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Numerical analysis of nonlinear parabolic problems with variable exponent and L^1 data

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ABSTRACT

In this paper, we make the numerical analysis of the mild solution which is also an entropy solution of parabolic problem involving the $p(x)-$Laplacian operator with L^1- data.

RESUMEN

En este artículo, realizamos el análisis numérico de la solución mild que también es una solución de entropía del problema parabólico involucrando el operador $p(x)-$Laplaciano con datos en L^1.

Keywords and Phrases: Elliptic-parabolic, numerical iterative method, variable exponent, mild solution, renormalized solution.

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1 Introduction

We consider a bounded open domain $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz boundary denoted by $\partial\Omega$. Let $T > 0$ and $p : \overline{\Omega} \to (1, \infty)$ be a continuous function. In this paper, one of our main goals is the numerical approximation of the mild solution of the following nonlinear parabolic problem involving the $p(x)-$Laplacian operator

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f & \text{in } Q \equiv \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$. The assumptions on the variable exponent $p(x)$ will be specified later.

Partial differential equations with nonlinearities involving non-constant exponents have attracted an increasing amount of attention on recent years. Their study is an interesting topic which raises many mathematical difficulties (see [1, 2, 14, 16, 27, 30]). There are many results devoted to questions on existence and uniqueness of solutions to problems like (1.1), we refer for example the reader to the bibliography [3, 4, 5, 9, 24, 29] and references therein. Many of these models have already been analyzed for constant exponents of nonlinearity (see the references therein), but it seems to be more realistic to assume the exponent to be variable. From numerical point of view, in the classical evolution problem case where $p(x) \equiv p$, the numerical analysis was firstly considered in [7, 22]. Afterward, Jäger and Kačur [18] and Kačur [20] studied the numerical approximation. Inspired by these works, Maitre [23] proposed a numerical scheme to approximate the mild solutions. On the other side, for problems with variable exponent, in recent years, there are some papers devoted to their numerical analysis (see for example [8, 10, 12, 13, 17, 19, 26]). Thus, in [13] the authors used a quasi-Newton minimization method to approach the solution of the $p(x)-$Lapacian problems; in [12], they present an inverse power method to compute the first homogeneous eigenpair. In [26], an interior penalty discontinuous Galerkin method has been used by the authors to approximate the minimizer of a variational problem related to the $p(x)-$Laplacian. Other authors use finite elements to approximate the solution (see [10]). Nevertheless, there are scarcely papers about the numerical analysis of nonlinear parabolic problems with variable exponent (see for example [11]).

The importance of investigating the problem (1.1) lies in their occurrence in modeling various physical problems involving strong anisotropic phenomena related to electrorheological fluids (an important class of non-Newtonian fluids, see [27]) which are characterized by their ability to change the mechanical properties under the influence of the exterior electromagnetic field. Other important applications are related to image processing, elasticity [30], the processes of filtration in complex media, stratigraphy problems and also mathematical biology. The study of problem (1.1) involves using of generalized Lebesgue and Sobolev spaces *i.e.*, $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ respectively (see [15]).

Throughout this paper we assume that the exponent $p(.)$ appearing in (1.1) is a continuous function $p : \overline{\Omega} \to (1, \infty)$ such that:

$$\begin{cases} \exists C > 0 \ : |p(x) - p(y)| \leq \dfrac{C}{-\log|x-y|} \text{ for every } x, y \in \Omega \text{ with } |x-y| \leq \dfrac{1}{2} \\ \\ \dfrac{2d}{d+2} < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p^+ := \max_{x\in\overline{\Omega}} p(x) < \infty. \end{cases} \tag{1.2}$$

The first condition says that $p(.)$ belongs to the class of log-Hölder continuous functions. These assumptions are used to obtain several regularity results for Sobolev spaces with variable exponents; in particular, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(.)}(\Omega)$ and $W_0^{1,p(.)}(\Omega) = W^{1,p(.)}(\Omega) \cap W_0^{1,1}(\Omega)$.

Our paper was inspired by the work of Maitre (see [23]) where the author studied the numerical analysis of an elliptic-parabolic problem in the context of constant exponent setting.

The rest of this paper is organized as follows: in Section 2, we give some results for the study of (1.1). In Section 3, we recall the notion of mild solution. In Section 4, we proceed to the numerical study, where we show the existence and uniqueness of solution of numerical scheme for the approximation of mild solution and the study of the convergence of this numerical scheme. We conclude this section by numerical tests.

2 Preliminaries

We first recall in what follows some definitions and basic properties of generalized Lebesgue-Sobolev spaces with variable exponent. We define the Lebesgue space with a variable exponent $p(.)$ by

$$L^{p(.)}(\Omega) = \left\{u : \Omega \to \mathbb{R};\ u \text{ is measurable with } \rho_{p(.)}(u) < \infty\right\},$$

where

$$\rho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)} dx,$$

is called a modular. We define a norm, the so-called *Luxemburg norm,* on this space by the formula

$$|u|_{p(.)} = \inf\left\{\mu > 0 :\ \rho_{p(.)}\left(\frac{u}{\mu}\right) \leq 1\right\}.$$

The space $(L^{p(.)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(.)} |v|_{p'(.)}$$

for all $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$.

We define also the variable Sobolev space

$$W^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) : \ |\nabla u| \in L^{p(.)}(\Omega) \right\}.$$

On $W^{1,p(.)}(\Omega)$ we may consider the following norm

$$\|u\|_{1,p(.)} = |u|_{p(.)} + |\nabla u|_{p(.)}.$$

The space $(W^{1,p(.)}(\Omega), \|u\|_{1,p(.)})$ is a separable and reflexive Banach space. Next, we define $W_0^{1,p(.)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$ under the norm

$$\|u\| := |\nabla u|_{p(.)}.$$

The space $(W_0^{1,p(.)}(\Omega), \|u\|)$ is a separable and reflexive Banach space. For the interested reader, more details about Lebesgue and Sobolev spaces with variable exponent can be found in [15] (see also [21]).

Since Ω is bounded and $p : \Omega \to (1, \infty)$ is log-Hölder continuous, the Poincaré inequality holds (see [28])

$$|u|_{p(.)} \leq C |\nabla u|_{p(.)}, \quad \forall\, u \in W_0^{1,p(.)}(\Omega),$$

where C is a constant which depends on Ω and on the function p.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by modular $\rho_{p(.)}$ of the space $L^{p(.)}$. We have the following result (see [28]).

Lemma 2.1. *If u_n, $u \in L^{p(.)}$ and $p^+ < \infty$, then the following relations hold:*

(1) $|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p^-} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p^+}$;

(2) $|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p^+} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p^-}$;

(3) $|u|_{p(.)} < 1$ (*respectively* $= 1; > 1$) $\Longleftrightarrow \rho_{p(.)}(u) < 1$ (*respectively* $= 1; > 1$);

(4) $|u|_{p(.)} \to 0$ (*respectively* $\to \infty$) $\Longleftrightarrow \rho_{p(.)}(u) \to 0$ (*respectively* $\to \infty$);

(5) $\rho_{p(.)}\left(u/|u|_{p(.)}\right) = 1$.

Following [4], we extend a variable exponent $p : \overline{\Omega} \to [1, +\infty)$ to $\overline{Q} = [0, T] \times \overline{\Omega}$ by setting $p(t, x) := p(x)$ for all $(t, x) \in \overline{Q}$. We also consider the generalized Lebesgue space

$$L^{p(.)}(Q) = \left\{ u : Q \to \mathbb{R} \text{ measurable such that } \iint_Q |u(x,t)|^{p(x)}\, d(x,t) < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^{p(.)}} := \inf \left\{ \mu > 0 : \ \iint_Q \left| \frac{u(x,t)}{\mu} \right|^{p(x)} d(x,t) < 1 \right\}$$

which shares the same properties as $L^{p(.)}(\Omega)$.

Now, we recall the main results for the study of (1.1).

In order to approximate the mild solution of (1.1), let us recall that Ouaro and Traoré have studied in [25] the existence and uniqueness of weak energy and entropy solutions of the following stationary problem associated to the problem (1.1)

$$\begin{cases} u - \operatorname{div} a(x, \nabla u) = f & \text{in} \quad \Omega, \\ \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary and $f \in L^1(\Omega)$. For the vector field $a(x, \xi) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, in addition to be Carathéodory, is the continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, *i.e.* $a(x, \xi) = \nabla_\xi A(x, \xi)$ such that:

$$A(x, 0) = 0 \quad \text{for almost every } x \in \Omega. \tag{2.2}$$

There exists a positive constant C_1 such that

$$|a(x, \xi)| \leq C_1(j(x) + |\xi|^{p(x)-1}), \tag{2.3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^d$ where j is a non-negative function in $L^{p'(.)}(\Omega)$, with $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.

The following inequalities hold

$$(a(x, \xi) - a(x, \eta)).(\xi - \eta) > 0, \tag{2.4}$$

for almost every $x \in \Omega$ and for every ξ, $\eta \in \mathbb{R}^d$, with $\xi \neq \eta$ and

$$\frac{1}{C}|\xi|^{p(x)} \leq a(x, \xi).\xi \leq Cp(x)A(x, \xi), \tag{2.5}$$

for almost every $x \in \Omega$, $C > 0$ and for every $\xi \in \mathbb{R}^d$.

The exponent appearing in (2.3) and (2.5) is defined as follows.

$$\begin{cases} p(.) : \Omega \to \mathbb{R} \;\; \text{is a measurable function such that} \\ \\ 1 < p^- := ess\inf_{x \in \Omega} p(x) \leq p^+ := ess\sup_{x \in \Omega} p(x) < \infty. \end{cases} \tag{2.6}$$

For more details, see [24, 25].

As example of models with respect to above assumptions, we can give the following.

Set $A(x, \xi) = \dfrac{1}{p(x)}|\xi|^{p(x)}$, $a(x, \xi) = |\xi|^{p(x)-2}\xi$. Then, we get the $p(x)-$Laplace operator

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u).$$

Note that the weak solution of (2.1) is defined as follows.

Definition 2.2. *A weak solution of* (2.1) *is a function* $u \in W_0^{1,1}(\Omega)$ *such that* $a(.,\nabla u) \in \left(L^1_{loc}(\Omega)\right)^d$ *and*

$$\int_\Omega a(.,\nabla u).\nabla\varphi\,dx + \int_\Omega u\varphi\,dx = \int_\Omega f\varphi\,dx, \tag{2.7}$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

A weak energy solution is a weak solution such that $u \in W_0^{1,p(.)}(\Omega)$.

Now, we recall one of main results.

Theorem 2.3. *Assume that* (2.2)–(2.6) *hold and* $f \in L^\infty(\Omega)$. *Then there exists a unique weak energy solution of* (2.1).

We also recall a useful result needed in this paper (see [23]).

Lemma 2.4 ([23])**.** *Let* X *be a Banach space and* C *a convex subset of* X, *containing* 0. *Let* $\bar{T}$ *be a non-expansive map on* C *such that* $\bar{T}(C) \subset C$, *admitting a unique fixed point* x^* *in* C. *Let* λ_k *be a sequence of* $(0,1)$ *verifying*

$$\lim_{k\to\infty} \lambda_k = 1, \quad \prod_{k\geq 0} \lambda_k = 0, \quad \sum_{k\geq 0} |\lambda_{k+1} - \lambda_k| < \infty.$$

Then the sequence (x^k) *generated by the iterative scheme*

$$x^0 \in C, \quad x^{k+1} = \lambda_{k+1}\bar{T}(x^k) \tag{2.8}$$

verifies $\lim_{k\to\infty} x^k - \bar{T}(x^k) = 0$. *Consequently, if all subsequences of* (x^k) *have in turn a subsequence converging to a point of* C, *then the whole sequence* (x^k) *converges toward* x^*.

Recall that a self-mapping $\bar{T}$ *of* C *is non-expansive if*

$$\|\bar{T}(x) - \bar{T}(y)\| \leq \|x - y\| \text{ for all } x, y \in C.$$

In the next section, we give the definition of mild solution.

3 Notion of mild solution

Let $f \in L^1(0,T;L^1(\Omega))$, $u_0 \in L^1(\Omega)$ and $\varepsilon > 0$ be given. We consider the time discretization of problem (1.1) by an implicit Euler scheme

$$\begin{cases} \dfrac{u^\varepsilon_{n+1} - u^\varepsilon_n}{t_{n+1} - t_n} - \operatorname{div}(|\nabla u^\varepsilon_{n+1}|^{p(x)-2}\nabla u^\varepsilon_{n+1}) = f^\varepsilon_{n+1} \text{ in } \mathcal{D}'(\Omega) \text{ for } n = 0,\ldots,N-1, \\ \\ u^\varepsilon_{n+1} \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega); \end{cases} \tag{3.1}$$

where

$$\begin{cases} N \in \mathbb{N}^*, 0 = t_0 < t_1 < \cdots < t_N \leq T \text{ is a partition of } [0,T]. \\ \\ f_n^\varepsilon \in L^\infty(\Omega) \text{ for } n = 1,\ldots,N \text{ such that } \displaystyle\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|f(t) - f_n^\varepsilon\|_{L^1(\Omega)} dt \to 0 \text{ as } \varepsilon \to 0, \\ \\ \max_{n=1,\ldots,N}(t_n - t_{n-1}) \to 0, T - t_N \to 0 \text{ as } \varepsilon \to 0,\ u_0^\varepsilon \in L^\infty(\Omega) \text{ such that} \\ \\ \|u_0 - u_0^\varepsilon\|_{L^1(\Omega)} \to 0 \text{ as } \varepsilon \to 0, \\ \\ \text{with } u^\varepsilon \text{ the piecewise constant function defined by} \\ \\ u^\varepsilon(t) = u_n^\varepsilon \text{ on } (t_{n-1}, t_n] \text{ with } n = 1,\ldots,N; \quad u^\varepsilon(0) = u_0^\varepsilon. \end{cases} \tag{3.2}$$

Definition 3.1. *A mild solution of* (1.1) *is a function* $u \in C([0,T]; L^1(\Omega))$ *with* $u(0) = u_0 \in L^1(\Omega)$ *such that, for all* $\varepsilon > 0$*, there exists* $(t_0, t_1, \ldots, t_N\,;\, f_1^\varepsilon, f_2^\varepsilon, \ldots, f_N^\varepsilon)$ *and* u_0^ε *verifying* (3.2)*; and for which there exists* $(u_1^\varepsilon, \ldots, u_N^\varepsilon)$ *verifying* (3.1) *such that* $\|u(t) - u_n^\varepsilon\|_{L^1(\Omega)} \leq \varepsilon$ *for all* $t \in (t_{n-1}, t_n]$, $n = 1, \ldots, N$.

Remark 3.2. *In this paper, for the sake of simplicity and readability, we chose to present the constant step subdivision algorithm, i.e. that we set* $t_{n+1} - t_n = h = \frac{T}{N}$ *for all* $n = 0, \ldots, N-1$*. However, the techniques developed thereafter can be adapted to a varying step subdivision without difficulty.*

Note that using the nonlinear semigroups theory [6], Ouaro and Ouédraogo have proved in [24] the existence and uniqueness of mild solutions of the following parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div} a(x, \nabla u) = f & \text{in } Q \equiv \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases}$$

where $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$. The assumptions on the vector field are the same than those given in (2.2)–(2.5) and those on the variable exponent $p(x)$ are the same as (2.6). Thanks to their paper, one has the existence and uniqueness of the mild solution of problem (1.1).

4 Numerical study

4.1 Numerical scheme

We are now interested in the numerical resolution of (3.1). Let $f_1, f_2, \ldots, f_N$, u_0 be some functions satisfying (3.2), we use the following iterative scheme (proposed by Maitre in [23]) to get u^ε_{n+1} from u^ε_n.

$$\begin{cases} \text{Let } u^{\varepsilon,0}_{n+1} = u^\varepsilon_n \in L^\infty(\Omega), \text{ solve for } k = 0, 1, \ldots, \\ \\ u^{\varepsilon,k+1}_{n+1} - \rho\,\mathrm{div}(|\nabla u^{\varepsilon,k+1}_{n+1}|^{p(x)-2}\nabla u^{\varepsilon,k+1}_{n+1}) = \lambda_k u^{\varepsilon,k}_{n+1} - \dfrac{\rho}{h}(\lambda_k u^{\varepsilon,k}_{n+1} - u^\varepsilon_n) + \rho f^\varepsilon_{n+1}, \end{cases} \tag{4.1}$$

where $\rho > 0$ is a given parameter and (λ_k) is a sequence of $(0,1)$ such that

$$\lim_{k\to\infty} \lambda_k = 1, \quad \prod_{k\geq 0} \lambda_k = 0, \quad \sum_{k\geq 0} |\lambda_{k+1} - \lambda_k| < \infty. \tag{4.2}$$

For example, we can take $\lambda_k = 1 - \dfrac{1}{k+1}$.

Remark 4.1. *For the sake of simplicity, we could take $\rho = h$, but in this paper our idea is to build a non-expansive map and use the Halpern algorithm to approach the solution of* (3.1)*. In the numerical simulation one will give examples where $\rho = h$.*

4.2 Existence and uniqueness of solution of (4.1)

In this section, we state and prove the well-posedness of our scheme.

Definition 4.2. *For any $n = 0, \ldots, N-1$, $\varepsilon > 0$ and $u^\varepsilon_n \in L^\infty(\Omega)$, a weak solution of* (4.1) *is a sequence $\left(u^{\varepsilon,k+1}_{n+1}\right)_{k\geq 0}$ such that $u^{\varepsilon,k+1}_{n+1} \in W^{1,p(.)}_0(\Omega) \cap L^\infty(\Omega)$ for all $k = 0, 1, \ldots,$ and*

$$\int_\Omega u^{\varepsilon,k+1}_{n+1}\varphi\,dx + \rho\int_\Omega |\nabla u^{\varepsilon,k+1}_{n+1}|^{p(x)-2}\nabla u^{\varepsilon,k+1}_{n+1}.\nabla\varphi\,dx = \int_\Omega g^\varepsilon_{n,k}\varphi\,dx, \tag{4.3}$$

for all $\varphi \in W^{1,p(.)}_0(\Omega)$, where

$$g^{\varepsilon,k}_n := \lambda_k u^{\varepsilon,k}_{n+1} - \frac{\rho}{h}(\lambda_k u^{\varepsilon,k}_{n+1} - u^\varepsilon_n) + \rho f^\varepsilon_{n+1}.$$

Theorem 4.3. *Let $\varepsilon > 0$. For any $n = 0, \ldots, N-1$ let $u^{\varepsilon,0}_{n+1} = u^\varepsilon_n \in L^\infty(\Omega)$ and $f^\varepsilon_{n+1} \in L^\infty(\Omega)$. Then, problem* (4.1) *admits a unique weak solution $u^{\varepsilon,k+1}_{n+1} \in W^{1,p(.)}_0(\Omega)$ for all $k = 0, 1, \ldots$*

Furthermore, for $k = 0, 1, \ldots,$ $u^{\varepsilon,k+1}_{n+1} \in L^\infty(\Omega)$.

Proof. Let $\varepsilon > 0$ and fix n. For $k = 0$ we rewrite problem (4.1) as

$$\begin{cases} u^{\varepsilon,1}_{n+1} - \rho\,\mathrm{div}(|\nabla u^{\varepsilon,1}_{n+1}|^{p(x)-2}\nabla u^{\varepsilon,1}_{n+1}) = g^{\varepsilon,0}_n & \text{in } \Omega \\ \\ u^{\varepsilon,1}_{n+1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

where

$$g_n^{\varepsilon,0} = \left[\lambda_0\left(1-\frac{\rho}{h}\right)+1\right]u_n^{\varepsilon} + \rho f_{n+1}^{\varepsilon}.$$

Consider the energy functional J_ρ on $W_0^{1,p(.)}(\Omega)$ associated to (4.4) given by

$$J_\rho(U) = \frac{1}{2}\int_\Omega U^2 dx + \rho\int_\Omega \frac{|\nabla U|^{p(x)}}{p(x)}dx - \int_\Omega g_n^{\varepsilon,0} U\, dx.$$

We will establish that $J_\rho(U)$ has a minimizer $u_{n+1}^{\varepsilon,1}$ in $W_0^{1,p(.)}(\Omega)$.

Note that J_ρ is well-defined and Gateaux differentiable on $W_0^{1,p(.)}(\Omega)$, since $W_0^{1,p(.)}(\Omega) \hookrightarrow L^2(\Omega)$ thanks to (1.2).

For $\|U\|_{W_0^{1,p(.)}(\Omega)} \geq 1$ we have from the continuous embedding of $W_0^{1,p(.)}(\Omega)$ in $L^{p^-}(\Omega)$ and $g_n^{\varepsilon,0} \in L^\infty(\Omega)$,

$$J_\rho(U) = \frac{1}{2}\int_\Omega U^2 dx + \rho\int_\Omega \frac{|\nabla U|^{p(x)}}{p(x)}dx - \int_\Omega g_n^{\varepsilon,0} U\, dx \geq \frac{\rho}{p^+}\|U\|^{p^-}_{W_0^{1,p(x)}(\Omega)} - C\|U\|_{W_0^{1,p(x)}(\Omega)}.$$

As $p^- > 1$, then J_ρ is coercive. $J_\rho(U)$ is lower bounded and furthermore weakly lower semi-continuous; therefore, admits a global minimizer $u_{n+1}^{\varepsilon,1} \in W_0^{1,p(.)}(\Omega)$ which is a weak solution to (4.4). The global minimizer $u_{n+1}^{\varepsilon,1}$ is also unique.

It remains to show that $u_{n+1}^{\varepsilon,1} \in L^\infty(\Omega)$. To do this, let us show that $\|u_{n+1}^{\varepsilon,1}\|_\infty \leq \|g_n^{\varepsilon,0}\|_\infty$.

As $u_{n+1}^{\varepsilon,1}$ is a weak solution of (4.4), we have

$$\int_\Omega u_{n+1}^{\varepsilon,1}\varphi\, dx + \rho\int_\Omega |\nabla u_{n+1}^{\varepsilon,1}|^{p(x)-2}\nabla u_{n+1}^{\varepsilon,1}.\nabla\varphi\, dx = \int_\Omega g_n^{\varepsilon,0}\varphi\, dx, \tag{4.5}$$

for all $\varphi \in W_0^{1,p(.)}(\Omega)$.

Let $\tau \in \mathbb{R}^+$. Then, $u_{n+1}^{\varepsilon,1} - \tau \in W_0^{1,p(.)}(\Omega)$ and $\left(u_{n+1}^{\varepsilon,1} - \tau\right)^+ \in W_0^{1,p(.)}(\Omega)$.
Note that for $r \in \mathbb{R}$, $r^+ := \max(r,0)$ and $r^- := \min(r,0)$.

Taking $\left(u_{n+1}^{\varepsilon,1} - \tau\right)^+$ as a test function, it follows from (4.5) that

$$\int_\Omega u_{n+1}^{\varepsilon,1}(u_{n+1}^{\varepsilon,1}-\tau)^+\, dx + \rho\int_\Omega |\nabla u_{n+1}^{\varepsilon,1}|^{p(x)-2}\nabla u_{n+1}^{\varepsilon,1}.\nabla(u_{n+1}^{\varepsilon,1}-\tau)^+\, dx = \int_\Omega g_n^{\varepsilon,0}(u_{n+1}^{\varepsilon,1}-\tau)^+\, dx.$$

Setting $A_\tau = \left\{x \in \Omega : u_{n+1}^{\varepsilon,1} \geq \tau\right\}$, we have

$$\begin{aligned}\rho\int_\Omega |\nabla u_{n+1}^{\varepsilon,1}|^{p(x)-2}\nabla u_{n+1}^{\varepsilon,1}.\nabla(u_{n+1}^{\varepsilon,1}-\tau)^+\, dx &= \rho\int_{A_\tau} |\nabla u_{n+1}^{\varepsilon,1}|^{p(x)-2}\nabla u_{n+1}^{\varepsilon,1}.\nabla(u_{n+1}^{\varepsilon,1}-\tau)\, dx \\ &= \rho\int_{A_\tau} |\nabla u_{n+1}^{\varepsilon,1}|^{p(x)}dx \geq 0.\end{aligned}$$

Therefore,

$$\int_\Omega u_{n+1}^{\varepsilon,1}(u_{n+1}^{\varepsilon,1}-\tau)^+\, dx \leq \int_\Omega g_n^{\varepsilon,0}(u_{n+1}^{\varepsilon,1}-\tau)^+\, dx.$$

As Ω is a bounded open domain, we have

$$\int_\Omega [(u^{\varepsilon,1}_{n+1}-\tau)^+]^2\,dx \le \int_\Omega (g^{\varepsilon,0}_n-\tau)(u^{\varepsilon,1}_{n+1}-\tau)^+\,dx.$$

Taking $\tau=\|g^{\varepsilon,0}_n\|_\infty$, then $g^{\varepsilon,0}_n-\tau\le 0$ a.e. in Ω.

Therefore, we have $(u^{\varepsilon,1}_{n+1}-\tau)^+=0$ a.e. in Ω for all $\tau=\|g^{\varepsilon,0}_n\|_\infty$ which is equivalent to saying

$$u^{\varepsilon,1}_{n+1}\le \|g^{\varepsilon,0}_n\|_\infty \quad \text{a.e. in } \Omega.$$

It remains to prove that $u^{\varepsilon,1}_{n+1}\ge -\|g^{\varepsilon,0}_n\|_\infty$ a.e. in Ω. To do this we take $(u^{\varepsilon,1}_{n+1}+\tau)^-$ as test function in (4.5) and use the same argument as previously. Thus, setting $C=\|g^{\varepsilon,0}_n\|_\infty$ implies that $u^{\varepsilon,1}_{n+1}\in L^\infty(\Omega)$.

In short $u^{\varepsilon,1}_{n+1}\in W^{1,p(.)}_0(\Omega)\cap L^\infty(\Omega)$.

By induction, we deduce in the same manner that the problem (4.1) has a unique weak solution $\left(u^{\varepsilon,k+1}_{n+1}\right)_{k\ge 0}$ such that $u^{\varepsilon,k+1}_{n+1}\in W^{1,p(.)}_0(\Omega)\cap L^\infty(\Omega)$ for all $k\in\mathbb{N}$. □

4.3 Study of the convergence

We begin with the following lemma which provides a crucial L^∞ uniform bound for the sequence $\left(u^{\varepsilon,k}_{n+1}\right)_{k\ge 0}$.

Lemma 4.4. *Let $\varepsilon>0$ and fix n. If $\rho\le h$, there exists $M>0$ independent of k such that $\|u^{\varepsilon,k}_{n+1}\|_\infty\le M$.*

Proof. Let $M=\max\left(\|u^{\varepsilon,0}_{n+1}\|_\infty, \|hf^\varepsilon_{n+1}+u^\varepsilon_n\|_\infty\right)$.

Now let us show by induction that $\|u^{\varepsilon,k}_{n+1}\|_\infty\le M$. We first note that $\|u^{\varepsilon,0}_{n+1}\|_\infty\le M$.
One assumes that $\|u^{\varepsilon,k}_{n+1}\|_\infty\le M$, and one shows that $\|u^{\varepsilon,k+1}_{n+1}\|_\infty\le M$.

As $u^{\varepsilon,k+1}_{n+1}\in L^\infty(\Omega)$ and verifies

$$u^{\varepsilon,k+1}_{n+1}-\operatorname{div}\left(\rho|\nabla u^{\varepsilon,k+1}_{n+1}|^{p(x)-2}\nabla u^{\varepsilon,k+1}_{n+1}\right)=\lambda_k u^{\varepsilon,k}_{n+1}-\frac{\rho}{h}(\lambda_k u^{\varepsilon,k}_{n+1}-u^\varepsilon_n)+\rho f^\varepsilon_{n+1},$$

then, from the previous proof, it is established that for all $k=1,2,\dots,$

$$\|u^{\varepsilon,k+1}_{n+1}\|_\infty\le \left\|\lambda_k u^{\varepsilon,k}_{n+1}-\frac{\rho}{h}(\lambda_k u^{\varepsilon,k}_{n+1}-u^\varepsilon_n)+\rho f^\varepsilon_{n+1}\right\|_\infty.$$

Since $\rho\le h$, we then obtain using the induction assumption

$$\|u^{\varepsilon,k+1}_{n+1}\|_\infty\le\left(1-\frac{\rho}{h}\right)M+\frac{\rho}{h}\|hf^\varepsilon_{n+1}+u^\varepsilon_n\|_\infty\le M. \quad \square$$

Thanks to M defined in the above proof we have the following convergence result.

Theorem 4.5. *Assume that conditions in Theorem 4.3 are satisfied. Then, for $\rho \leq h$, the iterative scheme* (4.1) *converges, i.e.*

$$u_{n+1}^{\varepsilon,k} \longrightarrow u_{n+1}^{\varepsilon} \quad \text{strongly in } L^1(\Omega) \text{ as } k \to +\infty,$$

where u_{n+1}^{ε} verifies (3.1).

Proof. Thanks to Lemma 4.4, we can write (4.1) as

$$\frac{1}{\lambda_{k+1}}\bar{u}_{n+1}^{\varepsilon,k+1} - \rho\,\mathrm{div}\left(|\nabla \frac{1}{\lambda_{k+1}}\bar{u}_{n+1}^{\varepsilon,k+1}|^{p(x)-2}\nabla \frac{1}{\lambda_{k+1}}\bar{u}_{n+1}^{\varepsilon,k+1}\right) = \bar{u}_{n+1}^{\varepsilon,k} - \frac{\rho}{h}(\bar{u}_{n+1}^{\varepsilon,k} - u_n^{\varepsilon}) + \rho f_{n+1}^{\varepsilon}, \quad (4.6)$$

where we put $\bar{u}_{n+1}^{\varepsilon,k} = \lambda_k u_{n+1}^{\varepsilon,k}$ and $\bar{u}_{n+1}^{\varepsilon,k+1} = \lambda_{k+1} u_{n+1}^{\varepsilon,k+1}$.

Let $A(u) = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$. We identify the operator $A : L^1(\Omega) \to L^1(\Omega)$ associated with the $p(x)-$Laplacian problem (1.1) with its graph *i.e.*

$$G(A) = \left\{(u,v) \in L^1(\Omega) \times L^1(\Omega);\ v \in A(u)\right\}.$$

Therefore, A is $T-$accretive as soon as u is an entropy solution of problem (2.1) where $a(x,\nabla u) = (|\nabla u|^{p(x)-2}\nabla u)$. For more details, see [6] and [24, Proposition 4.3]. A is called $T-$accretive if $\|(u-\hat{u})^+\|_1 \leq \|(u-\hat{u}+\rho(v-\hat{v}))^+)\|_1$, for any (u,v), $(\hat{u},\hat{v}) \in A$, $\rho > 0$; equivalently, if $\displaystyle\int_{\{u>\hat{u}\}} (v-\hat{v}) + \int_{\{u=\hat{u}\}} (v-\hat{v})^+ \geq 0$ for any (u,v), $(\hat{u},\hat{v}) \in A$.

Hence, (4.6) yields

$$(I+\rho A)\left(\frac{1}{\lambda_{k+1}}\bar{u}_{n+1}^{\varepsilon,k+1}\right) = \bar{u}_{n+1}^{\varepsilon,k} - \frac{\rho}{h}(\bar{u}_{n+1}^{\varepsilon,k} - u_n^{\varepsilon}) + \rho f_{n+1}^{\varepsilon}. \quad (4.7)$$

To complete the proof of Theorem 4.5, we use the following technical lemma.

Lemma 4.6. *Let $\rho \leq 2h$ and M defined in the above proof such that $C_M = \left\{u \in L^1(\Omega), \|u\|_\infty \leq M\right\}$. The iteration operator*

$$\tilde{T}(\bar{u}) = (I+\rho A)^{-1}\left(\bar{u} - \frac{\rho}{h}(\bar{u} - u_n^{\varepsilon}) + \rho f_{n+1}^{\varepsilon}\right)$$

is an L^1-non-expanding operator from C_M to C_M.

Proof. The fact that $\tilde{T}$ maps C_M to C_M is easily seen thanks to the proof of the Lemma 4.4 and (4.7). Now let $(\bar{u},\bar{v}) \in C_M^2$. One has from the $T-$accretiveness of A on $L^1(\Omega)$ that $(I+\rho A)^{-1}$ is a $T-$contraction in $L^1(\Omega)$ (see [6]), thus, a contraction. Therefore,

$$\begin{aligned}
\|\tilde{T}(\bar{u}) - \tilde{T}(\bar{v})\|_1 &= \Big\|(I+\rho A)^{-1}\left(\bar{u} - \frac{\rho}{h}(\bar{u} - u_n) + \rho f_{n+1}\right) \\
&- (I+\rho A)^{-1}\left(\bar{v} - \frac{\rho}{h}(\bar{v} - u_n) + \rho f_{n+1}\right)\Big\|_1 \\
&\leq \left\|\left(1 - \frac{\rho}{h}\right)\bar{u} - \left(1 - \frac{\rho}{h}\right)\bar{v}\right\|_1.
\end{aligned}$$

Since $\rho \leq 2h$, we obtain

$$\|\tilde{T}(\bar{u}) - \tilde{T}(\bar{v})\|_1 \leq \|\bar{u} - \bar{v}\|_1. \quad \square$$

Consequently, from (4.7) one has the iteration $\bar{u}_{n+1}^{\varepsilon,k+1} = \lambda_{k+1}\tilde{T}(\bar{u}_{n+1}^{\varepsilon,k})$ where $\tilde{T}$ is a non-expansive operator in $L^1(\Omega)$ defined as in Lemma 4.6. Now, we are going to apply the Lemma 2.4 with $X = L^1(\Omega)$ and $C = C_M$ which is clearly a convex subset of $L^1(\Omega)$ containing 0. The uniqueness of a fixed point is verified thanks to Theorem 2.3. Indeed a fixed point u^* of $\tilde{T}$ verifies

$$u^* - \rho \operatorname{div}(|\nabla u^*|^{p(x)-2}\nabla u^*) = u^* - \frac{\rho}{h}(u^* - u_n^\varepsilon) + \rho f_{n+1}^\varepsilon.$$

Thus, $u^* - h \operatorname{div}(|\nabla u^*|^{p(x)-2}\nabla u^*) = u_n^\varepsilon + h f_{n+1}^\varepsilon$. From Theorem 2.3 this equation has a unique solution and from the definition of mild solution it is u_{n+1}^ε.

To conclude the proof of convergence of (4.1), we point out that each subsequence of $\bar{u}_{n+1}^{\varepsilon,k}$ has a convergent subsequence to an element of C_M, using the L^∞ bound of $\bar{u}_{n+1}^{\varepsilon,k}$ and the monotonicity of $(|\nabla \bar{u}_{n+1}^{\varepsilon,k}|^{p(x)-2}\nabla \bar{u}_{n+1}^{\varepsilon,k})$, to the equation (4.6). Applying Lemma 2.4, we conclude that the sequence $\bar{u}_{n+1}^{\varepsilon,k}$ converges strongly in $L^1(\Omega)$ toward u_{n+1}^ε. The same occurs for $u_{n+1}^{\varepsilon,k} = \dfrac{1}{\lambda_k}\bar{u}_{n+1}^{\varepsilon,k}$. $\square$

4.4 Convergence when $\varepsilon \to 0$ toward a solution of (1.1)

Note that for a mild solution we do not need to show the convergence in time since it is included in its definition: once convergence in k is achieved for u_{n+1}^ε, then, by the definition of mild solution, u_{n+1}^ε approaches $u^\varepsilon(t)$ on $(t_n, t_{n+1}]$ up to ε. Thus, our scheme converges to the mild solution when ε goes to zero.

We can state also the following result.

