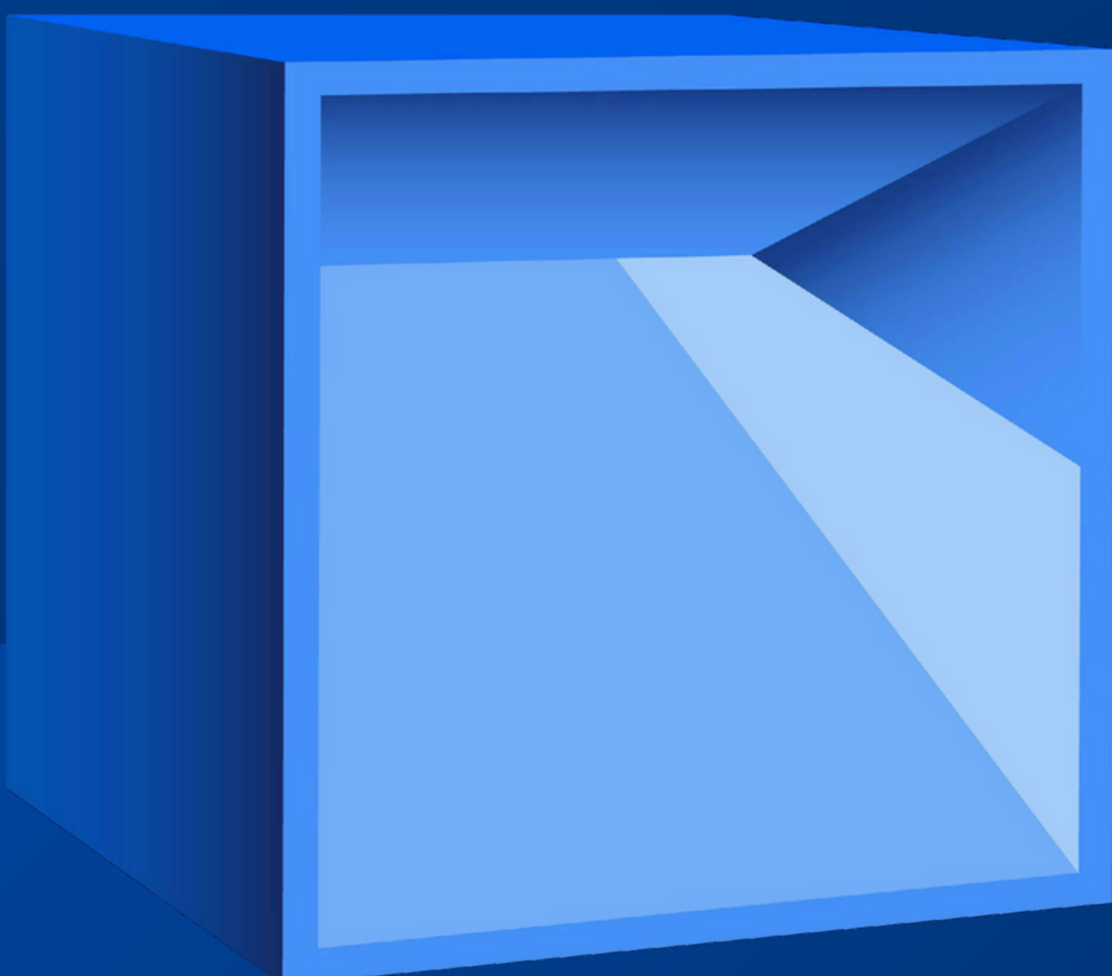
ISSN 0719-0646
ONLINE VERSION

UNIVERSIDAD
DE LA FRONTERA

VOLUME 23 · ISSUE 3
2021

Cubo
A Mathematical Journal



Departamento de Matemática y Estadística
Facultad de Ingeniería y Ciencias
Temuco - Chile

www.cubo.ufro.cl

CUBO
A Mathematical Journal

CUBO, A Mathematical Journal, is a scientific journal founded in 1985, and published by the Department of Mathematics and Statistics of the Universidad de La Frontera, Temuco, Chile. CUBO appears in three issues per year and is indexed in DOAJ, Latindex, MathSciNet, MIAR, REDIB, SciELO-Chile, Scopus and zbMATH. The journal publishes original results of research papers, preferably not more than 20 pages, which contain substantial results in all areas of pure and applied mathematics.

EDITORIAL BOARD

CUBO
A MATHEMATICAL JOURNAL
Universidad de La Frontera
Volume 23/N$^{\underline{\text{o}}}$3 – DECEMBER 2021

SUMMARY

– **Non-algebraic limit cycles in Holling type III zooplankton-phytoplankton models** 343
Homero G. Díaz-Marín and Osvaldo Osuna

– **Basic asymptotic estimates for powers of Wallis' ratios** 357
Vito Lampret

– **The structure of extended function groups** 369
Rubén A. Hidalgo

– **Entropy solution for a nonlinear parabolic problem with homogeneous Neumann boundary condition involving variable exponents** 385
U. Traoré

– **Independent partial domination** 411
L. Philo Nithya and J. Varghese Kureethara

– **Foundations of generalized Prabhakar-Hilfer fractional calculus with applications** 423
George A. Anastassiou

– **Existence and uniqueness of solutions to discrete, third-order three-point boundary value problems** 441
S. S. Almuthaybiri, Jagan Mohan Jonnalagadda and C. C. Tisdell

– **Some integral inequalities related to Wirtinger's result for p-norms** 457
S. S. Dragomir

– **On the periodic solutions for some retarded partial differential equations by the use of semi-Fredholm operators** 469
Abdelhai Elazzouzi, Khalil Ezzinbi and Mohammed Kriche

– **Extension of Exton's hypergeometric function K_{16}** 489
Ahmed Ali Atash and Maisoon Ahmed Kulib

Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 03, pp. 343–355, December 2021
DOI: 10.4067/S0719-06462021000300343

Non-algebraic limit cycles in Holling type III zooplankton-phytoplankton models

Homero G. Díaz-Marín[1]
Osvaldo Osuna[2]

[1] *Facultad de Ciencias Físico-Matemáticas, Universidad Michoacana, Edif. Alfa, Ciudad Universitaria, C.P. 58040, Morelia, Michoacán, México.*
homero.diaz@umich.mx

[2] *Instituto de Física y Matemáticas, Universidad Michoacana, Edif. C-3, Ciudad Universitaria, C.P. 58040, Morelia, Michoacán, México.*
osvaldo.osuna@umich.mx

ABSTRACT

We prove that for certain polynomial differential equations in the plane arising from predator-prey type III models with generalized rational functional response, any algebraic solution should be a rational function. As a consequence, limit cycles, which are unique for these dynamical systems, are necessarily trascendental ovals. We exemplify these findings by showing a numerical simulation within a system arising from zooplankton-phytoplankton dynamics.

RESUMEN

Probamos que para ciertas ecuaciones diferenciales polinomiales en el plano que aparecen a partir de modelos predador-presa de tipo III con respuesta funcional racional generalizada, toda solución algebraica debe ser una función racional. Como consecuencia, los ciclos límite, que son únicos para estos sistemas dinámicos, son necesariamente óvalos trascendentes. Ejemplificamos estos resultados mostrando una simulación numérica para un sistema que aparece en la dinámica de zooplancton-fitoplancton.

Keywords and Phrases: Predator-prey models, functional-response, Puiseux series, Newton polygon, limit cycles, invariant algebraic curve.

2020 AMS Mathematics Subject Classification: 34C25, 34M25, 34M35, 37N25, 92D25.

Accepted: 07 July, 2021
Received: 08 January, 2021

1 Introduction

We consider predator-prey model

$$\begin{aligned} \dot{u} &= ru\left(1 - \frac{u}{K}\right) - vp(u), \\ \dot{v} &= v\left(-D + \gamma p(u)\right), \end{aligned} \tag{1.1}$$

with functional response proposed in [13]

$$p(u) = mu^n/(a + u^n).$$

where the parameters are: the maximal feeding rate m; an affinity constant a, related to handling times, capture efficiencies, etc.; and the number of encounters $n \geq 1$ a predator must have with a prey item before becoming maximally efficient at utilizing the prey item as a resource. According to [13], this last parameter is derived from an analogy with Michaelis-Menten equation for enzymatic kinetics. Here n measures the amount of 'learning' exhibited by the predator. For $n > 1$, this functional response has Holling type III, while for $n = 1$ it has Holling type II, that is why this functional is also called *generalized functional response*. Increasing the attack exponent $1 < n < 2$ introduces the stability of simple consumer-resource population models, theoretical findings reveal that this increases biodiversity, see [14] and references therein. By fitting parameters, it is shown that $n \geq 2$ appear in certain models in ecology, where predator free-space is a component of the habitat structure, see [1]. Other theoretical models of biological relevance consider the specific attack exponent $n = 2$, see [15, 18].

For $1 < n < 2$ existence and uniqueness of limit cycles for predator-prey system (1.1) is proved in [16]. Existence and uniqueness for $0 \leq n \leq 1$, $n \geq 2$ also holds true under certain conditions, see [17]. Along this work we consider only integer values $n \geq 2$.

Existence of non-algebraic limit cycles for the Lotka-Volterra model were first exhibited by [12]. Since then, existence of trascendental ovals as limit cycles in system generalizing Lotka-Voltera models have been proved, see for instance [9, 5, 6]. Motivated by these results we explore this question for generalized functional responses.

Our main result is contained in Theorem 2.1 which asserts that limit cycles can not be algebraic ovals in the case of Holing type III predator-prey models. The proof uses Puiseux series at infinity in the variable x. We estimate the number of branches of solutions given by the Puiseux series. We perform calculation of the leading term and prove that there exists at most one determination or branch of such series. To see this we show how each coefficient c_n is completely specified by the parameters of the system. Thus, we conclude that any invariant algebraic curve must have at most degree one in y. Thus any algebraic invariant curve $y(x)$, should be a rational function.

For related works which also apply formal and Puiseux series to planar polynomial systems see [4, 7, 8].

2 Rational functions as invariant algebraic curves

Under a suitable change of variables, $x = u$, $y = -v/m$, and time reparametrization, $\frac{ds}{dt} = 1/(a + u^2)$, system (1.1) becomes

$$\begin{aligned} \dot{x} &= x\left((r - 1/K)a + (r - 1/K)x^n + x^{n-1}y\right), \\ \dot{y} &= y\left(-D + (\gamma m - D)x^n\right). \end{aligned} \tag{2.1}$$

Thus we study the algebraic system:

$$\begin{aligned} \dot{x} &= x\left(a_0 + a_n x^n + x^{n-1}y\right), \\ \dot{y} &= y\left(b_0 + b_n x^n\right), \quad a_n \neq b_n. \end{aligned} \tag{2.2}$$

Notice that the axes $x = 0$, $y = 0$ are algebraic solutions of (2.1).

Take the ODE defined by system (2.1) in the complex domain

$$\frac{dy}{dx} = \frac{y\left(b_0 + b_n x^n\right)}{x\left(a_0 + a_n x^n + x^{n-1}y\right)}. \tag{2.3}$$

Solutions are Riemann surfaces immersed in $\overline{\mathbb{C}}_x \times \overline{\mathbb{C}}_y$, where $\mathbb{C}_x \simeq \mathbb{C}$ and poles of solutions correspond to values $y = \infty$ in the compactification $\overline{\mathbb{C}}_y = \mathbb{C}_y \cup \{\infty\} \simeq \mathbb{CP}^1$.

If we ask for the existence of algebraic solutions $F(x, y) = 0$ for $F \in \mathbb{C}[x, y]$, of the dynamical system (2.1). Then, such algebraic curve should be rational.

Theorem 2.1. *Suppose that the following conditions hold,*

$$a_0 \neq b_0, \quad a_n \neq b_n. \tag{2.4}$$

If there exists an invariant algebraic curve $F(x, y) = 0$ of equation (2.3) with $x, y \nmid F(x, y)$, then $\deg_y F = 1$. Therefore, any algebraic (possibly multivalued) solution should also be a rational (univalued) solution, $y = \phi(x)$, provided we exclude the trivial solution, $y(x) \equiv 0$.

The following claim becomes of interest.

Corollary 2.2. *There can not exist algebraic limit cycles of the dynamical system (2.1) as a real vector field in $\mathbb{R}^2$, whenever conditions (2.4) hold true.*

For the proof of Theorem 2.1 we consider the Newton-Puiseux algorithm to describe explicitly the nature of solutions at the infinites $x = \infty$ and $y = \infty$. For further explanation of the Newton-Puiseux method for ODE, see [2, 10, 11]. The crucial step of the proof is to apply the following result.

Theorem 2.3 (Theorem 1.4 in [3])**.** *Let $G(z, w) = 0$ be an invariant algebraic curve, $\partial_w G \neq 0$ of the polynomial ODE*

$$P(z, w)\frac{dw}{dz} - Q(z, w) = 0. \tag{2.5}$$

Then $\deg_w G$ *is at most the number of Puiseux series*

$$w(z) = c_0 z^{\mu_0} + \sum_{l=1}^{\infty} c_l z^{\frac{l}{m_0}+\mu_0}, \tag{2.6}$$

solving (2.5)*, whenever the number of these series is finite. Here* $\mu_0 = l_0/m_0$ *with* m_0, l_0 *relatively prime integers* $m_0 \geq 0$*.*

Proof of Theorem 2.1. We proceed analyzing poles and algebraic branch points according to Painlevé methodology, see [10, 11]. Notice that under the blow-up change of coordinates $\xi = \frac{1}{x}$, equation (2.1) yields an equation at $x = \infty$ corresponding to $\xi = 0$

$$\frac{dy}{d\xi} = -\frac{y(a_0\xi^n + a_n)}{\xi(b_0\xi^n + b_n + \xi y)}, \tag{2.7}$$

At infinity the trivial solution $y \equiv 0$ yields a trivial solution which tends to $\xi = 0$. To find an expansion of non-trivial solutions along $\xi = 0$, with $\xi = 1/x$, in equation (2.7), we adopt the following Puiseux series expansion:

$$y(\xi) = c_0\xi^{\mu_0} + \sum_{l=1}^{\infty} c_l \xi^{\frac{l}{m_0}+\mu_0}, \tag{2.8}$$

where $\mu_0 = l_0/m_0$ and $-1/\mu_0$ is one of many possible slopes of the corresponding Newton polygon, and l_0, m_0 are relatively prime integers. For equation (2.7) the Newton polygon is a right-angled triangle whose only oblique side is the hypothenuse, see Fig. 1.

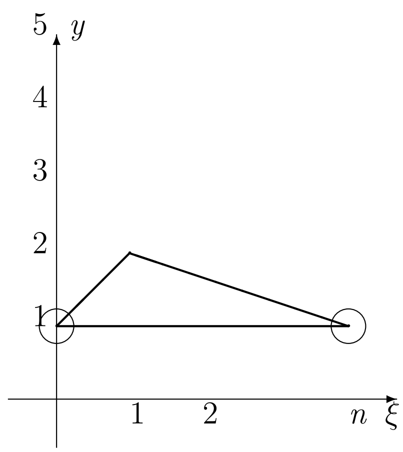


Figure 1: Newton polygon associated to the ODE (2.7) and used to calculate μ_0. Circled vertices correspond to monomials appearing in By' within the expression $A(\xi, y) + B(\xi, y)\frac{dy}{d\xi} = 0$.

Therefore, the only slope to consider is $-1/\mu_0 = 1$. Accordingly, $\mu_0 = -1$ and $c_0(b_n - a_n) - c_0^2 = 0$, with two possible roots: $c_0 = 0, b_n - a_n \in \mathbb{C}$. If we make a direct substitution $c_0 = 0$ of the Laurent expansion, $\sum_{l=0}^{\infty} c_l \xi^{-1+l}$. This yields the trivial solution, $y \equiv 0$. We claim that the remaining value,

$$c_0 = b_n - a_n \neq 0 \tag{2.9}$$

gives rise to a unique Laurent series of a simple pole at $x = \infty$, and therefore to just one branch of the Puiseux series. Indeed, under substitution

$$\xi_1 = \xi,\ y = c_0\xi_1^{-1} + y_1,$$

we obtain a Newton polygon for

$$(a_0\xi_1^n + b_n + \xi_1 y_1)\xi_1 \frac{dy_1}{d\xi_1} + (b_0\xi_1^n + a_n)y_1 + (a_n - b_n)(a_0 - b_0)\xi_1^{n-1} = 0 \tag{2.10}$$

which has two possible slopes and corresponding values $\mu_1 = 1, -1/(n-1)$. See Fig. 2.

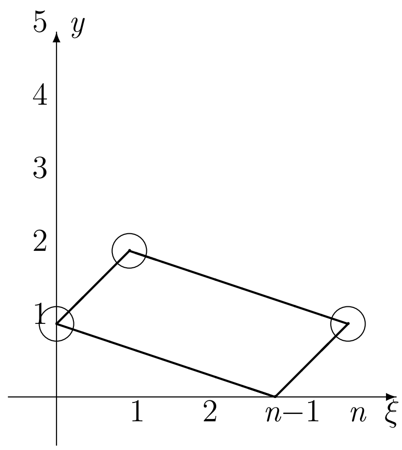


Figure 2: Newton polygon associated to the ODE (2.10) and used to calculate μ_1.

According to the algorithm given in [2], for positive slope $-1/\mu_1$, we choose as *principal side*, $\mu_1 = n - 1$. We have

$$c_n = \frac{(a_n - b_n)(a_0 - b_0)}{(1-n)b_n - a_n}. \tag{2.11}$$

In the following step, we have a principal side with $\mu_2 = 2n - 1$. See the corresponding Newton polygon used to calculate μ_2 in Fig. 3.

This determines c_{2n}. Therefore, there is a unique determination for the Puiseux-Laurent series:

$$y = c_0\xi^{-1} + \sum_{k=1}^{\infty} c_{kn}\xi^{kn-1} + \dots. \tag{2.12}$$

By direct substitution of the Puiseux-Laurent series in eq. (2.7) we can also verify that the middle coefficients vanish, i.e. for each $k = 0, 1, 2, \dots$, we have

$$c_l = 0, \quad \forall l = kn + 1, \dots, (k+1)n - 1.$$

Theorem 2.3 implies that $\deg_y F \leq 1$.

Thus, under the hypothesis of Theorem 2.1 we conclude that $y = \phi(x)$ is a rational function which cannot contain an algebraic limit cycle because of the uniqueness of its determination with respect to x. □

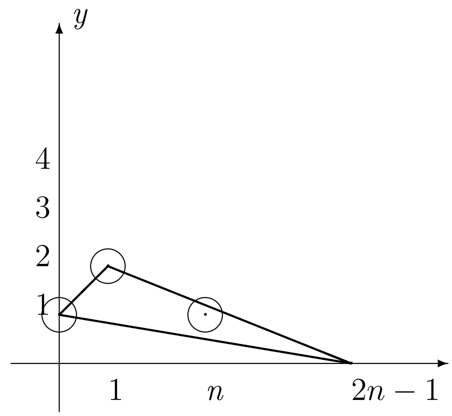


Figure 3: Newton polygon used to calculate μ_2.

Remark 2.4. *We have chosen Puiseux-Laurent series of the form $y = y(x)$ because there is a recognizable pattern in the successive Newton polygons, namely triangles with a moving low vertex. This yields a unique side with a unique slope. Therefore a unique μ_k yields a unique linear relation that allows us to compute all the coefficients c_k.*

3 On the degree with respect to x

We may ask whether the degree in x for an invariant curve can be estimated with the same methods. Notice that expression (2.12) suggests that $\deg_x F = nk$ for some $k \in \mathbb{N}$. We illustrate the difficulties to calculate an upper bound for $\deg_x F$ using the same techniques by considering $n = 3$.

If we take, $x = x(y)$ at $y = \infty$, then we may take the coordinate change $y = \frac{1}{\eta}$. Thus system (2.1) becomes

$$\frac{dx}{d\eta} = \frac{a_0 x\eta + a_n x^{n+1}\eta - x^{n-1}}{\eta^2(b_0 + b_n x^n \eta^2)}$$

The corresponding Newton polygon is shown in Fig. 4.

There are three posible cases for Puiseux-Laurent series

$$x(\eta) = c_0\eta^{\mu_0} + \sum_{l=1}^{\infty} c_l \eta^{\frac{l}{m_0}+\mu_0},$$

corresponding to slopes $-1/\mu_0$ equal to $1, \infty, -(n-1)$ which yield μ_0 equal to $-1, 0, \frac{1}{n-1}$. No infinite values $c_0 \in \mathbb{R}$ arise in each case. This can be verified as follows:

(1) Case $\mu_0 = -1$. Under substitution

$$\eta_1 = \eta, \quad x = c_0\eta_1^{\mu_0} + x_1,$$

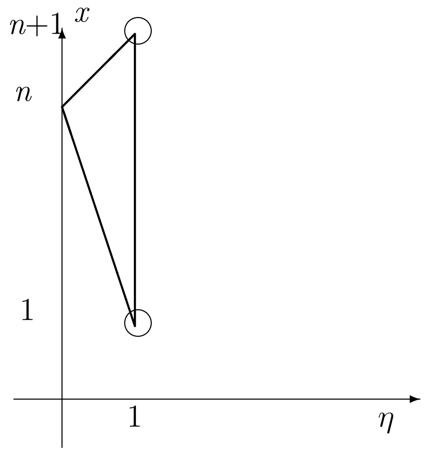


Figure 4: Newton polygon for $x(y)$ at infinity used to calculate $\mu_0 = \frac{1}{n-1}$.

we regard the least degree coefficient. There exists a unique Puiseux series where c_0 is determined by a linear relation

$$c_0^n(c_0(b_n - a_n) - 1) = 0,$$

therefore there exists just one branch.

(2) Case $\mu_0 = 0$. Corresponds to the trivial solution $x \equiv 0$ with $c_0 = 0$.

(3) Case $\mu_0 = \frac{1}{n-1}$. Puiseux-Laurent series arise as c_0 solve a relation:

$$a_0 c_0 + \frac{b_0 c_0}{n-1} + c_0^n = 0,$$

therefore there exists $n-1$ posible branches. Each branch corresponds to a $(n-1)-$th root

$$c_0 = \left(-a_0 - \frac{b_0}{n-1}\right)^{1/n-1}.$$

If we choose $\mu_0 = \frac{1}{n-1}$, then we get the ODE,

$$A(\eta_1, x_1) + B(\eta_1, x_1)\frac{dx_1}{d\eta_1},$$

with extended expression,

$$\begin{aligned}
&A_{(3,0)}\eta_1^3 && +A_{(1,1)}\eta_1 x_1 + A_{(5/2,1)}\eta_1^{5/2}x_1 \\
&+ A_{(1/2,2)}\eta_1^{1/2}x_1^2 + A_{(2,2)}\eta_1^2 x_1^2 && +A_{(0,3)}x_1^3 + A_{(3/2,1)}\eta_1^{3/2}x_1 \\
&+ A_{(1,4)}\eta_1 x_1^4 && \\
&+ \frac{dx_1}{d\eta_1} \times [B_{(1,1)}\eta_1^2 + B_{(5/2,1)}\eta^{7/2} && +B_{(2,2)}\eta_1^3 x_1 \\
&+ B_{(3/2,3)}\eta_1^{5/2}x_1^2 + B_{(1,4)}\eta_1^2 x_1^4] && = 0.
\end{aligned}$$

whose Newton polygon is shown in Fig 5. Circled vertices correspond to monomials appearing in Bx_1' within the ODE.

Under the same assumptions of Theorem 2.1, if there exists an invariant algebraic curve $F(x, y) = 0$ of (2.3) with $x, y \nmid F(x, y)$, then $\deg_x F$ has upper bound at least $n - 1$, provided we exclude the trivial solution, $y(x) \equiv 0$. This would require

$$a_0 + \frac{b_0}{n-1} \neq 0. \tag{3.1}$$

We still can not conclude that $\deg_x F \leq n - 1$, since the proof of this fact would require a suitable description of successive Newton polygons, as well as an effective calculation of the number of branches of the corresponding Puiseux-Laurent series. Two main difficulties arise: On one hand these Newton polygons may follow a complex pattern. On the other hand, we may have several different relations defining general coefficients c_k, $k > 0$ requiring enough conditions so that there is a finite number of branches rather than a continuum.

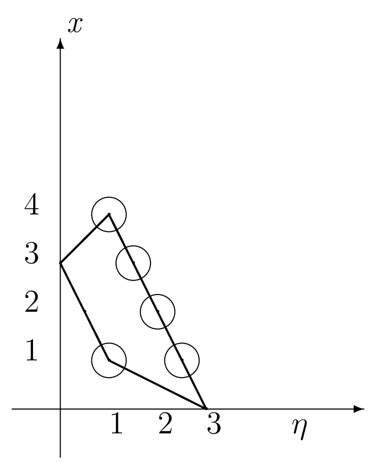


Figure 5: Newton polygon that determines $\mu_1 = 2$ with $n = 3$.

In the second step we have the possibility to choose either $\mu_1 = 2$ or $\mu_1 = 1/2$. If we choose $\mu_1 = 2$. Then, the corresponding relations arising from the least degree terms in the substitution

$$\eta_1 = \eta_2, \quad x_1 = c_1\eta_2^{\mu_1} + x_2$$

become,

$$4c_1(4a_0 - b_0) = 8a_3a_0^2 + 4a_0^2b_3 + 8a_3a_0b_0 + 4a_0b_3b_0 + 2a_3b_0^2 + b_3b_0^2,$$

Therefore, we would require the additional condition

$$4a_0 \neq b_0 \tag{3.2}$$

in order to be able to calculate c_1 and thus have a finite number of branches.

According to [2, Lemma 2], in order to have a finite number of branches, i.e. a *good side* of the Newton Polygon in its terminology, it is sufficient that the following conditions hold for the high and low vertices of such a side, (a, b), (a', b'), respectively:

(1) $B_{(a,b)} \neq 0$ and $\frac{A_{(a,b)}}{B_{(a,b)}} \notin \mathbb{Q}_{\geq \mu_1} = \{q \in \mathbb{Q} \,:\, q \geq \mu\}$,

(2) $A_{(a',b')} + \mu B_{(a',b')} \neq 0$,

where $A_{(a,b)}$, $B_{(a,b)}$ refer to the coefficients for the monomials in the equation $A + B\frac{dx_1}{d\eta_1} = 0$ associated to the vertex (a, b).

In our concrete example, in Fig. 5 we have chosen $\mu_1 = 2$ because it corresponds to the slope $-1/\mu_1$ of the unique good side which has vertices $(a, b) = (1, 1)$ and $(a', b') = (3, 0)$. Calculations yield

$$A_{(1,1)} = \frac{1}{8}(16a_0 + 12b_0), \quad B_{(1,1)} = -b_0,$$

$$A_{(3,0)} = -c_0^4\left(a_0 + \frac{b_0}{2}\right), \quad B_{(3,0)} = 0.$$

Recall that under our conventions, $B_{(3,0)} = 0$ is implied by the fact that the vertex $(3, 0)$ is not circled. Conditions for a good side which are sufficient to have a finite number of branches read as follows:

(1) $\frac{16a_0+12b_0}{8b_0} \notin \mathbb{Q}_{\geq \mu_1} = \{q \in \mathbb{Q} \,:\, q \geq \mu_1\}$. That is, either

$$\frac{a_0}{b_0} < \frac{1}{4} \quad \text{or} \quad \frac{a_0}{b_0} \geq \frac{1}{4} \quad \text{but} \quad \frac{a_0}{b_0} \notin \mathbb{Q}. \tag{3.3}$$

(2) $a_0 + \frac{b_0}{2} \neq 0$. We recover condition (3.1).

Notice that condition (3.3) is stronger than (3.2).

In the following step we choose $\mu_2 = 7/2$ by considering the slopes of the Newton polygon shown in Fig. 6 with the unique good side which has vertex (a', b') on the η_1-axis. Remark the increasing complexity of the polygon. Thus in the step $k \geq 2$, we can always choose the good side largest negative slope $-1/\mu_k$ with vertex $(a', b') = (a', 0)$ on the η_k-axis and vertex $(a, b) = (1, 1)$.

But even if in each step $k \geq 2$ we achieve linear relations to determine coefficients c_k, we still can not conclude that there is a finite number of determinations. An additional calculation needs to be done, namely to verify that no other side in the Newton polygon, yield a continuous indetermination $c_k \in \mathbb{C}$. Those additional sides are not good. To illustrate this difficulty suppose that we do not choose the unique good side in Fig. 5. Suppose that on the contrary we choose the side with vertices $(a, b) = (0, 3)$ and $(a', b') = (1, 1)$ which is not good. Further calculation yields

$$(a_0 + 2b_0)c_0 = 0.$$

Therefore, either $c_0 = 0$ or an indetermination $c_0 \in \mathbb{C}$ arises whenever the following relation does or does not hold:

$$a_0 + 2b_0 \neq 0. \tag{3.4}$$

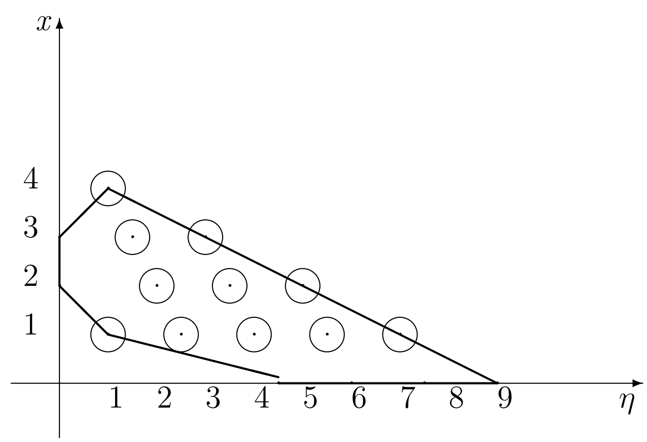


Figure 6: Newton polygon to determine $\mu_2 = 7/2$. Notice its increasing complexity with respect to Figs. 5 and 4.

Summarizing, if we follow the same strategy, to find an effective upper bound for $\deg_x F$ requires further calculations and a detailed and complete description of the conditions that allow a finite number of branches. We leave it for a future work.

4 Zooplankton-phytoplankton dynamics

We consider the dependence of a predator's (zooplankton) grazing rate on prey (phytoplankton) is taken as that of Holling type III response as in [15], instead of type II as in (2.67), (2.68) in [18]. Suppose that phytoplankton grows in logistic form whereas the zooplankton predation by fish is neglected. We then get the following system

$$\begin{aligned} \dot{u} &= u\,(1-u) - \frac{vu^2}{h+u^2}, \\ \dot{v} &= \frac{\gamma v u^2}{h+u^2} - \delta v, \end{aligned} \tag{4.1}$$

which yields a system similar to (2.1):

$$\begin{aligned} \dot{x} &= x\,(xy)\,, \\ \dot{y} &= y\left(-\delta + (\gamma-\delta)x^2\right). \end{aligned} \tag{4.2}$$

The criterion exposed in [17] for existence and uniqueness of a limit cycle adapted to system (1.1) with $m = 1$ and $n \geq 2$ states that

$$(nD - (n-2)\gamma) \cdot \sqrt[n]{\frac{aD}{\gamma - D}} < (pD - (p-1)\gamma)K,$$

which for (4.1) yields

$$2\delta\sqrt{\frac{h\delta}{\gamma - \delta}} < 2\delta - \gamma. \tag{4.3}$$

For the specific choice of parameters: $\delta = 0.25$, $\gamma = 0.35$, $h = 0.01$, condition (4.3) holds true. Therefore there exists a unique limit cycle. Indeed, numerical evidence for the existence of a limit cycle is given in Fig. 7.

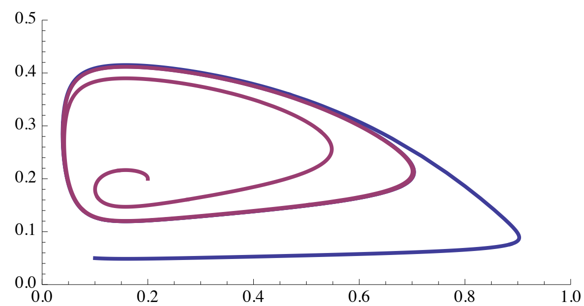


Figure 7: Solutions of system (4.1) with initial conditions $(0.1, 0.05)$ and $(0.2, 0.2)$ converge to a limit cycle.

On the other hand, due to Corollary 2.2 this limit cycle can not be an algebraic curve. Indeed, for an algebraic invariant curve $F(x, y) = 0$, there is only one branch of a simple pole at $x = \infty$. The corresponding Puiseux-Laurent series of this branch is

$$y = \sum_{k=0}^{\infty} c_{2k}\xi^{2k-1} = c_0\xi^{-1} + c_2\xi + c_4\xi^3 + \dots,$$

where $\xi = 1/x$. A straightforward calculation of the coefficients yields

$$c_0 = \gamma - \delta = 0.1,\ c_2 = \delta = 0.25,\ c_4 = 0 = c_{2k},\ k \geq 2.$$

Therefore, $y = c_0\xi^{-1} + c_2\xi$. Hence for an invariant algebraic curve, $\deg_y F = 1$, and $y = \phi(x)$, should be rational with $F(x, \phi(x)) = 0$. In this example $\deg_x F = 2$.

References

[1] D. Barrios-O'Neill, J. T. A. Dick, M. C. Emmerson, A. Ricciardi and H. J. MacIsaac, "Predator-free space, functional responses and biological invasions", Functional Ecology, vol. 29, no. 3, pp. 377–384, 2015.

[2] J. Cano, "An extension of the Newton-Puiseux polygon construction to give solutions of Pfaffian forms", Ann. Inst. Fourier (Grenoble), vol. 43, no. 1, pp. 125–142, 1993.

[3] M. V. Demina, "Novel algebraic aspects of Liouvillian integrability for two-dimensional polynomial dynamical systems". Phys. Lett. A, vol. 382, no. 20, pp. 353–1360, 2018.

[4] M. V. Demina, "Invariant algebraic curves for liénard dynamical systems revisited", Appl. Math. Lett., vol. 84, pp. 42–48, 2018.

[5] A. Ferragut and A. Gasull. "Non-algebraic oscillations for predator-prey models", Publ. Mat., vol. 58, suppl., pp. 195–207, 2014.

[6] J. Giné and M. Grau, "Coexistence of algebraic and non-algebraic limit cycles, explicitly given, using Riccati equations", Nonlinearity, vol. 19, no. 8, pp. 1939–1950, 2006.

[7] J. Giné and J. Llibre, "Strongly formal Weierstrass non-integrability for polynomial differential systems in $\mathbb{C}^2$", Electron. J. Qual. Theory Differ. Equ., no. 1, pp. 1–16, 2020.

[8] J. Giné and J. Llibre, "Formal Weierstrass nonintegrability criterion for some classes of polynomial differential systems in $\mathbb{C}^2$", Internat. J. Bifur. Chaos Appl. Sci. Engrg., vol. 30, no. 4, 7 pages, 2020.

[9] M. Hayashi, "On polynomial Liénard systems which have invariant algebraic curves", Funkcial. Ekvac., vol. 39, no. 3, pp. 403–408, 1996.

[10] E. Hille, *Ordinary Differential Equations in the Complex Domain*, Dover Publications, Inc., Mineola, NY, 1976.

[11] E. L. Ince, *Ordinary Differential Equations*, Dover Publications, New York, 1944.

[12] K. Odani, "The limit cycle of the van der Pol equation is not algebraic", J. Differential Equations, vol. 115, no. 1, pp. 146–152, 1995.

[13] L. A. Real, "The kinetics of functional response", The American Naturalist, vol. 111, no. 978, pp. 289–300, 1977.

[14] B. Rosenbaum and B. C. Rall, "Fitting functional responses: Direct parameter estimation by simulating differential equations", Methods in Ecology and Evolution, vol. 9, no. 10, pp. 2076–2090, 2018.

[15] V. A. Ryabchenko, M. J. R. Fasham, B. A. Kagan and E. E. Popova, “What causes short-term oscillations in ecosystem models of the ocean mixed layer?”, Journal of Marine Systems, vol. 13, no. 1, pp. 33–50, 1997.

[16] J. Sugie, “Uniqueness of limit cycles in a predator-prey system with Holling-type functional response”, Quart. Appl. Math., vol. 58, no. 3, pp. 577–590, 2000.

[17] J. Sugie, R. Kohno, and R. Miyazaki, “On a predator-prey system of Holling type”, Proc. Amer. Math. Soc., vol. 125, no. 7, pp. 2041–2050, 1997.

[18] R. K. Upadhyay and S. R. K. Iyengar, *Introduction to Mathematical Modeling and Chaotic Dynamics*, CRC Press, 2013.

Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 03, pp. 357–368, December 2021
DOI: 10.4067/S0719-06462021000300357

Basic asymptotic estimates for powers of Wallis' ratios

Vito Lampret[1]

[1] *University of Ljubljana, Ljubljana, 386 Slovenia, EU.*
vito.lampret@guest.arnes.si

ABSTRACT

For any $a \in \mathbb{R}$, for every $n \in \mathbb{N}$, and for n-th Wallis' ratio $w_n := \prod_{k=1}^{n} \frac{2k-1}{2k}$, the relative error $r_0(a,n) := \big(v_0(a,n) - w_n^a\big)/w_n^a$ of the approximation $w_n^a \approx v_0(a,n) := (\pi n)^{-a/2}$ is estimated as $\big|r_0(a,n)\big| < \frac{1}{4n}$. The improvement $w_n^a \approx v(a,n) := (\pi n)^{-a/2}\left(1 - \frac{a}{8n} + \frac{a^2}{128n^2}\right)$ is also studied.

RESUMEN

Para cualquier $a \in \mathbb{R}$, para todo $n \in \mathbb{N}$, y para el n-ésimo cociente de Wallis $w_n := \prod_{k=1}^{n} \frac{2k-1}{2k}$, el error relativo $r_0(a,n) := \big(v_0(a,n) - w_n^a\big)/w_n^a$ de la aproximación $w_n^a \approx v_0(a,n) := (\pi n)^{-a/2}$ se estima como $\big|r_0(a,n)\big| < \frac{1}{4n}$. También se estima la mejora $w_n^a \approx v(a,n) := (\pi n)^{-a/2}\left(1 - \frac{a}{8n} + \frac{a^2}{128n^2}\right)$.

Keywords and Phrases: approximation, asymptotic, estimate, inequality, power, Wallis' ratio.

2020 AMS Mathematics Subject Classification: 41A60, 65B10, 11Y60, 33E20, 33F05, 40A25.

Accepted: 09 July, 2021
Received: 16 January, 2021

1 Introduction

The sequence of Wallis[1] ratios

$$w_n := \prod_{k=1}^{n} \frac{2k-1}{2k} = \frac{(2n-1)!!}{(2n)!!} = 4^{-n}\binom{2n}{n} \tag{1.1}$$

is often encountered in pure and applied mathematics, in physics, and in several other exact sciences too. For example, the perimeter $P(a,b)$ of an ellipse having semi-axes of length a and $b \le a$ is given as

$$P(a,b) = 4a\int_0^{\pi/2} \sqrt{1-\varepsilon^2\sin^2(\tau)}\,\mathrm{d}\tau = 2a\pi\left(1-\sum_{k=1}^{\infty}\frac{w_k^2}{2k-1}\varepsilon^{2k}\right) \tag{1.2}$$

[20], where $\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$, the eccentricity of an ellipse.

Similarly, the period T of a simple pendulum, located in the gravitational field with the acceleration g and having the length L and the amplitude of the oscillation $\alpha \in (0,\pi)$, is determined by the formula

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{\mathrm{d}\tau}{\sqrt{1-\varepsilon^2\sin^2(\tau)}} = 2\pi\sqrt{\frac{L}{g}}\left(1+\sum_{k=1}^{\infty}w_k^2\varepsilon^{2k}\right) \tag{1.3}$$

[21, p. 26], where $\varepsilon = \sin(\alpha/2)$. Not only in mechanics, but also in other parts of physics, the Wallis ratio has several interesting roles, see for example [9] and [12].

In mathematics, the sequence of the Landau constants G_n, important in the theory of analytic functions, see [1], is also defined by the Wallis ratios as

$$G_n := \sum_{k=1}^{n} w_k^2 \qquad (n \in \mathbb{N}). \tag{1.4}$$

The Wallis ratio attracts mathematicians also because of its direct connections with Catalan numbers $c_n := \frac{1}{n+1}\binom{2n}{n}$, also important objects for pure and applied mathematics [15, 29]. In fact, the Wallis ratio, i.e. the sequence $n \mapsto w_n$, was investigated by many researches, see for example the papers [2, 3, 4, 5, 6, 7, 8, 11, 14, 16, 22, 23, 26, 27, 28, 29, 31, 33].

In 2007 was presented [33] aesthetically pleasing double inequality

$$\frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}} < w_n \le \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+16}}, \tag{1.5}$$

true for all $n \in \mathbb{N}$.

In 2013 was demonstrated [10] the estimate

$$\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n} < w_n \le \frac{4}{3}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n}, \tag{1.6}$$

[1] John Wallis, 1616 – 1703

true for $n \geq 2$.

In 2015 was derived [11] the inequalities

$$\left(\frac{2}{3}\right)^{3/2}\left(1-\frac{1}{2n}\right)^{n+1/2}\left(n-\frac{3}{2}\right)^{-1/2} \leq w_n < \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^{n+1/2}\left(n-\frac{3}{2}\right)^{-1/2}, \tag{1.7}$$

valid for $n \geq 2$. At the same time, in [28, Theorems 4.2 and 5.2] were presented the estimates

$$w_n > \sqrt{\frac{e}{\pi n}}\left(1-\frac{1}{2n}\right)^{n} \exp\left(\frac{1}{24n^2}+\frac{1}{48n^3}+\frac{1}{160n^4}+\frac{1}{960n^5}\right) \tag{1.8}$$

$$w_n > \sqrt{\frac{e}{\pi n}}\left(1-\frac{1}{2(n+1/3)}\right)^{n+1/3} \tag{1.9}$$

and

$$w_n < \sqrt{\frac{e}{\pi n}}\left(1-\frac{1}{2(n+1/3)}\right)^{n+1/3} \exp\left(\frac{1}{144n^3}\right), \tag{1.10}$$

all true for $n \geq 1$.

In the mentioned formulas for the perimeter of an ellipse and the period of a simple pendulum, as well as for the Landau sequence, see (1.2)–(1.4), we met the second powers of the Wallis ratios. This fact initiated our desire to approximate any power of the Wallis ratio. But, all the inequalities (1.5)–(1.10) are less suitable for estimating the power w_n^a for $a \in \mathbb{R}$. Fortunately, the approximation formula for the Wallis ratio, presented in [19], is more convenient for this task. In this contribution we shall show the first two steps how to approximate simply and accurately the powers of the Wallis ratios having real exponents.

2 Basic discussion

The sequence of Wallis' ratios was estimated recently [19] as

$$w_n = \frac{1}{\sqrt{\pi\, n}} \exp\big(-\widetilde{s}_r(n)+\delta_r(n)\big) \qquad (n \in \mathbb{N}), \tag{2.1}$$

where

$$\widetilde{s}_r(n) = \sum_{i=1}^{r} \frac{(1-4^{-i})B_{2i}}{i(2i-1)n^{2i-1}} \qquad (n, r \in \mathbb{N}) \tag{2.2}$$

and, for any $n, r \in \mathbb{N}$, the error $\delta_r(n)$ is estimated as

$$-\frac{\big|B_{2r+2}\big|}{(r+1)(2r+1)n^{2r+1}} < (-1)^r \delta_r(n) < \frac{\big|B_{2r+2}\big|}{2(r+1)(2r+1)(2n)^{2r+1}}\,. \tag{2.3}$$

Here $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $\ldots$ are the Bernoulli numbers, defined by the identity $\frac{x}{e^x-1} \equiv \sum_{j=0}^{\infty} B_j \frac{x^j}{j!} \quad (\,|x| < 2\pi\,)$.

We obtain the basic approximation by using $r = 1$,

$$w_n^a = (\pi n)^{-a/2} \exp\big(-a\widetilde{s}_1(n) + a\delta_1(n)\big) \qquad (a \in \mathbb{R},\, n \in \mathbb{N}), \tag{2.4}$$

with, for $n \in \mathbb{N}$,

$$\widetilde{s}_1(n) := \frac{1}{8n} > 0 \tag{2.5}$$

and

$$-\frac{1}{180\,n^3} < -\frac{1}{2880\,n^3} < \delta_1(n) < \frac{1}{180\,n^3}\,. \tag{2.6}$$

Thus, due to (2.5),

$$a\widetilde{s}_1(n) = \frac{a}{8n} \qquad (a \in \mathbb{R},\, n \in \mathbb{N})\,. \tag{2.7}$$

Moreover, thanks to (2.5)–(2.6), we estimate, for $n \in \mathbb{N}$,

$$-\widetilde{s}_1(n) \pm \left|\delta_1(n)\right| \ge -\widetilde{s}_1(n) - \left|\delta_1(n)\right| > -\frac{1}{8n} - \frac{1}{180n^3} > -\frac{1}{7n} \tag{2.8}$$

and

$$-\widetilde{s}_1(n) \pm \left|\delta_1(n)\right| \le -\widetilde{s}_1(n) + \left|\delta_1(n)\right| < -\frac{1}{8n} + \frac{1}{180n^3} < -\frac{1}{9n}\,. \tag{2.9}$$

Therefore, $-\frac{a}{7n} < a\big(-\widetilde{s}_1(n) \pm \delta_1(n)\big) < -\frac{a}{9n}$, for $a > 0$ and $-\frac{a}{7n} > a\big(-\widetilde{s}_1(n) \pm \delta_1(n)\big) > -\frac{a}{9n}$, for $a < 0$. Thus,

$$\min\left\{-\tfrac{a}{7n}, -\tfrac{a}{9n}\right\} < a\big(-\widetilde{s}_1(n) \pm \left|\delta_1(n)\right|\big) < \max\left\{-\tfrac{a}{7n}, -\tfrac{a}{9n}\right\} \qquad (a \neq 0,\, n \in \mathbb{N}). \tag{2.10}$$

Hence, considering (2.4), together with the equality $\min(-S) = -\max(S)$, for every $S \subseteq \mathbb{R}$, we derive the following theorem.

Theorem 2.1. *For $a \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$, the following double inequality holds:*

$$(\pi n)^{-a/2} \exp\big(-\max\left\{\tfrac{a}{7n}, \tfrac{a}{9n}\right\}\big) < w_n^a < (\pi n)^{-a/2} \exp\big(-\min\left\{\tfrac{a}{7n}, \tfrac{a}{9n}\right\}\big). \tag{2.11}$$

Figure 1 shows[2] the graphs of the function $a \mapsto w_2^a$ and its approximation (dashed line) $a \mapsto (\pi \cdot 2)^{-a/2}$.

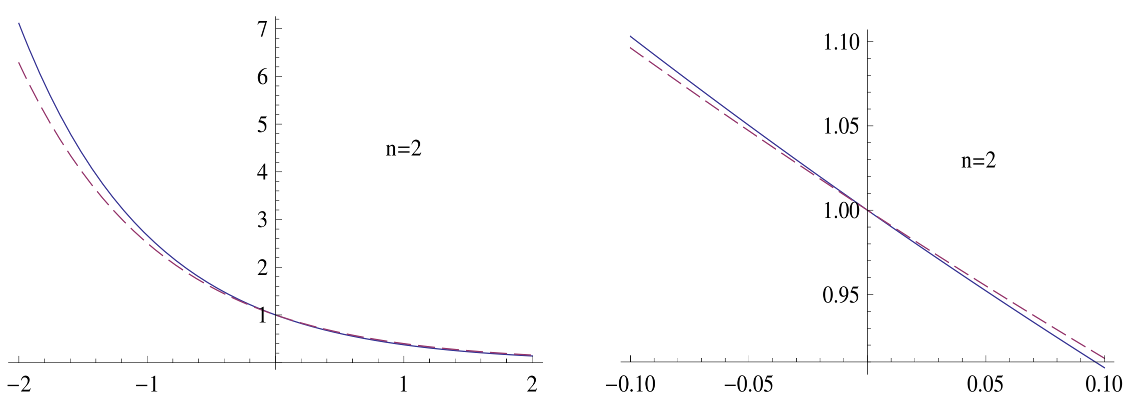


Figure 1: The graphs of the function $a \mapsto w_2^a$ and its approximation (dashed line) $a \mapsto (\pi \cdot 2)^{-a/2}$.

[2]All graphics and calculations in this paper are made using the Mathematica [32] computer system.

Example 2.2. *For any $n \in \mathbb{N}$ we have*

$$(\pi n)^{-50} \exp\left(-\tfrac{100}{7n}\right) < w_n^{100} < (\pi n)^{-50} \exp\left(-\tfrac{100}{9n}\right),$$
$$(\pi n)^{50} \exp\left(\tfrac{100}{9n}\right) < w_n^{-100} < (\pi n)^{50} \exp\left(\tfrac{100}{7n}\right).$$

From Theorem 2.1 there follows the next corollary.

Corollary 2.3. *For every $a \in \mathbb{R} \setminus \{0\}$ and for any positive integer $n \geq a$ we have*

$$w_n^a > \frac{6}{7}(\pi n)^{-a/2}. \tag{2.12}$$

Proof. For $k \geq a > 0$, using (2.11), we obtain[3] $w_k^a > (\pi k)^{-a/2} \exp\left(-\frac{a}{7k}\right) > (\pi k)^{-a/2}(1 - \frac{a}{7k}) \geq (\pi k)^{-a/2}(1 - \frac{1}{7}) = \frac{6}{7}(\pi k)^{-a/2}$. Furthermore, for $a < 0$, due to (2.11), we estimate $w_k^a > (\pi k)^{-a/2} \exp\left(-\frac{a}{9k}\right) = (\pi k)^{-a/2} \exp\left(\frac{|a|}{9k}\right) > (\pi k)^{-a/2} \cdot 1$. □

Lemma 2.4. *Let real numbers α, β, v and w satisfy the inequalities $\alpha\beta \geq 0$, $\beta \leq \frac{1}{2}$, $v > 0$ and $e^\alpha v < w < e^\beta v$. Then we have $|v - w| < \frac{3}{2} v \cdot \max\{|\alpha|, |\beta|\}$.*

Proof. Supposing that all conditions of Lemma 2.4 are satisfied, we have only two possibilities $\alpha < \beta \leq 0$ or $0 \leq \alpha < \beta$, together with the estimate

$$(e^\alpha - 1)v < w - v < (e^\beta - 1)v.$$

Therefore, in case $\alpha \leq 0$, we have $(1 - e^\alpha)v > v - w > (1 - e^\beta)v \geq 0$. Thus, see Footnote 3, $|v-w| < -\alpha = |\alpha|$. Additionally, using the first order Taylor's formula and the estimate $0 \leq \beta \leq \frac{1}{2}$, in case $\alpha \geq 0$, we obtain, $0 \leq (e^\alpha - 1)v < -(v - w) < (e^\beta - 1)v < \beta + \frac{1}{2}e^\beta \beta^2 \leq \beta + \frac{1}{2}e^{1/2}\frac{1}{2}\beta < \frac{3}{2}\beta$. Hence, in both cases we have $|v - w| < v \cdot \max\{|\alpha|, \frac{3}{2}|\beta|\}$. □

Corollary 2.5 (relative error)**.** *For every $a \in \mathbb{R} \setminus \{0\}$ and for any positive integer $n \geq a$ the relative error $r_0(a, n) := \big(w_n^a - v_0(a, n)\big)/w_n^a$ of the approximation $w_n^a \approx v_0(a, n) := (\pi n)^{-a/2}$ is roughly estimated as*

$$\big|r_0(a, n)\big| < \frac{1}{4n}.$$

Proof. Thanks to Theorem 2.1 and Lemma 2.4, using the notations $\alpha = -\max\left\{\frac{a}{7n}, \frac{a}{9n}\right\}$, $\beta = -\min\left\{\frac{a}{7n}, \frac{a}{9n}\right\}$, $v = v_0(a, n) = (\pi n)^{-a/2}$ and $w = w_n^a$, we obtain

$$\big|v_0(a, n) - w_n^a\big| < \frac{3}{2}(\pi n)^{-a/2} \cdot \max\Big\{ \big|\max\left\{\tfrac{a}{7n}, \tfrac{a}{9n}\right\}\big|, \big|\min\left\{\tfrac{a}{7n}, \tfrac{a}{9n}\right\}\big| \Big\}.$$

Thus, according to the identity $\max\big\{ |\max\{x, y\}|, |\min\{x, y\}| \big\} = \max\big\{|x|, |y|\big\}$, we get

$$\big|v_0(a, n) - w_n^a\big| < \frac{3}{2}(\pi n)^{-a/2} \cdot \frac{|a|}{7n}.$$

[3]considering the well-known estimate $e^x > 1 + x$, true for $x \in \mathbb{R} \setminus \{0\}$.

Hence, using Corollary 2.3,

$$\frac{\left|v_0(a,n)-w_n^a\right|}{w_n^a} < \frac{3}{2}(\pi n)^{-a/2}\frac{|a|}{7n}\cdot\frac{7}{6}(\pi n)^{a/2} = \frac{|a|}{4n}\,. \qquad \square$$

Figure 2 shows, on the left – the graph of the actual relative error function $a\mapsto r_0(a,n)$ and on the right – the graphs of the functions $a\mapsto r_0(a,n)$ and $a\mapsto \frac{|a|}{4\times 1000}$ (dashed line).

