, Tx_0).$$

(2) If ϕ satisfies the condition (A_2) and T has a fixed point, then the fixed point is unique.

(3) If ϕ satisfies the condition (A_3) and T has a fixed point x, then T is continuous at x.

Proof. (1) For each $x_0 \in X$ and $n \in \mathbb{N}$, put $x_{n+1} = Tx_n$. It follows from (2.1) and Lemma 2 that

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_{n+2}) &= S(Tx_n, Tx_n, Tx_{n+1}) \\ &\leq \phi\Big(S(x_n, x_n, x_{n+1}), S(x_n, x_n, Tx_n), S(x_{n+1}, x_{n+1}, Tx_{n+1}), \\ &\quad S(x_n, x_n, Tx_{n+1}), S(x_{n+1}, x_{n+1}, Tx_n)\Big) \\ &= \phi\Big(S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2}), \\ &\quad S(x_n, x_n, x_{n+2}), S(x_{n+1}, x_{n+1}, x_{n+1})\Big) \\ &= \phi\Big(S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2}), \\ &\quad S(x_n, x_n, x_{n+2}), 0\Big). \end{aligned} \tag{2.2}$$

By condition (CSM_3) and Lemma 2, we have

$$\begin{aligned} S(x_n, x_n, x_{n+2}) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_{n+2}, x_{n+2}, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}). \end{aligned} \tag{2.3}$$

Since ϕ satisfies the condition (A_1), there exists $k \in [0, 1)$ such that

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq kS(x_n, x_n, x_{n+1}) \leq k^{n+1}S(x_0, x_0, x_1). \tag{2.4}$$

Thus for all $n < m$, by using (CSM_3), Lemma 2 and equation (2.4), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\cdots \\ &\leq 2[k^n + \cdots + k^{m-1}]S(x_0, x_0, x_1) \\ &\leq \Big(\frac{2k^n}{1-k}\Big)S(x_0, x_0, x_1). \end{aligned}$$

This implies that

$$\|S(x_n, x_n, x_m)\| \leq \Big(\frac{2k^n K}{1-k}\Big)\|S(x_0, x_0, x_1)\|.$$

Taking the limit as $n, m \to \infty$, we get

$$\|S(x_n, x_n, x_m)\| \to 0,$$

since $0 < k < 1$. Thus, we have $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.

This shows that the sequence $\{x_n\}$ is a Cauchy sequence in the complete cone S-metric space (X, S). By the completeness of the space, we have $\lim_{n\to\infty} x_n = x \in X$. Moreover, taking the limit as $m \to \infty$ we get

$$S(x_n, x_n, x) \leq \Big(\frac{2k^{n+1}}{1-k}\Big)S(x_0, x_0, x_1).$$

It implies that

$$S(Tx_n, Tx_n, x) \leq \Big(\frac{2k^n}{1-k}\Big) S(x_0, x_0, Tx_0).$$

Now we prove that x is a fixed point of T. By using inequality (2.1) again we obtain

$$\begin{aligned}
S(x_{n+1}, x_{n+1}, Tx) &= S(Tx_n, Tx_n, Tx) \\
&\leq \phi\Big(S(x_n, x_n, x), S(x_n, x_n, Tx_n), S(x, x, Tx), \\
&\quad S(x_n, x_n, Tx), S(x, x, Tx_n)\Big) \\
&= \phi\Big(S(x_n, x_n, x), S(x_n, x_n, x_{n+1}), S(x, x, Tx), \\
&\quad S(x_n, x_n, Tx), S(x, x, x_{n+1})\Big).
\end{aligned}$$

Note that $\phi \in \psi$, then using Lemma 3 and taking the limit as $n \to \infty$, we get

$$S(x, x, Tx) \leq \phi\Big(0, 0, S(x, x, Tx), S(x, x, Tx), 0\Big).$$

Since ϕ satisfies the condition (A_1), then $S(x, x, Tx) \leq k.0 = 0$. This shows that $x = Tx$. Thus x is a fixed point of T.