Proposition 4.7. *Let $u_0 \in L^\infty(\Omega)$, $f \in L^\infty(Q)$ and u the unique mild solution of* (1.1)*. Then u is a weak solution of* (1.1)*. By a weak solution we understand a solution in the sense of distributions that belongs to the energy space, i.e.,*

$$\begin{aligned} &u \in V := \left\{ v \in L^{p^-}(0,T; W_0^{1,p(.)}(\Omega)); |\nabla v| \in L^{p(.)}(Q) \right\}, \\ &\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f \text{ in } \mathcal{D}'(Q), \quad u(.,0) = u_0. \end{aligned} \tag{4.8}$$

Remark 4.8. *Note that a proof of the above proposition exists in [24]. Here, we use L^∞ uniform boundedness and the strong convergence in $L^1(\Omega)$ of the solution of our numerical scheme to prove Proposition 4.7.*

Moreover, these two results lead to the L^∞ uniform boundedness of the weak solution.

Proof of Proposition 4.7. Let u be the mild solution of (1.1). For $n = 0, \ldots, N-1$, u_{n+1}^ε is the unique weak solution of (3.1). We have

$$\int_\Omega \frac{u_{n+1}^\varepsilon - u_n^\varepsilon}{h}\varphi\, dx + \int_\Omega |\nabla u_{n+1}^\varepsilon|^{p(x)-2}\nabla u_{n+1}^\varepsilon.\nabla\varphi\, dx = \int_\Omega f_{n+1}^\varepsilon \varphi\, dx, \tag{4.9}$$

$\forall \varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and

$$\begin{cases} \bullet \ \ 0 = t_0 < \cdots < t_N = T \text{ such that } t_n - t_{n-1} = h \leq \varepsilon \text{ for } n = 1, \ldots, N, \\ \bullet \ \ \displaystyle\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|f(t) - f_n^\varepsilon\|_{L^1(\Omega)}\, dt \leq \varepsilon \Rightarrow \|f_n^\varepsilon\|_{L^\infty(\Omega)} \leq \|f(t)\|_{L^\infty(\Omega)}, \\ \bullet \ \ \displaystyle\sum_{n=1}^{N} h \|f_n^\varepsilon\|_{L^\infty(\Omega)} \leq \int_0^T \|f(.,t)\|_{L^\infty(\Omega)}\, dt, \\ \bullet \ \ \|u_0 - u_0^\varepsilon\|_{L^1(\Omega)} \leq \varepsilon \Rightarrow \|u_0^\varepsilon\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}. \end{cases} \tag{4.10}$$

Note that relations in (4.10) are equivalent to relations in (3.2).

Let us set $u_\varepsilon(t) = u_{n+1}^\varepsilon \ \ \forall\, t \in (t_n, t_{n+1}]$, $u_\varepsilon(0) = u_0^\varepsilon$ and $f_\varepsilon(t) = f_{n+1}^\varepsilon$, $\forall\, t \in (t_n, t_{n+1}]$.

Lemma 4.4, Theorem 4.5 and the above relations in (4.10) imply that

$$\|u_\varepsilon\|_{L^\infty(Q)} \leq C(\|u_0\|_{L^\infty(\Omega)}; \|f\|_{L^\infty(Q)}). \tag{4.11}$$

Let ζ be the function defined by $\zeta(r) = \dfrac{r^2}{2}$ that satisfies $\zeta(r) - \zeta(\tilde{r}) \leq (r - \tilde{r})r$.

Taking $\varphi = u_{n+1}^\varepsilon$ as test function in (4.9) and integrating over $(t_n, t_{n+1}]$ and summing over $n = 0, \ldots, N-1$, we get

$$\int_\Omega \zeta(u_\varepsilon(t))\, dx + \int_Q |\nabla u_\varepsilon|^{p(x)}\, dx\, dt \leq \int_Q f_\varepsilon u_\varepsilon\, dx\, dt + \int_\Omega \zeta(u_0^\varepsilon)\, dx.$$

Thanks to the uniform boundedness of u_ε in ε and as $u_0^\varepsilon \in L^\infty(\Omega)$, we have

$$\int_Q |\nabla u_\varepsilon|^{p(x)}\, dx\, dt \leq C.$$

Moreover,

$$\int_0^T \|\nabla u_\varepsilon\|_{L^{p(.)}(\Omega)}^{p^-}\, dt \leq \int_0^T \max\left(\int_\Omega |\nabla u_\varepsilon|^{p(x)}; \left(\int_\Omega |\nabla u_\varepsilon|^{p(x)} \right)^{\frac{p^-}{p^+}} \right) dt.$$

Hence,

$$\int_0^T \|u_\varepsilon\|_{W_0^{1,p(.)}(\Omega)}^{p^-}\, dt \leq C.$$

As a consequence, there exists a subsequence still denoted $(u_\varepsilon)_{\varepsilon > 0}$, such that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u, \quad \text{weakly-* in } L^\infty(Q), \\ u_\varepsilon &\rightharpoonup u, \quad \text{weakly in } L^{p^-}(0, T; W_0^{1,p(.)}(\Omega)), \\ |\nabla u_\varepsilon|^{p(.)-2} \nabla u_\varepsilon &\rightharpoonup \Phi, \quad \text{weakly in } \left(L^{p'(.)}(Q) \right)^d. \end{aligned}$$

Using the monotonicity method we show that $\Phi = |\nabla u|^{p(.)-2} \nabla u$ a.e. in Q.

Now, let $\tilde{u}_\varepsilon$ be the piecewise linear function defined by

$$\tilde{u}_\varepsilon(t) = u_n^\varepsilon + \frac{t - t_n}{h}(u_{n+1}^\varepsilon - u_n^\varepsilon) \text{ for } t \in [t_n, t_{n+1}], \ n = 0, \ldots, N-1.$$

The function $\tilde{u}_\varepsilon$ verifies $(\tilde{u}_\varepsilon)_t(t) = \dfrac{u^\varepsilon_{n+1} - u^\varepsilon_n}{h}$ and $\tilde{u}_\varepsilon \to u$ in $L^\infty(0,T;L^1(\Omega))$. Hence,

$$u \in C([0,T];L^1(\Omega)).$$

Integrating (4.9) over (t_n, t_{n+1}) and summing over $n = 0,\dots,N-1$, we find

$$\begin{aligned} &- \int_0^T\int_\Omega \varphi_t \tilde{u}_\varepsilon \,dx\,dt - \int_\Omega \varphi(0) u_0^\varepsilon \,dx + \int_0^T\int_\Omega \left(|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\right).\nabla\varphi \,dx\,dt \\ &= \int_0^T\int_\Omega f_\varepsilon \varphi \,dx\,dt. \end{aligned} \tag{4.12}$$

Using the convergence results and passing to the limit in (4.12) as $\varepsilon \to 0$, we get the result. □

Remark 4.9. *For $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$ the unique mild solution u of* (1.1) *is also an entropy solution. Indeed, since L^∞ is dense in L^1, we consider two sequences of functions $(f_m)_{m\geq 1} \subset L^\infty(Q)$ and $(u_{0m})_{m\geq1} \subset L^\infty(\Omega)$ satisfying*

$$\begin{cases} f_m \to f \text{ in } L^1(Q), \qquad u_{0m} \to u_0 \text{ in } L^1(\Omega), \quad \text{as } m \to \infty, \\ \\ \|f_m\|_{L^1(Q)} \leq \|f\|_{L^1(Q)}, \quad \|u_{0m}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}. \end{cases} \tag{4.13}$$

Then, we get the following approximate problem of (1.1).

$$\begin{cases} \dfrac{\partial u_m}{\partial t} - \operatorname{div}(|\nabla u_m|^{p(x)-2}\nabla u_m) = f_m & \text{in } Q, \\ u_m = 0 & \text{on } \partial\Omega \times (0,T), \\ u_m(x,0) = u_{0m} & \text{in } \Omega. \end{cases} \tag{4.14}$$

Thanks to [24], for each $m = 1,2,\dots,$ we can find a unique mild solution $u_m \in C([0,T];L^1(\Omega))$ for problem (4.14) *which verifies the L^1-contraction principle, i.e. the following estimate holds for almost all $t \in (0,T)$,*

$$\begin{aligned} \|u_m(.,t)\|_{L^1(\Omega)} &\leq \|u_{0m}\|_{L^1(\Omega)} + \int_0^t \|f_m(.,s)\|_{L^1(\Omega)}\,ds \\ &\leq \|u_0\|_{L^1(\Omega)} + \int_0^t \|f(.,s)\|_{L^1(\Omega)}\,ds. \end{aligned}$$

By Proposition 4.7, and following the proof of [24, Theorem 5.1] we get the result.

Note that this entropy solution is equivalent to the renormalized solution of (1.1). Indeed, in [29], Zhang and Zhou have proved thanks to the assumptions (1.2) the existence and uniqueness of renormalized and entropy solutions of (1.1). In their paper, they have showed the equivalence between entropy and renormalized solutions.

4.5 Numerical tests

4.5.1 Implementation

We know that solving the equation (4.1) is equivalent to solve the following minimization problem for $n = 0, 1, \ldots, N-1$ and $k = 0, 1, \ldots$

$$u_{n+1}^{\varepsilon,k+1} = \operatorname{argmin}_{v \in \mathbb{W}} J(v), \tag{4.15}$$

where,

$$\mathbb{W} := \left\{ v \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega) \right\}$$

and the functional J is

$$\begin{aligned} J(v) &= \frac{1}{2}\int_\Omega v^2\,dx + \rho\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx - \left(1-\frac{\rho}{h}\right)\lambda_k \int_\Omega u_{n+1}^{\varepsilon,k} v - \frac{\rho}{h}\int_\Omega u_n^\varepsilon v\,dx \\ &\quad -\rho\int_\Omega f_{n+1}^\varepsilon v\,dx. \end{aligned} \tag{4.16}$$

We formulate a basic procedure for solving problem (4.15) following the split Bregman technique (see [17]). We solve the minimization problem by introducing an auxiliary variable b. We have

$$\begin{aligned} \min_v \Bigg\{ \frac{1}{2}\int_\Omega v^2\,dx + \rho\int_\Omega \frac{1}{p(x)}|b|^{p(x)}\,dx - \left(1-\frac{\rho}{h}\right)\lambda_k \int_\Omega u_{n+1}^{\varepsilon,k} v\,dx - \frac{\rho}{h}\int_\Omega u_n^\varepsilon v\,dx \\ - \rho\int_\Omega f_{n+1}^\varepsilon v\,dx \;\; \text{subject to } b = \nabla v \Bigg\}. \end{aligned} \tag{4.17}$$

By adding one quadratic penalty function term, we convert equation (4.17) to an unconstrained splitting formulation as follow.

$$\begin{aligned} \min_{v,b} \Bigg\{ \frac{1}{2}\int_\Omega v^2\,dx + \rho\int_\Omega \frac{1}{p(x)}|b|^{p(x)}\,dx + \frac{\gamma}{2}\int_\Omega |b-\nabla v|^2\,dx - \left(1-\frac{\rho}{h}\right)\lambda_k \int_\Omega u_{n+1}^{\varepsilon,k} v\,dx \\ -\frac{\rho}{h}\int_\Omega u_n^\varepsilon v\,dx - \rho\int_\Omega f_{n+1}^\varepsilon v\,dx \Bigg\}, \end{aligned} \tag{4.18}$$

where γ is a positive parameter which controls the weight of the penalty term. Similar to the split Bregman iteration, we propose the following scheme.

$$\begin{cases} (v^{l+1}, b^{l+1}) = \operatorname{argmin}_{v,b} \Bigg\{ \frac{1}{2}\int_\Omega v^2\,dx + \rho\int_\Omega \frac{1}{p(x)}|b|^{p(x)}\,dx + \frac{\gamma}{2}\int_\Omega |b-\nabla v-\delta^l|^2\,dx \\ \qquad - \left(1-\frac{\rho}{h}\right)\lambda_k \int_\Omega u_{n+1}^{\varepsilon,k} v\,dx - \frac{\rho}{h}\int_\Omega u_n^\varepsilon v\,dx - \rho\int_\Omega f_{n+1}^\varepsilon v\,dx \Bigg\}, \\ \delta^{l+1} = \delta^l + \nabla v^{l+1} - b^{l+1}. \end{cases} \tag{4.19}$$

Alternatively, this joint minimization problem can be solved by decomposing into several subproblems.

4.5.2 Subproblem v with fixed b and δ

Given the fixed variable b^l and δ^l, our aim is to find the solution of the following problem

$$\begin{aligned} v^{l+1} \quad = \quad & \operatorname{argmin}_v \left\{ \frac{1}{2} \int_\Omega v^2 \, dx + \frac{\gamma}{2} \int_\Omega |b^l - \nabla v - \delta^l|^2 \, dx - \left(1 - \frac{\rho}{h}\right) \lambda_k \int_\Omega u_{n+1}^{\varepsilon,k} v \, dx \right. \\ & \left. - \frac{\rho}{h} \int_\Omega u_n^\varepsilon v \, dx - \rho \int_\Omega f_{n+1}^\varepsilon v \, dx \right\}. \end{aligned} \tag{4.20}$$

We know that solve (4.20) is equivalent to solve the following optimality condition.

$$v - \gamma \Delta v = \gamma \nabla.(\delta^l - b^l) + \left(1 - \frac{\rho}{h}\right) \lambda_k u_{n+1}^{\varepsilon,k} + \frac{\rho}{h} u_n^\varepsilon + \rho f_{n+1}^\varepsilon. \tag{4.21}$$

Since the discrete system is strictly diagonally dominant with Neumann boundary condition, the most natural choice is the Gauss-Seidel method.

4.5.3 Subproblem b with fixed v and δ

Similarly, we solve

$$b^{l+1} = \operatorname{argmin}_b \left\{ \rho \int_\Omega \frac{1}{p(x)} |b|^{p(x)} \, dx + \frac{\gamma}{2} \int_\Omega |b - \nabla v^{l+1} - \delta^l|^2 \, dx \right\} \tag{4.22}$$

<u>In two dimensional space.</u>

Here, setting $b = (b_x, b_y)$ and $\delta = (\delta_x, \delta_y)$.

Then, the resolution of (4.22) is equivalent to solve the following optimality condition.

$$\begin{cases} \rho |b|^{p(x,y)-2} b_x + \gamma (b_x - \nabla_x v^{l+1} - \delta_x^l) & = 0 \\ \\ \rho |b|^{p(x,y)-2} b_y + \gamma (b_y - \nabla_y v^{l+1} - \delta_y^l) & = 0, \end{cases} \tag{4.23}$$

where $\nabla v = (\nabla_x v, \nabla_y v)$.

If b_x and b_y are not zero, then,

$$b_x = \frac{\nabla_x v^{l+1} + \delta_x^l}{\nabla_y v^{l+1} + \delta_y^l} b_y. \tag{4.24}$$

Substituting (4.24) into (4.23), we obtain

$$\operatorname{sign}(b_y) T |b_y|^{p(x,y)-1} + \gamma (b_y - \nabla_y v^{l+1} - \delta_y^l) = 0, \tag{4.25}$$

where $T = \rho \left(\left(\frac{\nabla_x v^{l+1} + \delta_x^l}{\nabla_y v^{l+1} + \delta_y^l} \right)^2 + 1 \right)^{\frac{p(x,y)-2}{2}}$. Here, sign is defined as follows.

$$\operatorname{sign}(\omega) := \begin{cases} 1 & \text{if} \quad \omega > 0, \\ 0 & \text{if} \quad \omega = 0, \\ -1 & \text{if} \quad \omega < 0. \end{cases}$$

Note that

$$\operatorname{sign}(b_x) = \operatorname{sign}(\nabla_x v^{l+1} + \delta_x^l) \tag{4.26}$$

and

$$\operatorname{sign}(b_y) = \operatorname{sign}(\nabla_y v^{l+1} + \delta_y^l). \tag{4.27}$$

So, (4.25) can be expressed as

$$\operatorname{sign}(\nabla_y v^{l+1} + \delta_y^l) T |b_y|^{p(x,y)-1} + \gamma(b_y - \nabla_y v^{l+1} - \delta_y^l) = 0. \tag{4.28}$$

Unfortunately, we cannot obtain the explicit solution of the equation (4.28). We can use Newton method to get an approximate solution. If b_y is solved, b_x can be easily determined using (4.24) and (4.26).

4.5.4 Applications

In the following numerical simulation the iteration process stops when the following condition is satisfied

$$\frac{\|u_{n+1}^{k+1} - u_{n+1}^k\|_2}{\|u_{n+1}^{k+1}\|_2} \leq \text{stop} := 10^{-5}, \tag{4.29}$$

where $\|.\|_2$ is the Euclidean norm and u_{n+1}^k the vector approaching, at iteration k, the space-discretization of u_{n+1}. After stopping the iterations at $k = k_{last}$, we denote $u_{n+1} = u_{n+1}^{k_{last}}$ and switch to the next time step.

Note that for implementation, finite difference method is used to approximate the partial derivatives. Moreover, for sake of simplicity, the domain Ω will be a square. The domain Ω will be subdivided into N_x^2 uniform squares.

For numerical simulation, we will use the following parameters

$$N_x = 80 \quad \text{and} \quad h = 0.02.$$

Let us recall that h is the time step. The space step is easily computed thanks to N_x and Ω.

Example 4.10. *In this example, we take* $\Omega = (0,1) \times (0,1)$, $T = 1$, $p(x,y) = 2$, *and* $f = xy(1-x)(1-y) + 2t((1-y)y + (1-x)x)$. *As initial condition, we set*

$$u_0(x,y) = 0.$$

Let us note that with these data p, u_0 *and* f, *the exact solution is*

$$u(x,y,t) = txy(1-x)(1-y).$$

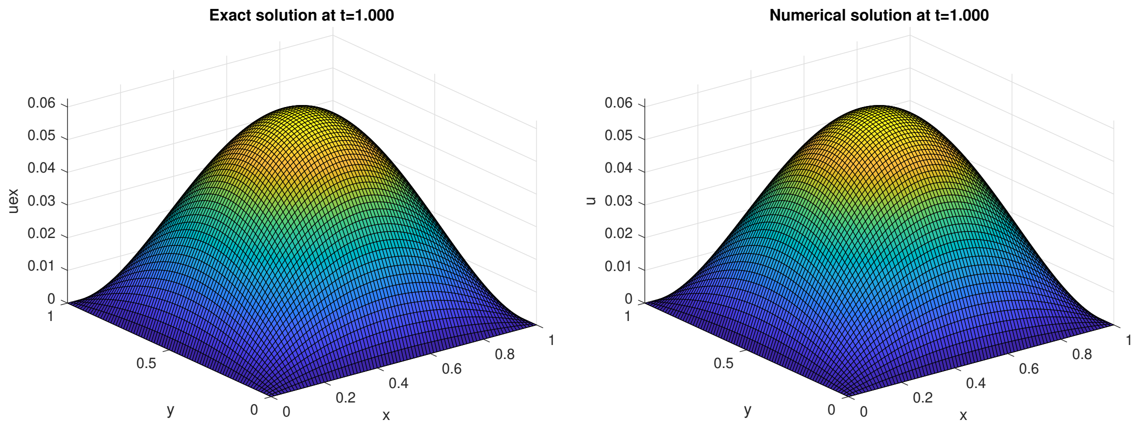

Figure 1: left: $u(x, y, t) = txy(1 - x)(1 - y)$ right: For $\rho = h$ and $\gamma = 0.02$

.

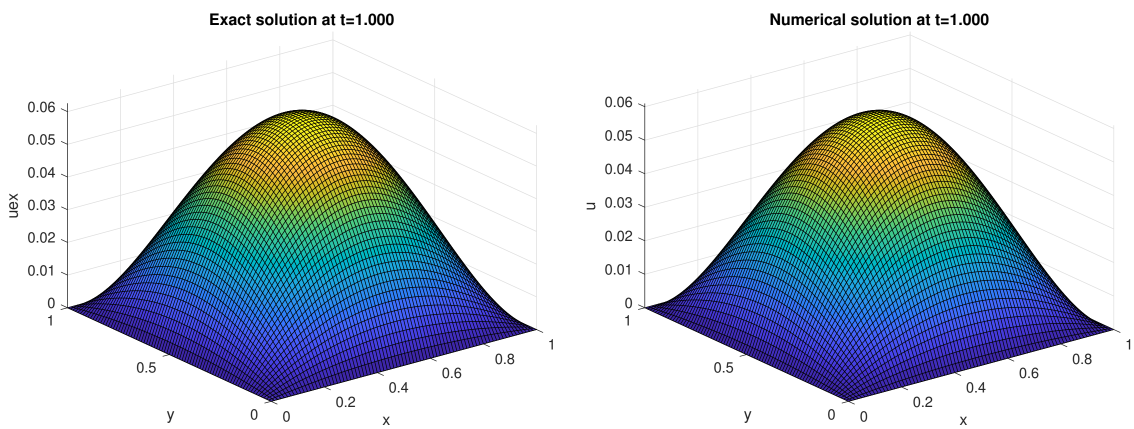

Figure 2: left: $u(x, y, t) = txy(1 - x)(1 - y)$ right: For $\rho = h/2$ and $\gamma = 0.02$

Figure 1 shows the exact solution and the numerical solution for $\gamma = 0.02$ and $\rho = h$. While, Figure 2 shows the exact solution and the numerical solution for $\gamma = 0.02$ and $\rho = h/2$.

As we can see, we always get a good numerical approximation of the solution even if ρ varies.

Denoting u_h the numerical solution and u the exact solution of Example 4.10, with $\rho = h$ and $\gamma = 0.02$, we get the following table of the error approximation.

t	0.1	0.2	0.3	0.4	0.5
$\|u_h - u\|_1$	$2.5099.10^{-5}$	$5.6941.10^{-5}$	$7.9789.10^{-5}$	$1.0717.10^{-4}$	$1.345.10^{-4}$

t	0.6	0.7	0.8	0.9	1
$\|u_h - u\|_1$	$1.6192.10^{-4}$	$1.8930.10^{-4}$	$2.1668.10^{-4}$	$2.4406.10^{-4}$	$2.7144.10^{-4}$

Example 4.11. *In this example, we set* $\Omega = (0,1) \times (0,1)$*,* $T = 5$*,* $p(x,y) = 2 + \dfrac{|x|}{2}$*, and* $f = 1$*. As initial condition we set*

$$u_0(x,y) = 0.$$

As parameters we set $\rho = h$ *and* $\gamma = 0.02$*.*

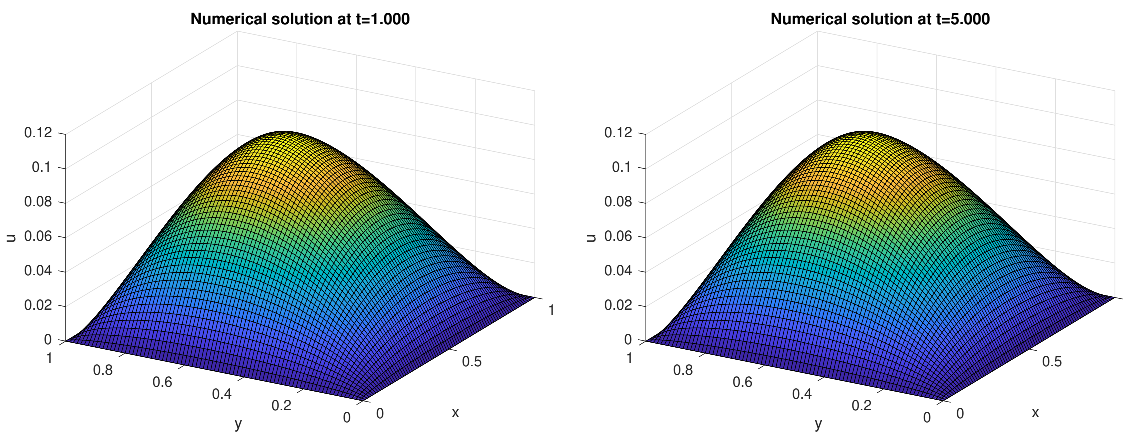

Figure 3: Numerical solution for $p(x,y) = 2 + \dfrac{|x|}{2}$, $\rho = h$ and $\gamma = 0.02$.

Figure 3 shows the numerical solution at $t = 1$ and at $t = 5$. One can see that both figures are the same.

Example 4.12. *In this example, we take* $\Omega = (-1,1) \times (-1,1)$*,* $T = 5$*,* $p(x,y) = \dfrac{9}{5} - \dfrac{x^2}{2}$ *and*

$$f = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0. \end{cases}$$

As the initial condition, we set

$$u_0(x,y) = e^{(1-x^2)(1-y^2)} - 1.$$

We use the same parameters ρ *and* γ *as previously.*

Figure 4 shows the numerical solution at $t = 1$ *and* $t = 5$*.*

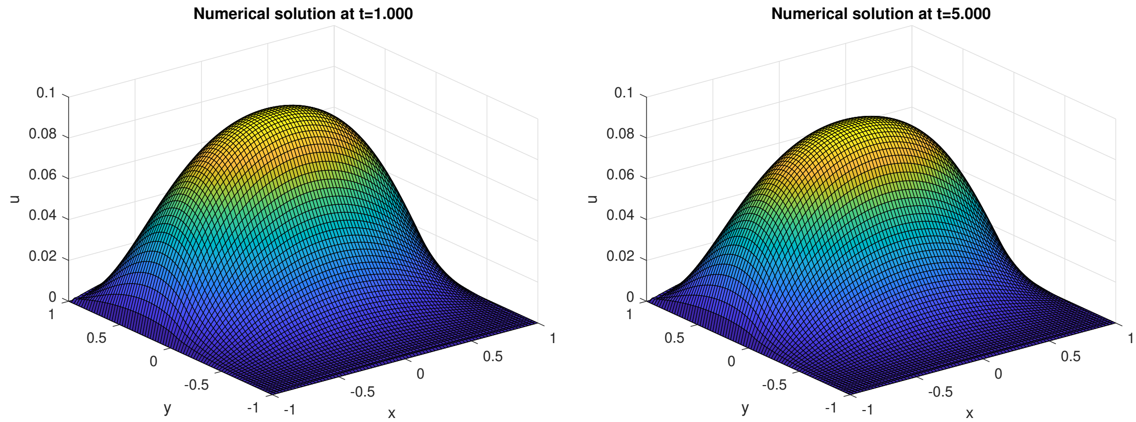

Figure 4: Numerical solution for $p(x,y) = \dfrac{9}{5} - \dfrac{x^2}{2}$, $\rho = h$ and $\gamma = 0.02$.

We remark that the exponents $p(x)$ considered in the three examples satisfy the condition 1.2. Also, note that the choice of γ results from the knowledge of the explicit solution of the Example 4.10. Indeed, knowing the explicit solution, we choose γ so as to obtain a better approximation of this explicit solution. This leads to the choice of $\gamma = 0.02$.

Conclusion and discussion

Inspired by the work of Maitre (see [23]), we have in this paper made a numerical analysis of the mild solution of parabolic problem involving the $p(x)-$Laplacian operator. Using the works of Zhang and Zhou (see [29]), and Ouaro and Ouédraogo (see [24]), we have shown that the mild solution is also an entropy solution which is equivalent to the renormalized solution. For the numerical tests, we have used the split Bregman iteration.

In a forthcoming paper, we will make a comparison of the solutions of our numerical scheme (4.1) to those of the classical backward Euler scheme.

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Vlasov-Poisson equation in weighted Sobolev space $W^{m,p}(w)$

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ABSTRACT

In this paper, we are concerned about the well-posedness of Vlasov-Poisson equation near vaccum in weighted Sobolev space $W^{m,p}(w)$. The most difficult part comes from estimates of the electronic term $\nabla_x\phi$. To overcome this difficulty, we establish the L^p-L^q estimates of the electronic term $\nabla_x\phi$; some weight is introduced as well to obtain the off-diagonal estimate. The weight is also useful when it comes to control the higher-order derivative term.

RESUMEN

En este artículo, estamos interesados en que la ecuación de Vlasov-Poisson está bien puesta cercana al vacío en el espacio de Sobolev $W^{m,p}(w)$ con peso. La parte más difícil proviene de estimaciones del término electrónico $\nabla_x\phi$. Para superar esta dificultad, establecemos las estimaciones L^p-L^q del término electrónico $\nabla_x\phi$; donde algún peso es también introducido para obtener la estimación fuera de la diagonal. El peso es también útil cuando se trata de controlar el término de la derivada de alto orden.

Keywords and Phrases: Vlasov-Poisson, L^p-Sobolev, weighted estimates, L^p-L^q estimates.

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1 Introduction

Understanding the evolution of a distribution of particles over time is a major research area of statistical physics. The Vlasov-Poisson equation is one of the key equations governing this evolution. Specifically, it models particle behaviors with long range interactions in a non-relativistic zero-magnetic field setting. Two principal types of long range interactions are Coulomb's forces, the electrostatic repulsion of similarly charged particles in a plasma, and Newtonian's forces, the gravitational attraction of stars in a galaxy. The general Cauchy's problem for the Vlasov-Poisson equation (VP equation) in n dimensional space is as follows:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = 0, \\ -\Delta_x \phi = \int_{\mathbb{R}^n} f \, dv, \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{1.1}$$

where $f(t,x,v)$ denotes the distribution function of particles, $x \in \mathbb{R}^n$ is the position, $v \in \mathbb{R}^n$ is the velocity, and $t > 0$ is the time and $n \geq 3$.

The Cauchy problem for the Vlasov-Poisson equation has been studied for several decades. The first paper on global existence is due to Arsen'ev [3]. He showed the global existence of weak solutions. Then in 1977 Batt [5] established the global existence for spherically symmetric data. In 1981 Horst [9] extended the global classical solvability to cylindrically symmetric data. Next, in 1985, Bardos and Degond [4] obtained the global existence for "small" data. Finally, in 1989 Pfaffelmoser [12] proved the global existence of a smooth solution with large data. Later, simpler proofs of the same results were published by Schaeffer [13], Horst [10], and Lions and Pertharne [11]. Nevertheless, most of them were concerned about solutions in L^∞ or continuous function spaces. Also, there are many papers studying Vlasov-Poisson-Boltzmman (Landau) equation in L^2 setting, see [2, 6, 7, 8] and the references therein. A natural question is whether we can obtain the solutions in L^p context, for example, $W^{m,p}$ spaces. This becomes our main theme in this paper.

In this paper, our aim is to construct the solution to (1.1) in $W^{m,p}$ space. The difficulty lies in the absence of L^p estimates of the electronic term $\nabla_x \phi$. To handle this issue, we establish the L^p-L^q off-diagonal estimates of $\nabla_x \phi$ which is highly important in estimating the higher order derivative term. Also, it is necessary to introduce a weight w in order to obtain this off-diagonal estimate. It is worthy to mention that this weight is crucial to deal with the higher-order derivative term.

2 Preliminaries and main theorem

2.1 Notations and definitions

We first would like to introduce some notations.

- Given a locally integrable function f, the maximal function Mf is defined by

$$(Mf)(x) = \sup_{\delta>0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y)|dy, \tag{2.1}$$

  where $|B(x,\delta)|$ is the volume of the ball of $B(x,\delta)$ with center x and radius δ.

- Weight $w(v) = \langle v\rangle^{\gamma}$, $\gamma \cdot \frac{p'}{p} > n$, $\frac{1}{p} + \frac{1}{p'} = 1$, n is the dimension.

- $\|f\|^p_{L^p_{x,v}(w)} =: \int_{\mathbb{R}^{2n}} |f|^p w\, dxdv.$

- Define the higher-order energy norm as

$$\mathcal{E}(f(t)) =: \|f\|_{W^{m,p}(w)} = \sum_{|\alpha|+|\beta|\leq m} \|\partial_x^{\alpha}\partial_v^{\beta} f(t)\|^p_{L^p_{x,v}(w)},$$

  and

$$\mathcal{E}(f_0) =: \mathcal{E}(f(0)) = \sum_{|\alpha|+|\beta|\leq m} \|\partial_x^{\alpha}\partial_v^{\beta} f_0\|^p_{L^p_{x,v}(w)},$$

  where $m \geq 5$ and $\frac{n}{3} < p < \frac{n}{2}$, $n \geq 3$. Here α and β denote multi-indices with length $|\alpha|$ and $|\beta|$, respectively. If each component of α_1 is not greater than that of α, we denote the condition by $\alpha_1 \leq \alpha$. We also define $\alpha_1 < \alpha$ if $\alpha_1 \leq \alpha$ and $|\alpha_1| < |\alpha|$. We also denote $\binom{\alpha_1}{\alpha}$ by $C_{\alpha}^{\alpha_1}$.

- $A \lesssim B$ means there exists a constant $c > 1$ independent of the main parameters such that $A \leq cB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

Now we are ready to state our main theorem.

Theorem 2.1. *For any sufficiently small $M > 0$, there exists $T^*(M) > 0$ such that if*

$$\mathcal{E}(f_0) = \sum_{|\alpha|+|\beta|\leq m} \|\partial_x^{\alpha}\partial_v^{\beta} f_0\|^p_{L^p_{x,v}(w)} \leq \frac{M}{2},$$

then there is a unique solution $f(t,x,v)$ to Vlasov-Poisson system (1.1) in $[0,T^*(M)) \times \mathbb{R}^n \times \mathbb{R}^n$ such that $\sup_{0\leq t\leq T^*} \mathcal{E}(f(t)) \leq M$, where $m > \frac{n}{p} + 1$ with $n \geq 3$ and $\frac{n}{3} < p < \frac{n}{2}$.

Remark 2.2.

- *One should pay attention to the differential index m in $W^{m,p}(\mathbb{R}^n)$ which represents the weak derivative, is not the classical derivative in $C^2(\mathbb{R}^n)$. Indeed, for the space $W^{4,1.4}(\mathbb{R}^6)$ in which we could obtain solutions that could not be embedded into $C(\mathbb{R}^6)$ (the continuous function space) or $L^\infty(\mathbb{R}^6)$, not to mention $C^2(\mathbb{R}^6)$ (the twice continuously differentiable function space) due to the fact $4 \cdot 1.4 < 6$, i.e. $W^{4,1.4}(\mathbb{R}^6) \not\hookrightarrow C^2(\mathbb{R}^6)$ which implies that the classical results in [3, 4] and [9]-[13] could not cover our results.*

- *In [4], C. Bardos and P. Degond also imposed the pointwise condition like*

$$0 \le u_{\alpha,0}(x,v) \le \frac{\epsilon}{(1+|x|)^4 \cdot (1+|v|)^4}.$$

  However, the polynomial decay in the x variable is not needed at all in our proofs.