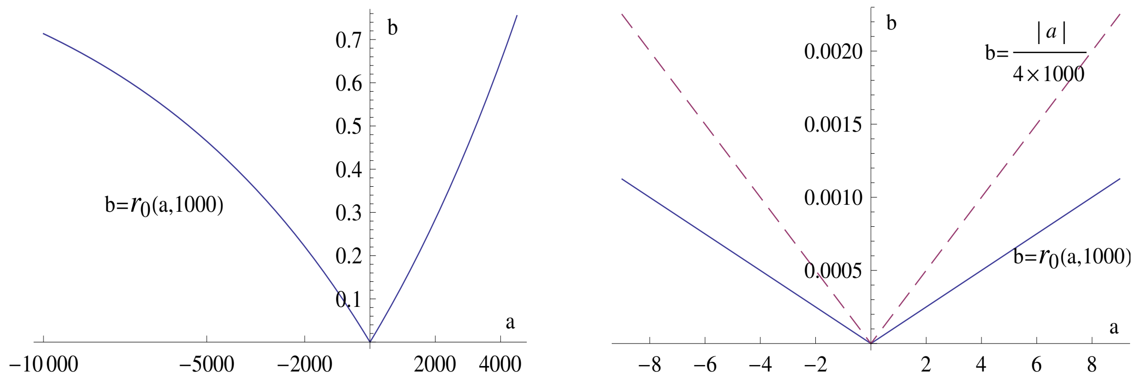


Figure 2: Left – the graph of the actual relative error function $a\mapsto r_0(a,1000)$; Right – the graphs of the actual relative error $a\mapsto r_0(a,1000)$ and its approximation (dashed line) $a\mapsto \frac{|a|}{4\times 1000}$.

3 Improvement

The relations (2.4)–(2.6) can be exploited more accurately to derive the next theorem.

Theorem 3.1. *For any $a\in\mathbb{R}$ and every integer $n\ge \frac{|a|}{8}$, we have*

$$w_n^a = v(a,n)+\varepsilon(a,n), \tag{3.1}$$

where

$$v(a,n) := (\pi n)^{-a/2}\left(1-\frac{a}{8n}+\frac{a^2}{128n^2}\right), \tag{3.2}$$

and the error $\varepsilon(a,n)$ is estimated as

$$\left|\varepsilon(a,n)\right| \le \varepsilon^*(a,n) := (\pi n)^{-a/2}\left[\frac{a^2}{100}+\frac{1}{18}\exp\left(-\min\left\{\frac{a}{7n},\frac{a}{9n}\right\}\right)\right]\frac{|a|}{10n^3} \tag{3.3}$$

$$\le (\pi n)^{-a/2}\left(\frac{a^2}{100}+\frac{1}{18}\exp\left(\frac{|a|}{7n}\right)\right)\frac{|a|}{10n^3}$$

$$\le \varepsilon^{**}(a,n) := (\pi n)^{-a/2}\left(a^2+\frac{35}{2}\right)\frac{|a|}{(10n)^3}\,. \tag{3.4}$$

Proof. Using the second order Taylor's formula, we have

$$\exp\left(-a\widetilde{s}_1(n)\right) = \exp\left(-\frac{a}{8n}\right) = 1-\frac{a}{8n}+\frac{1}{2}\left(-\frac{a}{8n}\right)^2+R_2(a,n) \tag{3.5}$$

with

$$R_2(a,n) = \frac{1}{6}\exp\Big(-\vartheta\cdot\frac{a}{8n}\Big)\Big(-\frac{a}{8n}\Big)^3, \qquad \text{for some } \vartheta = \vartheta(a,n)\in(0,1).$$

Therefore, for $a\in\mathbb{R}$ and $n \geq \frac{|a|}{8}$,

$$\big|R_2(a,n)\big| \leq \frac{1}{6}\exp\Big(\frac{|a|}{8n}\Big)\cdot\Big(\frac{|a|}{8n}\Big)^3 \leq \frac{e}{6}\cdot\frac{|a|^3}{512n^3} \leq \frac{|a|^3}{1000n^3}. \tag{3.6}$$

Similarly,

$$\exp\big(a\delta_1(n)\big) = 1 + \exp\big(\vartheta\cdot a\delta_1(n)\big)\cdot a\delta_1(n), \tag{3.7}$$

for some $\vartheta = \vartheta(a,n)\in(0,1)$.

Thanks to (3.7), (2.10) and (2.6), we estimate, using some $\theta = \theta(a,n)\in(0,1)$,

$$\begin{aligned}\Big|\underbrace{\exp\big(-a\widetilde{s}_1(n)+a\delta_1(n)\big)-\exp\big(-a\widetilde{s}_1(n)\big)}_{=\Delta(a,n)}\Big| &= \exp\big(-a\widetilde{s}_1(n)\big)\cdot\big|\exp\big(\theta\cdot a\delta_1(n)\big)\cdot a\delta_1(n)\big| \\ &\leq \exp\big(-a\widetilde{s}_1(n)\big)\cdot\exp\big(|a\delta_1(n)|\big)\cdot\big|a\delta_1(n)\big| \\ &= \exp\Big(a\big(-\widetilde{s}_1(n)\pm|\delta_1(n)|\big)\Big)|a|\big|\delta_1(n)\big| \\ &\underset{(2.6)}{\overset{(2.10)}{\leq}} \exp\Big(\max\Big\{-\frac{a}{7n},-\frac{a}{9n}\Big\}\Big)\cdot\frac{|a|}{180n^3}. \end{aligned} \tag{3.8}$$

Consequently, according to (2.4) and (3.5), we obtain

$$w_n^a \overset{(2.4)}{=} (\pi k)^{-a/2}\Big(\exp\big(-a\widetilde{s}_1(n)+a\delta_1(n)\big)\Big) \overset{(3.5)}{=} (\pi n)^{-a/2}\Big(\underbrace{1-\frac{a}{8n}+\frac{a^2}{128n^2}+R_2(a,n)}_{=\exp(-a\widetilde{s}_1(n))}+\Delta(a,n)\Big),$$

where, considering (3.6) and (3.8), for $a\in\mathbb{R}$ and $n\geq\frac{|a|}{8}$, we estimate the error $\varepsilon(a,n) := (\pi n)^{-a/2}\big(R_2(a,n)+\Delta(a,n)\big)$ as

$$\begin{aligned}\big|\varepsilon(a,n)\big| &\leq (\pi n)^{-a/2}\left[\frac{|a|^3}{1000n^3}+\exp\Big(-\min\Big\{\frac{a}{7n},\frac{a}{9n}\Big\}\Big)\frac{|a|}{180n^3}\right] \\ &= (\pi n)^{-a/2}\left[\frac{a^2}{100}+\frac{1}{18}\exp\Big(-\min\Big\{\frac{a}{7n},\frac{a}{9n}\Big\}\Big)\right]\frac{|a|}{10n^3} \\ &\leq (\pi n)^{-a/2}\left[\frac{a^2}{100}+\frac{1}{18}\exp\Big(\frac{|a|}{8n}\cdot\frac{8}{7}\Big)\right]\frac{|a|}{10n^3} \\ &\leq (\pi n)^{-a/2}\left(\frac{a^2}{100}+\frac{1}{18}\exp\Big(1\cdot\frac{8}{7}\Big)\right)\frac{|a|}{10n^3} \\ &\leq (\pi n)^{-a/2}\left(\frac{a^2}{100}+\frac{7}{40}\right)\frac{|a|}{10n^3}. \end{aligned}$$

□

Remark 3.2. *The sequence* $n\mapsto W_n := \frac{1}{2n+1}\Big(\prod_{k=1}^n\frac{2k}{2k-1}\Big)^2$*, called the Wallis sequence, is closely connected to the sequence of the Wallis ratios* w_n *by the identity* $W_n = w_n^{-2}/(2n+1)$*. So,* W_n *can be estimated easily using Theorem 3.1, e.g. its consequence* (3.14).

Remark 3.3. *According to Theorem 3.1, the constant* π *can be easily approximated using certain rational functions* $R_{\mp}(n)$*. For example, from* (3.14) *we get, for any* $n \in \mathbb{N}$,

$$\frac{1}{n}\left(w_n^{-2}-\frac{1}{5n^2}\right)\left(1+\frac{1}{4n}+\frac{1}{32n^2}\right)^{-1} < \pi < \frac{1}{n}\left(w_n^{-2}+\frac{1}{5n^2}\right)\left(1+\frac{1}{4n}+\frac{1}{32n^2}\right)^{-1}.$$

Directly from Theorem 3.1 and Corollary 2.3, from (3.4) and (2.12), we read the next corollary.

Corollary 3.4 (relative error)**.** *For every* $a \in \mathbb{R}$ *and for any positive integer* $n \geq |a|$ *the relative error of the approximation* $w_n^a \approx v(a,n)$,

$$r(a,n) := \frac{w_n^a - v(a,n)}{w_n^a}, \tag{3.9}$$

is a priori estimated as

$$\big|r(a,n)\big| \leq r^*(a,n) := \left(a^2+\frac{13}{2}\right)\frac{7|a|}{6(10n)^3}\,. \tag{3.10}$$

For any $a \in \mathbb{R}$ *and all integers* $n \geq |a|$ *the rough estimate* $r^*(a,n) < 8.2$ ‰ *holds true.*

Figure 3 shows the graphs of the actual relative error functions $a \mapsto r(a,n)$, for $n \in \{10,\ 100\}$.

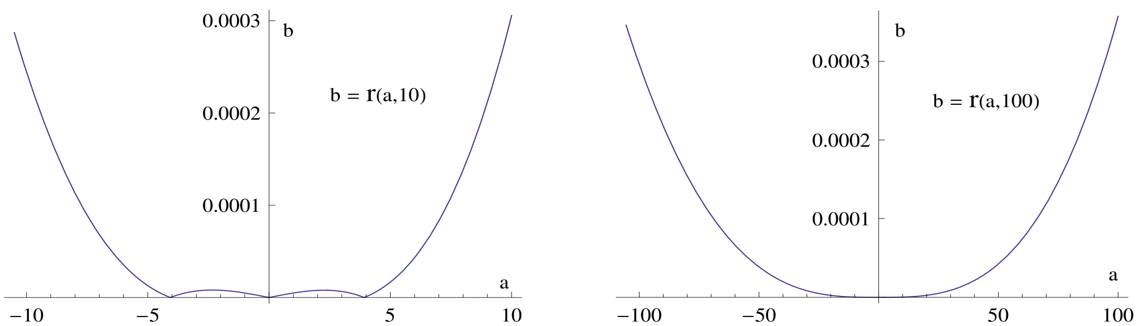


Figure 3: The graphs of the actual relative error functions $a \mapsto r(a,n)$ for $n \in \{10,\ 100\}$.

Figures 4–5 compare the actual relative error functions $a \mapsto r(a,n)$ and their approximations $a \mapsto r^*(a,n)$, for $n \in \{1,\ 3,\ 10,\ 100\}$.

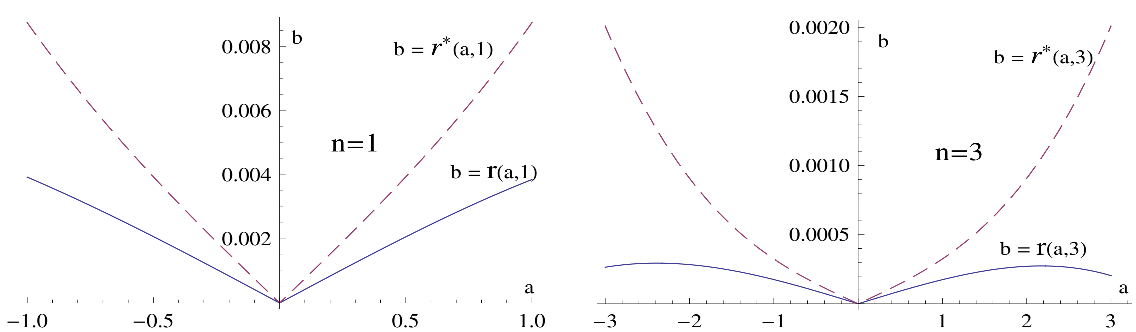


Figure 4: The graphs of the actual relative error functions $a \mapsto r(a,n)$ and their approximations $a \mapsto r^*(a,n)$, for $n \in \{1,\ 3\}$.

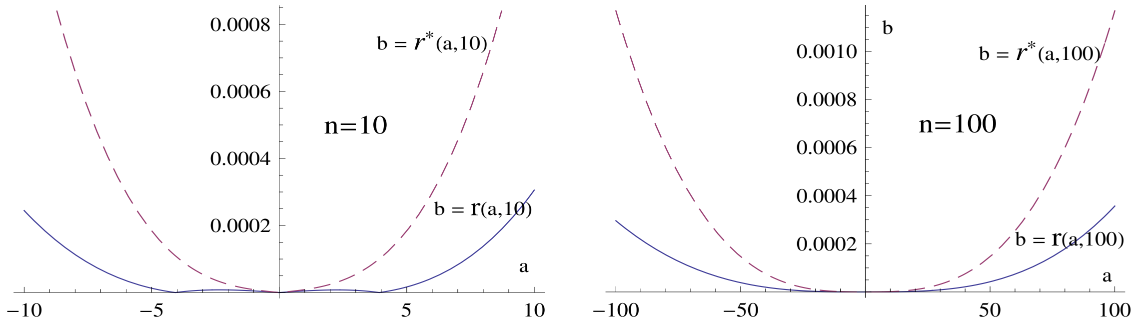


Figure 5: The graphs of the actual relative error functions $a \mapsto r(a,n)$ and their approximations $a \mapsto r^*(a,n)$, for $n \in \{10,\ 100\}$.

Using $a \in \{1, -1, 2, -2, \frac{1}{2}, \pi, -2\pi\}$ in Theorem 3.1, considering (3.1) and (3.4), we obtain several inequalities for Wallis' ratios, presented in the next corollary.

Corollary 3.5. *For every*[4] *positive integer n we have*

$$\frac{1}{\sqrt{\pi n}}\left(1-\tfrac{1}{8n}+\tfrac{1}{128n^2}\right)-\tfrac{1}{95\,n^{7/2}} < w_n < \frac{1}{\sqrt{\pi n}}\left(1-\tfrac{1}{8n}+\tfrac{1}{128n^2}\right)+\tfrac{1}{95\,n^{7/2}}\,, \tag{3.11}$$

$$\sqrt{\pi n}\left(1+\tfrac{1}{8n}+\tfrac{1}{128n^2}\right)-\tfrac{1}{30\,n^{5/2}} < \frac{1}{w_n} < \sqrt{\pi n}\left(1+\tfrac{1}{8n}+\tfrac{1}{128n^2}\right)+\tfrac{1}{30\,n^{5/2}}\,, \tag{3.12}$$

$$\frac{1}{\pi n}\left(1-\tfrac{1}{4n}+\tfrac{1}{32n^2}\right)-\tfrac{1}{73\,n^4} < w_n^2 < \frac{1}{\pi n}\left(1-\tfrac{1}{4n}+\tfrac{1}{32n^2}\right)+\tfrac{1}{73\,n^4}\,, \tag{3.13}$$

$$(\pi n)\left(1+\tfrac{1}{4n}+\tfrac{1}{32n^2}\right)-\tfrac{1}{7\,n^2} < \frac{1}{w_n^2} < (\pi n)\left(1+\tfrac{1}{4n}+\tfrac{1}{32n^2}\right)+\tfrac{1}{7\,n^2}\,, \tag{3.14}$$

$$\frac{1}{\sqrt[4]{\pi n}}\left(1-\tfrac{1}{16n}+\tfrac{1}{512n^2}\right)-\tfrac{1}{150n^{13/4}} < \sqrt{w_n} < \frac{1}{\sqrt[4]{\pi n}}\left(1-\tfrac{1}{16n}+\tfrac{1}{512n^2}\right)+\tfrac{1}{150n^{13/4}}\,, \tag{3.15}$$

$$\frac{1}{(\pi n)^{\pi/2}}\left(1-\tfrac{\pi}{8n}+\tfrac{\pi^2}{128n^2}\right)-\tfrac{1}{70n^{3+\pi/2}} < w_n^{\pi} < \frac{1}{(\pi n)^{\pi/2}}\left(1-\tfrac{\pi}{8n}+\tfrac{\pi^2}{128n^2}\right)+\tfrac{1}{70n^{3+\pi/2}}\,, \tag{3.16}$$

$$(\pi n)^{\pi}\left(1+\tfrac{\pi}{4n}+\tfrac{\pi^2}{32n^2}\right)-14\,n^{\pi-3} < w_n^{-2\pi} < (\pi n)^{\pi}\left(1+\tfrac{\pi}{4n}+\tfrac{\pi^2}{32n^2}\right)+14\,n^{\pi-3}. \tag{3.17}$$

Remark 3.6. *In case $a > 0$, the inequalities in Corollary 3.5 can be slightly improved using* (3.3) *instead of* (3.4)*. For example, due to* (3.3)*, we have, for $a \in \{1,\ 2\}$,*

$$|\varepsilon(1,n)| \le \varepsilon^*(1,n) = (\pi n)^{-1/2}\left(\tfrac{1}{100}+\tfrac{1}{18}\cdot 1\right)\tfrac{1}{10n^3} < \tfrac{1}{270n^{7/2}}$$

$$|\varepsilon(2,n)| \le \varepsilon^*(2,n) = (\pi n)^{-1}\left(\tfrac{1}{25}+\tfrac{1}{18}\cdot 1\right)\tfrac{2}{10n^3} < \tfrac{1}{164n^4}\,.$$

[4]For $1 \le n < |a|$ the inequalities are approved directly.

References

[1] H. Alzer, "Inequalities for the constants of Landau and Lebesgue", J. Comput. Appl. Math., vol. 139, no. 2, pp. 215–230, 2002.

[2] T. Burić and N. Elezović, "Bernoulli polynomials and asymptotic expansions of the quotient of gamma functions", J. Comput. Appl. Math., vol. 235, no. 11, pp. 3315–3331, 2011.

[3] T. Burić and N. Elezović, "New asymptotic expansions of the quotient of gamma functions", Integral Transforms and Spec. Funct., vol. 23, no. 5, pp. 355–368, 2012.

[4] C.-P. Chen and F. Qi, "The best bounds in Wallis' inequality", Proc. Amer. Math. Soc., vol. 133, no. 2, pp. 397–401, 2005.

[5] V. G. Cristea, "A direct approach for proving Wallis ratio estimates and an improvement of Zhang-Xu-Situ inequality", Studia Univ. Babeş-Bolyai Math., vol. 60, no. 2, pp. 201–209, 2015.

[6] S. Dumitrescu, "Estimates for the ratio of gamma functions using higher order roots", Stud. Univ. Babeş-Bolyai Math., vol. 60, pp. 173–181, 2015.

[7] N. Elezović, L. Lin and L. Vukšić, "Inequalities and asymptotic expansions of the Wallis sequence and the sum of the Wallis ratio", J. Math. Inequal., vol. 7, no. 4, pp. 679–695, 2013.

[8] N. Elezović, "Asymptotic expansions of gamma and related functions, binomial coefficient, inequalities and means", J. Math. Inequal., vol. 9, no. 4, pp. 1001–1054, 2015.

[9] T. Friedmann and C. R. Hagen, "Quantum mechanical derivation of the Wallis formula for π", J. Math. Phys., vol. 56, no. 11, 3 pages, 2015.

[10] S. Guo, J.-G. Xu and F. Qi, "Some exact constants for the approximation of the quantity in the Wallis' formula", J. Inequal. Appl., vol. 2013 , no. 67, 7 pages, 2013.

[11] S. Guo, Q. Feng, Y.-Q. Bi and Q.-M. Luo; "A sharp two-sided inequality for bounding the Wallis ratio", J. Inequal. Appl., vol. 2015, no. 43, 5 pages, 2015.

[12] P. Haggstrom, Quantum mechanical derivation of the Wallis formula for Pi, https://gotohaggstrom.com/QuantummechanicalderivationoftheWallisformulaforPi.pdf, 2020.

[13] M. D. Hirschhorn, "Comments on the paper: "Wallis sequence estimated through the Euler-Maclaurin formula: even from the Wallis product π could be computed fairly accurately" by V. Lampret", Austral. Math. Soc. Gaz., vol. 32, no. 3, pp. 194, 2005.

[14] D. K. Kazarinoff, "On Wallis' formula", Edinburgh Math. Notes, no. 40, pp. 19–21, 1956.

[15] T. Koshy, *Catalan numbers with applications*, New York: Oxford University Press, 2009.

[16] A. Laforgia and P. Natalini, "On the asymptotic expansion of a ratio of gamma functions", J. Math. Anal. Appl., vol. 389, no. 4, pp. 833–837, 2012.

[17] V. Lampret, "An asymptotic approximation of Wallis' sequence", Cent. Eur. J. Math., vol. 10, no. 2, pp. 775–787, 2012.

[18] V. Lampret, "Wallis' sequence estimated accurately using an alternating series", J. Number Theory, vol. 172, pp. 256–269, 2017.

[19] V. Lampret, "A simple asymptotic estimate of Wallis' ratio using Stirling's factorial formula", Bull. Malays. Math. Sci. Soc., vol. 42, no. 6, pp. 3213–3221, 2019.

[20] V. Lampret, "The perimeter of a flattened ellipse can be estimated accurately even from Maclaurin's series", Cubo, vol. 21, no. 2, pp. 51–64, 2019.

[21] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics: Mechanics*, 3ed, Oxford: Butterworth-Heinemann, 1986.

[22] C. Mortici, "Sharp inequalities and complete monotonicity for the Wallis ratio", Bull. Belg. Math. Math. Soc. Simon Stevin, vol. 17, no. 5, pp. 929–936, 2010.

[23] C. Mortici, "A new method for establishing and proving new bounds for the Wallis ratio", Math. Inequal. Appl., vol. 13, no. 4, pp. 803–815, 2010.

[24] C. Mortici, "Refinements of Gurland's formula for pi", Comput. Math. Appl., vol. 62, no. 6, pp. 2616–2620, 2011.

[25] C. Mortici, "Sharp bounds of the Landau constants", Math. Comp., vol. 80, no. 274, pp. 1011–1015, 2011.

[26] C. Mortici, "Completely monotone functions and the Wallis ratio", Appl. Math. Lett., vol. 25, no. 4, pp. 717–722, 2012.

[27] C. Mortici and V. G. Cristea, "Estimates for Wallis' ratio and related functions", Indian J. Pure Appl. Math., vol. 47, no. 3, pp. 437–447, 2016.

[28] F. Qi and C. Mortici, "Some best approximation formulas and the inequalities for the Wallis ratio", Appl. Math. Comput., vol. 253, pp. 363–368, 2015.

[29] F. Qi, "An improper integral, the beta function, the Wallis ratio, and the Catalan numbers", Probl. Anal. Issues Anal., vol. 7(25), no. 1, pp. 104–115, 2018.

[30] D. V. Slavić, "On inequalities for $\Gamma(x+1)/\Gamma(x+1/2)$", Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., no. 498–541, pp. 17–20, 1975.

[31] J.-S. Sun and C.-M. Qu, "Alternative proof of the best bounds of Wallis' inequality", Commun. Math. Anal., vol. 2, no. 1, pp. 23–27, 2007.

[32] S. Wolfram, *Mathematica*, version 7.0, Wolfram Research, Inc., 1988–2009.

[33] X.-M. Zhang, T. Q. Xu and L. B. Situ "Geometric convexity of a function involving gamma function and application to inequality theory", JIPAM. J. Inequal. Pure Appl. Math., vol. 8, no. 1, art. 17, 9 pages, 2007.

Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 03, pp. 369–384, December 2021
DOI: 10.4067/S0719-06462021000300369

The structure of extended function groups

R. A. Hidalgo[1]

[1] *Departamento de Matemática y Estadística, Universidad de La Frontera, Temuco, Chile.*
`ruben.hidalgo@ufrontera.cl`

ABSTRACT

Conformal (respectively, anticonformal) automorphisms of the Riemann sphere are provided by the Möbius (respectively, extended Möbius) transformations. A Kleinian group (respectively, an extended Kleinian group) is a discrete group of Möbius transformations (respectively, a discrete group of Möbius and extended Möbius transformations, necessarily containing extended ones).
A function group (respectively, an extended function group) is a finitely generated Kleinian group (respectively, a finitely generated extended Kleinian group) with an invariant connected component of its region of discontinuity.
A structural decomposition of function groups, in terms of the Klein-Maskit combination theorems, was provided by Maskit in the middle of the 70's. One should expect a similar decomposition structure for extended function groups, but it seems not to be stated in the existing literature. The aim of this paper is to state and provide a proof of such a decomposition structural picture.

RESUMEN

Los automorfismos conformes (respectivamente, anticonformes) de la esfera de Riemann son dados por las transformaciones de Möbius (respectivamente, Möbius extendidas). Un grupo Kleiniano (respectivamente, un grupo Kleiniano extendido) es un grupo discreto de transformaciones de Möbius (respectivamente, un grupo discreto de transformaciones de Möbius y transformaciones de Möbius extendidas, necesariamente conteniendo extendidas).
Un grupo función (respectivamente, un grupo función extendido) es un grupo Kleiniano finitamente generado (respectivamente, un grupo Kleiniano extendido finitamente generado) con una componente conexa invariante de su región de discontinuidad.
Una descomposición estructural de los grupos función, en términos de los teoremas de combinación de Klein-Maskit, fue dado por Maskit a mediados de los 70's. Se debiera esperar una estructura de descomposición similar para los grupos función extendidos, pero no parece estar enunciado en la literatura existente. El objetivo de este artículo es enunciar y dar una demostración de una tal descomposición estructural.

Keywords and Phrases: Kleinian groups, equivariant loop theorem.

2020 AMS Mathematics Subject Classification: 30F10, 30F40.

Accepted: 20 July, 2021
Received: 24 March, 2021

1 Introduction

The conformal (respectively, anticonformal) automorphisms of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ are provided by the Möbius (respectively, extended Möbius) transformations, that is, transformations of the form $T(z) = (az+b)/(cz+d)$ (respectively, $L(z) = (a\overline{z}+b)/(c\overline{z}+d)$) where $a, b, c, d \in \mathbb{C}$ are such that $ad - bc = 1$. The group of Möbius transformations $\mathbb{M}$ is isomorphic to the special projective linear group $\mathrm{PSL}_2(\mathbb{C})$ and the group of Möbius and extended Möbius transformations is $\widehat{\mathbb{M}} = \langle \mathbb{M}, J(z) = \overline{z} \rangle$.

A *Kleinian group* (respectively, an *extended Kleinian group*) is a discrete subgroup of $\mathbb{M}$ (respectively, a discrete subgroup of $\widehat{\mathbb{M}}$ necessarily *containing extended Möbius transformations*). The *region of discontinuity* of a (extended) Kleinian group K is the locus of points $p \in \widehat{\mathbb{C}}$ admitting an open neighborhood $p \in U \subset \widehat{\mathbb{C}}$ such that $k(U) \cap U \neq \emptyset$ only for finitely many elements $k \in K$. By definition, the region of discontinuity is an open set (it might be empty). The complement of the region of discontinuity is called the *limit set* and it is the place where the dynamics of the group action is chaotic. The history of Kleinian groups can be traced back to Poincaré [17] and a classical source is the book [13].

A *function group* is a finitely generated Kleinian group (with a non-empty region of discontinuity) admitting an invariant connected component of its region of discontinuity. Basic examples of function groups are provided by *elementary groups* (Kleinian groups with finite limit set), *quasifuchsian groups* (function groups whose limit set is a Jordan curve) and *totally degenerate groups* (non-elementary finitely generated Kleinian groups whose region of discontinuity is both connected and simply-connected). In a serie of papers, Maskit provided the following decomposition structure of function groups, in terms of the Klein-Maskit combination theorems [7, 8, 13].

Theorem 1.1 (Maskit's decomposition of function groups [6, 9, 10, 11])**.** *Every function group is constructed from elementary groups, quasifuchsian groups and totally degenerate groups by a finite number of applications of the Klein-Maskit combination theorems. Moreover, in the construction, the amalgamated free products and the HNN-extensions are realized along either (i) a finite cyclic group (including the trivial group) or (ii) a cyclic group generated by an accidental parabolic element.*

An *extended function group* is a finitely generated extended Kleinian group with an invariant connected component of its region of discontinuity. Basic examples of extended function groups are the *extended elementary groups* (extended Kleinian groups with finite limit set), *extended quasifuchsian groups* (finitely generated extended function groups whose limit set is a Jordan curve) and *extended totally degenerate groups* (non-elementary extended finitely generated Kleinian groups with connected and simply-connected region of discontinuity).

Note that the term "extended quasifuchsian group" used in this paper is different from the given by other authors in the sense that they refer it to Kleinian groups whose limit set is a Jordan curve and contains elements permuting the two discs bounded by it.

As it is for the case of function groups, one should expect a similar decomposition result for the extended function groups (see Theorem 1.2). It seems that such a result is missing in the literature. The aim of this paper is to provide such an structural decomposition of extended function groups.

Theorem 1.2 (Decomposition of extended function groups)**.** *Every extended function group is constructed from (extended) elementary groups, (extended) quasifuchsian groups and (extended) totally degenerate groups by a finite number of applications of the Klein-Maskit combination theorems. Moreover, in the construction, the amalgamated free products and the HNN-extensions are realized along either (i) a finite cyclic group (including the trivial group) or (ii) an infinite dihedral group generated by two reflections or (iii) a cyclic group generated by either an accidental parabolic element or by an accidental pseudo-parabolic element (i.e., its square is accidental parabolic).*

The above structure description is a consequence of Theorems 3.1 and 3.2 (which are new results); their statements are at the beginning of section 3. Their proofs build upon a sequence of lemmas 3.5, 3.7, 3.8, 3.9, which also are not found in the literature.

The idea of the proof is the following. Let K be an extended function group, with invariant connected component Δ. Its index two orientation preserving half $K^+ = K \cap \mathbb{M}$ is a function group with the same invariant component. As K^+ is finitely generated, Selberg's lemma [19] asserts the existence of a torsion free finite index normal subgroup G_1 of K^+ (which is again a function group). Since $K = \langle K^+, \tau \rangle$, where $\tau^2 \in K^+$, the group $G = G_1 \cap \tau G_1 \tau^{-1}$ is a finite index torsion free normal subgroup of K. By Ahlfors finiteness theorem [1], the quotient space $S = \Delta/G$ is an analytically finite Riemann surface, that is, $S = \widehat{S} - C$, where $\widehat{S}$ is a closed Riemann surface and $C \subset \widehat{S}$ is a finite set of points (it might be empty). The finite group $H = K/G$ is a group of conformal and anticonformal automorphisms of S. Maskit's decomposition of function groups may be applied to G. There are many possible decompositions, but in order to get one which can be used to obtain a decomposition of K, we must find one which is in some sense equivariant with respect to G. This is solved by Theorem 2.2 (equivariant theorem for function groups) obtained by Maskit and the author in [4]. This result permits us to obtain a first decomposition structural picture (see Theorem 3.1). In such a picture, there may appear (extended) B-groups as factors. A *B-group* (respectively, an *extended B-group*) is a function group (respectively, an extended function group) with a simply-connected invariant component in its region of discontinuity. A subtle modification to Maskit's arguments for the case of B-groups, to deal with these extended B-groups, is provided (see Theorem 3.2).

A. Haas's thesis [3] concerns with uniformizing groups of conformal and anticonformal automorphisms acting on plane domains. It leads naturally to extended function groups, but it seems that the above decomposition does not follow immediately from it.

2 Preliminaries

2.1 Riemann orbifolds

A *Riemann orbifold* $\mathcal{O}$ consists of a (possible non-connected) Riemann surface S (called the *underlying Riemann surface of the orbifold*), an isolated collection of points of S (called the *cone points of the orbifold*) and associated to each cone point an integer at least 2 (called the *cone order*). A connected Riemann orbifold is *analytically finite* if its underlying (connected) Riemann surface is the complement of a finite number of points of a closed Riemann surface and the number of cone points is also finite. We may think of a Riemann surface as a Riemann orbifold without cone points. A *conformal automorphism* (respectively, *anticonformal automorphism*) of the Riemann orbifold $\mathcal{O}$ is a conformal automorphism (respectively, anticonformal) of the underlying Riemann surface S which preserves both its set of cone points together with their cone orders (cone points can be permuted but preserving their cone orders). We denote by $\mathrm{Aut}(\mathcal{O})$ (respectively, $\mathrm{Aut}(S)$) the group of conformal/anticonformal automorphisms of $\mathcal{O}$ (respectively, S) and by $\mathrm{Aut}^+(\mathcal{O})$ (respectively, $\mathrm{Aut}^+(S)$) its subgroup of conformal automorphisms.

2.2 Kleinian and extended Kleinian groups

In the following, we recall some facts on (extended) Kleinian groups. A good source on the topic are the classical books [13, 14]. Let us start by observing that, if $K_1 < K_2 < \widehat{\mathbb{M}}$ and K_1 has finite index in K_2, then both are discrete if one of them is and, in the discreteness case, both have the same region of discontinuity.

Let $K < \widehat{\mathbb{M}}$ and set $K^+ := K \cap \mathbb{M}$. If $K \neq K^+$, then K^+ is called the *orientation-preserving half* of K and, in this case, K is an extended Kleinian group if and only if K^+ is a Kleinian group; in which case both have the same region of discontinuity. If moreover, K is an extended Kleinian group and K^+ is a function group, then either: (i) K is an extended function group or (ii) K^+ is a quasifuchsian group and there is an element of $K - K^+$ permuting both discs bounded by the limits set Jordan curve (so K is not an extended function group).

2.3 Accidental parabolic elements

A B-group is a function group K with a simply-connected invariant component Δ. Let us assume K is non-elementary (i.e., its limit set is not finite). By the Klein-Poincaré uniformization theorem [18], there is a bi-holomorphism $f : \mathbb{H}^2 \to \Delta$, where $\mathbb{H}^2$ denotes the hyperbolic upper-half plane. The group $\Gamma = f^{-1}Kf$ is a group of conformal automorphisms of $\mathbb{H}^2$, i.e., a fuchsian group of the first kind, in particular, a B-group with $\mathbb{H}^2$ as an invariant connected component of its region of discontinuity. In this case, $\mathbb{H}^2/\Gamma$ has finite hyperbolic area. It is known that f sends parabolic transformations to parabolic transformations, but it may send a hyperbolic transformation to a parabolic one. A parabolic element $P \in K$ is called *accidental* if $f^{-1}Pf$ is a hyperbolic transformation. In this case, the image under f of the axis of the hyperbolic transformation $f^{-1}Pf$ is called the *axis* of P (in Maskit's notation this is the true axis of P).

If K is an extended B-group, that is, an extended function group with a simply-connected invariant component, then we say that an element of K is *accidental pseudo-parabolic* if its square is an accidental parabolic element of K^+.

2.4 Klein-Maskit's decomposition theorems

Let K be a Kleinian group with region of discontinuity Ω and let H be a subgroup of K with limit set $\Lambda(H)$. A set $X \subset \widehat{\mathbb{C}}$ is called *precisely invariant under H in K* if $E(X) = X$, for every $E \in H$, and $T(X) \cap X = \emptyset$, for every $T \in K \setminus H$.

We will assume H to be either (i) the trivial group, (ii) a finite cyclic group or (iii) an infinite cyclic group generated by a parabolic transformation. If H is a cyclic subgroup, a *precisely invariant disc* B is the interior of a closed topological disc $\overline{B}$, where $\overline{B} - \Lambda(H) \subset \Omega$ is precisely invariant under H in K.

Theorem 2.1 (Klein-Maskit's combination theorems [7, 8])**.**

(1) (Amalgamated free products). For $j = 1, 2$, let K_j be a Kleinian group, let $H \leq K_1 \cap K_2$ be a cyclic subgroup (either trivial, finite or generated by a parabolic transformation), $H \neq K_j$, and let B_j be a precisely invariant disc under H in K_j. Assume that B_1 and B_2 have as a common boundary the simple loop Σ and that $B_1 \cap B_2 = \emptyset$. Then $K = \langle K_1, K_2 \rangle$ is a Kleinian group isomorphic to the free product of K_1 and K_2 amalgamated over H, that is, $K = K_1 *_H K_2$, and every elliptic or parabolic element of K is conjugated in K to an element of either K_1 or K_2. Moreover, if K_1 and K_2 are both geometrically finite, then K is also geometrically finite.

(2) (HNN extensions). Let K be a Kleinian group. For $j = 1, 2$, let B_j be a precisely invariant disc under the cyclic subgroup H_j (either trivial, finite or generated by a parabolic) in K, let Σ_j be the boundary loop of B_j and assume that $T(\overline{B}_1) \cap \overline{B}_2 = \emptyset$, for every $T \in K$. Let A be a loxodromic transformation such that $A(\Sigma_1) = \Sigma_2$, $A(B_1) \cap B_2 = \emptyset$, and $A^{-1}H_2A = H_1$. Then $K_A = \langle K, A \rangle$ is a Kleinian group, isomorphic to the HNN-extension $K*_{\langle A \rangle}$ (that is, every relation in K_A is consequence of the realtions in K and the relations $A^{-1}H_2A = H_1$). If each H_j, for $j = 1, 2$, is its own normalization in K, then every elliptic or parabolic element of K_A is conjugated to some element of K. Moreover, if K is geometrically finite, then K_A is also geometrically finite.

2.5 An equivariant loop theorem for function groups

Let K be a function group and Δ be a K-invariant connected component of its region of discontinuity. By the Alhfors' finiteness theorem [1, 2], the quotient $\mathcal{O} = \Delta/K$ turns out to be an analytically finite Riemann orbifold. Let $\mathcal{B} \subset \mathcal{O}$ be the (finite) collection of the cone points and let $\mathcal{G} \subset \mathcal{O} - \mathcal{B}$ be the collection of loops which lift to loops under the natural regular holomorphic covering $\pi : \Delta^0 \to \mathcal{O} - \mathcal{B}$, where Δ^0 is the open dense subset of Δ consisting of those points with trivial K-stabilizer. In [5], Maskit proved the existence of a finite subcollection $\mathcal{F} \subset \mathcal{G}$ of pairwise disjoint loops inside $\mathcal{O} - \mathcal{B}$, each one being a finite power of a simple loop, such that the cover π is determined as a highest regular planar cover for which the loops in $\mathcal{F}$ lift to loops (such a collection of loops is not unique). The collection $\mathcal{F}$ is called a *fundamental system of loops* of the above regular planar covering. Assume that there is a finite group $H < \mathrm{Aut}(\mathcal{O})$ whose elements lift to automorphisms of Δ under π. Then, in [4], Maskit and the author proved that there is a fundamental system of loops $\mathcal{F}$ being equivariant under H.

Theorem 2.2 (Equivariant loop theorem for function groups [4])**.** *Let K be a function group, with invariant connected component Δ in its region of discontinuity, $\mathcal{O} = \Delta/K$ (which is an analytically finite Riemann orbifold) and let $\mathcal{B}$ be the finite set of cone points of $\mathcal{O}$. Let $\pi : \Delta \to \mathcal{O}$ be the natural regular branched regular covering induced by K. Let $\mathcal{G}$ be the collection of loops in $\mathcal{O} - \mathcal{B}$ which lift to loops in Δ under π. If $H < Aut(\mathcal{O})$ lifts to a group of automorphisms of Δ, then there is a finite sub-collection $\mathcal{F} \subset \mathcal{G}$ such that:*

(1) *$\mathcal{F}$ consists of pairwise disjoint powers of simple loops;*

(2) *$\mathcal{F}$ is H-invariant; and*

(3) *every loop in $\mathcal{G}$ is homotopic to the product of finite powers of a finite loop in $\mathcal{F}$.*

The collection $\mathcal{F}$ is called a fundamental set of loops for the pair (K, H).

Remark 2.3. *The condition (3) above is equivalent to say that $\mathcal{F}$ is a fundamental system of loops for π. Also, if the function group K is torsion-free, then $\mathcal{O}$ is an analytically finite Riemann surface and each of the loops in the finite collection $\mathcal{F}$ turns out to be a simple loop.*

As a consequence of the above, one may write the following equivariant result for Kleinian groups.

Theorem 2.4 (Equivariant loop theorem for Kleinian groups)**.** *Let K be a Kleinian group with region of discontinuity $\Omega \neq \emptyset$, let Δ be a (non-empty) collection of connected components of Ω which is invariant under the action of K, let $\mathcal{O} = \Delta/K$, let $\mathcal{B}$ be the cone points of $\mathcal{O}$ and let $H < \mathrm{Aut}(\mathcal{O})$ be a finite group of automorphisms of $\mathcal{O}$. Let us assume that $\mathcal{O}$ consists of (may be infinitely many) analytically finite Riemann orbifolds. Fix some regular (branched) covering map $\pi : \Delta \to \mathcal{O}$ with K as its deck group. Let $\mathcal{G}$ be the collection of loops in $\mathcal{O} - \mathcal{B}$ which lift, with respect* *to π, to loops in Δ. If H lifts to a group of automorphisms of Δ, then there is a sub-collection $\mathcal{F} \subset \mathcal{G}$ such that:*

(1) *$\mathcal{F}$ consists of pairwise disjoint powers of simple loops;*

(2) *$\mathcal{F}$ is H-invariant; and*

(3) *every loop in $\mathcal{G}$ is homotopic to the product of finite powers of a finite sub-collection of loops in $\mathcal{F}$.*

Proof. Let us consider a maximal subcollection of non-equivalent components of Δ under the action of K, say Δ_j for $j \in J$. Let K_j be the K-stabilizer of Δ_j under the action of K. By Theorem 2.2, on $\mathcal{O}_j = \Delta_j/K_j$ there is a collection of loops, say $\mathcal{F}_j$, satisfying the properties on that theorem. Clearly the collection of fundamental loops $\mathcal{F} = \cup_{j \in J} \mathcal{F}_j$ is the required one. □

Remark 2.5. *The condition for $\mathcal{O} = \Delta/K$ to consist of analytically finite Riemann orbifolds is equivalent, by the Ahlfors finiteness theorem, for the K-stabilizer of each connected component in Δ to be finitely generated. In particular, if K is finitely generated, then $\mathcal{O}$ is a finite collection of analytically finite Riemann surfaces and $\mathcal{F}$ turns out to be a finite collection. If, in Theorem 2.4, we assume K to be torsion-free, then the loops in $\mathcal{F}$ will be simple loops.*

2.6 A connection to Kleinian 3-manifolds

Let K be a Kleinian group, with region of discontinuity $\Omega \subset \widehat{\mathbb{C}}$. There is a natural discrete action (by Poincaré extension) of K on the upper half-space $\mathbb{H}^3 = \{(z,t) : z \in \mathbb{C},\ t \in (0,+\infty)\}$, which is given by isometries in the hyperbolic metric $ds^2 = (|dz|^2 + dt^2)/t^2$. The quotient $M_K = (\mathbb{H}^3 \cup \Omega)/K$ carries the structure of a 3-orbifold, its interior $\mathbb{H}^3/K$ has a structure of a complete hyperbolic 3-orbifold and Ω/K the structure of a Riemann orbifold. In the case that K is torsion free, all the above are manifolds and we say that M_K is a Kleinian 3-manifold.

A direct consequence of Theorem 2.4 is the equivariant theorem for Kleinian 3-manifolds in the case that the conformal boundary is non-empty and it consists of analytically finite Riemann surfaces.

Corollary 2.6. *Let K be a torsion free Kleinian group, with non-empty region of discontinuity Ω, such that $S_K = \Omega/K$ is a collection (it might be infinitely many of them) of analytically finite Riemann surfaces. Let H be a finite group of automorphismsm of the Kleinian 3-manifold $M_K = (\mathbb{H}^3 \cup \Omega)/K$. If $\mathcal{G}$ is the collection of loops on S_K that are homotopically nontrivial in S_K but homotopically trivial in M_K, then there exists a collection of pairwise disjoint simple loops $\mathcal{F} \subset \mathcal{G}$, equivariant under the action of H, so that $\mathcal{G}$ is the smallest normal subgroup of $\pi_1(S_K)$ generated by $\mathcal{F}$.*

Remark 2.7. *Let K be a torsion free Kleinian group and let H be as in Corollary 2.6. Then the following hold. (1) If $\pi_1(M)$ is finitely generated, then the collection $\mathcal{F}$ is finite. (2) By lifting H to the universal cover space, one obtains a (extended) Kleinian group $\widehat{K}$ containing K as a finite index normal subgroup so that $H = \widehat{K}/K$. Corollary 2.6 may be used to obtain a geometric structure picture of $\widehat{K}$, in the sense of the Klein-Maskit combination theorems, in terms of the algebraic structure of H. (3) If M_K is compact, then the result follows from Meeks-Yau's equivariant loop theorem [15, 16], whose arguments are based on minimal surfaces theory. If K is not a purely loxodromic geometrically finite Kleinian group, then M_K is non-compact and the result is no longer a consequence of Meek's-Yau's equivariant theorem.*

3 Proof of Theorem 1.2

The proof of Theorem 1.2 is a direct consequence of Theorem 3.1, which is the main step, and Theorem 3.2 as described below. If the word "extended" is removed, the statements of these theorems are simply Maskit's original theorems (see [6, 9, 10, 11]).

Theorem 3.1 (First step in Maskit-type decomposition of an extended function group)**.** *Every extended function group is constructed, using the Klein-Maskit combination theorems, as amalgamated free products and HNN-extensions using a finite collection of (extended) B-groups. Moreover, the amalgamations and HNN-extensions are realized along either trivial or a finite cyclic group or a dihedral group generated by two reflections (this last one only in the amalgamated free product operation).*

The above result asserts that every extended function group is constructed from (extended) B-groups by applying the Klein-Maskit combination theorems. Maskit's results provide a geometrical decomposition of B-groups (see Theorem 3.2 below and delete the word "extended"). We now need to take care of the extended B-groups, which is exactly what the next result is about.

Theorem 3.2 (Decomposition of extended B-groups)**.** *Let K be an extended B-group with a simply-connected invariant component Δ. Then either (i) K is an elementary extended Kleinian group or (ii) K is an extended quasifuchsian group or (iii) K is an extended degenerate group or (iv) Δ is the only invariant component and K is constructed as amalgamated free products and HNN-extensions, by use of the Klein-Maskit combination theorems, using (extended) elementary groups, (extended) quasifuchsian groups and (extended) totally degenerate groups. The amalgamated free products and HNN-extensions are given along axes of accidental parabolic transformations.*

Remark 3.3. *We note for the reader that the proof of Theorem 3.1 includes Remarks 3.4 and 3.6 and Lemmas 3.5 and 3.7 and that the proof of Theorem 3.2 includes Lemmas 3.8 and 3.9.*

3.1 Proof of Theorem 3.1

Let K be an extended function group and let Δ be a K-invariant connected component of its region of discontinuity (we may assume K to be non-elementary). If there is another different invariant connected component of its region of discontinuity, then $K^+ = K \cap \mathbb{M} \neq K$ is known to be a quasifuchsian group [12]; so K is an extended quasifuchsian group. Let us assume, from now on, that Δ is the unique invariant connected component. By Selberg's lemma [19], there is a torsion free finite index normal subgroup G_1 of K^+. As $K = \langle K^+, \tau \rangle$, where $\tau^2 \in K^+$, one has that $G = G_1 \cap \tau G_1 \tau^{-1}$ is a torsion free finite index normal subgroup of K.

It follows that G is a function group with Δ as an invariant connected component of its region of discontinuity (the same as for K). Also, Δ is the only invariant connected component of G; otherwise G is a quasifuchsian group and K will have two different invariant connected components, which is a contradiction to our assumption on K.

Let $S = \Delta/G$ (an analytically finite Riemann surface by Ahlfors finiteness theorem) and consider a regular planar unbranched cover $P : \Delta \to S$ with G as its deck group. Set $H = K/G < \mathrm{Aut}(S)$, which is a non-trivial finite group (since $G \neq K$). Theorem 2.2 asserts the existence of a fundamental set of loops $\mathcal{F} \subset S$ for the pair (G, H). Such a collection of loops cuts S into some finite number of connected components and such a collection of components is invariant under H. The H-stabilizer of each of these connected components and each of the loops in $\mathcal{F}$ is a finite group.

Remark 3.4 (Decomposition structure of H)**.** *The H-equivariant fundamental system of loops $\mathcal{F}$ permits to obtain a structure of H as a finite iteration of amalgamated free products and HNN-extensions of certain subgroups of H as follows. Let us consider a maximal collection of components of $S - \mathcal{F}$, say $S_1 \ldots, S_n$, so that any two different components are not H-equivalent. Let us denote by H_j the H-stabilizer of S_j. It is possible to choose these surfaces so that, by adding some on the boundary loops, we obtain a planar surface S^* (containing each S_j in its interior). If two surfaces S_i and S_j have a common boundary in S^*, then $H_i \cap H_j$ is either trivial or a cyclic group (this being exactly the H-stabilizer of the common boundary loop). We perform the amalgamated free product of H_i and H_j along the trivial or cyclic group $H_i \cap H_j$. Set S_{ij} be the union of S_i, S_j with the common boundary loop in S^* and set H_{ij} the constructed group. Now, if S_k is another of the surfaces which has a common boundary loop in S^* with S_{ij}, then we again perform the amalgamated free product of H_{ij} and H_k along the trivial or cyclic group $H_{ij} \cap H_k$. Continuing with this procedure, we end with a group H^* obtained as amalgamated free product along finite cyclic groups or trivial groups. For each boundary of S^* we add a boundary loop, in order to stay with a planar compact surface (we are out of S in this part). If α is any of the boundary loops of S^*, there should be another boundary loop β of S^* and an element $h \in H$ so that $h(\alpha) = \beta$. By the choice of the surfaces S_j, we must have that $h(S^*) \cap S^* = \emptyset$. In particular, $\beta \neq \alpha$. If* *there is another element $k \in H - \{h\}$ so that $k(\alpha) = \beta$, then $k^{-1}h$ is a non-trivial element that stabilizes α and $k^{-1}h(S^*) \cap S^* \neq \emptyset$, a contradiction. Also, if there is another boundary loop γ of S^* (different from β) and an element $u \in H$ so that $u(\alpha) = \gamma$, then $uh^{-1} \in H - \{I\}$ satisfies that $uh^{-1}(S^*) \cap S^* \neq \emptyset$, which is again a contradiction. We may now perform the HHN-extension of H^* by the finite cyclic group generated by h. If $\alpha_1 = \alpha, \ldots, \alpha_m$ are the boundary loops of S^*, which are not H-equivalent, then we perform the HHN-extension with each of them. At the end, we obtain an isomorphic copy of H.*

We may assume the fundamental set of loops $\mathcal{F}$ to be minimal, that is, by deleting any non-empty subcollection of loops from it, we obtain a collection which fails to be a fundamental set of loops for (G, H). The minimality condition asserts that each connected component of $S - \mathcal{F}$ is different from either a disc or an annulus. By lifting $\mathcal{F}$ to Δ, under P, one obtains a collection $\widehat{\mathcal{F}} \subset \Delta$ of pairwise disjoint simple loops, so that $\widehat{\mathcal{F}}$ is invariant under the group K. Each of the loops in $\widehat{\mathcal{F}}$ is called a *structure loop* and each of the connected components of $\Delta - \widehat{\mathcal{F}}$ a *structure region*. These structure loops and regions are permuted by the action of K. The K-stabilizer (respectively, the G-stabilizer) of each structure loop and each structure region is called a structure subgroup of K (respectively, a structure subgroup of G).

If R is a structure region, then its K-stabilizer, denoted by K_R, is a finite extension of its G-stabilizer, denoted by G_R. Similarly, if α is a structure loop, then its K-stabilizer is a finite extension of its G-stabilizer.

Lemma 3.5. *Let α be a structure loop and let R be a structure region containing α on its border. Then the K_R-stabilizer of α is either trivial or a finite cyclic group or a dihedral group generated by two reflections (both circles of fixed points intersecting at two points, one inside of one of the two discs bounded by α and the other point contained inside the other disc). Moreover, the K-stabilizer of α is either equal to its K_R-stabilizer or it is generated by its K_R-stabilizer and an involution (conformal or anticonformal) that sends R to the other structure region containing α in its border.*

Proof. Let $\alpha \in \widehat{\mathcal{F}}$ be a structure loop. As α is contained in the region of discontinuity of K, the K-stabilizer of α is a finite group; so also its G-stabilizer is finite. Note that the K^+-stabilizer of α is either trivial, a finite cyclic group or a dihedral group. Moreover, in the dihedral case, one of the involutions interchanges both discs bounded by α. Let R be a structure region containing α as a boundary loop. Then the K_R^+-stabilizer of α is either trivial or a finite cyclic group. It follows that the K_R-stabilizer of α is either trivial or a finite cyclic group or a dihedral group generated by two reflections (both circles of fixed points intersecting at two points, one inside of one of the two discs bounded by α and the other point contained inside the other disc). The K-stabilizer of α is generated by the K_R-stabilizer and probably an extra involution (conformal or anticonformal) that interchanges both discs bounded by α. ☐

Remark 3.6. *We do not need this extra information for the rest of the proof, but it may help with a clarification of the gluing process at the Klein-Maskit combination theorems. It follows, from Lemma 3.5, that the K-stabilizer of $\alpha \in \widehat{\mathcal{F}}$ must be one of the following: (1) the trivial group, (2) a cyclic group generated by a reflection with α as its circle of fixed points (so it permutes both discs bounded by α), (3) a cyclic group generated by a reflection that keeps invariant each of the two discs bounded by α (the reflection has exactly two fixed points over α), (4) a cyclic group generated by an imaginary reflection (it permutes both discs bounded by α), (5) a cyclic group generated by an elliptic transformation of order two (permuting the two discs bounded by α), (6) a cyclic group generated by an elliptic transformation (preserving each of the two discs bounded by α), (7) a group generated by an elliptic transformation (preserving each of the two discs bounded by α) and a reflection whose circle of fixed points is α, (8) a group generated by an elliptic transformation (preserving each of the two discs bounded by α) and an imaginary reflection (permuting both discs bounded by α), (9) a group generated by an elliptic transformation of order two (permuting the two discs bounded by α) and an imaginary reflection that keeps α invariant (it permutes both discs bounded by α), (10) a dihedral group generated by two reflections (both circles of fixed points intersecting at two points, one inside of one of the two discs bounded by α and the other point contained inside the other disc), (11) a group generated by an elliptic transformation (preserving each of the two discs bounded by α) and an imaginary reflection that keeps α invariant (it permutes both discs bounded by α), (12) a group generated by a dihedral group of Möbius transformations and a reflection with α as circle of fixed points, (13) a group generated by a dihedral group generated by two reflections (both circles of fixed points intersecting at two points, one inside of one of the two discs bounded by α and the other point contained inside the other disc) and an elliptic transformation of order two that permutes both discs bounded by α, To obtain the above, we use the following fact. Let α be a loop which is invariant under (i) an elliptic transformation E, of order two that interchanges both discs bounded by it, and (ii) also invariant under an imaginary reflection τ. Then $E\tau$ is necessarily a reflection whose circle of fixed points is transversal to α.*

Let R be a structure region and let $\alpha \in \widehat{\mathcal{F}}$ be on the boundary of R. By Lemma 3.5, the K_R-stabilizer of α is some finite group; either trivial or a finite cyclic group or a dihedral group generated by two reflections (both circles of fixed points intersecting at two points, one inside of one of the two discs bounded by α and the other point contained inside the other disc). Let D_α be the topological disc bounded by α and disjoint from R. Clearly, the K_R-stabilizer of such a disc is contained in the K_R-stabilizer of α (each element of K_R that stabilizes D_α also stabilizes α), so D_α is contained in the region of discontinuity of K_R. It follows that K_R is a (extended) function group with an invariant connected component Δ_R of its region of discontinuity containing R and all the discs D_α, for every structure loop α on its boundary.