(2) Let x_1, x_2 be fixed points of T. We shall prove that $x_1 = x_2$. It follows from equation (2.1) and Lemma 2 that

$$\begin{aligned}
S(x_1, x_1, x_2) &= S(Tx_1, Tx_1, Tx_2) \\
&\leq \phi\Big(S(x_1, x_1, x_2), S(x_1, x_1, Tx_1), S(x_2, x_2, Tx_2), \\
&\quad S(x_1, x_1, Tx_2), S(x_2, x_2, Tx_1)\Big) \\
&= \phi\Big(S(x_1, x_1, x_2), S(x_1, x_1, x_1), S(x_2, x_2, x_2), \\
&\quad S(x_1, x_1, x_2), S(x_2, x_2, x_1)\Big) \\
&= \phi\Big(S(x_1, x_1, x_2), 0, 0, S(x_1, x_1, x_2), S(x_2, x_2, x_1)\Big) \\
&= \phi\Big(S(x_1, x_1, x_2), 0, 0, S(x_1, x_1, x_2), S(x_1, x_1, x_2)\Big).
\end{aligned}$$

Since ϕ satisfies the condition (A_2), then $S(x_1, x_1, x_2) = 0$. This shows that $x_1 = x_2$. Thus the fixed point of T is unique.

(3) Let x be the fixed point of T and $y_n \to x \in X$. By Lemma 4, we need to prove that

$Ty_n \to Tx$. It follows from inequality (2.1) and Lemma 2 that

$$\begin{aligned} S(x,x,Ty_n) &= S(Tx,Tx,Ty_n) \\ &\leq \phi\Big(S(x,x,y_n), S(x,x,Tx), S(y_n,y_n,Ty_n), \\ &\quad S(x,x,Ty_n), S(y_n,y_n,Tx)\Big) \\ &= \phi\Big(S(x,x,y_n), S(x,x,x), S(y_n,y_n,Ty_n), \\ &\quad S(x,x,Ty_n), S(y_n,y_n,x)\Big) \\ &= \phi\Big(S(x,x,y_n), 0, S(Ty_n,Ty_n,y_n), \\ &\quad S(Ty_n,Ty_n,x), S(x,x,y_n)\Big). \end{aligned}$$

Since ϕ satisfies the condition (A_3), by Lemma 2 and (CSM_3), we have

$$\begin{aligned} S(Ty_n,Ty_n,y_n) &\leq 2S(Ty_n,Ty_n,x) + S(y_n,y_n,x) \\ &= 2S(Ty_n,Ty_n,x) + S(x,x,y_n) \end{aligned}$$

then we have

$$\begin{aligned} S(x,x,Ty_n) &\leq \phi\Big(S(x,x,y_n),0,0,0,S(x,x,y_n)\Big) \\ &\quad +\phi\Big(0,0,2S(Ty_n,Ty_n,x),S(Ty_n,Ty_n,x),0\Big) \\ &\leq \phi\Big(S(x,x,y_n),0,0,0,S(x,x,y_n)\Big) \\ &\quad +kS(Ty_n,Ty_n,x) \\ &= \phi\Big(S(x,x,y_n),0,0,0,S(x,x,y_n)\Big) \\ &\quad +kS(x,x,Ty_n). \quad (\text{by Lemma 2}) \end{aligned}$$

Therefore

$$S(x,x,Ty_n) \leq \Big(\frac{1}{1-k}\Big)\phi\Big(S(x,x,y_n),0,0,0,S(x,x,y_n)\Big).$$

Note that $\phi \in \psi$, hence taking the limit as $n \to \infty$, we get $S(x,x,Ty_n) \to 0$. This shows that $Ty_n \to x = Tx$. This completes the proof. □

Next, we give some analogues of fixed point theorems in metric spaces for cone S-metric spaces by combining Theorem 1 with $\phi \in \psi$ and ϕ satisfies the conditions (A_1), (A_2) and (A_3). The following corollary is an analogue of Banach's contraction principle.