- *Our working space $W^{m,p}(\mathbb{R}^n)$ has more flexibility than $C^2(\mathbb{R}^n)$ because of the triplet (m,n,p) which implies that we can obtain the solutions in more spaces.*

Let us illustrate our strategies for proving Theorem 2.1. As is known, the routine to prove the existence of solution is to get a uniform-in-k estimate for the energy norm $\mathcal{E}(f^{k+1}(t))$. In this paper, we adopt the L^p version energy method, *i.e.* to do the dual with $|\partial_x^\alpha \partial_v^\beta f^{k+1}|^{p-2}(\partial_x^\alpha \partial_v^\beta f^{k+1})w$ (see (4.5)). We expect all the estimates J_i in Section 4 can be controlled by

$$\mathcal{E}(f(t)) =: \sum_{|\alpha|+|\beta|\le m} \|\partial_x^\alpha \partial_v^\beta f(t)\|^p_{L^p_{x,v}(w)}.$$

To achieve our goal, some estimates related to the electronic term $\nabla_x \phi$ are needed. The L^p-L^q estimate is established to deal with the higher-order derivative. For instance, when $|\alpha| = m$, the L^p-L^q estimate comes in to handle the highest order derivative term $\partial_x^\alpha \nabla_x \phi^k$:

$$\begin{aligned} &\Big\langle \partial_x^\alpha \nabla_x \phi^k \cdot \nabla_v f^{k+1},\ |\partial_x^\alpha f^{k+1}|^{p-2} \cdot \partial_x^\alpha f^{k+1} \cdot w \Big\rangle \\ &\lesssim \|\partial_x^\alpha \nabla_x \phi^k\|_{L^q_x} \|\nabla_v f^{k+1}\|_{L^n_{x,v}(w)} \|\partial_x^\alpha f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}. \end{aligned} \tag{2.2}$$

In turn, in order to get this L^p-L^q estimate involving $\nabla_x \phi$, we introduce weight w; surprisingly, this weight w also plays another crucial role to deal with the higher order derivative. More precisely, we do this trick when $|\alpha| + |\beta| = m$, w could "absorb" the extra derivative in ∇_v as follows:

$$\begin{aligned} &\Big\langle \nabla_x \phi^k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^{k+1},\ |\partial_x^\alpha \partial_v^\beta f^{k+1}|^{p-2}(\partial_x^\alpha \partial_v^\beta f^{k+1})w \Big\rangle \\ &\sim \Big\langle \nabla_x \phi^k \cdot \nabla_v |\partial_x^\alpha \partial_v^\beta f^{k+1}|^p,\ w \Big\rangle \sim -\Big\langle \nabla_x \phi^k \cdot |\partial_x^\alpha \partial_v^\beta f^{k+1}|^p,\ \nabla_v w \Big\rangle. \end{aligned} \tag{2.3}$$

Before we give the proof of the main theorem, we would like to establish the following L^p-L^q estimates.

3 L^p-L^q estimates

In this section, we are going to prove the L^p-L^q estimate which plays an essentially important role in our proofs.

Lemma 3.1. *Suppose* $1 < p < \frac{n}{2}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. *If* $-\Delta\phi = \int_{\mathbb{R}^n} f dv =: g$, *then it holds that*

$$\|\nabla_x \phi\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)}, \tag{3.1}$$

Proof. Note that $\nabla_x\phi = \nabla_x(I_2 * g)$, with $I_2(x) = \frac{1}{(n-2)\omega_{n-1}} \cdot \frac{1}{|x|^{n-2}}$, for more details, see the last section Appendix. Therefore there holds

$$\begin{aligned} \|\nabla_x\phi\|_{L^q(\mathbb{R}^n)} &= \|\nabla_x(I_2 * g)\|_{L^q(\mathbb{R}^n)} \\ &\lesssim \|(Mg)^{\frac{1}{2}} \cdot (I_2 * |g|)^{\frac{1}{2}}\|_{L^q(\mathbb{R}^n)} \\ &\lesssim \|(Mg)^{\frac{1}{2}}\|_{L^{q_1}(\mathbb{R}^n)} \cdot \|(I_2 * |g|)^{\frac{1}{2}}\|_{L^{q_2}(\mathbb{R}^n)} \\ &\lesssim \|Mg\|^{\frac{1}{2}}_{L^{\frac{q_1}{2}}(\mathbb{R}^n)} \cdot \|I_2 * |g|\|^{\frac{1}{2}}_{L^{\frac{q_2}{2}}(\mathbb{R}^n)}, \end{aligned}$$

where we applied (5.3) in the second line, and Hölder's inequality with

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}, \quad q_i > 1,$$

in the third line separately.

On the one hand, the boundedness of Hardy-Littlewood operator M as defined by identity (2.1) yields that

$$\|Mg\|_{L^{\frac{q_1}{2}}(\mathbb{R}^n)} \lesssim \|g\|_{L^{\frac{q_1}{2}}(\mathbb{R}^n)} = \|g\|_{L^p(\mathbb{R}^n)}, \tag{3.2}$$

since we require that $\frac{q_1}{2} = p$, *i.e.*

$$\frac{2}{q_1} = \frac{1}{p}. \tag{3.3}$$

On the other hand, by Lemma 5.3, we have

$$\|I_2 * |g|\|_{L^{\frac{q_2}{2}}(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)}, \tag{3.4}$$

where

$$\frac{2}{q_2} = \frac{1}{p} - \frac{2}{n}. \tag{3.5}$$

Consequently, $\|\nabla_x\phi\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|^{\frac{1}{2}}_{L^p(\mathbb{R}^n)} \cdot \|g\|^{\frac{1}{2}}_{L^p(\mathbb{R}^n)} = \|g\|_{L^p(\mathbb{R}^n)}$. □

A "derivative version" is immediate:

Corollary 3.2. *With the same assumptions as in Lemma 3.1, we have*

$$\|\partial_x^\alpha \nabla_x \phi\|_{L^q(\mathbb{R}^n)} \lesssim \|\partial_x^\alpha g\|_{L^p(\mathbb{R}^n)}.$$

Proof. One only needs to observe that

$$\nabla_x \partial_x^\alpha \phi = \partial_x^\alpha \nabla_x \phi = \partial_x^\alpha \nabla_x (I_2 * g) = \nabla_x (I_2 * \partial_x^\alpha g).$$

Applying Lemma 3.1 with ϕ and g replaced by $\partial^\alpha \phi$ and $\partial^\alpha g$ respectively, the desired result is immediate. □

Now we adapt Corollary 3.2 to the "kinetic version". To achieve this goal, we need to introduce a weight w.

Corollary 3.3. *Take* $g = \int_{\mathbb{R}^n} f dv$ *in Corollary 3.2, then we have*

$$\|\partial_x^\alpha \nabla_x \phi\|_{L_x^q(\mathbb{R}^n)} \lesssim \|\partial_x^\alpha f\|_{L^p_{x,v}(w)}.$$

Proof. Hölder's inequality leads to

$$\left|\int_{\mathbb{R}^n} \partial_x^\alpha f dv\right| \lesssim \left(\int_{\mathbb{R}^n} |\partial_x^\alpha f|^p w dv\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} w^{-\frac{p'}{p}} dv\right)^{\frac{1}{p'}}.$$

Note that $w = \langle v \rangle^\gamma$ and $\gamma \cdot \frac{p'}{p} > n$, which implies that

$$\left(\int_{\mathbb{R}^n} w^{-\frac{p'}{p}} dv\right)^{\frac{1}{p'}} \leq c.$$

Thus we end the proof of Corollary 3.3. □

An L^∞ estimate is also needed in the proof of the main Theorem 2.1.

Lemma 3.4. *Suppose* $-\Delta \phi = \int_{\mathbb{R}^n} f dv$. *If* $0 \leq |\alpha| \leq m-2, m \geq 3$, *then*

$$\|\partial_x^\alpha \nabla_x \phi\|_{L_x^\infty} \lesssim \sum_{|i| \leq 2} \|\partial_x^{i+\alpha} f\|_{L^p_{x,v}(w)}. \tag{3.6}$$

Proof. Choose a q such that $q > \frac{n}{2}$ and $p \leq q$, then $W^{2,q} \hookrightarrow L^\infty$. Thus we have

$$\|\partial_x^\alpha \nabla_x \phi\|_{L_x^\infty} \lesssim \|\partial_x^\alpha \nabla_x \phi\|_{W^{2,q}(\mathbb{R}_x^n)}.$$

Combining Corollary 3.2 and Corollary 3.3 leads to

$$\|\partial_x^\alpha \nabla_x \phi\|_{W^{2,q}} = \sum_{|i| \leq 2} \|\partial_x^i \partial_x^\alpha \nabla_x \phi\|_{L_x^q} \lesssim \sum_{|i| \leq 2} \|\partial_x^{i+\alpha} f\|_{L^p_{x,v}(w)},$$

i.e.

$$\|\partial_x^\alpha \nabla_x \phi\|_{L_x^\infty} \lesssim \sum_{|i| \leq 2} \|\partial_x^{i+\alpha} f\|_{L^p_{x,v}(w)}.$$ □

4 Proof of main theorem

Now we are in the position to prove Theorem 2.1. We split the proof into two parts which are existence and uniqueness.

Part I: Proof of existence. To prove the existence of the solution to (1.1), we adopt the L^p-version energy method and iteration method. In this process, we will apply the L^p-L^q estimate of electronic term $\nabla_x\phi$ proved in Lemma 3.1 to estimate J_3.

Proof. We consider the following iterating sequence for solving the Vlasov-Poisson system (1.1),

$$\begin{cases} \partial_t f^{k+1} + v\cdot\nabla_x f^{k+1} + \nabla_x\phi^k\cdot\nabla_v f^{k+1} = 0, & (4.1)\\ -\Delta\phi^k = \int_{\mathbb{R}^n} f^k dv, & (4.2)\\ f^{k+1}(0,x,v) = f_0(x,v). & (4.3)\end{cases}$$

Step 1. Applying $\partial_x^\alpha\partial_v^\beta$ to (4.1) with $\beta\neq 0, |\alpha|+|\beta|\le m$, starting with $f^0(t,x,v)=f_0(x,v)$, we have

$$\begin{aligned}&(\partial_t + v\cdot\nabla_x + \nabla_x\phi^k\cdot\nabla_v)\partial_x^\alpha\partial_v^\beta f^{k+1} + \sum_{\beta_1<\beta} C_\beta^{\beta_1}\partial_v^{\beta-\beta_1}v\cdot\partial_v^{\beta_1}\nabla_x\partial_x^\alpha f^{k+1}\\ &= -\sum_{0\neq\alpha_1\le\alpha} C_\alpha^{\alpha_1}\partial_x^{\alpha_1}\nabla_x\phi^k\cdot\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v f^{k+1}.\end{aligned} \tag{4.4}$$

Multiplying $|\partial_x^\alpha\partial_v^\beta f^{k+1}|^{p-2}(\partial_x^\alpha\partial_v^\beta f^{k+1})w$ on both sides of (4.4), and then integrating over $\mathbb{R}_x^n\times\mathbb{R}_v^n$ yields that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\partial_x^\alpha\partial_v^\beta f^{k+1}\|^p_{L^p_{x,v}(w)}\\ &+\underbrace{\sum_{\beta_1<\beta} C_\beta^{\beta_1}\left\langle\partial_v^{\beta-\beta_1}v\cdot\partial_v^{\beta_1}\nabla_x\partial_x^\alpha f^{k+1}, |\partial_x^\alpha\partial_v^\beta f^{k+1}|^{p-2}(\partial_x^\alpha\partial_v^\beta f^{k+1})w\right\rangle}_{J_1}\\ &=\underbrace{\left\langle\nabla_x\phi^k\cdot|\partial_x^\alpha\partial_v^\beta f^{k+1}|^p, \nabla_v w\right\rangle}_{J_2}\\ &-\underbrace{\sum_{0\neq\alpha_1\le\alpha} C_\alpha^{\alpha_1}\left\langle\partial_x^{\alpha_1}\nabla_x\phi^k\cdot\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v f^{k+1}, |\partial_x^\alpha\partial_v^\beta f^{k+1}|^{p-2}(\partial_x^\alpha\partial_v^\beta f^{k+1})w\right\rangle}_{J_3}.\end{aligned} \tag{4.5}$$

We now estimate (4.5) term by term.

For J_1, note that $|\partial_v^{\beta-\beta_1} v| \le c$, $\beta_1 < \beta$. Thus,

$$
\begin{aligned}
J_1 &\lesssim \sum_{\beta_1<\beta} \int_{\mathbb{R}^{2n}} |\partial_v^{\beta_1} \nabla_x \partial_x^{\alpha} f^{k+1}| w^{\frac{1}{p}} \cdot |\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}|^{p-1} w^{\frac{1}{p'}} \, dxdv \\
&\lesssim \sum_{\beta_1<\beta} \left(\int_{\mathbb{R}^{2n}} |\partial_v^{\beta_1} \nabla_x \partial_x^{\alpha} f^{k+1}|^p w \, dxdv \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^{2n}} |\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}|^{(p-1)p'} w \, dxdv \right)^{\frac{1}{p'}} \\
&\lesssim \sum_{\beta_1<\beta} \|\partial_v^{\beta_1} \nabla_x \partial_x^{\alpha} f^{k+1}\|_{L^p_{x,v}(w)} \cdot \|\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}\|^{p-1}_{L^p_{x,v}(w)},
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, *i.e.* $(p-1)p' = p$, $\frac{p}{p'} = p-1$.
For J_2, note $|\nabla_v w| \le w$, by Lemma 3.4, we have

$$
\begin{aligned}
J_2 &\lesssim \|\nabla_x \phi^k\|_{L^\infty_x} \|\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}\|^p_{L^p_{x,v}(w)} \\
&\lesssim \sum_{|i|\le 2} \|\partial_x^i f^k\|_{L^p_{x,v}(w)} \|\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}\|^p_{L^p_{x,v}(w)}.
\end{aligned}
$$

For J_3, we consider two cases individually.

Case 1: Recall $|\alpha| \le m-1$, if $0 < |\alpha_1| \le m-2, m \ge 3$, Lemma 3.4 leads to

$$
\|\partial_x^{\alpha_1} \nabla_x \phi^k\|_{L^\infty_x} \lesssim \sum_{|i|\le 2} \|\partial_x^{i+\alpha_1} f^k\|_{L^p_{x,v}(w)}.
$$

Note $|i| + |\alpha_1| \le m-2+2 = m$, the order of the derivatives does not exceed m, then we obtain,

$$
\begin{aligned}
J_3 &\lesssim \sum_{0<|\alpha_1|\le m-2} \int_{\mathbb{R}^n} \|\partial_x^{\alpha_1} \nabla_x \phi^k\|_{L^\infty_x} \|\partial_x^{\alpha-\alpha_1} \partial_v^{\beta} \nabla_v f^{k+1}\|_{L^p_x} \left\| |\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}|^{p-1} \right\|_{L^{p'}_x} w \, dv \\
&\lesssim \sum_{0<|\alpha_1|\le m-2} \sum_{|i|\le 2} \|\partial_x^{i+\alpha_1} f^k\|_{L^p_{x,v}(w)} \|\partial_x^{\alpha-\alpha_1} \partial_v^{\beta} \nabla_v f^{k+1}\|_{L^p_{x,v}(w)} \|\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}\|^{p-1}_{L^p_{x,v}(w)},
\end{aligned}
$$

where $|\alpha-\alpha_1| + |\beta| + 1 \le |\alpha| + |\beta| \le m$.

Case 2: $|\alpha_1| = m-1$, we have

$$
\begin{aligned}
J_3 &\lesssim \sum_{|\alpha_1|=m-1} \int_{\mathbb{R}^n} w(v) \|\partial_x^{\alpha_1} \nabla_x \phi^k\|_{L^q_x} \|\partial_x^{\alpha-\alpha_1} \partial_v^{\beta} \nabla_v f^{k+1}\|_{L^n_x} \left\| |\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}|^{p-1} \right\|_{L^{p'}_x} dv \\
&\lesssim \sum_{|\alpha_1|=m-1} \|\partial_x^{\alpha_1} \nabla_x \phi^k\|_{L^q_x} \int_{\mathbb{R}^n} w(v) \sum_{|i|\le m-2} \|\partial_x^i \partial_x^{\alpha-\alpha_1} \partial_v^{\beta} \nabla_v f^{k+1}\|_{L^p_x} \left\| |\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}|^{p-1} \right\|_{L^{p'}_x} dv \\
&\lesssim \sum_{|\alpha_1|=m-1} \|\partial_x^{\alpha_1} f^k\|_{L^p_{x,v}(w)} \sum_{|i|\le m-2} \|\partial_x^i \partial_x^{\alpha-\alpha_1} \partial_v^{\beta} \nabla_v f^{k+1}\|_{L^p_{x,v}(w)} \|\partial_x^{\alpha} \partial_v^{\beta} f^{k+1}\|^{p-1}_{L^p_{x,v}(w)},
\end{aligned}
$$

where in the first inequality, we applied Hölder's inequality with respect to x with

$$
\frac{1}{q} + \frac{1}{n} + \frac{1}{p'} = 1, \quad \frac{1}{p} + \frac{1}{p'} = 1.
$$

And in the second inequality, we used the embedding $W^{m-2,p} \hookrightarrow L^n$, with

$$
m > \frac{n}{p} + 1, \quad p \le n. \tag{4.6}
$$

In the third inequality, we applied Corollary 3.3 and Hölder's inequality in v.

Finally, plugging all the estimates of J_1, J_2, and J_3 into (4.5) yields that

$$\begin{aligned}\frac{d}{dt}\|\partial_x^\alpha\partial_v^\beta f^{k+1}\|^p_{L^p_{x,v}(w)} &\lesssim \sum_{\beta_1<\beta}\|\partial_v^{\beta_1}\nabla_x\partial_x^\alpha f^{k+1}\|_{L^p_{x,v}(w)}\cdot\|\partial_x^\alpha\partial_v^\beta f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}\\
&+\sum_{|i|\le 2}\|\partial_x^i f^k\|_{L^p_{x,v}(w)}\|\partial_x^\alpha\partial_v^\beta f^{k+1}\|^p_{L^p_{x,v}(w)}\\
&+\sum_{0<|\alpha_1|\le m-2}\sum_{|i|\le 2}\|\partial_x^{i+\alpha_1}f^k\|_{L^p_{x,v}(w)}\|\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v f^{k+1}\|_{L^p_{x,v}(w)}\|\partial_x^\alpha\partial_v^\beta f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}\\
&+\sum_{|\alpha_1|=m-1}\sum_{|i|\le m-2}\|\partial_x^{\alpha_1}f^k\|_{L^p_{x,v}(w)}\|\partial_x^i\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v f^{k+1}\|_{L^p_{x,v}(w)}\|\partial_x^\alpha\partial_v^\beta f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}.\end{aligned}\tag{4.7}$$

Step 2. $\beta = 0, |\alpha| \le m$, applying ∂_x^α to (4.1) on both sides, we have

$$(\partial_t + v\cdot\nabla_x + \nabla_x\phi^k\cdot\nabla_v)\partial_x^\alpha f^{k+1} = -\sum_{0\ne\alpha_1\le\alpha} C_\alpha^{\alpha_1}\partial_x^{\alpha_1}\nabla_x\phi^k\cdot\partial_x^{\alpha-\alpha_1}\nabla_v f^{k+1}.\tag{4.8}$$

We could completely repeat the process of step 1, the only difference is that we do not need to estimate J_1, thus we give the estimates as below but omit the process of proof in details.

$$\begin{aligned}\frac{d}{dt}\|\partial_x^\alpha f^{k+1}\|^p_{L^p_{x,v}(w)} &\lesssim \sum_{|i|\le 2}\|\partial_x^i f^k\|_{L^p_{x,v}(w)}\|\partial_x^\alpha f^{k+1}\|^p_{L^p_{x,v}(w)}\\
&+\sum_{0<|\alpha_1|\le m-2}\sum_{|i|\le 2}\|\partial_x^{i+\alpha_1}f^k\|_{L^p_{x,v}(w)}\|\partial_x^{\alpha-\alpha_1}\nabla_v f^{k+1}\|_{L^p_{x,v}(w)}\|\partial_x^\alpha f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}\\
&+\sum_{m-1\le|\alpha_1|\le m}\sum_{|i|\le m-2}\|\partial_x^{\alpha_1}f^k\|_{L^p_{x,v}(w)}\|\partial_x^i\partial_x^{\alpha-\alpha_1}\nabla_v f^{k+1}\|_{L^p_{x,v}(w)}\|\partial_x^\alpha f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}.\end{aligned}\tag{4.9}$$

Collecting the estimates of J_1, J_2 and J_3 and integrating over $[0,t]$ of (4.5), summing over $|\alpha|+|\beta|\le m$, we deduce from the definition of $\mathcal{E}(f(t))$ that

$$\mathcal{E}(f^{k+1}(t)) \le \mathcal{E}(f_0) + Ct\sup_{0\le s\le t}\mathcal{E}(f^{k+1}(s)) + Ct\sup_{0\le s\le t}(\mathcal{E}(f^k(s)))^{\frac{1}{p}}\cdot\sup_{0\le s\le t}\mathcal{E}(f^{k+1}(s)).$$

Inductively, assume $\sup\limits_{0\le s\le T^*(M)}\mathcal{E}(f^k(s)) \le M$, $T^*(M)$ and M are sufficiently small; note that $f^0(t,x,v)\equiv f_0(x,v)$, $\mathcal{E}(f_0)\le\frac{M}{2}$, we have

$$\mathcal{E}(f^{k+1}(t)) \le \frac{M}{2} + Ct\sup_{0\le s\le t}\mathcal{E}(f^{k+1}(s)) + CM^{\frac{1}{p}}\cdot t\sup_{0\le s\le t}\mathcal{E}(f^{k+1}(s)),$$

i.e.

$$(1 - CT^* - CM^{\frac{1}{p}}T^*(M))\sup_{0\le s\le T^*(M)}\mathcal{E}(f^{k+1}(s)) \le \frac{M}{2}.$$

Thus $\sup\limits_k\sup\limits_{0\le s\le T^*(M)}\mathcal{E}(f^k(s)) \le M$, *i.e.* we get a uniform-in-k estimate.

As a routine, let $k\to\infty$, we obtain the solution and complete the proof of existence.

Remark 4.1. *We summarize the indices as follows:*

$$\begin{cases} \dfrac{2}{q_1} = \dfrac{1}{p}, \quad 1 < p < \dfrac{n}{2}, \quad n \geq 3, & (4.10) \\ \dfrac{2}{q_2} = \dfrac{1}{p} - \dfrac{2}{n}, & (4.11) \\ \dfrac{1}{q_1} + \dfrac{1}{q_2} = \dfrac{1}{q}, \quad q > 1, \quad q_i > 1, \ i = 1, 2, & (4.12) \\ q > \dfrac{n}{2}, & (4.13) \\ m > \dfrac{n}{p} + 1, \quad m, n \in \mathbb{N}, & (4.14) \\ \dfrac{1}{q} + \dfrac{1}{p_1} = \dfrac{1}{p}, & (4.15) \\ \gamma > n(p-1). & (4.16) \end{cases}$$

In fact, for any given (m, n, p) satisfying

$$\begin{cases} m > \frac{n}{p} + 1, m \in \mathbb{N}, \\ n \geq 3, \\ \frac{n}{3} < p < \frac{n}{2}, \end{cases} \tag{4.17}$$

we could designate

$$\begin{cases} q_1 = 2p, \\ q_2 = \frac{2np}{n-2p}, \\ q = \frac{np}{n-1}. \end{cases} \tag{4.18}$$

□

Let us move on to proving the uniqueness.

Part II: Proof of uniqueness. The proof of the uniqueness is analogous to the existence part. However, we use a different energy norm $\mathcal{E}_1(f(t)) =: \sum_{|\alpha|+|\beta| \leq m-1} \|\partial_x^\alpha \partial_v^\beta f(t)\|_{L^p_{x,v}(w)}^p$ because of a difficult term $\tilde{J}_4$. In $\tilde{J}_4$, there is a term

$$\Big\langle \nabla_x(\phi_f - \phi_g) \cdot \partial_x^\alpha \partial_v^\beta \nabla_v g, \ |\partial_x^\alpha \partial_v^\beta (f-g)|^{p-2} \cdot \partial_x^\alpha \partial_v^\beta (f-g) w \Big\rangle.$$

If we still work with $\mathcal{E}(f(t)) = \sum_{|\alpha|+|\beta| \leq m} \|\partial_x^\alpha \partial_v^\beta f(t)\|_{L^p_{x,v}(w)}^p$ as in the existence part, the order of derivative of $\partial_x^\alpha \partial_v^\beta \nabla_v g$ will be $m+1$ which exceeds m when $|\alpha| + |\beta| = m$. This is the main reason we choose $\mathcal{E}_1(f(t))$ instead of $\mathcal{E}(f(t))$.

Proof. Assume another solution g exists such that $\sup_{0\le s\le T^*}\mathcal{E}(g(s))\le M$, taking the difference, we have

$$\begin{cases} (\partial_t + v\cdot\nabla_x + \nabla_x\phi_f\cdot\nabla_v)(f-g) + (\nabla_x\phi_f - \nabla_x\phi_g)\cdot\nabla_v g = 0, \\ -\Delta_x(\phi_f-\phi_g) = \displaystyle\int_{\mathbb{R}^n}(f-g)\,dv, \\ f(0,x,v) = g(0,x,v). \end{cases} \tag{4.19}$$

Step 1. Applying $\partial_x^\alpha\partial_v^\beta$ on both sides of $(4.19)_1$ with $\beta\neq 0$, $|\alpha|+|\beta|\le m-1$, we have

$$\begin{aligned} &(\partial_t + v\cdot\nabla_x + \nabla_x\phi_f\cdot\nabla_v)\cdot\partial_x^\alpha\partial_v^\beta(f-g) + \sum_{\beta_1<\beta} C_\beta^{\beta_1}\partial_v^{\beta-\beta_1}v\cdot\partial_v^{\beta_1}\nabla_x\partial_x^\alpha(f-g) \\ &= -\sum_{0\neq\alpha_1\le\alpha} C_\alpha^{\alpha_1}\partial_x^{\alpha_1}\nabla_x\phi_f\cdot\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v(f-g) \\ &\quad - \sum_{0\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\partial_x^{\alpha_1}(\nabla_x\phi_f-\nabla_x\phi_g)\cdot\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v g. \end{aligned} \tag{4.20}$$

Multiplying $|\partial_x^\alpha\partial_v^\beta(f-g)|^{p-2}\cdot\partial_x^\alpha\partial_v^\beta(f-g)w$ on both sides of (4.20), and then integrating over $\mathbb{R}_x^n\times\mathbb{R}_v^n$ yields that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|\partial_x^\alpha\partial_v^\beta(f-g)\|^p_{L^p_{x,v}(w)} \\ &\quad + \underbrace{\sum_{\beta_1<\beta} C_\beta^{\beta_1}\left\langle\partial_v^{\beta-\beta_1}v\cdot\partial_v^{\beta_1}\nabla_x\partial_x^\alpha(f-g), |\partial_x^\alpha\partial_v^\beta(f-g)|^{p-2}\cdot\partial_x^\alpha\partial_v^\beta(f-g)w\right\rangle}_{\tilde{J}_1} \\ &= \underbrace{\left\langle\nabla_x\phi_f\cdot|\partial_x^\alpha\partial_v^\beta(f-g)|^p, \nabla_v w\right\rangle}_{\tilde{J}_2} \\ &\quad - \underbrace{\sum_{0\neq\alpha_1\le\alpha} C_\alpha^{\alpha_1}\left\langle\partial_x^{\alpha_1}\nabla_x\phi_f\cdot\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v(f-g), |\partial_x^\alpha\partial_v^\beta(f-g)|^{p-2}\cdot\partial_x^\alpha\partial_v^\beta(f-g)w\right\rangle}_{\tilde{J}_3} \\ &\quad - \underbrace{\sum_{0\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\left\langle\partial_x^{\alpha_1}(\nabla_x\phi_f-\nabla_x\phi_g)\cdot\partial_x^{\alpha-\alpha_1}\partial_v^\beta\nabla_v g, |\partial_x^\alpha\partial_v^\beta(f-g)|^{p-2}\cdot\partial_x^\alpha\partial_v^\beta(f-g)w\right\rangle}_{\tilde{J}_4}. \end{aligned} \tag{4.21}$$

We could repeat the estimates in the proof of the existence except for some special term. Thus we would like to write down the estimates directly without the details.

For $\tilde{J}_1$, we have

$$\tilde{J}_1 \lesssim \sum_{\beta_1<\beta}\|\partial_v^{\beta_1}\nabla_x\partial_x^\alpha(f-g)\|_{L^p_{x,v}(w)}\cdot\|\partial_x^\alpha\partial_v^\beta(f-g)\|^{p-1}_{L^p_{x,v}(w)}.$$

For $\tilde{J}_2$, we get

$$\tilde{J}_2 \lesssim \sum_{|i|\le 2}\|\partial_x^i f\|_{L^p_{x,v}(w)}\cdot\|\partial_x^\alpha\partial_v^\beta(f-g)\|^p_{L^p_{x,v}(w)}.$$

For $\tilde{J}_3$, since $0<|\alpha_1|\le m-2$, we have

$$\tilde{J}_3 \lesssim \sum_{0<|\alpha_1|\le m-2}\sum_{|i|\le 2}\|\partial_x^{i+\alpha_1}f\|_{L^p_{x,v}(w)}\cdot\|\partial_x^{\alpha-\alpha_1}\partial_v^{\beta}\nabla_v(f-g)\|_{L^p_{x,v}(w)}\cdot\|\partial_x^{\alpha}\partial_v^{\beta}(f-g)\|^{p-1}_{L^p_{x,v}(w)},$$

where $|\alpha-\alpha_1|+|\beta|+1\le|\alpha|+|\beta|\le m-1$.

For $\tilde{J}_4$, note that $-\Delta_x(\phi_f-\phi_g)=\displaystyle\int_{\mathbb{R}^n}(f-g)\,dv$. We consider two cases separately.

Case 1: $0\le|\alpha_1|\le m-3$

$$\tilde{J}_4 \lesssim \sum_{0\le|\alpha_1|\le m-3}\sum_{|i|\le 2}\|\partial_x^{i+\alpha_1}(f-g)\|_{L^p_{x,v}(w)}\cdot\|\partial_x^{\alpha-\alpha_1}\partial_v^{\beta}\nabla_v g\|_{L^p_{x,v}(w)}\cdot\|\partial_x^{\alpha}\partial_v^{\beta}(f-g)\|^{p-1}_{L^p_{x,v}(w)},$$

where $|i|+|\alpha_1|\le 2+m-3=m-1$ and

$$|\alpha-\alpha_1|+|\beta|+1\le|\alpha|+|\beta|-|\alpha_1|+1\le m-1-|\alpha_1|+1\le m.$$

Case 2: $|\alpha_1|=m-2$

$$\tilde{J}_4 \lesssim \sum_{|\alpha_1|=m-2}\sum_{|i|\le m-2}\|\partial_x^{\alpha_1}(f-g)\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^{i}\partial_x^{\alpha-\alpha_1}\partial_v^{\beta}\nabla_v g\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^{\alpha}\partial_v^{\beta}(f-g)\|^{p-1}_{L^p_{x,v}(w)},$$

where $|i|+|\alpha-\alpha_1|+|\beta|+1\le m-2+|\alpha|-|\alpha_1|+|\beta|+1\le m$.

Collecting all the estimates of $\tilde{J}_j$, $j=1,2,3,4$, we have

$$\begin{aligned}
\frac{d}{dt}\|\partial_x^{\alpha}\partial_v^{\beta}(f-g)\|^p_{L^p_{x,v}(w)} &\lesssim \sum_{\beta_1<\beta}\|\partial_v^{\beta_1}\nabla_x\partial_x^{\alpha}(f-g)\|_{L^p_{x,v}(w)}\cdot\|\partial_x^{\alpha}\partial_v^{\beta}(f-g)\|^{p-1}_{L^p_{x,v}(w)}\\
&+\sum_{|i|\le 2}\|\partial_x^{i}f\|_{L^p_{x,v}(w)}\cdot\|\partial_x^{\alpha}\partial_v^{\beta}(f-g)\|^{p}_{L^p_{x,v}(w)}\\
&+\sum_{0<|\alpha_1|\le m-2}\ \sum_{|i|\le 2}\|\partial_x^{i+\alpha_1}f\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^{\alpha-\alpha_1}\partial_v^{\beta}\nabla_v(f-g)\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^{\alpha}\partial_v^{\beta}(f-g)\|^{p-1}_{L^p_{x,v}(w)}\\
&+\sum_{0\le|\alpha_1|\le m-3}\ \sum_{|i|\le 2}\|\partial_x^{i+\alpha_1}(f-g)\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^{\alpha-\alpha_1}\partial_v^{\beta}\nabla_v g\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^{\alpha}\partial_v^{\beta}(f-g)\|^{p-1}_{L^p_{x,v}(w)}\\
&+\sum_{|\alpha_1|=m-2}\ \sum_{|i|\le m-2}\|\partial_x^{\alpha_1}(f-g)\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^{i}\partial_x^{\alpha-\alpha_1}\partial_v^{\beta}\nabla_v g\|_{L^p_{x,v}(w)}\cdot\ \|\partial_x^{\alpha}\partial_v^{\beta}(f-g)\|^{p-1}_{L^p_{x,v}(w)}.
\end{aligned}\tag{4.22}$$

Step 2. $\beta=0$, $|\alpha|\le m-1$, applying ∂_x^{α} on both sides of $(4.19)_1$ yields

$$\begin{aligned}
(\partial_t+v\cdot\nabla_x+\nabla_x\phi_f\cdot\nabla_v)\partial_x^{\alpha}(f-g)=&-\sum_{0\ne\alpha_1\le\alpha}C_{\alpha}^{\alpha_1}\partial_x^{\alpha_1}\nabla_x\phi_f\cdot\partial_x^{\alpha-\alpha_1}\nabla_v(f-g)\\
&-\sum_{0\le\alpha_1\le\alpha}C_{\alpha}^{\alpha_1}\partial_x^{\alpha_1}(\nabla_x\phi_f-\nabla_x\phi_g)\cdot\partial_x^{\alpha-\alpha_1}\nabla_v g.
\end{aligned}\tag{4.23}$$

Repeating the process of step 1, we get

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x^\alpha (f-g)\|^p_{L^p_{x,v}(w)} \lesssim \sum_{|i|\le 2} \|\partial_x^i f\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^\alpha (f-g)\|^p_{L^p_{x,v}(w)} \\
&+ \sum_{0<|\alpha_1|\le m-2} \ \sum_{|i|\le 2} \|\partial_x^{i+\alpha_1} f\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^{\alpha-\alpha_1}\nabla_v (f-g)\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^\alpha (f-g)\|^{p-1}_{L^p_{x,v}(w)} \\
&+ \sum_{|\alpha_1|=m-1} \ \sum_{|i|\le m-2} \|\partial_x^{\alpha_1} f\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^i\partial_x^{\alpha-\alpha_1}\nabla_v (f-g)\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^\alpha (f-g)\|^{p-1}_{L^p_{x,v}(w)} \\
&+ \sum_{0<|\alpha_1|\le m-3} \ \sum_{|i|\le 2} \|\partial_x^{i+\alpha_1} (f-g)\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^{\alpha-\alpha_1}\nabla_v g\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^\alpha \partial_v^\beta (f-g)\|^{p-1}_{L^p_{x,v}(w)} \\
&+ \sum_{m-2\le|\alpha_1|\le m-1} \ \sum_{|i|\le m-2} \|\partial_x^{\alpha_1} (f-g)\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^i\partial_x^{\alpha-\alpha_1}\nabla_v g\|_{L^p_{x,v}(w)} \cdot \ \|\partial_x^\alpha (f-g)\|^{p-1}_{L^p_{x,v}(w)}.
\end{aligned} \tag{4.24}$$

Note $f(0,x,v) = g(0,x,v)$,

$$\sup_{0\le s\le t} \|\partial_x^{i+\alpha_1} f(s)\|_{L^p_{x,v}(w)} \le M, \quad \sup_{0\le s\le t} \|\partial_x^{\alpha-\alpha_1}\partial_v^\beta \nabla_v g(s)\|_{L^p_{x,v}(w)} \le M,$$

and

$$\sup_{0\le s\le t} \|\partial_x^i\partial_x^{\alpha-\alpha_1}\partial_v^\beta \nabla_v g(s)\|_{L^p_{x,v}(w)} \le M, \quad \sup_{0\le s\le t} \|\partial_x^i f(s)\|_{L^p_{x,v}(w)} \le M.$$

Integrating (4.22) and (4.24) over $[0,t]$, then summing over $|\alpha|+|\beta|\le m-1$, we deduce

$$\mathcal{E}_1((f-g)(t)) \lesssim (1+M)\int_0^t \mathcal{E}_1((f-g)(s))\, ds,$$

where

$$\mathcal{E}_1(f(t)) =: \sum_{|\alpha|+|\beta|\le m-1} \|\partial_x^\alpha\partial_v^\beta f(t)\|^p_{L^p_{x,v}(w)}.$$

By Gronwall's inequality, we have $\mathcal{E}_1((f-g)(t)) \equiv 0$ implying $f\equiv g$, which completes the proof of uniqueness. Thus we end the proof of Theorem 2.1. □

Remark 4.2. *All in all, we improved the results in [4] to the more general function space $W^{m,p}(\mathbb{R}^n)$ which does not have to be $C^2(\mathbb{R}^n)$ (too strong). Our results also shed light on exploring solutions in Sobolev spaces. We are very confident that our method could be applied in fractional Sobolev spaces, even the supercritical spaces which are far from being understood yet.*

5 Appendix

For the sake of completeness, we cite some known results about the estimate for the Riesz potential.