Lemma 3.7. *Δ_R is simply-connected.*

Proof. If Δ_R is not simply-connected, then there is a simple loop $\beta \subset R$ bounding two topological discs, each one containing limit points of K_R (so limit points of K). The projection on S of β produces a loop $\widetilde{\beta} \subset S$ which lifts to a loop under P. But, we know that $\widetilde{\beta}$ is homotopic to the product of finite powers of the simple loops on the boundary of the finite domain $P(R) \subset S$. It follows that β must be homotopic to the product of finite powers of a finite collection of structure loops on the boundary of R. As each of these boundary loops bounds a disc containing no limit points, we get a contradiction for β to bound two discs, each one containing limit points. □

We may follow the same lines as described in Remark 3.4 to obtain that K is constructed, using the Klein-Maskit combination theorems [13, 7], as amalgamated free products and HNN-extensions using a finite collection of the structure subgroups of K_R (which, by Lemma 3.7, are extended B-groups with invariant simply-connected component Δ_R). By Lemma 3.5, the amalgamations and HNN-extensions are realized along either trivial or a finite cyclic group or a dihedral group generated by two reflections. This ends the proof of Theorem 3.1. □

3.2 Proof of Theorem 3.2

We proceed to describe the subtle modifications in Maskit's arguments in the decomposition of B-groups [10, 11] adapted to the case of extended B-groups (see also chapter IX.H. in [13]). Let us assume that K is an extended B-group and that it is neither an (extended) elementary group or a (extended) quasifuchsian group or a (extended) degenerate group. Let Δ be the simply-connected invariant component of the region of discontinuity of K. Every other connected component of the region of discontinuity of K is simply-connected (see Proposition IX.D.2. in [13]). By our assumptions on K, we have that K^+ is neither elementary nor degenerate Kleinian group. It may be, even if K is not an extended quasifuchsian, that K^+ is a quasifuchsian. But in this case, we have that K is just a HNN-extension of a quasifuchsian group along a cyclic group. So, from now on, we assume that K^+ is neither a quasifuchsian group.

As K is non-elementary, we may consider a bi-holomorphism $f : \mathbb{H}^2 \to \Delta$ and consider the fuchsian group $f^{-1}Kf$. As it is well known that no rank two parabolic subgroup can preserve a disc in the Riemann sphere, it follows that $f^{-1}K^+f$ does not contain rank two parabolic subgroup, in particular, K^+ neither does contain a rank two parabolic subgroup. Theorem IX.D.21 in [13] states that K^+ is either quasifuchsian or totally degenerate or it contains accidental parabolics. By our assumptions on K and K^+, we note that K^+ necessarily must have accidental parabolic transformations. Moreover, there is a finite number of conjugacy classes of primitive accidental parabolic transformations in K^+. Let us consider a collection of accidental parabolic transformations in K^+, say P_1,..., P_m, so that P_j is not K^+-conjugate to $P_r^{\pm 1}$ if $j \neq r$, and P_j is primitive, that is it is not of the form Q^a for some $Q \in K$ and $a \geq 2$. Let us denote by $L_j \subset \Delta$ the axis of P_j (note that L_j is a geodesic for the hyperbolic metric of Δ and that P_j keeps it invariant acting by a translation on it).

Lemma 3.8. *(1) If $j \neq r$, then the K^+-translates of L_j do not intersect the K^+-translates of L_r. (2) For each fixed j, any K^+-translates of L_j is either disjoint from L_j or it coincides with it.*

Proof. Let us consider a Riemann map $f : \mathbb{H}^2 \to \Delta$, where $\mathbb{H}^2$ is the upper half-plane with the hyperbolic metric $ds^2 = |dz|^2/\text{Im}(z)^2$. It is well known that any two different geodesics in $\mathbb{H}^2$ are either disjoint of they intersect at exactly one point. The push-forward of the hyperbolic metric in $\mathbb{H}^2$ provides the hyperbolic metric of Δ. It follows that any K^+-translate of L_j and any K^+ translate of L_r (for j not necessarily different from r) are either disjoint or they intersect exactly at one point or they are the same. Let us first prove (1), that is, we assume $j \neq r$. If there are K^+-translates of L_j and L_r which are the same, as P_j and P_r are primitive parabolic, share the same fixed point and K^+ is discrete, then P_j is conjugate to either $P_r^{\pm 1}$, a contradiction. If there are K^+-translates of L_j and L_r which intersect at a point, then the planarity of Δ asserts that the non-empty intersection only may happen if a K^+ conjugate of P_j and a K^+-conjugate of P_r share their unique fixed point. The discreteness of K^+ asserts that K^+ must contain a rank two parabolic subgroup, a contradiction. Let us now prove (2), that is, we assume $j = r$. This follows the same lines a the previous case to see that either the translates are either disjoint or equal. □

Lemma 3.9. *If $T \in K - K^+$, then T preserves the collection of K^+-translates of $\{L_1, ..., L_m\}$.*

Proof. T acts as an isometry on Δ and must permute the accidental parabolic transformations. As the axis is unique for each accidental parabolic, we are done. □

Let $\widehat{L}_j$ be equal to L_j together the corresponding fixed point of P_j. Then the collection $\mathcal{F}$ given by the K^+-translates of $\{\widehat{L}_1, ..., \widehat{L}_m\}$ consists of pairwise disjoint simple loops; each one is called a *structure loop* for the group K. Such a collection of structure loops is still invariant for any $T \in K - K^+$ by Lemma 3.9. The structure loops cut Ω (the region of discontinuity of K) and Δ into regions; called *structure regions* for K. These are different from our previous definitions of structure loops and regions as these ones are not completely contained in the region of discontinuity.

Let $\alpha \in \mathcal{F}$ be a structure loop and let R_1 and R_2 be the two structure regions containing α in their common boundary. Let $K_j < K$ be the K-stabilizer of R_j, let K_α be the K-stabilizer of α and let $P \in K$ be the primitive accidental parabolic transformation whose axis is α (which is then K-conjugated to some of the P_j's). Clearly, $\langle P \rangle$ is contained in K_j, $\langle P \rangle < K_\alpha$ and either (i) $\langle P \rangle = K_\alpha$ or (ii) $\langle P \rangle$ has index two in K_α or (iii) $\langle P \rangle$ has index four in K_α (this last case means that $\langle P \rangle$ has index two inside the K_j-stabilizer of α). The region R_{3-j} is contained in a disc D_{3-j}, whose K_j-stabilizer is equal to the K_j-stabilizer of the loop α; this is either the cyclic group generated by P or it contains it as an index two subgroup. It follows that D_{3-j} is contained in the region of discontinuity Ω_j of K_j and that there is an invariant connected component $\Delta_j \subset \Omega_j$ containing Δ. Lemma IX.H.10 in [13] states that K_j^+ is a B-group, with Δ_j as invariant simply-connected component, without accidental parabolic transformations. It follows that K_j^+ is either elementary or quasifuchsian or totally degenerate, in particular, that K_j is either (extended) elementary or (extended) quasifuchsian or (extended) totally degenerate. One possibility is that K_α is an extension of degree two of the K_j-stabilizer of α. In this case, there is an element $Q \in K_\alpha$ that permutes R_1 with R_2 (Q is either a pseudo-parabolic whose square is P or an involution). In this case, $\langle K_1, K_2 \rangle$ is the HNN-extension of K_1 by Q (in the sense of the second Klein-Maskit combination theorem). The other possibility is that K_α is equal to $K_1 \cap K_2$ (either the cyclic group generated by the parabolic P or a group generated by two reflections sharing as a common fixed point the fixed point of P). In this case, $\langle K_1, K_2 \rangle$ is the free product of K_1 and K_2 amalgamated over $K_1 \cap K_2$ (in the sense of the first Klein-Maskit combination theorem).

Now, following the same ideas in [10, 11], one obtains a decomposition of K as an amalgamated free products and HNN-extensions, by use of the Klein-Maskit combination theorems, using (extended) elementary groups, (extended) quasifuchsian groups and (extended) totally degenerate groups. $\square$

Acknowledgment

The author would like to thank the referees for their valuable comments, suggestions and corrections to the previous versions.

References

[1] L. V. Ahlfors, “Finitely generated Kleinian groups”, Amer. J. of Math., vol. 86, pp. 413–429, 1964.

[2] L. V. Ahlfors, “Correction to “Finitely generated Kleinian groups”. Amer. J. Math., vol. 87, p. 759, 1965.

[3] A. Haas, “Linearization and mappings onto pseudocircle domains”, Trans. Amer. Math. Soc., vol. 282, no. 1, pp. 415–429, 1984.

[4] R. A. Hidalgo and B. Maskit, “A Note on the Lifting of Automorphisms” in *Geometry of Riemann Surfaces*, Lecture Notes of the London Mathematics Society, vol. 368, New York: Cambridge University Press, 2009, pp. 260–267.

[5] B. Maskit, “A theorem on planar covering surfaces with applications to 3-manifolds”, Ann. of Math. vol. 81, no. 2, pp. 341–355, 1965.

[6] B. Maskit, “Construction of Kleinian groups,” in *Proceedings of the Conference on Complex Analysis*, Minneapolis: Springer-Verlag, 1965, pp. 281–296.

[7] B. Maskit, “On Klein’s combination theorem III”, in *Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969)*, Ann. of Math. Studies, vol. 66, Princeton: Princeton Univ. Press, 1971, pp. 297–316;

[8] B. Maskit, “On Klein’s combination theorem. IV”, Trans. Amer. Math. Soc., vol. 336, no. 1, pp. 265–294, 1993.

[9] B. Maskit, “Decomposition of certain Kleinian groups”, Acta Math., vol. 130, pp. 243–263, 1973.

[10] B. Maskit, “On the classification of Kleinian Groups I. Koebe groups”, Acta Math., vol. 135, pp. 249–270, 1975.

[11] B. Maskit, “On the classification of Kleinian Groups II. Signatures”, Acta Math., vol. 138, no. 1-2, pp. 17–42, 1976.

[12] B. Maskit, “On extended quasifuchsian groups”, Ann. Acad. Sci. Fenn. Ser. A I Math., vol. 15, no. 1, pp. 53–64, 1990.

[13] B. Maskit, *Kleinian Groups*, Grundlehren der Mathematischen Wissenschaften, Berlin: Springer-Verlag, 1988.

[14] K. Matsuzaki and M. Taniguchi, *Hyperbolic Manifolds and Kleinian Groups*, Oxford Mathematical Monographs, New York: The Clarendon Press, Oxford University Press, 1998.

[15] W. Meeks and S.-T. Yau, "The equivariant Dehn's lemma and loop theorem", Comment. Math. Helvetici, vol. 56, no. 2, pp. 225–239, 1981.

[16] W. Meeks and S.-T.Yau, "The equivariant loop theorem for three-dimensional manifolds and a review of the existence theorems for minimal surfaces", in *The Smith conjecture (New York, 1979)*, Pure Appl. Math., vol. 112, Orlando, FL: Academic Press, 1984, pp.153–163.

[17] H. Poincaré, "Mémoire: Les groupes Kleinéens", Acta Math., vol. 3, pp. 193–294, 1982.

[18] H. Poincaré, "Sur l'uniformisation des fonctions analytiques", Acta Math., vol. 31, no. 1, pp. 1–63, 1908.

[19] A. Selberg, "On discontinuous groups in higher dimensional symmetric spaces" in *Contributions to function theory*, Internat. Colloq. Function Theory, Bombay: Tata Institute of Fundamental Research, 1960, pp. 147–164.

CUBO, A Mathematical Journal
Vol. 23, no. 03, pp. 385–409, December 2021
DOI: 10.4067/S0719-06462021000300385

Entropy solution for a nonlinear parabolic problem with homogeneous Neumann boundary condition involving variable exponents

U. Traoré[1]

[1] *Laboratoire de Mathématiques et Informatique (LAMI), Université Joseph KI-ZERBO 03, BP 7021 Ouaga 03, Ouagadougou, Burkina Faso.*
`urbain.traore@yahoo.fr`

ABSTRACT

In this paper we prove the existence and uniqueness of an entropy solution for a non-linear parabolic equation with homogeneous Neumann boundary condition and initial data in L^1. By a time discretization technique we analyze the existence, uniqueness and stability questions. The functional setting involves Lebesgue and Sobolev spaces with variable exponents.

RESUMEN

En este artículo probamos la existencia y unicidad de una solución de entropía para una ecuación parabólica no lineal con condiciones de borde Neumann homogéneas y data inicial en L^1. Usando una técnica de discretización del tiempo, analizamos las preguntas de existencia, unicidad y estabilidad. El contexto funcional involucra espacios de Lebesgue y Sobolev con exponentes variables.

Keywords and Phrases: Nonlinear parabolic problem, variable exponents, entropy solution, Neumann-type boundary conditions, semi-discretization.

2020 AMS Mathematics Subject Classification: 35K55, 35K61, 35J60, 35Dxx.

Accepted: 12 August, 2021
Received: 19 December, 2020

1 Introduction and main result

Let Ω be a smooth bounded open domain of $\mathbb{R}^d$, $(d \geq 3)$ with Lipschitz boundary $\partial\Omega$, T is a fixed positive number, in this paper we study the existence and uniqueness of an entropy solution for the following nonlinear parabolic problem

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{div} a\,(x, \nabla u) + b\,(u) = f \text{ in } Q_T =]0, T[\times\Omega, \\ \\ a\,(x, \nabla u)\,.\eta = 0 \text{ on } \sum_T =]0, T[\times\partial\Omega, \\ \\ u(0, .) = u_0 \text{ in } \Omega, \end{cases}$$

where $f \in L^1(Q_T)$, $b : \mathbb{R} \to \mathbb{R}$, $a(x, \xi) : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is Carathéodory function and verifying some assumptions which will be given later, η denotes the unit vector normal on $\partial\Omega$.

The usual weak formulations of parabolic problems in the case where the initial data are in L^1 do not ensure existence and uniqueness of solutions. For this reason, new formulations and types of solutions are given in order to obtain existence and uniqueness. For that, three notions of solution have been adopted: solutions named SOLA (Solution Obtained as the Limit of Approximations) defined by A. Dall'Aglio (see [10]); renormalized solutions defined by R. DiPerna and P.-L. Lions (see [12]); and entropy solutions defined by Ph. Bénilan *et al.* in [8]. In this paper, we will be interested in the entropy formulation.

The stationary version of the problem for the problem (P) has been already studied by Bonzi *et al.* (cf. [9]), where they proved the existence and uniqueness of an entropy solution for the initial data in L^1.

The study of parabolic equations with variable exponents is a very active field (see [1, 2, 20, 21, 23, 27, 29]), in these papers, the authors consider the homogeneous Dirichlet boundary conditions, which permit them to use many results in the generalized Sobolev space $W^{1,p(.)}(\Omega)$ and the many results concerned the differential equation in the literature to achieve there works. In particular in the case of $p(x)$-Laplace, where $b \equiv 0$, Bendahmane *et al.* (see [6]) have proved the existence and uniqueness of renormalized solution. We can also point out that the well-posedness of triply nonlinear degenerate elliptic- parabolic-hyperbolic problems: $b(u)_t - \mathrm{div} a(x, \nabla\phi(u)) + \psi(u) = f$ in a bounded domain with homogeneous Dirichlet boundary conditions by K. H. Karlsen *et al.* in [3].

Unfortunately, in this paper, due to the Neumann boundary condition, we cannot use the ideas developed in these papers and also some functional analysis results which play and important role in the a priori estimation, in particular the famous Poincaré inequality.

To overcome these difficulties we apply a time discretization of given continuous problem by the Euler forward scheme. Let's recall that this method has been used in the literature for the study of some nonlinear parabolic problems, we refer for example to [7, 13, 16, 17] for some details. This scheme is usually used to prove existence of solutions as well as to compute numerical approximations.

In this paper, our assumptions are the following:

$$\begin{cases} p\,(.) : \overline{\Omega} \;\; \rightarrow \;\; \mathbb{R} \quad \text{is a continuous function such that} \\ 1 < p_- \;\; \leq \;\; p_+ < +\infty, \end{cases} \tag{1.1}$$

where $p_- := ess \inf_{x\in\Omega} p(x)$ and $p_+ := ess \sup_{x\in\Omega} p(x)$ and

$$b : \overline{\Omega} \rightarrow \mathbb{R} \quad \text{is a continuous, nondecreasing function, surjective such that } b(0) = 0. \tag{1.2}$$

Also, we assume that $a(x, \xi) : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is Carathéodory such that:

- there exists a positive constant C_1 with

$$|a(x, \xi)| \leq C_1 \left(j(x) + |\xi|^{p(x)-1} \right) \tag{1.3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where j is a nonnegative function in $L^{p'(.)}(\Omega)$ with $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$;

- there exists a positive constant C_2 such that for every $x \in \Omega$ and every $\xi_1, \xi_2 \in \mathbb{R}^d$ with $\xi_1 \neq \xi_2$, the following two inequalities hold

$$(a(x, \xi_1) - a(x, \xi_2)) \,.\, (\xi_1 - \xi_2) > 0 \tag{1.4}$$

$$a(x, \xi) \,.\, \xi \geq C_2 |\xi|^{p(x)}. \tag{1.5}$$

The rest of the paper is organized as follows: after some preliminary results in Section 2, we introduce the Euler forward scheme associated with the problem (P) in Section 3. We analyze the stability of the discretized problem and we study the existence of an entropy solution to the parabolic problem (P) in the Section 4.

2 Preliminaries

We define the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ (see [11]) as the set of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}\, dx$$

is finite.

If the exponent is bounded, *i.e.*, if $p_+ < +\infty$, then the expression

$$\|u\|_{p(.)} := \inf \left\{ \lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1 \right\}$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxembourg norm.
The space $\left(L^{p(.)}(\Omega), \|.\|_{p(.)}\right)$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.
Finally, we have the Hölder type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p_-} + \frac{1}{(p_-)'}\right) \|u\|_{p(.)} \|v\|_{p'(.)}, \tag{2.1}$$

for all $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$.
Let

$$W^{1,p(.)}(\Omega) := \left\{u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega)\right\},$$

which is Banach space equipped with the following norm

$$\|u\|_{1,p(.)} := \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

The space $\left(W^{1,p(.)}(\Omega), \|.\|_{1,p(.)}\right)$ is a separable and reflexive Banach space.
An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(.)}(\Omega)$. We have the following result.

Proposition 2.1 (see [14, 28])**.** *If $u_n, u \in L^{p(.)}(\Omega)$ and $p_+ < \infty$, the following properties hold true:*

(i) $\|u\|_{p(.)} > 1 \Rightarrow \|u\|_{p(.)}^{p_-} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p_+}$;

(ii) $\|u\|_{p(.)} < 1 \Rightarrow \|u\|_{p(.)}^{p_+} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p_-}$;

(iii) $\|u\|_{p(.)} < 1$ $(respectively = 1; > 1) \Leftrightarrow \rho_{p(.)}(u) < 1$ $(respectively = 1; > 1)$;

(iv) $\|u_n\|_{p(.)} \to 0$ $(respectively \to +\infty) \Leftrightarrow \rho_{p(.)}(u_n) < 1$ $(respectively \to +\infty)$;

(v) $\rho_{p(.)}\left(u/\|u\|_{p(.)}\right) = 1$.

For a measurable function $u : \Omega \to \mathbb{R}$ we introduce the following notation:

$$\rho_{1,p(.)}(u) = \int_\Omega |u|^{p(x)} dx + \int_\Omega |\nabla u|^{p(x)} dx.$$

Proposition 2.2 (see [25, 26])**.** *If $u \in W^{1,p(.)}(\Omega)$, the following properties hold true:*

(i) $\|u\|_{1,p(.)} > 1 \Rightarrow \|u\|_{1,p(.)}^{p_-} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p_+}$;

(ii) $\|u\|_{1,p(.)} < 1 \Rightarrow \|u\|_{1,p(.)}^{p_+} < \rho_{p(.)}(u) < \|u\|_{1,p(.)}^{p_-}$;

(iii) $\|u\|_{1,p(.)} < 1$ $(respectively = 1; > 1) \Leftrightarrow \rho_{1,p(.)}(u) < 1$ $(respectively = 1; > 1)$.

Put

$$p^{\partial}(x) := (p(x))^{\partial} = \begin{cases} \dfrac{(N-1)\,p(x)}{N-p(x)}, & \text{if} \quad p(x) < N \\ \infty, & \text{if} \quad p(x) \geq N. \end{cases}$$

Proposition 2.3 (see [26])**.** *Let $p \in C\left(\bar{\Omega}\right)$ and $p_- > 1$. If $q \in C(\partial\Omega)$ satisfies the condition*

$$1 < q(x) < p^{\partial}(x) \quad \forall x \in \partial\Omega,$$

then, there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega)$.
In particular, there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\partial\Omega)$.

Following [29], we extend a variable exponent $p : \overline{\Omega} \to [1,+\infty)$ to $\overline{Q_T} = [0,T] \times \overline{\Omega}$ by setting $p(t,x) = p(x)$ for all $(t,x) \in \overline{Q_T}$.
We may also consider the generalized Lebesgue space

$$L^{p(.)}(Q) = \left\{ u : Q \to \mathbb{R} \text{ measurable such that } \iint_Q |u(t,x)|^{p(x)}\, d(t,x) < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^{p(.)}(Q_T)} := \inf\left\{ \lambda > 0,\ \iint_{Q_T} \left| \frac{u(t,x)}{\lambda} \right|^{p(x)} d(t,x) < 1 \right\},$$

which share the same properties as $L^{p(.)}(\Omega)$.

For a measurable set U in $\mathbb{R}^d$, meas(U) denotes its measure, C_i and C will denote various positive constants. For a Banach space X and $a < b$, $L^q(a,b;X)$ is the space of measurable functions $u : [a,b] \to X$ such that

$$\left(\int_a^b \|u\|_X^q \, dt \right)^{\frac{1}{q}} := \|u\|_{L^q(a,b;X)} < \infty. \tag{2.2}$$

For a given constant $k > 0$ we define the cut-off function $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(s) := \begin{cases} s & \text{if} \quad |s| \leq k \\ k \text{ sign}(s) & \text{if} \quad |s| > k \end{cases}$$

with

$$\text{sign}(s) := \begin{cases} 1 & \text{if} \quad s > 0 \\ 0 & \text{if} \quad s = 0 \\ -1 & \text{if} \quad s < 0. \end{cases}$$

Let $J_k : \mathbb{R} \to \mathbb{R}^+$ defined by

$$J_k(x) = \int_0^x T_k(s)ds$$

(J_k is a primitive of T_k). We have (see [15])

$$\left\langle \frac{\partial v}{\partial t}, T_k(s) \right\rangle = \frac{d}{dt}\left(\int_\Omega J_k(v)dx \right) \text{ in } L^1(]0,T[)$$

which implies that

$$\int_0^t \left\langle \frac{\partial v}{\partial t}, T_k(s) \right\rangle = \int_\Omega J(v(t))dx - \int_\Omega J(v(0))dx$$

For all $u \in W^{1,p(.)}(\Omega)$ we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense.
In the sequel, we will identify at the boundary, u and $\tau(u)$.
Set

$$\mathcal{T}^{1,p(.)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}, \text{ measurable such that } T_k(u) \in W^{1,p(.)}(\Omega), \text{ for any } k > 0 \right\}.$$

Proposition 2.4 (see [8])**.** *Let $u \in \mathcal{T}^{1,p(.)}(\Omega)$. Then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, for all $k > 0$. The function v is denoted by ∇u. Moreover, if $u \in W^{1,p(.)}(\Omega)$ then $v \in \left(L^{p(.)}(\Omega)\right)^N$ and $v = \nabla u$ in the usual sense.*

We denote by $\mathcal{T}_{tr}^{1,p(.)}(\Omega)$ (*cf.* [4, 5, 18, 19]) the set of functions $u \in \mathcal{T}^{1,p(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p(.)}(\Omega)$ satisfying the following conditions:

i) $u_n \to u$ a.e. in Ω.

ii) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $\left(L^1(\Omega)\right)^N$ for any $k > 0$.

iii) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [4, 5]. In the sequel, the trace of $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. If $u \in W^{1,p(.)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$ and if $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ then $(u - \varphi) \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ and $tr(u - \varphi) = tr(u) - tr(\varphi)$.

3 The semi-discrete problem

In this section, we study the Euler forward scheme associated with the problem (P):

$$(P_n) \begin{cases} U^n - \tau \mathrm{div}\, a(x, \nabla U^n) + \tau b(U^n) = \tau f^n + U^{n-1} \text{ in } \Omega \\ \\ a(x, \nabla U^n).\eta = 0 \text{ on } \partial\Omega, \\ \\ U^0 = u_0 \text{ in } \Omega \end{cases}$$

where $N\tau = T$, $0 < \tau < 1$, $1 \le n \le N$ and

$$f_n(.) \quad = \quad \frac{1}{\tau}\int_{(n-1)\tau}^{n\tau} f(s,.)ds \quad \text{in } \Omega.$$

Definition 3.1. *An entropy solution to the discretized problems* (P_n) *is a sequence* $(U^n)_{0\leq n\leq N}$ *such that* $U^0 = u_0 \in L^1(\Omega)$ *and* U^n *is defined by induction as an entropy solution to the problem*

$$\begin{cases} U^n - \tau \operatorname{div} a(x, \nabla U^n) + \tau b(U^n) = \tau f_n + U^{n-1} \ in\ \Omega \\ \\ a(x, \nabla U^n).\eta = 0 \ on\ \partial\Omega \end{cases}$$

i.e. $U^n \in \mathcal{T}^{1,p(.)}_{tr}(\Omega)$, $b(U^n) \in L^1(\Omega)$, *and for every* $k > 0$

$$\tau \int_\Omega a(x, \nabla U^n).\nabla T_k(U^n - \varphi)dx + \int_\Omega (\tau b(U^n) + U^n)T_k(U^n - \varphi)dx \leq \int_\Omega (\tau f_n + U^{n-1})T_k(U^n - \varphi)dx \tag{3.1}$$

for all $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

We have the following result

Lemma 3.2. *Let hypotheses* (1.3) – (1.5) *be satisfied. If* $(U^n)_{0\leq n\leq N}$ *is an entropy solution of problems* (P_n), *then* $U^n \in L^1(\Omega)$ *for all* $n = 1, \ldots, N$.

Proof. For $n = 1$, we take $\varphi = 0$ in (3.1), to get,

$$\tau \int_\Omega a(x, \nabla U^1).\nabla T_k(U^1)dx + \int_\Omega (\tau b(U^1) + U^1)T_k(U^1)dx \leq \int_\Omega (\tau f_1 + u_0)T_k(U^1)dx,$$

which is equivalent to

$$\tau \int_\Omega a(x, \nabla T_k(U^1))\nabla T_k(U^1)dx + \int_\Omega \tau b(U^1)T_k(U^1)dx + \int_\Omega U^1 T_k(U^1)dx \leq \int_\Omega (\tau f_1 + u_0)T_k(U^1)dx, \tag{3.2}$$

By the assumption (1.5) and the properties of the function b, we have

$$\tau \int_\Omega a(x, \nabla T_k(U^1))\nabla T_k(U^1)dx + \int_\Omega \tau b(U^1)T_k(U^1)dx \geq 0,$$

then it follows that

$$\int_\Omega U^1 T_k(U^1)dx \leq k\tau \|f_1\|_1 + k\|u_0\|_1 .$$

Since

$$\sum_{n=1}^N \tau \|f_n\|_1 \leq \|f\|_1 .$$

Then, it follows that

$$\int_\Omega U^1 T_k(U^1)dx \leq k(\|f\|_1 + \|u_0\|_1). \tag{3.3}$$

Since

$$\lim_{k\to 0} U^1 \frac{T_k(U^1)}{k} = |U^1|.$$

Then dividing (3.3) by k and letting $k \to 0$; we deduce by Fatou's lemma that

$$\|U^1\|_1 \leq (\|f\|_1 + \|u_0\|_1) \tag{3.4}$$

□

Theorem 3.3. *Let hypotheses* (1.3) − (1.5) *be satisfied. Then for all* $N \in \mathbb{N}$, *the problems* (P_n) *have unique entropy solution* $U^n \in \mathcal{T}_{tr}^{1,p(.)}(\Omega) \cap L^1(\Omega)$ *for all* $n = 1, \dots, N$.

Proof. The problem (P_1) can be rewritten in the following form

$$-\tau \mathrm{div} a\left(x, \nabla u\right) + \overline{b}(u) = F_1 \text{ in } \Omega$$

$$a(x, \nabla u).\eta = 0 \text{ on } \partial\Omega$$

with

$$\overline{b}(s) := \tau b(s) + s, \quad F_1 := \tau f_1 + u_0.$$

From the assumption (H_2), we have $F_1 \in L^1(\Omega)$, and using the properties of b, we obtain $\overline{b}$ is a continuous, nondecreasing function, surjective such that $b(0) = 0$. Hence, using [9, Theorem 4.3], we have the existence of unique entropy solution $U^1 \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$, $b\left(U^1\right) \in L^1(\Omega)$.

Thanks to Lemma 3.2, by induction, we deduce that for $n = 2, \dots, N$, the problem

$$\begin{aligned} &u - \tau \mathrm{div} a\left(x, \nabla u\right) + \tau\alpha\left(u\right) = \tau f_n + U^{n-1} \text{ in } \Omega \\ &a\left(x, \nabla u\right).\eta = 0 \text{ on } \partial\Omega, \end{aligned}$$

has an unique entropy solution $U^n \in \mathcal{T}_{tr}^{1,p(.)}(\Omega) \cap L^1(\Omega)$, $b\left(U^n\right) \in L^1(\Omega)$. □

4 Stability

This section is devoted to the a priori estimates for the discrete entropy solution $(U^n)_{1\le n\le N}$. These result are essentials for the study of the convergence of the Euler forward scheme.

Theorem 4.1. *Let hypotheses* (1.3)−(1.5) *be satisfied. Then there exist positive constants* $C(u_0, f)$, $C(u_0, f, \Omega)$ *depending on the data but not on* N *such that for all* $n = 1, \dots, N$, *we have*

1. $\|U^n\|_1 \le C(u_0, f)$
2. $\tau \sum_{i=1}^n \left\|b(U^i)\right\|_1 \le C(u_0, f)$
3. $\sum_{i=1}^n \left\|U^i - U^{i-1}\right\|_1 \le C(, u_0, f)$
4. $\tau \sum_{i=1}^n \rho_{p(.)}(\nabla T_k(U^i)) \le kC(u_0, f)$
5. $\tau \sum_{i=1}^n \int_{\{|U^i|\le k\}} |\nabla U^i|^{p_-} dx \le kC(u_0, f, \Omega)$

Proof. 1 and 2. For $\varphi = 0$ as a test function in (3.1), we have

$$\begin{aligned} &\frac{\tau}{k}\int_\Omega a(x, \nabla T_k(U^i))\nabla T_k(U^i)dx + \int_\Omega U^i \frac{T_k(U^i)}{k}dx + \int_\Omega \tau b(U^i)\frac{T_k(U^i)}{k}dx \\ &\le \tau \left\|f_i\right\|_1 + \left\|U^{i-1}\right\|_1 dx. \end{aligned}$$

Since

$$\int_\Omega a(x, \nabla T_k(U^i))\nabla T_k(U^i)dx \geq 0.$$

Then, it follows that

$$\int_\Omega U^i \frac{T_k(U^i)}{k}dx + \int_\Omega \tau b(U^i)\frac{T_k(U^i)}{k}dx \leq \tau \|f_i\|_1 + \|U^{i-1}\|_1 .$$

Then letting $k \to 0$ and using Fatou's lemma, we deduce that

$$\|U^i\|_1 + \tau \|b(U^i)\|_1 \leq \tau \|f_i\|_1 + \|U^{i-1}\|_1 . \tag{4.1}$$

Now, we sum (4.1) from $i = 1$ to n to obtain

$$\|U^n\|_1 + \tau \sum_{i=1}^{n} \|b(U^i)\|_1 \leq \|f\|_1 + \|u_0\|_1 \tag{4.2}$$

which give, the inequalities 1 and 2.

3. For $k \geq 1$, we take $\varphi = T_h(U^i - \text{sign}(U^i - U^{i-1}))$, $(h > 1)$ as a test function in (3.1), then letting $h \to \infty$, for $k \geq 1$, we obtain,

$$\tau \lim_{h\to\infty} \mathcal{I}(k, h) + \|U^i - U^{i-1}\|_1 \leq \tau \left(\|f_i\|_1 + \|b(U^i)\|_1\right)$$

where

$$\begin{aligned} \mathcal{I}(k, h) &:= \int_\Omega a(x, \nabla U^i)\nabla T_k(U^i - T_h(U^i - \text{sign}(U^i - U^{i-1})))dx \\ &= \int_{\Omega_{k,h}\cap\overline{\Omega(k)}} a(x, \nabla U^i)\nabla U^i dx \end{aligned}$$

and

$$\begin{aligned} \Omega_{k,h} &:= \left\{|U^i - T_h(U^i - \text{sign}(U^i - U^{i-1}))| \leq k\right\} \\ \overline{\Omega(k)} &= \left\{|U^i - \text{sign}(U^i - U^{i-1})| > h\right\}. \end{aligned}$$

Then by the hypothesis (1.3) , we have

$$\lim_{h\to\infty} \mathcal{I}(k, h) \geq 0.$$

Then, it follows that

$$\|U^i - U^{i-1}\|_1 \leq k\tau \left(\|f_i\|_1 + \|b(U^i)\|_1\right). \tag{4.3}$$

Then, summing (4.3) from $i = 1$ to n and by the stability result 2, we obtain the stability result 3.

4. We take $\varphi = 0$ as a test function in 3.1 to get

$$\tau \left(\int_\Omega |a(x, \nabla T_k(U^i))\nabla T_k(U^i)dx\right) \leq k\tau(\|f_i\|_1 + \|b(U^i)\|_1) + k \|U^i - U^{i-1}\|_1 .$$

Therefore, using the assumption (1.5) it follows that

$$\tau\rho_{p(x)}(\nabla T_k(U^i)) \leq C_3[k\tau(\|f_i\|_1 + \left\|b(U^i)\right\|_1) + k\left\|U^i - U^{i-1}\right\|_1]. \tag{4.4}$$

Now, summing (4.4) from $i = 1$ to n and using the stability results 1, 2, 3, we get

$$\begin{aligned} \tau\sum_{i=1}^{n}\rho_{p(x)}(\nabla T_k(U^i)) &\leq C_3 k\left[\|f\|_1 + \tau\sum_{i=1}^{n}\left\|b(U^i)\right\|_1 + \sum_{i=1}^{n}\left\|U^i - U^{i-1}\right\|_1\right] \\ &\leq kC(f, u_0). \end{aligned} \tag{4.5}$$

5. According to (4.5), we get from the above estimate

$$\tau\sum_{i=1}^{n}\int_{\{|U^i|\leq k\}}|\nabla U^i|^{p(x)}dx \leq kC(u_0, f). \tag{4.6}$$

Now, note that

$$\begin{aligned} \int_{\{|U^i|\leq k\}}|\nabla U^i|^{p_-}dx &= \int_{\left\{|U^i|\leq k,\ |\nabla U^i|>\frac{1}{N}\right\}}|\nabla U^i|^{p_-}dx + \int_{\left\{|U^i|\leq k\ |\nabla U^i|\leq\frac{1}{N}\right\}}|\nabla U^i|^{p_-}dx \\ &\leq \int_{\left\{|U^i|\leq k,\ |\nabla U^i|>\frac{1}{N}\right\}}|\nabla U^i|^{p_-}dx + \frac{1}{N}\text{meas}(\Omega) \\ &\leq \int_{\{|U^i|\leq k\}}|\nabla U^i|^{p(x)}dx + \frac{1}{N}\text{meas}(\Omega). \end{aligned}$$

By the inequalities above, thanks to (4.6), we obtain

$$\begin{aligned} \tau\sum_{i=1}^{n}\int_{\{|U^i|\leq k\}}|\nabla U^i|^{p_-}dx &\leq kC(u_0, f) + \frac{n}{N}\text{meas}(\Omega) \\ &\leq kC(u_0, f) + \text{meas}(\Omega) \leq k(C(u_0, f) + \text{meas}(\Omega)) \end{aligned} \tag{4.7}$$

for all $k \geq 1$. □

5 Convergence and existence result

In this section, we prove the existence of an entropy solution of problem (P). First of all, we introduce the appropriate spaces for the entropy formulation of the nonlinear parabolic problem (P).

We define the space:

$$V = \left\{v \in L^{p_-}(0, T; W^{1,p(\cdot)}(\Omega)) : \nabla v \in (L^{p(\cdot)}(Q_T))^d\right\},$$

and

$$\begin{aligned} \mathcal{T}^{1,p(\cdot)}(Q_T) &= \Big\{u : \Omega\times(0, T];\text{measurable} \mid T_k(u) \in L^{p_-}(0, T; W^{1,p(\cdot)}(\Omega)) \\ &\qquad \text{with } \nabla T_k(u) \in (L^{p(\cdot)}(Q_T))^d \ \text{ for every } k > 0\Big\}. \end{aligned}$$

Definition 5.1. *An entropy solution to problem (P) is a function $u \in \mathcal{T}^{1,p(\cdot)}(Q_T) \cap C(0,T;L^1(\Omega))$ such that and for all $k>0$ we have*

$$\begin{aligned}&\int_0^t\int_\Omega a(x,\nabla u)\nabla T_k(u-\varphi)+\int_0^t\int_\Omega b(u)T_k(u-\varphi)\\&\leq -\int_0^t\left\langle \frac{\partial\varphi}{\partial s},T_k(u-\varphi)\right\rangle+\int_\Omega J_k(u(0)-\varphi(0))-\int_\Omega J_k(u(t)-\varphi(t))\\&+\int_0^t\int_\Omega fT_k(u-\varphi)\end{aligned}$$

for all $\varphi \in L^\infty(Q)\cap V\cap W^{1,1}(0,T;L^1(\Omega))$ and $t\in[0,T]$.

Our main result is

Theorem 5.2. *Let hypotheses $(H1)-(H3)$ be satisfied. Then the nonlinear parabolic problem (P) has an entropy solution.*

Proof. The proof is divided into two steps

Step 1: The Rothe function. We introduce a piecewise linear extension:

$$\begin{cases} u^N(0) & := & u_0,\\ \\ u^N(t) & := & U^{n-1}+(U^n-U^{n-1})\frac{t-t^{n-1}}{\tau}\end{cases}\tag{5.1}$$

for all $t\in]t^{n-1},t^n]$, $n=1,\cdots,N$, in Ω and a piecewise constant function

$$\begin{cases} \overline{u}^N(0) & := & u_0,\\ \\ \overline{u}^N(t) & := & U^n,\ \forall t\in]t^{n-1},t^n],\ n=1,\cdots,N,\text{ in }\Omega,\end{cases}\tag{5.2}$$

where $t^n := n\tau$ and $(U^n)_{1\leq n\leq N}$ an entropy solution of (P_n).

By Theorem 3.3, for any $N\in\mathbb{N}$; the solution $(U^n)_{N\in\mathbb{N}}$ of problems (P_n) is unique. Thus, u^N and $\overline{u}^N$ are uniquely defined. Consequently, by the Theorem 4.1, we deduce the existence of a constant $C(T,u_0,f)$ not depending on N such that for all $N\in\mathbb{N}$, we have

$$\begin{aligned}&\left\|\overline{u}^N-u^N\right\|_{L^1(Q_T)}\leq\frac{1}{N}C(T,u_0,f)\\&\left\|u^N\right\|_{L^1(Q_T)}\leq C(T,u_0,f)\\&\left\|\overline{u}^N\right\|_{L^1(Q_T)}\leq C(T,u_0,f)\\&\left\|\frac{\partial u^N}{\partial t}\right\|_{L^1(Q_T)}\leq C(T,u_0,f)\\&\left\|b(\overline{u}^N)\right\|_{L^1(Q_T)}\leq C(T,u_0,f)\end{aligned}\tag{5.3}$$

Moreover combining Proposition 2.1 and Young inequality, we get

$$\begin{aligned}
\left\|\nabla T_k(U^N)\right\|_{p(x)}^{p_-} &\leq \max\left\{\rho_{p(x)}(\nabla T_k(U^N)), \rho_{1,p(x)}(\nabla T_k U^N)^{\frac{p_-}{p_+}}\right\} \\
&\leq \rho_{p(x)}(\nabla T_k(U^N)) + \rho_{1,p(x)}(\nabla T_k U^N)^{\frac{p_-}{p_+}} \\
&\leq \rho_{p(x)}(\nabla T_k(U^N)) + \frac{p_-}{p_+}\rho_{p(x)}(\nabla T_k(U^N)) + 1 - \frac{p_-}{p_+} \qquad (5.4) \\
&\leq 2\rho_{p(x)}(\nabla T_k(U^N)) + 1.
\end{aligned}$$

Thanks to Poincaré-Wirtinger inequality, we have

$$\left\|T_k(U^N)\right\|_{p(x)} \leq C\text{meas}(\Omega)\left\|\nabla T_k(U^N)\right\|_{p(x)} + k\left\|1\right\|_{p(x)},$$

which implies that

$$\left\|T_k(U^N)\right\|_{p(x)}^{p_-} \leq 2^{p_- - 1}\left((C\text{meas}(\Omega))^{p_-}\left\|\nabla T_k(U^N)\right\|_{p(x)}^{p_-} + k^{p_-}\left\|1\right\|_{p(x)}^{p_-}\right), \qquad (5.5)$$

then it follows that,

$$\begin{aligned}
\left\|T_k(U^N)\right\|_{1,p(x)}^{p_-} &\leq 2^{p_- - 1}\left[(C\text{meas}(\Omega))^{p_-}\left(2\rho_{p(x)}(\nabla T_k(U^N)) + 1\right) + k^{p_-}\left\|1\right\|_{p(x)}^{p_-}\right] \qquad (5.6) \\
&\quad + 2\rho_{p(x)}(\nabla T_k(U^N)) + 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^T \left\|T_k(U^N)\right\|_{1,p(.)}^{p_-} dt &\leq 2^{p_- - 1}\left[(C\text{meas}(\Omega))^{p_-}\left(2\int_0^T \rho_{p(.)}(\nabla T_k(U^N))dt + T\right)\right. \\
&\quad \left. + Tk^{p_-}\left\|1\right\|_{p(x)}^{p_-}\right] + 2\int_0^T \rho_{p(.)}(\nabla T_k(U^N))dt + T \\
&\leq 2^{p_- - 1}\left[(C\text{meas}(\Omega))^{p_-}\left(2\sum_{n=1}^N \int_{(n-1)\tau}^{n\tau} \rho_{p(.)}(\nabla T_k(U^N))dt + T\right)\right. \\
&\quad \left. + Tk^{p_-}\left\|1\right\|_{p(.)}^{p_-}\right] + 2\sum_{n=1}^N \int_{(n-1)\tau}^{n\tau} \rho_{p(.)}(\nabla T_k(U^N))dt + T \qquad (5.7) \\
&\leq 2^{p_- - 1}\left[(C\text{meas}(\Omega))^{p_-}\left(2\sum_{n=1}^N \tau\rho_{p(.)}(\nabla T_k(U^n)) + T\right)\right. \\
&\quad \left. + Tk^{p_-}\left\|1\right\|_{p(.)}^{p_-}\right] + 2\sum_{n=1}^N \tau\rho_{1,p(.)}(T_k(U^n)) + T.
\end{aligned}$$

Consequently from stability result 4 it follows that

$$\left\|T_k(\overline{u}^N)\right\|_{L^{p_-}(0,T;W^{1,p(x)}(\Omega))} \leq C(T, k, u_0, f, p_-). \qquad (5.8)$$

Lemma 5.3. *Let hypotheses* (1.3) – (1.5) *be satisfied. Then the sequence* $(\overline{u}^N)_{N\in\mathbb{N}}$ *converges in measure and a.e. in* Q_T.

Proof. Let ε, r, k be positive numbers. For $N, M \in \mathbb{N}$, we have the inclusion

$$\begin{aligned}\left\{|\overline{u}^N - \overline{u}^M| > r\right\} &\subset \left\{|\overline{u}^N| > k\right\} \cup \left\{|\overline{u}^M| > k\right\} \\ &\cup \left\{|\overline{u}^N| \leq k,\, |\overline{u}^M| \leq k, |\overline{u}^N - \overline{u}^M| > r\right\}.\end{aligned}$$

Firstly, we have

$$\text{meas}\left\{|\overline{u}^N| > k\right\} \leq \frac{1}{k}\left\|\overline{u}^N\right\|_{L^1(Q_T)} \leq \frac{1}{k}C(T, u_0, f). \tag{5.9}$$

Similarly, we have

$$\text{meas}\left\{|\overline{u}^M| > k\right\} \leq \frac{1}{k}\left\|\overline{u}^N\right\|_{L^1(Q_T)} \leq \frac{1}{k}C(T, u_0, f). \tag{5.10}$$

Therefore, for k large enough, we have

$$\text{meas}(\left\{|\overline{u}^M| > k\right\} \cup \left\{|\overline{u}^M| > k\right\}) \leq \frac{\varepsilon}{2}. \tag{5.11}$$

Secondly, by the Proposition 2.1 and Young inequality, we have

$$\begin{aligned}\left\|\nabla T_k(\overline{u}^N)\right\|_{L^{p(.)}(Q_T)} &\leq \max\left\{\left(\int_0^T\int_\Omega |\nabla T_k(\overline{u}^N)|^{p(x)}dxdt\right)^{\frac{1}{p_-}}; \left(\int_0^T\int_\Omega |\nabla T_k(\overline{u}^N)|^{p(x)}dxdt\right)^{\frac{1}{p_+}}\right\} \\ &\leq \left(\int_0^T\int_\Omega |\nabla T_k(\overline{u}^N)|^{p(x)}dxdt\right)^{\frac{1}{p_-}} + \left(\int_0^T\int_\Omega |\nabla T_k(\overline{u}^N)|^{p(x)}dxdt\right)^{\frac{1}{p_+}}\end{aligned}$$

and also, we have

$$\begin{aligned}\int_0^T\int_\Omega |\nabla T_k(\overline{u}^N)|^{p(x)}dxdt = \int_0^T \rho_{p(.)}(T_k(\nabla\overline{u}^N)) &= \sum_{n=1}^N \int_{(n-1)\tau}^{n\tau} \rho_{p(.)}(\nabla T_k(U^N))dt \\ &\leq \sum_{n=1}^N \tau\rho_{p(.)}(\nabla T_k(U^n)).\end{aligned}$$

Therefore, using the stability result 4 and Proposition 2.1, it follows

$$\left\|\nabla T_k(\overline{u}^N)\right\|_{(L^{p(x)}(Q_T))^d} \leq (kC(u_0, f))^{\frac{1}{p_-}} + (kC(u_0, f))^{\frac{1}{p_+}}. \tag{5.12}$$

Since by the Poincaré-Wirtinger inequality, we have

$$\left\|T_k(\overline{u}^N)\right\|_{L^{p(x)}(Q_T)} \leq C\text{meas}(\Omega)\left\|\nabla T_k(\overline{u}^N)\right\|_{L^{p(x)}(Q_T)} + k\left\|1\right\|_{L^{p(x)}(Q_T)},$$

then by (5.12), we get

$$\left\|T_k(\overline{u}^N)\right\|_{L^{p(x)}(Q_T)} \leq C\text{meas}(\Omega)\left((kC(u_0, f))^{\frac{1}{p_-}} + (kC(u_0, f))^{\frac{1}{p_+}}\right) + k\left\|1\right\||_{L^{p(x)}(Q_T)}. \tag{5.13}$$

Hence, the sequences $(T_k(\overline{u}^N))_{N\in\mathbb{N}}$ are bounded in $L^{p(.)}(Q_T)$. Then, there exists a subsequence, still denoted by $(T_k(\overline{u}^N))_{N\in\mathbb{N}}$, that is a Cauchy sequence in $L^{p(.)}(Q_T)$ and in measure. Thus, there exists $N_0 \in \mathbb{N}$ such that for all $N, M \geq N_0$, we have

$$\text{meas}\left(\left\{|\overline{u}^N| \leq k,\, |\overline{u}^M| \leq k, |\overline{u}^N - \overline{u}^M| > r\right\}\right) < \frac{\varepsilon}{2}. \tag{5.14}$$

Then, by (5.11) and (5.14), $(\overline{u}^N)_{N\in\mathbb{N}}$ converges in measure. Therefore there exists an element $u \in M(Q_T)$ such that

$$\overline{u}^N \to u \ \text{ a.e. in}\, Q_T.$$

□

Now, by (5.12)

$$(\nabla T_k(\overline{u}^N))_{N\in\mathbb{N}} \text{ is uniformly bounded in, } (L^{p(.)}(Q_T))^d. \tag{5.15}$$

Hence there exists a subsequence, still denoted by

$$(\nabla T_k(\overline{u}^N))_{N\in\mathbb{N}} \quad \text{converges weakly to an element } \; V \text{ in } L^{p(.)}(Q_T).$$

Since

$$T_k(\overline{u}^N) \quad \text{converges weakly to } \; T_k(u) \text{ in } L^{p(.)}(Q_T).$$

Then

$$\nabla T_k(\overline{u}^N) \quad \text{converges weakly to } \; \nabla T_k(u) \text{ in } (L^{p(.)}(Q_T))^d. \tag{5.16}$$

and by (5.8) we conclude that

$$T_k(u) \in L^{p_-}(0,T;W^{1,p(.)}(\Omega)) \; \text{ for all } \; k>0.$$

In the sequel, we need the following Lemma (see [22]).

Lemma 5.4. *Let $(v_n)_{n\geq 1}$ be a sequence of measurable functions in Ω. If $(v_n)_{n\geq 1}$ converges in measure to v and is uniformly bounded in $L^{p(.)}(\Omega)$ for some $1 << p(.) \in L^\infty(\Omega)$, then $(v_n)_{n\geq 1} \to v$ strongly in $L^1(\Omega)$.*

Now, we have the following result

Lemma 5.5. *Let hypotheses $(1.3)-(1.5)$ be satisfied. Then*

(i) $(\nabla T_k(\overline{u}^N))_{N\in\mathbb{N}}$ *converges in measure to* $\nabla T_k(u)$;

(ii) $(a(x,T_k(\overline{u}^N)))_{N\in\mathbb{N}}$ *converges strongly to* $a(x,\nabla T_k(u))$ *in* $(L^1(Q_T))^d$ *and weakly in* $(L^{p'(.)}(Q_T))^d$.

Proof. (i) Let $h\geq 1$, from the Hölder type inequality, we have

$$\begin{aligned}
\text{meas}\left\{|\nabla T_k(\overline{u}^N)-\nabla T_k(u)|>h\right\} &\leq \frac{1}{h}\int_{Q_T}|\nabla T_k(\overline{u}^N)-\nabla T_k(u)|dxds \\
&\leq \frac{1}{h}\left(\frac{1}{p_-}+\frac{1}{p_+}\right)\left\|\nabla T_k(\overline{u}^N)-\nabla T_k(u)\right\|_{p(.)}\|1\|_{p'(.)} \qquad (5.17)\\
&\leq \frac{1}{h}\left(\frac{1}{p_-}+\frac{1}{(p_-)'}\right)\left(\left\|\nabla T_k(\overline{u}^N)\right\|_{p(.)}+\|\nabla T_k(u)\|_{p(.)}\right)\|1\|_{p'(.)}.
\end{aligned}$$

So by (5.15), $\text{meas}\left\{|\nabla T_k(\overline{u}^N)-\nabla T_k(u)|>h\right\}\to 0$ as $h\to\infty$ for any fixed $k>0$ and the proof of (i) is complete.

As a consequence of (i), up to a subsequence, we can assume that $\nabla T_k(\overline{u}^N)\to\nabla T_k(u)$ a.e in Q_T.

(ii) Since $a(x,\xi)$ is continuous with respect to $\xi\in\mathbb{R}^N$, then by (i) we deduce that

$$(a(x,T_k(\overline{u}^N)))_{N\in\mathbb{N}} \text{ converges in measure to } a(x,\nabla T_k(u)) \text{ and a.e. in } Q_T.$$

Moreover, using the hypotheses (1.3) and (5.12) one shows that $(a(x,\nabla T_k(\overline{u}^N)))_{N\in\mathbb{N}}$ is uniformly bounded in $(L^{p'(.)}(Q_T))^d$.

Consequently, in the one part thanks to Lemma 5.4 it follows that $(a(x,T_k(\overline{u}^N)))_{N\in\mathbb{N}} \to a(x,\nabla T_k(u))$ strongly in $\left(L^1(Q_T)\right)^d$.

On the other part, we can extract a subsequence still denoted by $(a(x,\nabla T_k(\overline{u}^N)))_{N\in\mathbb{N}}$ such that $a(x,\nabla T_k(\overline{u}^N)) \rightharpoonup \zeta_k$ in $(L^{p'(.)}(Q_T))^d$. Since each of the convergence implies the weak L^1-convergence, ζ_k can be identified with $a(x,\nabla T_k(u))$, thus $a(x,\nabla T_k(u)) \in (L^{p'(.)}(Q_T))^d$. This completes the proof. ☐

Lemma 5.6. $(\overline{u}^N)_{N\in\mathbb{N}}$ *converges a.e. in* Σ_T.