Corollary 1. *Let (X,S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$S(Tx,Tx,Ty) \leq h\,S(x,x,y)$$

for all $x, y \in X$, where $h \in [0, 1)$ is a constant. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 1 with $\phi(x, y, z, s, t) = hx$ for some $h \in [0, 1)$ and all $x, y, z, s, t \in \mathbb{R}_+$. □

The following corollary is an analogue of R. Kannan's result [10].

Corollary 2. *Let (X, S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose that the mapping $T : X \to X$ satisfies the following condition:*

$$S(Tx, Tx, Ty) \le q\,[S(x, x, Tx) + S(y, y, Ty)]$$

for all $x, y \in X$, where $q \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 1 with $\phi(x, y, z, s, t) = q(y + z)$ for some $q \in [0, \frac{1}{2})$ and all $x, y, z, s, t \in \mathbb{R}_+$. Indeed, ϕ is continuous. First, we have $\phi(x, x, y, z, 0) = q(x + y)$. So, if $y \le \phi(x, x, y, z, 0)$ with $z \le 2x + y$, then $y \le \Big(\frac{q}{1-q}\Big)x$ with $\Big(\frac{q}{1-q}\Big) < 1$. Thus, T satisfies the condition (A_1).

Next, if $y \le \phi(y, 0, 0, y, y)$, then $y = 0$. Thus, T satisfies the condition (A_2).

Finally, if $x_i \le y_i + z_i$ for $i \le 5$, then

$$\begin{aligned}
\phi(x_1, \dots, x_5) &= q(x_2 + x_3) \\
&\le q[(y_2 + z_2) + (y_3 + z_3)] \\
&= q(y_2 + y_3) + q(z_2 + z_3) \\
&= \phi(y_1, \dots, y_5) + \phi(z_1, \dots, z_5).
\end{aligned}$$

Moreover

$$\phi(0, 0, 2y, y, 0) = q(0 + 2y) = 2qy$$

where $2q < 1$. Thus, T satisfies the condition (A_3). □

The following corollary is an analogue of S. K. Chatterjae's result [4].

Corollary 3. *Let (X, S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose that the mapping $T : X \to X$ satisfies the following condition:*

$$S(Tx, Tx, Ty) \le p\,[S(x, x, Ty) + S(y, y, Tx)]$$

for all $x, y \in X$, where $p \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 1 with $\phi(x,y,z,s,t) = p(s+t)$ for some $p \in [0, \frac{1}{2})$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed, ϕ is continuous. First, we have $\phi(x,x,y,z,0) = p(z+0)$. So, if $y \leq \phi(x,x,y,z,0)$ with $z \leq 2x+y$, then $y \leq \left(\frac{2p}{1-p}\right)x$ with $\left(\frac{2p}{1-p}\right) < 1$. Thus, T satisfies the condition (A_1).

Next, if $y \leq \phi(y,0,0,y,y) = 2py$, then $y = 0$ since $p < \frac{1}{2}$. Thus, T satisfies the condition (A_2).

Finally, if $x_i \leq y_i + z_i$ for $i \leq 5$, then

$$\begin{aligned}
\phi(x_1,\ldots,x_5) &= p(x_4+x_5)\\
&\leq p[(y_4+z_4)+(y_5+z_5)]\\
&= p(y_4+y_5)+p(z_4+z_5)\\
&= \phi(y_1,\ldots,y_5)+\phi(z_1,\ldots,z_5).
\end{aligned}$$

Moreover

$$\phi(0,0,2y,y,0) = p(y+0) = py$$

where $p < 1$. Thus, T satisfies the condition (A_3). □

The following corollary is an analogue of S. Reich's result [21].