First of all, we give the pointwise estimate of the Riesz potential, for more details, see chapter 3, section 1, page 57 in [1].

Proposition 5.1 ([1]). *For any multi-index ξ with $|\xi| < \alpha < n$, there is a constant A such that for any $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and almost every x, we have*

$$|D^\xi(I_\alpha * f(x))| \leq AMf(x)^{\frac{|\xi|}{\alpha}} \cdot (I_\alpha * |f|(x))^{1-\frac{|\xi|}{\alpha}}, \tag{5.1}$$

where $I_\alpha = \dfrac{\gamma_\alpha}{|x|^{n-\alpha}}$, $\gamma_\alpha = \dfrac{\Gamma(n-\frac{\alpha}{2})}{\pi^{\frac{n}{2}}2^\alpha\Gamma(\frac{\alpha}{2})}$.

Remark 5.2. *In our paper, we consider* $-\Delta\phi = \displaystyle\int_{\mathbb{R}^n} f dv =: g$, $n \geq 3$. *Thus, in our context,* I_α *can be taken*

$$I_2(x) = \frac{1}{(n-2)\omega_{n-1}} \cdot \frac{1}{|x|^{n-2}}, \quad i.e. \quad \alpha = 2, \tag{5.2}$$

where $\omega_{n-1} = \dfrac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ *is the* $(n-1)-$*dimensional area of the unit sphere in* $\mathbb{R}^n$, *then we have*

$$|D^\xi(I_2 * g(x))| \leq cMg(x)^{\frac{|\xi|}{2}} \cdot (I_2 * |g|(x))^{1-\frac{|\xi|}{2}}. \tag{5.3}$$

Next, we give the off-diagonal estimate of the Riesz potential I_2. For the details, see chapter V, section 1 and page 119 in [14].

Lemma 5.3 ([14]). *If* $-\Delta\phi = g \in L^p(\mathbb{R}^n)$, *then* $\phi = I_2 * g$ *and*

$$\|I_2 * g\|_{L^{\tilde{q}}(\mathbb{R}^n)} \leq c\|g\|_{L^p(\mathbb{R}^n)}, \tag{5.4}$$

where $1 < p < \frac{n}{2}, c = c(p, \tilde{q})$ *and*

$$\frac{1}{\tilde{q}} = \frac{1}{p} - \frac{2}{n}. \tag{5.5}$$

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Variational methods to second-order Dirichlet boundary value problems with impulses on the half-line

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ABSTRACT

In this paper, the existence of solutions for a second-order impulsive differential equation with a parameter on the half-line is investigated. Applying Lax-Milgram Theorem, we deal with a linear Dirichlet impulsive problem, while the nonlinear case is established by using standard results of critical point theory.

RESUMEN

En este artículo, se investiga la existencia de soluciones de una ecuación diferencial de segundo orden impulsiva con un parámetro en la semi-recta. Aplicando el Teorema de Lax-Milgram, tratamos un problema lineal impulsivo de Dirichlet, mientras que el caso no lineal es establecido usando resultados estándar de teoría de punto crítico.

Keywords and Phrases: Dirichlet boundary value problem, half-line, Lax-Milgram theorem, critical points, impulsive differential equation.

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1 Introduction

In recent years, many researchers have extensively applied variational methods to study boundary value problems (BVPs) for impulsive differential equations on the finite intervals. More precisely, employing critical point theory, Nieto and O'Regan [8] studied a linear Dirichlet boundary value problem with impulses

$$\begin{cases} -u''(t)+\lambda u(t) &= \sigma(t), \quad \text{a.e. } t\in[0,T],\\ \triangle u'(t_j) &= d_j, \quad j\in\{1,2,\dots,l\},\\ u(0)=u(T) &= 0, \end{cases} \tag{1.1}$$

and a nonlinear impulsive problem

$$\begin{cases} -u''(t)+\lambda u(t) &= f(t,u(t)), \quad \text{a.e. } t\in[0,T],\\ \triangle u'(t_j) &= I_j(u(t_j^-)), \quad j\in\{1,2,\dots,l\},\\ u(0)=u(T) &= 0, \end{cases} \tag{1.2}$$

where λ is a positive parameter.

Moreover, the study of solutions for impulsive BVPs on the infinite intervals by using variational methods has received considerably more attention, see for example [1, 2, 3, 9, 10], and the references therein.

In the present paper, our aim is to improve some assumptions made in [8] in order to extend problems (1.1) and (1.2) on the half-line via variational approach.

This paper is organized as follows. In Section 2 we state some preliminaries. In Section 3 we consider the linear Dirichlet problem with impulses in the derivative. Due to the Lax-Milgram Theorem, we show the existence of weak solutions that are precisely the critical points of some functionals. The last section is to deal with the nonlinear Dirichlet problem. To investigate the existence of solutions, we use standard results of critical point theory. Also, some examples are given to illustrate our main results.

2 Preliminaries

We cite some basic and celebrated theorems from critical point theory which are crucial tools in the proof of our main results.

Let H be a Hilbert space.

Theorem 2.1 (Lax-Milgram [4, 5]). *Let $a: H\times H\to\mathbb{R}$ be a bounded bilinear form. If a is coercive, i.e., there exists $\alpha>0$ such that $a(u,u)\geq\alpha\|u\|^2$ for every $u\in H$, then for any $\sigma\in H'$ (the conjugate space of H) there exists a unique $u\in H$ such that*

$$a(u,v)=(\sigma,v), \quad \textit{for every} \quad v\in H.$$

Moreover, if a is also symmetric, then the functional $\varphi : H \to \mathbb{R}$ defined by

$$\varphi(v) = \frac{1}{2}a(v,v) - (\sigma, v)$$

attains its minimum at u.

Theorem 2.2 ([7]). *If φ is weakly lower semi-continuous (w.l.s.c.) on a reflexive Banach space X and has a bounded minimizing sequence, then φ has a minimum on X.*

Now, let us recall some necessary concepts that will be needed in our argument. Let us define the following reflexive Banach space

$$H_0^1(0,\infty) = \left\{u : [0,\infty) \to \mathbb{R} \text{ is absolutely continuous, } u, u' \in L^2(0,\infty),\ u(0) = u(\infty) = 0\right\},$$

equipped with the norm

$$\|u\| = \left(\int_0^{+\infty} |u(t)|^2 dt + \int_0^{+\infty} |u'(t)|^2 dt \right)^{\frac{1}{2}}.$$

Set the space

$$C_{l,p}[0,+\infty) = \{u \in C([0,+\infty),\mathbb{R}) : \lim_{t\to\infty} p(t)u(t) \text{ exists}\}$$

with the norm

$$\|u\|_{\infty,p} = \sup_{t\in[0,+\infty)} p(t)|u(t)|,$$

where the function $p : [0;+\infty) \to (0,+\infty)$ is continuously differentiable and bounded, satisfying

$$C = 2\max(\|p\|_{L^2}, \|p'\|_{L^2}) < +\infty.$$

Concerning the above spaces, we get the following vital embeddings.

Lemma 2.3 ([6]). *The space $H_0^1(0,\infty)$ embeds continuously in $C_{l,p}[0,\infty)$, more precisely $\|u\|_{\infty,p} \leq C\|u\|$ for every $u \in H_0^1(0,\infty)$.*

Lemma 2.4 ([6]). *The embedding $H_0^1(0,\infty) \hookrightarrow C_{l,p}[0,\infty)$ is compact.*

3 Impulsive linear problem

We consider the following linear Dirichlet boundary value problem with impulses in the derivative at the prescribed instants t_j, $j \in \mathbb{N}^* = \{1,2,3,\dots\}$

$$\begin{cases} -u''(t) + \lambda u(t) = \sigma(t), & \text{a.e. } t \in [0,\infty),\ t \neq t_j, \\ \triangle u'(t_j) = d(t_j), & j \in \mathbb{N}^*, \\ u(0) = u(+\infty) = 0, \end{cases} \tag{3.1}$$

where $\lambda \in \mathbb{R}$, $\sigma \in L^2(0,\infty)$, $0 = t_0 < t_1 < t_2 < \cdots < t_j < \cdots < t_m \to \infty$, as $m \to \infty$, are the impulse points, $d : [0,\infty) \to \mathbb{R}$ satisfies $\sum_{j=1}^{\infty} \frac{d(t_j)}{p(t_j)} < \infty$ and $\triangle u'(t_j) = u'(t_j^+) - u'(t_j^-)$ for $u'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} u'(t)$.

Now, multiply the equation in problem (3.1) by $v \in H_0^1(0,\infty)$, and then integrate over $(0,+\infty)$, we obtain

$$-\int_0^{+\infty} u''v + \lambda \int_0^{+\infty} uv = \int_0^{+\infty} \sigma v.$$

We have

$$-\int_0^{+\infty} u''v = -\sum_{j=0}^{\infty} \int_{t_j}^{t_{j+1}} u''v,$$

and

$$\int_{t_j}^{t_{j+1}} u''v = u'(t_{j+1}^-)v(t_{j+1}^-) - u'(t_j^+)v(t_j^+) - \int_{t_j}^{t_{j+1}} u'v'.$$

Consequently,

$$\begin{aligned} -\int_0^{+\infty} u''v &= \sum_{j=1}^{\infty} \triangle u'(t_j)v(t_j) + u'(0)v(0) - u'(\infty)v(\infty) + \int_0^{+\infty} u'v' \\ &= \sum_{j=1}^{\infty} d(t_j)v(t_j) + \int_0^{+\infty} u'v'. \end{aligned}$$

This leads to define the bilinear form $a : H_0^1(0,\infty) \times H_0^1(0,\infty) \to \mathbb{R}$, by

$$a(u,v) = \int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv, \tag{3.2}$$

and the linear operator $l : H_0^1(0,\infty) \to \mathbb{R}$ by

$$l(v) = \int_0^{+\infty} \sigma v - \sum_{j=1}^{\infty} d(t_j)v(t_j). \tag{3.3}$$

Definition 3.1. *We say that a function u is a weak solution of the impulsive problem (3.1) if $u \in H_0^1(0,\infty)$ such that $a(u,v) = l(v)$ is valid for any $v \in H_0^1(0,\infty)$.*

In what follows we refer to problem (3.1) as (LP).

It is easily verified that a and l defined by (3.2), (3.3) respectively are continuous, and a is coercive if $\lambda > 0$.

Consider the functional $\varphi : H_0^1(0,\infty) \to \mathbb{R}$, defined by

$$\varphi(u) = \frac{1}{2} \int_0^{+\infty} u'^2 + \frac{\lambda}{2} \int_0^{+\infty} u^2 - \int_0^{+\infty} \sigma u + \sum_{j=1}^{\infty} d(t_j)u(t_j). \tag{3.4}$$

It is clear that φ is differentiable at any $u \in H_0^1(0,\infty)$ and

$$\varphi'(u)v \quad = \quad \int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv - \int_0^{+\infty} \sigma v + \sum_{j=1}^{\infty} d(t_j)v(t_j) = a(u,v) - l(v).$$

Thus, a critical point of (3.4) gives us a weak solution of the problem (LP).

Definition 3.2. *We mean by a classical solution of the problem (LP) a function $u \in H^2(t_j, t_{j+1})$ for all $j \in \mathbb{N}^*$, where*

$$H^2(t_j, t_{j+1}) = \left\{u : [0,\infty) \to \mathbb{R} \quad \textit{is absolutely continuous}, \ u', u'' \in L^2(t_j, t_{j+1})\right\},$$

and u satisfies the first equation of (3.1) a.e. on $[0,\infty)$ with $u(0) = u(\infty) = 0$, the limits $u'(t_j^+)$, $u'(t_j^-)$, $j \in \mathbb{N}^*$ exist and the impulse conditions hold.

Lemma 3.3. *If $u \in H_0^1(0,\infty)$ is a weak solution of (LP), then u is a classical solution of (LP).*

Proof. Since $u \in H_0^1(0,\infty)$, it is evident that $u(0) = u(\infty) = 0$.

For $j \in \{1, 2, \dots\}$, choose any $v \in H_0^1(0,\infty)$ such that $v(t) = 0$ for $t \in [0, t_j] \cup [t_{j+1}, +\infty)$. Then

$$\int_{t_j}^{t_{j+1}} u'v' + \lambda \int_{t_j}^{t_{j+1}} uv = \int_{t_j}^{t_{j+1}} \sigma v.$$

Hence, $-u'' + \lambda u = \sigma$ a.e. on (t_j, t_{j+1}). So, $u \in H^2(t_j, t_{j+1})$ and satisfies the previous equation a.e. on $[0,\infty)$.

Multiplying $-u'' + \lambda u = \sigma$ by $v \in H_0^1(0,\infty)$ and integrating over $[0,\infty)$, we get

$$\sum_{j=1}^{\infty} \triangle u'(t_j)v(t_j) = \sum_{j=1}^{\infty} d(t_j)v(t_j).$$

Therefore, $\triangle u'(t_j) = d(t_j)$ for every $j \in \mathbb{N}^*$, and the impulsive conditions are satisfied. □

Lemma 3.4. *If $u \in H_0^1(0,\infty)$ is a critical point of φ defined by (3.4), then u is a weak solution of the impulsive Dirichlet problem (LP).*

Proof. Let $u \in H_0^1(0,\infty)$. The assumption that u is a critical point of φ means that $\varphi'(u)v = 0$, for all $v \in H_0^1(0,\infty)$. Thus,

$$\int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv - \int_0^{+\infty} \sigma v + \sum_{j=1}^{\infty} d(t_j)v(t_j) = 0, \quad \forall v \in H_0^1(0,\infty).$$

Hence,

$$\int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv = \int_0^{+\infty} \sigma v - \sum_{j=1}^{\infty} d(t_j)v(t_j), \quad \forall v \in H_0^1(0,\infty).$$

This implies that $a(u,v) = l(v)$ is valid for any $v \in H_0^1(0,\infty)$. As a result, u is a weak solution of the (LP). □

In view of Lax-Milgram theorem, we formulate the following main result.

Theorem 3.5. *If $\lambda > 0$, then the Dirichlet impulsive problem (LP) has a weak solution $u \in H_0^1(0,\infty)$ for any $\sigma \in L^2(0,\infty)$. Moreover, $u \in H^2(0,\infty)$ and u is a classical solution and minimizes the functional (3.4) and hence it is a critical point of (3.4).*

Proof. For $\lambda > 0$, it follows that the bilinear a is coercive. The fact that a is continuous, by applying Theorem 2.1, for any $\sigma \in L^2(0,\infty)$, there exists a unique $u \in H_0^1(0,\infty)$ such that $a(u,v) = l(v)$ for all $v \in H_0^1(0,\infty)$. So, the problem (LP) has a weak solution $u \in H_0^1(0,\infty)$.

Owing to Lemma 3.3, a weak solution of (LP) is a classical solution. In addition, a is symmetric, then the functional φ attains its minimum at u which is exactly a critical point of φ since it is differentiable. □

Example 3.6. *As an example, let $\lambda = 1$ and $p(t) = \frac{1}{1+t^2}$.*

This impulsive boundary value problem

$$\begin{cases} -u''(t) + u(t) &= \frac{1}{1+t}, \quad a.e. \;\; t \in [0,\infty), \\ \triangle u'(j) &= e^{-j}, \quad j \in \mathbb{N}^*, \\ u(0) = u(+\infty) &= 0, \end{cases} \tag{3.5}$$

has a solution.

4 Impulsive nonlinear problem

In the nonlinear situation we consider the following impulsive boundary value problem

$$\begin{cases} -u''(t) + \lambda u(t) &= f(t,u(t)), \quad \text{a.e. } \; t \in [0,\infty), \; t \neq t_j, \\ \triangle u'(t_j) &= g(t_j) I_j(u(t_j^-)), \quad j \in \mathbb{N}^*, \\ u(0) = u(+\infty) &= 0, \end{cases} \tag{4.1}$$

where λ is a positive parameter, the functions $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $I_j : \mathbb{R} \to \mathbb{R}$, $j \in \mathbb{N}^*$, and $g : [0,\infty) \to [0,\infty)$ are continuous with $\sum_{j=1}^{\infty} g(t_j) < \infty$.

We refer to problem (4.1) as (NP).

Definition 4.1. *A weak solution of (NP) is a function $u \in H_0^1(0,\infty)$ such that*

$$\int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv + \sum_{j=1}^{\infty} g(t_j) I_j(u(t_j)) v(t_j) - \int_0^{+\infty} f(t,u(t))dt = 0,$$

for every $v \in H_0^1(0,\infty)$.

Setting $F(t,u)=\int_0^u f(t,s)ds$, we define the functional $\varphi : H_0^1(0,\infty)\to\mathbb{R}$ by

$$\varphi(u)=\frac{1}{2}\int_0^{+\infty} u'^2(t)dt+\frac{\lambda}{2}\int_0^{+\infty} u^2(t)dt+\sum_{j=1}^{\infty} g(t_j)\int_0^{u(t_j)} I_j(s)ds-\int_0^{+\infty} F(t,u(t))dt. \tag{4.2}$$

Now we present our principal results for this part.

Theorem 4.2. *Suppose that the following conditions hold:*

(H_1) *There exists a positive bounded function* $M\in L^1(0,+\infty)$ *with* $\frac{M}{p}\in L^1(0,+\infty)$ *such that*

$$|f(t,u)|\le M(t) \quad \textit{for} \quad (t,u)\in[0,+\infty)\times\mathbb{R}.$$

(I_1) *There exist* $M_j>0$, $j\in\mathbb{N}^*$, *satisfying* $\sum_{j=1}^{\infty} M_j g(t_j)<\infty$ *and* $\sum_{j=1}^{\infty}\frac{M_j g(t_j)}{p(t_j)}<\infty$, *such that the impulsive functions* I_j *are bounded i.e.,*

$$|I_j(u)|\le M_j \quad \textit{for every} \quad u\in\mathbb{R},\ j\in\{1,2,\dots\}.$$

Then there is a critical point of φ, *and* (NP) *has at least one solution.*

Proof. Claim 1. φ *is weakly lower semi-continuous (w.l.s.c).*

Let $(u_n)\subset H_0^1(0,\infty)$ be a sequence such that $u_n\rightharpoonup u$ in $H_0^1(0,\infty)$, when $n\to\infty$. Then,

$$\|u\|\le\liminf_{n\to\infty}\|u_n\|,$$

and by Lemma 2.4 we have that (u_n) converges to u in $C_{l,p}[0,\infty)$, hence $u_n(t)$ converges to $u(t)$ for all $t\in[0,\infty)$.

From (H_1) and (I_1), using the continuity of f and I_j, $j\in\mathbb{N}^*$, together with the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}\liminf_{n\to+\infty}\varphi(u_n)&=\liminf_{n\to+\infty}\left[\frac{1}{2}\int_0^{+\infty} u_n'^2+\frac{\lambda}{2}\int_0^{+\infty} u_n^2+\sum_{j=1}^{\infty} g(t_j)\int_0^{u_n(t_j)} I_j(s)ds-\int_0^{+\infty} F(t,u_n(t))dt\right]\\ &\ge\frac{1}{2}\int_0^{+\infty} u'^2+\frac{\lambda}{2}\int_0^{+\infty} u^2+\sum_{j=1}^{\infty} g(t_j)\int_0^{u(t_j)} I_j(s)ds-\int_0^{+\infty} F(t,u(t))dt=\varphi(u).\end{aligned}$$

Thus, φ is w.l.s.c.

Claim 2. φ *is coercive.*

For any $u\in H_0^1(0,\infty)$, the fact that $\lambda>0$, there exists $\alpha>0$ such that

$$\varphi(u)\ge\alpha\|u\|^2+\sum_{j=1}^{\infty} g(t_j)\int_0^{u(t_j)} I_j(s)ds-\int_0^{+\infty} F(t,u(t))dt.$$

Using conditions (H_1), (I_1) and Lemma 2.3, we have

$$\begin{aligned}\varphi(u) &\geq \alpha\|u\|^2 - \sum_{j=1}^{\infty}\frac{M_j g(t_j)}{p(t_j)}p(t_j)|u(t_j)| - \int_0^{+\infty}\frac{M(t)}{p(t)}p(t)|u(t)|dt \\ &\geq \alpha\|u\|^2 - \|u\|_{\infty,p}\sum_{j=1}^{\infty}\frac{M_j g(t_j)}{p(t_j)} - \|u\|_{\infty,p}\int_0^{+\infty}\frac{M(t)}{p(t)}dt \\ &\geq \alpha\|u\|^2 - C\|u\|\sum_{j=1}^{\infty}\frac{M_j g(t_j)}{p(t_j)} - C\|u\|\left\|\frac{M}{p}\right\|_{L^1} \\ &\geq \alpha\|u\|^2 - C\left(\sum_{j=1}^{\infty}\frac{M_j g(t_j)}{p(t_j)} + \left\|\frac{M}{p}\right\|_{L^1}\right)\|u\|,\end{aligned}$$

for some $C > 0$. Then, the above inequality implies that $\lim_{\|u\|\to+\infty}\varphi(u) = +\infty$. Hence, φ is coercive.

Applying Theorem 2.2, φ possesses a minimum which is a critical point of φ. Finally, by (H_1) and (I_1), it is easy to check that φ is continuous and differentiable for any $u \in H_0^1(0,\infty)$ and that

$$\varphi'(u)v = \int_0^{+\infty} u'v' + \lambda\int_0^{+\infty} uv + \sum_{j=1}^{\infty} g(t_j)I_j(u(t_j))v(t_j)dt - \int_0^{+\infty} f(t,u(t))v(t)dt. \tag{4.3}$$

Therefore, a critical point of φ is a weak solution of the problem (NP). □

Remark 4.3. *Assume $M \in L^2(0,\infty)$ in (H_1), then it is easy to see that a weak solution u is in $H^2(0,\infty)$.*

Example 4.4. *Take $\lambda = 1$, $p(t) = e^{-t}$, $M(t) = e^{-2t}$, $g(t) = e^{-2t}$, $M_j = \frac{1}{j}$ and $I_j(s) = \frac{1}{j+s^2}$, $j \in \mathbb{N}^*$.*

The following IBVP:

$$\begin{cases} -u''(t) + u(t) &= e^{-3t}, \quad a.e. \ t \in [0,\infty), \\ \triangle u'(j) &= \dfrac{e^{-2j}}{j+u^2(j)}, \quad j \in \mathbb{N}^*, \\ u(0) = u(+\infty) &= 0, \end{cases}$$

has at least one solution. (See Figure 1)

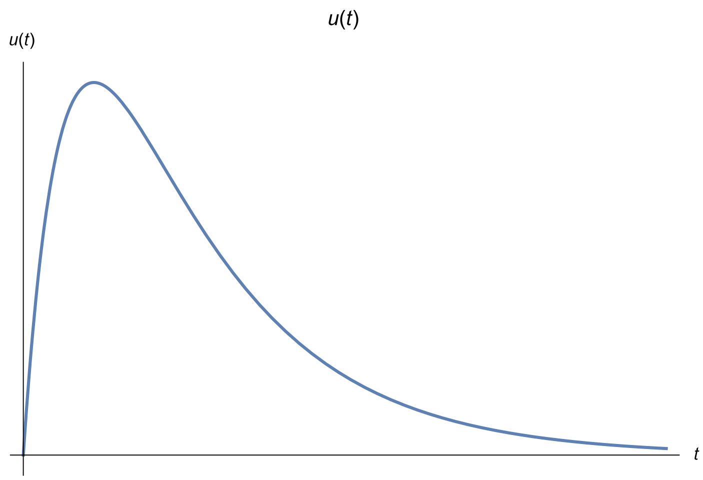

Figure 1

Theorem 4.5. *Assume the following conditions are satisfied:*

(H_2) *The function f is sublinear i.e., there exist a constant $\gamma \in [0,1)$ and positive functions $a, b \in L^1(0,\infty)$ with $\frac{a}{p}, \frac{b}{p^\gamma}, \frac{b}{p^{\gamma+1}} \in L^1[0,\infty)$ such that*

$$|f(t,u)| \leq a(t) + b(t)|u|^\gamma \quad \text{for} \quad (t,u) \in [0,+\infty) \times \mathbb{R}.$$

(I_2) *There exist constants $\delta \in [0,1)$ and $a_j, b_j > 0$, $j \in \{1,2,\dots\}$ with $\sum_{j=1}^{\infty} a_j g(t_j)$, $\sum_{j=1}^{\infty} \frac{a_j g(t_j)}{p(t_j)}$, $\sum_{j=1}^{\infty} \frac{b_j g(t_j)}{p^\delta(t_j)}$, $\sum_{j=1}^{\infty} \frac{b_j g(t_j)}{p^{\delta+1}(t_j)}$ are convergent series, such that the impulsive functions I_j have sublinear growths i.e.,*

$$|I_j(u)| \leq a_j + b_j|u|^\delta \quad \text{for every} \quad u \in \mathbb{R},\ j \in \{1,2,\dots\}.$$

Then there is a critical point of φ, and (NP) has at least one solution.

Proof. Claim 1. φ is weakly lower semi-continuous.

Under (H_2) and (I_2), arguing analogously to the proof of Theorem 4.2, we find the weak lower semi-continuity of φ.

Claim 2. φ is coercive.

In view of conditions (H_2), (I_2) and (4.2), for any $u \in H_0^1(0,\infty)$, we have

$$\varphi(u) = \frac{1}{2}\int_0^{+\infty} u'^2 + \frac{\lambda}{2}\int_0^{+\infty} u^2 + \sum_{j=1}^{\infty} g(t_j) \int_0^{u(t_j)} I_j(s)ds - \int_0^{+\infty} F(t,u(t))dt$$

$$\geq \alpha\|u\|^2 - \sum_{j=1}^{\infty} g(t_j) \int_0^{u(t_j)} (a_j + b_j|s|^\delta)ds - \int_0^{+\infty} \left(a(t)|u(t)| + \frac{b(t)}{\gamma+1}|u(t)|^{\gamma+1} \right) dt$$

$$
\begin{aligned}
&\geq \alpha\|u\|^2 - \sum_{j=1}^{\infty} g(t_j)\left(\frac{a_j}{p(t_j)}p(t_j)|u(t_j)| + \frac{b_j}{(\delta+1)p^{\delta+1}(t_j)}|p(t_j)u(t_j)|^{\delta+1}\right)\\
&- \int_0^{+\infty} \frac{a(t)}{p(t)}p(t)|u(t)|dt - \frac{1}{(\gamma+1)}\int_0^{+\infty}\frac{b(t)}{p^{\gamma+1}(t)}|p(t)u(t)|^{\gamma+1}dt\\
&\geq \alpha\|u\|^2 - \|u\|_{\infty,p}\sum_{j=1}^{\infty}\frac{a_j g(t_j)}{p(t_j)} - \|u\|_{\infty,p}^{\delta+1}\sum_{j=1}^{\infty}\frac{b_j g(t_j)}{p^{\delta+1}(t_j)} - \|u\|_{\infty,p}\left\|\frac{a}{p}\right\|_{L^1}\\
&- \|u\|_{\infty,p}^{\gamma+1}\left\|\frac{b}{p^{\gamma+1}}\right\|_{L^1}.
\end{aligned}
$$

Hence, by Lemma 2.3, we get

$$
\begin{aligned}
\varphi(u) &\geq \alpha\|u\|^2 - C\|u\|\sum_{j=1}^{\infty}\frac{a_j g(t_j)}{p(t_j)} - C^{\delta+1}\|u\|^{\delta+1}\sum_{j=1}^{\infty}\frac{b_j g(t_j)}{p^{\delta+1}(t_j)} - C\|u\|\left\|\frac{a}{p}\right\|_{L^1}\\
&- C^{\gamma+1}\|u\|^{\gamma+1}\left\|\frac{b}{p^{\gamma+1}}\right\|_{L^1}\\
&\geq \alpha\|u\|^2 - C\left(\left\|\frac{a}{p}\right\|_{L^1} + \sum_{j=1}^{\infty}\frac{a_j g(t_j)}{p(t_j)}\right)\|u\| - C^{\delta+1}\left(\sum_{j=1}^{\infty}\frac{b_j g(t_j)}{p^{\delta+1}(t_j)}\right)\|u\|^{\delta+1}\\
&- C^{\gamma+1}\left\|\frac{b}{p^{\gamma+1}}\right\|_{L^1}\|u\|^{\gamma+1}.
\end{aligned}
$$

Since $\delta,\gamma \in [0,1)$, then $\lim\limits_{\|u\|\to+\infty}\varphi(u) = +\infty$. This means, φ is coercive. Using Theorem 2.2, φ has a minimum, which is a critical point of φ. Finally, from (H_2) and (I_2), we get the differentiability of φ such that its differentiable is defined by (4.3). Consequently, (NP) has at least one solution. □

Remark 4.6. *In (H_2), assume $a, \dfrac{b}{p^{\gamma}} \in L^2(0,\infty)$, then a weak solution u is in $H^2(0,\infty)$.*

Example 4.7. *Consider the following problem*

$$
\begin{cases}
-u''(t)+u(t) = e^{-2t}\sqrt{|u(t)|}+e^{-3t}, & a.e.\ \ t\in[0,\infty),\\
\triangle u'(j) = e^{-2j}\left(\dfrac{1}{j^2}+\dfrac{|s|^{\frac{1}{4}}}{j}\right), & j\in\mathbb{N}^*,\\
u(0)=u(+\infty) = 0,
\end{cases}
$$

where $\lambda = 1$, $p(t) = e^{-t}$, $g(t) = e^{-2t}$, $a_j = \dfrac{1}{j^2}$, $b_j = \dfrac{1}{j}$ and $I_j(s) = \dfrac{1}{j^2} + \dfrac{|s|^{\frac{1}{4}}}{j}$, $j \in \mathbb{N}^*$.
By simple calculations, all conditions in Theorem 4.5 are satisfied, then (4.1) has at least one solution.

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On an *a priori* L^∞ estimate for a class of Monge-Ampère type equations on compact almost Hermitian manifolds

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ABSTRACT

We investigate Monge-Ampère type equations on almost Hermitian manifolds and show an *a priori* L^∞ estimate for a smooth solution of these equations.

RESUMEN

Investigamos ecuaciones de tipo Monge-Ampère en variedades casi Hermitianas y mostramos una estimación L^∞ *a priori* para una solución suave de estas ecuaciones.

Keywords and Phrases: Monge-Ampère type equation, almost Hermitian manifold, Chern connection.

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1 Introduction

Let (M^{2n}, J, ω) be a compact almost Hermitian manifold of real dimension $2n$ with $n \geq 2$. Let χ be a smooth real $(1,1)$-form on M. We define for a function $u \in C^2(M)$,

$$\chi_u := \chi + \sqrt{-1}\partial\bar{\partial}u$$

and

$$[\chi] := \{\chi_u | u \in C^2(M)\}, \quad [\chi]^+ := \{\chi' \in [\chi] | \chi' > 0\}, \quad \mathcal{H}(M,\chi) := \{u \in C^2(M) | \chi_u > 0\}$$

and

$$\mathcal{C}_\alpha(\psi) := \{[\chi] | \exists \chi' \in [\chi]^+, n\chi'^{n-1} > (n-\alpha)\psi\chi'^{n-\alpha-1} \wedge \omega^\alpha\}.$$

We consider the following fully nonlinear Monge-Ampère type equations, which are called the $(n, n-\alpha)$-quotient equations for $1 \leq \alpha \leq n$:

$$\chi_u^n = \psi\chi_u^{n-\alpha} \wedge \omega^\alpha \text{ with } \chi_u > 0, \tag{1.1}$$

where ψ is a smooth positive function. We will call a function $u \in C^2(M)$ admissible if it satisfies that $u \in \mathcal{H}(M,\chi)$. When solutions u are admissible, the equations (1.1) are elliptic. Since the equation (1.1) is invariant under the addition of constants to u, we may assume that u satisfies the normalized condition such that

$$\sup_M u = 0. \tag{1.2}$$

W. Sun has studied a class of fully nonlinear elliptic equations on closed Hermitian manifolds and derived some *a priori* estimates for these equations (cf. [5, 6]). In [5], W. Sun has proven a uniform *a priori* C^∞ estimates of a smooth solution of the equation (1.1) and shown the existence of a solution of (1.1) on a closed Hermitian manifold. In [12], J. Zhang has shown that on a compact almost Hermitian manifold (M^{2n}, J, ω), if there exists an admissible $\mathcal{C}$-subsolution and an admissible supersolution for the equation (1.1) for $\chi = \omega$, there exists a pair of (u, b) with $b \in \mathbb{R}$ such that $u \in \mathcal{H}(M,\omega)$, $\sup_M u = 0$, $\omega_u^n = e^b\psi\omega^{n-\alpha} \wedge \omega^\alpha$ for $1 \leq \alpha \leq n$ on M. L. Chen has studied a Hessian equation with its structure as a combination of elementary symmetric functions on a closed Kähler manifold and Chen has provided a sufficient and necessary condition for the solvability of this equation in [1]. Q. Tu and N. Xiang have investigated the Dirichlet problem for a class of Hessian type equation with its structure as a combination of elementary symmetric functions on a closed Hermitian manifold with smooth boundary and they have derived *a priori* estimates for the complex mixed Hessian equation in [9].

In this paper, we show that we have the *a priori* L^∞ estimate for a smooth solution of the equation (1.1) on general almost Hermitian manifolds.

Theorem 1.1. *Let (M, J, ω) be a compact almost Hermitian manifold of real dimension $2n$ with $n \geq 2$ and u be a smooth admissible solution to* (1.1). *Suppose that $\chi \in \mathcal{C}_\alpha(\psi)$. Then there is a uniform a priori L^∞ estimate for u depending only on (M, J, ω), χ, ψ.*

This paper is organized as follows: in section 2, we recall some basic definitions and computations on an almost Hermitian manifold (M, J, ω). In section 3, for an arbitrary chosen smooth function φ on M, we show the result that $\partial\partial\bar{\partial}\varphi$ and $\bar{\partial}\partial\bar{\partial}\varphi$ depend only on the first derivative of φ and some geometric quantities of (M, J, ω). In section 4, we give a proof for Theorem 1.1. Notice that we assume the Einstein convention omitting the symbol of sum over repeated indexes in all this paper.

2 Preliminaries

2.1 The Nijenhuis tensor of the almost complex structure

Let M be a $2n$-dimensional smooth differentiable manifold. An almost complex structure on M is an endomorphism J of TM, $J \in \Gamma(\mathrm{End}(TM))$, satisfying $J^2 = -Id_{TM}$, where TM is the real tangent vector bundle of M. The pair (M, J) is called an almost complex manifold. Let (M, J) be an almost complex manifold. We define a bilinear map on $C^\infty(M)$ for $X, Y \in \Gamma(TM)$ by

$$4N(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \tag{2.1}$$

which is the Nijenhuis tensor of J. The Nijenhuis tensor N satisfies $N(X, Y) = -N(Y, X)$, $N(JX, Y) = -JN(X, Y)$, $N(X, JY) = -JN(X, Y)$, $N(JX, JY) = -N(X, Y)$. For any $(1, 0)$-vector fields W and V, $N(V, W) = -[V, W]^{(0,1)}$, $N(V, \bar{W}) = N(\bar{V}, W) = 0$ and $N(\bar{V}, \bar{W}) = -[\bar{V}, \bar{W}]^{(1,0)}$ since we have $4N(V, W) = -2([V, W] + \sqrt{-1}J[V, W])$, $4N(\bar{V}, \bar{W}) = -2([\bar{V}, \bar{W}] - \sqrt{-1}J[\bar{V}, \bar{W}])$. An almost complex structure J is called integrable if $N = 0$ on M. Giving a complex structure to a differentiable manifold M is equivalent to giving an integrable almost complex structure to M (cf. [4]). A Riemannian metric g on M is called J-invariant if J is compatible with g, *i.e.*, for any $X, Y \in \Gamma(TM)$, $g(X, Y) = g(JX, JY)$. In this case, the pair (J, g) is called an almost Hermitian structure.