Proof. We know that the trace operator is compact from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, then there exists a constant C such that

$$\int_0^T \left\|T_k(\overline{u}^N(t)) - T_k(u(t))\right\|_{L^1(\partial\Omega)} dt \leq C\int_0^T \left\|T_k(\overline{u}^N(t)) - T_k(u(t))\right\|_{W^{1,1}(\Omega)} dt.$$

Since $W^{1,p(.)}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ for all $p(.) \geq 1$, then by the Hölder type inequality, we deduce that

$$T_k(\overline{u}^N(t)) \to T_k(u) \,\text{in}\, L^1(\Sigma_T) \text{ and a.e. on } \Sigma_T.$$

So, there exists $A \subset \Sigma_T$ such that $T_k(\overline{u}^N(t))$ converges to $T_k(u(t))$ on $\Sigma_T \setminus A$ with meas$(A) = 0$. For every $k > 0$, we set

$$A_k = \{(t,x) \in \Sigma_T : |T_k(u(t))| < k\}, \quad \text{and } B = \Sigma_T \setminus \bigcup_{k=1}^{\infty} A_k.$$

We have, by Hölder's inequality

$$\begin{aligned}
\text{meas}(B) &\leq \frac{1}{k}\int_B |T_k(u)|\, d\sigma \\
&\leq \frac{1}{k}\int_0^T \|T_k(u)\|_{L^1(\partial\Omega)} dt \\
&\leq \frac{1}{k}\int_0^T \|T_k(u)\|_{W^{1,1}(\Omega)} dt \qquad (5.18)\\
&\leq \frac{1}{k}\int_0^T\int_\Omega (|T_k(u)| + |\nabla T_k(u)|) \\
&\leq \frac{1}{k}\left(\frac{1}{p_-} + \frac{1}{(p_-)'}\right)\|1\|_{L^{p'(x)}(Q_T)}\left(\|T_k(u)\|_{L^{p(x)}(Q_T)} + \|\nabla T_k(u)\|_{(L^{p(x)}(Q_T))^d}\right).
\end{aligned}$$

Thanks to (5.12) and (5.13), for all $k > 0$, we have

$$\begin{aligned}
\left\|T_k(\overline{u}^N)\right\|_{L^{p(x)}(Q)} + \left\|\nabla T_k(\overline{u}^N)\right\|_{(L^{p(x)}(Q))^d} \leq\ & 2\left(k^{\frac{1}{p_-}} + k^{\frac{1}{p_+}}\right) \qquad (5.19)\\
& \times \max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}}, C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\}
\end{aligned}$$

We now use the Fatou's lemma in (5.19) to get

$$\begin{aligned}\|T_k(u)\|_{L^{p(x)}(Q)} + \|\nabla T_k(u)\|_{(L^{p(x)}(Q))^d} \quad &\leq \quad 2\left(k^{\frac{1}{p_-}} + k^{\frac{1}{p_+}}\right) \\ &\times \max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}}, C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\},\end{aligned}$$

and (5.18) becomes

$$\operatorname{meas}(B) \leq 2\left(\frac{1}{k^{1-\frac{1}{p_-}}} + \frac{1}{k^{1-\frac{1}{p_+}}}\right) \max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}}, C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\}. \tag{5.20}$$

Therefore, we get by letting $k \to \infty$ in (5.20) that $\operatorname{meas}(B) = 0$.
Let us now define on $\partial\Omega$, the function v by

$$v(t,x) = T_k(u(t))(x) \;\text{ if }\; (x,t) \in A_k.$$

We take $(x,t) \in \Sigma_T \setminus (A \cup B)$; then there exists $k > 0$ such that $(x,t) \in A_k$ and we have

$$\overline{u}^N(t,x) - v(t,x) = (\overline{u}^N(t,x) - T_k(\overline{u}^N(t))(x)) + (T_k(\overline{u}^N(t))(x) - T_k(u(t))(x)).$$

Since $(x,t) \in A_k$, we have $\left|T_k(\overline{u}^N(t))(x)\right| < k$ from which we deduce that $T_k(\overline{u}^N(t))(x) = \overline{u}^N(t,x)$. Therefore,

$$\overline{u}^N(t,x) - v(t,x) = (T_k(\overline{u}^N(t))(x) - T_k(u(t))(x)) \to 0, \text{ as } N \to \infty.$$

This means that $\left(\overline{u}^N\right)$ converges to v a.e. on Σ_T. ☐

Lemma 5.7. *The sequence $(\overline{u}^N)_{N\in\mathbb{N}}$ converges to u in $C(0,T;L^1(\Omega))$.*

Proof. Let $(t^n = n\tau_N)_{n=1}^N$ and $(t^m = m\tau_M)_{n=1}^M$ be two partitions of the interval $[0,T]$ and let $(u^N(t), \overline{u}^N(t))$, $(u^M(t); \overline{u}^M(t))$; be the semi-discrete solutions defined by (5.1), (5.2) and corresponding to the respective partitions. Let $\varphi \in L^\infty(\Omega) \cap V \cap W^{1,1}(0,T;L^1(\Omega))$. We rewrite (3.1) in the forms

$$\begin{aligned}&\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_k(\overline{u}^N - \varphi)\right\rangle ds + \int_0^t\int_\Omega a(x,\nabla\overline{u}^N).\nabla T_k(\overline{u}^N - \varphi)dxds \\ &+ \int_0^t\int_\Omega b(\overline{u}^N)T_k(\overline{u}^N - \varphi)dxds \\ &\leq \int_0^t\int_\Omega f_N T_k(\overline{u}^N - \varphi)dxds\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}&\int_0^t \left\langle \frac{\partial u^M}{\partial s}, T_k(\overline{u}^M - \varphi)\right\rangle ds + \int_0^t\int_\Omega a(x,\nabla\overline{u}^M).\nabla T_k(\overline{u}^M - \varphi)dxds \\ &+ \int_0^t\int_\Omega b(\overline{u}^M)T_k(\overline{u}^M - \varphi)dxds \\ &\leq \int_0^t\int_\Omega f_M T_k(\overline{u}^M - \varphi)dxds\end{aligned} \tag{5.22}$$

where

$$f_N(t,x) = f_n(x) \qquad \forall t \in]t^{n-1}, t^n]$$

$$f_M(t,x) = f_m(x) \qquad \forall t \in]t^{m-1}, t^m]$$

Let $h > 1$, in inequality (5.21) we take $\varphi = T_h(\overline{u}^M)$ and in inequality (5.22) we take $\varphi = T_h(\overline{u}^N)$. Summing both inequalities, we get, for $k = 1$,

$$\begin{aligned}
& \int_0^t \left\langle \frac{\partial(u^N - u^M)}{\partial s}, T_1(u^N - u^M) \right\rangle ds + I_{N,M}(h) \\
& + \int_0^t \int_\Omega b(\overline{u}^N) T_1(\overline{u}^N - T_h(\overline{u}^M)) dxds \\
& + \int_0^t \int_\Omega b(\overline{u}^M) T_1(\overline{u}^M - T_h(\overline{u}^N)) dxds \\
\leq \; & \int_0^t \left\langle \frac{\partial(u^N - u^M)}{\partial s}, T_1(u^N - u^M) \right\rangle - \left\langle \frac{\partial u^N}{\partial s}, T_1(\overline{u}^N - T_h(\overline{u}^M)) \right\rangle ds \qquad (5.23) \\
& - \int_0^t \left\langle \frac{\partial u^M}{\partial s}, T_1(\overline{u}^M - T_h(\overline{u}^N)) \right\rangle ds \\
& + \int_0^t \int_\Omega [f_N T_1(\overline{u}^N - T_h(\overline{u}^M)) + f_M T_1(\overline{u}^M - T_h(\overline{u}^N))] dxds
\end{aligned}$$

where

$$\begin{aligned}
I_{N,M}(h) \; = \; & \int_0^t \int_\Omega a(x, \nabla \overline{u}^N).\nabla T_1(\overline{u}^N - T_h(\overline{u}^M)) dxds \\
& + \int_0^t \int_\Omega a(x, \nabla \overline{u}^M).\nabla T_1(\overline{u}^M - T_h(\overline{u}^N)) dxds.
\end{aligned}$$

We have

$$\begin{aligned}
\left| \int_0^t \left\langle \frac{\partial(u^N - u^M)}{\partial s}, T_1(u^N - u^M) \right\rangle ds \right| \; & \leq \; \left\| \frac{\partial(u^N - u^M)}{\partial s} \right\|_{L^1(Q_T)} \left\| T_1(u^N - u^M) \right\|_{L^\infty(Q_T)} \\
& \leq \; 2C(T, f, u_0) \left\| T_1(u^N - u^M) \right\|_{L^\infty(Q_T)}.
\end{aligned}$$

Since

$$\lim_{N,M \to \infty} \left\| T_1(u^N - u^M) \right\|_{L^\infty(Q_T)} = 0.$$

Then it follows that

$$\lim_{h \to \infty} \lim_{N,M \to \infty} \int_0^t \left\langle \frac{\partial(u^N - u^M)}{\partial s}, T_1(u^N - u^M) \right\rangle ds = 0. \qquad (5.24)$$

Similarly, we show that

$$\lim_{h \to \infty} \lim_{N,M \to \infty} \left(\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_1(\overline{u}^N - T_h(\overline{u}^M)) \right\rangle + \left\langle \frac{\partial u^M}{\partial s}, T_1(\overline{u}^M - T_h(\overline{u}^N)) \right\rangle ds \right) = 0$$

$$\lim_{h \to \infty} \lim_{N,M \to \infty} \int_0^t \int_\Omega [f_N T_1(\overline{u}^N - T_h(\overline{u}^M)) + f_M T_1(\overline{u}^M - T_h(\overline{u}^N))] dxds = 0$$

and

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_\Omega b(\overline{u}^N)T_1(\overline{u}^N-T_h(\overline{u}^M))dxds+\int_0^t\int_\Omega b(\overline{u}^M)T_1(\overline{u}^M-T_h(\overline{u}^N))dxds=0.$$

Then, letting N, $M\to\infty$ and $h\to\infty$, in (5.23)we get

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\left\langle\frac{\partial(u^N-u^M)}{\partial s},T_1(u^N-u^M)\right\rangle ds+\lim_{h\to\infty}\lim_{N,M\to\infty}I_{N,M}(h)\leq 0. \tag{5.25}$$

Since

$$\left\langle\frac{\partial v}{\partial t},T_k(v)\right\rangle=\frac{d}{dt}\int_\Omega J_k(v)\quad\text{in } L^1(]0,T[),$$

inequality (5.25) becomes

$$\lim_{N,M\to\infty}\int_\Omega J_1(u^N(t)-u^M(t))dx+\lim_{h\to\infty}\lim_{N,M\to\infty}I_{N,M}(h)\leq 0. \tag{5.26}$$

Now, we show that

$$\lim_{h\to\infty}\lim_{N,M\to\infty}I_{N,M}(h)\geq 0.$$

We consider the decomposition

$$I_{N,M}(h)=\sum_{i=1}^4 L_i(h),$$

where

$$\begin{aligned}L_i(h)=&\int_0^t\int_{\Omega_i(h)}a(x,\nabla\overline{u}^N).\nabla T_1(\overline{u}^N-T_h(\overline{u}^M))dxds\\&+\int_0^t\int_{\Omega_i(h)}a(x,\nabla\overline{u}^M).\nabla T_1(\overline{u}^M-T_h(\overline{u}^N))dxds\end{aligned}$$

and

$$\begin{aligned}\Omega_1(h)&=\left\{|\overline{u}^N|\leq h,\,|\overline{u}^M|\leq h\right\}\qquad \Omega_2(h)=\left\{|\overline{u}^N|\leq h,\,|\overline{u}^M|>h\right\}\\ \Omega_3(h)&=\left\{|\overline{u}^N|>h,\,|\overline{u}^M|\leq h\right\}\qquad \Omega_4(h)=\left\{|\overline{u}^N|>h,\,|\overline{u}^M|>h\right\}.\end{aligned}$$

On the one hand, thanks to assumption (1.4) we have

$$L_1(h)=\int_0^t\int_{\Omega_1^1(h)}[a(x,\nabla\overline{u}^N)-a(x,\nabla\overline{u}^M)].\nabla(\overline{u}^N-\overline{u}^M)dxds\geq 0.$$

Therefore

$$\lim_{h\to\infty}\lim_{N,M\to\infty}L_1(h)\geq 0.$$

On the other hand, we have

$$\begin{aligned}L_2(h)\;&=\;\int_0^t\int_{\Omega_2^1(h)}a(x,\nabla\overline{u}^N).\nabla\overline{u}^N dxds\\&\quad+\int_0^t\int_{\Omega_2^2(h)}a(x,\nabla\overline{u}^M).\nabla(\overline{u}^M-\overline{u}^N)dxds\\&\geq\;-\int_0^t\int_{\Omega_2^2(h)}a(x,\nabla\overline{u}^M).\nabla\overline{u}^N dxds,\end{aligned}$$

where

$$\begin{aligned}
\Omega_2^1(h) &= \left\{|\overline{u}^N| \le h,\, |\overline{u}^M| > h, |\overline{u}^N - h\text{sign}(\overline{u}^M)| \le 1\right\},\\
\Omega_2^2(h) &= \left\{|\overline{u}^N| \le h,\, |\overline{u}^M| > h, |\overline{u}^N - \overline{u}^M| \le 1\right\}.
\end{aligned}$$

Now, taking $\varphi = T_h(\overline{u}^N)$ in (5.21), we deduce that

$$\lim_{h\to\infty}\lim_{N\to\infty}\int_0^t\int_{\left\{h\le|\overline{u}^N|\le h+k\right\}} a(x,\nabla\overline{u}^N).\nabla\overline{u}^N = 0.$$

This implies

$$\lim_{h\to\infty}\lim_{N\to\infty}\int_0^t\int_{\left\{h\le|\overline{u}^N|\le h+k\right\}} |\nabla\overline{u}^N|^{p(x)} = 0, \quad k>0. \tag{5.27}$$

By the Young inequality, we have

$$\begin{aligned}
&\left|\int_0^t\int_{\Omega_2^2(h)} a(x,\nabla\overline{u}^M).\nabla\overline{u}^N dxds\right|\\
\le\ & \int_0^t\int_{\Omega_2^2(h)} |\nabla\overline{u}^M|^{p(x)-1}|\nabla\overline{u}^N|dxds\\
\le\ & \int_0^t\int_{\left\{h\le|\overline{u}^M|\le h+1\right\}}\frac{1}{p'(x)}|\nabla\overline{u}^M|^{p(x)}dxds + \int_0^t\int_{\left\{h-1\le|\overline{u}^N|\le h\right\}}\frac{1}{p(x)}|\nabla\overline{u}^M|^{p(x)}dxds\\
\le\ & \int_0^t\int_{\left\{h\le|\overline{u}^M|\le h+1\right\}}\frac{1}{p'_-}|\nabla\overline{u}^M|^{p(x)}dxds + \int_0^t\int_{\left\{h-1\le|\overline{u}^N|\le h\right\}}\frac{1}{p_-}|\nabla\overline{u}^M|^{p(x)}dxds.
\end{aligned}$$

Thus (5.27) gives

$$\lim_{N,M\to\infty}\int_0^t\int_0^t\int_{\Omega_2^2(h)} a(x,\nabla\overline{u}^M).\nabla\overline{u}^N dxds = 0,$$

which implies that

$$\lim_{h\to\infty}\lim_{N,M\to\infty} L_2(h) \ge 0.$$

Similarly, we show that

$$\lim_{h\to\infty}\lim_{N,M\to\infty}(L_3(h)+L_4(h)) \ge 0.$$

Therefore

$$\lim_{h\to\infty}\lim_{N,M\to\infty} I_{N,M}(h) \ge 0.$$

Thus (5.26) becomes

$$\lim_{N,M\to\infty}\int_\Omega J_1(u^N(t)-u^M(t))dx = 0. \tag{5.28}$$

Since

$$\frac{1}{2}\int_{\{|u^N-u^M|\le1\}}|u^N(t)-u^M(t)|^2dx + \int_{\{|u^N-u^M|\ge1\}}|u^N(t)-u^M(t)|dx \le \int_\Omega J_1(u^N(t)-u^M(t));$$

we have

$$\begin{aligned}
&\int_{\{|u^N-u^M|\geq 1\}} |u^N(t)-u^M(t)|dx \\
= \;& \int_{\{|u^N-u^M|\leq 1\}} |u^N(t)-u^M(t)|dx + \int_{\{|u^N-u^M|\geq 1\}} |u^N(t)-u^M(t)|dx \\
\leq \;& C_\Omega \left(\int_{\{|u^N-u^M|\leq 1\}} |u^N(t)-u^M(t)|^2 dx \right)^{\frac{1}{2}} + \int_{\{|u^N-u^M|\geq 1\}} |u^N(t)-u^M(t)|dx \\
\leq \;& C_2(\Omega) \left(\int_\Omega J_1(u^N(t)-u^M(t))dx \right)^{\frac{1}{2}} + \int_\Omega J_1(u^N(t)-u^M(t))dx.
\end{aligned}$$

By (5.26), we deduce that $(u^N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $C(0,T;L^1(\Omega))$. Hence $(u^N)_{N\in\mathbb{N}}$ converges to u in $C(0,T;L^1(\Omega))$. □

Step 2: Existence of entropy solution. Now, we prove that the limit function u is an entropy solution of the problem (P). Since $u^N(0) = U^0 = u_0$ for all $N \in \mathbb{N}$, we have $u(0,.) = u_0$, and inequality (5.21) implies

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_k(\overline{u}^N-\varphi) - T_k(u^N-\varphi) \right\rangle ds + \int_0^t \int_\Omega a(x,\nabla \overline{u}^N).\nabla T_k(\overline{u}^N-\varphi)dxds \\
&+ \int_0^t \int_\Omega b(\overline{u}^N) T_k(\overline{u}^N-\varphi)dxds \qquad (5.29)\\
&\leq \int_0^t \left\langle \frac{\varphi}{\partial s}, T_k(u^N-\varphi) - T_k(u^N-\varphi) \right\rangle ds + \int_\Omega J_k(u^N(0)-\varphi(0))dx - \int_\Omega J_k(u^N(t)-\varphi(t))dx \\
&+ \int_0^t \int_\Omega f_N T_k(\overline{u}^N-\varphi)dxds.
\end{aligned}$$

Let $\overline{k} = k + \|\varphi\|_\infty$. Then

$$\begin{aligned}
\int_0^t \int_\Omega a(x,\nabla \overline{u}^N).\nabla T_k(\overline{u}^N-\varphi)dxds &= \int_0^t \int_\Omega a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla T_k(T_{\overline{k}}(\overline{u}^N)-\varphi)dxds \\
&= \int_0^t \int_\Omega [a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla T_{\overline{k}}(\overline{u}^N) \\
&\quad - a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla\varphi]\mathbf{1}_{Q(N,k)}dxds,
\end{aligned}$$

where $Q(N,k) = \left\{|T_{\overline{k}}(\overline{u}^N)-\varphi| \leq k\right\}$. Thus, the inequality (5.29) becomes

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_k(\overline{u}^N-\varphi) - T_k(u^N-\varphi) \right\rangle ds - \int_0^t \int_\Omega a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla\varphi \mathbf{1}_{Q(N,k)} \\
&+ \int_0^t \int_\Omega [a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla T_{\overline{k}}(\overline{u}^N)]\mathbf{1}_{Q(N,k)} + \int_0^t \int_\Omega b(\overline{u}^N) T_k(\overline{u}^N-\varphi)dxds \qquad (5.30)\\
\leq \;& -\int_0^t \left\langle \frac{\partial\varphi}{\partial s}, T_k(u^N-\varphi) \right\rangle ds + \int_\Omega J_k(u^N(0)-\varphi(0))dx - \int_\Omega J_k(u^N(t)-\varphi(t))dx \\
&+ \int_0^t \int_\Omega f_N T_k(\overline{u}^N-\varphi)dxds.
\end{aligned}$$

On the one hand, thanks to Lemma 5.5 $a(x,\nabla T_{\overline{k}}(\overline{u}^N))$ converges weakly to $a(x,\nabla T_{\overline{k}}(u))$ in $\left(L^{p'(.)}(\Omega)\right)^d$. Therefore,

$$\lim_{N\to\infty}\int_0^t\int_\Omega a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla\varphi\mathbf{1}_{Q(N,k)}=\int_0^t\int_\Omega a(x,\nabla T_{\overline{k}}(u)).\nabla\varphi\mathbf{1}_{Q(k)}, \tag{5.31}$$

where $Q(k)=\left\{|T_{\overline{k}}(u)-\varphi|\le k\right\}$. Moreover, $a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla T_{\overline{k}}(\overline{u}^N)$ is nonnegative and converges a.e. in Q_T to $a(x,\nabla T_{\overline{k}}(u)).\nabla T_{\overline{k}}(u)$ (see Lemma 5.5). Therefore by Fatou's lemma, we obtain

$$\liminf_{N\to\infty}\int_0^t\int_0^t\int_\Omega[a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla T_{\overline{k}}(\overline{u}^N)]\mathbf{1}_{Q(N,k)}dxds\ge\int_0^t\int_0^t\int_\Omega[a(x,\nabla T_{\overline{k}}(u)).\nabla T_{\overline{k}}(u)]\mathbf{1}_{Q(k)}dxds.$$

For the fourth term of (5.30), we have

$$\int_0^t\int_\Omega b(\overline{u}^N)T_k(\overline{u}^N-\varphi)dxds=\int_0^t\int_\Omega(b(\overline{u}^N)-b(\varphi))T_k(\overline{u}^N-\varphi)dxds+\int_0^t\int_\Omega b(\varphi)T_k(\overline{u}^N-\varphi)dxds.$$

The quantity $(b(\overline{u}^N)-b(\varphi))T_k(\overline{u}^N-\varphi)$ is is nonnegative and since for all $s\in\mathbb{R}$, $s\mapsto b(s)$ is continuous, we obtain

$$(b(\overline{u}^N)-b(\varphi))T_k(\overline{u}^N-\varphi)\to(b(u)-b(\varphi))T_k(\overline{u}^N-\varphi)\quad\text{a.e. in }\ \Omega.$$

Then, it follows by Fatou's lemma that

$$\liminf_{N\to\infty}\int_0^t\int_\Omega(b(\overline{u}^N)-b(\varphi))T_k(\overline{u}^N-\varphi)dxds\ge\int_0^t\int_\Omega(b(u)-b(\varphi))T_k(u-\varphi)dxds.$$

We have $b(\varphi)\in L^1(Q_T)$. Since $T_k(\overline{u}^N-\varphi)$ converges weakly$-*$ to $T_k(u-\varphi)$ and $b(\varphi)\in L^1(Q_T)$, it follows that

$$\liminf_{N\to\infty}\int_0^t\int_\Omega b(\varphi)T_k(\overline{u}^N-\varphi)dxds\ge\int_0^t\int_\Omega b(\varphi)T_k(u-\varphi)dxds.$$

By Lemma 5.7 , we deduce that $u^N(t)\to u(t)$ in $L^1(\Omega)$ for all $t\in[0,T]$, which implies that

$$\int_\Omega J_k(u^N(t)-\varphi(t))dx\to\int_\Omega J_k(u(t)-\varphi(t))dx\quad\forall t\in[0,T]. \tag{5.32}$$

We follow the method used in the proof of equality (5.24) to show that

$$\lim_{N\to\infty}\int_0^t\left\langle\frac{\partial u^N}{\partial s},T_k(\overline{u}^N-\varphi)-T_k(u^N-\varphi)\right\rangle ds=0. \tag{5.33}$$

Finally, letting $N\to\infty$ and using the above results, the continuity of b and the facts that

$$f_N\to f\quad\text{in }L^1(Q_T),$$
$$T_{\overline{k}}(\overline{u}^N-\varphi)\to T_{\overline{k}}(u-\varphi)\quad\text{in }L^\infty(Q_T),$$

we deduce that u is an entropy solution of the nonlinear parabolic problem (P). □

6 Conclusion

In this paper we prove the existence and uniqueness of an entropy solution for a non- linear parabolic equation with homogeneous Neumann boundary conditions and initial data in L^1 by a time discretization technique.

This method turns out to be better suited for the study of parabolic problems under Neumann-type boundary conditions. However, this technique assumes that the associated elliptic problem is well posed. This study opens up new perspectives, we could always in the context of the Sobolev space with variable exponents look at the problem with measure data or consider the function b as maximal monotone graph.

References

[1] M. Abdellaoui, E. Azroul, S. Ouaro and U. Traoré, "Nonlinear parabolic capacity and renormalized solutions for PDEs with diffuse measure data and variable exponent", An. Univ. Craiova Ser. Mat. Inform., vol. 46, no. 2, pp. 269–297, 2019.

[2] M. Abdellaoui and E. Azroul, "Nonlinear parabolic equations with data soft measure data", J. Nonlinear Evol. Equ. Appl., vol. 2019, no. 7, pp. 115–133, 2020.

[3] B. Andreianov, M. Bendahmane, K. H. Karlsenc and S. Ouaro, "Well-posedness results for triply nonlinear degenerate parabolic equations.", vol. 247, no. 1, pp. 277–302, 2009.

[4] F. Andreu, J. M. Mazón, S. Segura de Léon and J. Toledo, "Existence and uniqueness for a degenerate parabolic equation with L^1 data", Trans. Amer. Math. Soc., vol. 351, no. 1, pp. 285–306, 1999.

[5] F. Andreu, N. Igbida, J. M. Mazón and J. Toledo, "L^1 existence and uniqueness results for quasi-linear elliptic equations with nonlinear boundary conditions", Ann. Inst. H. Poincaré Anal. Non Linéaire, vol. 20, no. 1, pp. 61–89, 2017.

[6] M. Bendahmane, P. Wittbold and A. Zimmermann, "Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1 data", J. Differential Equations, vol. 249, no. 6, pp. 1483–1515, 2010.

[7] F. Benzekri and A. El Hachimi, "Doubly nonlinear parabolic equations related to the p-Laplacian operator: Semi-discretization", Electron. J. Differential Equations, vol. 2003, no. 113, pp. 1–14, 2003.

[8] P. Bénilan, L. Boccardo, T. Gallouët, M. Pierre and J. L. Vazquez, "An L^1 theory of existence and uniqueness of nonlinear elliptic equations", Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), vol. 22, no. 2, pp. 214–273, 1995.

[9] B. K. Bonzy, I. Nyanquini and S. Ouaro, "Existence and uniqueness of weak and entropy solutions for homogeneous Neumann boundary-value problems involving variable exponents", Electron. J. Differential Equations, vol. 2012, no. 12, pp. 1–19, 2012.

[10] A. Dall'Aglio, "Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations", Ann. Mat. Pura Appl.(4), vol. 170, pp. 207–240, 1996.

[11] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, New York: Springer-Verlag Heidelberg, 2011.

[12] R. J. DiPerna and P.-L. Lions, "On the Cauchy problem for Boltzmann equations: global existence and weak stability", Ann. of Math. (2), vol. 130, no. 2, pp. 321–366, 1989.

[13] A. Eden, B. Michaux and J.-M. Rakotoson, "Semi-discretized nonlinear evolution equations as discrete dynamical systems and error analysis", Indiana Univ. Math. J., vol. 39, no. 3, pp. 737–783, 1990.

[14] X. L. Fan and D. Zhao, "On the generalized Orlicz-Sobolev space $W^{k,p(x)}(\Omega)$", Journal of Gansu Education College, vol. 12, no. 1, pp. 1–6, 1998.

[15] G. Gagneux and M. Madaune-Tort, *Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière*, Mathématiques et Applications 22, Berlin: Springer, 1996.

[16] A. El Hachimi and M. R. Sidi Ammi, "Thermistor problem: a nonlocal parabolic problem", in Differential Equations and Mechanics. Electron. J. Diff. Eqns., vol. 2004, no. 11, 2004, pp. 117–128.

[17] A. El Hachimi, J. Igbida and A. Jamea, "Existence result for nonlinear parabolic problems with L^1-data", Appl. Math. (Warsaw), vol. 37, no. 4, pp. 483–508, 2010.

[18] S. Ouaro, S. Soma, "Weak and entropy solutions to nonlinear Neumann boundary value-problems with variable exponents", Complex Var. Elliptic Equ., vol. 56, nos. 7-9, pp. 829–851, 2011.

[19] S. Ouaro, A. Tchousso, "Well-posedness result for a nonlinear elliptic problem involving variable exponent and Robin type boundary condition", Afr. Diaspora J. Math., vol. 11, no. 2, pp. 36–64, 2011.

[20] S. Ouaro and A. Ouédraogo, "Nonlinear parabolic problems with variable exponent and L^1-data", Electron. J. Differential Equations, vol. 2017, no. 32, pp. 1–32, 2017.

[21] S. Ouaro and U. Traoré, "Nonlinear parabolic problem with variable exponent and measure data", J. Nonlinear Evol. Equ. Appl., vol. 2020, no. 5, pp. 65–93, 2020.

[22] M. Sanchón and J. M. Urbano, "Entropy solution for $p(x)$-Laplace equation", Trans. Amer. Math. Soc., vol. 361, no. 2, pp. 6387–6404, 2009.

[23] H. Redwane, "Nonlinear parabolic equation with variable exponents and diffuse measure data", J. Nonl. Evol. Equ. Appl., vol. 2019, no. 6, pp. 95–114, 2020.

[24] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, vol. 1748, Berlin: Springer-Verlag, 2000.

[25] L.-L. Wang, Y.-H. Fan, W.-G. Ge, “Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator”, Nonlinear Anal., vol. 71, no. 9, pp. 4259–4270, 2009.

[26] J. Yao, “Solutions for Neumann boundary value problems involving $p(x)$-Laplace operator”, Nonlinear Anal., vol. 68, no. 5, pp. 1271–1283, 2008.

[27] C. Zhang, S. Zhou, “Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data”, Journal of Differential Equations, vol. 248, no. 6, pp. 1376–1400, 2010.

[28] D. Zhao, W. J. Qiang, X. L. Fan, “On generalized Orlicz spaces $L^{p(x)}(\Omega)$”, J. Gansu Sci., vol. 9, no. 2, pp. 1–7, 1997 [in Chinese].

[29] A. Zimmermann, “Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1 data”, Ph. D. Thesis, T. U. Berlin, 2010.

CUBO, A Mathematical Journal
Vol. 23, no. 03, pp. 411–421, December 2021
DOI: 10.4067/S0719-06462021000300411

Independent partial domination

L. Philo Nithya[1]
Joseph Varghese Kureethara[1]

[1] *Department of Mathematics, Christ University, Bengaluru, Karnataka, India.*
philo.nithya@res.christuniversity.in
frjoseph@christuniversity.in

ABSTRACT

For $p \in (0,1]$, a set $S \subseteq V$ is said to p-dominate or partially dominate a graph $G = (V, E)$ if $\frac{|N[S]|}{|V|} \geq p$. The minimum cardinality among all p-dominating sets is called the p-domination number and it is denoted by $\gamma_p(G)$. Analogously, the independent partial domination $(i_p(G))$ is introduced and studied here independently and in relation with the classical domination. Further, the partial independent set and the partial independence number $\beta_p(G)$ are defined and some of their properties are presented. Finally, the partial domination chain is established as $\gamma_p(G) \leq i_p(G) \leq \beta_p(G) \leq \Gamma_p(G)$.

RESUMEN

Para $p \in (0,1]$, un conjunto $S \subseteq V$ se dice que p-domina o parcialmente domina un grafo $G = (V, E)$ si $\frac{|N[S]|}{|V|} \geq p$. La cardinalidad mínima entre todos los conjuntos p-dominantes se llama el número de p-dominación y se denota por $\gamma_p(G)$. Análogamente, la dominación parcial independiente $(i_p(G))$ es introducida y estudiada independientemente y en relación con la dominación clásica. Más aún el conjunto independiente parcial y el número de independencia parcial $\beta_p(G)$ se definen y se presentan algunas de sus propiedades. Finalmente, se establece la cadena de dominación partial como $\gamma_p(G) \leq i_p(G) \leq \beta_p(G) \leq \Gamma_p(G)$.

Keywords and Phrases: Domination chain, independent partial dominating set, partial independent set.

2020 AMS Mathematics Subject Classification: 05C30, 05C69.

Accepted: 13 September, 2021
Received: 10 November, 2020

1 Introduction

The theory of domination is one of the profusely researched areas in graph theory. Recently a new domination parameter called partial domination number was introduced simultaneously in [3], [4] and [6], and studied in [12, 13, 14, 15]. We extend the concept of partial domination to independent domination in graphs. In [9], the concept of independent partial domination has been defined in the context of partial domination that was defined in [4]. But our work is based on the definition of partial domination in [3, 6] and we concentrate on partial domination chain. Domination addresses the issue of the number of vertices that are dominating all the vertices in a graph. As the set of all vertices of a graph dominates itself, the mathematical adventure is in finding the least number of vertices that can dominate the entire graph. This number is the domination number of a graph.

Finding the domination number of a graph is a well known NP-complete decision problem [11]. In the case of large graphs with a good number of small-degree vertices, the domination number shoots up. Hence, instead of finding the dominating set that dominates the entire graph, it might be convenient to study the set of vertices that dominates the graph partially. This also could be treated as the density problem. By identifying the vertices with large degrees, we can find dense structures in the graph. The vertices that are contributing to the high density neighbourhoods are likely to dominate the major section of vertices of a graph. Hence, domination problem and its variations could also be interpreted as density problems. We follow the popular nomenclature domination and study the structures that are partially dominating a graph.

Domination has been addressed in many different ways by imposing conditions on the dominating set or on its complement or on both. The relations between various domination parameters thus developed aroused mathematical curiosity. The domination chain proposed by Cockayne *et al.* is mathematically profound and aesthetically appealing (see Section 5). A recent survey by Bazgan *et al.* lists the most important results regarding the domination chain parameters [2]. In this paper, we partially address the problem raised by Case *et al.* in [3].

The paper is structured as follows. In Section 2, we present all the preliminary concepts required for this paper. In Section 3, we define independent partial domination number and study some of their properties. In Section 4, we explore some relations between independent dominating set and independent partial dominating set. In Section 5, we define partial independence number and investigate some of its properties which in turn lead to a part of the partial domination chain.

2 Preliminaries

Let G be a simple, finite and undirected graph with $V(G)$ as its set of vertices and $E(G)$ as its edge set. A set $S \subseteq V(G)$ is an independent set of vertices if no two vertices of S are adjacent to each other. An independent set S of vertices is said to be maximal if no superset $T \supset S$ is independent. The maximum cardinality of an independent set in G is called its vertex independence number denoted by $\beta(G)$ and the corresponding vertex set is called the β-set of G. For every vertex u in G, the set $N(u)$ of all vertices adjacent to u is called the open neighbourhood of u. The set $N(u)$ taken together with $\{u\}$ is called the closed neighbourhood of u and is denoted by $N[u]$.

A set D of vertices is called a dominating set of G if every vertex outside D is adjacent to at least one vertex in D. A dominating set D is minimal if no proper subset of D is a dominating set. The minimum cardinality of a minimal dominating set is called the domination number of G denoted by $\gamma(G)$ and the maximum cardinality of a minimal dominating set is called the upper domination number denoted by $\Gamma(G)$. If a dominating set is independent, it becomes an independent dominating set and the minimum cardinality of such a set is called the independent domination number of G denoted by $i(G)$. For any graph $G = (V, E)$ and proportion $p \in (0, 1]$, a set $S \subseteq V$ is a p-dominating or partial dominating set if $\frac{|N[S]|}{|V|} \geq p$. The p-domination or partial domination number $\gamma_p(G)$ equals the minimum cardinality of a p-dominating set in G.

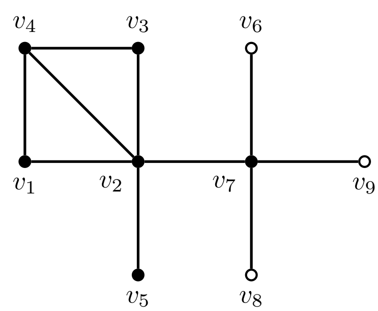


Figure 1: Partial domination by white vertices

In the light of definitions of the neighbourhoods, it is obvious that, for a dominating set S, $N[S] = V$. So a partial dominating set, when compared with a dominating set, dominates a proportion 'p' of the vertex set, which is not necessarily the whole set and hence partially dominates G. The set of all white vertices in Figure 1 dominates exactly 4 vertices and hence is a $\frac{4}{9}$-dominating set. As $\{v_7\}$ is enough to dominate 4 vertices, $\gamma_{\frac{4}{9}} = 1$ for the above graph. For all the other graph theoretic parameters and the notations that are used in this paper, one can refer to [11].

3 Independent partial domination

In this section, we define independent partial domination number and present some observations and some of the basic results. Since the observations are obvious, we present them without proofs.

Definition 3.1. *Suppose $G = (V, E)$ is a simple graph and $p \in (0, 1]$. A subset S of V is called* *an independent p-dominating set (IPD-set) if S is a p-dominating set and is independent.*

Definition 3.2. *The minimum cardinality of an independent p-dominating set is called the independent p-domination number (IPD-number) and is denoted by $i_p(G)$.*

Observations:

(i) For $p \in (0,1]$, $\gamma_p(G) \leq i_p(G)$.

(ii) For any n-vertex graph G and for $p \in (0, \frac{\Delta+1}{n}]$, $i_p(G) = 1$.

(iii) For all $p \in (0,1]$, $i_p(G) = 1$ if and only if $i(G) = 1$.

(iv) For $p \in \left(\frac{n-1}{n}, 1\right]$, $i_p(G) = i(G)$.

(v) For all $p \in (0,1]$, $i_p(G) \leq i(G)$.

We proceed to find the IPD-numbers of paths, cycles and complete bipartite graphs.

Proposition 3.3. *Suppose P_n and C_n are paths and cycles respectively on n-vertices. Then for $n \geq 3$, $i_p(C_n) = i_p(P_n) = \lceil \frac{np}{3} \rceil$.*

Proof. Consider C_n for $n \geq 3$. Let S be a γ_p-set of C_n. Then $|S| = \gamma_p = \lceil \frac{np}{3} \rceil$. If we can choose S in such a way that S is independent, then $i_p(C_n) = \lceil \frac{np}{3} \rceil$. For this, consider $C_n = (v_1, v_2, v_3, ..., v_{3r}, v_{3r+1}, v_{3r+2})$.

Here three cases arise *viz.*, (i) $n = 3r$, (ii) $n = 3r+1$, (iii) $n = 3r+2$ where $r \geq 1$.

Let $S_1 = \{v_2, v_5, ..., v_{3r-1}\}, S_2 = \{v_2, v_5, ..., v_{3r-1}, v_{3r+1}\}$ and $S_3 = \{v_2, v_5, ..., v_{3r-1}, v_{3r+2}\}$. We can see that $|S_1| = |S_2| = |S_3| = \lceil \frac{n}{3} \rceil$ and S_i is independent for $1 \leq i \leq 3$. For cases (i), (ii) and (iii) we can choose our set of $\lceil \frac{np}{3} \rceil$ vertices from S_1, S_2 and S_3. Hence, $i_p(C_n) = \lceil \frac{np}{3} \rceil$. This proof holds for P_n also. □

Proposition 3.4. *For $m \leq n$,*
$$i_p(K_{m,n}) = \begin{cases} 1, & \text{for } p \in (0, \frac{n+1}{m+n}] \\ i+1, & \text{for } p \in (\frac{n+i}{m+n}, \frac{n+(i+1)}{m+n}] \text{ where } 1 \leq i \leq m-1. \end{cases}$$
Also $i_p(K_{m,n}) \leq m$.

Proof. Consider $K_{m,n}$ for $m \leq n$. Let $V_1 = \{v_1, v_2, ..., v_m\}$ and $V_2 = \{u_1, u_2, ..., u_n\}$ be the two partite sets of $K_{m,n}$, where each of V_1 and V_2 is an independent set.

Now, $v_1 \in V_1$ dominates $\frac{n+1}{m+n}$ vertices. Consequently, of the remaining $m-1$ vertices in V_1, each $v_i \in V_1$ dominates $\frac{n+i}{m+n}$ vertices. Thus $i_p(K_{m,n}) \leq m$. □

4 Independent domination and independent partial domination

Allan and Laskar described the relation between the domination number and the independent domination number of a graph in [1]. Partial domination is all about dominating a proportion p of the vertices of G. So a natural question which arises is that: whether this proportion p has any role in relating the partial domination and the original domination parameters. In this section, we do say 'yes' to that question by giving an upper bound for IPD-numbers in terms of p and independent domination numbers. We also give some results, which relate independent dominating sets [10] with that of partial independent dominating sets.

Theorem 4.1. *For any graph G with independent domination number i(G) and $p \in (0,1]$, $i_p(G) \leq \lceil p.i(G) \rceil$.*

Proof. Let $D = \{v_1, v_2, ..., v_i\}$ be an i-set of G. Partition V into sets $V_1, V_2, ..., V_i$ such that for each $1 \leq j \leq i$, $V_j \subseteq N[v_j]$. Without loss of generality, let us assume that $|V_j| \geq |V_{j+1}|$ for $1 \leq j \leq i$.

Consider $D' = \{v_1, v_2, ..., v_{\lceil p.i \rceil}\}$.

Claim: D' is an IPD-set of G.

Proof of the Claim

Our construction yields,

$$\left|\bigcup_{j=1}^{i} V_j\right| = |V| \implies \left|\bigcup_{j=1}^{\lceil p.i \rceil} V_j\right| + \left|\bigcup_{j=\lceil p.i \rceil}^{i} V_j\right| = |V| \implies \frac{\left|\bigcup_{j=1}^{\lceil p.i \rceil} V_j\right|}{\lceil p.i \rceil} \geq \frac{\left|\bigcup_{j=1}^{i} V_j\right|}{i} = \frac{|V|}{i}$$
$$\implies \left|\bigcup_{j=1}^{\lceil p.i \rceil} V_j\right| \geq \frac{|V|.\lceil p.i \rceil}{i}.$$

Hence $|N[D']| \geq p.|V|$. We have thus proved the claim.

Thus using the claim, we have $i_p(G) \leq |D'| = \lceil p.i(G) \rceil$. □

Proposition 4.2. *Let G be any graph with independent domination number i(G) and $p \in (0,1]$. Then $i_p(G) + i_{1-p}(G) \leq i(G) + 1$.*

Proof. By Theorem 5.7, $i_p(G) \leq \lceil p.i \rceil < p.i + 1$ and $i_{1-p}(G) \leq \lceil (1-p)\,.i \rceil < (1-p).i + 1$, then $i_p(G) + i_{1-p}(G) < i + 2 \leq i + 1$. □

Proposition 4.3. *Let S be any independent dominating set of G. If $p = \frac{|N[H]|}{|V|}$, for some $H \subset S$, then $S - H$ is a $1-p$ independent dominating set in G.*

Proof. It can be easily proved that, $N[S] - N[H] \subseteq N[S - H]$. Therefore, $\frac{|N[S-H]|}{|V|} \geq 1 - p$ since $N[S] = V$. $\square$

The following result provides us with an algorithm that develops a minimal independent dominating set from a minimal IPD-set.

Proposition 4.4. *Every minimal IPD-set can be extended to form a minimal independent dominating set.*

Proof. Let I be a minimal IPD-set for any $p \in (0, 1]$. The following algorithm extends I to I', a minimal independent dominating set and gives m, the cardinality of I'.

Procedure 1 Algorithm to construct I' from I

```
Input: V(G), I, N[I], N[u] ∀u ∈ V(G) − N[I]
Output: I', m
1: I' = I, m = |I'|, M = {}
2: M = V(G) − N[I']
3: if M = φ then
4:     return I', m
5: else
6:     I' = I' ∪ {u} for any u ∈ M
7:     N[I'] = N[I'] ∪ N[u]
8:     m = m + 1
9:     go to 2
10: end if
```

$\square$

When a graph is claw-free, it has been already proved in [1], that its domination number coincides with that of its independent domination number. We found that to be true in the context of partial domination also.

Proposition 4.5. *If a graph G is claw-free, then $\gamma_p(G) = i_p(G)$.*

Proof. Let S be a γ_p-set of G, for any $p \in (0, 1]$. Since G is claw-free, $< N[S] >$ is also claw-free. Hence, $\gamma(< N[S] >) = i(< N[S] >)$. This implies that $\gamma_p(G) \geq i_p(G)$. But in general, $\gamma_p(G) \leq i_p(G)$. Thus $\gamma_p(G) = i_p(G)$. $\square$

Corollary 4.6. *If $L(G)$ is the line graph of a graph G, then $\gamma_p(L(G)) = i_p(L(G))$.*

5 Partial domination chain

A chain of inequalities involving domination numbers, independence numbers and irredundance numbers of the form

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G)$$

was first observed in 1978 (see [5]). This type of chain was observed in the case of many other domination parameters like α-domination [7] and k-dependent domination [8]. Also one of the open questions posed by Case *et al.* in [3] was to find out, what relationship the above parameters have amongst themselves in the context of partial domination. Hence we try to establish a similar kind of chain involving partial domination and partial independence parameters. Having already defined independent partial domination, we now define partial independence number of a graph.

Definition 5.1. *Suppose $G = (V, E)$ is a graph and $p \in (0, 1]$. A set S of independent vertices is called a p-independent set in G if $N[S] \subseteq V(H)$ for some induced subgraph H of G with $|V(H)| \geq np$. A p-independent set S is said to be p-maximal if S is a maximal independent set in $V(H)$. A maximal p-independent set S is said to be min-max p-independent set if $T \subset S$ is not p-maximal. Partial independence number or p-independence number is the maximum cardinality of a min-max p-independent set and is denoted by $\beta_p(G)$ and the associated induced subgraph H is denoted by H_p.*

For the graph in Figure 1, the set of all white vertices form a $\beta_{\frac{4}{9}}$-set. For the same graph, the set $\{v_5, v_8\}$ is a min-max $\frac{4}{9}$-independent set, but it is not of maximum cardinality and hence is not a $\beta_{\frac{4}{9}}$-set.

5.1 Partial independent sets

This section explores some of the properties of partial independent sets, thereby proceeding towards the suggested partial domination chain.

In light of the above definition, it may be noted that, for every maximal p-independent set S, $N[S] = V(H)$ of the proposed induced subgraph H of G and hence S is a p-dominating set. Thus independent p-domination number is the minimum cardinality of a maximal p-independent set and we have the following inequality.

Proposition 5.2. *For $p \in (0, 1]$, $\gamma_p(G) \leq i_p(G) \leq \beta_p(G)$.*

Proposition 5.3. *If $p_1 \leq p_2$, then $\beta_{p_1}(G) \leq \beta_{p_2}(G)$.*

Proof. Let $S \subseteq V$ be such that $|S| = \beta_{p_2}(G)$. Then S is also maximal p_1-independent set. Also the cardinality of every min-max p_1-independent set $\leq |S|$. Thus $\beta_{p_1}(G) \leq \beta_{p_2}(G)$. $\square$

We now proceed to relate partial independence number β_p with that of upper p-domination number, Γ_p which is the maximum cardinality of a minimal p-dominating set.

Proposition 5.4. *Every min-max p-independent set is a minimal p-dominating set.*

Proof. Let S be a min-max p-independent set and H_p be an induced subgraph associated with it. Then by definition, S is p-dominating in G.

Suppose S is not minimal p-dominating. Then $\exists\ u \in S$ such that $S - \{u\}$ is p-dominating. Then $\exists\ v \in S - \{u\}$, such that $uv \in E(H)$ which is a contradiction since S is an independent set.

Thus S is a minimal p-dominating set. □

Corollary 5.5. *For $p \in (0, 1]$, $\beta_p(G) \leq \Gamma_p(G)$.*

From Proposition 5.2 and Corollary 5.5 we obtain the following chain of inequalities:

For $p \in (0, 1]$, $\gamma_p(G) \leq i_p(G) \leq \beta_p(G) \leq \Gamma_p(G)$.

We present some more properties of independent sets, which in turn lead us to a method, by which one can deduce β_p-sets for some 'p' values from the existing β-set of a graph.

Lemma 5.6. *Suppose S is a β-set of a graph G and $T \subset S$. Then T is a min-max $\frac{|N[T]|}{n}$-independent set.*

Proof. By definition T is a maximal $\frac{|N[T]|}{n}$-independent set. It is also min-max since $R \subset T$ is not $\frac{|N[T]|}{n}$ maximal. Suppose R is maximal then $(S-T) \cup R$ is a dominating set which is a contradiction as S is a minimal dominating set of G. □

Theorem 5.7. *Let $\mathcal{B}_i$ denote the set of all i-element subsets of a β-set of a graph G for $1 \leq i \leq \beta(G)$. Let $B_i \in \mathcal{B}_i$ be such that $|N[B_i]| = min\{|N[X]|/X \in \mathcal{B}_i\}$. Then*

(i) B_i is a β_p-set for $p = \frac{|N[B_i]|}{n}$.

(ii) For $0 < p \leq \frac{|N[B_1]|}{n}$, $\beta_p = 1$ and B_1 is a β_p-set.

Proof. For $1 \leq i \leq \beta(G)$ let B_i be chosen by the given method. By the previous Lemma (5.6) B_i is a min-max $\frac{|N[B_i]|}{n}$ independent set. Suppose B_i is not of maximum cardinality amongst all $\frac{|N[B_i]|}{n}$ independent sets, then for $j > i$ there exists a $Y \in \mathcal{B}_j$ such that Y is a min-max $\frac{|N[B_i]|}{n}$ independent set. Also Y is min-max $\frac{|N[Y]|}{n}$ independent set and thus Y is a maximal independent set in both $< N[B_i] >$ and $< N[Y] >$ and also $|N[B_i]| = |N[Y]|$. But by the definition of B_is, $|N[Y]| \geq |N[B_j]|$ which implies that $|N[B_j]| \leq |N[B_i]|$ which is a contradiction since for $j > i$, $|N[B_j]| > |N[B_i]|$.

Suppose $|N[B_j]| \leq |N[B_i]|$ for some $j > i$, choose R such that $R \subset B_j$ and $|R| = i$. Then $|N[R]| < |N[B_j]|$ which implies that $|N[R]| < |N[B_i]|$ by our assumption. This contradicts our definition of B_i. □

6 Conclusion

Partial domination has a lot to promise. One of the striking features of the concept of partial domination is its nature of accommodation. Domination with conditions are studied extensively. In the case of partial domination, the imperfect situations are addressed. Hence, it is worth exploring the partial domination in all the numerous types of dominations. In this context we could establish the partial domination chain. Future beckons with great hope of the explorations of partial domination in the areas of distance domination, stratified domination, Roman domination etc., but not exclusively.

Acknowledgement

We thank all the referees for their reviews of this paper and their creative suggestions.

References

[1] R. B. Allan and R. Laskar, "On Domination and Independent Domination Numbers of a Graph", Discrete Math., vol. 23, no. 2, pp. 73–76, 1978.

[2] C. Bazgan, L. Brankovic, K. Casel and H. Fernau, "Domination chain: Characterisation, classical complexity, parameterised complexity and approximability", Discrete Appl. Math., vol. 280, pp. 23–42, 2020.

[3] B. M. Case, S. T. Hedetniemi, R. C. Laskar and D. J. Lipman, "Partial domination in graphs", Congr. Numer., vol. 228, pp. 85–96, 2017.

[4] Y. Caro and A. Hansberg, "Partial domination–the isolation number of a graph", Filomat, vol. 31, no. 12, pp. 3925–3944, 2017.

[5] E. J. Cockayne, S. T. Hedetniemi and D. J. Miller, "Properties of hereditary hypergraphs and middle graphs", Canad. Math. Bull., vol. 21, no. 4, pp. 461–468, 1978.

[6] A. Das, "Partial domination in graphs", Iran. J. Sci. Technol. Trans. A Sci., vol. 43, no. 4, pp. 1713–1718, 2019.

[7] J. E. Dunbar, D. G. Hoffman, R. C. Laskar and L. R. Markus, α-Domination, Discrete Math., vol. 211, no. 1–3, pp. 11–26, 2000.

[8] O. Favaron, S. M. Hedetniemi, S. T. Hedetniemi and D. F. Rall, "On k-dependent domination", Discrete Math., vol. 249, nos. 1–3, pp. 83–94, 2002.

[9] O. Favaron and P. Kaemawichanurat, "Inequalities between the K_k-isolation number and the Independent K_k-isolation number of a graph", Discrete Appl. Math., vol. 289, pp. 93–97, 2021.

[10] W. Goddard and M. A. Henning, "Independent domination in graphs: a survey and recent results", Discrete Math., vol. 313, no. 7, pp. 839–854, 2013.

[11] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, 464, CRC Press, Boca Raton, 1998.

[12] R. D. Macapodi and R. T. Isla, "Total partial domination in graphs under some binary operations", Eur. J. Pure Appl. Math., vol. 12, no. 4, pp. 1643–1655, 2019.

[13] R. D. Macapodi, R. I. Isla and S. R. Canoy, "Partial domination in the join, corona, lexicographic and cartesian products of graphs", Adv. Appl. Discrete Math., vol. 20, no. 2, pp. 277–293, 2019.

[14] L. P. Nithya and J. V. Kureethara, "On Some Properties of Partial Dominating Sets", AIP Conference Proceedings, vol. 2236, no. 1, 060004, 2020.

[15] L. P. Nithya and J. V. Kureethara, "Partial domination in prisms of graphs", Ital. J. Pure Appl. Math., to be published.

Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 03, pp. 423–440, December 2021
DOI: 10.4067/S0719-06462021000300423

Foundations of generalized Prabhakar-Hilfer fractional calculus with applications

George A. Anastassiou[1]

[1] *Department of Mathematical Sciences*
University of Memphis, Memphis,
TN 38152, U.S.A.
ganastss@memphis.edu

ABSTRACT

Here we introduce the generalized Prabhakar fractional calculus and we also combine it with the generalized Hilfer calculus. We prove that the generalized left and right side Prabhakar fractional integrals preserve continuity and we find tight upper bounds for them. We present several left and right side generalized Prabhakar fractional inequalities of Hardy, Opial and Hilbert-Pachpatte types. We apply these in the setting of generalized Hilfer calculus.