Corollary 4. *Let (X,S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$S(Tx,Tx,Ty) \leq a\,S(x,x,y) + b\,S(x,x,Tx) + c\,S(y,y,Ty)$$

for all $x,y \in X$, where $a,b,c \geq 0$ are constants with $a+b+c < 1$. Then T has a unique fixed point in X. Moreover, if $c < \frac{1}{2}$, then T is continuous at the fixed point.

Proof. The assertion follows using Theorem 1 with $\phi(x,y,z,s,t) = ax+by+cz$ for some $a,b,c \geq 0$ are constants with $a+b+c < 1$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed, ϕ is continuous. First, we have $\phi(x,x,y,z,0) = ax+bx+cy$. So, if $y \leq \phi(x,x,y,z,0)$ with $z \leq 2x+y$, then $y \leq \left(\frac{a+b}{1-c}\right)x$ with $\left(\frac{a+b}{1-c}\right) < 1$. Thus, T satisfies the condition (A_1).

Next, if $y \leq \phi(y,0,0,y,y) = ay$, then $y = 0$ since $a < 1$. Thus, T satisfies the condition (A_2).

Finally, if $x_i \leq y_i + z_i$ for $i \leq 5$, then

$$\begin{aligned}
\phi(x_1,\ldots,x_5) &= ax_1+bx_2+cx_3\\
&\leq a(y_1+z_1)+b(y_2+z_2)+c(y_3+z_3)\\
&= (ay_1+by_2+cy_3)+(az_1+bz_2+cz_3)\\
&= \phi(y_1,\ldots,y_5)+\phi(z_1,\ldots,z_5).
\end{aligned}$$

Moreover

$$\phi(0,0,2y,y,0) = a.0 + b.0 + c.2y = 2cy$$

where $2c < 1$. Thus, T satisfies the condition (A_3). □

The following corollary is an analogue of L. B. Ćirić's result [5].

Corollary 5. *Let (X,S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$\begin{aligned} S(Tx,Tx,Ty) \leq h \max\Big\{& S(x,x,y), S(x,x,Tx), S(y,y,Ty), \\ & S(x,x,Ty), S(y,y,Tx)\Big\} \end{aligned}$$

for all $x,y \in X$, where $h \in [0,\frac{1}{3})$ is a constant. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 1 with $\phi(x,y,z,s,t) = h \max\{x, y,z,s,t\}$ for some $h \in [0,\frac{1}{3})$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed, ϕ is continuous. First, we have $\phi(x,x,y,z,0) = h \max\{x,x,y,z,0\}$. So, if $y \leq \phi(x,x,y,z,0)$ with $z \leq 2x+y$, then $y \leq hx$ or $y \leq hz \leq h(2x+y)$. Then $y \leq kx$ with $k = \max\Big\{h, \frac{2h}{1-h}\Big\} < 1$. Thus, T satisfies the condition (A_1).

Next, if $y \leq \phi(y,0,0,y,y) = h \max\{y,0,0,y,y\} = hy$, then $y = 0$ since $h < \frac{1}{3}$. Thus, T satisfies the condition (A_2).

Finally, if $x_i \leq y_i + z_i$ for $i \leq 5$, then

$$\begin{aligned} \phi(x_1,\dots,x_5) &= h \max\{x_1,\dots,x_5\} \\ &\leq h \max\{y_1+z_1,\dots,y_5+z_5\} \\ &\leq h \max\{y_1,\dots,y_5\} + h \max\{z_1,\dots,z_5\} \\ &= \phi(y_1,\dots,y_5) + \phi(z_1,\dots,z_5). \end{aligned}$$

Moreover

$$\phi(0,0,2y,y,0) = h \max\{0,0,2y,y,0\} = 2hy$$

where $2h < 1$. Thus, T satisfies the condition (A_3). □

Example 8. *Let $E = \mathbb{R}^2$, the Euclidean plane, $P = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ a normal cone in E and $X = \mathbb{R}$. Then the function $S\colon X^3 \to E$ defined by $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in X$. Then (X,S) is a cone S-metric space. Now, we consider the mapping $T\colon X \to X$ by $T(x) = \frac{x}{2}$ and $\{x_n\} = \{\frac{1}{2^n}\}$ for all $n \in \mathbb{N}$ is a sequence converging to zero.*