The complexified tangent vector bundle is given by $T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ for the real tangent vector bundle TM. By extending J $\mathbb{C}$-linearly and g $\mathbb{C}$-bilinearly to $T^{\mathbb{C}}M$, they are also defined on $T^{\mathbb{C}}M$ and we observe that the complexified tangent vector bundle $T^{\mathbb{C}}M$ can be decomposed as $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$, $T^{0,1}M$ are the eigenspaces of J corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively:

$$T^{1,0}M = \{X - \sqrt{-1}JX \big| X \in TM\}, \quad T^{0,1}M = \{X + \sqrt{-1}JX \big| X \in TM\}. \tag{2.2}$$

Let $\Lambda^r M = \bigoplus_{p+q=r} \Lambda^{p,q} M$ for $0 \leq r \leq 2n$ denote the decomposition of complex differential r-forms into (p,q)-forms, where $\Lambda^{p,q} M = \Lambda^p(\Lambda^{1,0} M) \otimes \Lambda^q(\Lambda^{0,1} M)$,

$$\Lambda^{1,0} M = \{\eta + \sqrt{-1} J\eta \big| \eta \in \Lambda^1 M\}, \quad \Lambda^{0,1} M = \{\eta - \sqrt{-1} J\eta \big| \eta \in \Lambda^1 M\} \tag{2.3}$$

and $\Lambda^1 M$ denotes the dual of $T^{\mathbb{C}} M$.

Let $\{Z_r\}$ be a local $(1,0)$-frame on (M,J) with an almost Hermitian metric g and let $\{\zeta^r\}$ be a local associated coframe with respect to $\{Z_r\}$, *i.e.*, $\zeta^i(Z_j) = \delta^i_j$ for $i,j = 1, \ldots, n$. Since g is almost Hermitian, its components satisfy $g_{ij} = g_{\bar{i}\bar{j}} = 0$ and $g_{i\bar{j}} = g_{\bar{j}i} = \bar{g}_{\bar{i}j}$. Using these local frame $\{Z_r\}$ and coframe $\{\zeta^r\}$, we have

$$N(Z_{\bar{i}}, Z_{\bar{j}}) = -[Z_{\bar{i}}, Z_{\bar{j}}]^{(1,0)} =: N^k_{\bar{i}\bar{j}} Z_k, \quad N(Z_i, Z_j) = -[Z_i, Z_j]^{(0,1)} = \overline{N^k_{\bar{i}\bar{j}}} Z_{\bar{k}},$$

and

$$N = \frac{1}{2} \overline{N^k_{\bar{i}\bar{j}}} Z_{\bar{k}} \otimes (\zeta^i \wedge \zeta^j) + \frac{1}{2} N^k_{\bar{i}\bar{j}} Z_k \otimes (\zeta^{\bar{i}} \wedge \zeta^{\bar{j}}). \tag{2.4}$$

Let (M,J,g) be an almost Hermitian manifold with $\dim_{\mathbb{R}} M = 2n$. An affine connection D on $T^{\mathbb{C}} M$ is called almost Hermitian connection if $Dg = DJ = 0$. For the almost Hermitian connection, we have the following Lemma (cf. [10, 13]).

Lemma 2.1. *Let (M,J,g) be an almost Hermitian manifold with $\dim_{\mathbb{R}} M = 2n$. Then for any given vector valued $(1,1)$-form $\Theta = (\Theta^i)_{1 \leq i \leq n}$, there exists a unique almost Hermitian connection ∇ on (M,J,g) such that the $(1,1)$-part of the torsion is equal to the given Θ.*

If the $(1,1)$-part of the torsion of an almost Hermitian connection vanishes everywhere, then the connection is called the second canonical connection or the Chern connection. We will refer the connection as the Chern connection and denote it by ∇. Now let ∇ be the Chern connection on M. We denote the structure coefficients of Lie bracket by

$$[Z_i, Z_j] = B^r_{ij} Z_r + B^{\bar{r}}_{ij} Z_{\bar{r}}, \quad [Z_i, Z_{\bar{j}}] = B^r_{i\bar{j}} Z_r + B^{\bar{r}}_{i\bar{j}} Z_{\bar{r}}, \quad [Z_{\bar{i}}, Z_{\bar{j}}] = B^r_{\bar{i}\bar{j}} Z_r + B^{\bar{r}}_{\bar{i}\bar{j}} Z_{\bar{r}}.$$

We have $B^k_{ij} = -B^k_{ji}$ since $[Z_i, Z_j] = -[Z_j, Z_i]$. Notice that J is integrable if and only if the $B^{\bar{r}}_{ij}$'s vanish.

For any p-form ψ, there holds that

$$\begin{aligned} d\psi(X_1, \ldots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} X_i(\psi(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j} \psi([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{p+1}) \end{aligned} \tag{2.5}$$

for any vector fields $X_1, \ldots, X_{p+1}$ on M (cf. [13]). We directly compute that

$$d\zeta^s = -\frac{1}{2} B^s_{kl} \zeta^k \wedge \zeta^l - B^s_{k\bar{l}} \zeta^k \wedge \zeta^{\bar{l}} - \frac{1}{2} B^s_{\bar{k}\bar{l}} \zeta^{\bar{k}} \wedge \zeta^{\bar{l}}. \tag{2.6}$$

For any real $(1,1)$-form $\eta = \sqrt{-1}\eta_{i\bar{j}}\zeta^i \wedge \zeta^{\bar{j}}$, we have

$$\partial\eta = \frac{\sqrt{-1}}{2}\Big(Z_i(\eta_{j\bar{k}}) - Z_j(\eta_{i\bar{k}}) - B^s_{ij}\eta_{s\bar{k}} - B^{\bar{s}}_{i\bar{k}}\eta_{j\bar{s}} + B^{\bar{s}}_{j\bar{k}}\eta_{i\bar{s}}\Big)\zeta^i \wedge \zeta^j \wedge \zeta^{\bar{k}}, \tag{2.7}$$

$$\bar{\partial}\eta = \frac{\sqrt{-1}}{2}\Big(Z_{\bar{j}}(\eta_{k\bar{i}}) - Z_{\bar{i}}(\eta_{k\bar{j}}) - B^s_{k\bar{i}}\eta_{s\bar{j}} + B^s_{k\bar{j}}\eta_{s\bar{i}} + B^{\bar{s}}_{\bar{i}\bar{j}}\eta_{k\bar{s}}\Big)\zeta^k \wedge \zeta^{\bar{i}} \wedge \zeta^{\bar{j}}. \tag{2.8}$$

We can split the exterior differential operator $d : \Lambda^p M \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^{p+1} M \otimes_{\mathbb{R}} \mathbb{C}$, into four components

$$d = A + \partial + \bar{\partial} + \bar{A}$$

with

$$\partial : \Lambda^{p,q} M \to \Lambda^{p+1,q} M, \quad \bar{\partial} : \Lambda^{p,q} M \to \Lambda^{p,q+1} M,$$

$$A : \Lambda^{p,q} M \to \Lambda^{p+2,q-1} M, \quad \bar{A} : \Lambda^{p,q} M \to \Lambda^{p-1,q+2} M.$$

In terms of these components, the condition $d^2 = 0$ can be written as

$$A^2 = 0, \quad \partial A + A\partial = 0, \quad \bar{\partial}\bar{A} + \bar{A}\bar{\partial} = 0, \quad \bar{A}^2 = 0,$$

$$A\bar{\partial} + \partial^2 + \bar{\partial}A = 0, \quad A\bar{A} + \partial\bar{\partial} + \bar{\partial}\partial + \bar{A}A = 0, \quad \partial\bar{A} + \bar{\partial}^2 + \bar{A}\partial = 0. \tag{2.9}$$

A direct computation yields for any $\varphi \in C^\infty(M, \mathbb{R})$,

$$\sqrt{-1}\partial\bar{\partial}\varphi = \frac{1}{2}(dJd\varphi)^{(1,1)} = \sqrt{-1}(Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})(\varphi)\zeta^i \wedge \zeta^{\bar{j}}, \tag{2.10}$$

so we write locally

$$\partial_i\partial_{\bar{j}}\varphi = (Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})\varphi. \tag{2.11}$$

2.2 The torsion and the curvature on almost complex manifolds

Since the Chern connection ∇ preserves J, we have

$$\nabla_i Z_j := \nabla_{Z_i} Z_j = \Gamma^r_{ij} Z_r, \quad \nabla_i Z_{\bar{j}} := \nabla_{Z_i} Z_{\bar{j}} = \Gamma^{\bar{r}}_{i\bar{j}} Z_{\bar{r}},$$

where $\Gamma^r_{ij} = g^{r\bar{s}} Z_i(g_{j\bar{s}}) - g^{r\bar{s}} g_{j\bar{l}} B^{\bar{l}}_{i\bar{s}}$. We can obtain that $\Gamma^{\bar{r}}_{i\bar{j}} = B^{\bar{r}}_{i\bar{j}}$ since the $(1,1)$-part of the torsion of the Chern connection vanishes everywhere.

Note that the mixed derivatives $\nabla_i Z_{\bar{j}}$ do not depend on g (cf. [10]). Let $\{\gamma^i_j\}$ be the connection form, which is defined by $\gamma^i_j = \Gamma^i_{sj}\zeta^s + \Gamma^i_{\bar{s}j}\zeta^{\bar{s}}$. The torsion T of the Chern connection ∇ is given by $T^i = d\zeta^i - \zeta^p \wedge \gamma^i_p$, $T^{\bar{i}} = d\zeta^{\bar{i}} - \zeta^{\bar{p}} \wedge \gamma^{\bar{i}}_{\bar{p}}$, which has no $(1,1)$-part and the only non-vanishing components are as follows:

$$T^s_{ij} = \Gamma^s_{ij} - \Gamma^s_{ji} - B^s_{ij}, \quad T^{\bar{s}}_{ij} = -B^{\bar{s}}_{ij}.$$

These tell us that $T = (T^i)$ splits into $T = T' + T''$, where $T' \in \Gamma(\Lambda^{2,0}M \otimes T^{1,0}M)$, $T'' \in \Gamma(\Lambda^{0,2}M \otimes T^{1,0}M)$.

We denote by Ω the curvature of the Chern connection ∇. We can regard Ω as a section of $\Lambda^2 M \otimes \Lambda^{1,1}M$, $\Omega \in \Gamma(\Lambda^2 M \otimes \Lambda^{1,1}M)$ and Ω splits in $\Omega = \mathcal{H} + \mathcal{R} + \bar{\mathcal{H}}$, where $\mathcal{R} \in \Gamma(\Lambda^{1,1}M \otimes \Lambda^{1,1}M)$, $\mathcal{H} \in \Gamma(\Lambda^{2,0}M \otimes \Lambda^{1,1}M)$. The curvature form can be expressed by $\Omega^i_j = d\gamma^i_j + \gamma^i_s \wedge \gamma^s_j$.

In terms of Z_r's, we have

$$\mathcal{R}_{i\bar{j}k}{}^r = \Omega^r_k(Z_i, Z_{\bar{j}}) = Z_i(\Gamma^r_{\bar{j}k}) - Z_{\bar{j}}(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{ik} - B^s_{i\bar{j}}\Gamma^r_{sk} + B^{\bar{s}}_{\bar{j}i}\Gamma^r_{\bar{s}k} = -\mathcal{R}_{\bar{j}ik}{}^r, \quad (2.12)$$

$$\mathcal{H}_{ijk}{}^r = \Omega^r_k(Z_i, Z_j) = Z_i(\Gamma^r_{jk}) - Z_j(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik} - B^s_{ij}\Gamma^r_{sk} - B^{\bar{s}}_{ij}\Gamma^r_{\bar{s}k} = -\mathcal{H}_{jik}{}^r, \quad (2.13)$$

$$\mathcal{H}_{\bar{i}\bar{j}k}{}^r = \Omega^r_k(Z_{\bar{i}}, Z_{\bar{j}}) = Z_{\bar{i}}(\Gamma^r_{\bar{j}k}) - Z_{\bar{j}}(\Gamma^r_{\bar{i}k}) + \Gamma^r_{\bar{i}s}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{\bar{i}k} - B^s_{\bar{i}\bar{j}}\Gamma^r_{sk} - B^{\bar{s}}_{\bar{i}\bar{j}}\Gamma^r_{\bar{s}k} = -\mathcal{H}_{\bar{j}\bar{i}k}{}^r. \quad (2.14)$$

Lemma 2.2 (The first Bianchi identity for the Chern curvature)**.** *For any $X, Y, Z \in T^{\mathbb{C}}M$,*

$$\sum \Omega(X,Y)Z = \sum \Big(T(T(X,Y),Z) + \nabla_X T(Y,Z)\Big),$$

where the sum is taken over all cyclic permutations.

This identity induces the following formulae:

$$\mathcal{R}_{i\bar{j}k}{}^l = \mathcal{R}_{k\bar{j}i}{}^l - T^{\bar{r}}_{ik}T^l_{\bar{r}\bar{j}} + \nabla_{\bar{j}}T^l_{ki} = \mathcal{R}_{k\bar{j}i}{}^l - B^{\bar{r}}_{ik}B^l_{\bar{r}\bar{j}} + \nabla_{\bar{j}}T^l_{ki}, \quad (2.15)$$

$$\mathcal{H}_{ijk}{}^l = T^{\bar{r}}_{ji}T^{\bar{k}}_{\bar{r}\bar{l}} + \nabla_{\bar{l}}T^{\bar{k}}_{ji} = -B^{\bar{r}}_{ji}T^{\bar{k}}_{\bar{r}\bar{l}} + \nabla_{\bar{l}}T^{\bar{k}}_{ji}, \quad (2.16)$$

where used that $\mathcal{R}_{ij\bar{k}\bar{l}} = \mathcal{R}_{\bar{i}\bar{j}kl} = \mathcal{H}_{j\bar{l}ik} = \mathcal{H}_{\bar{j}l\bar{i}\bar{k}} = \mathcal{H}_{\bar{l}ijk} = \mathcal{H}_{l\bar{i}\bar{j}\bar{k}} = 0$.

Let $\{Z_r\}$ be a local unitary $(1,0)$-frame with respect to g around a fixed point $p \in M$. Note that unitary frames always exist locally since we can take any frame and apply the Gram-Schmidt process. Then with respect to a local g-unitary frame, we have $g_{i\bar{j}} = \delta_{ij}$ for any $i,j,k = 1,\dots,n$, and the Christoffel symbols satisfy

$$\Gamma^k_{ij} = -\Gamma^{\bar{j}}_{i\bar{k}}, \quad \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = -\Gamma^j_{\bar{i}k},$$

since we have

$$\Gamma^k_{ij} = g(\nabla_i Z_j, Z_{\bar{k}}) = Z_i(g_{j\bar{k}}) - g(Z_j, \nabla_i Z_{\bar{k}}) = -\Gamma^{\bar{j}}_{i\bar{k}},$$
$$\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = g(Z_k, \nabla_{\bar{i}} Z_{\bar{j}}) = Z_{\bar{i}}(g_{k\bar{j}}) - g(\nabla_{\bar{i}} Z_k, Z_{\bar{j}}) = -\Gamma^j_{\bar{i}k}.$$

And also we have

$$\begin{aligned}
\mathcal{R}_{i\bar{j}k}{}^r &= Z_i(\Gamma^r_{\bar{j}k}) - Z_{\bar{j}}(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{ik} - B^s_{i\bar{j}}\Gamma^r_{sk} + B^{\bar{s}}_{\bar{j}i}\Gamma^r_{\bar{s}k} \\
&= -Z_i(\Gamma^{\bar{k}}_{\bar{j}\bar{r}}) + Z_{\bar{j}}(\Gamma^{\bar{k}}_{i\bar{r}}) + \Gamma^{\bar{s}}_{i\bar{r}}\Gamma^{\bar{k}}_{\bar{j}\bar{s}} - \Gamma^{\bar{s}}_{\bar{j}\bar{r}}\Gamma^{\bar{k}}_{i\bar{s}} + B^s_{i\bar{j}}\Gamma^{\bar{k}}_{s\bar{r}} - B^{\bar{s}}_{\bar{j}i}\Gamma^{\bar{k}}_{\bar{s}\bar{r}} \\
&= -\mathcal{R}_{i\bar{j}\bar{r}}{}^{\bar{k}}, \qquad (2.17)
\end{aligned}$$

$$
\begin{aligned}
{\mathcal{H}_{ijk}}^{r} &= Z_i(\Gamma^r_{jk}) - Z_j(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik} - B^s_{ij}\Gamma^r_{sk} - B^{\bar{s}}_{ij}\Gamma^r_{\bar{s}k} \\
&= -Z_i(\Gamma^{\bar{k}}_{j\bar{r}}) - Z_j(\Gamma^{\bar{k}}_{i\bar{r}}) + \Gamma^{\bar{s}}_{i\bar{r}}\Gamma^{\bar{k}}_{j\bar{s}} - \Gamma^{\bar{s}}_{j\bar{r}}\Gamma^{\bar{k}}_{i\bar{s}} + B^s_{ij}\Gamma^{\bar{k}}_{s\bar{r}} + B^{\bar{s}}_{ij}\Gamma^{\bar{k}}_{\bar{s}\bar{r}} \\
&= -{\mathcal{H}_{ij\bar{r}}}^{\bar{k}}
\end{aligned} \tag{2.18}
$$

and

$$
\begin{aligned}
\overline{{\mathcal{R}_{i\bar{j}k}}^{r}} &= Z_{\bar{i}}(\Gamma^{\bar{r}}_{j\bar{k}}) - Z_j(\Gamma^{\bar{r}}_{\bar{i}\bar{k}}) + \Gamma^{\bar{r}}_{\bar{i}\bar{s}}\Gamma^{\bar{s}}_{j\bar{k}} - \Gamma^{\bar{r}}_{j\bar{s}}\Gamma^{\bar{s}}_{\bar{i}\bar{k}} - B^{\bar{s}}_{\bar{i}j}\Gamma^{\bar{r}}_{\bar{s}\bar{k}} + B^{s}_{j\bar{i}}\Gamma^{\bar{r}}_{s\bar{k}} \\
&= Z_j(\Gamma^{k}_{\bar{i}r}) - Z_{\bar{i}}(\Gamma^{k}_{jr}) + \Gamma^{s}_{\bar{i}r}\Gamma^{k}_{js} - \Gamma^{s}_{jr}\Gamma^{k}_{\bar{i}s} - B^{s}_{j\bar{i}}\Gamma^{k}_{sr} + B^{\bar{s}}_{\bar{i}j}\Gamma^{k}_{\bar{s}r} \\
&= {\mathcal{R}_{j\bar{i}r}}^{k},
\end{aligned} \tag{2.19}
$$

$$
\begin{aligned}
\overline{{\mathcal{H}_{ijk}}^{r}} &= Z_{\bar{i}}(\Gamma^{\bar{r}}_{\bar{j}\bar{k}}) - Z_{\bar{j}}(\Gamma^{\bar{r}}_{\bar{i}\bar{k}}) + \Gamma^{\bar{r}}_{\bar{i}\bar{s}}\Gamma^{\bar{s}}_{\bar{j}\bar{k}} - \Gamma^{\bar{r}}_{\bar{j}\bar{s}}\Gamma^{\bar{s}}_{\bar{i}\bar{k}} - B^{\bar{s}}_{\bar{i}\bar{j}}\Gamma^{\bar{r}}_{\bar{s}\bar{k}} - B^{s}_{\bar{i}\bar{j}}\Gamma^{\bar{r}}_{s\bar{k}} \\
&= -Z_{\bar{i}}(\Gamma^{k}_{\bar{j}r}) + Z_{\bar{j}}(\Gamma^{k}_{\bar{i}r}) + \Gamma^{s}_{\bar{i}r}\Gamma^{k}_{\bar{j}s} - \Gamma^{s}_{\bar{j}r}\Gamma^{k}_{\bar{i}s} - B^{\bar{s}}_{\bar{j}\bar{i}}\Gamma^{k}_{\bar{s}r} - B^{s}_{\bar{j}\bar{i}}\Gamma^{k}_{sr} \\
&= {\mathcal{H}_{\bar{j}\bar{i}r}}^{k}.
\end{aligned} \tag{2.20}
$$

Hence we obtain $\mathcal{R}_{i\bar{j}k\bar{r}} = -\mathcal{R}_{i\bar{j}\bar{r}k}$, $\mathcal{H}_{ijk\bar{r}} = -\mathcal{H}_{ij\bar{r}k}$ and $\overline{\mathcal{R}_{i\bar{j}k\bar{r}}} = \mathcal{R}_{j\bar{i}r\bar{k}}$, $\overline{\mathcal{H}_{ijk\bar{r}}} = \mathcal{H}_{\bar{j}\bar{i}r\bar{k}}$ by using a local unitary $(1,0)$-frame with respect to g.

3 Some results for a smooth function on almost Hermitian manifolds

Let (M, J, g) be an almost Hermitian manifold. Here note that $B^{\bar{q}}_{j\bar{b}}$, $B^{q}_{\bar{j}b}$'s do not depend on the metric g, which depend only on the almost complex structure J since the mixed derivatives $\nabla_j Z_{\bar{b}}$, $\nabla_{\bar{j}} Z_b$ do not depend on g. Since we have $B^{q}_{b\bar{j}} = -B^{q}_{\bar{j}b}$, we have that $B^{q}_{b\bar{j}}$, $B^{\bar{q}}_{\bar{b}j}$'s also do not depend on g (cf. [10]). Also note that $B^{\bar{s}}_{ri}$, $B^{s}_{\bar{r}\bar{i}}$ do not depend on g, depend only on J. We can choose a local unitary frame $\{Z_r\}$ around an arbitrary chosen point $p_0 \in M$ such that $g_{i\bar{j}}(p_0) = \delta_{ij}$ and $\nabla Z(p_0) = 0$ (cf. [11]). Then we have $\Gamma^k_{ij}(p_0) = 0$ since $\nabla_i Z_j(p_0) = \Gamma^k_{ij}(p_0) Z_k = 0$, also we obtain that

$$
[Z_i, Z_{\bar{j}}](p_0) = \nabla_i Z_j(p_0) - \nabla_j Z_i(p_0) - T(Z_i, Z_{\bar{j}})(p_0) = 0 \quad \text{for all } i, j = 1, \ldots, n. \tag{3.1}
$$

Then we have that $0 = [Z_i, Z_{\bar{j}}](p_0) = B^k_{i\bar{j}}(p_0) Z_k + B^{\bar{k}}_{i\bar{j}}(p_0) Z_{\bar{k}}$, which gives that $B^k_{i\bar{j}}(p_0) = 0$ for all $i, j, k = 1, \ldots, n$ and that $B^{\bar{k}}_{i\bar{j}}(p_0) = 0$ for all $i, j, k = 1, \ldots, n$. By choosing such a local unitary frame around a point p_0, we have that the torsion tensor T' satisfies that $T^k_{ij}(p_0) = -B^k_{ij}(p_0)$ for all $i, j, k = 1, \ldots, n$, and for instance from the formula (2.11), we have that $\varphi_{i\bar{j}}(p_0) = \partial_i \partial_{\bar{j}} \varphi(p_0) = Z_i Z_{\bar{j}}(\varphi)(p_0) = Z_{\bar{j}} Z_i(\varphi)(p_0) = \varphi_{\bar{j}i}(p_0)$ for a smooth real-valued function φ. We show the following critical lemma for proving the main result. We choose and fix a local unitary frame $\{Z_r\}$ around an arbitrary chosen point $p_0 \in M$ such that $g_{i\bar{j}}(p_0) = \delta_{ij}$ and $\nabla Z(p_0) = 0$. Our computations will be done at the point p_0.

We introduce some results for a smooth function on almost Hermitian manifolds. We write that $\varphi_s := \nabla_s\varphi = \partial\varphi(Z_s) = Z_s(\varphi)$.

Lemma 3.1. *One has for a smooth real-valued function φ on M,*

$$\partial\partial\bar{\partial}\varphi(Z_k, Z_j, Z_{\bar{i}}) = \bar{\partial}(B_{kj}^{\bar{s}})(Z_{\bar{i}})\bar{\partial}\varphi(Z_{\bar{s}}). \tag{3.2}$$

Proof. We compute that from (2.7),

$$\begin{aligned}
\partial\partial\bar{\partial}\varphi(Z_k, Z_j, Z_{\bar{i}}) &= Z_k(\varphi_{j\bar{i}}) - Z_j(\varphi_{k\bar{i}}) - B_{kj}^{s}\varphi_{s\bar{i}} - B_{k\bar{i}}^{\bar{s}}\varphi_{j\bar{s}} + B_{j\bar{i}}^{\bar{s}}\varphi_{k\bar{s}} \\
&= Z_k(Z_jZ_{\bar{i}}(\varphi) - B_{j\bar{i}}^{\bar{s}}\varphi_{\bar{s}}) - Z_j(Z_kZ_{\bar{i}}(\varphi) - B_{k\bar{i}}^{\bar{s}}\varphi_{\bar{s}}) - B_{kj}^{s}(Z_sZ_{\bar{i}}(\varphi) - B_{s\bar{i}}^{\bar{r}}\varphi_{\bar{r}}) \\
&= Z_kZ_jZ_{\bar{i}}(\varphi) - Z_jZ_kZ_{\bar{i}}(\varphi) - B_{kj}^{s}Z_sZ_{\bar{i}}(\varphi) - Z_k(B_{j\bar{i}}^{\bar{s}})\varphi_{\bar{s}} + Z_j(B_{k\bar{i}}^{\bar{s}})\varphi_{\bar{s}} \\
&= [Z_k, Z_j]Z_{\bar{i}}(\varphi) - B_{kj}^{s}Z_sZ_{\bar{i}}(\varphi) - Z_k(B_{j\bar{i}}^{\bar{s}})\varphi_{\bar{s}} + Z_j(B_{s\bar{i}}^{\bar{r}})\varphi_{\bar{r}} \\
&= B_{kj}^{\bar{s}}Z_{\bar{s}}Z_{\bar{i}}(\varphi) - Z_k(B_{j\bar{i}}^{\bar{s}})\varphi_{\bar{s}} + Z_j(B_{k\bar{i}}^{\bar{s}})\varphi_{\bar{s}} \\
&= B_{kj}^{\bar{s}}[Z_{\bar{s}}, Z_{\bar{i}}](\varphi) + B_{kj}^{\bar{s}}Z_{\bar{i}}Z_{\bar{s}}(\varphi) - \Big\{Z_k(\Gamma_{j\bar{i}}^{\bar{s}}) - Z_j(\Gamma_{k\bar{i}}^{\bar{s}})\Big\}\varphi_{\bar{s}} \\
&= B_{kj}^{\bar{s}}B_{\bar{s}\bar{i}}^{r}\varphi_r + B_{kj}^{\bar{s}}B_{\bar{s}\bar{i}}^{\bar{r}}\varphi_{\bar{r}} + B_{kj}^{\bar{s}}Z_{\bar{i}}Z_{\bar{s}}(\varphi) - \overline{\mathcal{H}_{\bar{k}\bar{j}i}{}^{s}\varphi_s},
\end{aligned} \tag{3.3}$$

where we have used that $\Gamma_{i\bar{j}}^{\bar{k}}(p_0) = B_{i\bar{j}}^{\bar{k}}(p_0) = 0$, $\Gamma_{\bar{i}j}^{k}(p_0) = B_{ij}^{k}(p_0) = 0$, $\Gamma_{ij}^{k}(p_0) = 0$ for all $i, j, k = 1, \ldots, n$, and that from (2.14),

$$\begin{aligned}
\mathcal{H}_{\bar{k}\bar{j}i}{}^{s}(p_0) &= \Big\{Z_{\bar{k}}(\Gamma_{\bar{j}i}^{s}) - Z_{\bar{j}}(\Gamma_{\bar{k}i}^{s}) + \Gamma_{\bar{k}r}^{s}\Gamma_{\bar{j}i}^{r} - \Gamma_{\bar{j}r}^{s}\Gamma_{\bar{k}i}^{r} - B_{\bar{k}\bar{j}}^{r}\Gamma_{ri}^{s} - B_{\bar{k}\bar{j}}^{\bar{r}}\Gamma_{\bar{r}i}^{s}\Big\}(p_0) \\
&= Z_{\bar{k}}(\Gamma_{\bar{j}i}^{s})(p_0) - Z_{\bar{j}}(\Gamma_{\bar{k}i}^{s})(p_0).
\end{aligned}$$

We compute that

$$\begin{aligned}
B_{kj}^{\bar{s}}Z_{\bar{i}}Z_{\bar{s}}(\varphi) &= Z_{\bar{i}}(B_{kj}^{\bar{s}}Z_{\bar{s}}(\varphi)) - Z_{\bar{i}}(B_{kj}^{\bar{s}})Z_{\bar{s}}(\varphi) \\
&= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)) - Z_{\bar{i}}(B_{kj}^{\bar{s}})\bar{\partial}\varphi(Z_{\bar{s}}) \\
&= \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) - Z_{\bar{i}}(B_{kj}^{\bar{s}})\bar{\partial}\varphi(Z_{\bar{s}}),
\end{aligned} \tag{3.4}$$

where we used that

$$\begin{aligned}
\partial^2\varphi(Z_k, Z_j) &= Z_kZ_j(\varphi) - Z_jZ_k(\varphi) - B_{kj}^{s}Z_s(\varphi) \\
&= [Z_k, Z_j](\varphi) - B_{kj}^{s}Z_s(\varphi) \\
&= B_{kj}^{\bar{s}}Z_{\bar{s}}(\varphi),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) &= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)) - \partial^2\varphi([Z_{\bar{i}}, Z_k], Z_j) + \partial^2\varphi([Z_{\bar{i}}, Z_j], Z_k) \\
&= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)) - B_{sj}^{\bar{r}}B_{\bar{i}k}^{s}\varphi_{\bar{r}} + B_{sk}^{\bar{r}}B_{\bar{i}j}^{s}\varphi_{\bar{r}} \\
&= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)).
\end{aligned}$$

By combining (3.3) with (3.4), we obtain

$$\begin{aligned}
\partial\partial\bar{\partial}\varphi(Z_k, Z_j, Z_{\bar{i}}) &= \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) + B_{kj}^{\bar{s}}B_{\bar{s}\bar{i}}^{r}\partial\varphi(Z_r) + \Big\{B_{kj}^{\bar{r}}B_{\bar{r}\bar{i}}^{\bar{s}} - Z_{\bar{i}}(B_{kj}^{\bar{s}}) - \overline{\mathcal{H}_{\bar{k}\bar{j}i}{}^{s}}\Big\}\bar{\partial}\varphi(Z_{\bar{s}}) \\
&= \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) + B_{kj}^{\bar{s}}B_{\bar{s}\bar{i}}^{r}\partial\varphi(Z_r),
\end{aligned}$$

where we have used that from (2.16) and (2.20),

$$\begin{aligned}\overline{\mathcal{H}_{\bar{k}\bar{j}i}{}^{s}} &= \mathcal{H}_{jks}{}^{i}\\ &= -B^{\bar{r}}_{kj}T^{\bar{s}}_{\bar{r}\bar{i}} + \nabla_{\bar{i}}T^{\bar{s}}_{kj}\\ &= B^{\bar{r}}_{kj}B^{\bar{s}}_{\bar{r}\bar{i}} - Z_{\bar{i}}(B^{\bar{s}}_{kj}).\end{aligned} \tag{3.6}$$

We compute by using (3.5),

$$\begin{aligned}\partial\partial\bar{\partial}\varphi(Z_k, Z_j, Z_{\bar{i}}) &= \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) + B^{\bar{s}}_{kj}B^{r}_{\bar{s}\bar{i}}\partial\varphi(Z_r)\\ &= \bar{\partial}(B^{\bar{s}}_{kj}\bar{\partial}\varphi(Z_{\bar{s}}))(Z_{\bar{i}}) + T^{\bar{s}}_{kj}T^{r}_{\bar{s}\bar{i}}\partial\varphi(Z_r)\\ &= \bar{\partial}(B^{\bar{s}}_{kj})(Z_{\bar{i}})\bar{\partial}\varphi(Z_{\bar{s}}) + B^{\bar{s}}_{kj}\bar{\partial}^2\varphi(Z_{\bar{i}}, Z_{\bar{s}}) + T^{\bar{s}}_{kj}T^{r}_{\bar{s}\bar{i}}\partial\varphi(Z_r)\\ &= \bar{\partial}(B^{\bar{s}}_{kj})(Z_{\bar{i}})\bar{\partial}\varphi(Z_{\bar{s}}) - T^{\bar{s}}_{kj}B^{r}_{\bar{i}\bar{s}}\partial\varphi(Z_r) + T^{\bar{s}}_{kj}T^{r}_{\bar{s}\bar{i}}\partial\varphi(Z_r)\\ &= \bar{\partial}(B^{\bar{s}}_{kj})(Z_{\bar{i}})\bar{\partial}\varphi(Z_{\bar{s}}),\end{aligned}$$

where we have used that $B^{r}_{\bar{i}\bar{s}} = -T^{r}_{\bar{i}\bar{s}} = T^{r}_{\bar{s}\bar{i}}$. □

Lemma 3.2. *One has for a smooth real-valued function φ on M,*

$$\bar{\partial}\partial\bar{\partial}\varphi(Z_{\bar{k}}, Z_i, Z_{\bar{j}}) = \partial(B^{s}_{\bar{k}\bar{j}})(Z_i)\partial\varphi(Z_s). \tag{3.7}$$