RESUMEN

Introducimos el cálculo fraccionario generalizado de Prabhakar y también lo combinamos con el cálculo generalizado de Hilfer. Demostramos que las integrales fraccionarias generalizadas de Prabhakar izquierda y derecha preservan la continuidad y encontramos cotas superiores ajustadas para ellas. Presentamos diversas desigualdades fraccionarias generalizadas de Prabhakar izquierda y derecha de tipos Hardy, Opial y Hilbert-Pachpatte. Aplicamos estos resultados en el contexto del cálculo generalizado de Hilfer.

Keywords and Phrases: Prabhakar fractional calculus, Hilfer fractional calculus, fractional Hardy, Opial and Hilbert-Pachpatte inequalities.

2020 AMS Mathematics Subject Classification: 26A33, 26D10, 26D15.

Accepted: 13 September, 2021
Received: 08 April, 2021

1 Background

During the last 50 years fractional calculus due to its wide applications to many applied sciences has become a main trend in mathematics. Its predominant kinds are the old Riemann-Liouville fractional calculus and the newer one of Caputo type. Around these two versions have been built a plethora of other variants and all of these involve singular kernels. More recently researchers presented also new fractional calculi of non singular kernels.

The recent Hilfer fractional calculus unifies the Riemann-Liouville and Caputo fractional calculi and the Prabhakar fractional calculus unifies both singular and non-singular kernel fractional calculi.

Finally the newer Hilfer-Prabhakar fractional calculus is the most general one unifying all trends and for different values of its parameters we get the particular fractional calculi. In this article we present and employ unifying advanced and generalized versions of Prabhakar and Hilfer-Prabhakar fractional calculi and we establish related unifying fractional integral inequalities of the following types: Hardy, Opial and Hilbert-Pachpatte. The advantage of this unification is the uniform action taken in describing the various natural phenomena.

We are inspired by [7], [6] and [1]. We start by introducing our own generalized ψ-Prabhakar type of fractional calculus, then mixing it with the ψ-Hilfer fractional calculus. Then, we prove a variety of generalized Hardy, Opial and Hilbert-Pachpatte type left and right fractional integral inequalities related to ψ-Hilfer ([8]) and ψ-Prabhakar fractional calculi. We involve several functions.

We consider the Prabhakar function (also known as the three parameter Mittag-Leffler function), (see [4, p. 97]; [3])

$$E_{\alpha,\beta}^{\gamma}\left(z\right)=\sum_{k=0}^{\infty}\frac{\left(\gamma\right)_{k}}{k!\Gamma\left(\alpha k+\beta\right)}z^{k}, \tag{1.1}$$

where Γ is the gamma function; $\alpha,\beta,\gamma\in\mathbb{R}:\alpha,\beta>0$, $z\in\mathbb{R}$, and $\left(\gamma\right)_{k}=\gamma\left(\gamma+1\right)\cdots\left(\gamma+k-1\right)$. It is $E_{\alpha,\beta}^{0}\left(z\right)=\dfrac{1}{\Gamma\left(\beta\right)}$.

Let $a,b\in\mathbb{R}$, $a<b$ and $x\in[a,b]$; $f\in C\left([a,b]\right)$. Let also $\psi\in C^{1}\left([a,b]\right)$ which is increasing. The left and right Prabhakar fractional integrals with respect to ψ are defined as follows:

$$\left(e_{\rho,\mu,\omega,a+}^{\gamma;\psi}f\right)(x)=\int_{a}^{x}\psi'(t)\left(\psi(x)-\psi(t)\right)^{\mu-1}E_{\rho,\mu}^{\gamma}\left[\omega\left(\psi(x)-\psi(t)\right)^{\rho}\right]f(t)\,dt, \tag{1.2}$$

and

$$\left(e_{\rho,\mu,\omega,b-}^{\gamma;\psi}f\right)(x)=\int_{x}^{b}\psi'(t)\left(\psi(t)-\psi(x)\right)^{\mu-1}E_{\rho,\mu}^{\gamma}\left[\omega\left(\psi(t)-\psi(x)\right)^{\rho}\right]f(t)\,dt, \tag{1.3}$$

where $\rho,\mu>0$; $\gamma,\omega\in\mathbb{R}$.

Functions (1.2) and (1.3) are continuous, see Theorem 3.1.

Next, additionally, assume that $\psi'(x)\neq 0$ over $[a,b]$.

Let $\psi,f\in C^{N}\left([a,b]\right)$, where $N=\lceil\mu\rceil$, ($\lceil\cdot\rceil$ is the ceiling of the number), $0<\mu\notin\mathbb{N}$. We define the

ψ-Prabhakar-Caputo left and right fractional derivatives of order μ as follows ($x \in [a,b]$):

$$\left({}^{C}D^{\gamma;\psi}_{\rho,\mu,\omega,a+} f \right)(x) = \int_a^x \psi'(t) (\psi(x) - \psi(t))^{N-\mu-1}$$

$$E^{-\gamma}_{\rho,N-\mu} \left[\omega (\psi(x) - \psi(t))^{\rho} \right] \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^N f(t)\, dt, \tag{1.4}$$

and

$$\left({}^{C}D^{\gamma;\psi}_{\rho,\mu,\omega,b-} f \right)(x) = (-1)^N \int_x^b \psi'(t) (\psi(t) - \psi(x))^{N-\mu-1}$$

$$E^{-\gamma}_{\rho,N-\mu} \left[\omega (\psi(t) - \psi(x))^{\rho} \right] \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^N f(t)\, dt. \tag{1.5}$$

One can write (see (1.4), (1.5))

$$\left({}^{C}D^{\gamma;\psi}_{\rho,\mu,\omega,a+} f \right)(x) = \left(e^{-\gamma;\psi}_{\rho,N-\mu,\omega,a+} f^{[N]}_{\psi} \right)(x), \tag{1.6}$$

and

$$\left({}^{C}D^{\gamma;\psi}_{\rho,\mu,\omega,b-} f \right)(x) = (-1)^N \left(e^{-\gamma;\psi}_{\rho,N-\mu,\omega,b-} f^{[N]}_{\psi} \right)(x), \tag{1.7}$$

where

$$f^{[N]}_{\psi}(x) = f^{(N)}_{\psi} f(x) := \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^N f(x), \tag{1.8}$$

$\forall\ x \in [a,b]$.

Functions (1.6) and (1.7) are continuous on $[a,b]$.

Next we define the ψ-Prabhakar-Riemann-Liouville left and right fractional derivatives of order μ as follows ($x \in [a,b]$):

$$\left({}^{RL}D^{\gamma;\psi}_{\rho,\mu,\omega,a+} f \right)(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^N \int_a^x \psi'(t) (\psi(x) - \psi(t))^{N-\mu-1}$$

$$E^{-\gamma}_{\rho,N-\mu} \left[\omega (\psi(x) - \psi(t))^{\rho} \right] f(t)\, dt, \tag{1.9}$$

and

$$\left({}^{RL}D^{\gamma;\psi}_{\rho,\mu,\omega,b-} f \right)(x) = \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^N \int_x^b \psi'(t) (\psi(t) - \psi(x))^{N-\mu-1}$$

$$E^{-\gamma}_{\rho,N-\mu} \left[\omega (\psi(t) - \psi(x))^{\rho} \right] f(t)\, dt. \tag{1.10}$$

That is we have

$$\left({}^{RL}D^{\gamma;\psi}_{\rho,\mu,\omega,a+} f \right)(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^N \left(e^{-\gamma;\psi}_{\rho,N-\mu,\omega,a+} f \right)(x), \tag{1.11}$$

and

$$\left({}^{RL}D^{\gamma;\psi}_{\rho,\mu,\omega,b-} f \right)(x) = \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^N \left(e^{-\gamma;\psi}_{\rho,N-\mu,\omega,b-} f \right)(x), \tag{1.12}$$

$\forall\ x \in [a,b]$.

We define also the ψ-Hilfer-Prabhakar left and right fractional derivatives of order μ and type $0 \leq \beta \leq 1$, as follows

$$\left({}^{H}\mathbb{D}^{\gamma,\beta;\psi}_{\rho,\mu,\omega,a+}f\right)(x) = e^{-\gamma\beta;\psi}_{\rho,\beta(N-\mu),\omega,a+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N} e^{-\gamma(1-\beta);\psi}_{\rho,(1-\beta)(N-\mu),\omega,a+}f(x), \tag{1.13}$$

and

$$\left({}^{H}\mathbb{D}^{\gamma,\beta;\psi}_{\rho,\mu,\omega,b-}f\right)(x) = e^{-\gamma\beta;\psi}_{\rho,\beta(N-\mu),\omega,b-}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N} e^{-\gamma(1-\beta);\psi}_{\rho,(1-\beta)(N-\mu),\omega,b-}f(x), \tag{1.14}$$

$\forall\, x \in [a,b]$.

When $\beta = 0$, we get the Riemann-Liouville version, and when $\beta = 1$, we get the Caputo version.

We call $\xi = \mu + \beta(N-\mu)$, we have that $N-1 < \mu \leq \mu + \beta(N-\mu) \leq \mu + N - \mu = N$, hence $\lceil \xi \rceil = N$.

We can easily write that

$$\left({}^{H}\mathbb{D}^{\gamma,\beta;\psi}_{\rho,\mu,\omega,a+}f\right)(x) = e^{-\gamma\beta;\psi}_{\rho,\xi-\mu,\omega,a+}\ {}^{RL}D^{\gamma(1-\beta);\psi}_{\rho,\xi,\omega,a+}f(x), \tag{1.15}$$

and

$$\left({}^{H}\mathbb{D}^{\gamma,\beta;\psi}_{\rho,\mu,\omega,b-}f\right)(x) = e^{-\gamma\beta;\psi}_{\rho,\xi-\mu,\omega,b-}\ {}^{RL}D^{\gamma(1-\beta);\psi}_{\rho,\xi,\omega,b-}f(x), \tag{1.16}$$

$\forall\, x \in [a,b]$.

2 Main results

We start with a left ψ-Prabhakar fractional Hardy type integral inequality involving several functions.

Theorem 2.1. *Here $i = 1,\ldots,m$; $f_i \in C([a,b])$, $\psi \in C^1([a,b])$ and ψ is increasing. Let $\rho_i, \mu_i > 0$, $\gamma_i, \omega_i \in \mathbb{R}$. Also let $r_1, r_2, r_3 > 1 : \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$, and assume that $\mu_i > \frac{1}{r_2} + \frac{1}{r_3}$, for all $i = 1,\ldots,m$.*

Then

$$\left\| \prod_{i=1}^{m} e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+} f_i \right\|_{L_{r_1}([a,b],\psi)} \leq$$

$$\frac{(\psi(b)-\psi(a))^{\left[\sum_{i=1}^{m}\mu_i - m + \frac{m}{r_1} + \frac{1}{r_1} - \frac{1}{r_1 r_2}\right]}}{\left(r_1 r_3\left(\sum_{i=1}^{m}\mu_i - m\right) + m r_3 + 1\right)^{\frac{1}{r_1 r_3}}\left(\prod_{i=1}^{m}\left(r_1(\mu_i - 1) + 1\right)\right)^{\frac{1}{r_1}}}$$

$$\left\{\int_a^b \left[\prod_{i=1}^{m}\left(\int_a^x \left|E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(x)-\psi(t))^{\rho_i}\right]\right|^{r_2} d\psi(t)\right)\right]^{r_1} d\psi(x)\right\}^{\frac{1}{r_1 r_2}}$$

$$\left(\prod_{i=1}^{m}\|f_i\|_{L_{r_3}([a,b],\psi)}\right). \tag{2.1}$$

Proof. By (1.2) we have

$$\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+}f_i\right)(x) = \int_a^x \psi'(t)\left(\psi(x)-\psi(t)\right)^{\mu_i-1} E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i\left(\psi(x)-\psi(t)\right)^{\rho_i}\right] f_i(t)\,dt, \tag{2.2}$$

$i = 1, \ldots, m$; $\forall\ x \in [a,b]$.

By Hölder's inequality and (2.2) we obtain

$$\left|\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+}f_i\right)(x)\right| \leq$$

$$\int_a^x \psi'(t)\left(\psi(x)-\psi(t)\right)^{\mu_i-1}\left|E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i\left(\psi(x)-\psi(t)\right)^{\rho_i}\right]\right| |f_i(t)|\,dt \leq$$

$$\left(\int_a^x \left(\psi(x)-\psi(t)\right)^{r_1(\mu_i-1)} d\psi(t)\right)^{\frac{1}{r_1}}$$

$$\left(\int_a^x \left|E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i\left(\psi(x)-\psi(t)\right)^{\rho_i}\right]\right|^{r_2} d\psi(t)\right)^{\frac{1}{r_2}} \left(\int_a^x |f_i(t)|^{r_3}\,d\psi(t)\right)^{\frac{1}{r_3}} \leq \tag{2.3}$$

$$\frac{\left(\psi(x)-\psi(a)\right)^{\mu_i-1+\frac{1}{r_1}}}{\left(r_1(\mu_i-1)+1\right)^{\frac{1}{r_1}}}$$

$$\left(\int_a^x \left|E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i\left(\psi(x)-\psi(t)\right)^{\rho_i}\right]\right|^{r_2} d\psi(t)\right)^{\frac{1}{r_2}} \|f_i\|_{L_{r_3}([a,b],\psi)}\,.$$

So far we have

$$\left|\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+}f_i\right)(x)\right| \leq \frac{\left(\psi(x)-\psi(a)\right)^{\mu_i-1+\frac{1}{r_1}}}{\left(r_1(\mu_i-1)+1\right)^{\frac{1}{r_1}}}$$

$$\left(\int_a^x \left|E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i\left(\psi(x)-\psi(t)\right)^{\rho_i}\right]\right|^{r_2} d\psi(t)\right)^{\frac{1}{r_2}} \|f_i\|_{L_{r_3}([a,b],\psi)}\,, \tag{2.4}$$

$\forall\ x \in [a,b]$, with $\mu_i > \frac{1}{r_2} + \frac{1}{r_3}$, for any $i = 1, \ldots, m$.

Hence it holds

$$\left(\prod_{i=1}^m \left|\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+}f_i\right)(x)\right|\right)^{r_1} \leq \frac{\left(\psi(x)-\psi(a)\right)^{r_1\sum\limits_{i=1}^m \mu_i - mr_1+m}}{\left(\prod\limits_{i=1}^m \left(r_1(\mu_i-1)+1\right)\right)}$$

$$\left[\prod_{i=1}^m \left(\int_a^x \left|E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i\left(\psi(x)-\psi(t)\right)^{\rho_i}\right]\right|^{r_2} d\psi(t)\right)\right]^{\frac{r_1}{r_2}} \left(\prod_{i=1}^m \|f_i\|_{L_{r_3}([a,b],\psi)}\right)^{r_1}, \tag{2.5}$$

$\forall\ x \in [a,b]$.

Therefore we obtain

$$\int_a^b \left(\prod_{i=1}^m \left|\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+}f_i\right)(x)\right|\right)^{r_1} d\psi(x) \leq \frac{\left(\prod\limits_{i=1}^m \|f_i\|_{L_{r_3}([a,b],\psi)}\right)^{r_1}}{\left(\prod\limits_{i=1}^m \left(r_1(\mu_i-1)+1\right)\right)} \tag{2.6}$$

$$\left[\int_a^b \left(\psi(x)-\psi(a)\right)^{r_1\sum\limits_{i=1}^m \mu_i - mr_1+m}\right.$$

$$\left[\prod_{i=1}^{m}\left(\int_a^x \left|E_{\rho_i,\mu_i}^{\gamma_i}\left[\omega_i\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\rho_i}\right]\right|^{r_2} d\psi\left(t\right)\right)\right]^{\frac{r_1}{r_2}} d\psi\left(x\right)\Bigg]$$

(again by Hölder's inequality)

$$\leq \frac{\left(\prod_{i=1}^{m}\|f_i\|_{L_{r_3}([a,b],\psi)}\right)^{r_1}}{\left(\prod_{i=1}^{m}\left(r_1\left(\mu_i-1\right)+1\right)\right)}\left(\int_a^b \left(\psi\left(x\right)-\psi\left(a\right)\right)^{r_1 r_3\sum_{i=1}^{m}\mu_i - m r_1 r_3 + m r_3} d\psi\left(x\right)\right)^{\frac{1}{r_3}}$$

$$\left\{\int_a^b\left[\prod_{i=1}^{m}\left(\int_a^x \left|E_{\rho_i,\mu_i}^{\gamma_i}\left[\omega_i\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\rho_i}\right]\right|^{r_2} d\psi\left(t\right)\right)\right]^{r_1} d\psi\left(x\right)\right\}^{\frac{1}{r_2}}\left(\psi\left(b\right)-\psi\left(a\right)\right)^{\frac{1}{r_1}} =$$

$$\frac{\left(\prod_{i=1}^{m}\|f_i\|_{L_{r_3}([a,b],\psi)}\right)^{r_1}\left(\psi\left(b\right)-\psi\left(a\right)\right)^{r_1\sum_{i=1}^{m}\mu_i - m r_1 + m + 1 - \frac{1}{r_2}}}{\left(\prod_{i=1}^{m}\left(r_1\left(\mu_i-1\right)+1\right)\right)\left(r_1 r_3\sum_{i=1}^{m}\mu_i - m r_1 r_3 + m r_3 + 1\right)^{\frac{1}{r_3}}} \tag{2.7}$$

$$\left\{\int_a^b\left[\prod_{i=1}^{m}\left(\int_a^x \left|E_{\rho_i,\mu_i}^{\gamma_i}\left[\omega_i\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\rho_i}\right]\right|^{r_2} d\psi\left(t\right)\right)\right]^{r_1} d\psi\left(x\right)\right\}^{\frac{1}{r_2}},$$

where $\mu_i > \frac{1}{r_2} + \frac{1}{r_3}$, $i = 1, \ldots, m$.

The claim is proved. ☐

We continue with a right ψ-Prabhakar fractional Hardy type integral inequality involving several functions.

Theorem 2.2. *All as in Theorem 2.1. It holds*

$$\left\|\prod_{i=1}^{m} e_{\rho_i,\mu_i,\omega_i,b-}^{\gamma_i;\psi} f_i\right\|_{L_{r_1}([a,b],\psi)} \leq$$

$$\frac{\left(\psi\left(b\right)-\psi\left(a\right)\right)^{\left[\sum_{i=1}^{m}\mu_i - m + \frac{m}{r_1} + \frac{1}{r_1} - \frac{1}{r_1 r_2}\right]}}{\left(r_1 r_3\left(\sum_{i=1}^{m}\mu_i - m\right) + m r_3 + 1\right)^{\frac{1}{r_1 r_3}}\left(\prod_{i=1}^{m}\left(r_1\left(\mu_i-1\right)+1\right)\right)^{\frac{1}{r_1}}}$$

$$\left\{\int_a^b\left[\prod_{i=1}^{m}\left(\int_x^b \left|E_{\rho_i,\mu_i}^{\gamma_i}\left[\omega_i\left(\psi\left(t\right)-\psi\left(x\right)\right)^{\rho_i}\right]\right|^{r_2} d\psi\left(t\right)\right)\right]^{r_1} d\psi\left(x\right)\right\}^{\frac{1}{r_1 r_2}}$$

$$\left(\prod_{i=1}^{m}\|f_i\|_{L_{r_3}([a,b],\psi)}\right). \tag{2.8}$$

Proof. Similar to the proof of Theorem 2.1 and omitted. ☐

Next we apply Theorems 2.1, 2.2.

We give the related Hardy type inequalities:

Theorem 2.3. *Here* $i = 1, \ldots, m$; $f_i \in C^{N_i}([a,b])$, *where* $N_i = \lceil \mu_i \rceil$, $0 < \mu_i \notin \mathbb{N}$; $\theta := \max\{N_1, \ldots, N_m\}$, $\psi \in C^{\theta}([a,b])$ *with* $\psi' \neq 0$ *and increasing. Let* $\rho_i > 0$, $\gamma_i, \omega_i \in \mathbb{R}$. *Also let* $r_1, r_2, r_3 > 1 : \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$, *and assume that* $N_i - \mu_i > \frac{1}{r_2} + \frac{1}{r_3}$, *for all* $i = 1, \ldots, m$. *Then*

i)

$$\left\| \prod_{i=1}^{m} {}^{C}D^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+} f_i \right\|_{L_{r_1}([a,b],\psi)} \leq$$

$$\frac{(\psi(b) - \psi(a))^{\left[\sum_{i=1}^{m}(N_i-\mu_i) - m + \frac{m}{r_1} + \frac{1}{r_1} - \frac{1}{r_1 r_2}\right]}}{\left(r_1 r_3 \left(\sum_{i=1}^{m}(N_i - \mu_i) - m\right) + m r_3 + 1\right)^{\frac{1}{r_1 r_3}} \left(\prod_{i=1}^{m}(r_1(N_i - \mu_i - 1) + 1)\right)^{\frac{1}{r_1}}}$$

$$\left\{ \int_a^b \left[\prod_{i=1}^{m} \left(\int_a^x \left| E^{-\gamma_i}_{\rho_i, N_i - \mu_i}\left[\omega_i (\psi(x) - \psi(t))^{\rho_i}\right] \right|^{r_2} d\psi(t) \right) \right]^{r_1} d\psi(x) \right\}^{\frac{1}{r_1 r_2}}$$

$$\left(\prod_{i=1}^{m} \left\| f^{[N_i]}_{i\psi} \right\|_{L_{r_3}([a,b],\psi)} \right), \quad (2.9)$$

and

ii)

$$\left\| \prod_{i=1}^{m} {}^{C}D^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,b-} f_i \right\|_{L_{r_1}([a,b],\psi)} \leq$$

$$\frac{(\psi(b) - \psi(a))^{\left[\sum_{i=1}^{m}(N_i-\mu_i) - m + \frac{m}{r_1} + \frac{1}{r_1} - \frac{1}{r_1 r_2}\right]}}{\left(r_1 r_3 \left(\sum_{i=1}^{m}(N_i - \mu_i) - m\right) + m r_3 + 1\right)^{\frac{1}{r_1 r_3}} \left(\prod_{i=1}^{m}(r_1(N_i - \mu_i - 1) + 1)\right)^{\frac{1}{r_1}}}$$

$$\left\{ \int_a^b \left[\prod_{i=1}^{m} \left(\int_x^b \left| E^{-\gamma_i}_{\rho_i, N_i - \mu_i}\left[\omega_i (\psi(t) - \psi(x))^{\rho_i}\right] \right|^{r_2} d\psi(t) \right) \right]^{r_1} d\psi(x) \right\}^{\frac{1}{r_1 r_2}}$$

$$\left(\prod_{i=1}^{m} \left\| f^{[N_i]}_{i\psi} \right\|_{L_{r_3}([a,b],\psi)} \right). \quad (2.10)$$

Proof. By (1.6), (1.7) and Theorems 2.1, 2.2. □

We also present other Hardy type related inequalities:

Theorem 2.4. *Here* $i = 1, \ldots, m$; $f_i \in C^{N_i}([a,b])$, *where* $N_i = \lceil \mu_i \rceil$, $0 < \mu_i \notin \mathbb{N}$; $\theta := \max\{N_1, \ldots, N_m\}$, $\psi \in C^{\theta}([a,b])$, $\psi' \neq 0$, *and* ψ *is increasing. Let* $\rho_i > 0$, $\gamma_i, \omega_i \in \mathbb{R}$, $0 \leq \beta_i \leq 1$, $\xi_i = \mu_i + \beta_i(N_i - \mu_i)$. *Also let* $r_1, r_2, r_3 > 1 : \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$, *and assume that* $\xi_i - \mu_i > \frac{1}{r_2} + \frac{1}{r_3}$, *for all* $i = 1, \ldots, m$.

Also assume that ${}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,a+} f_i$, ${}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,b-} f_i \in C([a,b])$, $i = 1, \ldots, m$.

Then

i)

$$\left\| \prod_{i=1}^{m} {}^{H}\mathbb{D}^{\gamma_i,\beta_i;\psi}_{\rho_i,\mu_i,\omega_i,a+} f_i \right\|_{L_{r_1}([a,b],\psi)} \leq$$

$$\frac{(\psi(b)-\psi(a))^{\left[\sum_{i=1}^{m}(\xi_i-\mu_i)-m+\frac{m}{r_1}+\frac{1}{r_1}-\frac{1}{r_1 r_2}\right]}}{\left(r_1 r_3\left(\sum_{i=1}^{m}(\xi_i-\mu_i)-m\right)+mr_3+1\right)^{\frac{1}{r_1 r_3}}\left(\prod_{i=1}^{m}(r_1(\xi_i-\mu_i-1)+1)\right)^{\frac{1}{r_1}}} \tag{2.11}$$

$$\left\{\int_a^b \left[\prod_{i=1}^{m}\left(\int_a^x \left|E^{-\gamma_i\beta_i}_{\rho_i,\xi_i-\mu_i}\left[\omega_i(\psi(x)-\psi(t))^{\rho_i}\right]\right|^{r_2} d\psi(t)\right)\right]^{r_1} d\psi(x)\right\}^{\frac{1}{r_1 r_2}}$$

$$\left(\prod_{i=1}^{m}\left\| {}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,a+} f_i\right\|_{L_{r_3}([a,b],\psi)}\right),$$

and

ii)

$$\left\| \prod_{i=1}^{m} {}^{H}\mathbb{D}^{\gamma_i,\beta_i;\psi}_{\rho_i,\mu_i,\omega_i,b-} f_i \right\|_{L_{r_1}([a,b],\psi)} \leq$$

$$\frac{(\psi(b)-\psi(a))^{\left[\sum_{i=1}^{m}(\xi_i-\mu_i)-m+\frac{m}{r_1}+\frac{1}{r_1}-\frac{1}{r_1 r_2}\right]}}{\left(r_1 r_3\left(\sum_{i=1}^{m}(\xi_i-\mu_i)-m\right)+mr_3+1\right)^{\frac{1}{r_1 r_3}}\left(\prod_{i=1}^{m}(r_1(\xi_i-\mu_i-1)+1)\right)^{\frac{1}{r_1}}} \tag{2.12}$$

$$\left\{\int_a^b \left[\prod_{i=1}^{m}\left(\int_x^b \left|E^{-\gamma_i\beta_i}_{\rho_i,\xi_i-\mu_i}\left[\omega_i(\psi(t)-\psi(x))^{\rho_i}\right]\right|^{r_2} d\psi(t)\right)\right]^{r_1} d\psi(x)\right\}^{\frac{1}{r_1 r_2}}$$

$$\left(\prod_{i=1}^{m}\left\| {}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,b-} f_i\right\|_{L_{r_3}([a,b],\psi)}\right).$$

Proof. By (1.15), (1.16) and Theorems 2.1, 2.2. □

From now on all entities are according and respectively to Section 1. Background.

Next we give Opial type inequalities related to Prabhakar fractional calculus.

A left side one follows:

Theorem 2.5. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_a^x \left|\left(e^{\gamma;\psi}_{\rho,\mu,\omega,a+} f\right)(w)\right| |f(w)| \psi'(w)\, dw \leq 2^{-\frac{1}{q}}$$

$$\left[\int_a^x \left\{\int_a^w (\psi(w)-\psi(t))^{p(\mu-1)} \left|E^{\gamma}_{\rho,\mu}\left[\omega(\psi(w)-\psi(t))^{\rho}\right]\right|^p dt\right\} dw\right]^{\frac{1}{p}}$$

$$\left(\int_a^x |f(w)|^q (\psi'(w))^q\, dw\right)^{\frac{2}{q}}, \tag{2.13}$$

$\forall\ x \in [a,b]$.

Proof. By (1.2), using Hölder's inequality, we have

$$\left|\left(e^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right)(x)\right| \leq \int_a^x \psi'(t)\left(\psi(x)-\psi(t)\right)^{\mu-1}\left|E^{\gamma}_{\rho,\mu}\left[\omega\left(\psi(x)-\psi(t)\right)^{\rho}\right]\right||f(t)|\,dt$$

$$\leq \left(\int_a^x \left(\psi(x)-\psi(t)\right)^{p(\mu-1)}\left|E^{\gamma}_{\rho,\mu}\left[\omega\left(\psi(x)-\psi(t)\right)^{\rho}\right]\right|^p dt\right)^{\frac{1}{p}}$$

$$\left(\int_a^x \left(\psi'(t)|f(t)|\right)^q dt\right)^{\frac{1}{q}}. \tag{2.14}$$

Call

$$\phi(x) = \int_a^x \left(\psi'(t)|f(t)|\right)^q dt, \quad \phi(a)=0. \tag{2.15}$$

Thus

$$\phi'(x) = \left(\psi'(x)|f(x)|\right)^q \geq 0, \tag{2.16}$$

and

$$\left(\phi'(x)\right)^{\frac{1}{q}} = \psi'(x)|f(x)| \geq 0, \quad \forall\, x\in[a,b].$$

Consequently, we get

$$\left|\left(e^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right)(w)\right|\psi'(w)|f(w)| \leq$$

$$\left(\int_a^w \left(\psi(w)-\psi(t)\right)^{p(\mu-1)}\left|E^{\gamma}_{\rho,\mu}\left[\omega\left(\psi(w)-\psi(t)\right)^{\rho}\right]\right|^p dt\right)^{\frac{1}{p}}$$

$$\left(\phi(w)\,\phi'(w)\right)^{\frac{1}{q}}, \quad \forall\, w\in[a,b]. \tag{2.17}$$

Then, by applying again Hölder's inequality:

$$\int_a^x \left|\left(e^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right)(w)\right||f(w)|\,\psi'(w)\,dw \leq \tag{2.18}$$

$$\int_a^x \left\{\int_a^w \left(\psi(w)-\psi(t)\right)^{p(\mu-1)}\left|E^{\gamma}_{\rho,\mu}\left[\omega\left(\psi(w)-\psi(t)\right)^{\rho}\right]\right|^p dt\right\}^{\frac{1}{p}}$$

$$\left(\phi(w)\,\phi'(w)\right)^{\frac{1}{q}} dw \leq$$

$$\left[\int_a^x \left\{\int_a^w \left(\psi(w)-\psi(t)\right)^{p(\mu-1)}\left|E^{\gamma}_{\rho,\mu}\left[\omega\left(\psi(w)-\psi(t)\right)^{\rho}\right]\right|^p dt\right\} dw\right]^{\frac{1}{p}}$$

$$\left(\int_a^x \phi(w)\,d\phi(w)\right)^{\frac{1}{q}} =$$

$$\left[\int_a^x \left\{\int_a^w \left(\psi(w)-\psi(t)\right)^{p(\mu-1)}\left|E^{\gamma}_{\rho,\mu}\left[\omega\left(\psi(w)-\psi(t)\right)^{\rho}\right]\right|^p dt\right\} dw\right]^{\frac{1}{p}}$$

$$\left(\frac{\phi^2(x)}{2}\right)^{\frac{1}{q}} = 2^{-\frac{1}{q}}$$

$$\left[\int_a^x \left\{\int_a^w \left(\psi(w)-\psi(t)\right)^{p(\mu-1)}\left|E^{\gamma}_{\rho,\mu}\left[\omega\left(\psi(w)-\psi(t)\right)^{\rho}\right]\right|^p dt\right\} dw\right]^{\frac{1}{p}}$$

$$\left(\int_a^x \left(\psi'(w)|f(w)|\right)^q dw\right)^{\frac{2}{q}}. \tag{2.19}$$

The theorem is proved. ☐

The right side Opial inequality follows:

Theorem 2.6. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_x^b \left| \left(e^{\gamma;\psi}_{\rho,\mu,\omega,b-} f \right)(w) \right| |f(w)| \, \psi'(w) \, dw \leq 2^{-\frac{1}{q}}$$

$$\left[\int_x^b \left\{ \int_w^b (\psi(t) - \psi(w))^{p(\mu-1)} \left| E^{\gamma}_{\rho,\mu} \left[\omega (\psi(t) - \psi(w))^{\rho} \right] \right|^p dt \right\} dw \right]^{\frac{1}{p}}$$

$$\left(\int_x^b |f(w)|^q (\psi'(w))^q \, dw \right)^{\frac{2}{q}}, \tag{2.20}$$

$\forall\, x \in [a, b]$.

Proof. As it is similar to the proof of Theorem 2.5, is omitted. □

We continue with more interesting Opial type Prabhakar-Caputo fractional inequalities:

Theorem 2.7. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

i)

$$\int_a^x \left| \left({}^C D^{\gamma;\psi}_{\rho,\mu,\omega,a+} f \right)(w) \right| \left| f^{[N]}_{\psi}(w) \right| \psi'(w) \, dw \leq 2^{-\frac{1}{q}}$$

$$\left[\int_a^x \left\{ \int_a^w (\psi(w) - \psi(t))^{p(N-\mu-1)} \left| E^{-\gamma}_{\rho,N-\mu} \left[\omega (\psi(w) - \psi(t))^{\rho} \right] \right|^p dt \right\} dw \right]^{\frac{1}{p}}$$

$$\left(\int_a^x \left| f^{[N]}_{\psi}(w) \right|^q (\psi'(w))^q \, dw \right)^{\frac{2}{q}}, \tag{2.21}$$

and

ii)

$$\int_x^b \left| \left({}^C D^{\gamma;\psi}_{\rho,\mu,\omega,b-} f \right)(w) \right| \left| f^{[N]}_{\psi}(w) \right| \psi'(w) \, dw \leq 2^{-\frac{1}{q}}$$

$$\left[\int_x^b \left\{ \int_w^b (\psi(t) - \psi(w))^{p(N-\mu-1)} \left| E^{-\gamma}_{\rho,N-\mu} \left[\omega (\psi(t) - \psi(w))^{\rho} \right] \right|^p dt \right\} dw \right]^{\frac{1}{p}}$$

$$\left(\int_x^b \left| f^{[N]}_{\psi}(w) \right|^q (\psi'(w))^q \, dw \right)^{\frac{2}{q}}, \tag{2.22}$$

$\forall\, x \in [a, b]$.

Proof. By Theorems 2.5, 2.6 and (1.6)-(1.8). □

Next come ψ-Hilfer-Prabhakar left and right Opial type fractional inequalities:

Theorem 2.8. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Additionally here assume that*

$$^{RL}D^{\gamma(1-\beta);\psi}_{\rho,\xi,\omega,a+}f, ^{RL}D^{\gamma(1-\beta);\psi}_{\rho,\xi,\omega,b-}f \in C([a,b]).$$

Then

i)

$$\int_a^x \left|\left({}^H\mathbb{D}^{\gamma,\beta;\psi}_{\rho,\mu,\omega,a+}f\right)(w)\right|\left|\left({}^{RL}D^{\gamma(1-\beta);\psi}_{\rho,\xi,\omega,a+}f\right)(w)\right|\psi'(w)\,dw \leq 2^{-\frac{1}{q}}$$

$$\left[\int_a^x\left\{\int_a^w (\psi(w)-\psi(t))^{p(\xi-\mu-1)}\left|E^{-\gamma\beta}_{\rho,\xi-\mu}\left[\omega(\psi(w)-\psi(t))^{\rho}\right]\right|^p dt\right\}dw\right]^{\frac{1}{p}}$$

$$\left(\int_a^x\left|\left({}^{RL}D^{\gamma(1-\beta);\psi}_{\rho,\xi,\omega,a+}f\right)(w)\right|^q(\psi'(w))^q\,dw\right)^{\frac{2}{q}}, \tag{2.23}$$

and

ii)

$$\int_x^b \left|\left({}^H\mathbb{D}^{\gamma,\beta;\psi}_{\rho,\mu,\omega,b-}f\right)(w)\right|\left|\left({}^{RL}D^{\gamma(1-\beta);\psi}_{\rho,\xi,\omega,b-}f\right)(w)\right|\psi'(w)\,dw \leq 2^{-\frac{1}{q}}$$

$$\left[\int_x^b\left\{\int_w^b (\psi(t)-\psi(w))^{p(\xi-\mu-1)}\left|E^{-\gamma\beta}_{\rho,\xi-\mu}\left[\omega(\psi(t)-\psi(w))^{\rho}\right]\right|^p dt\right\}dw\right]^{\frac{1}{p}}$$

$$\left(\int_x^b\left|\left({}^{RL}D^{\gamma(1-\beta);\psi}_{\rho,\xi,\omega,b-}f\right)(w)\right|^q(\psi'(w))^q\,dw\right)^{\frac{2}{q}}, \tag{2.24}$$

$\forall\ x \in [a,b]$.

Proof. By Theorems 2.5, 2.6 and (1.15), (1.16). ☐

Next we give several Prabhakar Hilbert-Pachpatte fractional inequalities. We start with a left side one.

Theorem 2.9. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $i = 1, 2$. Let $[a_i, b_i] \subset \mathbb{R}$, $\psi_i \in C^1([a_i, b_i])$ and strictly increasing, $f_i \in C([a_i, b_i])$; $\rho_i, \mu_i > 0$, $\gamma_i, \omega_i \in \mathbb{R}$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\left|\left(e^{\gamma_1;\psi_1}_{\rho_1,\mu_1,\omega_1,a_1+}f_1\right)(x_1)\right|\left|\left(e^{\gamma_2;\psi_2}_{\rho_2,\mu_2,\omega_2,a_2+}f_2\right)(x_2)\right|dx_1dx_2}{\left\{\frac{\left[\int_{a_1}^{x_1}\left\{(\psi_1(x_1)-\psi_1(t_1))^{\mu_1-1}\left|E^{\gamma_1}_{\rho_1,\mu_1}[\omega_1(\psi_1(x_1)-\psi_1(t_1))^{\rho_1}]\right|\right\}^p dt_1\right]}{p}+\frac{\left[\int_{a_2}^{x_2}\left\{(\psi_2(x_2)-\psi_2(t_2))^{\mu_2-1}\left|E^{\gamma_2}_{\rho_2,\mu_2}[\omega_2(\psi_2(x_2)-\psi_2(t_2))^{\rho_2}]\right|\right\}^q dt_2\right]}{q}\right\}}$$

$$\leq (b_1-a_1)(b_2-a_2)\left\|\psi_1'f_1\right\|_q\left\|\psi_2'f_2\right\|_p. \tag{2.25}$$

Proof. We have that $(i = 1, 2)$

$$\left(e^{\gamma_i;\psi_i}_{\rho_i,\mu_i,\omega_i,a_i+}f_i\right)(x_i) \overset{(1.2)}{=}$$

$$\int_{a_i}^{x_i}\psi_i'(t_i)(\psi_i(x_i)-\psi_i(t_i))^{\mu_i-1}E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi_i(x_i)-\psi_i(t_i))^{\rho_i}\right]f_i(t_i)\,dt_i, \tag{2.26}$$

$\forall\ x_i \in [a_i, b_i]$, where $\rho_i, \mu_i > 0$; $\gamma_i, \omega_i \in \mathbb{R}$.

Then

$$\left|\left(e^{\gamma_i;\psi_i}_{\rho_i,\mu_i,\omega_i,a_i+}f_i\right)(x_i)\right| \leq$$
$$\int_{a_i}^{x_i} \psi_i'(t_i)(\psi_i(x_i)-\psi_i(t_i))^{\mu_i-1}\left|E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi_i(x_i)-\psi_i(t_i))^{\rho_i}\right]\right||f_i(t_i)|\,dt_i, \tag{2.27}$$

$i = 1, 2$, $\forall\ x_i \in [a_i, b_i]$.

By appying Hölder's inequality twice we get:

$$\left|\left(e^{\gamma_1;\psi_1}_{\rho_1,\mu_1,\omega_1,a_1+}f_1\right)(x_1)\right| \leq$$
$$\left[\int_{a_1}^{x_1}\left\{(\psi_1(x_1)-\psi_1(t_1))^{\mu_1-1}\left|E^{\gamma_1}_{\rho_1,\mu_1}\left[\omega_1(\psi_1(x_1)-\psi_1(t_1))^{\rho_1}\right]\right|\right\}^p dt_1\right]^{\frac{1}{p}}$$
$$\left(\int_{a_1}^{x_1}(\psi_1'(t_1)|f_1(t_1)|)^q\,dt_1\right)^{\frac{1}{q}}, \tag{2.28}$$

$\forall\ x_1 \in [a_1, b_1]$, and

$$\left|\left(e^{\gamma_2;\psi_2}_{\rho_2,\mu_2,\omega_2,a_2+}f_2\right)(x_2)\right| \leq$$
$$\left[\int_{a_2}^{x_2}\left\{(\psi_2(x_2)-\psi_2(t_2))^{\mu_2-1}\left|E^{\gamma_2}_{\rho_2,\mu_2}\left[\omega_2(\psi_2(x_2)-\psi_2(t_2))^{\rho_2}\right]\right|\right\}^q dt_2\right]^{\frac{1}{q}}$$
$$\left(\int_{a_2}^{x_2}(\psi_2'(t_2)|f_2(t_2)|)^p\,dt_2\right)^{\frac{1}{p}}, \tag{2.29}$$

$\forall\ x_2 \in [a_2, b_2]$.

Hence we have (by (2.28), (2.29))

$$\left|\left(e^{\gamma_1;\psi_1}_{\rho_1,\mu_1,\omega_1,a_1+}f_1\right)(x_1)\right|\left|\left(e^{\gamma_2;\psi_2}_{\rho_2,\mu_2,\omega_2,a_2+}f_2\right)(x_2)\right| \leq$$
$$\left[\int_{a_1}^{x_1}\left\{(\psi_1(x_1)-\psi_1(t_1))^{\mu_1-1}\left|E^{\gamma_1}_{\rho_1,\mu_1}\left[\omega_1(\psi_1(x_1)-\psi_1(t_1))^{\rho_1}\right]\right|\right\}^p dt_1\right]^{\frac{1}{p}}$$
$$\left[\int_{a_2}^{x_2}\left\{(\psi_2(x_2)-\psi_2(t_2))^{\mu_2-1}\left|E^{\gamma_2}_{\rho_2,\mu_2}\left[\omega_2(\psi_2(x_2)-\psi_2(t_2))^{\rho_2}\right]\right|\right\}^q dt_2\right]^{\frac{1}{q}}$$
$$\|\psi_1'f_1\|_q\|\psi_2'f_2\|_p \leq \tag{2.30}$$

(using Young's inequality for $a, b \geq 0$, $a^{\frac{1}{p}}b^{\frac{1}{q}} \leq \frac{a}{p}+\frac{b}{q}$)

$$\left\{\frac{\left[\int_{a_1}^{x_1}\left\{(\psi_1(x_1)-\psi_1(t_1))^{\mu_1-1}\left|E^{\gamma_1}_{\rho_1,\mu_1}\left[\omega_1(\psi_1(x_1)-\psi_1(t_1))^{\rho_1}\right]\right|\right\}^p dt_1\right]}{p}+\right.$$
$$\left.\frac{\left[\int_{a_2}^{x_2}\left\{(\psi_2(x_2)-\psi_2(t_2))^{\mu_2-1}\left|E^{\gamma_2}_{\rho_2,\mu_2}\left[\omega_2(\psi_2(x_2)-\psi_2(t_2))^{\rho_2}\right]\right|\right\}^q dt_2\right]}{q}\right\}$$
$$\|\psi_1'f_1\|_q\|\psi_2'f_2\|_p,$$

$\forall\ x_i \in [a_i, b_i]\,,\ i = 1, 2.$

So far we have

$$\frac{\left|\left(e^{\gamma_1;\psi_1}_{\rho_1,\mu_1,\omega_1,a_1+}f_1\right)(x_1)\right|\left|\left(e^{\gamma_2;\psi_2}_{\rho_2,\mu_2,\omega_2,a_2+}f_2\right)(x_2)\right|}{\left\{\frac{\left[\int_{a_1}^{x_1}\left\{(\psi_1(x_1)-\psi_1(t_1))^{\mu_1-1}\left|E^{\gamma_1}_{\rho_1,\mu_1}[\omega_1(\psi_1(x_1)-\psi_1(t_1))^{\rho_1}]\right|\right\}^p dt_1\right]}{p}+\frac{\left[\int_{a_2}^{x_2}\left\{(\psi_2(x_2)-\psi_2(t_2))^{\mu_2-1}\left|E^{\gamma_2}_{\rho_2,\mu_2}[\omega_2(\psi_2(x_2)-\psi_2(t_2))^{\rho_2}]\right|\right\}^q dt_2\right]}{q}\right\}} \leq \left\|\psi_1' f_1\right\|_q \left\|\psi_2' f_2\right\|_p, \tag{2.31}$$

$\forall\ x_i \in [a_i, b_i]\,,\ i = 1, 2.$

The denominator in (2.31) can be zero only when $x_1 = a_1$ and $x_2 = a_2$. Therefore we obtain (2.25) by integrating (2.31) over $[a_1, b_1] \times [a_2, b_2]$. □

It follows the corresponding to (2.25) right side inequality.

Theorem 2.10. *All as in Theorem 2.9. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\left|\left(e^{\gamma_1;\psi_1}_{\rho_1,\mu_1,\omega_1,b_1-}f_1\right)(x_1)\right|\left|\left(e^{\gamma_2;\psi_2}_{\rho_2,\mu_2,\omega_2,b_2-}f_2\right)(x_2)\right|dx_1dx_2}{\left\{\frac{\left[\int_{x_1}^{b_1}\left\{(\psi_1(t_1)-\psi_1(x_1))^{\mu_1-1}\left|E^{\gamma_1}_{\rho_1,\mu_1}[\omega_1(\psi_1(t_1)-\psi_1(x_1))^{\rho_1}]\right|\right\}^p dt_1\right]}{p}+\frac{\left[\int_{x_2}^{b_2}\left\{(\psi_2(t_2)-\psi_2(x_2))^{\mu_2-1}\left|E^{\gamma_2}_{\rho_2,\mu_2}[\omega_2(\psi_2(t_2)-\psi_2(x_2))^{\rho_2}]\right|\right\}^q dt_2\right]}{q}\right\}}$$

$$\leq (b_1 - a_1)(b_2 - a_2)\left\|\psi_1' f_1\right\|_q \left\|\psi_2' f_2\right\|_p. \tag{2.32}$$

Proof. As similar to the proof of Theorem 2.9 is omitted. □

We continue with applications of Theorems 2.9, 2.10.

Theorem 2.11. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $i = 1, 2$. Let $[a_i, b_i] \subset \mathbb{R}$, $\psi_i \in C^{\max(N_1,N_2)}([a_i, b_i])$, $\psi_i' \neq 0$, and strictly increasing; $f_i \in C^{N_i}([a_i, b_i])$, where $N_i = \lceil \mu_i \rceil$, $0 < \mu_i \notin \mathbb{N}$. Here $\rho_i > 0$; $\gamma_i, \omega_i \in \mathbb{R}$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\left|\left({}^C D^{\gamma_1;\psi_1}_{\rho_1,\mu_1,\omega_1,a_1+}f_1\right)(x_1)\right|\left|\left({}^C D^{\gamma_2;\psi_2}_{\rho_2,\mu_2,\omega_2,a_2+}f_2\right)(x_2)\right|dx_1dx_2}{\left\{\frac{\left[\int_{a_1}^{x_1}\left\{(\psi_1(x_1)-\psi_1(t_1))^{N_1-\mu_1-1}\left|E^{-\gamma_1}_{\rho_1,N_1-\mu_1}[\omega_1(\psi_1(x_1)-\psi_1(t_1))^{\rho_1}]\right|\right\}^p dt_1\right]}{p}+\frac{\left[\int_{a_2}^{x_2}\left\{(\psi_2(x_2)-\psi_2(t_2))^{N_2-\mu_2-1}\left|E^{-\gamma_2}_{\rho_2,N_2-\mu_2}[\omega_2(\psi_2(x_2)-\psi_2(t_2))^{\rho_2}]\right|\right\}^q dt_2\right]}{q}\right\}}$$

$$\leq (b_1 - a_1)(b_2 - a_2)\left\|\psi_1' f^{[N_1]}_{1\psi_1}\right\|_q \left\|\psi_2' f^{[N_2]}_{2\psi_2}\right\|_p. \tag{2.33}$$

Proof. By Theorem 2.9 and (1.2), (1.6). □

We also give

Theorem 2.12. *All as in Theorem 2.11. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\left|\left({}^{C}D^{\gamma_1;\psi_1}_{\rho_1,\mu_1,\omega_1,b_1-}f_1\right)(x_1)\right|\left|\left({}^{C}D^{\gamma_2;\psi_2}_{\rho_2,\mu_2,\omega_2,b_2-}f_2\right)(x_2)\right|dx_1dx_2}{\left\{\frac{\left[\int_{x_1}^{b_1}\left\{(\psi_1(t_1)-\psi_1(x_1))^{N_1-\mu_1-1}\left|E^{-\gamma_1}_{\rho_1,N_1-\mu_1}[\omega_1(\psi_1(t_1)-\psi_1(x_1))^{\rho_1}]\right|\right\}^p dt_1\right]}{p}+\frac{\left[\int_{x_2}^{b_2}\left\{(\psi_2(t_2)-\psi_2(x_2))^{N_2-\mu_2-1}\left|E^{-\gamma_2}_{\rho_2,N_2-\mu_2}[\omega_2(\psi_2(t_2)-\psi_2(x_2))^{\rho_2}]\right|\right\}^q dt_2\right]}{q}\right\}}$$

$$\leq (b_1-a_1)(b_2-a_2)\left\|\psi_1' f^{[N_1]}_{1\psi_1}\right\|_q\left\|\psi_2' f^{[N_2]}_{2\psi_2}\right\|_p. \tag{2.34}$$

Proof. By Theorem 2.10 and (1.3), (1.7). ☐

We present

Theorem 2.13. *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$; $i=1,2$. Let $[a_i,b_i]\subset\mathbb{R}$, $\psi_i\in C^{\max(N_1,N_2)}([a_i,b_i])$, $\psi_i'\neq 0$, and strictly increasing; $f_i\in C^{N_i}([a_i,b_i])$, where $N_i=\lceil\mu_i\rceil$, $0<\mu_i\notin\mathbb{N}$. Here $\rho_i>0$; $\gamma_i,\omega_i\in\mathbb{R}$ and $\xi_i=\mu_i+\beta_i(N_i-\mu_i)$, $i=1,2$, where $0\leq\beta_i\leq 1$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\left|\left({}^{H}\mathbb{D}^{\gamma_1,\beta_1;\psi_1}_{\rho_1,\mu_1,\omega_1,a_1+}f_1\right)(x_1)\right|\left|\left({}^{H}\mathbb{D}^{\gamma_2,\beta_2;\psi_2}_{\rho_2,\mu_2,\omega_2,a_2+}f_2\right)(x_2)\right|dx_1dx_2}{\left\{\frac{\left[\int_{a_1}^{x_1}\left\{(\psi_1(x_1)-\psi_1(t_1))^{\xi_1-\mu_1-1}\left|E^{-\gamma_1\beta_1}_{\rho_1,\xi_1-\mu_1}[\omega_1(\psi_1(x_1)-\psi_1(t_1))^{\rho_1}]\right|\right\}^p dt_1\right]}{p}+\frac{\left[\int_{a_2}^{x_2}\left\{(\psi_2(x_2)-\psi_2(t_2))^{\xi_2-\mu_2-1}\left|E^{-\gamma_2\beta_2}_{\rho_2,\xi_2-\mu_2}[\omega_2(\psi_2(x_2)-\psi_2(t_2))^{\rho_2}]\right|\right\}^q dt_2\right]}{q}\right\}}$$

$$\leq (b_1-a_1)(b_2-a_2)\left\|\psi_1'\ {}^{RL}D^{\gamma_1(1-\beta_1);\psi_1}_{\rho_1,\xi_1,\omega_1,a_1+}f_1\right\|_q\left\|\psi_2'\ {}^{RL}D^{\gamma_2(1-\beta_2);\psi_2}_{\rho_2,\xi_2,\omega_2,a_2+}f_2\right\|_p. \tag{2.35}$$

Proof. By Theorem 2.9 and (1.15). ☐

We also give

Theorem 2.14. *All as in Theorem 2.13. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\left|\left({}^{H}\mathbb{D}^{\gamma_1,\beta_1;\psi_1}_{\rho_1,\mu_1,\omega_1,b_1-}f_1\right)(x_1)\right|\left|\left({}^{H}\mathbb{D}^{\gamma_2,\beta_2;\psi_2}_{\rho_2,\mu_2,\omega_2,b_2-}f_2\right)(x_2)\right|dx_1dx_2}{\left\{\frac{\left[\int_{x_1}^{b_1}\left\{(\psi_1(t_1)-\psi_1(x_1))^{\xi_1-\mu_1-1}\left|E^{-\gamma_1\beta_1}_{\rho_1,\xi_1-\mu_1}[\omega_1(\psi_1(t_1)-\psi_1(x_1))^{\rho_1}]\right|\right\}^p dt_1\right]}{p}+\frac{\left[\int_{x_2}^{b_2}\left\{(\psi_2(t_2)-\psi_2(x_2))^{\xi_2-\mu_2-1}\left|E^{-\gamma_2\beta_2}_{\rho_2,\xi_2-\mu_2}[\omega_2(\psi_2(t_2)-\psi_2(x_2))^{\rho_2}]\right|\right\}^q dt_2\right]}{q}\right\}}$$

$$\leq (b_1-a_1)(b_2-a_2)\left\|\psi_1'\ {}^{RL}D^{\gamma_1(1-\beta_1);\psi_1}_{\rho_1,\xi_1,\omega_1,b_1-}f_1\right\|_q\left\|\psi_2'\ {}^{RL}D^{\gamma_2(1-\beta_2);\psi_2}_{\rho_2,\xi_2,\omega_2,b_2-}f_2\right\|_p. \tag{2.36}$$

Proof. By Theorem 2.10 and (1.16). ☐

3 Appendix

We give the following important fundamental results:

Theorem 3.1. *Let* $\rho, \mu > 0$; $\gamma, \omega \in \mathbb{R}$; *and* $\psi \in C^1([a,b])$ *increasing,* $f \in C([a,b])$. *Then* $\left(e^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right), \left(e^{\gamma;\psi}_{\rho,\mu,\omega,b-}f\right) \in C([a,b])$.