Result Analysis

(1) Taking $x = x_{n-1}$ *and* $y = x_n$ *in inequality (2.1) and using* (CSM_3)*, we have*

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq \phi\Big(S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, Tx_{n-1}), S(x_n, x_n, Tx_n), \\ &\quad S(x_{n-1}, x_{n-1}, Tx_n), S(x_n, x_n, Tx_{n-1})\Big) \\ &= \phi\Big(S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ &\quad S(x_{n-1}, x_{n-1}, x_{n+1}), S(x_n, x_n, x_n)\Big) \\ &= \phi\Big(S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ &\quad S(x_{n-1}, x_{n-1}, x_{n+1}), 0\Big) \\ &\leq \phi\Big(S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ &\quad 2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1}), 0\Big). \end{aligned}$$

Since ϕ *satisfies the condition* (A_1)*, so there exists* $k \in [0, 1)$ *such that*

$$S(x_n, x_n, x_{n+1}) \leq k\, S(x_{n-1}, x_{n-1}, x_n)$$

or

$$2\big(x_n - x_{n+1})\big) \leq k\,.2\big(x_{n-1} - x_n\big)$$

or

$$\Big(\frac{1}{2^n} - \frac{1}{2^{n+1}}\Big)\big) \leq k\,\Big(\frac{1}{2^{n-1}} - \frac{1}{2^n}\Big)$$

or

$$k \geq \frac{1}{2}.$$

If we take $0 < k < 1$*, then inequality (2.1) is satisfied. Thus all the conditions of Theorem 1 are satisfied. Hence by Theorem 1,* T *has a unique fixed point. Here, note that* $'0'$ *is the unique fixed point of* T*.*

(2) Let $\{y_n\} = \{\frac{1}{3^n}\}$ *be a sequence in* X *converging to the fixed point* $z = 0$*, then we have to show that* $Ty_n \to z$ *as* $n \to \infty$*, that is,* T *is continuous at the fixed point of* T*, we have*

$$\lim_{n\to\infty} Ty_n = T(\lim_{n\to\infty} y_n) = T(0) = 0 = z.$$

That is,

$$Ty_n \to z \text{ as } n \to \infty.$$

Thus, T is continuous at the fixed point of T.

Example 9. *Let $E = \mathbb{R}^2$, the Euclidean plane, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ a normal cone in E and $X = \mathbb{R}$. Then the function $S\colon X^3 \to E$ defined by $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in X$. Then (X, S) is a cone S-metric space. Now, we consider the mapping $T\colon X \to X$ by $T(x) = \frac{x}{3}$. Then*

$$\begin{aligned} S(Tx, Tx, Ty) &= |Tx - Ty| + |Tx - Ty| \\ &= 2|Tx - Ty| = 2\Big|\Big(\frac{x}{3}\Big) - \Big(\frac{y}{3}\Big)\Big| \\ &= \frac{2}{3}|x - y| \\ &= \frac{1}{3}\big(2|x - y|\big) \\ &\leq \frac{1}{2}\big(2|x - y|\big) \\ &= h\, S(x, x, y) \end{aligned}$$

where $h = \frac{1}{2} < 1$. Thus T satisfies all the conditions of Corollary 1 and clearly $0 \in X$ is the unique fixed point of T.

3 Conclusion

In this paper, we establish some fixed point theorems using implicit relation in the framework of complete cone S-metric spaces. Our results extend, unify and generalize several results from the existing literature. Especially, they extend the corresponding results of Sedghi and Dung [24] from complete S-metric spaces to the setting of complete cone S-metric spaces. However, these results have vast potential in solving various nonlinear problems in functional analysis, differential and integral equations, computer science and engineering.

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