Proof. We compute that from (2.8), using $B^{s}_{i\bar{j}}(p_0) = 0$ for all $i, j, s = 1, \ldots, n$, $B^{\bar{r}}_{i\bar{s}}(p_0) = 0$ for all $i, r, s = 1, \ldots, n$ and $[Z_{\bar{k}}, Z_i](p_0) = 0$ for all $i, k = 1, \ldots, n$,

$$\begin{aligned}\bar{\partial}\partial\bar{\partial}\varphi(Z_{\bar{k}}, Z_i, Z_{\bar{j}}) &= Z_{\bar{k}}(\varphi_{i\bar{j}}) - Z_{\bar{j}}(\varphi_{i\bar{k}}) - B^{s}_{i\bar{j}}\varphi_{s\bar{k}} + B^{s}_{i\bar{k}}\varphi_{s\bar{j}} + B^{\bar{s}}_{\bar{j}\bar{k}}\varphi_{i\bar{s}}\\ &= Z_{\bar{k}}(Z_iZ_{\bar{j}}(\varphi) - B^{\bar{s}}_{i\bar{j}}\varphi_{\bar{s}}) - Z_{\bar{j}}(Z_iZ_{\bar{k}}(\varphi) - B^{\bar{s}}_{i\bar{k}}\varphi_{\bar{s}}) + B^{\bar{s}}_{\bar{j}\bar{k}}\varphi_{i\bar{s}}\\ &= Z_{\bar{k}}Z_iZ_{\bar{j}}(\varphi) - Z_{\bar{j}}Z_iZ_{\bar{k}}(\varphi) + B^{\bar{s}}_{\bar{j}\bar{k}}(Z_iZ_{\bar{s}}(\varphi) - B^{\bar{r}}_{i\bar{s}}\varphi_{\bar{r}})\\ &\quad -Z_{\bar{k}}(B^{\bar{s}}_{i\bar{j}})\varphi_{\bar{s}} + Z_{\bar{j}}(B^{\bar{s}}_{i\bar{k}})\varphi_{\bar{s}}\\ &= Z_iZ_{\bar{k}}Z_{\bar{j}}(\varphi) + [Z_{\bar{k}}, Z_i]Z_{\bar{j}}(\varphi) - Z_iZ_{\bar{j}}Z_{\bar{k}}(\varphi) - [Z_{\bar{j}}, Z_i]Z_{\bar{k}}(\varphi)\\ &\quad +B^{\bar{s}}_{\bar{j}\bar{k}}Z_iZ_{\bar{s}}(\varphi) - Z_{\bar{k}}(B^{\bar{s}}_{i\bar{j}})\varphi_{\bar{s}} + Z_{\bar{j}}(B^{\bar{s}}_{i\bar{k}})\varphi_{\bar{s}}\\ &= Z_i[Z_{\bar{k}}, Z_{\bar{j}}](\varphi) - B^{\bar{s}}_{\bar{k}\bar{j}}Z_iZ_{\bar{s}}(\varphi) - Z_{\bar{k}}(\Gamma^{\bar{s}}_{i\bar{j}})\varphi_{\bar{s}} + Z_{\bar{j}}(\Gamma^{\bar{s}}_{i\bar{k}})\varphi_{\bar{s}}\\ &= B^{s}_{\bar{k}\bar{j}}Z_iZ_s(\varphi) + Z_i(B^{s}_{\bar{k}\bar{j}})\varphi_s + Z_i(B^{\bar{s}}_{\bar{k}\bar{j}})\varphi_{\bar{s}} - Z_{\bar{k}}(\Gamma^{\bar{s}}_{i\bar{j}})\varphi_{\bar{s}} + Z_{\bar{j}}(\Gamma^{\bar{s}}_{i\bar{k}})\varphi_{\bar{s}}\\ &= B^{s}_{\bar{k}\bar{j}}Z_iZ_s(\varphi) - Z_i(T^{s}_{\bar{k}\bar{j}})\varphi_s - \Big\{Z_i(\Gamma^{\bar{s}}_{\bar{k}\bar{j}}) - Z_i(\Gamma^{\bar{s}}_{\bar{j}\bar{k}}) - Z_i(B^{\bar{s}}_{\bar{k}\bar{j}})\Big\}\varphi_{\bar{s}}\\ &\quad -\Big\{Z_{\bar{k}}(\Gamma^{\bar{s}}_{i\bar{j}}) - Z_i(\Gamma^{\bar{s}}_{\bar{k}\bar{j}})\Big\}\varphi_{\bar{s}} + \Big\{Z_{\bar{j}}(\Gamma^{\bar{s}}_{i\bar{k}}) - Z_i(\Gamma^{\bar{s}}_{\bar{j}\bar{k}})\Big\}\varphi_{\bar{s}}\\ &= B^{s}_{\bar{k}\bar{j}}Z_iZ_s(\varphi) - Z_i(T^{s}_{\bar{k}\bar{j}})\varphi_s - Z_i(T^{\bar{s}}_{\bar{k}\bar{j}})\varphi_{\bar{s}} - \overline{\mathcal{R}_{k\bar{i}j}{}^{s}}\varphi_{\bar{s}} + \overline{\mathcal{R}_{j\bar{i}k}{}^{s}}\varphi_{\bar{s}},\end{aligned} \tag{3.8}$$

where we have used that $B^{\bar{s}}_{\bar{k}\bar{j}} = -B^{\bar{s}}_{\bar{j}\bar{k}}$ and that

$$\begin{aligned}Z_iZ_{\bar{k}}Z_{\bar{j}}(\varphi) - Z_iZ_{\bar{j}}Z_{\bar{k}}(\varphi) &= Z_i[Z_{\bar{k}}, Z_{\bar{j}}](\varphi)\\ &= Z_i(B^{s}_{\bar{k}\bar{j}}Z_s + B^{\bar{s}}_{\bar{k}\bar{j}}Z_{\bar{s}})(\varphi)\\ &= Z_i(B^{s}_{\bar{k}\bar{j}})\varphi_s + B^{s}_{\bar{k}\bar{j}}Z_iZ_s(\varphi) + Z_i(B^{\bar{s}}_{\bar{k}\bar{j}})\varphi_{\bar{s}} + B^{\bar{s}}_{\bar{k}\bar{j}}Z_iZ_{\bar{s}}(\varphi),\end{aligned}$$

and from (2.12),

$$\begin{aligned}\mathcal{R}_{k\bar{i}j}{}^{s}(p_0) &= \Big\{Z_k(\Gamma^s_{\bar{i}j}) - Z_{\bar{i}}(\Gamma^s_{kj}) + \Gamma^s_{kr}\Gamma^r_{\bar{i}j} - \Gamma^s_{\bar{i}r}\Gamma^r_{kj} - B^r_{k\bar{i}}\Gamma^s_{rj} + B^{\bar{r}}_{\bar{i}k}\Gamma^s_{\bar{r}j}\Big\}(p_0)\\ &= Z_k(\Gamma^s_{\bar{i}j})(p_0) - Z_{\bar{i}}(\Gamma^s_{kj})(p_0).\end{aligned}$$

We compute that

$$\begin{aligned}B^s_{\bar{k}\bar{j}}Z_iZ_s(\varphi) &= Z_i(B^s_{\bar{k}\bar{j}}Z_s(\varphi)) - Z_i(B^s_{\bar{k}\bar{j}})Z_s(\varphi)\\ &= Z_i(\bar{\partial}^2\varphi(Z_{\bar{k}}, Z_{\bar{j}})) - Z_i(B^s_{\bar{k}\bar{j}})\partial\varphi(Z_s)\\ &= \partial\bar{\partial}^2\varphi(Z_i, Z_{\bar{k}}, Z_{\bar{j}}) + Z_i(T^s_{\bar{k}\bar{j}})\partial\varphi(Z_s),\end{aligned} \tag{3.9}$$

where we used that

$$\begin{aligned}\bar{\partial}^2\varphi(Z_{\bar{k}}, Z_{\bar{j}}) &= Z_{\bar{k}}Z_{\bar{j}}(\varphi) - Z_{\bar{j}}Z_{\bar{k}}(\varphi) - B^{\bar{s}}_{\bar{k}\bar{j}}Z_{\bar{s}}(\varphi)\\ &= [Z_{\bar{k}}, Z_{\bar{j}}](\varphi) - B^{\bar{s}}_{\bar{k}\bar{j}}Z_{\bar{s}}(\varphi)\\ &= B^s_{\bar{k}\bar{j}}Z_s(\varphi),\end{aligned} \tag{3.10}$$

$$\begin{aligned}\partial\bar{\partial}^2\varphi(Z_i, Z_{\bar{k}}, Z_{\bar{j}}) &= Z_i(\bar{\partial}^2\varphi(Z_{\bar{k}}, Z_{\bar{j}})) - \bar{\partial}^2\varphi([Z_i, Z_{\bar{k}}], Z_{\bar{j}}) + \bar{\partial}^2\varphi([Z_i, Z_{\bar{j}}], Z_{\bar{k}})\\ &= Z_i(\partial^2\varphi(Z_{\bar{k}}, Z_{\bar{j}})) - B^r_{\bar{s}\bar{j}}B^{\bar{s}}_{i\bar{k}}\varphi_r + B^r_{\bar{s}\bar{k}}B^{\bar{s}}_{i\bar{j}}\varphi_r\\ &= Z_i(\partial^2\varphi(Z_{\bar{k}}, Z_{\bar{j}})).\end{aligned}$$

Combining (3.8) with (3.9), we obtain that

$$\begin{aligned}\bar{\partial}\partial\bar{\partial}\varphi(Z_{\bar{k}}, Z_i, Z_{\bar{j}}) &= \partial\bar{\partial}^2\varphi(Z_i, Z_{\bar{k}}, Z_{\bar{j}}) + \Big\{\overline{\mathcal{R}_{j\bar{i}k}{}^{s}} - \overline{\mathcal{R}_{k\bar{i}j}{}^{s}} - Z_i(T^{\bar{s}}_{\bar{k}\bar{j}})\Big\}\bar{\partial}\varphi(Z_{\bar{s}})\\ &= \partial\bar{\partial}^2\varphi(Z_i, Z_{\bar{k}}, Z_{\bar{j}}) + T^r_{\bar{k}\bar{j}}T^{\bar{s}}_{ri}\bar{\partial}\varphi(Z_{\bar{s}}),\end{aligned}$$

where we have used that from (2.15),

$$\begin{aligned}\overline{\mathcal{R}_{j\bar{i}k}{}^{s} - \mathcal{R}_{k\bar{i}j}{}^{s}} &= \overline{-B^r_{\bar{j}\bar{k}}B^{\bar{s}}_{ri} + \nabla_{\bar{i}}T^s_{kj}}\\ &= T^r_{\bar{k}\bar{j}}T^{\bar{s}}_{ri} + Z_i(T^{\bar{s}}_{\bar{k}\bar{j}}).\end{aligned}$$

We compute that by applying (3.5) and (3.10),

$$\begin{aligned}\bar{\partial}\partial\bar{\partial}\varphi(Z_{\bar{k}}, Z_i, Z_{\bar{j}}) &= \partial\bar{\partial}^2\varphi(Z_i, Z_{\bar{k}}, Z_{\bar{j}}) + T^r_{\bar{k}\bar{j}}T^{\bar{s}}_{ri}\bar{\partial}\varphi(Z_{\bar{s}})\\ &= \partial(B^s_{\bar{k}\bar{j}}\partial\varphi(Z_s))(Z_i) + T^r_{\bar{k}\bar{j}}T^{\bar{s}}_{ri}\bar{\partial}\varphi(Z_{\bar{s}})\\ &= \partial(B^s_{\bar{k}\bar{j}})(Z_i)\partial\varphi(Z_s) + B^r_{\bar{k}\bar{j}}\partial^2\varphi(Z_i, Z_r) + T^r_{\bar{k}\bar{j}}T^{\bar{s}}_{ri}\bar{\partial}\varphi(Z_{\bar{s}})\\ &= \partial(B^s_{\bar{k}\bar{j}})(Z_i)\partial\varphi(Z_s) - T^r_{\bar{k}\bar{j}}B^{\bar{s}}_{ir}\bar{\partial}\varphi(Z_{\bar{s}}) + T^r_{\bar{k}\bar{j}}T^{\bar{s}}_{ri}\bar{\partial}\varphi(Z_{\bar{s}})\\ &= \partial(B^s_{\bar{k}\bar{j}})(Z_i)\partial\varphi(Z_s),\end{aligned}$$

where we have used that $B^{\bar{s}}_{ir} = -T^{\bar{s}}_{ir} = T^{\bar{s}}_{ri}$. □

Lemma 3.3. *One has for a smooth real-valued function* φ *on* M,

$$\partial\bar{\partial}\partial\bar{\partial}\varphi(Z_l, Z_{\bar{k}}, Z_i, Z_{\bar{j}}) \;=\; -\partial^2(T^s_{\bar{k}\bar{j}})(Z_l, Z_i)\partial\varphi(Z_s) + \partial(T^s_{\bar{k}\bar{j}})(Z_i)T^{\bar{r}}_{ls}\bar{\partial}\varphi(Z_{\bar{r}}). \tag{3.11}$$

Proof. By applying (3.5) and (3.6), we have that

$$\begin{aligned}\partial\bar{\partial}\partial\bar{\partial}\varphi(Z_l, Z_{\bar{k}}, Z_i, Z_{\bar{j}}) &= \partial^2(B^s_{\bar{k}\bar{j}})(Z_l, Z_i)\partial\varphi(Z_s) + \partial(B^s_{\bar{k}\bar{j}})(Z_i)\partial^2\varphi(Z_l, Z_s)\\ &= -\partial^2(T^s_{\bar{k}\bar{j}})(Z_l, Z_i)\partial\varphi(Z_s) + \partial(T^s_{\bar{k}\bar{j}})(Z_i)T^{\bar{r}}_{ls}\bar{\partial}\varphi(Z_{\bar{r}}). \quad \square\end{aligned}$$

In order to avoid a notational quagmire, we adopt the following $*$-convention $\mathcal{C}_1 * \mathcal{C}_2$ between two geometric quantities $\mathcal{C}_1$ and $\mathcal{C}_2$ with respect to a metric g:

(1) Summation over pairs of maching upper and lower indices.

(2) Contraction on upper indices with respect to the metric.

(3) Contraction on lower indices with respect to the dual metrics.

Since the point p_0 was chosen arbitrary, the computations in Lemma 3.1–3.3 hold globally on an almost Hermitian manifold M for any real-valued smooth function φ, which implies that we can write (3.2), (3.7), and (3.11) globally on M as follows:

$$\partial\bar{\partial}\partial\bar{\partial}\varphi =: \mathcal{T}_1 * \partial\varphi + \mathcal{T}_2 * \bar{\partial}\varphi, \quad \partial^2\bar{\partial}\varphi =: \mathcal{T}_3 * \bar{\partial}\varphi, \quad \bar{\partial}\partial\bar{\partial}\varphi =: \mathcal{T}_4 * \partial\varphi. \tag{3.12}$$

4 Proof of Theorem 1.1

Let (M^{2n}, J, ω) be a compact almost Hermitian manifold of real dimension $2n$ with $n \geq 2$ in this whole section. Let u be a smooth solution of (1.1). As in [5], we let $S_k(\lambda)$ denote the k-th elementary symmetric polynomial of $\lambda \in \mathbb{R}^n$:

$$S_k(\lambda) := \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}.$$

For a square matrix U, we define $S_\alpha(U) := S_\alpha(\lambda(U))$, where $\lambda(U)$ denote the eigenvalues of the matrix U. Locally, we can write the equation (1.1) in the following form:

$$\frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)} = \frac{\psi}{C_n^\alpha}, \tag{4.1}$$

where $C_n^\alpha := \frac{n!}{(n-\alpha)!\alpha!}$. We need the following generalized Newton-MacLaurin inequality.

Lemma 4.1 (cf. [1, Proposition 3], [9, Proposition 2.1])**.** *For* $\lambda \in \Gamma_k := \{\lambda \in \mathbb{R}^n : S_k(\lambda) > 0,$ $\forall\, 1 \leq i \leq k\}$ *and* $0 \leq l < k \leq n$, $0 \leq s < r$, $r \leq k$, $s \leq l$, *we have*

$$\left[\frac{\frac{S_k(\lambda)}{C_n^k}}{\frac{S_l(\lambda)}{C_n^l}}\right]^{\frac{1}{k-l}} \leq \left[\frac{\frac{S_r(\lambda)}{C_n^r}}{\frac{S_s(\lambda)}{C_n^s}}\right]^{\frac{1}{r-s}}. \tag{4.2}$$

In this section, the positive constant C may be changed from line to line, but it depends on the allowed data.

Proof of Theorem 1.1. It suffices to show the following key inequality:

$$\int_M |\partial e^{-\frac{p}{2}u}|_g^2 \omega^n \leq Cp \int_M e^{-pu}\omega^n \tag{4.3}$$

for p large enough.

Lemma 4.2. *Let u be a smooth admissible solution to the Monge-Ampère type equation (1,1). Then, there are uniform constants C, p_0 such that for any $p \geq p_0$, we have the inequality* (4.3)*.*

Proof. Without loss of generality, we may assume that

$$n\chi^{n-1} > (n-\alpha)\psi\chi^{n-\alpha-1} \wedge \omega^\alpha, \tag{4.4}$$

and there exist uniform positive constants $\lambda, \Lambda > 0$ such that

$$\lambda\omega \leq \chi \leq \Lambda\omega. \tag{4.5}$$

As the local expression (4.1):

$$\frac{\chi_u^n}{\chi_u^{n-\alpha} \wedge \omega^\alpha} = C_n^\alpha \frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)} = \psi,$$

we locally have that

$$C_{n-1}^\alpha \frac{S_{n-1}(\chi_u)}{S_{n-\alpha-1}(\chi_u)} = \frac{\chi_u^{n-1}}{\chi_u^{n-\alpha-1} \wedge \omega^\alpha}$$

and which implies that the following inequality

$$n\chi_u^{n-1} > (n-\alpha)\psi\chi_u^{n-\alpha-1} \wedge \omega^\alpha \tag{4.6}$$

is equivalent to

$$\frac{S_{n-1}(\chi_u)}{S_{n-\alpha-1}(\chi_u)} > \frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)} \tag{4.7}$$

since we have locally that

$$\frac{n-\alpha}{n} \cdot \psi = \frac{n-\alpha}{n} \cdot C_n^\alpha \frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)} = C_{n-1}^\alpha \frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)}.$$

Note that we may apply Lemma 4.1 to χ_u since $\chi_u > 0$. Applying the inequality (4.2), we have

$$\left[\frac{\frac{S_n(\chi_u)}{C_n^n}}{\frac{S_{n-\alpha}(\chi_u)}{C_n^{n-\alpha}}}\right]^{\frac{1}{\alpha}} \leq \left[\frac{\frac{S_{n-1}(\chi_u)}{C_n^{n-1}}}{\frac{S_{n-\alpha-1}(\chi_u)}{C_n^{n-\alpha-1}}}\right]^{\frac{1}{\alpha}},$$

which can be written by

$$\begin{aligned} \frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)} &\leq \frac{C_n^{n-\alpha-1}}{C_n^{n-1}C_n^{n-\alpha}} \frac{S_{n-1}(\chi_u)}{S_{n-\alpha-1}(\chi_u)} \\ &= \frac{n-\alpha}{n(\alpha+1)} \frac{S_{n-1}(\chi_u)}{S_{n-\alpha-1}(\chi_u)} \\ &< \frac{S_{n-1}(\chi_u)}{S_{n-\alpha-1}(\chi_u)}, \end{aligned}$$

where we used that $\frac{n-\alpha}{n(\alpha+1)} < 1$. Therefore, the inequality (4.7) holds and as a consequence, we have the inequality (4.6).

We estimate that

$$\begin{aligned} I &:= \int_M e^{-pu}((\chi_u^n - \chi^n) - \psi(\chi_u^{n-\alpha}\wedge\omega^\alpha - \chi^{n-\alpha}\wedge\omega^\alpha)) \\ &= \int_M e^{-pu}\Big(\frac{\chi_u^n}{\chi_u^{n-\alpha}\wedge\omega^\alpha} - \frac{\chi^n}{\chi^{n-\alpha}\wedge\omega^\alpha}\Big)\chi^{n-\alpha}\wedge\omega^\alpha \\ &\leq C\int_M e^{-pu}\omega^n. \end{aligned} \tag{4.8}$$

On the other hand, we have that by Stokes' theorem,

$$\begin{aligned} I &= \int_0^1\int_M e^{-pu}\frac{d}{dt}(\chi_{tu}^n - \psi\chi_{tu}^{n-\alpha}\wedge\omega^\alpha)dt \\ &= \int_0^1\int_M e^{-pu}\sqrt{-1}\partial\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &= \int_0^1\int_M d(e^{-pu}\sqrt{-1}\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha))dt \\ &\quad - \int_0^1\int_M \sqrt{-1}\partial e^{-pu}\wedge\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &\quad + \int_0^1\int_M \sqrt{-1}e^{-pu}\bar\partial u\wedge\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &= p\int_0^1\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &\quad -\frac{1}{p}\int_0^1\int_M \sqrt{-1}\bar\partial e^{-pu}\wedge\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &= p\int_0^1\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &\quad -\frac{1}{p}\int_0^1\int_M d(\sqrt{-1}e^{-pu}\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha))dt \\ &\quad +\frac{1}{p}\int_0^1\int_M e^{-pu}\sqrt{-1}\bar\partial\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha))dt \\ &= p\int_0^1\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &\quad -\frac{1}{p}\int_0^1\int_M e^{-pu}\sqrt{-1}\partial\bar\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt, \end{aligned} \tag{4.9}$$

where we have used that $d = A + \partial + \bar\partial + \bar A$,

$$\bar\partial(e^{-pu}\sqrt{-1}\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)) = 0,$$

$$A(e^{-pu}\sqrt{-1}\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)) = 0,$$

$$\bar A(e^{-pu}\sqrt{-1}\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)) = 0,$$

$$\partial(\sqrt{-1}e^{-pu}\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)) = 0,$$

$$A(\sqrt{-1}e^{-pu}\partial(n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha}))=0,$$
$$\bar{A}(\sqrt{-1}e^{-pu}\partial(n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha}))=0$$

and from (2.9),

$$\begin{aligned}\bar{\partial}\partial(n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha}) &= -(\partial\bar{\partial}+A\bar{A}+\bar{A}A)(n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha})\\ &= -\partial\bar{\partial}(n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha})\end{aligned}$$

since we have

$$A(n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha})=\bar{A}(n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha})=0.$$

We compute that for $0\leq t\leq 1$,

$$\begin{aligned}
& -\frac{1}{p}\int_M e^{-pu}\sqrt{-1}\partial\bar{\partial}(n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha})\\
= & -\frac{1}{p}\int_M e^{-pu}\sqrt{-1}\partial\left(n(n-1)\chi_{tu}^{n-2}\wedge(\bar{\partial}\chi+\sqrt{-1}t\bar{\partial}\partial\bar{\partial}u)-(n-\alpha)\bar{\partial}\psi\wedge\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha}\right.\\
& \left.-(n-\alpha)(n-\alpha-1)\psi\chi_{tu}^{n-\alpha-2}\wedge(\bar{\partial}\chi+\sqrt{-1}t\bar{\partial}\partial\bar{\partial}u)\wedge\omega^{\alpha}-\alpha(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha-1}\wedge\bar{\partial}\omega\right)\\
= & -\frac{1}{p}\int_M \sqrt{-1}e^{-pu}\Big\{n(n-1)(n-2)\chi_{tu}^{n-3}\wedge(\partial\chi+t\sqrt{-1}\partial\partial\bar{\partial}u)\wedge(\bar{\partial}\chi+t\sqrt{-1}\bar{\partial}\partial\bar{\partial}u)\\
& +n(n-1)\chi_{tu}^{n-2}\wedge(\partial\bar{\partial}\chi+t\sqrt{-1}\partial\bar{\partial}\partial\bar{\partial}u)-(n-\alpha)\partial\bar{\partial}\psi\wedge\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha}\\
& +(n-\alpha)(n-\alpha-1)\bar{\partial}\psi\wedge\chi_{tu}^{n-\alpha-2}\wedge(\partial\chi+t\sqrt{-1}\partial\partial\bar{\partial}u)\wedge\omega^{\alpha}\\
& +\alpha(n-\alpha)\bar{\partial}\psi\wedge\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha-1}\wedge\partial\omega\\
& -(n-\alpha)(n-\alpha-1)\partial\psi\wedge\chi_{tu}^{n-\alpha-2}\wedge(\bar{\partial}\chi+t\sqrt{-1}\bar{\partial}\partial\bar{\partial}u)\wedge\omega^{\alpha}\\
& -(n-\alpha)(n-\alpha-1)(n-\alpha-2)\psi\chi_{tu}^{n-\alpha-3}\wedge(\partial\chi+t\sqrt{-1}\partial\partial\bar{\partial}u)\wedge(\bar{\partial}\chi+t\sqrt{-1}\bar{\partial}\partial\bar{\partial}u)\wedge\omega^{\alpha}\\
& -(n-\alpha)(n-\alpha-1)\psi\chi_{tu}^{n-\alpha-2}\wedge(\partial\bar{\partial}\chi+t\sqrt{-1}\partial\bar{\partial}\partial\bar{\partial}u)\wedge\omega^{\alpha}\\
& -\alpha(n-\alpha)(n-\alpha-1)\psi\chi_{tu}^{n-\alpha-2}\wedge(\bar{\partial}\chi+t\sqrt{-1}\bar{\partial}\partial\bar{\partial}u)\wedge\omega^{\alpha-1}\wedge\partial\omega\\
& -\alpha(n-\alpha)\partial\psi\wedge\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha-1}\wedge\bar{\partial}\omega\\
& -\alpha(n-\alpha)(n-\alpha-1)\psi\chi_{tu}^{n-\alpha-2}\wedge(\partial\chi+t\sqrt{-1}\partial\partial\bar{\partial}u)\wedge\omega^{\alpha-1}\wedge\bar{\partial}\omega\\
& -\alpha(n-\alpha)(\alpha-1)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha-2}\wedge\partial\omega\wedge\bar{\partial}\omega\\
& -\alpha(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha-1}\wedge\partial\bar{\partial}\omega\Big\}\\
\geq & -\frac{C}{p}\int_M e^{-pu}\chi_{tu}^{n-3}\wedge\omega^3-\frac{C}{p}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-3}\wedge\omega^2-C\int_M e^{-pu}\chi_{tu}^{n-2}\wedge\omega^2\\
& -\frac{C}{p}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-2}\wedge\omega\\
& -\frac{C}{p}\int_M e^{-pu}\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha+1}-\frac{C}{p}\int_M e^{-pu}\chi_{tu}^{n-\alpha-2}\wedge\omega^{\alpha+2}\\
& -\frac{C}{p}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-\alpha-2}\wedge\omega^{\alpha+1}-\frac{C}{p}\int_M e^{-pu}\chi_{tu}^{n-\alpha-3}\wedge\omega^{\alpha+3}\\
& -\frac{C}{p}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-\alpha-3}\wedge\omega^{\alpha+2},
\end{aligned}\tag{4.10}$$

where we have used that for instance, by applying (3.12),

$$\begin{aligned}
&\int_M \sqrt{-1}e^{-pu}\chi_{tu}^{n-2}\wedge t\sqrt{-1}\partial\bar\partial\partial\bar\partial u\\
=\ &\int_M \sqrt{-1}e^{-pu}\chi_{tu}^{n-2}\wedge t\sqrt{-1}(\mathcal{T}_1 * \partial u+\mathcal{T}_2 * \bar\partial u)\\
\leq\ & C\int_M e^{-pu}\chi_{tu}^{n-2}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega + C\int_M e^{-pu}\chi_{tu}^{n-2}\wedge\omega^2, \qquad (4.11)
\end{aligned}$$

$$\begin{aligned}
&\int_M \bar\partial\psi\wedge\chi_{tu}^{n-\alpha-2}\wedge t\sqrt{-1}\partial\partial\bar\partial u\wedge\omega^\alpha\\
=\ &\int_M \bar\partial\psi\wedge\chi_{tu}^{n-\alpha-2}\wedge t\sqrt{-1}\mathcal{T}_3 * \bar\partial u\wedge\omega^\alpha\\
\leq\ & C\int_M \chi_{tu}^{n-\alpha-2}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega^{\alpha+1} + C\int_M \chi_{tu}^{n-\alpha-2}\wedge\omega^{\alpha+2}, \qquad (4.12)
\end{aligned}$$

$$\begin{aligned}
&\int_M \sqrt{-1}e^{-pu}\chi_{tu}^{n-3}\wedge\partial\chi\wedge t\sqrt{-1}\bar\partial\partial\bar\partial u\wedge\omega\\
=\ &\int_M \sqrt{-1}e^{-pu}\chi_{tu}^{n-3}\wedge\partial\chi\wedge t\sqrt{-1}\mathcal{T}_4 * \partial u\wedge\omega\\
\leq\ & C\int_M e^{-pu}\chi_{tu}^{n-3}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega + C\int_M e^{-pu}\chi_{tu}^{n-3}\wedge\omega^3. \qquad (4.13)
\end{aligned}$$

Since we have assumed that $\chi, \chi_u > 0$, then we have that $\chi_{tu} > 0$ for any $0 \leq t \leq 1$. Now we introduce the following crucial inequalities (cf. [6]):

Lemma 4.3. *For any $0 < t \leq 1$, $1 < l \leq n$, one has that*

$$\frac{l}{l-1}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-1}\wedge\omega^{n-l}ds \geq \lambda\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-2}\wedge\omega^{n-l+1}ds, \quad (4.14)$$

and for any $0 < t \leq 1$, $1 \leq k \leq n$, one has that

$$\frac{k+1}{k}\int_0^t \chi_{su}^k\wedge\omega^{n-k}ds \geq \lambda\int_0^t \chi_{su}^{k-1}\wedge\omega^{n-k+1}ds, \quad (4.15)$$

where $\lambda > 0$ is the uniform constant in (4.5).

Proof. By using integration by parts and Gårding's inequality as in [6, (2.22)], we have that by using $\chi \geq \lambda\omega$,

$$\begin{aligned}
&\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-1}\wedge\omega^{n-l}ds\\
=\ &\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-2}\wedge(\chi+s\sqrt{-1}\partial\bar\partial u)\wedge\omega^{n-l}ds\\
\geq\ &\lambda\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-2}\wedge\omega^{n-l+1}ds\\
&+\frac{1}{l-1}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge s\frac{d}{ds}\chi_{su}^{l-1}\wedge\omega^{n-l}ds\\
\geq\ &\lambda\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-2}\wedge\omega^{n-l+1}ds\\
&-\frac{1}{l-1}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-1}\wedge\omega^{n-l}ds, \qquad (4.16)
\end{aligned}$$

where we used that

$$\begin{aligned}
&\int_0^t \int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge s\frac{d}{ds}\chi_{su}^{l-1} \wedge \omega^{n-l} ds \\
= \quad & t\int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{l-1} \wedge \omega^{n-l} - \int_0^t \int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \chi_{su}^{l-1} \wedge \omega^{n-l} ds \\
\geq \quad & -\int_0^t \int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \chi_{su}^{l-1} \wedge \omega^{n-l} ds.
\end{aligned}$$

The inequality (4.16) gives the desired one (4.14). Next we compute that by using integration by parts and Gårding's inequality as in [6, (3.7)], for $1 \leq k \leq n$, using $\chi \geq \lambda\omega$,

$$\begin{aligned}
\int_0^t \chi_{su}^k \wedge \omega^{n-k} ds &= \int_0^t \chi_{su}^{k-1} \wedge (\chi + s\sqrt{-1}\partial\bar{\partial}u) \wedge \omega^{n-k} ds \\
&\geq \lambda \int_0^t \chi_{su}^{k-1} \wedge \omega^{n-k+1} ds + \frac{1}{k}\int_0^t s\frac{d}{ds}(\chi_{su}^k \wedge \omega^{n-k}) ds \\
&= \lambda \int_0^t \chi_{su}^{k-1} \wedge \omega^{n-k+1} ds + \frac{t}{k}\chi_{tu}^k \wedge \omega^{n-k} - \frac{1}{k}\int_0^t \chi_{su}^k \wedge \omega^{n-k} ds \\
&\geq \lambda \int_0^t \chi_{su}^{k-1} \wedge \omega^{n-k+1} ds - \frac{1}{k}\int_0^t \chi_{su}^k \wedge \omega^{n-k} ds,
\end{aligned}$$

which implies the inequality (4.15). □

By applying these inequalities (4.14) and (4.15) for $t = 1$ to the estimate (4.10), we obtain that

$$\begin{aligned}
&-\frac{1}{p}\int_0^1 \int_M e^{-pu}\sqrt{-1}\partial\bar{\partial}(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1} \wedge \omega^\alpha) dt \\
\geq \quad & -\frac{C}{p}\int_0^1 \int_M e^{-pu}\chi_{tu}^{n-1} \wedge \omega dt - \frac{C}{p}\int_0^1 \int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{n-1} dt. \qquad (4.17)
\end{aligned}$$

Combining (4.17) with (4.9), we have that

$$\begin{aligned}
I \quad \geq \quad & p\int_0^1 \int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \left\{\left(n - \frac{C}{p^2}\right)\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1} \wedge \omega^\alpha\right\} dt \\
& -\frac{C}{p}\int_0^1 \int_M e^{-pu}\chi_{tu}^{n-1} \wedge \omega dt. \qquad (4.18)
\end{aligned}$$

By the concavity of hyperbolic polynomials, for $0 < \tau < 1$, $1 \leq k \leq n$, we have (cf. [6, (2.13)])

$$\frac{1}{\tau}S_k^{\frac{1}{k}}(\chi_{\tau tu}) + \left(1 - \frac{1}{\tau}\right)S_k^{\frac{1}{k}}(\chi) \geq S_k^{\frac{1}{k}}(\chi_{tu}),$$

which gives

$$S_k(\chi_{\tau tu}) \geq \tau^k S_k(\chi_{tu}).$$

For $\tau = \frac{1}{2}$, $k = n-1$, we obtain that

$$\begin{aligned}
\int_0^1 \int_M e^{-pu}\chi_{tu}^{n-1} \wedge \omega dt &\leq 2^{n-1}\int_0^1 \int_M e^{-pu}\chi_{\frac{tu}{2}}^{n-1} \wedge \omega dt \\
&= 2^n \int_0^{\frac{1}{2}} \int_M e^{-pu}\chi_{tu}^{n-1} \wedge \omega dt. \qquad (4.19)
\end{aligned}$$

By combining (4.8), (4.18) and (4.19), we have that

$$\begin{aligned} & p\int_0^1\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\Big\{\Big(n-\frac{C}{p^2}\Big)\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha\Big\}dt \\ \leq\ & C\int_M e^{-pu}\omega^n+\frac{C}{p}\int_0^{\frac12}\int_M e^{-pu}\chi_{tu}^{n-1}\wedge\omega dt. \end{aligned} \tag{4.20}$$

Since we have $\chi_{tu}>0$ and

$$n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha>0$$

for any $0\leq t\leq 1$, we can choose a sufficiently large p so that

$$n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha-\frac{C}{p^2}\chi_{tu}^{n-1}>0.$$

Then we have that by the concavity of the quotient equation, for some $0<\delta<1$, we have (cf. [6, (3.10)])

$$n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha>n\Big\{1-\frac{1}{(1+\delta-t\delta)^\alpha}\Big\}\chi_{tu}^{n-1},$$

hence for sufficiently large p,

$$\begin{aligned} & \int_0^1\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\Big\{\Big(n-\frac{C}{p^2}\Big)\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha\Big\}dt \\ \geq\ & \int_0^{\frac12}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\Big\{\Big(n-\frac{C}{p^2}\Big)\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha\Big\}dt \\ \geq\ & \int_0^{\frac12} n\Big\{1-\frac{C}{np^2}-\frac{1}{(1+\delta-t\delta)^\alpha}\Big\}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{tu}^{n-1}dt \\ \geq\ & n\Big\{1-\frac{C}{np^2}-\frac{1}{(1+\frac{\delta}{2})^\alpha}\Big\}\int_0^{\frac12}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{tu}^{n-1}dt. \end{aligned} \tag{4.21}$$