Proof. We only prove $\left(e^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right) \in C([a,b])$. We skip the proof for the other is similar.

We consider the power series

$$\overline{E}^{\gamma}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{|(\gamma)_k|}{k!\Gamma(\rho k+\mu)(\rho k+\mu)} z^k, \quad z \in \mathbb{R}. \tag{3.1}$$

We form

$$\overline{R}^{-1} := \lim_{k\to\infty} \frac{\frac{|(\gamma)_{k+1}|}{(k+1)!\Gamma(\rho(k+1)+\mu)(\rho(k+1)+\mu)}}{\frac{|(\gamma)_k|}{k!\Gamma(\rho k+\mu)(\rho k+\mu)}} = \lim_{k\to\infty} \frac{\frac{|\gamma+k|}{(k+1)\Gamma(\rho(k+1)+\mu)(\rho(k+1)+\mu)}}{\frac{1}{\Gamma(\rho k+\mu)(\rho k+\mu)}} = \tag{3.2}$$

$$\lim_{k\to\infty} \frac{|\gamma+k|\,\Gamma(\rho k+\mu)(\rho k+\mu)}{(k+1)\Gamma(\rho(k+1)+\mu)(\rho(k+1)+\mu)} =$$

$$\lim_{k\to\infty} \left(\frac{|\gamma+k|\,\Gamma(\rho k+\mu)}{(k+1)\Gamma(\rho(k+1)+\mu)}\right) \lim_{k\to\infty} \left(\frac{\rho k+\mu}{(\rho k+\mu)+\rho}\right) =: (\Xi). \tag{3.3}$$

Notice that

$$\lim_{k\to\infty} \left(\frac{\rho k+\mu}{(\rho k+\mu)+\rho}\right) = 1. \tag{3.4}$$

From (1.1) we have that its radius of convergence is

$$R = \lim_{k\to\infty} \frac{\frac{|(\gamma)_k|}{k!\Gamma(\rho k+\mu)}}{\frac{|(\gamma)_{k+1}|}{(k+1)!\Gamma(\rho(k+1)+\mu)}} = \lim_{k\to\infty} \frac{\frac{1}{\Gamma(\rho k+\mu)}}{\frac{|\gamma+k|}{(k+1)\Gamma(\rho(k+1)+\mu)}} = \lim_{k\to\infty} \frac{(k+1)\Gamma(\rho(k+1)+\mu)}{|\gamma+k|\,\Gamma(\rho k+\mu)} = \infty,$$

because (1.1) is an entire function.

Therefore, we have that

$$\lim_{k\to\infty} \frac{|\gamma+k|\,\Gamma(\rho k+\mu)}{(k+1)\Gamma(\rho(k+1)+\mu)} = 0.$$

Consequently by (3.3), (3.4), we get that $(\Xi) = 0$. Thus $\overline{R}^{-1} = 0$ and the radius of convergence of $\overline{E}^{\gamma}_{\rho,\mu}(z)$, see (3.1), is $\overline{R} = \infty$, hence (3.1) is convergent everywhere.

Consequently it holds

$$\sum_{k=0}^{\infty} \frac{|(\gamma)_k|\left(|\omega|(\psi(x)-\psi(a))^{\rho}\right)^k}{k!\Gamma(\rho k+\mu)(\rho k+\mu)} < \infty, \tag{3.5}$$

$\forall\, x \in [a,b]$.

We notice that

$$\sum_{k=0}^{\infty} \frac{|(\gamma)_k|\,|\omega|^k}{k!\Gamma(\rho k+\mu)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{(\rho k+\mu)-1}|f(t)|\,dt \leq$$

$$\|f\|_{\infty}\sum_{k=0}^{\infty}\frac{|(\gamma)_k|\,|\omega|^k}{k!\Gamma(\rho k+\mu)}\frac{(\psi(x)-\psi(a))^{\rho k+\mu}}{\rho k+\mu}\leq \tag{3.6}$$

$$\|f\|_{\infty}(\psi(b)-\psi(a))^{\mu}\sum_{k=0}^{\infty}\frac{|(\gamma)_k|\left(|\omega|(\psi(x)-\psi(a))^{\rho}\right)^k}{k!\Gamma(\rho k+\mu)(\rho k+\mu)}\overset{(3.5)}{<}\infty.$$

Consequently, by [5, p. 175], we derive

$$\begin{aligned}\left(e^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right)(x) &\overset{(1.2)}{=} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\mu-1}\left(\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(\rho k+\mu)}\left(\omega(\psi(x)-\psi(t))^{\rho}\right)^k\right)f(t)\,dt\\ &= \sum_{k=0}^{\infty}\frac{(\gamma)_k\,\omega^k}{k!\Gamma(\rho k+\mu)}\int_a^x\psi'(t)(\psi(x)-\psi(t))^{(\rho k+\mu)-1}f(t)\,dt, \end{aligned}\tag{3.7}$$

$\forall\ x\in[a,b]$.

By [2, p. 98], we obtain that the function

$$\lambda^{(k)}_{\rho,\mu}(f,x)=\int_a^x\psi'(t)(\psi(x)-\psi(t))^{(\rho k+\mu)-1}f(t)\,dt,$$

$x\in[a,b]$, is absolutely continuous for $\rho k+\mu\geq 1$ and continuous for $\rho k+\mu\in(0,1)$; $\psi\in C^1([a,b])$ and increasing.

That is always $\lambda^{(k)}_{\rho,\mu}(|f|,x)\in C([a,b])$, for all $k=0,1,\ldots$

By (3.5), one can derive that

$$\sum_{k=0}^{\infty}\frac{|(\gamma)_k|\,|\omega|^k}{k!\Gamma(\rho k+\mu)}\lambda^{(k)}_{\rho,\mu}(|f|,x)\leq$$

$$\|f\|_{\infty}(\psi(b)-\psi(a))^{\mu}\sum_{k=0}^{\infty}\frac{|(\gamma)_k|\left(|\omega|(\psi(b)-\psi(a))^{\rho}\right)^k}{k!\Gamma(\rho k+\mu)(\rho k+\mu)}<\infty. \tag{3.8}$$

Notice that

$$\begin{aligned}\left|\lambda^{(k)}_{\rho,\mu}(f,x)\right|\leq\lambda^{(k)}_{\rho,\mu}(|f|,x) &= \int_a^x\psi'(t)(\psi(x)-\psi(t))^{(\rho k+\mu)-1}|f(t)|\,dt\\ &\leq \|f\|_{\infty}\frac{(\psi(b)-\psi(a))^{(\rho k+\mu)}}{(\rho k+\mu)},\ k=0,1,\ldots\end{aligned}\tag{3.9}$$

And even more we get:

$$\frac{|(\gamma)_k|\,|\omega|^k}{k!\Gamma(\rho k+\mu)}\left|\lambda^{(k)}_{\rho,\mu}(f,x)\right|\leq\frac{|(\gamma)_k|\,|\omega|^k}{k!\Gamma(\rho k+\mu)}\lambda^{(k)}_{\rho,\mu}(|f|,x)\leq$$

$$\left(\frac{|(\gamma)_k|\,|\omega|^k}{k!\Gamma(\rho k+\mu)}\right)\frac{\|f\|_{\infty}(\psi(b)-\psi(a))^{(\rho k+\mu)}}{(\rho k+\mu)}=:M_k,\quad k=0,1,\ldots; \tag{3.10}$$

and by (3.8) that $\sum_{k=0}^{\infty}M_k<\infty$, converges.

By Weierstrass M-test we get that $\sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k}{k!\Gamma(\rho k+\mu)}\lambda^{(k)}_{\rho,\mu}(f,x)$ is uniformly and absolutely convergent for $x\in[a,b]$.

Consequently by (3.7) we derive that $\left(e^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right)\in C([a,b])$. The proof is completed. $\square$

We finish with

Corollary 3.2. *All as in Theorem 3.1. We have that*

$$\left\| e^{\gamma;\psi}_{\rho,\mu,\omega,a+(b-)} f \right\|_{\infty} \leq \left(\sum_{k=0}^{\infty} \frac{\left|(\gamma)_k\right| \left|\omega\right|^k \left(\psi\left(b\right) - \psi\left(a\right)\right)^{\rho k + \mu}}{k! \Gamma\left(\rho k + \mu + 1\right)} \right) \|f\|_{\infty} < +\infty. \tag{3.11}$$

That is $e^{\gamma;\psi}_{\rho,\mu,\omega,a+(b-)}$ *are bounded linear operators and positive operators if* $\gamma, \omega > 0$.

Proof. By (3.7), (3.8). □

References

[1] G. A. Anastassiou, *Fractional differentiation inequalities*, New York: Springer-Verlag, 2009.

[2] G. A. Anastassiou, *Intelligent Computations: abstract fractional calculus inequalities, approximations*, Cham: Springer, 2018.

[3] A. Giusti, I. Colombaro, R. Garra, R. Garrappa, F. Polito, M. Popolizio and F. Mainardi, "A practical guide to Prabhakar fractional calculus", Fract. Calc. Appl. Anal., vol. 23, no. 1, pp. 9–54, 2020.

[4] R. Gorenflo, A. Kilbas, F. Mainardi and S. Rogosin, *Mittag-Leffler functions, related topics and applications*, Heidelberg: Springer, 2014.

[5] E. Hewith and K. Stromberg, *Real and abstract analysis. A modern treatment of the theory of functions of a real variable*, New York: Springer, 1965.

[6] F. Polito and Ž. Tomovski, "Some properties of Prabhakar-type fractional calculus operators", Fract. Differ. Calc., vol. 6, no. 1, pp. 73–94, 2016.

[7] T. R. Prabhakar, "A singular integral equation with a generalized Mittag-Leffler function in the kernel", Yokohama Math. J., vol. 19, pp. 7–15, 1971.

[8] J. Vanterler da C. Sousa, E. Capelas de Oliveira, "On the ψ-Hilfer fractional derivative", Commun. Nonlinear Sci. Numer. Simul., vol. 60, pp. 72–91, 2018.

CUBO, A Mathematical Journal
Vol. 23, no. 03, pp. 441–455, December 2021
DOI: 10.4067/S0719-06462021000300441

Existence and uniqueness of solutions to discrete, third-order three-point boundary value problems

Saleh S. Almuthaybiri[1,2]
Jagan Mohan Jonnalagadda[3]
Christopher C. Tisdell[1]

[1] *School of Mathematics and Statistics, The University of New South Wales, UNSW, Sydney, NSW, 2052, Australia.*
cct@unsw.edu.au

[2] *Department of Mathematics, College of Science and Arts in Uglat Asugour, Qassim University, Buraydah, Kingdom of Saudi Arabia.*
s.almuthaybiri@qu.edu.sa

[3] *Department of Mathematics, Birla Institute of Technology and Science Pilani, Hyderabad, 500078, Telangana, India.*
j.jaganmohan@hotmail.com

ABSTRACT

The purpose of this article is to move towards a more complete understanding of the qualitative properties of solutions to discrete boundary value problems. In particular, we introduce and develop sufficient conditions under which the existence of a unique solution for a third-order difference equation subject to three-point boundary conditions is guaranteed. Our contributions are realized in the following ways. First, we construct the corresponding Green's function for the problem and formulate some new bounds on its summation. Second, we apply these properties to the boundary value problem by drawing on Banach's fixed point theorem in conjunction with interesting metrics and appropriate inequalities. We discuss several examples to illustrate the nature of our advancements.

RESUMEN

El propósito de este artículo es avanzar hacia un entendimiento más completo de las propiedades cualitativas de las soluciones a problemas discretos de valor en la frontera. En particular, introducimos y desarrollamos condiciones suficientes bajo las cuales se garantiza la existencia de una única solución para una ecuación en diferencias de tercer orden sujeta a condiciones de borde en tres puntos. Nuestras contribuciones son de dos tipos. En primer lugar, construimos las funciones de Green correspondientes para el problema y formulamos nuevas cotas para su suma. En segundo lugar, aplicamos estas propiedades al problema de valor en la frontera usando el teorema del punto fijo de Banach junto con métricas interesantes y desigualdades apropiadas. Discutimos varios ejemplos para ilustrar la naturaleza de nuestros avances.

Keywords and Phrases: Forward difference, boundary value problem, Green's function, contraction, fixed point, existence, uniqueness.

2020 AMS Mathematics Subject Classification: 39A12.

Accepted: 14 September, 2021
Received: 09 January, 2021

1 Introduction

Discrete boundary value problems are of significant interest to scientific and technical communities. For instance, their perceived utility is partly due to their ability to act as a mathematical framework to model purely discrete processes and phenomena that arise in various fields of science and engineering. In addition, developing a theory of discrete boundary value problems has the potential to inform our understanding of continuous boundary value problems. For example, discrete boundary value problems can arise as approximations to "continuous" boundary value problems that involve differential equations, where the numerical aspects of solutions are of importance. Furthermore, it is also possible to construct a theory of differential equations by only using difference equations [7].

Although discrete problems have enjoyed continued interest, the mathematics community is yet to reach a complete understanding of the qualitative and quantitative properties of their solutions. This includes, for example, discrete boundary value problems of the third order, which have not been advanced to the same degree as their "continuous cousins" or to the same extent as discrete problems of the second order. Moreover, we are yet to achieve a total comprehension of the mathematical similarities and distinctions between such continuous and discrete problems.

Motivated by the above discussion, the purpose of the current paper is to make progress towards a more complete theory concerning the existence and uniqueness of solutions to discrete boundary value problems of the third order. "Knowing an equation has a unique solution is important from both a modelling and theoretical point of view" [19, p. 794] as it informs our mathematical understanding from applied and pure perspectives. For example, by developing a deeper understanding of the existence and uniqueness of solutions to discrete boundary value problems we are simultaneously expanding capacity and knowledge of the associated models and the mathematical frameworks that attempt to describe them.

For any $a,\ b \in \mathbb{R}$ such that $(b-a) \in \mathbb{N}$, we will denote $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}$. Let Δ denote the usual forward difference operator defined by

$$\big(\Delta u\big)(t) = u(t+1) - u(t), \quad t \in \mathbb{N}_0^{T+2},$$

Herein we will consider the following third-order, three-point discrete boundary value problem

$$\begin{cases} \big(\Delta^3 u\big)(t-2) + f(t, u(t)) = 0, \quad t \in \mathbb{N}_2^{T+2}, \\ u(0) = \big(\Delta u\big)(0) = 0, \quad u(T+3) = ku(\eta) \end{cases} \tag{1.1}$$

where f is a continuous function from $\mathbb{N}_0^{T+3} \times \mathbb{R}$ to $\mathbb{R}$ which we denote via $f \in C\left[\mathbb{N}_0^{T+3} \times \mathbb{R}, \mathbb{R}\right]$. In addition, $T \in \mathbb{N}_1$, $k \in \mathbb{R}$ and $\eta \in \mathbb{N}_1^{T+2}$.

Let us briefly outline recent and relevant literature to situate and contextualize our work. Agarwal and Henderson [2] initiated the study of positive solutions to the third-order three-point discrete boundary value problem

$$\begin{cases} \big(\Delta^3 u\big)(t-2) + a(t)g(u(t)) = 0, \quad t \in \mathbb{N}_2^{T+2}, \\ u(0) = u(1) = u(T+3) = 0, \end{cases} \tag{1.2}$$

where $a : \mathbb{N}_0^{T+2} \to \mathbb{R}^+$ and $g \in C\,[\mathbb{R}^+, \mathbb{R}^+]$. Following this work, Anderson [5] and Anderson and Avery [6] examined the existence of multiple solutions to third-order, three-point discrete focal boundary value problems. Positive solutions to discrete, third-order problems have been shown to exist using fixed point theory in cones [13]. In addition, several authors have discussed various qualitative properties of different classes of third-order three-point discrete boundary value problems and a detailed discussion can be found in [11, 12, 23, 24, 25, 13] and the references therein.

Motivated by the recent work [4, 15], where the differential equation version of (1.1) was analyzed, in the present article we investigate the discrete boundary value problem (1.1). When compared with the ideas in [4, 15] our methods and results herein are different; and they reveal some thought-provoking distinctions and connections between the sets of works. For example, the present work develops alternative bounds on the Green's functions to those in [4, 15] and we employ purely discrete ways of working. In particular, we observe that some of our bounds for the discrete case are sharp, while others are rougher. The bounds are different from those developed for the continuous case [4]. This highlights some of the interesting distinctions between the discrete and the continuous in terms of results and methods within the domain of third order problems.

Our article is organized as follows: In Section 2, we construct the Green's function corresponding to the boundary value problem (1.1) and establish new bounds on its summation. In Section 3 we apply the properties of the Green's function to the boundary value problem (1.1) in conjunction with Banach's contraction mapping theorem to establish sufficient conditions for the existence of a unique solution. We provide a discussion of examples in Section 4 to illustrate how our ideas can be put into practice and the relationships between them. Finally, we conclude with some ideas for further work in Section 5.

For more on discrete problems, see the monographs [1, 8, 9, 10, 14].

2 Green's function and its properties

In order to develop the Green's function for the three-point case, we first analyze the two-point discrete boundary value problem

$$\begin{cases} \big(\Delta^3 v\big)(t-2) + h(t) = 0, \quad t \in \mathbb{N}_2^{T+2}, \\ v(0) = \big(\Delta v\big)(0) = v(T+3) = 0, \end{cases} \tag{2.1}$$

where $h \in C\left[\mathbb{N}_2^{T+2}, \mathbb{R}\right]$. The boundary value problem (2.1) can be equivalently rewritten as

$$\begin{cases} \left(\Delta^3 v\right)(t-2) + h(t) = 0, & t \in \mathbb{N}_2^{T+2}, \\ v(0) = v(1) = v(T+3) = 0. \end{cases} \tag{2.2}$$

Yang and Weng [25] derived a Green's function for the boundary value problem (2.2) and also investigated its sign. The following two results are found therein and will be helpful in our present analysis.

Lemma 2.1 ([25])**.** *The unique solution of the boundary value problem* (2.2) *(or* (2.1)*) is given by*

$$v(t) = \sum_{s=2}^{T+2} H(t,s)h(s), \quad t \in \mathbb{N}_0^{T+3}, \tag{2.3}$$

where

$$H(t,s) = \begin{cases} \dfrac{t(t-1)(T+3-s)(T+4-s)}{2(T+3)(T+2)} - \dfrac{(t-s)(t-s+1)}{2}, & s \in \mathbb{N}_0^{t-1}, \\ \dfrac{t(t-1)(T+3-s)(T+4-s)}{2(T+3)(T+2)}, & s \in \mathbb{N}_t^{T+3}. \end{cases} \tag{2.4}$$

Lemma 2.2 ([25])**.** *The Green's function $H(t,s)$ in* (2.4) *satisfies $H(t,s) \geq 0$ for all $(t,s) \in \mathbb{N}_0^{T+3} \times \mathbb{N}_2^{T+2}$.*

Now let us construct the Green's function for the boundary value problem

$$\begin{cases} \left(\Delta^3 u\right)(t-2) + h(t) = 0, & t \in \mathbb{N}_2^{T+2}, \\ u(0) = \left(\Delta u\right)(0) = 0, & u(T+3) = ku(\eta) \end{cases} \tag{2.5}$$

to form the following new result.

Lemma 2.3. *Let $h \in C\left[\mathbb{N}_2^{T+2}, \mathbb{R}\right]$ and assume*

$$(T+2)(T+3) \neq k\eta(\eta-1).$$

The unique solution to the boundary value problem (2.5) *is given by*

$$u(t) = \sum_{s=2}^{T+2} G(t,s)h(s), \quad t \in \mathbb{N}_0^{T+3}, \tag{2.6}$$

where

$$G(t,s) = H(t,s) + \frac{kt(t-1)}{(T+2)(T+3) - k\eta(\eta-1)} H(\eta,s). \tag{2.7}$$

Proof. Assume the solution of the boundary value problem (2.5) can be expressed as

$$u(t) = v(t) + \left[C_0 + C_1 t + C_2 t(t-1)\right] v(\eta), \tag{2.8}$$

where C_0, C_1 and C_2 are constants to be determined and v be the unique solution of the boundary value problem (2.1). When $v(\eta) = 0$, the $u(t)$ defined by (2.8) is the same as $v(t)$ and it is actually the solution of (2.5). In what follows, we assume that $v(\eta) \neq 0$.

It follows from (2.8) that

$$\big(\Delta u\big)(t) = \big(\Delta v\big)(t) + [C_1 + 2C_2 t]\, v(\eta). \tag{2.9}$$

From (2.1), (2.5), (2.8) and (2.9), we have

$$u(0) = 0 \Rightarrow v(0) + C_0 v(\eta) = 0 \Rightarrow C_0 = 0, \tag{2.10}$$

$$\big(\Delta u\big)(0) = 0 \Rightarrow \big(\Delta v\big)(0) + C_1 v(\eta) = 0 \Rightarrow C_1 = 0, \tag{2.11}$$

and

$$\begin{aligned} & u(T+3) = ku(\eta) \\ \Rightarrow\ & v(T+3) + C_2(T+2)(T+3)v(\eta) = k\left[v(\eta) + C_2\eta(\eta-1)v(\eta)\right] \\ \Rightarrow\ & C_2 = \frac{k}{(T+2)(T+3) - k\eta(\eta-1)}. \end{aligned} \tag{2.12}$$

Using (2.3) and (2.10) – (2.12) in (2.8) and rearranging the terms, we obtain (2.6) and (2.7). The proof is complete. □

Let us now establish new bounds on the summation of the Green's functions $H(t,s)$ and $G(t,s)$ via the following result, which is of interest in its own right, for example, the bounds may prove useful in areas beyond the scope of this paper, such as in the application of topological ways of working with fixed point theory. We will draw on it to establish the main existence and uniqueness results of Section 3. The bounds will be formulated in terms of T, k, η. To assist with notation, we define the following constant Λ (that depends on the form of T) that we will use below. For $n \in \mathbb{N}_1$ we define

$$\Lambda = \begin{cases} \dfrac{n(n+2)(2n+1)}{3}, & \text{if } T = 3n, \\[2ex] \dfrac{n(n+1)(2n+1)}{3}, & \text{if } T = 3n-1, \\[2ex] \dfrac{n(n+1)(2n-1)}{3}, & \text{if } T = 3n-2. \end{cases} \tag{2.13}$$

Lemma 2.4. *The Green's function $G(t,s)$ in (2.7) satisfies*

$$\sum_{s=2}^{T+2} |G(t,s)| \leq \Gamma,$$

where Γ depends on T, k, η and is explicitly given by

$$\Gamma = \Lambda + \left|\frac{k}{(T+2)(T+3) - k\eta(\eta-1)}\right| (T+2)(T+3)\left[\frac{\eta(\eta-1)(T+7)}{6} + \eta\right] \tag{2.14}$$

and Λ is defined in (2.13).

Proof. Consider

$$\begin{aligned}
\sum_{s=2}^{T+2} H(t,s) &= \sum_{s=2}^{t-1}\left[\frac{t(t-1)(T+3-s)(T+4-s)}{2(T+3)(T+2)} - \frac{(t-s)(t-s+1)}{2}\right] \\
&\quad + \sum_{s=t}^{T+2}\left[\frac{t(t-1)(T+3-s)(T+4-s)}{2(T+3)(T+2)}\right] \\
&= \sum_{s=2}^{T+2}\left[\frac{t(t-1)(T+3-s)(T+4-s)}{2(T+3)(T+2)}\right] - \sum_{s=2}^{t-1}\left[\frac{(t-s)(t-s+1)}{2}\right] \\
&= \frac{t(t-1)}{2(T+3)(T+2)}\sum_{s=1}^{T+1} s(s+1) - \frac{1}{2}\sum_{s=1}^{t-2} s(s+1) \\
&= \frac{t(t-1)}{2(T+3)(T+2)}\left[\sum_{s=1}^{T+1} s^2 + \sum_{s=1}^{T+1} s\right] - \frac{1}{2}\left[\sum_{s=1}^{t-2} s^2 + \sum_{s=1}^{t-2} s\right] \\
&= \frac{t(t-1)}{2(T+3)(T+2)}\left[\frac{(T+1)(T+2)(2T+3)}{6} + \frac{(T+1)(T+2)}{2}\right] \\
&\quad - \frac{1}{2}\left[\frac{(t-2)(t-1)(2t-3)}{6} + \frac{(t-2)(t-1)}{2}\right] \\
&= \frac{t(t-1)(T+1) - t(t-1)(t-2)}{6} = \frac{t(t-1)(T+3-t)}{6}.
\end{aligned}$$

Clearly,

$$\sum_{s=2}^{T+2} H(0,s) = \sum_{s=2}^{T+2} H(1,s) = 0.$$

Now we wish to maximize $\sum_{s=2}^{T+2} H(t,s)$ for $t \in \mathbb{N}_2^{T+3}$. Denote by

$$g(t) = \frac{t(t-1)(T+3-t)}{6}, \quad t \in \mathbb{N}_2^{T+3}.$$

The first forward difference of g with respect to t is given by

$$(\Delta g)(t) = \frac{t\,(2T-3t+5)}{6}.$$

In this expression, the term $t/6$ is positive for all $t \in \mathbb{N}_2^{T+3}$. The equation $2T-3t+5=0$ has the solution $t = \frac{2T+5}{3}$, so we consider $t = \lfloor \frac{2T+5}{3} \rfloor \in \mathbb{N}_2^{T+3}$. If $t \leq \lfloor \frac{2T+5}{3} \rfloor$, the difference $2T-3t+5$ is positive, and thus g is increasing. If $t > \lfloor \frac{2T+5}{3} \rfloor$, the quantity $2T-3t+5$ is negative, and thus g is decreasing. Hence, the maximum value of g occurs at $t = \lfloor \frac{2T+5}{3} \rfloor$. We observe that, for $n \in \mathbb{N}_1$,

$$\left\lfloor \frac{2T+5}{3} \right\rfloor = \begin{cases} 2n+1, & \text{if } T = 3n, \\ 2n+1, & \text{if } T = 3n-1, \\ 2n, & \text{if } T = 3n-2. \end{cases}$$

Therefore,

$$\max_{t \in \mathbb{N}_0^{T+3}} \sum_{s=2}^{T+2} H(t,s) = \max_{t \in \mathbb{N}_2^{T+3}} g(t) = g\left(\left\lfloor \frac{2T+5}{3} \right\rfloor\right) = \Lambda. \tag{2.15}$$

Consider

$$\begin{aligned}
\sum_{s=2}^{T+2} H(\eta,s) &= \sum_{s=2}^{t-1}\left[\frac{\eta(\eta-1)(T+3-s)(T+4-s)}{2(T+3)(T+2)} - \frac{(\eta-s)(\eta-s+1)}{2}\right] \\
&\quad + \sum_{s=t}^{T+2}\left[\frac{\eta(\eta-1)(T+3-s)(T+4-s)}{2(T+3)(T+2)}\right] \\
&= \sum_{s=2}^{T+2}\left[\frac{\eta(\eta-1)(T+3-s)(T+4-s)}{2(T+3)(T+2)}\right] - \sum_{s=2}^{t-1}\left[\frac{(\eta-s)(\eta-s+1)}{2}\right] \\
&= \frac{\eta(\eta-1)}{2(T+3)(T+2)}\sum_{s=1}^{T+1} s(s+1) - \frac{1}{2}\sum_{s=1}^{t-2}(\eta-s)(\eta-s-1) \\
&= \frac{\eta(\eta-1)}{2(T+3)(T+2)}\left[\sum_{s=1}^{T+1} s^2 + \sum_{s=1}^{T+1} s\right] - \frac{1}{2}\left[\eta(\eta-1)\sum_{s=1}^{t-2} 1 - (2\eta-1)\sum_{s=1}^{t-2} s + \sum_{s=1}^{t-2} s^2\right] \\
&= \frac{\eta(\eta-1)}{2(T+3)(T+2)}\left[\frac{(T+1)(T+2)(2T+3)}{6} + \frac{(T+1)(T+2)}{2}\right] \\
&\quad - \frac{1}{2}\left[\eta(\eta-1)(t-2) - (2\eta-1)\frac{(t-2)(t-1)}{2} + \frac{(t-2)(t-1)(2t-3)}{6}\right] \\
&= \frac{\eta(\eta-1)(T-3t+7) + (3\eta-t)(t-1)(t-2)}{6}.
\end{aligned}$$

Now we wish to maximize $\sum_{s=2}^{T+2} H(\eta,s)$ for $t \in \mathbb{N}_0^{T+3}$. Denote by

$$h(t) = \frac{\eta(\eta-1)(T-3t+7) + (3\eta-t)(t-1)(t-2)}{6}, \quad t \in \mathbb{N}_0^{T+3}.$$

The first forward difference of h with respect to t is given by

$$\big(\Delta h\big)(t) = -\frac{t^2 - (2\eta+1)t + \eta(\eta+1)}{2}.$$

We observe that

$$\big(\Delta h\big)(t) \begin{cases} < 0, & \text{for } t \in \mathbb{N}_0^{\eta-1}, \\ = 0, & \text{for } t = \eta, \\ = 0, & \text{for } t = \eta+1, \\ < 0, & \text{for } t \in \mathbb{N}_{\eta+2}^{T+3}, \end{cases}$$

implying that

$$\max_{t \in \mathbb{N}_0^{T+3}} h(t) = h(0) = \frac{\eta(\eta-1)(T+7)}{6} + \eta.$$

That is,

$$\max_{t \in \mathbb{N}_0^{T+3}} \sum_{s=2}^{T+2} H(\eta,s) = \frac{\eta(\eta-1)(T+7)}{6} + \eta.$$

Now, consider

$$\begin{aligned}\sum_{s=2}^{T+2}|G(t,s)| &= \sum_{s=2}^{T+2}\left|H(t,s)+\frac{kt(t-1)}{(T+2)(T+3)-k\eta(\eta-1)}H(\eta,s)\right| \\ &\le \sum_{s=2}^{T+2}|H(t,s)|+\left|\frac{k}{(T+2)(T+3)-k\eta(\eta-1)}\right|t(t-1)\sum_{s=2}^{T+2}|H(\eta,s)| \\ &= \sum_{s=2}^{T+2}H(t,s)+\left|\frac{k}{(T+2)(T+3)-k\eta(\eta-1)}\right|t(t-1)\sum_{s=2}^{T+2}H(\eta,s) \\ &\le \Lambda+\left|\frac{k}{(T+2)(T+3)-k\eta(\eta-1)}\right|(T+2)(T+3)\left[\frac{\eta(\eta-1)(T+7)}{6}+\eta\right] \\ &= \Gamma.\end{aligned}$$

The proof is complete. □

Remark 2.5. *From the proof of Lemma 2.4 we see that* (2.15) *implies that the bound* Λ *on*

$$\sum_{s=2}^{T+2}H(t,s), \quad t\in N_0^{T+3}$$

is sharp therein. If we compare this sharp bound with the sharp bound for the integral of the corresponding Green's function in the continuous case of $(T+3)^3/81$ *in [4] then we see the bounds between the discrete and continuous cases are different. This is partly due to the differing forms of the Green's function for the discrete and continuous problems. However, it is possible to establish a connection between the two theories by forming a new bound that is common to both problems simply by choosing the larger of the two bounds. The price to pay for this unity in this situation is that one of the bounds will no longer be sharp. Thus we see a trade-off between unification and sharpness in this situation.*

3 Application of Banach's theorem

In this section we establish sufficient conditions on the existence of a unique solution for the boundary value problem (1.1) using Banach's fixed point theorem. "The field of fixed point theory aims to establish conditions under which certain classes of problems will admit one, or more, fixed points [21, 20]" [16, C16]. First let us recall the statement of this theorem.

Theorem 3.1 ([3]). *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a contraction mapping, that is, there is an* α, $0\le\alpha<1$, *such that*

$$d(Tx,Ty)\le \alpha d(x,y),$$

for all x, y *in* X. *Then* T *has a unique fixed point* z *in* X, *that is,* $Tz=z$.

Every solution of the boundary value problem (1.1) can be treated as a $(T+4)$-tuple real vector. Denote the set $X=\mathbb{R}^{T+4}$ and consider the following metrics defined on X:

$$d(u,v)=\max_{t\in\mathbb{N}_0^{T+3}}|u(t)-v(t)|,$$

$$\delta(u,v)=\left(\sum_{t=0}^{T+3}|u(t)-v(t)|^p\right)^{\frac{1}{p}},\quad p>1,$$

for all u, $v\in X$. The pair (X,d) forms a complete metric space, and the pair (X,δ) also forms a complete metric space. Define the operator $T:X\to X$ by

$$\big(Tu\big)(t)=\sum_{s=2}^{T+2}G(t,s)f(s,u(s)),\quad t\in\mathbb{N}_0^{T+3}.$$

Note that u is a solution of the boundary value problem (1.1) if and only if u is a fixed point of T. We apply Theorem 3.1 to show that T has a unique fixed point in X with the ideas manifested in the following two new theorems.

Theorem 3.2. *Let $f\in C\left[\mathbb{N}_0^{T+3}\times\mathbb{R},\mathbb{R}\right]$, let $f(t,0)\neq 0$ for all $t\in\mathbb{N}_0^{T+3}$, let $(T+2)(T+3)\neq k\eta(\eta-1)$ and let Γ be defined in (2.14). If f satisfies a Lipschitz condition with respect to the second variable on $\mathbb{N}_0^{T+3}\times\mathbb{R}$ with Lipschitz constant K, that is, there is a nonnegative constant K, such that*

$$|f(t,x)-f(t,y)|\leq K|x-y|,\quad \text{for all}\quad t\in\mathbb{N}_0^{T+3}\quad \text{and all}\quad x,y\in\mathbb{R}$$

and

$$K\Gamma<1, \tag{3.1}$$

then the boundary value problem (1.1) has a unique nontrivial solution.

Proof. For u, $v\in X$ and $t\in\mathbb{N}_0^{T+3}$, consider

$$\begin{aligned}\big|\big(Tu\big)(t)-\big(Tv\big)(t)\big|&=\left|\sum_{s=2}^{T+2}G(t,s)f(s,u(s))-\sum_{s=2}^{T+2}G(t,s)f(s,v(s))\right|\\&\leq\sum_{s=2}^{T+2}|G(t,s)|\,|f(s,u(s))-f(s,v(s))|\\&\leq K\sum_{s=2}^{T+2}|G(t,s)|\,|u(s)-v(s)|\\&\leq Kd(u,v)\sum_{s=2}^{T+2}|G(t,s)|\\&\leq K\Gamma d(u,v),\end{aligned}$$

implying that

$$d(Tu,Tv)\leq\alpha d(u,v),$$

where $\alpha = K\Gamma < 1$. Thus, T is a contraction mapping on X. Hence, by Theorem 3.1, our T has a unique fixed point in X. This is equivalent to the boundary value problem (1.1) admitting a unique nontrivial solution. The proof is complete. □

The following result sharpens the inequality (3.1) in Theorem 3.2 through the strategic use of a different metric.

Theorem 3.3. *Let the conditions of Theorem 3.2 hold, with the assumption* (3.1) *removed. If there are constants* $p > 1$ *and* $q > 1$ *such that* $1/p + 1/q = 1$ *and*

$$K\left(\sum_{t=0}^{T+3}\left(\sum_{s=2}^{T+2}|G(t,s)|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} < 1, \tag{3.2}$$

then the boundary value problem (1.1) *has a unique nontrivial solution.*

Proof. We apply Theorem 3.1 to show that T has a unique fixed point in X where X is defined in the proof of Theorem 3.2 but is now coupled with the metric

$$\delta(u,v) := \left(\sum_{t=0}^{T+3}|u(t)-v(t)|^p\right)^{\frac{1}{p}}.$$

Consider

$$\begin{aligned}\big|(Tu)(t)-(Tv)(t)\big| &= \left|\sum_{s=2}^{T+2}G(t,s)f(s,u(s)) - \sum_{s=2}^{T+2}G(t,s)f(s,v(s))\right| \\ &\le \sum_{s=2}^{T+2}|G(t,s)|\,|f(s,u(s))-f(s,v(s))| \\ &\le K\sum_{s=2}^{T+2}|G(t,s)|\,|u(s)-v(s)|. \end{aligned} \tag{3.3}$$

By Holder's inequality, we have

$$\sum_{s=2}^{T+2}|G(t,s)|\,|u(s)-v(s)| \le \left(\sum_{s=2}^{T+2}|u(s)-v(s)|^p\right)^{\frac{1}{p}}\left(\sum_{s=2}^{T+2}|G(t,s)|^q\right)^{\frac{1}{q}}. \tag{3.4}$$

Thus,

$$\begin{aligned}\big|(Tu)(t)-(Tv)(t)\big| &\le K\left(\sum_{s=2}^{T+2}|u(s)-v(s)|^p\right)^{\frac{1}{p}}\left(\sum_{s=2}^{T+2}|G(t,s)|^q\right)^{\frac{1}{q}} \\ &\le K\left(\sum_{s=2}^{T+2}|G(t,s)|^q\right)^{\frac{1}{q}}\delta(u,v)\end{aligned} \tag{3.5}$$

and so, we have

$$\left(\sum_{t=0}^{T+3}\big|(Tu)(t)-(Tv)(t)\big|^p\right)^{\frac{1}{p}} \le K\left(\sum_{t=0}^{T+3}\left(\sum_{s=2}^{T+2}|G(t,s)|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}\delta(u,v),$$

implying that

$$\delta(Tu, Tv) \leq \gamma\delta(u, v),$$

where

$$\gamma = K\left(\sum_{t=0}^{T+3}\left(\sum_{s=2}^{T+2}|G(t,s)|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} < 1.$$

Thus, the conditions of Theorem 3.1 hold. Hence, by Theorem 3.1, our T has a unique fixed point in X. This is equivalent to the boundary value problem (1.1) furnishing a unique nontrivial solution. The proof is complete. □

For the choices $p = q = 2$, Theorem 3.3 takes the following form:

Theorem 3.4. *Let the conditions of Theorem 3.2 hold, with the assumption* (3.1) *removed. If*

$$K\left(\sum_{t=0}^{T+3}\left(\sum_{s=2}^{T+2}|G(t,s)|^2\right)\right)^{\frac{1}{2}} < 1, \tag{3.6}$$

then the boundary value problem (1.1) *has a unique nontrivial solution.*

4 Discussion of examples

Let us discuss two examples to illustrate the nature of our new theorems and the relationship between them.

Example 4.1. *Consider the following discrete boundary value problem*

$$\begin{cases} \big(\Delta^3 u\big)(t-2) + \frac{1}{150}\cos(u(t)) = 0, & t \in \mathbb{N}_2^{11}, \\ u(0) = \big(\Delta u\big)(0) = 0, \quad u(12) = u(6). \end{cases} \tag{4.1}$$

We claim that this problem admits a unique solution.

Proof. Observe that (4.1) is a special case of (1.1) with $T = 9$, $k = 1$, $\eta = 6$ and $f(t, u) = f(u) = (\cos(u))/150$.

We show that the conditions of Theorem 3.2 are satisfied.

Since T is a multiple of 3 we have $n = 3$ and so $\Lambda = 35$. Furthermore, appropriate calculations reveal $\Gamma \approx 146.294 < 147$.

Our f satisfies a Lipschitz condition due to the property that its derivative with respect to u is uniformly bounded by $1/150$ and we may choose this bound to be the Lipschitz constant, that is, on $\mathbb{R}$ we have

$$|\partial f/\partial u| = |-\sin(u)|/150 \leq 1/150 = K.$$

Finally, we see that $K\Gamma < 147/150 < 1$. Thus, all of the conditions of Theorem 3.2 hold and we conclude that the discrete boundary value problem (4.1) admits a unique solution. □

Let us now discuss an example that illustrates Theorem 3.3 and its distinction from Theorem 3.2.

Example 4.2. *Consider the following discrete boundary value problem*

$$\begin{cases} (\Delta^3 u)(t-2) + \frac{1}{54}\tan^{-1}(u(t)) + t^2 + 1 = 0, & t \in \mathbb{N}_2^{11}, \\ u(0) = (\Delta u)(0) = 0, & u(12) = u(6). \end{cases} \tag{4.2}$$

We claim that this problem admits a unique solution.

Proof. Observe that (4.2) is a special case of (1.1) with $T = 9$, $k = 1$, $\eta = 6$ and $f(t,u) = (\tan^{-1}(u))/54 + t^2 + 1$.

We show that the conditions of Theorem 3.3 are satisfied with $p = 2 = q$, that is, Theorem 3.4 will hold.

Appropriate calculations using Maple reveal

$$\left(\sum_{t=0}^{T+3}\left(\sum_{s=2}^{T+2}|G(t,s)|^2\right)\right)^{\frac{1}{2}} \approx 52.3839 < 53.$$

Our f satisfies a Lipschitz condition due to the property that its derivative with respect to u is uniformly bounded by $1/54$ and we may choose this bound to be the Lipschitz constant, that is, on $\mathbb{R}$ we have

$$|\partial f/\partial u| = |1/(54(u^2+1))| \leq 1/54 = K.$$

Finally, we see that (3.6) holds since

$$K\left(\sum_{t=0}^{T+3}\left(\sum_{s=2}^{T+2}|G(t,s)|^2\right)\right)^{\frac{1}{2}} < 53/54 < 1.$$

Thus, all of the conditions of Theorem 3.3 hold with $p = 2 = q$ (that is, Theorem 3.4 holds) and we conclude that the discrete boundary value problem (4.2) admits a unique solution. □

Remark 4.3. *We note that Theorem 3.2 cannot be directly applied to the boundary value problem* (4.2) *in Example 4.2. The reason is because the condition* $K\Gamma < 1$ *is not satisfied in this situation. Thus, we observe that Theorem 3.4 is more general than Theorem 3.2.*

5 Concluding remarks and further work

This paper deepened our understanding of the existence and uniqueness of solutions to discrete boundary value problems. We showed that a larger class of problems admitted a unique solution and achieved this by drawing on fixed-point theory and the use of new bounds. Our results add to the recent literature on discrete boundary value problems and difference equations [17, 18] and move us closer to a more complete understanding of the underlying theory and application.

Although our bound on the summation of $H(t, s)$ herein is sharp, the corresponding bound involving $G(t, s)$ remains rough and a natural question for further work is: can this bound be sharpened?

One of the main limitations with many fixed point theorems is the very nature of their assumptions. Because sufficient conditions are involved, it may be the case that the conditions of these theorems do not hold, yet the problem under consideration does actually admit a unique solution (or solutions). Thus it is important to also look beyond these types of sufficient assumptions and the development of new methods and altenative perspectives in mathematics are needed [21, 22] to advance the associated existence and uniqueness theory.

References

[1] R. P. Agarwal, *Difference equations and inequalities. Theory, methods, and applications*, Second edition, Monographs and Textbooks in Pure and Applied Mathematics, vol. 228. New York: Marcel Dekker, 2000.

[2] R. P. Agarwal and J. Henderson, "Positive solutions and nonlinear eigenvalue problems for third-order difference equations", Comput. Math. Appl., vol. 36, nos. 10–12, pp. 347–355, 1998.

[3] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed point theory and applications*, Cambridge Tracts in Mathematics, vol. 141, Cambridge: Cambridge University Press, 2001.

[4] S. S. Almuthaybiri and C. C. Tisdell, "Sharper existence and uniqueness results for solutions to third-order boundary value problems", Math. Model. Anal., vol. 25, no. 3, pp. 409–420, 2020.

[5] D. R. Anderson, "Discrete third-order three-point right-focal boundary value problems", Comput. Math. Appl., vol. 45, nos. 6–9, pp. 861–871, 2003.

[6] D. R. Anderson and R. I. Avery, "Multiple positive solutions to a third-order discrete focal boundary value problem", Comput. Math. Appl., vol. 42, nos. 3–5, pp. 333–340, 2001.

[7] D. R. Anderson and C. C. Tisdell, "Discrete approaches to continuous boundary value problems: existence and convergence of solutions", Abstr. Appl. Anal., vol. 2016, Article ID 3910972, 6 pages, 2016.

[8] M. Bohner, A. Peterson, *Dynamic equations on time scales. An introduction with applications*, Boston: Birkhäuser Boston-Springer, 2001.

[9] S. Elaydi, *An introduction to difference equations*, Third edition, Undergraduate Texts in Mathematics, New York: Springer, 2005.

[10] C. Goodrich and A. C. Peterson, *Discrete fractional calculus*, Cham: Springer, 2015.

[11] J. Ji and B. Yang, "Positive solutions of discrete third-order three-point right focal boundary value problems", J. Difference Equ. Appl., vol. 15, no. 2, pp. 185–195, 2009.

[12] J. Ji and B. Yang, "Computing the positive solutions of the discrete third-order three-point right focal boundary-value problems", Int. J. Comput. Math., vol. 91, no. 5, pp. 996–1004, 2014.

[13] I. Y. Karaca, "Discrete third-order three-point boundary value problem", J. Comput. Appl. Math., vol. 205, no. 1, pp. 458–468, 2007.

[14] W. G. Kelley and A. C. Peterson, *Difference equations. An introduction with applications*, Second edition, San Diego-CA: Harcourt/Academic Press, 2001.

[15] S. Smirnov, "Green's function and existence of a unique solution for a third-order three-point boundary value problem", Math. Model. Anal., vol. 24, no. 2, pp. 171–178, 2019.

[16] C. P. Stinson, S. S. Almuthaybiri and C. C. Tisdell, "A note regarding extensions of fixed point theorems involving two metrics via an analysis of iterated functions", ANZIAM J. (EMAC 2019), vol. 61 (2019), pp. C15–C30, 2020.

[17] C. C. Tisdell, "On first-order discrete boundary value problems", J. Difference Equ. Appl., vol. 12, no. 12, pp. 1213–1223, 2006.

[18] C. C. Tisdell, "A note on improved contraction methods for discrete boundary value problems", J. Difference Equ. Appl., vol. 18, no. 10, pp. 1173–1777, 2012.

[19] C. C. Tisdell, "Rethinking pedagogy for second-order differential equations: a simplified approach to understanding well-posed problems", Internat. J. Math. Ed. Sci. Tech., vol. 48, no. 5, pp. 794–801, 2017.

[20] C. C. Tisdell, "Improved pedagogy for linear differential equations by reconsidering how we measure the size of solutions", Internat. J. Math. Ed. Sci. Tech., vol. 48, no. 7, pp. 1087–1095, 2017.

[21] C. C. Tisdell, "Critical perspectives of pedagogical approaches to reversing the order of integration in double integrals", Internat. J. Math. Ed. Sci. Tech., vol. 48, no. 8, pp. 1285–1292, 2017.

[22] C. C. Tisdell, "On Picard's iteration method to solve differential equations and a pedagogical space for otherness", Internat. J. Math. Ed. Sci. Tech., vol. 50, no. 5, pp. 788–799, 2019.

[23] J. Wang and Ch. Gao, "Positive solutions of discrete third-order boundary value problems with sign-changing Green's function", Adv. Difference Equ., vol. 2015, 10 pages, 2015.

[24] Y. Xu, W. Tian and Ch. Gao, "Existence of positive solutions of discrete third-order three-point BVP with sign-changing Green's function", Adv. Difference Equ., vol. 2019, no. 206, 19 pages, 2019.

[25] Ch. Yang and P. Weng, "Green functions and positive solutions for boundary value problems of third-order difference equations", Comput. Math. Appl., vol. 54, no. 4, pp. 567–578, 2007.

Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 03, pp. 457–468, December 2021
DOI: 10.4067/S0719-06462021000300457

Some integral inequalities related to Wirtinger's result for p-norms

S. S. Dragomir[1,2]

[1] *Mathematics, College of Engineering & Science, Victoria University, PO Box 14428 Melbourne City, MC 8001, Australia.*
sever.dragomir@vu.edu.au

[2] *DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.*

ABSTRACT

In this paper we establish several natural consequences of some Wirtinger type integral inequalities for p-norms. Applications related to the trapezoid unweighted inequalities, of Grüss' type inequalities and reverses of Jensen's inequality are also provided.

RESUMEN

En este artículo establecemos varias consecuencias naturales de algunas desigualdades integrales de tipo Wirtinger para p-normas. También se entregan aplicaciones relacionadas a desigualdades trapezoidales sin peso, desigualdades de tipo Grüss y reversos de la desigualdad de Jensen.

Keywords and Phrases: Wirtinger's inequality, trapezoid inequality, Grüss' inequality, Jensen's inequality.

2020 AMS Mathematics Subject Classification: 26D15; 26D10.

Accepted: 05 October, 2021
Received: 02 May, 2021

1 Introduction

The following Wirtinger type inequalities are well known

$$\int_a^b u^2(t)dt \leq \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

provided $u \in C^1([a,b],\mathbb{R})$ and $u(a) = u(b) = 0$ with equality holding if and only if $u(t) = K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant K, and, similarly, if $u \in C^1([a,b],\mathbb{R})$ satisfies $u(a) = 0$, then

$$\int_a^b u^2(t)dt \leq \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt. \tag{1.2}$$

The equality holds in (1.2) if and only if $u(t) = K\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant K.

For $p > 1$ put $\pi_p := \frac{2\pi}{p}\sin\left(\frac{\pi}{p}\right)$. In [11], J. Jaroš obtained, as a particular case of a more general inequality, the following generalization of (1.1)

$$\int_a^b |u(t)|^p\,dt \leq \frac{(b-a)^p}{(p-1)\pi_p^p}\int_a^b |u'(t)|^p\,dt \tag{1.3}$$

provided $u \in C^1([a,b],\mathbb{R})$ and $u(a) = u(b) = 0$, with equality if and only if $u(t) = K\sin_p\left[\frac{\pi_p(t-a)}{b-a}\right]$ for some $K \in \mathbb{R}$, where $\sin_p$ is the $2\pi_p$-periodic generalized sine function, see [18] or [5].

If $u(a) = 0$ and $u \in C^1([a,b],\mathbb{R})$, then

$$\int_a^b |u(t)|^p\,dt \leq \frac{[2(b-a)]^p}{(p-1)\pi_p^p}\int_a^b |u'(t)|^p\,dt \tag{1.4}$$

with equality iff $u(t) = K\sin_p\left[\frac{\pi_p(t-a)}{2(b-a)}\right]$ for some $K \in \mathbb{R}$.

The inequalities (1.3) and (1.4) were obtained for $a = 0$, $b = 1$ and $q = p > 1$ in [17] by the use of an elementary proof.

For some related Wirtinger type integral inequalities see [1, 2, 4, 8, 9, 11, 12] and [15]-[17].

These inequalities are used in various fields of Mathematical Analysis, Approximation Theory, Integral Operator Theory and Analytic Inequalities Theory since they provide connections between the Lebesgue norms of a function and the corresponding Lebesgue norms of the derivative under some natural assumptions at the endpoints.

Motivated by the above results, in this paper we establish some natural consequences of the Wirtinger type integral inequalities for p-norms (1.3) and (1.4). Applications related to the trapezoid unweighted inequalities, of Grüss' type inequalities and reverses of Jensen's inequality are also provided.