On the other hand, we compute by Stokes' theorem,

$$\begin{aligned} & \frac1p\int_0^{\frac12}\int_M e^{-pu}\chi_{tu}^{n-1}\wedge\omega dt \\ =\ & \frac1p\int_0^{\frac12}\int_0^t\frac{d}{ds}\Big(\int_M e^{-pu}\chi_{su}^{n-1}\wedge\omega\Big)dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \\ =\ & \frac{n-1}{p}\int_0^{\frac12}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \\ =\ & \frac{n-1}{p}\int_0^{\frac12}\int_0^t\int_M d(e^{-pu}\sqrt{-1}\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega)dsdt \\ & -\frac{n-1}{p}\int_0^{\frac12}\int_0^t\int_M\sqrt{-1}\partial e^{-pu}\wedge\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega dsdt \\ & +\frac{n-1}{p}\int_0^{\frac12}\int_0^t\int_M e^{-pu}\sqrt{-1}\bar\partial u\wedge\partial(\chi_{su}^{n-2}\wedge\omega)dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \end{aligned}$$

$$
\begin{aligned}
&= (n-1)\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega dsdt \\
&\quad -\frac{n-1}{p^2}\int_0^{\frac{1}{2}}\int_0^t\int_M \sqrt{-1}\bar\partial e^{-pu}\wedge\partial(\chi_{su}^{n-2}\wedge\omega)dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \\
&= (n-1)\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega dsdt \\
&\quad -\frac{n-1}{p^2}\int_0^{\frac{1}{2}}\int_0^t\int_M d(\sqrt{-1}e^{-pu}\partial(\chi_{su}^{n-2}\wedge\omega))dsdt \\
&\quad +\frac{n-1}{p^2}\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\bar\partial\partial(\chi_{su}^{n-2}\wedge\omega)dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \\
&= (n-1)\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega dsdt \\
&\quad -\frac{n-1}{p^2}\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial\bar\partial(\chi_{su}^{n-2}\wedge\omega)dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega, \qquad (4.22)
\end{aligned}
$$

where we used that as in the computation in (4.9),

$$
d(e^{-pu}\sqrt{-1}\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega)=(\partial+\bar\partial+A+\bar A)(e^{-pu}\sqrt{-1}\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega)=\partial(e^{-pu}\sqrt{-1}\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega),
$$

$$
d(\sqrt{-1}e^{-pu}\wedge\partial(\chi_{su}^{n-2}\wedge\omega))=(\partial+\bar\partial+A+\bar A)(\sqrt{-1}e^{-pu}\wedge\partial(\chi_{su}^{n-2}\wedge\omega))=\bar\partial(\sqrt{-1}e^{-pu}\wedge\partial(\chi_{su}^{n-2}\wedge\omega)),
$$

and

$$
\bar\partial\partial(\chi_{su}^{n-2}\wedge\omega)=-(\partial\bar\partial+A\bar A+\bar A A)(\chi_{su}^{n-2}\wedge\omega)=-\partial\bar\partial(\chi_{su}^{n-2}\wedge\omega).
$$

Applying (3.12), we estimate that as in (4.11)-(4.13) such as

$$
\begin{aligned}
&\int_M \sqrt{-1}e^{-pu}\chi_{su}^{n-3}\wedge s\sqrt{-1}\partial\bar\partial\partial\bar\partial u\wedge\omega \\
&\le C\int_M e^{-pu}\chi_{su}^{n-3}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega^2+C\int_M e^{-pu}\chi_{su}^{n-3}\wedge\omega^3, \qquad (4.23)
\end{aligned}
$$

$$
\begin{aligned}
&\int_M e^{-pu}\chi_{su}^{n-4}\wedge s\sqrt{-1}\partial\partial\bar\partial u\wedge\bar\partial\chi\wedge\omega \\
&\le C\int_M e^{-pu}\chi_{su}^{n-4}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega^3+C\int_M e^{-pu}\chi_{su}^{n-4}\wedge\omega^4, \qquad (4.24)
\end{aligned}
$$

$$
\begin{aligned}
&\int_M e^{-pu}\chi_{su}^{n-4}\wedge\partial\chi\wedge s\sqrt{-1}\bar\partial\partial\bar\partial u\wedge\omega^2 \\
&\le C\int_M e^{-pu}\chi_{su}^{n-4}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega^3+C\int_M e^{-pu}\chi_{su}^{n-4}\wedge\omega^4. \qquad (4.25)
\end{aligned}
$$

Then we estimate that by applying these estimates (4.23)-(4.25) and the inequalities (4.14)-(4.15),

$$
\begin{aligned}
&\frac{n-1}{p^2}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial\bar\partial(\chi_{su}^{n-2}\wedge\omega)ds \\
&= \frac{n-1}{p^2}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial((n-2)\chi_{su}^{n-3}\wedge(\bar\partial\chi+s\sqrt{-1}\bar\partial\partial\bar\partial u)\wedge\omega)ds \\
&= \frac{n-1}{p^2}\int_0^t\int_M e^{-pu}\sqrt{-1}\Big\{(n-2)(n-3)\chi_{su}^{n-4}\wedge(\partial\chi+s\sqrt{-1}\partial\partial\bar\partial u)\wedge(\bar\partial\chi+s\sqrt{-1}\bar\partial\partial\bar\partial u)\wedge\omega \\
&\quad +(n-2)\chi_{su}^{n-3}\wedge(\partial\bar\partial\chi+s\sqrt{-1}\partial\bar\partial\partial\bar\partial u)\wedge\omega+(n-2)\chi_{su}^{n-3}\wedge(\bar\partial\chi+s\sqrt{-1}\bar\partial\partial\bar\partial u)\wedge\partial\omega
\end{aligned}
$$

$$
\begin{aligned}
& +(n-2)\chi_{su}^{n-3}\wedge(\partial\chi+s\sqrt{-1}\partial\partial\bar\partial u)\wedge\bar\partial\omega+\chi_{su}^{n-2}\wedge\partial\bar\partial\omega\Big\}ds \\
\leq\ & \frac{C}{p^2}\int_0^t\int_M e^{-pu}\chi_{su}^{n-4}\wedge\omega^4 ds+\frac{C}{p^2}\int_0^t\int_M e^{-pu}\chi_{su}^{n-4}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega^3 ds \\
& +\frac{C}{p^2}\int_0^t\int_M e^{-pu}\chi_{su}^{n-3}\wedge\omega^3 ds+\frac{C}{p^2}\int_0^t\int_M e^{-pu}\chi_{su}^{n-3}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega^2 ds \\
& +\frac{C}{p^2}\int_0^t\int_M e^{-pu}\chi_{su}^{n-2}\wedge\omega^2 ds \\
\leq\ & \frac{C_1}{p^2}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega ds+\frac{C_2}{p^2}\int_0^t\int_M e^{-pu}\chi_{su}^{n-2}\wedge\omega^2 ds. \qquad (4.26)
\end{aligned}
$$

By choosing p sufficiently large such that $\frac{C_1}{p^2}<n-1$, $\frac{C_2}{p}<\lambda\cdot\frac{n-1}{n}$, by combining (4.22) with (4.26), and applying (4.15) for $t=\frac{1}{2}$, $k=n-1$ such that

$$
\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-2}\wedge\omega^2 dt\leq\frac{1}{\lambda}\cdot\frac{n}{n-1}\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-1}\wedge\omega dt,
$$

we obtain that for $0\leq t\leq\frac{1}{2}$,

$$
\begin{aligned}
& \frac{1}{p}\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-1}\wedge\omega dt \\
\leq\ & (n-1)\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega ds dt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \\
& +\frac{C_1}{p^2}\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega ds dt+\frac{C_2}{p^2}\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\chi_{su}^{n-2}\wedge\omega^2 ds dt \\
\leq\ & (n-1)\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{tu}^{n-2}\wedge\omega dt \\
& +\frac{1}{2p}\cdot\frac{\lambda(n-1)}{n}\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-2}\wedge\omega^2 dt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \\
\leq\ & (n-1)\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{tu}^{n-2}\wedge\omega dt \\
& +\frac{1}{2p}\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-1}\wedge\omega dt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \qquad (4.27)
\end{aligned}
$$

which implies that we have by applying (4.14) for $t=\frac{1}{2}$, $l=n$,

$$
\begin{aligned}
& \frac{1}{2p}\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-1}\wedge\omega dt \\
\leq\ & (n-1)\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{tu}^{n-2}\wedge\omega dt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \\
\leq\ & \frac{n}{\lambda}\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{tu}^{n-1}dt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega. \qquad (4.28)
\end{aligned}
$$

Therefore, by combining (4.28) with (4.20), (4.21), we obtain that

$$
\begin{aligned}
& \Big[np\Big\{1-\frac{C}{np^2}-\frac{1}{(1+\frac{\delta}{2})^\alpha}\Big\}-C\frac{2n}{\lambda}\Big]\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{tu}^{n-1}dt \\
\leq\ & C\int_M e^{-pu}\omega^n+\frac{C}{p}\int_M e^{-pu}\chi^{n-1}\wedge\omega\leq C\int_M e^{-pu}\omega^n. \qquad (4.29)
\end{aligned}
$$

We choose p sufficiently large such that

$$\Big[n\Big\{1-\frac{C}{np^2}-\frac{1}{(1+\frac{\delta}{2})^\alpha}\Big\}-C\frac{2n}{\lambda p}\Big]>0.$$

By applying (4.14) for $t=\frac{1}{2}$ repeatedly, we obtain

$$\begin{aligned}\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-1}dt &\geq \lambda\frac{n-1}{n}\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-2}\wedge\omega dt\\ &\geq \lambda^2\frac{n-1}{n}\frac{n-2}{n-1}\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-3}\wedge\omega^2 dt\\ &\cdots\\ &\geq \frac{\lambda^{n-1}}{n}\frac{1}{2}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\omega^{n-1}. \end{aligned}\tag{4.30}$$

By combining (4.29) with (4.30), we finally obtain that for sufficiently large p,

$$p\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\omega^{n-1}\leq C\int_M e^{-pu}\omega^n,$$

which tells us that there exists a sufficiently large p_0 such that for all $p\geq p_0$, the desired inequality (4.3) holds. □

The rest of the proof is similar to the ones in [7, 8]. In the following, we give a brief proof for reader's convenience. We introduce the definition of Gauduchon metrics on almost complex manifolds.

Definition 4.4. *Let (M^{2n},J) be an almost complex manifold. A metric g is called a Gauduchon metric on M if g is an almost Hermitian metric whose associated real $(1,1)$-form $\omega=\sqrt{-1}g_{i\bar{j}}\zeta^i\wedge\zeta^{\bar{j}}$ satisfies $d^*(Jd^*\omega)=0$, where d^* is the adjoint of d with respect to g, which is equivalent to $d(Jd(\omega^{n-1}))=0$, or $\partial\bar{\partial}(\omega^{n-1})=0$.*

One has the following well-known result.

Proposition 4.5 (cf. [2, Theorem 2.1], [3])**.** *Let (M^{2n},J,ω) be a compact almost Hermitian manifold with $n\geq 2$. Then there exists a smooth function σ, unique up to addition of a constant, such that the conformal almost Hermitian metric $e^\sigma\omega$ is Gauduchon.*

Thanks to Proposition 4.5, there exists a smooth function $\sigma: M\to\mathbb{R}$ with $\sup_M\sigma=0$ such that $\omega_G:=e^\sigma\omega$ is Gauduchon on M.

Lemma 4.6 (cf. [8, Lemma 2.3])**.** *Let M be a compact almost complex manifold of real dimension $2n$ $(n\geq 2)$ with a Gauduchon metric ω_G. If ϕ is a smooth nonnegative function on M with $\Delta_G\phi\geq -C_0$, where Δ_G is the Laplacian operator with respect to ω_G, then there exists a positive constant C_1, C_2 depending only on (M,ω_G) and C_0 such that*

$$\int_M|\partial\phi^{\frac{p+1}{2}}|^2_{\omega_G}\omega_G^n\leq C_1p\int_M\phi^p\omega_G^n\tag{4.31}$$

for any $p\geq 1$, and

$$\sup_M\phi\leq C_2\max\Big\{\int_M\phi\omega_G^n,1\Big\}.\tag{4.32}$$

Proof. We compute for $p \geq 1$, by Stokes' theorem,

$$\begin{aligned}
\int_M |\partial\phi^{\frac{p+1}{2}}|^2_{\omega_G}\omega_G^n &= n\int_M \sqrt{-1}\partial\phi^{\frac{p+1}{2}} \wedge \bar\partial\phi^{\frac{p+1}{2}} \wedge \omega_G^{n-1} \\
&= \frac{n(p+1)^2}{4}\int_M \sqrt{-1}\phi^{p-1}\partial\phi \wedge \bar\partial\phi \wedge \omega_G^{n-1} \\
&= \frac{n(p+1)^2}{4p}\int_M \sqrt{-1}\partial(\phi^p) \wedge \bar\partial\phi \wedge \omega_G^{n-1} \\
&= \frac{n(p+1)^2}{4p}\int_M \sqrt{-1}(\partial + \bar\partial + A + \bar A)(\phi^p\bar\partial\phi \wedge \omega_G^{n-1}) \\
&\quad -\frac{n(p+1)^2}{4p}\int_M \phi^p\sqrt{-1}\partial\bar\partial\phi \wedge \omega_G^{n-1} + \frac{n(p+1)}{4p}\int_M \sqrt{-1}\bar\partial(\phi^{p+1}) \wedge \partial\omega_G^{n-1} \\
&= -\frac{(p+1)^2}{4p}\int_M \phi^p n\frac{\sqrt{-1}\partial\bar\partial\phi \wedge \omega_G^{n-1}}{\omega_G^n}\omega_G^n \\
&\quad +\frac{n(p+1)}{4p}\int_M \sqrt{-1}(\partial + \bar\partial + A + \bar A)(\phi^{p+1}\partial\omega_G^{n-1}) \\
&\quad -\frac{n(p+1)}{4p}\int_M \phi^{p+1}\sqrt{-1}\bar\partial\partial\omega_G^{n-1} \\
&= \frac{(p+1)^2}{4p}\int_M \phi^p(-\Delta_G\phi)\omega_G^n \\
&\leq C_1 p\int_M \phi^p\omega_G^n, \qquad (4.33)
\end{aligned}$$

where we used that $(\bar\partial + A + \bar A)(\phi^p\bar\partial\phi \wedge \omega_G^{n-1}) = 0$, $(\partial + A + \bar A)(\phi^{p+1}\partial\omega_G^{n-1}) = 0$, and that

$$\bar\partial\partial\omega_G^{n-1} = -(\partial\bar\partial + A\bar A + \bar A A)\omega_G^{n-1} = -\partial\bar\partial\omega_G^{n-1} = 0$$

since we have $A\omega_G^{n-1} = \bar A\omega_G^{n-1} = 0$.

We apply the Sobolev inequality: for $\beta := \frac{n}{n-1} > 1$, and for any smooth function f,

$$\Big(\int_M f^{2\beta}\omega^n\Big)^{\frac{1}{\beta}} \leq C\Big(\int_M |\partial f|^2_g\omega^n + \int_M f^2\omega^n\Big). \qquad (4.34)$$

Taking $\omega = \omega_G$ and $f = \phi^{\frac{q}{2}}$, where we put $q := p+1$, then for $q \geq 2$, we have that

$$\Big(\int_M \phi^{q\beta}\omega_G^n\Big)^{\frac{1}{\beta}} \leq Cq\max\Big\{\int_M \phi^q\omega_G^n, 1\Big\}.$$

By repeatedly replacing q by $q\beta$ and iterating, after setting $q = 2$, then we obtain that

$$\sup_M \phi \leq C\max\Big\{\Big(\int_M \phi^2\omega_G^n\Big)^{\frac{1}{2}}, 1\Big\} \leq C\max\Big\{\Big(\sup_M \phi\Big)^{\frac{1}{2}}\Big(\int_M \phi\omega_G^n\Big)^{\frac{1}{2}}, 1\Big\},$$

which gives us the desired estimate (4.32). □

By applying the inequality (4.3) and the Sobolev inequality (4.34), for any $p \geq p_0$, we obtain that

$$\|e^{-u}\|_{L^{p\beta}} \leq C^{\frac{1}{p}}p^{\frac{1}{p}}\|e^{-u}\|_{L^p},$$

and by the standard iteration, we have that

$$e^{-p_0 \inf_M u} \leq C \int_M e^{-p_0 u} \omega^n. \tag{4.35}$$

We need the following lemma, whose proof goes in the same way as in the Hermitian case.

Lemma 4.7 (cf. [7, Lemma 3.2], [8, Lemma 2.2]). *Let f be a smooth function on a compact almost Hermitian manifold (M, J, ω). Write $d\mu := \frac{\omega^n}{\int_M \omega^n}$. If there exists a constant C_1 such that*

$$e^{-\inf_M f} \leq e^{C_1} \int_M e^{-f} d\mu, \tag{4.36}$$

then

$$|\{f \leq \inf_M f + C_1 + 1\}| \geq \frac{e^{-C_1}}{4}, \tag{4.37}$$

where $|\cdot|$ denotes the volume of the set with respect to $d\mu$.

We apply Lemma 4.6 to $f = p_0 u$, and then since we have the inequality (4.35), there exist uniform constants C, $\delta > 0$ such that

$$|\{u \leq \inf_M u + C\}| \geq \delta. \tag{4.38}$$

Now, we define $\phi := u - \inf_M u$. Since it satisfies that $\Delta_G \phi = e^{-\sigma} \Delta \phi > -C$, where Δ is the Laplacian operator with respect to ω, we may apply Lemma 4.3 to the function ϕ. From the Poincaré inequality and the estimate (4.31) with $p = 1$, we obtain that

$$\|\phi - \underline{\phi}\|_{L^2} \leq C \Big(\int_M |\partial \phi|^2_{\omega_G} \omega_G^n \Big)^{\frac{1}{2}} \leq C \|\phi\|^{\frac{1}{2}}_{L^1}, \tag{4.39}$$

where we put $\underline{\phi} := \frac{1}{\int_M \omega_G^n} \int_M \phi \omega_G^n$.

By making use of (4.38), the set $S := \{\phi \leq C\}$ satisfies that $|S|_G \geq \delta$, where $|\cdot|_G$ denotes the volume of a set with respect to ω_G^n. Therefore, we obtain that

$$\delta \underline{\phi} \leq \int_S \underline{\phi} \omega_G^n \leq \int_S (|\phi - \underline{\phi}| + C) \omega_G^n \leq \int_M |\phi - \underline{\phi}| \omega_G^n + C,$$

which gives that by applying (4.39),

$$\|\phi\|_{L^1} \leq C(\|\phi - \underline{\phi}\|_{L^1} + 1) \leq C(\|\phi - \underline{\phi}\|_{L^2} + 1) \leq C(\|\phi\|^{\frac{1}{2}}_{L^1} + 1).$$

Hence, ϕ is uniformly bounded in L^1, and from (4.32) and (1.2), we obtain a uniform bound of u in the L^∞ norm. □

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Perfect matchings in inhomogeneous random bipartite graphs in random environment

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ABSTRACT

In this note we study inhomogeneous random bipartite graphs in random environment. These graphs can be thought of as an extension of the classical Erdős-Rényi random bipartite graphs in a random environment. We show that the expected number of perfect matchings obeys a precise asymptotic.

RESUMEN

En esta nota estudiamos grafos aleatorios bipartitos inhomogéneos en un ambiente aleatorio. Estos grafos pueden ser pensados como una extensión de los grafos bipartitos aleatorios clásicos de Erdős-Rényi en un ambiente aleatorio. Mostramos que el número esperado de pareos obedece un comportamiento asintótico preciso.

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1 Introduction

In their seminal paper [7], Erdős and Rényi studied a certain type of *random graphs*, which in the case of bipartite graphs correspond to the following. Consider a bipartite graph with set of vertices given by $W = \{w_1, \ldots, w_n\}$ and $M = \{m_1, \ldots, m_n\}$. Let $p \in [0,1]$, Σ be a probability space and consider the independent random variables $X_{(ij)}$ defined on Σ with law

$$X_{(ij)}(x) = \begin{cases} 1 & \text{with probability } p; \\ 0 & \text{with probability } 1-p, \end{cases}$$

for $x \in \Sigma$. Denote by $G_n(x)$ the bipartite graph with vertex set $W \cup M$ and edges $E(x)$, where the edge (w_i, m_j) belongs to $E(x)$ if and only if $X_{(ij)}(x) = 1$. Let $\mathrm{pm}(G_n(x))$ be the number of *perfect matchings* of the graph $G_n(x)$ (see Sec. 3 for precise definitions). Erdős and Rényi [8, p. 460] observed that the mean of the number of perfect matchings was given by

$$\mathbb{E}(\mathrm{pm}(G_n(x))) = n!p^n. \tag{1.1}$$

This number has been also studied by Bollobás and McKay [5, Theorem 1] in the context of $k-$regular random graphs and by O'Neil [11, Theorem 1] for random graphs having a fixed (large enough) proportion of edges. We refer to the text by Bollobás [4] for further details on the subject of random graphs.

This paper is devoted to study certain sequences of inhomogeneous random bipartite graphs $G_{n,\omega}$ in a random environment $\omega \in \Omega$ (definitions are given in Sec. 2). Inhomogeneous random graphs have been intensively studied over the last years (see [6], where non-bipartite graphs are also considered). Our main result (see Theorem 3.2 for precise statement) is that there exists a constant $c \in (0,1)$ such that for almost every environment $\omega \in \Omega$ and for large $n \in \mathbb{N}$

$$\mathbb{E}_{n,\omega}(\mathrm{pm}(G_{n,\omega}(x))) \asymp n!c^n, \tag{1.2}$$

where the meaning of the asymptotic $\asymp$ will be explained later. Moreover, we have an explicit formula for the number c.

The result in equation (1.2) should be understood in the sense that the mean number of perfect matchings for inhomogeneous random bipartite graphs in a random environment is asymptotically the same as the one of Erdős-Rényi bipartite graphs in which $p = c$. Note that p is a constant that does not depend on n. The number c is the so-called *scaling mean* of a function related to the random graphs. Scaling means were introduced, in more a general setting, in [2] and are described in Sec. 3.

2 Inhomogeneous random bipartite graphs in random environment

Consider the following generalization of the Erdős-Rényi bipartite graphs. Let $W = \{w_1, \dots, w_n\}$ and $M = \{m_1, \dots, m_n\}$ be two disjoint sets of vertices. For every pair $1 \leq i, j \leq n$, let $a_{ij} \in [0,1]$ and consider the independent random variables $X_{(ij)}$, with law

$$X_{(ij)}(x) = \begin{cases} 1 & \text{with probability } a_{(ij)}; \\ 0 & \text{with probability } 1 - a_{(ij)}. \end{cases}$$

Denote by $G_n(x)$ the bipartite graph with vertices W, M and edges $E(x)$, where the edge (w_i, m_j) belongs to $E(x)$ if and only if $X_{(ij)}(x) = 1$. As it is clear from the definition all vertex of the graph do not play the same role. This contrasts with the (homogenous) Erdős-Renyi graphs (see [6] for details). We remark that in relation to the graphs we are considering it is possible to include the stochastic block model (see [10]) that is used, for example, in problems of community detection, in the context of machine learning. In this note we consider inhomogeneous random bipartite graphs in random environments, that is, the laws of $X_{(ij)}$ (and hence the numbers $a_{(ij)}$) are randomly chosen following an exterior environment law. This approach to stochastic processes has developed since the groundbreaking work by Solomon [12] on Random Walks in Random Environment and subsequent work of a large community (see [3] for a survey on the subject).

The model we propose is to consider the vertex sets W, M as the environment and to consider that the number $a_{(ij)}$, which is the probability that the edge connecting w_i with m_j occurs in the graph, is a random variable depending on w_i and m_j. We now describe precisely this model.

The space of environments is as follows. Fix $\alpha \in \mathbb{N}$ and a stochastic vector $(p_1, p_2, \dots, p_\alpha)$. Endow the set $\{1, \dots, \alpha\}$ with the probability measure P_W defined by $P_W(\{i\}) = p_i$. Denote by Ω_W the product space $\prod_{i=1}^{\infty}\{1, 2, \dots, \alpha\}$ and by μ_W the corresponding product measure. Let (Ω_M, μ_M) be the analogous probability measure space for the set $\{1, 2, \dots, \beta\}$ and the stochastic vector $(q_1, q_2, \dots, q_\beta)$. The *space of environments* is $\Omega = \Omega_W \times \Omega_M$ with the measure $\mu_\Omega = \mu_W \times \mu_M$ and an *environment* is an element $\omega \in \Omega$. Note that every environment defines two sequences

$$W(\omega) = (w_1, w_2, \dots) \in \Omega_W \quad \text{and} \quad M(\omega) = (m_1, m_2, \dots) \in \Omega_M.$$

For each environment $\omega \in \Omega$ we now define the edge distribution $X_{\omega,(ij)}$. Let $F = [f_{sr}]$ be a $\alpha \times \beta$ matrix with entries f_{sr} satisfying $0 \leq f_{sr} \leq 1$ and let $f : \{1, 2, \dots, \alpha\} \times \{1, 2, \dots, \beta\} \to [0,1]$ be the function defined by $f(w, m) = f_{wm}$. For each $\omega \in \Omega$ let

$$a_{(ij)}(\omega) := f\left(w_i(\omega), m_j(\omega)\right) = f_{w_i(\omega), m_j(\omega)}. \tag{2.1}$$

Given an environment $\omega \in \Omega$ the corresponding *edge distributions* are the random variables $X_{\omega,(ij)}$

with laws

$$X_{\omega,(ij)}(x) = \begin{cases} 1 & \text{with probability } a_{(ij)}(\omega); \\ 0 & \text{with probability } 1 - a_{(ij)}(\omega). \end{cases}$$

Given an environment $\omega \in \Omega$, we construct a sequence of random bipartite graphs $G_{n,\omega}$ considering the sets of vertices

$$W_{n,\omega} = (w_1(\omega), \ldots, w_n(\omega)) \quad \text{and} \quad M_{n,\omega} = (m_1(\omega), \ldots, m_n(\omega)),$$

and edge distributions $X_{\omega,(ij)}$ given by the values of $a_{(ij)}(\omega)$ as in (2.1). We denote by $\mathbb{P}_{n,\omega}$ the law of the random graph $G_{n,\omega}$.

Example 2.1. *Given a choice of an environment $\omega \in \Omega$, the probability that the bipartite graph $G_{n,\omega}(x)$ equals the complete bipartite graph $K_{n,n}$, using independence of the edge variables, is*

$$\mathbb{P}_{n,\omega}\left(G_{n,\omega}(x) = K_{n,n}\right) = \prod_{1 \le i,j \le n} \mathbb{P}_{n,\omega}(X_{\omega,(ij)} = 1) = \prod_{1 \le i,j \le n} a_{(ij)}(\omega).$$

3 Counting Perfect Matchings

Recall that a perfect matching of a graph G is a subset of edges containing every vertex exactly once. We denote by $\mathrm{pm}(G)$ the number of perfect matchings of G. When the graph G is bipartite, and the corresponding bipartition of the vertices has the form $W = \{w_1, w_2, \ldots, w_n\}$ and $M = \{m_1, m_2, \ldots, m_n\}$, a perfect matching can be identified with a bijection between W and M, and hence with a permutation $\sigma \in S_n$. From this, the total number of perfect matchings can be computed as

$$\mathrm{pm}(G) = \sum_{\sigma \in S_n} x_{1\sigma(1)} x_{2\sigma(2)} \cdots x_{n\sigma(n)}, \tag{3.1}$$

where x_{ij} are the entries of the incidence matrix X_G of G, that is $x_{ij} = 1$ if (w_i, m_j) is an edge of G and $x_{ij} = 0$ otherwise. Of course, the right hand side of (3.1) is the *permanent*, $\mathrm{per}(X_G)$, of the matrix X_G.

In the framework of Section 2, we estimate the number of perfect matchings for the sequence of inhomogeneous random bipartite graphs $G_{n,\omega}$, for a given environment $\omega \in \Omega$. More precisely, we obtain estimates for the growth of the mean of

$$\mathrm{pm}(G_{n,\omega}(x)) = \mathrm{per}(X_{G_{n,\omega}(x)}) = \sum_{\sigma \in S_n} X_{\omega,(1\sigma(1))} \cdots X_{\omega,(n\sigma(n))}. \tag{3.2}$$

Denote by $\mathbb{E}_{n,\omega}$ the expected value with respect to the probability $\mathbb{P}_{n,\omega}$. Since the edges are independent and $\mathbb{E}_{n,\omega}(X_{\omega,(ij)}) = a_{ij}(\omega)$ we have

$$\begin{aligned}\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right) &= \mathbb{E}_{n,\omega}\left(\sum_{\sigma\in S_n} X_{\omega,(1\sigma(1))}\cdots X_{\omega,(n\sigma(n))}\right)\\ &= \sum_{\sigma\in S_n} a_{(1\sigma(1))}(\omega)\cdots a_{(n\sigma(n))}(\omega)\\ &= \mathrm{per}(A_n(\omega)),\end{aligned}$$

where the entries of the matrix are $(A_n(\omega))_{ij} = a_{(ij)}(\omega)$. The main result of this note describes the growth of this expected number for perfect matchings.

The following number is a particular case of a quantity introduced by the authors in a more general setting in [2].

Definition 3.1. *Let F be an $\alpha\times\beta$ matrix with non-negative entries (f_{rs}). Let $\vec{p} = (p_1,\dots,p_\alpha)$ and $\vec{q} = (q_1,\dots,q_\beta)$ be two stochastic vectors. The* scaling mean *of F with respect to $\vec{p}$ and $\vec{q}$ is defined by*

$$\mathrm{sm}_{\vec{p},\vec{q}}(F) := \inf_{(x_r)\in\mathbb{R}_+^\alpha,(y_s)\in\mathbb{R}_+^\beta}\left(\prod_{r=1}^{\alpha} x_r^{-p_r}\right)\left(\prod_{s=1}^{\beta} y_s^{-q_s}\right)\left(\sum_{r=1}^{\alpha}\sum_{s=1}^{\beta} x_r f_{rs} y_s p_r q_s\right).$$

The scaling mean is increasing with respect to the entries of the matrix and lies between the minimum and the maximum of the entries (see [2] for details and more properties). We stress that the scaling mean can be exponentially approximated using a simple iterative process (see Section 5).

The main result in this note is the following,

Theorem 3.2 (Main Theorem)**.** *Let $(G_{n,\omega})_{n\geq 1}$ be a sequence of random bipartite graphs on a random environment $\omega\in\Omega$. If for every pair (r,s) we have $f_{rs}>0$ then the following pointwise convergence holds*

$$\lim_{n\to\infty}\left(\frac{\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right)}{n!}\right)^{1/n} = \mathrm{sm}_{\vec{p},\vec{q}}(F), \tag{3.3}$$

for $\mu_W\times\mu_M$-almost every environment $\omega\in\Omega$.

Remark 3.3. *As discussed in the introduction Theorem 3.2 shows that there exists a constant $c\in(0,1)$, such that for almost every environment $\omega\in\Omega$ and for $n\in\mathbb{N}$ sufficiently large*

$$\mathbb{E}_{n,\omega}(\mathrm{pm}(G_{n,\omega}(x))) \asymp n!c^n.$$

Namely $c = \mathrm{sm}_{\vec{p},\vec{q}}(F)$. This result should be compared with the corresponding one obtained by Erdős and Rényi for their class of random graphs, that is

$$\mathbb{E}(\mathrm{pm}(G_n(x))) = n!p^n.$$

Thus, we have shown that for large values of n the growth of the number of perfect matchings for random graphs in a random environment behaves like the simpler model studied by Erdős and Rényi with $p = \mathrm{sm}_{\vec{p},\vec{q}}(F)$.

Remark 3.4. *Theorem 3.2 shows that the expected number of perfect matchings is a quenched variable, in the sense of that it does not depend on the environment ω (see for instance [P]).*

Remark 3.5. *Using the Stirling formula, the limit in (3.3) can be stated as*

$$\lim_{n\to\infty}\left(\frac{1}{n}\log\left(\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right)\right)-\log n\right)=\log \mathrm{sm}_{\vec{p},\vec{q}}(F)-1,$$

which gives a quenched result for the growth of the perfect matching entropy for the sequence of graphs $G_{\omega,n}$ (see [1]).

Remark 3.6. *Note that we assume a uniform ellipticity condition on the values of the probabilities $a_{(ij)}$ as in (2.1). A similar assumption appears in the setting of Random Walks in Random Environment (see [3, p. 355]).*

We now present some concrete examples.

Example 3.7. *Let $\alpha = \beta = 2$ and $p_1 = p_2 = q_1 = q_2 = 1/2$. Therefore, the space of environments is the direct product of two copies of the full shift on two symbols endowed with the $(1/2, 1/2)-$Bernoulli measure. The edge distribution matrix F is a 2×2 matrix with entries belonging to $(0, 1)$. In [2, Example 2.11], it was shown that*

$$\mathrm{sm}_{\vec{p},\vec{q}}\begin{pmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{pmatrix}=\frac{\sqrt{f_{11}f_{22}}+\sqrt{f_{12}f_{21}}}{2}.$$

Therefore, Theorem 3.2 implies that

$$\lim_{n\to\infty}\left(\frac{\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right)}{n!}\right)^{1/n}=\frac{\sqrt{f_{11}f_{22}}+\sqrt{f_{12}f_{21}}}{2},$$

for almost every environment $\omega \in \Omega$.

Example 3.8. *More generally let $\alpha \in \mathbb{N}$ with $\alpha \geq 2$ and $\beta = 2$. Consider the two stochastic vectors $\vec{p} = (p_1, p_2, \ldots, p_\alpha)$ and $\vec{q} = (q_1, q_2)$. The space of environments is the direct product of a full shift on α symbols endowed with the $\vec{p}$-Bernoulli measure with a full shift on two symbols endowed with the $\vec{q}$-Bernoulli measure. The edge distribution matrix F is a $\alpha \times 2$ matrix with entries $f_{r1}, f_{r2} \in (0,\infty)$, where $r \in \{1, \ldots, \alpha\}$. Denote by $\chi \in \mathbb{R}^+$ the unique positive solution of the equation*

$$\sum_{r=1}^{\alpha}\frac{p_r f_{r1}}{f_{r1}+f_{r2}\chi}=q_1.$$

Then

$$\mathrm{sm}_{\vec{p},\vec{q}}(F)=\mathrm{sm}_{\vec{p},\vec{q}}\begin{pmatrix} f_{11} & f_{12}\\ \vdots & \vdots\\ f_{\alpha 1} & f_{\alpha 2}\end{pmatrix}=q_1^{q_1}\left(\frac{q_2}{\chi}\right)^{q_2}\prod_{r=1}^{\alpha}\left(f_{r1}+f_{r2}\chi\right)^{p_r}.$$

Therefore, Theorem 3.2 implies that

$$\lim_{n\to\infty}\left(\frac{\mathbb{E}_{n,\omega}\left(\mathrm{pm}(\mathrm{G}_{\mathrm{n},\omega})\right)}{n!}\right)^{1/n} = q_1^{q_1}\left(\frac{q_2}{\chi}\right)^{q_2}\prod_{r=1}^{\alpha}\left(f_{r1}+f_{r2}\chi\right)^{p_r},$$

for almost every environment $\omega \in \Omega$. The quantity in the right hand side first appeared in the work by Halász and Székely in 1976 *[9], in their study of symmetric means. In [2, Theorem 5.1] using a completely different approach we recover their result.*

4 Proof of the Theorem

The *shift map* $\sigma_W : \Omega_W \to \Omega_W$ is defined by

$$\sigma_W(w_1, w_2, w_3, \dots) = (w_2, w_3, \dots).$$

The shift map σ_W is a μ_W-preserving, that is, $\mu_W(\Lambda) = \mu_W(\sigma_W^{-1}(\Lambda))$ for every measurable set $\Lambda \subset \Omega_W$, and it is ergodic, that is, if $\Lambda = \sigma_W^{-1}(\Lambda)$ then $\mu_W(\Lambda)$ equals 1 or 0. Analogously for σ_M and μ_M. We define a function $\Phi : \Omega_W \times \Omega_M \to \mathbb{R}$ by

$$\Phi(\vec{w}, \vec{m}) = f_{w_1 m_1}.$$

Thus

$$\Phi(\sigma_W^{i-1}(\vec{w}), \sigma_M^{j-1}(\vec{m})) = f_{w_i m_j} = a_{(ij)}(\omega).$$

That is, the matrix $A_n(\omega)$ has entries $a_{(ij)}(\omega) = \Phi(\sigma_W^{i-1}(\vec{w}), \sigma_M^{j-1}(\vec{m}))$. We are in the exact setting in order to apply the Law of Large Permanents see [2, Theorem 4.1].