2 Some applications for trapezoid inequality

We have:

Proposition 2.1. *Let $g \in C^1([a,b],\mathbb{R})$. Then for $p>1$ we have the trapezoid inequality*

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right| \le \frac{b-a}{2(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b |g'(t)-g'(a+b-t)|^p\,dt\right)^{1/p}. \quad (2.1)$$

In particular, for $p=2$, we have [7]

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right| \le \frac{b-a}{2\pi}\left(\frac{1}{b-a}\int_a^b |g'(t)-g'(a+b-t)|^2\,dt\right)^{1/2}. \quad (2.2)$$

Proof. If $g \in C^1([a,b],\mathbb{R})$, then by taking

$$u(t):=\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2},\ t\in[a,b]$$

we have $u(a)=u(b)=0$ and by (1.3) we have

$$\int_a^b \left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p dt \le \frac{(b-a)^p}{(p-1)2^p\pi_p^p}\int_a^b |g'(t)-g'(a+b-t)|^p\,dt, \quad (2.3)$$

namely

$$\left(\int_a^b \left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p dt\right)^{1/p} \le \frac{(b-a)}{2(p-1)^{1/p}\pi_p}\left(\int_a^b |g'(t)-g'(a+b-t)|^p\,dt\right)^{1/p}. \quad (2.4)$$

By Hölder's integral inequality we have for $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$

$$\begin{aligned}\int_a^b &\left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right| dt \\ &\le \left(\int_a^b dt\right)^{1/q}\left(\int_a^b \left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p dt\right)^{1/p} \\ &= (b-a)^{1/q}\left(\int_a^b \left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p dt\right)^{1/p} \\ &= (b-a)^{1-1/p}\left(\int_a^b \left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p dt\right)^{1/p}. \quad (2.5)\end{aligned}$$

By making use of the properties of modulus and integral, we also have

$$\int_a^b \left| \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right| dt \geq \left| \int_a^b \left[\frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right] dt \right|$$
$$= \left| \int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2}(b-a) \right|. \quad (2.6)$$

By making use of (2.4)-(2.6) we get the desired result (2.1). □

Further, we have:

Proposition 2.2. *Let $g \in C^1([a,b],\mathbb{R})$. Then for $p>1$ we have the trapezoid inequality*

$$\left| \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt \right|$$
$$\leq \frac{b-a}{(p-1)^{1/p}\pi_p} \left(\frac{1}{b-a}\int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^p dt \right)^{1/p}. \quad (2.7)$$

In particular, for $p=2$, we have [7]

$$\left| \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt \right| \leq \frac{b-a}{\pi} \left(\frac{1}{b-a}\int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^2 dt \right)^{1/2}. \quad (2.8)$$

Proof. If $g \in C^1([a,b],\mathbb{R})$, then by taking

$$u(t) := g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a}, \ t \in [a,b]$$

we have $u(a) = u(b) = 0$ and by (1.3) we have

$$\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^p dt \leq \frac{(b-a)^p}{(p-1)\pi_p^p} \int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^p dt, \quad (2.9)$$

which gives that

$$\left(\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^p dt \right)^{1/p}$$
$$\leq \frac{b-a}{(p-1)^{1/p}\pi_p} \left(\int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^p dt \right)^{1/p}. \quad (2.10)$$

By Hölder's integral inequality we have for p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ that

$$\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right| dt$$
$$\leq \left(\int_a^b dt \right)^{1/q} \left(\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^p dt \right)^{1/p}$$
$$= (b-a)^{1/q} \left(\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^p dt \right)^{1/p}. \quad (2.11)$$

By making use of the properties of modulus and integral, we also have

$$\int_a^b \left| g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right| dt \geq \left| \int_a^b \left[g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right] dt \right|$$
$$= \left| \int_a^b g(t)\, dt - \frac{g(a) + g(b)}{2}(b-a) \right|. \tag{2.12}$$

By making use of (2.10)-(2.12) we get the desired result (2.7). □

We also have:

Proposition 2.3. *Let $g \in C([a,b],\mathbb{R})$. Then for $p > 1$ we have the inequality*

$$\left| \frac{b+a}{2} \int_a^b g(s)\, ds - \int_a^b t g(t)\, dt \right|$$
$$\leq \frac{(b-a)^2}{(p-1)^{1/p} \pi_p} \left(\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right|^p dt \right)^{1/p}. \tag{2.13}$$

In particular, for $p = 2$, we have [7]

$$\left| \frac{b+a}{2} \int_a^b g(s)\, ds - \int_a^b t g(t)\, dt \right|$$
$$\leq \frac{(b-a)^2}{\pi} \left[\frac{1}{b-a} \int_a^b g^2(t)\, dt - \left(\frac{1}{b-a} \int_a^b g(s)\, ds \right)^2 \right]^{1/2}. \tag{2.14}$$

Proof. Assume that $g : [a,b] \to \mathbb{C}$ is continuous, then by taking

$$u(t) := \int_a^t g(s)\, ds - \frac{t-a}{b-a} \int_a^b g(s)\, ds, \ t \in [a,b]$$

we have $u(a) = u(b) = 0$, $u \in C^1([a,b],\mathbb{C})$ and by (1.3) we get

$$\int_a^b \left| \int_a^t g(s)\, ds - \frac{t-a}{b-a} \int_a^b g(s)\, ds \right|^p dt \leq \frac{(b-a)^p}{(p-1)\pi_p^p} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right|^p dt.$$

This is equivalent to

$$\left(\int_a^b \left| \int_a^t g(s)\, ds - \frac{t-a}{b-a} \int_a^b g(s)\, ds \right|^p dt \right)^{1/p}$$
$$\leq \frac{b-a}{(p-1)^{1/p} \pi_p} \left(\int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right|^p dt \right)^{1/p}. \tag{2.15}$$

By Hölder's integral inequality we also have for $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$ that

$$(b-a)^{1/q}\left(\int_a^b\left|\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^p dt\right)^{1/p}$$
$$\geq\int_a^b\left|\left(\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right)\right|dt$$
$$\geq\left|\int_a^b\left(\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right)dt\right|. \quad (2.16)$$

Observe that, integrating by parts, we have

$$\int_a^b\left(\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right)dt=\int_a^b\left(\int_a^t g(s)\,ds\right)dt-\frac{b-a}{2}\int_a^b g(s)\,ds$$
$$=b\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt-\frac{b-a}{2}\int_a^b g(s)\,ds$$
$$=\frac{b+a}{2}\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt. \quad (2.17)$$

By making use of (2.15)-(2.17) we get the desired result (2.13). □

3 Inequalities for the Čebyšev functional

For two *Lebesgue integrable* functions $f,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \quad (3.1)$$

In 1935, Grüss [10] showed that

$$|C(f,g)|\leq\frac{1}{4}(M-m)(N-n), \quad (3.2)$$

provided that there exist real numbers $m,\ M,\ n,\ N$ such that

$$m\leq f(t)\leq M \quad\text{and}\quad n\leq g(t)\leq N \quad\text{for a. e. } t\in[a,b]. \quad (3.3)$$

The constant $\frac{1}{4}$ is the best possible in (3.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [3], states that

$$|C(f,g)|\leq\frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty, \quad (3.4)$$

provided that $f',\ g'$ exist and are continuous on $[a,b]$ and $\|f'\|_\infty=\sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (3.4) also holds if $f,\, g : [a, b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and $f',\, g' \in L_\infty [a, b]$ while $\|f'\|_\infty = ess\ sup_{t\in[a,b]} |f'(t)|$.

A mixture between Grüss' result (3.2) and Čebyšev's one (3.4) is the following inequality obtained by Ostrowski in 1970, [14]:

$$|C(f, g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{3.5}$$

provided that f is *Lebesgue integrable* and satisfies (3.3) while g is absolutely continuous and $g' \in L_\infty [a, b]$. The constant $\frac{1}{8}$ is the best possible in (3.5).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [13] in which he proved that

$$|C(f, g)| \le \frac{1}{\pi^2}(b-a)\|f'\|_2 \|g'\|_2, \tag{3.6}$$

provided that $f,\, g$ are absolutely continuous and $f',\, g' \in L_2 [a, b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

We have:

Theorem 3.1. *If $f : [a, b] \to \mathbb{R}$ is continuous, $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $g : [a, b] \to \mathbb{C}$ is absolutely continuous with $g' \in L_q [a, b]$, then*

$$|C(f, g)| \le \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p} \times \left(\int_a^b |g\prime(t)|^q\,dt\right)^{1/q}. \tag{3.7}$$

In particular, for $p = q = 2$, we get

$$|C(f, g)| \le \frac{(b-a)^{1/2}}{\pi}\left(\frac{1}{b-a}\int_a^b f^2(t)\,dt - \left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)^2\right)^{1/2} \times \left(\int_a^b |g\prime(t)|^2\,dt\right)^{1/2}. \tag{3.8}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \left(\int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(s)\,ds\right) g\prime(x)\,dx \\
&= \frac{1}{b-a}\left[\left(\int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(s)\,ds\right) g(x)\Big|_a^b - \int_a^b g(x)\left(f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds\right)dx\right] \\
&= -\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{b-a}\int_a^b f(s)\,ds \frac{1}{b-a}\int_a^b g(x)\,dx,
\end{aligned}$$

which gives that

$$C(f,g)=\frac{1}{b-a}\int_a^b\left(\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right)g\prime(x)\,dx. \tag{3.9}$$

Using Hölder's integral inequality for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\begin{aligned}
|C(f,g)| &= \left|\frac{1}{b-a}\int_a^b\left(\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right)g\prime(x)\,dx\right| \\
&\le \frac{1}{b-a}\int_a^b\left|\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right||g\prime(x)|\,dx \\
&\le \frac{1}{b-a}\left(\int_a^b\left|\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right|^p dx\right)^{1/p}\left(\int_a^b|g\prime(x)|^q\,dx\right)^{1/q} =: I
\end{aligned} \tag{3.10}$$

Using (2.15) we have

$$\begin{aligned}
I &\le \frac{1}{b-a}\left(\int_a^b\left|\int_a^t f(s)\,ds-\frac{t-a}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^b|g\prime(x)|^q\,dx\right)^{1/q} \\
&\le \frac{1}{b-a}\frac{b-a}{(p-1)^{1/p}\pi_p}\left(\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^b|g\prime(x)|^q\,dx\right)^{1/q} \\
&= \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^b|g\prime(x)|^q\,dx\right)^{1/q}
\end{aligned} \tag{3.11}$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, which proves (3.7).

Now, if we put $p=q=2$ and take into account that

$$\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2 dt=\frac{1}{b-a}\int_a^b f^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)^2,$$

then by (3.7) we derive (3.8). □

This results can be used to obtain various inequalities by taking particular examples of functions f and g as follows.

We have the following trapezoid type inequality:

Proposition 3.2. *Assume that $g:[a,b]\to\mathbb{C}$ has an absolutely continuous derivative with $g''\in L_q[a,b]$, where $p,\ q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right|\le\frac{(b-a)^{1+1/p}}{2(p-1)^{1/p}(p+1)^{1/p}\pi_p}\left(\int_a^b|g''(t)|^q\,dt\right)^{1/q}. \tag{3.12}$$

Proof. We use the following identity that can be proved integrating by parts

$$\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)g'(t)\,dt=C\left(\ell-\frac{a+b}{2},g'\right),$$

where $\ell(t) = t,\ t \in [a,b]$.

Using (3.7) we have

$$\begin{aligned}
&\left| C\left(\ell - \frac{a+b}{2}, g'\right)\right| \\
&\le \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b \left|t - \frac{a+b}{2} - \frac{1}{b-a}\int_a^b\left(s - \frac{a+b}{2}\right)ds\right|^p dt\right)^{1/p}\left(\int_a^b |g''(x)|^q\,dx\right)^{1/q} \\
&= \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b \left|t - \frac{a+b}{2}\right|^p dt\right)^{1/p}\left(\int_a^b |g''(x)|^q\,dx\right)^{1/q} \\
&= \frac{(b-a)^{1+1/p}}{2(p-1)^{1/p}(p+1)^{1/p}\pi_p}\left(\int_a^b |g''(x)|^q\,dx\right)^{1/q},
\end{aligned}$$

which proves the desired inequality (3.12). □

Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m,M) and $f : [a,b] \to [m,M]$ be absolutely continuous so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L[a,b]$. If $f' \in L_\infty[a,b]$, then we have the Jensen's reverse inequality [6]

$$\begin{aligned}
0 &\le \frac{1}{b-a}\int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\
&\le \frac{1}{b-a}\int_a^b (\Phi' \circ f)(t) f(t)\,dt - \frac{1}{b-a}\int_a^b \Phi' \circ f(t)\,dt \frac{1}{b-a}\int_a^b f(t)\,dt = C(\Phi' \circ f, f). \quad (3.13)
\end{aligned}$$

We have the following reverse of Jensen's inequality:

Proposition 3.3. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m,M) and $f : [a,b] \to [m,M]$ be absolutely continuous so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L[a,b]$.*

(i) If $f' \in L_q[a,b]$, $\Phi' \circ f \in L_p[a,b]$ with $p,\ q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned}
0 &\le \frac{1}{b-a}\int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\
&\le \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b \left|(\Phi' \circ f)(t) - \frac{1}{b-a}\int_a^b (\Phi' \circ f)(s)\,ds\right|^p dt\right)^{1/p} \\
&\qquad \times \left(\int_a^b |f'(t)|^q\,dt\right)^{1/q}. \quad (3.14)
\end{aligned}$$

(ii) If Φ *is twice differentiable and* $(\Phi'' \circ f) f' \in L_q [a, b]$ *with* $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$0 \le \frac{1}{b-a} \int_a^b (\Phi \circ f)(t)\, dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\, dt\right) \le \frac{(b-a)^{1/p}}{(p-1)^{1/p} \pi_p} \left(\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\, ds\right|^p dt\right)^{1/p} \times \left(\int_a^b |(\Phi'' \circ f)(t) f'(t)|^q dt\right)^{1/q}. \quad (3.15)$$

The proof follows by Theorem 3.1 for $C(\Phi' \circ f, f)$ and the inequality (3.13).

We have the following mid-point type inequalities:

Corollary 3.4. *Let* $\Phi : [a, b] \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable convex function on* (a, b).

(i) If $\Phi' \in L_p [a, b]$ *with* $p > 1$, *then*

$$0 \le \frac{1}{b-a} \int_a^b \Phi(t)\, dt - \Phi\left(\frac{a+b}{2}\right) \le \frac{b-a}{(p-1)^{1/p} \pi_p} \left(\frac{1}{b-a}\int_a^b \left|\Phi'(t) - \frac{\Phi(b) - \Phi(a)}{b-a}\right|^p dt\right)^{1/p}. \quad (3.16)$$

(ii) If Φ *is twice differentiable and* $\Phi'' \in L_q [a, b]$ *with* $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$0 \le \frac{1}{b-a} \int_a^b \Phi(t)\, dt - \Phi\left(\frac{a+b}{2}\right) \le \frac{(b-a)^{1+1/p}}{2(p-1)^{1/p}(p+1)^{1/p} \pi_p} \left(\int_a^b |\Phi''(t)|^q dt\right)^{1/q}. \quad (3.17)$$

Acknowledgement

The author would like to thank the anonymous referees for valuable suggestions that have been implemented in the final version of the manuscript.

References

[1] M. W. Alomari, "On Beesack–Wirtinger Inequality", Results Math., vol. 73, no. 2, pp. 1213–1225, 2017.

[2] P. R. Beesack, "Extensions of Wirtinger's inequality", Trans. R. Soc. Can., vol. 53, pp. 21–30, 1959.

[3] P. L. Chebyshev, "Sur les expressions approximatives des intégrales dêfinis par les autres prises entre les même limites", Proc. Math. Soc. Charkov, vol. 2, pp. 93–98, 1882.

[4] J. B. Diaz and F. T. Metcalf, "Variations on Wirtinger's inequality", in *Inequalities*, New York: Academic Press, 1967, pp. 79–103.

[5] P. Drábek and R. Manásevich, "On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian", Differential Integral Equations, vol. 12, no. 6, pp. 773–788, 1999.

[6] S. S. Dragomir, "A Grüss type inequality for isotonic linear functionals and applications", Demonstratio Math., vol. 36, no. 3, pp. 551–562, 2003.

[7] S. S. Dragomir, "Some integral inequalities related to Wirtinger's result", Preprint, RGMIA Res. Rep. Coll., vol. 21, art. 60, 2018.

[8] R. Giova, "An estimate for the best constant in the L^p-Wirtinger inequality with weights", J. Func. Spaces Appl., vol. 6, no. 1, pp. 1–16, 2008.

[9] R. Giova and T. Ricciardi, "A sharp weighted Wirtinger inequality and some related functional spaces", Bull. Belg. Math. Soc. Simon Stevin, vol. 17, no. 2, pp. 209–218, 2010.

[10] G. Grüss, "Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int\limits_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int\limits_a^b f(x)dx\int\limits_a^b g(x)dx$", Math. Z., vol. 39, no. 1, pp. 215–226, 1935.

[11] J. Jaroš, "On an integral inequality of the Wirtinger type", Appl. Math. Letters, vol. 24, no. 8, pp. 1389–1392, 2011.

[12] C. F. Lee, C. C. Yeh, C. H. Hong and R. P. Agarwal, "Lyapunov and Wirtinger inequalities", Appl. Math. Lett., vol. 17, no. 7, pp. 847–853, 2004.

[13] A. Lupaş, "The best constant in an integral inequality", Mathematica (Cluj), vol. 15, no. 38, pp. 219–222, 1973.

[14] A. M. Ostrowski, "On an integral inequality", Aequationes Math., vol. 4, pp. 358–373, 1970.

[15] T. Ricciardi, "A sharp weighted Wirtinger inequality", Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8), vol. 8, no. 1, pp. 259–267, 2005.

[16] C. A. Swanson, “Wirtinger’s inequality”, SIAM J. Math. Anal., vol. 9, no. 3, pp. 484–491, 1978.

[17] S.-E. Takahasi, T. Miura and T. Hayata, “On Wirtinger’s inequality and its elementary proof”, Math. Inequal. Appl., vol. 10, no. 2, pp. 311–319, 2007.

[18] S. Takeuchi, “Generalized elliptic functions and their application to a nonlinear eigenvalue problem with p-Laplacian”, J. Math. Anal. Appl., vol. 385 , no. 1, pp. 24–35, 2012.

Cubo A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 03, pp. 469–487, December 2021
DOI: 10.4067/S0719-06462021000300469

On the periodic solutions for some retarded partial differential equations by the use of semi-Fredholm operators

Abdelhai Elazzouzi[1]
Khalil Ezzinbi[1,2]
Mohammed Kriche[1]

[1] *Département de Mathématiques, Laboratoire des Sciences de l'Ingénieur (LSI), Faculté Polydisciplinaire de Taza, Université Sidi Mohamed Ben Abdellah (USMBA) - Fes, BP. 1223, Taza, Morocco.*
abdelhai.elazzouzi@usmba.ac.ma
mohammed.kriche@usmba.ac.ma

[2] *Département de Mathématiques, Faculté des Sciences Semlalia, Université Cadi Ayyad, B.P. 2390, Marrakesh, Morocco.*
ezzinbi@uca.ac.ma

ABSTRACT

The main goal of this work is to examine the periodic dynamic behavior of some retarded periodic partial differential equations (PDE). Taking into consideration that the linear part realizes the Hille-Yosida condition, we discuss the Massera's problem to this class of equations. Especially, we use the perturbation theory of semi-Fredholm operators and the Chow and Hale's fixed point theorem to study the relation between the boundedness and the periodicity of solutions for some inhomogeneous linear retarded PDE. An example is also given at the end of this work to show the applicability of our theoretical results.

RESUMEN

El principal objetivo de este trabajo es examinar el comportamiento dinámico periódico de algunas ecuaciones diferenciales parciales (EDP) periódicas con retardo. Tomando en consideración que la parte lineal cumple la condición de Hille-Yosida, discutimos el problema de Massera para esta clase de ecuaciones. Especialmente usamos la teoría de perturbaciones de operadores semi-Fredholm y el teorema de punto fijo de Chow y Hale para estudiar la relación entre el acotamiento y la periodicidad de soluciones para algunas EDP no homogéneas lineales con retardo. Se entrega un ejemplo al final de este trabajo para mostrar la aplicabilidad de los resultados teóricos.

Keywords and Phrases: Hille-Yosida condition, Integral solutions, Semigroup, Semi-Fredholm operators, Periodic solution, Poincaré map.

2020 AMS Mathematics Subject Classification: 34K14, 34K30, 35B10, 35B40.

Accepted: 07 October, 2021
Received: 25 March, 2021

1 Introduction

Along this work, we investigate the periodicity of solutions of the following inhomogeneous linear retarded PDE

$$\begin{cases} \frac{d}{dt}y(t) &= \mathcal{A}y(t) + L(y_t) + h(t) \quad \text{for} \quad t \geq 0 \\ y(t) &= \psi(t) \quad \text{for} \quad -r \leq t \leq 0. \end{cases} \tag{1.1}$$

We assume that the generator $\mathcal{A}$ is not necessarily dense on a Banach space $\mathbb{E}$ and realizes the following Hille-Yosida condition:

(i) there exist $\overline{M} \geq 1$, $\widehat{\omega} \in \mathbb{R}$ such that $(\widehat{\omega}, \infty) \subset \rho(\mathcal{A})$,

(ii) the operator $\mathcal{A}$ satisfies for $n \in \mathbb{N},\ \lambda > \widehat{\omega}$, the following inequality

$$|\ (\lambda I - \mathcal{A})^{-n}\ | \quad \leq \frac{\overline{M}}{(\lambda - \widehat{\omega})^n},$$

where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$. The history function $y_t : [-r, 0] \to \mathbb{E}$ defined for each $\theta \in [-r, 0]$ by $y_t(\theta) = y(t+\theta)$, belongs to $\mathcal{C}([-r, 0], \mathbb{E})$ the space of continuous functions equipped with the supremum norm. $L : \mathcal{C} \to \mathbb{E}$ is a linear bounded operator and h is a continuous function from $\mathbb{R}$ to $\mathbb{E}$.

Almost periodic and periodic solutions remain the most interesting subject in the qualitative analysis of PDE in view of their important applications in many real phenomena and fields. Recall that the concept of almost periodic is more general than the one of periodicity. It was introduced by Bochner and studied by many authors. For more details on almost periodic function we refer to [9, 16, 17, 18]. For the periodicity, there is an extensive literature related to this topic, see for example [10, 11, 25] for more details. Moreover, the choice of a suitable fixed point theorem is a fundamental tool to establish the periodicity of solutions for different classes of differential equations, in fact, to find a fixed point of the well known Poincaré map is equivalent to find the initial data of the periodic solution of the equation. After a long period of research and development, Massera's theorem [24] is the first result explaining the relation between the existence of bounded and periodic solutions for periodic differential equations. In finite dimensional spaces, several works have been developed on this subject. The authors in [4, 12] proved the periodicity of solutions when the solutions of periodic system are just bounded and ultimately bounded by the use of the Horn's fixed point theorem. Especially, in infinite dimensional spaces, the authors in [8], used the Poincaré map approach to get the periodicity of solutions for a class of retarded differential equation, they supposed that the infinitesimal generator satisfies the Hille-Yosida condition and generates a compact semigroup $(\mathcal{T}(t))_{t\geq 0}$ by applying an appropriate fixed point theorem. In [22], the authors proved the periodicity for a nonhomogeneous linear differential equation when the linear part generates a C_0-semigroup on $\mathbb{E}$ and they obtained the existence and uniqueness of periodic solutions for this class of equations.

The present work would be a continuation and extension of the work [8] for inhomogeneous linear retarded PDE, we establish the periodicity of solution for Equation (1.1) by using the perturbation theory of semi-Fredholm operators and without considering the compactness condition of $(\mathcal{T}(t))_{t\geq 0}$. To achieve this goal, we suppose that Equation (1.1) admits a bounded solution on the positive real line and under suitable estimations on the norm of the operator L, we derive periodic solution of Equation (1.1) from bounded ones on the positive real line by using the perturbation theory of semi-Fredholm operators and the Chow and Hale's fixed point theorem.

This work is treated as follows, in Section 2, we give some definitions and results about integral solutions of Equation (1.1). Moreover, we give some definitions and properties concerning the semi-Fredholm operators. Section 3 is devoted to prove and introduce some useful estimations on the integral solutions of Equation (1.1). In Section 4, we discuss the problem of existence of periodic solutions of Equation (1.1). Finally, we apply our theoretical results to an equation appearing in physical systems.

2 Preliminary results

In this article, we assume that:

$(\mathbf{H_0})$ $\mathcal{A}$ satisfies the Hille-Yosida condition.

We consider the following definition and results.

Definition 2.1 ([1])**.** *A continuous function $y : [0, +\infty) \to \mathbb{E}$ is said to be an integral solution of Equation (1.1) if the following conditions hold:*

(i) $y : [0, +\infty) \to \mathbb{E}$ *is continuous, such that* $y_0 = \psi$,

(ii) $\displaystyle\int_0^t y(s)\,ds \in D(\mathcal{A}) \quad \textit{for} \quad t \geq 0,$

(iii) $\displaystyle y(t) = \psi(0) + \mathcal{A}\int_0^t y(s)\,ds + \int_0^t \left(L(y_s) + h(s)\right)\,ds \quad \textit{for} \quad t \geq 0.$

We can deduce from the continuity of the integral solution y that $y(t) \in \overline{D(\mathcal{A})}$, for all $t \geq 0$. Moreover $\psi(0) \in \overline{D(\mathcal{A})}$. In the next we define the part $\mathcal{A}_0$ of the operator $\mathcal{A}$ in $\overline{D(\mathcal{A})}$ as follows

$$D(\mathcal{A}_0) = \{y \in D(\mathcal{A}) : \mathcal{A}y \in \overline{D(\mathcal{A})}\},$$

and

$$\mathcal{A}_0 y = \mathcal{A}y \qquad \text{for} \quad y \in D(\mathcal{A}_0).$$

Lemma 2.2 ([2])**.** *The operator $\mathcal{A}_0$ is the infinitesimal generator of a strongly continuous semi-group denoted by $(\mathcal{T}_0(t))_{t\geq 0}$ on $\overline{D(\mathcal{A})}$.*

Theorem 2.3 ([1]). *Under the assumption* $(\mathbf{H_0})$. *For all* $\psi \in \mathcal{C}$ *such that* $\psi(0) \in \overline{D(\mathcal{A})}$, *Equation (1.1) has a unique integral solution* $y(.)$ *on* $[-r, +\infty)$. *Furthermore,* $y(.)$ *is given by*

$$y(t) = \mathcal{T}_0(t)\psi(0) + \lim_{\lambda \to +\infty} \int_0^t \mathcal{T}_0(t-s)\lambda R(\lambda, \mathcal{A}) \left(L(y_s) + h(s)\right) ds \qquad \text{for} \quad t \geq 0.$$

Through this work, the integral solutions of Equation (1.1) are called plainly solutions. Let $y(., \psi, L, h)$ be the solution of Equation (1.1).

We define $\mathcal{C}_0$ the phase space of Equation (1.1) as $\mathcal{C}_0 = \{\psi \in \mathcal{C} : \psi(0) \in \overline{D(\mathcal{A})}\}$.

Let $\mathcal{X}(t)$ be the linear operator defined on $\mathcal{C}_0$ for each $t \geq 0$, by

$$\mathcal{X}(t)\psi = y_t(., \psi, 0, 0),$$

where $y_t(., \psi, 0, 0)$ is the solution of the following equation

$$\begin{cases} \dfrac{d}{dt} y(t) &= Ay(t) \quad \text{for} \quad t \geq 0, \\ y_0 &= \psi. \end{cases}$$

Theorem 2.4 ([1]). $(\mathcal{X}(t))_{t \geq 0}$ *is a linear strongly continuous semigroup on* $\mathcal{C}_0$*:*

(i) for all $t \geq 0$, $\mathcal{X}(t)$ *is a bounded linear operator on* $\mathcal{C}_0$ *such that* $\mathcal{X}(0) = I$ *and* $\mathcal{X}(t+s) = \mathcal{X}(t)\mathcal{X}(s)$ *for all* $t, s \geq 0$,

(ii) for $t \geq 0$ *and* $\theta \in [-r, 0]$, $(\mathcal{X}(t))_{t \geq 0}$ *satisfies:*

$$[\mathcal{X}(t)\phi](\theta) = \begin{cases} [\mathcal{X}(t+\theta)\phi](0), & \text{if} \quad t + \theta \geq 0, \\ \phi(t+\theta), & \text{if} \quad t + \theta \leq 0. \end{cases}$$

(iii) for all $\psi \in \mathcal{C}_0$ *and* $t \geq 0$, $\mathcal{X}(t)\psi$ *is a continuous function with values in* $\mathcal{C}_0$.

Theorem 2.5 ([1]). *Under the assumption* $(\mathbf{H_0})$. *The solution* $\mathcal{Y}(t)\psi = y_t(., \psi, L, 0)$ *of Equation (1.1) with* $h = 0$ *can be decomposed as follows:*

$$\mathcal{Y}(t)\psi = \mathcal{X}(t)\psi + \mathcal{Z}(t)\psi \qquad \text{for} \quad t \geq 0,$$

where $\mathcal{Z}(t)$, *is a bounded linear operator defined on* $\mathcal{C}_0$, *by*

$$[\mathcal{Z}(t)\psi](\theta) = \begin{cases} \lim\limits_{\lambda \to +\infty} \displaystyle\int_0^{t+\theta} \mathcal{T}_0(t+\theta-s)\lambda R(\lambda, \mathcal{A}) L(y_s(\psi))\, ds & t + \theta \geq 0, \\ 0 & t + \theta \leq 0. \end{cases} \tag{2.1}$$

for each $t \geq 0$.

To discuss the existence of periodic solutions, we use the theory of semi-Fredholm operators. We consider some definitions and propositions which are taken from [21].

Definition 2.6. *Let $\mathbb{E}$, $\mathbb{F}$ be two Banach spaces. A bounded linear operator F from $\mathbb{E}$ to $\mathbb{F}$, denoted by $F \in \mathcal{L}(\mathbb{E},\mathbb{F})$, is said to be semi-Fredholm from $\mathbb{E}$ to $\mathbb{F}$ if*

(i) $\dim\ker(F) < \infty$, *where* $\ker(F)$ *is the null space.*

(ii) $Im(F)$ *the range of F is closed in $\mathbb{F}$.*

We designate by $\Phi_+(\mathbb{E},\mathbb{F})$ the set of all semi-Fredholm operators and $\Phi_+(\mathbb{E}) = \Phi_+(\mathbb{E},\mathbb{E})$. Now we recall some well known theorems for the closed range operators. Let $F \in \mathcal{B}(\mathbb{E},\mathbb{F})$. Then the quotient space $\mathbb{E}/\ker(F)$ is a Banach space equipped with the norm

$$|[x]| = \inf\{|x+m| : m \in \ker(F)\},$$

where

$$[x] = x + \ker(F) := \{x+m : m \in \ker(F)\}.$$

Furthermore, if $\dim\ker(F) < \infty$, then there exists a closed subspace M of $\mathbb{E}$ such that

$$\mathbb{E} = \ker(F) \oplus M.$$

Theorem 2.7 ([21])**.** *Let F be a bounded linear operator in $\mathbb{E}$. Then, $Im(F)$ is closed if and only if there exists a constant $\widetilde{c}$ such that*

$$|[x]| \leq \widetilde{c}\,\|Fx\| \quad \textit{for all} \quad x \in \mathbb{E}.$$

Theorem 2.8 ([21])**.** *Let F be a bounded linear operator in $\mathbb{E}$ such that $\dim\ker(F) < \infty$. Then, the following assertions are equivalent.*

(i) $F \in \Phi_+(\mathbb{E})$.

(ii) There exists a constant $\widetilde{c}$ such that

$$|[x]| \leq \widetilde{c}\,\|Fx\| \quad \textit{for all} \quad x \in \mathbb{E}.$$

(iii) There exists a constant b such that

$$\|(I-P)x\| \leq b\,\|Fx\| \quad \textit{for all} \quad x \in \mathbb{E},$$

where P is the projection operator onto $\ker(F)$ along M.

We present now a result for bounded perturbation of Semi-Fredholm operators.

Theorem 2.9 ([21])**.** *Let F be an operator in $\Phi_+(\mathbb{E},\mathbb{F})$. If $S \in \mathbf{L}(\mathbb{E})$ satisfying*

$$\|S\| < \frac{1}{2b},$$

where b is the constant given in Theorem 2.8. Then,

$$F + S \in \Phi_+(\mathbb{E},\mathbb{F}) \quad \textit{with} \quad \dim\ker(F+S) \leq \dim\ker(F).$$

Now, we need to introduce some well known definitions and results about the spectral theory. Let $\mathcal{J}$ be a linear bounded operator on $\mathbb{F}$, we define the measure of Kuratowski of noncompactness of the operator $\mathcal{J}$ as follows

$$|\mathcal{J}|_\alpha = \inf\{\epsilon > 0 : \alpha(\mathcal{J}(B)) \leq \epsilon\alpha(B), \quad \text{for every bounded set } B \subset \mathbb{F}\},$$

where $\alpha(.)$ is the measure of Kuratowski of noncompactness of bounded sets $B \subset \mathbb{F}$ defined by

$$\alpha(J) = \inf\{\epsilon > 0 : B \text{ has a finite cover of ball of diameter } < \epsilon\}.$$

The essential radius $r_{ess}(\mathcal{J})$ is characterized by the following Nussbaum Formula introduced in [19]:

$$r_{ess}(\mathcal{J}) = \lim_{n\to+\infty} [|\mathcal{J}^n|_\alpha]^{1/n}.$$

Moreover, if $\mathcal{J}$ is bounded and $r_{ess}(\mathcal{J}) < 1$, then $I - \mathcal{J} \in \Phi_+(\mathbb{F})$.

Let us define the essential growth bound of a strongly continuous semigroup $\mathcal{S} := (\mathcal{S}(t))_{t\geq 0}$ on a Banach space $\mathbb{F}$ as

$$\omega_{ess}(\mathcal{S}) := \lim_{t\to+\infty} \frac{1}{t} \log |\mathcal{S}(t)|_\alpha.$$

It is well know that

$$r_{ess}(\mathcal{S}(t)) = \exp\left(t\omega_{ess}(\mathcal{S})\right), \quad t > 0.$$

3 Several estimates

Before discussing the periodicity of solution of Equation (1.1), we need some preparatory estimates.

Proposition 3.1. *Suppose that* $|\mathcal{T}_0(t)| \leq M_0\, e^{\omega_0 t}$ *for* $t \geq 0$. *Then*

$$\|\mathcal{Z}(t)\|_{\mathcal{C}} \leq M_0\, e^{\omega_0^+ t}(e^{M_0|L|\overline{M}\, t} - 1) \text{ for } t \geq 0,$$

where $\omega_0^+ = \max\{\omega_0, 0\}$.

To prove the above Proposition, we need to introduce the following Lemma.

Lemma 3.2. *Let* $|\mathcal{T}_0(t)| \leq M_0 e^{\omega_0 t}$ *for* $t \geq 0$. *Then, the solution of Equation (1.1) in the case where* $h = 0$ *is estimated as*

$$|\mathcal{Y}(t)| \leq M_0\, e^{(\omega_0^+ + M_0|L|\overline{M})t}.$$

Proof. For $t \geq 0$, $\theta \in [-r, 0]$, one has

$$\|\mathcal{Y}(t)\psi\|_{\mathcal{C}} = \sup_{\theta\in[-r,0]} |y(t+\theta,\psi)| = \sup_{\xi\in[t-r,t]} |y(\xi,\psi)|.$$

Then for $t \geq r$,

$$\begin{aligned}\sup_{\xi\in[t-r,t]}|y(\xi,\psi)| &\leq \sup_{\xi\in[t-r,t]}|\mathcal{T}_0(\xi)\psi(0)| + \sup_{\xi\in[t-r,t]}\left|\lim_{\lambda\to+\infty}\lambda\int_0^{\xi}\mathcal{T}_0(\xi-s)R(\lambda,\mathcal{A})L(\mathcal{Y}(s)\psi)\,ds\right| \\ &\leq M_0e^{\omega_0^+t}\|\psi\|_{\mathcal{C}} + M_0|L|\overline{M}\left(\int_0^t e^{\omega_0^+(t-s)}\,\|\mathcal{Y}(s)\psi\|_{\mathcal{C}}\,ds\right).\end{aligned}$$

For $t < r$,

$$\begin{aligned}\sup_{\xi\in[t-r,t]}|y(\xi,\psi)| &= \max\left\{\sup_{\xi\in[t-r,0]}|y(\xi,\psi)|;\ \sup_{\xi\in[0,t]}|y(\xi,\psi)|\right\} \\ &\leq \max\left\{\|\psi\|_{\mathcal{C}};\ \sup_{\xi\in[0,t]}|y(\xi,\psi)|\right\},\end{aligned}$$

and

$$\sup_{\xi\in[0,t]}|y(\xi,\psi)| \leq M_0e^{\omega_0^+t}\|\psi\|_{\mathcal{C}} + M_0|L|\overline{M}\left(\int_0^t e^{\omega_0^+(t-s)}\,\|\mathcal{Y}(s)\psi\|_{\mathcal{C}}ds\right).$$

Finally, we obtain that

$$\|\mathcal{Y}(t)\psi\|_{\mathcal{C}} \leq M_0e^{\omega_0^+\,t}\,|\psi|_{\mathcal{C}} + M_0\,|L|\,\overline{M}\int_0^t e^{\omega_0^+(t-s)}\,\|\mathcal{Y}(s)\psi\|_{\mathcal{C}}\,ds.$$

Hence,

$$\|e^{-\omega_0^+\,t}\,\mathcal{Y}(t)\psi\|_{\mathcal{C}} \leq M_0\,\|\psi\|_{\mathcal{C}} + M_0\,|L|\,\overline{M}\int_0^t e^{-\omega_0^+\,s}\,\|\mathcal{Y}(s)\psi\|_{\mathcal{C}}\,ds.$$

Gronwall's inequality implies that

$$\|e^{-\omega_0^+\,t}\,\mathcal{Y}(t)\psi\|_{\mathcal{C}} \leq M_0\,e^{M_0\,|L|\,\overline{M}\,t}\,\|\psi\|_{\mathcal{C}},$$

and then

$$\|\mathcal{Y}(t)\psi\|_{\mathcal{C}} \leq M_0\,e^{(\omega_0^+ + M_0\,|L|\,\overline{M})\,t}\,\|\psi\|_{\mathcal{C}}$$ □

Proof of Proposition 3.1. Let $t \geq 0,\ t+\theta \geq 0.$ Then

$$\begin{aligned}\|\mathcal{Z}(t)\psi\|_{\mathcal{C}} &= \sup_{\theta\in[-r,0]}|(\mathcal{Z}(t)\psi)(\theta)| \\ &\leq M_0|L|\overline{M}\left(\int_0^t e^{\omega_0^+\,(t-s)}\|\mathcal{Y}(s)\psi\|_{\mathcal{C}}\,ds\right).\end{aligned}$$

From Lemma 3.2, we have

$$\begin{aligned}\|\mathcal{Z}(t)\psi\|_{\mathcal{C}} &\leq M_0^2|L|\overline{M}\,e^{\omega_0^+\,t}\left(\int_0^t e^{M_0|L|\,\overline{M}\,s}\,ds\right)\|\psi\|_{\mathcal{C}} \\ &\leq M_0\,e^{\omega_0^+t}(e^{M_0|L|\,\overline{M}\,t}-1)\,\|\psi\|_{\mathcal{C}}.\end{aligned}$$

This implies our inequality. □

Proposition 3.3. *A function* $\phi \in \ker(I-\mathcal{X}(\omega))$ *if and only if* $\phi(0) \in \ker(I-\mathcal{T}_0(\omega))$*, furthermore*

$$\dim\,\ker(I-\mathcal{X}(\omega)) = \dim\,\ker(I-\mathcal{T}_0(\omega)).$$

Proof. Let $\phi \in \ker(I - \mathcal{X}(\omega))$. Then,

$$\mathcal{X}(\omega)\phi = \phi \quad \text{and} \quad (\mathcal{X}(\omega)\phi)(\theta) = \phi(\theta) \quad \text{for} \quad \theta \in [-r, 0].$$

Since

$$(\mathcal{X}(\omega)\phi)(0) = \mathcal{T}_0(\omega)\phi(0),$$

then

$$\phi(0) = \mathcal{T}_0(\omega)\phi(0),$$

and hence

$$\phi(0) \in \ker(I - \mathcal{T}_0(\omega)).$$

Conversely, let $x \in \ker(I - \mathcal{T}_0(\omega))$ and $\phi_n(\theta) = \mathcal{T}_0(n\omega + \theta)x$ for $n \geq [\frac{r}{\omega}] + 1$, where $[.]$ denotes the integer part. Then,

$$\mathcal{T}_0(t + \omega)x = \mathcal{T}_0(t)\mathcal{T}_0(\omega)x = \mathcal{T}_0(t)x \quad \text{for} \quad t \geq 0,$$

and $\phi_n(\theta)$ is independent of the integer n and then

$$\phi_n(\theta) = \mathcal{T}_0(\omega + \theta)x = \phi(\theta) \quad \text{and} \quad \phi(0) = x.$$

If $\omega + \theta \geq 0$, then

$$(\mathcal{X}(\omega)\phi)(\theta) = \mathcal{T}_0(\omega + \theta)\phi(0) = \mathcal{T}_0(\omega + \theta)x = \phi(\theta).$$

If $\omega + \theta \leq 0$, then

$$\begin{aligned}(\mathcal{X}(\omega)\phi)(\theta) &= \phi(\omega + \theta) = \phi_n(\omega + \theta) \\ &= \mathcal{T}_0(\theta + \omega + n\omega)x = \mathcal{T}_0(n\omega + \theta)x \\ &= \phi_n(\theta) = \phi(\theta),\end{aligned}$$

hence,

$$\mathcal{X}(\omega)\phi = \phi,$$

which implies that $\phi \in \ker(I - \mathcal{X}(\omega))$. Moreover, $\ker(I - \mathcal{T}_0(\omega))$ is mapped bijectively onto the space $\ker(I - \mathcal{X}(\omega))$. Therefore,

$$\dim \ker(I - \mathcal{X}(\omega)) = \dim \ker(I - \mathcal{T}_0(\omega)).$$ □

Let us define the constant m_ω by

$$m_\omega = \sup_{0 \leq t \leq \omega} |\mathcal{T}_0(t)|.$$

As it is shown in [22], the proof is omitted here, if $I - \mathcal{T}_0(\omega)$ is semi-Fredholm on $\overline{D(\mathcal{A})}$, then, the operator defined by

$$S_M := I - \mathcal{T}_0(\omega) : M \to Im(I - \mathcal{T}_0(\omega)),$$

is bijective, such that M is a subset of $\mathbb{E}$ and $\overline{D(\mathcal{A})}$ is decomposed as

$$\overline{D(\mathcal{A})} = \ker(I - \mathcal{T}_0(\omega)) \oplus M.$$

Let S_M^{-1} be the inverse operator of S_M and let $k \in \mathbb{N}^*$ such that

$$(k-1)\omega < r \leq k\omega.$$

We set

$$I_k = [-r, -(k-1)\omega) \quad \text{and} \quad I_p = [-p\omega, -(p-1)\omega) \quad \text{for} \quad p = 1, 2, \ldots, k-1 \quad \text{with} \quad k \geq 2.$$

Let $\mathcal{G}$ be a linear operator defined from $D(\mathcal{G})$ to $\mathcal{C}_0$ by

$$(\mathcal{G}\phi)(\theta) = \sum_{j=0}^{p-1} \phi(\theta + j\omega) + \mathcal{T}_0(\theta + p\omega) S_M^{-1} \phi(0) \quad \text{for} \quad \theta \in I_p,$$

with

$$D(\mathcal{G}) = \{\phi \in \mathcal{C}_0 : \phi(0) \in Im(I - \mathcal{T}_0(\omega))\}.$$

Clearly, for $\theta \in I_p, p = 1, 2, \ldots, k$,

$$\|\mathcal{G}\phi\|_{\mathcal{C}} = \sup_{\theta \in [-r,0]} |(\mathcal{G}\phi)(\theta)| \leq \sum_{j=0}^{p-1} \|\phi\|_{\mathcal{C}} + \sup_{s \in [0,\omega]} |\mathcal{T}_0(s)||S_M^{-1}||\phi(0)|.$$

Then

$$\|\mathcal{G}\phi\|_{\mathcal{C}} \leq \left(k + m_\omega |S_M^{-1}|\right) \|\phi\|_{\mathcal{C}}. \tag{3.1}$$

Consequently, we have the following Theorem.

Theorem 3.4. *$I - \mathcal{T}_0(\omega)$ is semi-Fredholm on $\overline{D(\mathcal{A})}$ if and only if $I - \mathcal{X}(\omega)$ is semi-Fredholm on $\mathcal{C}_0$.*

To prove Theorem 3.4, we need the following Lemma

Lemma 3.5 ([14]).

$$Im(I - \mathcal{X}(\omega)) = D(\mathcal{G}).$$

Proof of Theorem 3.4. Suppose that $Im(I - \mathcal{T}_0(\omega))$ is closed, let $\phi_n \in D(\mathcal{G})$ such that $\phi_n \to \phi$ as $n \to \infty$. Then

$$\phi_n(0) \in Im(I - \mathcal{T}_0(\omega)) \quad \text{and} \quad \phi_n(0) \to \phi(0) \in Im(I - \mathcal{T}_0(\omega)),$$

which implies that $\phi \in D(\mathcal{G})$ and $D(\mathcal{G})$ is closed. Lemma 3.5 implies that $Im(I - \mathcal{X}(\omega))$ is closed. Now, we suppose that $Im(I - \mathcal{X}(\omega))$ is closed and $x_n \in Im(I - \mathcal{T}_0(\omega))$ such that $x_n \to x$ as $n \to \infty$. Let $\phi_n, \phi \in \mathcal{C}_0$ be such that $\phi_n(\theta) = x_n$ and $\phi(\theta) = x$. It is clear that $\phi_n \to \phi$ as $n \to \infty$ and by Lemma 3.5 we have that $(\phi_n) \in Im(I - \mathcal{X}(\omega))$. Then $\phi \in Im(I - \mathcal{X}(\omega))$ and $\phi(0) = x \in Im(I - \mathcal{T}_0(\omega))$. Consequently, $Im(I - \mathcal{T}_0(\omega))$ is closed. Therefore, by the use of Proposition 3.3 we obtain the desired result. □

In the nondensely defined case, we can prove the following result as in [22], the proof is omitted here.

Theorem 3.6. *Suppose that* $I-\mathcal{T}_0(\omega)$ *is semi-Fredholm on* $\overline{D(\mathcal{A})}$ *such that* $\dim\ker(I-\mathcal{T}_0(\omega))=n$. *If*

$$|\mathcal{Z}(\omega)| < \frac{1}{2\widetilde{c}\,(1+\sqrt{n})}.$$

Then,

$$I-\mathcal{Y}(\omega)\in\Phi_+(\mathcal{C}_0) \quad \textit{and} \quad \dim\ker(I-\mathcal{Y}(\omega))\leq n.$$

Proposition 3.7. *Suppose that* $I-\mathcal{T}_0(\omega)$ *is semi-Fredholm on* $\overline{D(\mathcal{A})}$. *If* $|L|$ *satisfies*

$$|L| < \frac{\log\left(\dfrac{e^{-\omega_0^+\omega}}{2M_0\widetilde{c}\,(1+\sqrt{n})}+1\right)}{M_0\overline{M}\omega}. \tag{3.2}$$

Then,

$$I-\mathcal{Y}(\omega)\in\Phi_+(\mathcal{C}_0) \quad \textit{and} \quad \dim\ker(I-\mathcal{Y}(\omega))\leq n.$$

Proof. By the inequality (3.2), it follows that

$$M_0\,e^{\omega_0^+\omega}(e^{M_0\overline{M}|L|\,\omega}-1) < \frac{1}{2\widetilde{c}\,(1+\sqrt{n})},$$

and by Proposition 3.1, one has

$$|\mathcal{Z}(\omega)| < \frac{1}{2\widetilde{c}\,(1+\sqrt{n})}.$$

Theorem 3.6 gives that

$$I-\mathcal{Y}(\omega)\in\Phi_+(\mathcal{C}_0) \quad \text{and} \quad \dim\ker(I-\mathcal{Y}(\omega))\leq n. \qquad \square$$

4 Periodic solutions for Equation (1.1)

To discuss the existence of periodic solutions of Equation (1.1), we introduce the following fixed point Theorem for a linear affine map T defined from $\mathbb{E}$ to $\mathbb{E}$ by

$$Tu=\overline{T}u+v \quad \text{for} \quad u\in\mathbb{E},$$

where $\overline{T}\in B(\mathbb{E})$ and $v\in\mathbb{E}$ is fixed. Let $\mathcal{F}_T$ be the set of all fixed points of the map T.

Theorem 4.1 ([5])**.** *Let* T *be a linear affine map on a Banach space* $\mathbb{E}$ *such that the range* $Im(I-\overline{T})$ *is closed. If there is an* $u_0\in\mathbb{E}$ *such that* $\{T^k u_0, k\in\mathbb{N}\}$ *is bounded in* $\mathbb{E}$, *then* $\mathcal{F}_T\neq\varnothing$.

If there exists some $v\in\mathcal{F}_T$, then

$$\mathcal{F}_T=v+\ker(I-\overline{T}).$$

$\dim \mathcal{F}_T$ is defined as

$$\dim \mathcal{F}_T = \dim \ker(I - \overline{T}).$$

If $I - \overline{T} \in \Phi_+(\mathbb{X})$. Then, Theorem 4.1 is refined as follows

Theorem 4.2 ([22]). *Let T be a linear affine map on a Banach space $\mathbb{E}$. If $I - \overline{T} \in \Phi_+(\mathbb{E})$ and if there exists an $u_0 \in \mathbb{E}$ such that $\{T^k u_0, k \in \mathbb{N}\}$ is bounded, then $\mathcal{F}_T \neq \varnothing$ and* $\dim$ *$\mathcal{F}_T$ is finite.*

Through the rest of this work, we suppose that

$(\mathbf{H_1})$ h is an ω-periodic function.

Furthermore, by property $(\mathcal{R})$ we mean the following equivalence:

Equation (1.1) has an $\omega-$periodic solution if and only if it has a bounded one on the positive real line. Then, we have the following result.

Theorem 4.3. *Under assumptions $(\mathbf{H_0})$ and $(\mathbf{H_1})$. If $I - \mathcal{T}_0(\omega)$ is semi-Fredholm on $\overline{D(\mathcal{A})}$ and if the operator L satisfies the following estimate*

$$|L| < \frac{\log\left(\dfrac{e^{-\omega_0^+\omega}}{2M_0\widetilde{c}\,(1+\sqrt{n})} + 1\right)}{M_0\overline{M}\omega},$$

where $\widetilde{c}$ and n are the constants given in Theorem 3.6. Then, Equation (1.1) satisfies the property $(\mathcal{R})$.

Proof. Let $y(.,\psi,h)$ be the solution of Equation (1.1). We introduce the Poincaré map $\mathcal{P}$ defined from $\mathcal{C}_0$ to $\mathcal{C}_0$ as follows

$$\mathcal{P}_\omega(\psi) = y_\omega(.,\psi,h),$$

Then,

$$\mathcal{P}_\omega\psi = y_\omega(.,\psi,0) + y_\omega(.,0,h),$$

and hence $\mathcal{P}_\omega$ is an affine map such that

$$\mathcal{P}_\omega\psi = \overline{P}\psi + \varphi,$$

with

$$\overline{P}\psi = y_\omega(.,\psi,0) \quad \text{and} \quad \varphi = y_\omega(.,0,h).$$

According to the second section, $\overline{P}$ is decomposed as

$$\overline{P} = \mathcal{X}(\omega) + \mathcal{Z}(\omega).$$

Moreover, Proposition 3.7 gives that

$$I - \overline{P} \in \Phi_+(\mathcal{C}_0).$$

Now, let $y(.,\psi,h)$ denote the bounded solution of Equation (1.1) on $\mathbb{R}^+$. Thus, for each $n\in\mathbb{N}$, we have

$$\mathcal{P}_\omega^n\psi = y_{n\omega}(.,\psi,h),$$

and then $(\mathcal{P}_\omega^n\psi)_{n\geq 0}$ is a bounded sequence in $\mathcal{C}_0$. All conditions of Theorem 4.2 are satisfied and then $\mathcal{F}_{\mathcal{P}_\omega}\neq\varnothing$, which yields an ω-periodic solution of Equation (1.1). □

Corollary 4.4. *Under assumptions* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$*. If* $I-\mathcal{T}_0(\omega)$ *is semi-Fredholm in* $\overline{D(\mathcal{A})}$ *and if* $|L|$ *satisfies the following inequality*

$$|L| < \frac{\log\left(\dfrac{e^{-\omega_0^+\omega}}{2M_0\left(k+m_\omega|S_M^{-1}|\right)(1+\sqrt{n})}+1\right)}{M_0\overline{M}\omega}.$$

Then, Equation (1.1) satisfies the property $(\mathcal{R})$*.*

To establish the proof, we need the following Lemma.

Lemma 4.5 ([14])**.** *Suppose that* $I-\mathcal{T}_0(\omega)$ *is semi-Fredholm on* $\overline{D(\mathcal{A})}$*. If there exists a constant* $\widetilde{c}>0$ *such that*

$$\|\mathcal{G}\psi\|_{\mathcal{C}}\leq\widetilde{c}\|\psi\|_{\mathcal{C}}\quad \textit{for all}\quad \psi\in D(\mathcal{G}).$$

Then,

$$|[\varphi]|\leq\widetilde{c}\|(I-\mathcal{X}(\omega))\varphi\|_{\mathcal{C}}\quad \textit{for all}\quad \varphi\in\mathcal{C}_0.$$

Proof of Corollary 4.4: Since,

$$|L| < \frac{\log\left(\dfrac{e^{-\omega_0^+\omega}}{2M_0\left(k+m_\omega|S_M^{-1}|\right)(1+\sqrt{n})}+1\right)}{M_0\overline{M}\omega}.$$

it follows that,

$$(k+m_\omega|S_M^{-1}|)(e^{M_0\overline{M}\omega|L|}-1) < \frac{e^{-\omega_0^+\omega}}{2M_0\,(1+\sqrt{n})}.$$

Lemma 4.5 and estimation (3.1) implies that

$$\widetilde{c}\leq k+m_\omega|S_M^{-1}|,$$

and then

$$\widetilde{c}(e^{M_0\overline{M}\omega|L|}-1) < \frac{e^{-\omega_0^+\omega}}{2M_0\,(1+\sqrt{n})}.$$

Finally

$$|L| < \frac{\log\left(\dfrac{e^{-\omega_0^+\omega}}{2M_0\widetilde{c}\,(1+\sqrt{n})}+1\right)}{M_0\overline{M}\omega}.$$

Now, Theorem 4.3 shows that Equation (1.1) satisfies the property $(\mathcal{R})$. □

In the particular case where the semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ is exponentially stable, we have the following Theorem.

Theorem 4.6. *Under assumptions* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$*. If the semigroup* $(\mathcal{T}_0(t))_{t\geq 0}$ *is exponentially stable and if the operator* L *satisfies the following inequality*

$$|L| < \frac{\log\left(\dfrac{1}{2M_0\left(k+m_\omega|S_M^{-1}|\right)}+1\right)}{M_0\overline{M}\omega}.$$

Then, Equation (1.1) satisfies the property $(\mathcal{R})$*.*

Proof. From the exponential stability of $(\mathcal{T}_0(t))_{t\geq 0}$, we have

$$\omega_{ess}(\mathcal{T}_0) = \lim_{t\to+\infty}\frac{1}{t}\log|\mathcal{T}_0(t)|_\alpha \leq \lim_{t\to+\infty}\frac{1}{t}\log|\mathcal{T}_0(t)| = -\omega_0 < 0.$$

Consequently,

$$r_{ess}(\mathcal{T}_0(\omega)) = \exp\left(\omega\omega_{ess}(\mathcal{T}_0)\right) < 1.$$

Which implies that $Im(I-\mathcal{T}_0(\omega))$ is closed. On the other hand, one has

$$|\mathcal{T}_0(\omega)^n| = |\mathcal{T}_0(n\omega)| \leq M_0 e^{-\omega_0 n\omega}$$

and

$$|\mathcal{T}_0(n\omega)|^{\frac{1}{n}} \leq M_0^{\frac{1}{n}} e^{-\omega_0\omega},$$

which implies that the spectral radius is estimated as

$$r(\mathcal{T}_0(\omega)) = \lim_{n\to+\infty}|\mathcal{T}_0(\omega)^n|^{\frac{1}{n}} \leq e^{-\omega_0\omega}\lim_{n\to+\infty}M_0^{\frac{1}{n}} < \lim_{n\to+\infty}M_0^{\frac{1}{n}} < 1.$$

Consequently

$$1 \notin \sigma(\mathcal{T}_0(\omega)) \quad \text{and} \quad n = \dim\ker(I-\mathcal{T}_0(\omega)) = 0.$$

All conditions of Corollary 4.4 are satisfied with $n=0$. Then, Equation (1.1) satisfies the property $(\mathcal{R})$. □

5 Application

In order to apply our theoretical results, we consider the following delayed partial differential equation:

$$\begin{cases} \dfrac{\partial}{\partial t}y(t,\zeta) = \dfrac{\partial^2}{\partial\zeta^2}y(t,\zeta) - ay(t,\zeta) + by(t-r,\zeta) + g(t,\zeta) & \text{for } t\in\mathbb{R}^+ \text{ and } \zeta\in\mathbb{R}, \\ y(\theta,\zeta) = \psi_0(\theta,\zeta) & \text{for } \theta\in[-r,0] \text{ and } \zeta\in\mathbb{R}, \end{cases} \tag{5.1}$$

where a and b are positive constants, $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\phi:[-r,0]\times\mathbb{R}\to\mathbb{R}$ are continuous functions where $\phi(\theta,\zeta)$ has a finite limit at $\pm\infty$.

Note that Equation (5.1) can be written in the form of Equation (1.1). In fact: we set $\overline{\mathbb{R}} := [-\infty, +\infty]$ and we say that $z \in \mathbf{C}^k(\overline{\mathbb{R}})$ if $z \in \mathbf{C}^k(\mathbb{R})$ and all derivatives of z up to the order k have finite limits at $\pm\infty$. Then, the space of continuous functions on $\overline{\mathbb{R}}$, denoted by $\mathbb{E} = \mathbf{C}(\overline{\mathbb{R}})$, endowed with the norm

$$\|z\|_\infty = \sup_{-\infty<\zeta<+\infty} |z(\zeta)|$$

becomes a Banach space. we consider the linear operator Δ defined from $D(\Delta) \subset \mathbb{E}$ to $\mathbb{E}$ by

$$\begin{cases} D(\Delta) &= \left\{ z \in \mathbf{C}^2\left(\overline{\mathbb{R}}\right) : \lim_{\zeta \to \pm\infty} z(\zeta) = 0 \right\}, \\ \Delta z &= z''. \end{cases}$$

Then, we have

Lemma 5.1 ([7]).

$$(0, +\infty) \subset \rho(\Delta)$$

and for each $\lambda > 0$

$$\left|(\lambda I - \Delta)^{-1}\right| \leq \frac{1}{\lambda}.$$

Clearly

$$\overline{D(\Delta)} = \left\{ z \in \mathbf{C}\left(\overline{\mathbb{R}}\right) : \lim_{\zeta \to \pm\infty} z(\zeta) = 0 \right\}.$$

We write the part Δ_0 of Δ in $\overline{D(\Delta)}$ as

$$\begin{cases} D(\Delta_0) &= \left\{ z \in \mathbf{C}^2\left(\overline{\mathbb{R}}\right) : \lim_{\zeta \to \pm\infty} z(\zeta) = \lim_{\zeta \to \pm\infty} z''(\zeta) = 0 \right\}, \\ \Delta_0 z &= z''. \end{cases}$$

Lemma 5.2 ([7]). Δ_0 *is the infinitesimal generator of a strongly continuous semigroup* $(\mathcal{T}_{\Delta_0}(t))_{t\geq 0}$ *on* $\overline{D(\Delta)}$. *Furthermore,*

$$|\mathcal{T}_{\Delta_0}(t)| \leq 1 \quad \textit{for} \quad t \geq 0.$$

Let $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{E} \to \mathbb{E}$ defined by:

$$\begin{cases} D(\mathcal{A}) = \left\{ z \in \mathbf{C}^2\left(\overline{\mathbb{R}}\right) : \lim_{\zeta \to \pm\infty} z(\zeta) = 0 \right\}, \\ \\ \mathcal{A}z = z'' - az. \end{cases}$$

By Lemma 5.1, it is clear that

Lemma 5.3.

$$(-a, +\infty) \subset \rho(\mathcal{A})$$

and for each $\lambda > -a$

$$\left|(\lambda I - \mathcal{A})^{-1}\right| \leq \frac{1}{\lambda + a}.$$

Lemma 5.3 guarantees that the assumption $(\mathbf{H}_0)$ is satisfied with $\widehat{\omega} = -a$ and $\overline{M} = 1$. Moreover,

$$\overline{D(\mathcal{A})} = \left\{ z \in \mathbf{C}\left(\overline{\mathbb{R}}\right) : \lim_{\zeta \to \pm\infty} z(\zeta) = 0 \right\} \subsetneq \mathbb{E}.$$

Moreover, we write the part $\mathcal{A}_0$ of the linear operator $\mathcal{A}$ in $\overline{D(\mathcal{A})}$ as:

$$\begin{cases} D(\mathcal{A}_0) = \left\{ z \in \mathbf{C}^2\left(\overline{\mathbb{R}}\right) : \lim\limits_{\zeta \to \pm\infty} z(\zeta) = 0 = \lim\limits_{\zeta \to \pm\infty} z''(\zeta) = 0 \right\}, \\ \mathcal{A}_0 z = z'' - az. \end{cases}$$

Lemma 5.4. *$\mathcal{A}_0$ is the infinitesimal generator of an exponentially stable continuous semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ on $\overline{D(\mathcal{A})}$. Moreover, for $t \geq 0$, we have*

$$|\mathcal{T}_0(t)| \leq e^{-at}.$$

Consider the following notations:

$$\begin{cases} y(t)(\zeta) &= y(t,\zeta) \quad \text{for} \quad t \in \mathbb{R}^+, \quad \zeta \in \mathbb{R}, \\ \psi(\theta)(\zeta) &= \psi_0(\theta,\zeta) \quad \text{for} \quad \theta \in [-r,0], \quad \zeta \in \mathbb{R}, \end{cases}$$

and define the function $L : \mathcal{C} \to \mathbb{E}$ as follows

$$L(\phi)(\zeta) = b\phi(-r)(\zeta) \quad \text{for} \quad \zeta \in \mathbb{R} \quad \text{and} \quad \phi \in \mathcal{C}.$$

$h : \mathbb{R} \longrightarrow \mathbb{E}$ is defined by

$$h(t)(\zeta) = g(t,\zeta) \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad \zeta \in \mathbb{R}.$$

Clearly, L is a linear bounded operator from $\mathcal{C}$ to $\mathbb{E}$. Then, Equation (5.1) can be written in $\mathbb{E}$ as follows

$$\begin{cases} \frac{d}{dt}y(t) &= \mathcal{A}y(t) + L(y_t) + h(t) \quad \text{for} \quad t \geq 0, \\ y_0 &= \psi \in \mathcal{C}. \end{cases} \tag{5.2}$$

We suppose that $\lim\limits_{\zeta \to \pm\infty} \psi_0(0,\zeta) = 0$, then Equation (5.2) has a unique integral solution y on $[-r, +\infty)$.

To get the periodicity of solutions of Equation (5.2), we suppose that

$(\mathbf{H_2})$ $b < a$.

Let $\rho = 1 + \dfrac{|h|_\infty}{a-b}$ where $|h|_\infty = \sup\limits_{0 \leq t \leq \omega} |h(t)|$. Then, we have

Lemma 5.5. *Under assumption $(\mathbf{H_2})$. For every $\psi \in \mathcal{C}$ such that $\|\psi\|_{\mathcal{C}} < \rho$, the solution of Equation (5.2) is bounded by ρ on $\mathbb{R}^+$.*

Proof. We proceed by contradiction. Let

$$t_* = \inf\{t > 0 : |y(t,\psi)| > \rho\}.$$

From the continuity of y, one has

$$|y(t_*,\psi)| = \rho,$$

and there is $\alpha > 0$, with

$$|y(t,\psi)| > \rho \quad \text{for each} \quad t \in (t_*, t_* + \alpha).$$

Applying the variation-of-constants formula for Equation (5.2) with the initial value ψ,

$$y(t) = \mathcal{T}_0(t)\psi(0) + \lim_{\lambda\to+\infty} \lambda \int_0^t \mathcal{T}_0(t-s)R(\lambda,\mathcal{A})\left(L(y_s) + h(s)\right) ds.$$

Then, for $t \geq 0$

$$|y(t_*,\psi)| \leq |\mathcal{T}_0(t_*)|\,|\psi(0)| + \int_0^{t_*} |\mathcal{T}_0(t_*-s)|\,\left(|L(y_s)| + |h(s)|\right) ds.$$

Since for $0 < s < t_*$, it follows that $-r \leq s - r \leq t_* - r < t_*$ and then

$$|L(y_s)| = b|y(s-r)| \leq b\,\rho,$$

hence

$$\begin{aligned} |y(t_*,\psi)| &\leq \rho e^{-at_*} + (b\,\rho + |h|_\infty) \int_0^{t_*} e^{-a(t_*-s)}\, ds \\ &\leq \rho e^{-at_*} + \frac{(1-e^{-at_*})}{a}\,(b\,\rho + |h|_\infty)\,. \end{aligned}$$

Consequently,

$$\begin{aligned} |y(t_*,\psi)| &\leq \rho e^{-at_*} + (b\rho + (a-b)(\rho-1))\,\frac{(1-e^{-at_*})}{a} \\ &\leq \rho e^{-at_*} + \left(\rho - 1 + \frac{b}{a}\right)(1-e^{-at_*}) \\ &\leq \rho e^{-at_*} + \rho(1-e^{-at_*}) \\ &\leq \rho, \end{aligned}$$

which contradicts the definition of t_*, and we deduce that

$$|y(t,\psi)| \leq \rho \qquad \text{for} \quad t \geq 0.$$ □

To discuss the periodicity of solutions of Equation (5.2), we assume that:

($\mathbf{H_3}$) h is an ω-periodic function in t.

Theorem 5.6. *Suppose that* ($\mathbf{H_2}$) *and* ($\mathbf{H_3}$) *hold true. If*

$$|L| < \omega^{-1} \log\left(\frac{(1-e^{-a\omega})(1+2k)+2}{2+2k(1-e^{-a\omega})}\right),$$

then, Equation (5.2) has an ω*-periodic solution.*

Proof. Let m_ω be the constant defined by

$$m_\omega = \sup_{0\le t\le\omega} |\mathcal{T}_0(t)|.$$

Then

$$m_\omega \le \sup_{0\le t\le\omega} e^{-at} = 1.$$

Moreover, since $|\mathcal{T}_0(\omega)| < 1$, one has

$$\begin{aligned} |S_M^{-1}| = |(I-\mathcal{T}_0(\omega))^{-1}| &\le \frac{1}{I-|\mathcal{T}_0(\omega)|} \\ &\le \frac{1}{1-e^{-a\omega}}. \end{aligned}$$

Thus,

$$k + m_\omega |S_M^{-1}| \le k + \frac{1}{1-e^{-a\omega}},$$

and

$$\begin{aligned} |L| &< \omega^{-1} \log\left(\frac{(1-e^{-a\omega})(1+2k)+2}{2+2k(1-e^{-a\omega})}\right) \\ &< \omega^{-1} \log\left(\frac{1}{2\left(k+m_\omega|S_M^{-1}|\right)}+1\right). \end{aligned}$$

All condition of Theorem 4.6 are satisfied. Then, Lemma 5.5 implies that Equation (5.2) has an $\omega-$periodic solution. □

Acknowledgement

The authors would like to thank the anonymous referees for their constructive comments and valuable suggestions, which are helpful to improve the quality of this paper.

References

[1] M. Adimy and K. Ezzinbi, "Local existence and linearized stability for partial functional differential equations", Dyn. Syst. Appl., vol. 7, no. 3, pp. 389–404, 1998.

[2] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, *Vector valued Laplace transforms and Cauchy problems*, Monographs in Mathematics, vol. 96, Basel: Birkhäuser Verlag, 2001.

[3] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, B. Neubrander and U. Schlotterbeck, *One-parameter semigroup of positive operators*, Lecture Notes in Mathematics, vol. 1184, Berlin: Springer-Verlag, 1984.

[4] T. Burton, *Stability and periodic solutions of ordinary differential equation and functional differential equations*, New York: Academic Press, 1985.

[5] S. N. Chow and J. K. Hale, "Strongly limit-compact maps", Funkcial. Ekvac., vol. 7, pp. 31–38, 1974.

[6] G. Da Prato and E. Sinestrari, "Differential operators with nondense domains", Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), vol. 14, no. 2, pp. 285–344, 1987.

[7] N. Dunford and J. T. Schwartz, *Linear operators, Part I*, New York: John Wiley & Sons Inc, 1987.

[8] A. Elazzouzi and K. Ezzinbi, "Ultimate boundedness and periodicity for some partial functional differential equations with infinite delay", J. Math. Anal. Appl., vol. 329, no. 1, pp. 498–514, 2007.

[9] K. Ezzinbi, "A survey on new methods for partial functional differential equations and applications", Afr. Mat., vol. 31, no. 1, pp. 89–113, 2020.

[10] K. Ezzinbi and M. Taoudi, "Periodic solutions and attractiveness for some partial functional differential equations with lack of compactness", Proc. Amer. Math. Soc., vol. 149, no. 3, pp. 1165–1174, 2021.

[11] K. J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, New York: Springer-Verlag, 2000.

[12] J. Hale and O. Lopes, "Fixed point theorems and dissipative processes", J. Differential Equations, vol. 13, no. 2, pp. 391–402, 1973.

[13] H. R. Henríquez, "Periodic solutions of quasi-linear partial functional differential equations with unbounded delay", Funkcial. Ekvac., vol. 37, no. 2, pp. 329–343, 1994.

[14] Y. Hino, T. Naito, N. Van Minh and J. Son Shin, *Almost periodic solutions of differential equations in Banach spaces*, Stability and Control: Theory, Methods and Applications, vol. 15, London: Taylor & Francis, 2002.

[15] Y. Hino, S. Murakami and T. Yoshizawa, "Existence of almost periodic solutions of some functional differential equations in a Banach space", Tohoku Math. J., vol. 49, no. 1, pp. 133–147, 1997.

[16] G. M. N'Guérékata, Hui-Sheng Ding and W. Long, "Existence of pseudo almost periodic solutions for a class of partial functional differential equations", Electron. J. Differential Equations, vol. 2013, no. 104, 14 pages, 2013.

[17] M. Kostić, *Almost periodic and almost automorphic type solutions to integro-differential equations*, Berlin: W. de Gruyter, 2019.

[18] B. M. Levitan, *Almost periodic functions* (in Russian), Moscow: Gosudarstv. Izdat. Tehn.-Teor. Lit., 1953.

[19] R. D. Nussbaum, "The radius of essential spectrum", Duke Math. J., vol. 37, pp. 473–478, 1970.

[20] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, New York: Springer-Verlag, 1983.

[21] M. Schechter, *Principles of functional analysis*, New York: Academic Press, 1971.

[22] J. S. Shin and T. Naito, "Semi-Fredholm operators and periodic solution for linear functional-differential equations", J. Differential Equations, vol. 153, no. 2, pp. 407–441, 1999.

[23] C. C. Travis and G. F. Webb, "Existence and stability for partial functional differential equations", Trans. Amer. Math. Soc., vol. 200, pp. 395–418, 1974.

[24] J. Massera, "The existence of periodic solutions of differential equations", Duke Math. J., vol. 17, pp. 457–475, 1950.

[25] J. Wu, *Theory and applications of partial functional differential equations*, Applied Mathematical Sciences, vol. 119, New York: Springer-Verlag, 1996.

[26] T. Yoshizawa, *Stability theory by Liapunov's second method*, Publications of the Mathematical Society of Japan, vol. 9, Tokyo: Math. Soc. Japan, 1966.

Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 23, no. 03, pp. 489–501, December 2021
DOI: 10.4067/S0719-06462021000300489

Extension of Exton's hypergeometric function K_{16}

Ahmed Ali Atash[1]
Maisoon Ahmed Kulib[2]

[1] *Department of Mathematics, Faculty of Education Shabwah, Aden University, Aden, Yemen.*
ah-a-atash@hotmail.com

[2] *Department of Mathematics, Faculty of Engineering, Aden University, Aden, Yemen.*
maisoonahmedkulib@gmail.com

ABSTRACT

The purpose of this article is to introduce an extension of Exton's hypergeometric function K_{16} by using the extended beta function given by Özergin *et al.* [11]. Some integral representations, generating functions, recurrence relations, transformation formulas, derivative formula and summation formulas are obtained for this extended function. Some special cases of the main results of this paper are also considered.

RESUMEN

El propósito de este artículo es introducir una extensión de la función hipergeométrica de Exton K_{16} usando la función beta extendida dada por Özergin *et al.* [11]. Se obtienen algunas representaciones integrales, funciones generatrices, relaciones de recurrencia, fórmulas de transformación, fórmulas de derivadas y fórmulas de sumación para esta función extendida. Se consideran también algunos casos especiales de los resultados principales de este artículo.

Keywords and Phrases: Extended beta function, Extended Exton's function, Integral representations, Generating functions, Recurrence relation, Transformation formula, Derivative formula, Summation formula.

2020 AMS Mathematics Subject Classification: 33B15, 33C05, 33C15, 33C65.

Accepted: 12 October, 2021
Received: 31 March, 2021

1 Introduction

In recent years, some extensions of beta function and Gauss hypergeometric function have been considered by several authors (see [3, 5, 6, 11]). The following extended beta function and extended Gauss hypergeometric function are introduced by Özergin *et al.* [11]:

$$B_p^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt, \tag{1.1}$$

$$(\Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(p)\geq 0,\ \Re(x)>0,\ \Re(y)>0)$$

and

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.2}$$

$$(\Re(c)>\Re(b)>0,\ |z|<1).$$

They [11] presented the following integral representation:

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt, \tag{1.3}$$

$$\Re(p)>0;\ p=0 \quad \text{and} \quad |arg(1-z)|<\pi;\ \Re(c)>\Re(b)>0.$$

Clearly, we have

$$B_0^{(\alpha,\beta)}(x,y) = B(x,y)$$

and

$$F_0^{(\alpha,\beta)}(a,b;c;z) = {}_2F_1(a,b;c;z),$$

where $B(x,y)$ and ${}_2F_1(z,b;c;z)$ are the classical beta function and Gauss hypergeometric function defined by (see [13])

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad \Re(x)>0, \quad \Re(y)>0 \tag{1.4}$$

and

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \quad c\neq 0,-1,-2,\ldots, \tag{1.5}$$

where $(\lambda)_n$ $(n\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ denotes the Pochhammer's symbol defined by [13]

$$(\lambda)_n = \begin{cases} 1, & n=0 \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & n\in\mathbb{N}. \end{cases} \tag{1.6}$$

Many authors have considered certain interesting extensions of some hypergeometric functions of two and three variables (see [1, 2, 8, 10]). By using the extended beta function in (1.1), Liu [8] defined the extended Appell's function F_1 as follows:

$$F_{1,p}^{(\alpha,\beta)}(a,b,c;d;x,y) = \sum_{m,n=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n,d-a)(b)_m(c)_n}{B(a,d-a)}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.7}$$

and obtained the following integral representation:

$$F_{1,p}^{(\alpha,\beta)}(a,b,c;d;x,y) = \frac{1}{B(a,d-a)} \times \int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right) dt. \quad (1.8)$$

Observe that

$$F_{1,0}^{(\alpha,\beta)}(a,b,c;d;x,y) = F_1(a,b,c;d;x,y),$$

where $F_1(a,b,c;d;x,y)$ is Appell's hypergeometric function [13]

$$F_1(a,b,c;d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \quad (1.9)$$

The Exton's hypergeometric function K_{16} is defined by [7] as follows:

$$K_{16}(a_1,a_2,a_3,a_4;b;x,y,z,t) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+p}(a_3)_{n+q}(a_4)_{p+q}\ \ x^m\ y^n\ z^p\ t^q}{(b)_{m+n+p+q}\ m!\ n!\ p!\ q!}. \quad (1.10)$$

In this paper, we use the extended beta function given in (1.1) to define extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ as follows:

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) = \sum_{m,n,r,s=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a+m+n,e-a+r+s)(b)_{m+r}(c)_{n+s}(d)_{r+s}\ x^m y^n z^r u^s}{B(a,e-a)(e-a)_{r+s}\ \ m!\ n!\ r!\ s!}. \quad (1.11)$$

The extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ given in (1.11) can be written as follows:

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) = \sum_{r,s=0}^{\infty} \frac{(d)_{r+s}(b)_r(c)_s}{(e)_{r+s}} F_{1,p}^{(\alpha,\beta)}(a,b+r,c+s;e+r+s;x,y) \frac{z^r\ u^s}{r!\ s!}. \quad (1.12)$$

Observe that:

- The special case $d = e-a$ of (1.11) yields the following extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}$:

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u) = \sum_{m,n,r,s=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a+m+n,e-a+r+s)(b)_{m+r}(c)_{n+s}\ x^m\ y^n\ z^r\ u^s}{B(a,e-a)\ m!\ n!\ r!\ s!}. \quad (1.13)$$

- The special case $p = 0$ of (1.11) yields the Exton's hypergeometric function K_{16}

$$K_{16,0}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) = K_{16}(a,b,c,d;e;x,y,z,u). \quad (1.14)$$

2 Integral representations

In this section, we present some integral representations for the extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ in (1.11).

Theorem 2.1. *The integral representations (2.1), (2.4), (2.5) of $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ hold for $\Re(p)>0$, $\Re(e)>\Re(a)>0$; $|x|+|z|<1$, $|y|+|u|<1$ and the others hold for $\Re(p)>0$, $\Re(e)>\Re(a)>\Re(d)>0$; $|x|+|z|<1$, $|y|+|u|<1$:*

$$\begin{aligned} &K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) \\ &\quad = \frac{1}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c} \\ &\quad \times F_1\left(d,b,c;e-a;\frac{z(1-t)}{1-xt},\frac{u(1-t)}{1-yt}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt \end{aligned} \tag{2.1}$$

$$\begin{aligned} &K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) \\ &\quad = \frac{1}{B(a,e-a)}\frac{1}{B(d,e-a-d)}\int_0^1\int_0^1 t^{a-1}s^{d-1}(1-t)^{e-a-1}(1-s)^{e-a-d-1} \\ &\quad \times (1-xt-zs(1-t))^{-b}(1-yt-us(1-t))^{-c}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dtds \end{aligned} \tag{2.2}$$

$$\begin{aligned} &K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) = \frac{1}{B(a,e-a)\ B(d,e-a-d)} \\ &\times \int_0^1\int_0^1 t^{a-1}s^{d-1}(1-t)^{e-a-1}(1-s)^{e-a-d-1}(1-zs)^{-b}(1-us)^{-c} \\ &\times \left(1-\left(\frac{x-zs}{1-zs}\right)t\right)^{-b}\left(1-\left(\frac{y-us}{1-us}\right)t\right)^{-c}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dtds \end{aligned} \tag{2.3}$$

$$\begin{aligned} &K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) = \frac{2}{B(a,e-a)} \\ &\times \int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta\cos^{2e-2a-1}\theta(1-x\sin^2\theta)^{-b}(1-y\sin^2\theta)^{-c} \\ &\times F_1\left(d,b,c;e-a;\frac{z\cos^2\theta}{1-x\sin^2\theta},\frac{u\cos^2\theta}{1-y\sin^2\theta}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{\sin^2\theta\cos^2\theta}\right)d\theta \end{aligned} \tag{2.4}$$

$$\begin{aligned} &K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) = \frac{1}{B(a,e-a)} \\ &\times \int_0^{\infty} \xi^{a-1}(1+\xi)^{c+b-e}(1+(1-x)\xi)^{-b}(1+(1-y)\xi)^{-c} \\ &\times F_1\left(d,b,c;e-a;\frac{z}{1+(1-x)\xi},\frac{u}{1+(1-y)\xi}\right){}_1F_1\left(\alpha;\beta;-\frac{p(1+\xi)^2}{\xi}\right)d\xi. \end{aligned} \tag{2.5}$$

Proof of (2.1). Using (1.1) in (1.11) and interchanging the order of summation and integration, we have

$$\begin{aligned}
&K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u) = \frac{1}{B(a,e-a)}\\
&\times \int_0^1 t^{a-1}(1-t)^{e-a-1} \sum_{r,s=0}^{\infty} \frac{(d)_{r+s}(b)_r(c)_s(z(1-t))^r(u(1-t))^s}{(e-a)_{r+s}\ r!\ s!}\\
&\times {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)\left(\sum_{m=0}^{\infty}\frac{(b+r)_m(xt)^m}{m!}\right)\left(\sum_{n=0}^{\infty}\frac{(c+s)_n(yt)^n}{n!}\right)dt\\
&= \frac{1}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c}\\
&\times {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)\left\{\sum_{r,s=0}^{\infty}\frac{(d)_{r+s}(b)_r(c)_s}{(e-a)_{r+s}\ r!\ s!}\left(\frac{z(1-t)}{1-xt}\right)^r\left(\frac{u(1-t)}{1-yt}\right)^s\right\}dt,
\end{aligned}$$

which by applying the definition of Appell hypergeometric function F_1 (1.9), we have the desired result (2.1). The integral representation (2.2) can be obtained easily from (2.1) by using the following integral representation of F_1 [12]:

$$F_1(a,b,c;d;x,y) = \frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}dt. \tag{2.6}$$

Also the integral representation (2.3) can be obtained directly from (2.2) if we use the following relation:

$$(1-xt-z(1-t))^{-a} = (1-z)^{-a}\left(1-\frac{(x-z)t}{1-z}\right)^{-a}. \tag{2.7}$$

Finally, the integral representations (2.4) and (2.5) can be easily obtained by taking the transformations $t=\sin^2\theta$ and $t=\frac{\xi}{1+\xi}$ in (2.1), respectively. This completes the proof of theorem 2.1. □

The special case $d=e-a$ of (2.1), (2.4) and (2.5), yields the following results:

Corollary 2.2.

$$\begin{aligned}
&K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)\\
&\quad= \frac{1}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt-z(1-t))^{-b}(1-yt-u(1-t))^{-c}\\
&\quad\times {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u) = \frac{2}{B(a,e-a)}\\
&\times \int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta\cos^{2e-2a-1}\theta(1-x\sin^2\theta-z\cos^2\theta)^{-b}(1-y\sin^2\theta-u\cos^2\theta)^{-c}\\
&\times {}_1F_1\left(\alpha;\beta;-\frac{p}{\sin^2\theta\cos^2\theta}\right)d\theta
\end{aligned} \tag{2.9}$$

and

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u) = \frac{1}{B(a,e-a)} \times \int_0^\infty \xi^{a-1}(1+\xi)^{c+b-e}(1+(1-x)\xi-z)^{-b}(1+(1-y)\xi-u)^{-c} \times {}_1F_1\left(\alpha;\beta;-\frac{p(1+\xi)^2}{\xi}\right)d\xi. \tag{2.10}$$

3 Generating functions

In this section, we derive certain generating functions for the extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ in (1.11).

Theorem 3.1. *The following generating functions holds true:*

$$\sum_{k=0}^{\infty}\frac{(b)_k t^k}{k!}K_{16,p}^{(\alpha,\beta)}(a,b+k,c,d;e;x,y,z,u) = (1-t)^{-b}K_{16,p}^{(\alpha,\beta)}\left(a,b,c,d;e;\frac{x}{1-t},y,\frac{z}{1-t},u\right) \tag{3.1}$$

$$\sum_{k=0}^{\infty}\frac{(c)_k t^k}{k!}K_{16,p}^{(\alpha,\beta)}(a,b,c+k,d;e;x,y,z,u) = (1-t)^{-c}K_{16,p}^{(\alpha,\beta)}\left(a,b,c,d;e;x,\frac{y}{1-t},z,\frac{u}{1-t}\right) \tag{3.2}$$

$$\sum_{k=0}^{\infty}\frac{(d)_k t^k}{k!}K_{16,p}^{(\alpha,\beta)}(a,b,c,d+k;e;x,y,z,u) = (1-t)^{-d}K_{16,p}^{(\alpha,\beta)}\left(a,b,c,d;e;x,y,\frac{z}{1-t},\frac{u}{1-t}\right). \tag{3.3}$$

Proof of (3.1). Using (1.11) in the L.H.S. of equation (3.1), we get

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(b)_k t^k}{k!}K_{16,p}^{(\alpha,\beta)}(a,b+k,c,d;e;x,y,z,u)\\
&= \sum_{m,n,r,s,k=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n,e-a+r+s)(b)_{m+r+k}(c)_{n+s}(d)_{r+s}\ x^m y^n z^r u^s t^k}{B(a,e-a)(e-a)_{r+s}\ m!\ n!\ r!\ s!\ k!}\\
&= \sum_{m,n,r,s=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n,e-a+r+s)(b)_{m+r}(c)_{n+s}(d)_{r+s}\ x^m y^n z^r u^s}{B(a,e-a)(e-a)_{r+s}\ m!\ n!\ r!\ s!}\sum_{k=0}^{\infty}\frac{(b+m+r)_k t^k}{k!}\\
&= (1-t)^{-b}K_{16,p}^{(\alpha,\beta)}\left(a,b,c,d;e;\frac{x}{1-t},y,\frac{z}{1-t},u\right).
\end{aligned}$$

This completes the proof of (3.1). The generating functions (3.2) and (3.3) can be proved by a similar method as in the proof of (3.1). □

Setting $p=0$ in (3.1), (3.2) and (3.3), we get known results [4].

Theorem 3.2. *The following generating functions holds true:*

$$\sum_{k=0}^{\infty}\frac{(\lambda)_k t^k}{k!}K_{16,p}^{(\alpha,\beta)}(a,b,c,-k;e;x,y,z,u) = (1-t)^{-\lambda}K_{16,p}^{(\alpha,\beta)}\left(a,b,c,\lambda;e;x,y,\frac{-zt}{1-t},\frac{-ut}{1-t}\right) \tag{3.4}$$

$$\sum_{k=0}^{\infty}\frac{(\lambda)_k t^k}{k!}K_{16,p}^{(\alpha,\beta)}(a,b,-k,d;e;x,y,z,u)=(1-t)^{-\lambda}K_{16,p}^{(\alpha,\beta)}\left(a,b,\lambda,d;e;x,\frac{-yt}{1-t},z,\frac{-ut}{1-t}\right) \quad (3.5)$$

$$\sum_{k=0}^{\infty}\frac{(\lambda)_k t^k}{k!}K_{16,p}^{(\alpha,\beta)}(a,-k,c,d;e;x,y,z,u)=(1-t)^{-\lambda}K_{16,p}^{(\alpha,\beta)}\left(a,\lambda,c,d;e;\frac{-xt}{1-t},y,\frac{-zt}{1-t},u\right). \quad (3.6)$$

Proof of (3.4). Using (1.11) in the L.H.S. of equation (3.4) and using the result [13]

$$(-k)_r=\frac{(-1)^r k!}{(k-r)!},\quad 0\le r\le k, \quad (3.7)$$

we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(\lambda)_k t^k}{k!}K_{16,p}^{(\alpha,\beta)}(a,b,c,-k;e;x,y,z,u)\\
&=\sum_{m,n,k=0}^{\infty}\sum_{r=0}^{k}\sum_{s=0}^{k-r}\frac{B_p^{(\alpha,\beta)}(a+m+n,e-a+r+s)(b)_{m+r}(c)_{n+s}(\lambda)_k\ x^m y^n(-z)^r(-u)^s t^k}{B(a,e-a)(e-a)_{r+s}\ m!\ n!\ r!\ s!\ (k-r-s)!}\\
&=\sum_{m,n,r,s,k=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n,e-a+r+s)(b)_{m+r}(c)_{n+s}(\lambda)_{k+r+s}\ x^m y^n(-zt)^r(-ut)^s t^k}{B(a,e-a)(e-a)_{r+s}\ m!\ n!\ r!\ s!\ k!}\\
&=\sum_{m,n,r,s=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n,e-a+r+s)(b)_{m+r}(c)_{n+s}(\lambda)_{r+s}\ x^m y^n(-zt)^r(-ut)^s}{B(a,e-a)(e-a)_{r+s}\ m!\ n!\ r!\ s!}\sum_{k=0}^{\infty}\frac{(\lambda+r+s)_k t^k}{k!}\\
&=(1-t)^{-\lambda}K_{16,p}^{(\alpha,\beta)}\left(a,b,c,\lambda;e;x,y,\frac{-zt}{1-t},\frac{-ut}{1-t}\right).
\end{aligned}$$

This completes the proof of (3.4). The generating functions (3.5) and (3.6) can be proved by a similar method as in the proof of (3.4). □

4 Recurrence relations

In this section, we deduce some recurrence relations for the extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ in (1.11) by using the recurrence relations of the confluent function ${}_1F_1$ and Appell's function F_1.

Theorem 4.1. *The following recurrence relation holds true:*

$$\begin{aligned}
&K_{16,p}^{(\alpha,\beta)}(a,b,c,d+1;e;x,y,z,u)-K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)\\
&-\frac{bz}{e}K_{16,p}^{(\alpha,\beta)}(a,b+1,c,d+1;e+1;x,y,z,u)-\frac{cu}{e}K_{16,p}^{(\alpha,\beta)}(a,b,c+1,d+1;e+1;x,y,z,u)=0
\end{aligned} \quad (4.1)$$

Proof. To prove Theorem 4.1, we consider the following recurrence relation of Appell's function F_1 [14]:

$$\begin{aligned}
F_1(\alpha+1,\beta_1,\beta_2;\gamma;x,y)-F_1(\alpha,\beta_1,\beta_2;\gamma;x,y)-\frac{x\beta_1}{\gamma}F_1(\alpha+1,\beta_1+1,\beta_2;\gamma+1;x,y)\\
-\frac{y\beta_2}{\gamma}F_1(\alpha+1,\beta_1,\beta_2+1;\gamma+1;x,y)=0
\end{aligned} \quad (4.2)$$

In (4.2) replacing α, β_1, β_2, γ, x, y by d, b, c, $e-a$, $\frac{z(1-t)}{1-xt}$, $\frac{u(1-t)}{1-yt}$ respectively, multiplying both sides by $\frac{1}{B(a,e-a)} t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c} {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)$ and integrating the resulting equation with respect to t between the limits 0 to 1, we get

$$\begin{aligned}
&\frac{1}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c}\\
&\times F_1\left(d+1,b,c;e-a;\frac{z(1-t)}{1-xt},\frac{u(1-t)}{1-yt}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt\\
&-\frac{1}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c}\\
&\times F_1\left(d,b,c;e-a;\frac{z(1-t)}{1-xt},\frac{u(1-t)}{1-yt}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt\\
&-\frac{bz}{(e-a)B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a}(1-xt)^{-b-1}(1-yt)^{-c}\\
&\times F_1\left(d+1,b+1,c;e-a+1;\frac{z(1-t)}{1-xt},\frac{u(1-t)}{1-yt}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt\\
&-\frac{cu}{(e-a)B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a}(1-xt)^{-b}(1-yt)^{-c-1}\\
&\times F_1\left(d+1,b,c+1;e-a+1;\frac{z(1-t)}{1-xt},\frac{u(1-t)}{1-yt}\right){}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt=0.
\end{aligned}$$

Finally, using the integral representation (2.1), we get the desired result (4.1). □

Theorem 4.2. *The following recurrence relations hold true:*

(i)

$$\begin{aligned}
&(\beta-\alpha)K_{16,p}^{(\alpha-1,\beta)}(a,b,c,e-a;e;x,y,z,u)-\alpha K_{16,p}^{(\alpha+1,\beta)}(a,b,c,e-a;e;x,y,z,u)\\
&+(2\alpha-\beta)K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)\\
&+\frac{pB(a-1,e-a-1)}{B(a,e-a)}K_{16,p}^{(\alpha,\beta)}(a-1,b,c,e-a-1;e-2;x,y,z,u)=0
\end{aligned}\tag{4.3}$$

(ii)

$$\begin{aligned}
&\beta(\beta-1)K_{16,p}^{(\alpha,\beta-1)}(a,b,c,e-a;e;x,y,z,u)-\beta(\beta-1)K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)\\
&-\frac{\beta pB(a-1,e-a-1)}{B(a,e-a)}K_{16,p}^{(\alpha,\beta)}(a-1,b,c,e-a-1;e-2;x,y,z,u)\\
&+\frac{p(\alpha-\beta)B(a-1,e-a-1)}{B(a,e-a)}K_{16,p}^{(\alpha,\beta+1)}(a-1,b,c,e-a-1;e-2;x,y,z,u)=0
\end{aligned}\tag{4.4}$$

(iii)

$$\begin{aligned}
&\alpha\beta K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)-\alpha\beta K_{16,p}^{(\alpha+1,\beta)}(a,b,c,e-a;e;x,y,z,u)\\
&+\frac{p\beta B(a-1,e-a-1)}{B(a,e-a)}K_{16,p}^{(\alpha,\beta)}(a-1,b,c,e-a-1;e-2;x,y,z,u)\\
&-\frac{p(\beta-\alpha)B(a-1,e-a-1)}{B(a,e-a)}K_{16,p}^{(\alpha,\beta+1)}(a-1,b,c,e-a-1;e-2;x,y,z,u)=0
\end{aligned}\tag{4.5}$$

(iv)

$$\beta K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)-\beta K_{16,p}^{(\alpha-1,\beta)}(a,b,c,e-a;e;x,y,z,u)$$
$$+\frac{pB(a-1,e-a-1)}{B(a,e-a)}K_{16,p}^{(\alpha,\beta+1)}(a-1,b,c,e-a-1;e-2;x,y,z,u)=0 \tag{4.6}$$

(v)

$$(\beta-\alpha-1)K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)+\alpha K_{16,p}^{(\alpha+1,\beta)}(a,b,c,e-a;e;x,y,z,u)$$
$$-(\beta-1)K_{16,p}^{(\alpha,\beta-1)}(a,b,c,e-a;e;x,y,z,u)=0 \tag{4.7}$$

(vi)

$$(\alpha-1)K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)$$
$$+\frac{pB(a-1,e-a-1)}{B(a,e-a)}K_{16,p}^{(\alpha,\beta)}(a-1,b,c,e-a-1;e-2;x,y,z,u)$$
$$+(\beta-\alpha)K_{16,p}^{(\alpha-1,\beta)}(a,b,c,e-a;e;x,y,z,u)$$
$$-(\beta-1)K_{16,p}^{(\alpha,\beta-1)}(a,b,c,e-a;e;x,y,z,u)=0. \tag{4.8}$$

Proof. To prove our results in Theorem 4.2, we require the following recurrence relations of the confluent function ${}_1F_1$ [9]:

$$(\beta-\alpha){}_1F_1(\alpha-1;\beta;z)-\alpha\ {}_1F_1(\alpha+1;\beta;z)+(2\alpha-\beta+z){}_1F_1(\alpha;\beta;z)=0 \tag{4.9}$$

$$\beta(\beta-1){}_1F_1(\alpha;\beta-1;z)-\beta(\beta-1+z){}_1F_1(\alpha;\beta;z)+(\beta-\alpha)z{}_1F_1(\alpha;\beta+1;z)=0 \tag{4.10}$$

$$\beta(\alpha+z){}_1F_1(\alpha;\beta;z)-\alpha\ \beta\ {}_1F_1(\alpha+1;\beta;z)-(\beta-\alpha)z\ {}_1F_1(\alpha;\beta+1;z)=0 \tag{4.11}$$

$$\beta\ {}_1F_1(\alpha;\beta;z)-\beta\ {}_1F_1(\alpha-1;\beta;z)-z\ {}_1F_1(\alpha;\beta+1;z)=0 \tag{4.12}$$

$$(\beta-\alpha-1){}_1F_1(\alpha;\beta;z)+\alpha\ {}_1F_1(\alpha+1;\beta;z)-(\beta-1){}_1F_1(\alpha;\beta-1;z)=0 \tag{4.13}$$

$$(\alpha+z-1){}_1F_1(\alpha;\beta;z)+(\beta-\alpha){}_1F_1(\alpha-1;\beta;z)-(\beta-1){}_1F_1(\alpha;\beta-1;z)=0. \tag{4.14}$$

Proof of (4.3). In (4.9) replacing z by $-\frac{p}{t(1-t)}$, multiplying both sides by $t^{a-1}(1-t)^{e-a-1}(1-xt-z(1-t))^{-b}(1-yt-u(1-t))^{-c}/B(a,e-a)$ and integrating the resulting equation with respect to t between the limits 0 to 1, we get

$$\frac{\beta-\alpha}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt-z(1-t))^{-b}(1-yt-u(1-t))^{-c}{}_1F_1\left(\alpha-1;\beta;-\frac{p}{t(1-t)}\right)dt$$
$$-\frac{\alpha}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt-z(1-t))^{-b}(1-yt-u(1-t))^{-c}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt$$
$$+\frac{2\alpha-\beta}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt-z(1-t))^{-b}(1-yt-u(1-t))^{-c}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt$$
$$+\frac{p}{B(a,e-a)}\int_0^1 t^{a-2}(1-t)^{e-a-2}(1-xt-z(1-t))^{-b}(1-yt-u(1-t))^{-c}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt=0$$

Finally, using the integral representation (2.8), we get the desired result (4.3). □

The results (4.4)-(4.8) can be proved by a similar method as in the proof of (4.3) and we use here the recurrence relations (4.10)-(4.14). □

5 Transformation, differentiation and summation formulas

In this section, we derive certain transformation, derivative and summation formulas for the extended Exton's hypergeometric function $K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)$ in (1.11).

Theorem 5.1. *The following transformation formula of* $K_{16,p}^{\alpha,\beta}(a,b,c,d;e;x,y,z,u)$ *holds true:*

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)=(1-z)^{-b}(1-u)^{-c}F_{1,p}^{(\alpha,\beta)}\left(a,b,c;e;\frac{x-z}{1-z},\frac{y-u}{1-u}\right). \tag{5.1}$$

Proof. Using (2.7) in (2.8), we have

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,e-a;e;x,y,z,u)=\frac{(1-z)^{-b}(1-u)^{-c}}{B(a,e-a)}$$
$$\times\int_0^1 t^{a-1}(1-t)^{e-a-1}\left(1-\left(\frac{x-z}{1-z}\right)t\right)^{-b}\left(1-\left(\frac{y-u}{1-u}\right)t\right)^{-c}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt,$$

which by using (1.8), we get the desired result (5.1). □

Setting $p=0$ in (5.1), we get a known result [7]

$$K_{16}(a,b,c,e-a;e;x,y,z,u)=(1-z)^{-b}(1-u)^{-c}F_1\left(a,b,c;e;\frac{x-z}{1-z},\frac{y-u}{1-u}\right). \tag{5.2}$$

Theorem 5.2. *The following derivative formula holds true:*

$$\frac{d^{m+n+r+s}}{dx^m\,dy^n\,dz^r\,du^s}\left\{K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)\right\}=\frac{(a)_{m+n}(b)_{m+r}(c)_{n+s}(d)_{r+s}}{(e)_{m+n+r+s}}$$
$$\times K_{16,p}^{(\alpha,\beta)}(a+m+n,b+m+r,c+n+s,d+r+s;e+m+n+r+s;x,y,z,u). \tag{5.3}$$

Proof. Differentiating (1.11) with respect to x,y,z and u, we have

$$\frac{d}{dx\,dy\,dz\,du}\left\{K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)\right\}$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{B_p(a+m+n,e-a+r+s)(b)_{m+r}(c)_{n+s}(d)_{r+s}x^{m-1}y^{n-1}z^{r-1}u^{s-1}}{B(a,e-a)(e-a)_{r+s}(m-1)!(n-1)!(r-1)!(s-1)!}$$

setting $m\to m+1,\ n\to n+1,\ r\to r+1,\ s\to s+1$ and using the following identities:

$$B(a,e-a)=\frac{e}{a}B(a+1,e-a)=\frac{e(e+1)}{a(a+1)}B(a+2,e-a),$$
$$(a)_{p+q+2}=a(a+1)(a+2)_{p+q},$$

we obtain

$$\frac{d}{dx\,dy\,dz\,du}\left\{K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)\right\}=\frac{(a)_2(b)_2(c)_2(d)_2}{(e)_2(e-a)_2}$$
$$\times\sum_{m,n,r,s=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(a+m+n+2,e-a+r+s+2)(b+2)_{m+r}(c+2)_{n+s}(d+2)_{r+s}x^my^nz^ru^s}{B(a+2,e-a)(e-a+2)_{r+s}\ m!\ n!\ r!\ s!}$$

Now using

$$B(a+2,e-a)=\frac{(e)_4}{(e)_2(e-a)_2}B(a+2,e-a+2),$$

we have

$$\frac{d}{dx\,dy\,dz\,du}\left\{K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;x,y,z,u)\right\}=\frac{(a)_2(b)_2(c)_2(d)_2}{(e)_4}$$
$$\times K_{16,p}^{(\alpha,\beta)}(a+2,b+2,c+2,d+2;e+4;x,y,z,u).$$

Thus by repeatedly differentiating, we find that the result (5.3) can be derived by induction. □

Theorem 5.3. *The following summation formulas hold true:*

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;1,1,1,1)=\frac{\Gamma(e)\Gamma(e-a-b-c-d)}{\Gamma(a)\Gamma(e-a-d)\Gamma(e-a-b-c)}B_p^{(\alpha,\beta)}(a,e-a-b-c) \tag{5.4}$$

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,d;1+a+b+d-c;1,1,1,-1)$$
$$=\frac{\Gamma(1-c)\Gamma(1+\frac{1}{2}d)\Gamma(1+a+b+d-c)}{\Gamma(a)\Gamma(1+d)\Gamma(1+b-c)\Gamma(1+\frac{1}{2}d-c)}B_p^{(\alpha,\beta)}(a,d-2c+1). \tag{5.5}$$

Proof. Setting $x=y=z=u=1$ in (2.1) and using the following formula:

$$F_1(a,b,c;d;1,1)=\frac{\Gamma(d)\Gamma(d-a-b-c)}{\Gamma(d-a)\Gamma(d-b-c)}, \tag{5.6}$$

we get

$$K_{16,p}^{(\alpha,\beta)}(a,b,c,d;e;1,1,1,1)=\frac{\Gamma(e)\Gamma(e-a-b-c-d)}{\Gamma(a)\Gamma(e-a-d)\Gamma(e-a-b-c)}$$
$$\times\int_0^1 t^{a-1}(1-t)^{e-a-b-c-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt \tag{5.7}$$

Now, by using (1.1) in (5.7), we obtain the desired result (5.4). The summation formula (5.5) can be obtained easily by putting $e=1+a+b+d-c,\ x=y=z=1,\ u=-1$ in (2.1) and using the formula

$$F_1(a,b,c;1+a+b-c;1,-1)=\frac{\Gamma(1-c)\Gamma(1+\frac{1}{2}a)\Gamma(1+a+b-c)}{\Gamma(1+a)\Gamma(1+b-c)\Gamma(1+\frac{1}{2}a-c)}. \tag{5.8}$$

This completes the proof of the theorem (5.3). □

Setting $p=0$ in (5.4) and (5.5), we get respectively the following summation formulas of Exton's hypergeometric function K_{16}:

$$K_{16}(a,b,c,d;e;1,1,1,1)=\frac{\Gamma(e)\Gamma(e-a-b-c-d)}{\Gamma(e-a-d)\Gamma(e-b-c)} \tag{5.9}$$

and

$$K_{16}(a,b,c,d;1+a+b+d-c;1,1,1,-1)=\frac{\Gamma(1-c)\Gamma(1+\frac{1}{2}d)\Gamma(1+a+b+d-c)\Gamma(d-2c+1)}{\Gamma(1+d)\Gamma(1+b-c)\Gamma(1+\frac{1}{2}d-c)\Gamma(a+d-2c+1)}. \tag{5.10}$$

6 Conclusion

In this paper, we have introduced the extended Exton's hypergeometric function $K_{16,p}^{\alpha,\beta}(a,b,c,d;e;x,y,z,u)$ by using the extended beta function $B_p^{\alpha,\beta}(x,y)$ given by Özergin *et al.* [11]. For this function we have presented some integral representations, generating functions, recurrence relations, transformation formulas, derivative formula and summation formulas. We have also established some a known and new generating functions, transformation formulas, and summation formulas for the classical Exton's hypergeometric function $K_{16}(a,b,c,d;e;x,y,z,u)$.

Acknowledgements

The authors are thankful to the referees for useful comments and suggestions towards the improvement of this paper.

References

[1] P. Agarwal, J. Choi and S. Jain, "Extended hypergeometric functions of two and three variables", Commun. Korean Math. Soc., vol. 30, no. 4, pp. 403–414, 2015.

[2] R. P. Agarwal, M. J. Luo and P. Agarwal, "On the extended Appell-Lauricella hypergeometric functions and their applications", Filomat, vol. 31, no. 12, pp. 3693–3713, 2017.

[3] A. Çetinkaya, I. O. Kıymaz, P. Agarwal and R. Agarwal, "A comparative study on generating function relations for generalized hypergeometric functions via generalized fractional operators", Adv. Difference Equ., vol. 2018, paper no. 156, pp. 1–11, 2018.

[4] R. C. Singh Chandel and A. Tiwari, "Generating relations involving hypergeometric functions of four variables", Pure Appl. Math. Sci., vol. 36, no. 1-2, pp. 15–25, 1991.

[5] M. A. Chaudhry, A. Qadir, M. Rafique and S. M. Zubair, "Extension of Euler's beta function", J. Comp. Appl. Math., vol. 78, no. 1, pp. 19–32, 1997.

[6] M. A. Chaudhry, A. Qadir, H. M. Srivastava and R. B. Paris, "Extended hypergeometric and confluent hypergeometric functions", Appl. Math. Comp., vol. 159, no. 2, pp. 589–602, 2004.

[7] H. Exton, *Multiple hypergeometric functions and applications*, New York: Halsted Press, 1976.

[8] H. Liu, "Some generating relations for extended Appell's and Lauricella's hypergeometric functions", Rocky Mountain J. Math., vol. 44, no. 6, pp. 1987–2007, 2014.

[9] Y. L. Luke, *The Special Functions and Their Approximations*, New York: Academic Press, 1969.

[10] M. A. Özarslan and E. Özergin, "Some generating relations for extended hypergeometric functions via generalized fractional derivative operator", Math. Comput. Modelling, vol. 52, no. 9-10, pp. 1825–1833, 2010.

[11] E. Özergin, M. A. Özarslan, and A. Altin, "Extension of gamma, beta and hypergeometric functions", J. Comp. Appl. Math., vol. 235, no. 16, pp. 4601–4610, 2011.

[12] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian hypergeometric Series*, New York: Halsted Press, 1985.

[13] H. M. Srivastava and H. L. Manocha, *A treatise on generating functions*, New York: Halsted Press, 1984.

[14] X. Wang, "Recursion formulas for Appell functions", Integral Transforms Spec. Funct., vol. 23, no. 6, pp. 421–433, 2012.

CUBO

A Mathematical Journal

All papers submitted to CUBO are pre-evaluated by the Editorial Board, who can decide to reject those articles considered imprecise, unsuitable or lacking in mathematical soundness. These manuscripts will not continue the editorial process and will be returned to their author(s).

Those articles that fulfill CUBO's editorial criteria will proceed to an external evaluation. These referees will write a report with a recommendation to the editors regarding the possibility that the paper may be published. The referee report should be received within 90 days. If the paper is accepted, the authors will have 15 days to apply all modifications suggested by the editorial board.

The final acceptance of the manuscripts is decided by the Editor-in-chief and the Managing editor, based on the recommendations by the referees and the corresponding Associate editor. The author will be formally communicated of the acceptance or rejection of the manuscript by the Editor-in-chief.

All opinions and research results presented in the articles are of exclusive responsibility of the authors.

Submitting: By submitting a paper to this journal, authors certify that the manuscript has not been previously published nor is it under consideration for publication by another journal or similar publication. Work submitted to CUBO will be refereed by specialists appointed by the Editorial Board of the journal.

Manuscript: Manuscripts should be written in English and submitted in duplicate to cubo@ufrontera.cl. The first page should contain a short descriptive title, the name(s) of the author(s), and the institutional affiliation and complete address (including e-mail) of each author. Papers should be accompanied by a short abstract, keywords and the 2020 AMS Mathematical Subject Classification codes corresponding to the topic of the paper. References are indicated in the text by consecutive Arabic numerals enclosed in square brackets. The full list should be collected and typed at the end of the paper in numerical order.

Press requirement: The abstract should be no longer than 250 words. CUBO strongly encourages the use of LaTeX for manuscript preparation. References should be headed numerically, alphabetically organized and complete in all its details. Authors' initials should precede their names; journal title abbreviations should follow the style of Mathematical Reviews.

For technical questions about CUBO, please send an e-mail to cubo@ufrontera.cl.

Cubo

A Mathematical Journal

343 **Non-algebraic limit cycles in Holling type III zooplankton-phytoplankton models**
Homero G. Díaz-Marín, Osvaldo Osuna

357 **Basic asymptotic estimates for powers of Wallis' ratios**
Vito Lampret

369 **The structure of extended function groups**
Rubén A. Hidalgo

385 **Entropy solution for a nonlinear parabolic problem with homogeneous Neumann boundary condition involving variable exponents**
U. Traoré

411 **Independent partial domination**
L. Philo Nithya, Joseph Varghese Kureethara

423 **Foundations of generalized Prabhakar-Hilfer fractional calculus with applications**
George A. Anastassiou

441 **Existence and uniqueness of solutions to discrete, third-order three-point boundary value problems**
Saleh S. Almuthaybiri, Jagan Mohan Jonnalagadda, Christopher C. Tisdell

457 **Some integral inequalities related to Wirtinger's result for p-norms**
S. S. Dragomir

469 **On the periodic solutions for some retarded partial differential equations by the use of semi-Fredholm operators**
Abdelhai Elazzouzi, Khalil Ezzinbi, Mohammed Kriche

489 **Extension of exton's hypergeometric function K_{16}**
Ahmed Ali Atash, Maisoon Ahmed Kulib

VOLUME 23 · ISSUE 3 · DECEMBER 2021

UNIVERSIDAD DE LA FRONTERA