Theorem (Law of Large Permanents). Let (X, μ), (Y, ν) be Lebesgue probability spaces, let $T\colon X \to X$ and $S\colon Y \to Y$ be ergodic measure preserving transformations, and let $g\colon X \times Y \to \mathbb{R}$ be a positive measurable function essentially bounded away from zero and infinity. Then for $\mu \times \nu$-almost every $(x, y) \in X \times Y$, the $n \times n$ matrix

$$M_n(x,y) = \begin{pmatrix} g(x,y) & g(Tx,y) & \cdots & g(T^{n-1}x,y) \\ g(x,Sy) & g(Tx,Sy) & \cdots & g(T^{n-1}x,Sy) \\ \vdots & \vdots & & \vdots \\ g(x,S^{n-1}y) & g(Tx,S^{n-1}y) & \cdots & g(T^{n-1}x,S^{n-1}y) \end{pmatrix}$$

verifies

$$\lim_{n\to\infty}\left(\frac{\mathrm{per}\left(M_n(x,y)\right)}{n!}\right)^{1/n} = \mathrm{sm}_{\mu,\nu}(g)$$

pointwise, where $\mathrm{sm}_{\mu,\nu}(g)$ is the scaling mean of g defined as

$$\mathrm{sm}_{\mu,\nu}(g) = \inf_{\phi,\psi}\frac{\iint_{X\times Y}\phi(x)g(x,y)\psi(y)d\mu d\nu}{\exp\left(\int_X \log\phi(x)d\mu\right)\exp\left(\int_Y \log\psi(y)d\nu\right)},$$

where the functions ϕ and ψ are assumed to be measurable, positive and such that their logarithms are integrable.

We apply this Law of Large Permanents setting $X = \Omega_W, Y = \Omega_M$, $T = \sigma_W, S = \sigma_M$, $g = \Phi$ and recalling that $f_{rs} > 0$. We have

$$\mathrm{sm}_{\mu_W,\mu_M}(\Phi) = \mathrm{sm}_{\vec{p},\vec{q}}(F)$$

as a consequence of an alternative characterization of the scaling mean given in (see [2, Proposition 3.5]). This concludes the proof of the Main Theorem. ■

Remark 4.1. *We have chosen to present our result in the simplest possible setting. That is, the environment space being products of full-shifts endowed with Bernoulli measures. Using the general form of the Law of Large Permanent above our results can be extended for inhomogeneous random graphs in more general random environments.*

5 An algorithm to compute the scaling mean

The purpose of this section is to show that the scaling mean is the unique fixed point of a contraction. Therefore it can be computed, or approximated, using a suitable iterative process. It should be stressed that, on the other hand, it has been shown that no such algorithm exists to compute the permanent.

Denote by $\mathcal{B}^\alpha \subset \mathbb{R}^\alpha$ and by $\mathcal{B}^\beta \subset \mathbb{R}^\beta$ the positive cones. Define the following maps forming a (non-commutative) diagram:

$$\begin{array}{ccc} \mathcal{B}^\alpha & \xrightarrow{\mathtt{I}_1} & \mathcal{B}^\alpha \\ {\scriptstyle \mathtt{K}_1}\uparrow & & \downarrow{\scriptstyle \mathtt{K}_2} \\ \mathcal{B}^\beta & \xleftarrow[\mathtt{I}_2]{} & \mathcal{B}^\beta \end{array}$$

by the formulas:

$$(\mathtt{I}_1(\vec{x}))_i := \frac{1}{x_i}, \quad (\mathtt{I}_2(\vec{y}))_i := \frac{1}{y_i},$$

$$(\mathtt{K}_1(\vec{x}))_j := \sum_{i=1}^{\beta} f_{ij} x_i p_i, \quad (\mathtt{K}_2(\vec{y}))_j := \sum_{j=1}^{\alpha} f_{ij} y_j q_j.$$

Let $\mathtt{T} : \mathcal{B}^\alpha \mapsto \mathcal{B}^\alpha$ be the map defined by $\mathtt{T} := \mathtt{K}_1 \circ \mathtt{I}_2 \circ \mathtt{K}_2 \circ \mathtt{I}_1$. The map $\mathtt{T}$ is a contraction for a suitable Hilbert metric. Indeed, for $\vec{x}, \vec{z} \in \mathcal{B}^\alpha$ define the following (pseudo)-metric

$$d(\vec{x}, \vec{z}) := \log\left(\frac{\max_i x_i/z_i}{\min_i x_i/z_i}\right).$$

It was proven in [2, Lemma 3.4]

Lemma 5.1. *We have that*

$$d(T(\vec{x}), T(\vec{z})) \leq \left(\tanh\frac{\delta}{4}\right)^2 d(\vec{x}, \vec{z}),$$

where

$$\delta \leq 2\log\left(\frac{\max_{i,j} f_{ij}}{\min_{ij} f_{ij}}\right) < \infty.$$

The following results was proved in [2, Lemma 3.3]

Lemma 5.2. *The map* $\mathtt{T}$ *has a unique (up to positive scaling) fixed point* $\vec{x}_{\mathtt{T}} \in \mathcal{B}^{\alpha}$. *Moreover, defining* $\vec{y}_{\mathtt{T}} := \mathtt{K}_2 \circ \mathtt{I}_1(\vec{x}_{\mathtt{T}})$ *one has that*

$$\mathrm{sm}(f) = \prod_{i=1}^{\alpha} x_i^{p_i} \prod_{j=1}^{\beta} y_j^{q_j}.$$

Therefore, it possible to find good approximations of the scaling mean using an iterative process.

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On existence results for hybrid $\psi-$Caputo multi-fractional differential equations with hybrid conditions

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ABSTRACT

In this paper, we study the existence and uniqueness results of a fractional hybrid boundary value problem with multiple fractional derivatives of $\psi-$Caputo with different orders. Using a useful generalization of Krasnoselskii's fixed point theorem, we have established results of at least one solution, while the uniqueness of solution is derived by Banach's fixed point. The last section is devoted to an example that illustrates the applicability of our results.

RESUMEN

En este artículo, estudiamos los resultados de existencia y unicidad de un problema de valor en la frontera fraccional híbrido con múltiples derivadas fraccionarias de $\psi-$Caputo con diferentes órdenes. Usando una generalización útil del teorema del punto fijo de Krasnoselskii, establecemos resultados de al menos una solución, mientras que la unicidad de dicha solución se obtiene a partir del punto fijo de Banach. La última sección está dedicada a un ejemplo que ilustra la aplicabilidad de nuestros resultados.

Keywords and Phrases: $\psi-$fractional derivative, fractional differential equation, hybrid conditions, fixed point, existence, uniqueness.

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1 Introduction

Fractional differential equations have received great attention of many researchers working in different disciplines of science and technology, especially, since they have found that certain thermal [3], electrochemical [4] and viscoelastic [16] systems are governed by fractional differential equations. Recently some publications show the importance of fractional differential equations in the mathematical modeling of many real-world phenomena. For example ecological models [10], economic models [20], physics [12], fluid mechanics [21]. There are many studies on fractional differential equations with distinct kinds of fractional derivatives in the literature, such as Riemann-Liouville fractional derivative, Caputo fractional derivative, and Grunwald Letnikov fractional derivative, etc. For example, see [11, 14, 15]. Very recently, a new kind of fractional derivative the $\psi-$Caputo's derivative, was introduced by Almeida in [1], the main advantage of this derivative is the freedom of choices of the kernels of the derivative by choosing different functions ψ, which gives us some well known fractional derivatives such Caputo, Caputo-Erdelyi-Koper and Caputo Hadamard derivative. For more details on the $\psi-$Caputo and fractional differential equation involving $\psi-$Caputo, we refer the reader to a series of papers [1, 2, 7] and the references cited therein.

Nowadays, many researchers have shown the interest of quadratic perturbations of nonlinear differential equations, these kind of differential equations are known under the name of hybrid differential equations. Some recent works regarding hybrid differential equations can be found in [8, 13, 17, 23] and the references cited therein. Dhage and Lakshmikantham [6] discussed the existence and uniqueness theorems of the solution to the ordinary first-order hybrid differential equation with perturbation of the first type

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{u(t)}{g(t,u(t))}\right) = f(t,u(t)), \quad \text{a.e.} \quad t \in [t_0, t_0+T], \\ \\ u(t_0) = u_0, \quad u_0 \in \mathbb{R}, \end{cases}$$

where $t_0, T \in \mathbb{R}$ with $T > 0$, $g : [t_0, t_0+T] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $f : [t_0, t_0+T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. By using the fixed point theorem in Banach algebra, the authors obtained the existence results.

In [9], Dong *et al.*, established the existence and the uniqueness of solutions for the following implicit fractional differential equation

$$\begin{cases} {}^cD^p u(t) = f(t, u(t), {}^cD^p u(t)), \quad t \in J := [0,T], \quad 0 < p \leq 1, \\ \\ u(0) = u_0, \end{cases}$$

where ${}^cD^p$ is the Caputo fractional derivative, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Sitho *et al.* [17] studied existence results for the initial value problems of hybrid fractional sequential integro-differential equations:

$$\begin{cases} D^p\left[\dfrac{D^q u(t)-\displaystyle\sum_{i=1}^{m} I^{\eta_i} g_i(t,u(t))}{h(t,x(t))}\right]=f(t,u(t),I^{\gamma}x(t)), \quad t\in J,\\ \\ u(0)=0, \qquad D^q u(0)=0, \end{cases}$$

where D^p, D^q denotes the Riemann-Liouville fractional derivative of order p, q respectively and $0<p,q\leq 1$, I^{η_i} is the Riemann-Liouville fractional integral of order $\eta_i>0$, $h\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $f\in C(J\times\mathbb{R}^2,\mathbb{R})$ and $g_i\in C(J\times\mathbb{R},\mathbb{R})$ with $g_i(0,0)=0$, $i=1,\ldots,m$.

In 2019, Derbazi *et al.* [8] proved the existence of solutions for the fractional hybrid boundary value problem

$$^cD^p\left[\frac{u(t)-g(t,u(t))}{h(t,u(t))}\right]=f(t,u(t)), \quad t\in J,$$

with the fractional hybrid boundary value conditions

$$\begin{cases} a_1\left[\dfrac{u(t)-g(t,u(t))}{h(t,u(t))}\right]_{t=0}+b_1\left[\dfrac{u(t)-g(t,u(t))}{h(t,u(t))}\right]_{t=T}=\upsilon_1,\\ \\ a_2\,{}^cD^{\delta}\left[\dfrac{u(t)-g(t,u(t))}{h(t,u(t))}\right]_{t=\xi}+b_2\,{}^cD^{\delta}\left[\dfrac{u(t)-g(t,u(t))}{h(t,u(t))}\right]_{t=T}=\upsilon_2\ ,\ \xi\in J, \end{cases}$$

where $1<p\leq 2$, $0<\delta\leq 1$, $\xi\in J$ and $a_1,a_2,b_1,b_2,\upsilon_1,\upsilon_2$ are real constants. Moreover, two fractional derivatives of Caputo type appeared in the above problem.

Motivated by these works, we mainly investigate the existence and uniqueness of solutions for a class of hybrid differential equations of arbitrary fractional order of the form

$$^cD^{p;\psi}\left[\frac{^cD^{q;\psi}u(t)-\displaystyle\sum_{i=1}^{m} I^{\eta_i;\psi} g_i(t,u(t))}{h(t,u(t))}\right]= f\left(t,u(t),{}^cD^{p;\psi}\left[\frac{^cD^{q;\psi}u(t)-\displaystyle\sum_{i=1}^{m} I^{\eta_i;\psi} g_i(t,u(t))}{h(t,u(t))}\right]\right), \quad t\in J, \tag{1.1}$$

endowed with the hybrid fractional integral boundary conditions

$$\begin{cases} u(0) = 0, \quad {}^cD^{q;\psi}u(0) = 0, \\ a_1 \left[\dfrac{{}^cD^{q;\psi}u(t) - \sum_{i=1}^{m} I^{\eta_i;\psi} g_i(t, u(t))}{h(t, u(t))} \right]_{t=0} + b_1 \left[\dfrac{{}^cD^{q;\psi}u(t) - \sum_{i=1}^{m} I^{\eta_i;\psi} g_i(t, u(t))}{h(t, u(t))} \right]_{t=T} = \upsilon_1, \\ a_2\, {}^cD^{\delta;\psi} \left[\dfrac{{}^cD^{q;\psi}u(t) - \sum_{i=1}^{m} I^{\eta_i;\psi} g_i(t, u(t))}{h(t, u(t))} \right]_{t=\xi} + \\ \qquad b_2\, {}^cD^{\delta;\psi} \left[\dfrac{{}^cD^{q;\psi}u(t) - \sum_{i=1}^{m} I^{\eta_i;\psi} g_i(t, u(t))}{h(t, u(t))} \right]_{t=T} = \upsilon_2\ , \quad \xi \in J, \end{cases} \tag{1.2}$$

where $J := [0, T]$, $D^{p;\psi}$, $D^{q;\psi}$ and $D^{\delta;\psi}$ denote the $\psi-$Caputo fractional derivative of order $2 < p \leq 3$ and $0 < q, \delta \leq 1$ respectively, $I^{\eta_i;\psi}$ is the $\psi-$Riemann-Liouville fractional integral of order $\eta_i > 0$, $h \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ and $g_i \in C(J \times \mathbb{R}, \mathbb{R})$ with $g_i(0, 0) = 0$, $i = 1, \ldots, m$, a_1, a_2, b_1, b_2, υ_1, υ_2 are real constants such that $b_1 \neq 0$ and

$$2\big(a_2\Psi_0^{2-\delta}(\xi) + b_2\Psi_0^{2-\delta}(T)\big) - \Psi_0^1(T)(2 - \delta)\big(a_2\Psi_0^{1-\delta}(\xi) + b_2\Psi_0^{1-\delta}(T)\big) \neq 0.$$

The rest of the paper is arranged as follows. Section 2 gives some background material needed in this paper, such as fractional differential equations and fixed point theorems. Section 3 treats the main results concerning the existence and uniqueness results of the solution for the given problem (1.1)-(1.2) by employing hybrid fixed point theorem for a sum of two operators in Banach algebra space and Banach's fixed point. In the last section, an example is presented to illustrate our results.

2 Preliminaries

In this section, we introduce some preliminaries and lemmas that will be used throughout this paper. We will prove an auxiliary lemma, which plays a key role in defining a fixed point problem associated with the given problem.

Let $\psi : J \to \mathbb{R}$ an increasing function satisfying $\psi'(t) \neq 0$ for all $t \in J$. For the sake of simplicity, we set $\Psi_0^r(t) = (\psi(t) - \psi(0))^r$.

Definition 2.1 ([2])**.** *The $\psi-$Riemann-Liouville fractional integral of order $(p > 0)$ of an integrable* *function* $g : [0, \infty) \to \mathbb{R}$ *is defined by*

$$I^{p;\psi} g(t) = \frac{1}{\Gamma(p)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{p-1} g(s) ds, \qquad 0 < s < t.$$

Definition 2.2 ([2]). *The* $\psi-$*Caputo fractional derivative of order* p $(n - 1 < p < n \in \mathbb{N})$ *of a function* $g \in C^n[0, \infty)$ *is defined by*

$${}^cD^{p;\psi} g(t) = \frac{1}{\Gamma(p - n)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{p-n-1} D^n_\psi \; g(s) ds, \qquad 0 < s < t,$$

where $n = [p] + 1$ *and* $D^n_\psi = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n$. *In case, if* $2 < p \le 3$, *we have*

$${}^cD^{p;\psi} g(t) = \frac{1}{\Gamma(p - 3)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{p-4} D^3_\psi g(s) ds, \qquad 0 < s < t.$$

Lemma 2.3 ([2]). *Let* $p > 0$. *The following hold*

- *If* $g \in C(J, \mathbb{R})$, *then*

$${}^cD^{p;\psi} I^{p;\psi} g(t) = g(t), \quad t \in J.$$

- *If* $g \in C^n(J, \mathbb{R})$, $n - 1 < p < n$, *then*

$$I^{p;\psi\, c} D^{p;\psi} g(t) = g(t) - \sum_{k=0}^{n-1} c_k \Psi^k_0(t), \quad t \in J,$$

where $c_k = \frac{D^k_\psi g(0)}{k!}$.

Lemma 2.4. *Let* $2 < p < 3$, $0 < q < 1$. *For any functions* $F \in C(J, \mathbb{R})$, $H \in C(J, \mathbb{R} \setminus \{0\})$ *and* $G_i \in C(J, \mathbb{R})$ *with* $G_i(0) = 0$, $i = 1, \ldots, m$, *the following linear fractional boundary value problem*

$$D^{p;\psi} \left[\frac{{}^cD^{q;\psi} u(t) - \sum_{i=1}^m I^{\eta_i;\psi} G_i(t)}{H(t)} \right] = F(t), \quad 2 < p \le 3, \quad 0 < q \le 1, \quad t \in J, \tag{2.1}$$

supplemented with the following conditions

$$
\begin{cases}
u(0) = 0, \quad {}^cD^{q;\psi}u(0) = 0, \\
a_1 \left[\dfrac{{}^cD^{q;\psi}u(t) - \sum_{i=1}^{m} I^{\eta_i;\psi}G_i(t)}{H(t)} \right]_{t=0} + b_1 \left[\dfrac{{}^cD^{q;\psi}u(t) - \sum_{i=1}^{m} I^{\eta_i;\psi}G_i(t)}{H(t)} \right]_{t=T} = \upsilon_1, \\
a_2\, {}^cD^{\delta;\psi} \left[\dfrac{{}^cD^{q;\psi}u(t) - \sum_{i=1}^{m} I^{\eta_i;\psi}G_i(t)}{H(t)} \right]_{t=\xi} + \\
\qquad b_2\, {}^cD^{\delta;\psi} \left[\dfrac{{}^cD^{q;\psi}u(t) - \sum_{i=1}^{m} I^{\eta_i;\psi}G_i(t)}{H(t)} \right]_{t=T} = \upsilon_2, \quad \xi \in J,
\end{cases} \tag{2.2}
$$

has a unique solution, which is given by

$$
\begin{aligned}
u(t) &= I^{q;\psi}\Big(H(s)I^{p;\psi}F(s)\Big)(t) + \sum_{i=1}^{m} I^{\eta_i+q;\psi}G_i(s)(t) \\
&+ I^{q;\psi}\left(H(s)\Big(\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\Big)\left(\frac{\upsilon_1}{b_1} - I^{p;\psi}F(s)\right)\right)(t) \\
&+ \Omega_1\Big(\upsilon_2 - a_2 I^{p-\delta;\psi}F(\xi) - b_2 I^{p-\delta;\psi}F(T)\Big) I^{q;\psi}\left(H(s)\Big(\Psi_0^2(s) - \Psi_0^1(T)\Psi_0^1(s)\Big)\right)(t),
\end{aligned} \tag{2.3}
$$

where

$$
\Omega_1 = \frac{\Gamma(3-\delta)}{2\big(a_2\Psi_0^{2-\delta}(\xi) + b_2\Psi_0^{2-\delta}(T)\big) - \Psi_0^1(T)(2-\delta)\big(a_2\Psi_0^{1-\delta}(\xi) + b_2\Psi_0^{1-\delta}(T)\big)},
$$

$$
\Omega_2 = \frac{a_2\Psi_0^{1-\delta}(\xi) + b_2\Psi_0^{1-\delta}(T)}{\Gamma(2-\delta)\Omega_1}, \quad \Omega_3 = 1 + \Omega_2\Psi_0^1(T).
$$

Proof. Applying the $\psi-$Caputo fractional integral of order p to both sides of equation in (2.1) and using Lemma 2.3, we get

$$
\frac{{}^cD^{q;\psi}u(t) - \sum_{i=1}^{m} I^{\eta_i;\psi}G_i(t)}{H(t)} = I^{p;\psi}F(t) + c_0 + c_1\Psi_0^1(t) + c_2\Psi_0^2(t), \tag{2.4}
$$

where $c_0, c_1, c_2 \in \mathbb{R}$.

Next, applying the $\psi-$Caputo fractional integral of order q to both sides (2.4), we get

$$
\begin{aligned}
u(t) = {} & I^{q;\psi}\Big(H(s)I^{p;\psi}F(s)\Big)(t) + \sum_{i=1}^{m} I^{\eta_i+q;\psi}G_i(s)(t) \\
& + I^{q;\psi}\Big(H(s)\Big(c_0 + c_1\Psi_0^1(s) + c_2\Psi_0^2(s)\Big)\Big)(t) + c_3, \qquad c_3 \in \mathbb{R}.
\end{aligned} \tag{2.5}
$$

With the help of conditions $u(0) = 0$ and ${}^cD^{q;\psi}u(0) = 0$, we find, $c_3 = 0$ and $c_0 = 0$ respectively.

Applying the boundary conditions (2.2), and from (2.4), we obtain

$$c_1\Psi_0^1(T) + c_2\Psi_0^2(T) = \frac{\upsilon_1}{b_1} - I^{p;\psi}F(T),$$

and

$$
\begin{aligned}
\frac{c_1}{\Gamma(2-\delta)}\Big(a_2\Psi_0^{1-\delta}(\xi) + b_2\Psi_0^{1-\delta}(T)\Big) &+ \frac{2c_2}{\Gamma(3-\delta)}\Big(a_2\Psi_0^{2-\delta}(\xi) + b_2\Psi_0^{2-\delta}(T)\Big) \\
&= \upsilon_2 - a_2 I^{p-\delta;\psi}F(\xi) - b_2 I^{p-\delta;\psi}F(T).
\end{aligned}
$$

Solving the resulting equations for c_1 and c_2, we find that

$$c_1 = \Big(\frac{\upsilon_1}{b_1} - I^{p;\psi}F(T)\Big)\Omega_3 - \Big(\upsilon_2 - a_2 I^{p-\delta;\psi}F(\xi) - b_2 I^{p-\delta;\psi}F(T)\Big)\Omega_1\Psi_0^1(T),$$

$$c_2 = \Big(\upsilon_2 - a_2 I^{p-\delta;\psi}F(\xi) - b_2 I^{p-\delta;\psi}F(T)\Big)\Omega_1 - \Big(\frac{\upsilon_1}{b_1} - I^{p;\psi}F(T)\Big)\Omega_2.$$

Inserting c_1 and c_2 in (2.5), which leads to the solution system (2.3). □

Let $E = C(J, \mathbb{R})$ be the Banach space of continuous real-valued functions defined on J. We define in E a norm $\|\cdot\|$ by

$$\|u\| = \sup_{t\in J} |u(t)|,$$

and a multiplication by

$$(uv)(t) = u(t)v(t), \quad \forall t \in J.$$

Clearly E is a Banach algebra with above defined supremum norm and multiplication.

Lemma 2.5 ([5]). *Let S be a nonempty, convex, closed, and bounded set such that $S \subseteq E$, and let $A : E \to E$ and $B : S \to E$ be two operators which satisfy the following:*

(1) A is contraction,

(2) B is completely continuous, and

(3) $u = Au + Bv$, for all $v \in S \Rightarrow u \in S$.

Then the operator equation $u = Au + Bu$ has at least one solution in S.

Theorem 2.6 ([18]). *Let S be a non-empty closed convex subset of a Banach space E, then any contraction mapping A of S into itself has a unique fixed point.*

3 Main result

In this section, we derive conditions for the existence and uniqueness of a solution for the problem (1.1)-(1.2).

The following assumptions are necessary in obtaining the main results.

(H1) The functions $h \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ are continuous, and there exist bounded functions $L, M : J \to [0, \infty)$, such that

$$|h(t, u(t)) - h(t, v(t))| \leq L(t)|u(t) - v(t)|,$$

and

$$|f(t, u(t), v(t)) - f(t, \overline{u}(t), \overline{v}(t))| \leq M(t)\big(|u(t) - \overline{u}(t)| + |v(t) - \overline{v}(t)|\big),$$

for $t \in J$ and $u, v, \overline{u}, \overline{v} \in \mathbb{R}$.

(H2) There exist functions $\vartheta, \chi, \varphi_i \in C(J, [0, \infty))$ such that

$$|f(t, u(t), v(t))| \leq \vartheta(t) \quad \text{for each } t, u \in J \times \mathbb{R},$$

$$|h(t, u(t))| \leq \chi(t) \quad \text{for each } t, u \in J \times \mathbb{R},$$

$$|g_i(t, u(t))| \leq \varphi_i(t) \quad \text{for each } t, u \in J \times \mathbb{R}, i = 1, \ldots, m,$$

for $t \in J$ and $u \in \mathbb{R}$.

(H3) The functions $g_i \in C(J \times \mathbb{R}, \mathbb{R})$ are continuous, and there exist bounded functions $K_i : J \to (0, \infty)$, such that

$$|g_i(t, u(t)) - g_i(t, v(t))| \leq K_i(t)|u(t) - v(t)|.$$

We set $L^* = \sup_{t\in J} |L(t)|$, $M^* = \sup_{t\in J} |M(t)|$, $\chi^* = \sup_{t\in J} |\chi(t)|$, $\vartheta^* = \sup_{t\in J} |\vartheta(t)|$ and $\varphi_i^* = \sup_{t\in J} |\varphi_i(t)|$, $K_i^* = \sup_{t\in J} |K_i(t)|$, $i = 1, 2, \ldots, m$.

3.1 Existence of solutions

In this subsection, we prove the existence of a solution for the problem (1.1)–(1.2) by applying a generalization of Krasnoselskii's fixed point theorem.

Theorem 3.1. *Assume that hypotheses (H1)–(H2) hold and if*

$$\begin{aligned} \Lambda =& \frac{\Psi_0^p(T)}{\Gamma(p+1)}\left(\frac{\chi^* M^*}{1-M^*} + \vartheta^* L^*\right)\left(\frac{\Psi_0^q(T)}{\Gamma(q+1)} + \frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)}\right) \\ &+ |\Omega_1|(q+4)\frac{\Psi_0^{q+2}(T)}{\Gamma(q+3)}\left(|\upsilon_2|L^* + \frac{|a_2|\Psi_0^{p-\delta}(\xi) + |b_2|\Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)}\right. \\ &\left.\times\left(\frac{\chi^* M^*}{1-M^*} + \vartheta^* L^*\right)\right) + \frac{|\upsilon_1|L^*}{|b_1|}\left(\frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)}\right) < 1. \end{aligned} \tag{3.1}$$

Then the problem (1.1)–(1.2) *has at least one solution on* J.

Proof. First, we choose $r > 0$ such that

$$\begin{aligned} r \geq & \chi^*\vartheta^*\frac{\Psi_0^{p+q}(T)}{\Gamma(p+1)\Gamma(q+1)} + \chi^*\Big(\frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)}\Big)\Big(\frac{|\upsilon_1|}{|b_1|} + \frac{\Psi_0^{p}(T)}{\Gamma(p+1)}\vartheta^*\Big) \\ & + \chi^*|\Omega_1|\frac{(q+4)\Psi_0^{q+2}(T)}{\Gamma(q+3)}\Big(|\upsilon_2| + \vartheta^*\frac{|a_2|\Psi_0^{p-\delta}(\xi) + |b_2|\Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)}\Big) \\ & + \sum_{i=1}^{n}\varphi_i^*\frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)}. \end{aligned}$$

Set

$$B_r = \{u \in E : \|u\| \leq r\}.$$

Clearly B_r is a closed, convex and bounded subset of the Banach space E.

Let $u(t)$ be a solution of the problem (1.1)–(1.2). Define

$$F_u(t) := f\left(t, u(t), {}^cD^{p;\psi}\left[\frac{{}^cD^{q;\psi}u(t) - \sum_{i=1}^{m} I^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]\right).$$

Then

$${}^cD^{p;\psi}\left[\frac{{}^cD^{q;\psi}u(t) - \sum_{i=1}^{m} I^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right] = F_u(t),$$

supplemented with the conditions (1.2), then by Lemma 2.4, we get

$$\begin{aligned} u(t) = & I^{q;\psi}\Big(h(s,u(s))I^{p;\psi}F_u(s)\Big)(t) + \sum_{i=1}^{m} I^{\eta_i+q;\psi}g_i(s,u(s))(t) + \\ & + I^{q;\psi}\Big(h(s,u(s))\Big(\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\Big)\Big(\frac{\upsilon_1}{b_1} - I^{p;\psi}F_u(s)\Big)\Big)(t) \\ & + \Omega_1\Big(\upsilon_2 - a_2I^{p-\delta;\psi}F_u(\xi) - b_2I^{p-\delta;\psi}F_u(T)\Big)I^{q;\psi}\Big(h(s,u(s))\Big(\Psi_0^2(s) - \Psi_0^1(T)\Psi_0^1(s)\Big)\Big)(t), \end{aligned}$$

Let us define three operators $C_p, C_{p-\delta} : E \to E$ and $D : E \to E$ such that

$$\begin{aligned} C_pu(t) &= \frac{1}{\Gamma(p)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1}F_u(s)ds, \quad t \in J, \\ C_{p-\delta}u(t) &= \frac{1}{\Gamma(p-\delta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-\delta-1}F_u(s)ds, \quad t \in J, \end{aligned}$$

and

$$Du(t) = h(t,u(t)), \quad t \in J.$$

Then, using assumptions (H1)–(H2) , we have

$$|C_pu(t) - C_pv(t)| \leq \frac{1}{\Gamma(p)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1}|F_u(s) - F_v(s)|ds, \tag{3.2}$$

and

$$\begin{aligned} |F_u(t) - F_v(t)| &\leq |f(t, u(t), F_u(t)) - f(t, v(t), F_v(t))| \\ &\leq M(t)\big(|u(t) - v(t)| + |F_u(t) - F_v(t)|\big) \\ &\leq \frac{M(t)}{1 - M(t)}\|u(\cdot) - v(\cdot)\|. \end{aligned} \tag{3.3}$$

By replacing (3.3) in (3.2), we obtain

$$|C_p u(t) - C_p v(t)| \leq \frac{M^* \Psi_0^p(T)}{(1 - M^*)\Gamma(p+1)}\|u(\cdot) - v(\cdot)\|,$$

and

$$|Du(t) - Dv(t)| \leq L^* \|u(\cdot) - v(\cdot)\|,$$

$$|C_p u(t)| \leq \frac{\Psi_0^p(T)}{\Gamma(p+1)}\vartheta^*,$$

and

$$|Du(t)| \leq \chi^*.$$

Now we define two more operators $A : E \to E$ and $B : B_r \to E$ such that

$$\begin{aligned} Au(t) = I^{q;\psi}\Big(Du(s)C_p u(s)\Big)(t) + I^{q;\psi}\Big(Du(s)\Big(\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\Big)\Big(\frac{\upsilon_1}{b_1} - C_p u(s)\Big)\Big)(t) \\ + \Omega_1\Big(\upsilon_2 - a_2 C_{p-\delta}u(\xi) - b_2 C_{p-\delta}u(T)\Big) I^{q;\psi}\Big(Du(s)\Big(\Psi_0^2(s) - \Psi_0^1(T)\Psi_0^1(s)\Big)\Big)(t), \end{aligned}$$

and

$$Bu(t) = \sum_{i=1}^{m} I^{\eta_i + q;\psi} g_i(s, u(s))(t).$$

We need to show that the two operators A and B satisfy all conditions of Lemma 2.5. This can be achieved in the following steps.

Step 1. First we show that A is a contraction mapping. Let $u(t), v(t) \in B_r$, then we have

$$\begin{aligned} &|Au(t) - Av(t)| \\ &\leq I^{q;\psi}\Big(\big|Du(s)C_p u(s) - Dv(s)C_p v(s)\big|\Big(1 + \big|\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\big|\Big)\Big)(t) \\ &+ I^{q;\psi}\Big(\frac{|\upsilon_1|}{|b_1|}\big|\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\big|\big|Du(s) - Dv(s)\big|\Big)(t) \\ &+ |\Omega_1| I^{q;\psi}\Big(\big|\Psi_0^2(s) - \Psi_0^1(T)\Psi_0^1(s)\big|\Big(|\upsilon_2|\big|Du(s) - Dv(s)\big| + |a_2|\big|Du(s)C_{p-\delta}u(\xi) \\ &- Dv(s)C_{p-\delta}v(\xi)\big| + |b_2|\big|Du(s)C_{p-\delta}u(T) - Dv(s)C_{p-\delta}v(T)\big|\Big)\Big)(t) \end{aligned}$$

$$
\begin{aligned}
&\le I^{q;\psi}\Big(\Big(\big|Du(s)\big|\big|C_pu(s)-C_pv(s)\big|+\big|C_pv(s)\big|\big|Du(s)-Dv(s)\big|\Big)\\
&\times\Big(1+\big|\Psi_0^1(s)\Omega_3-\Psi_0^2(s)\Omega_2\big|\Big)\Big)(t)+I^{q;\psi}\Big(\frac{|\upsilon_1|}{|b_1|}\big|\Psi_0^1(s)\Omega_3-\Psi_0^2(s)\Omega_2\big|\big|Du(s)-Dv(s)\big|\Big)(t)\\
&+|\Omega_1|I^{q;\psi}\Big(\big|\Psi_0^2(s)-\Psi_0^1(T)\Psi_0^1(s)\big|\Big(\big|Du(s)-Dv(s)\big|\Big(|\upsilon_2|+|a_2|\big|C_{p-\delta}v(\xi)\big|+|b_2|\big|C_{p-\delta}v(T)\big|\Big)\\
&+\big|Du(s)\big|\Big(|a_2|\big|C_{p-\delta}u(\xi)-C_{p-\delta}v(\xi)\big|+|b_2|\big|C_{p-\delta}u(T)-C_{p-\delta}v(T)\big|\Big)\Big)\Big)(t)
\end{aligned}
$$

Using the hypotheses (H1)–(H2) and taking the supremum over t, we get

$$\|Au(\cdot)-Av(\cdot)\|\le\Lambda\|u(\cdot)-v(\cdot)\|. \tag{3.4}$$

Therefore from (3.1), we conclude that the operator A is a contraction mapping.

Step 2. Next, we prove that the operator B satisfies condition (2) of Lemma 2.5, that is, the operator B is compact and continuous on B_r. Therefore first, we show that the operator B is continuous on B_r.

Let $u_n(t)$ be a sequence of functions in B_r converging to a function $u(t)\in B_r$. Then, by the Lebesgue dominant convergence theorem, for all $t\in J$, we have

$$
\begin{aligned}
\lim_{n\to\infty}Bu_n(t)&=\lim_{n\to\infty}\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\eta_i+q-1}g_i(s,u_n(s))ds\\
&=\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\eta_i+q-1}\lim_{n\to\infty}g_i(s,u_n(s))ds\\
&=\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\eta_i+q-1}g_i(s,u(s))ds.
\end{aligned}
$$

Hence $\lim_{n\to\infty}Bu_n(t)=Bu(t)$. Thus B is a continuous operator on B_r.

Further, we show that the operator B is uniformly bounded on B_r. For any $u\in B_r$, we have

$$
\begin{aligned}
\|Bu(\cdot)\|&\le\sup_{t\in J}\Big\{\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\eta_i+q-1}|g_i(s,u(s))|ds\Big\}\\
&\le\sum_{i=1}^{m}\frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)}\varphi_i^*\le r.
\end{aligned}
$$

Therefore $Bu(t)\le r$, for all $t\in J$, which shows that B is uniformly bounded on B_r.

Now, we show that the operator B is equi-continuous. Let $t_1,t_2\in J$ with $t_1>t_2$. Then for any

$u(t) \in B_r$, we have

$$
\begin{aligned}
&|Bu(t_1) - Bu(t_2)| \\
&\leq \sum_{i=1}^{m} \frac{1}{\Gamma(\eta_i + q)} \left| \int_0^{t_2} \psi'(s)\Big((\psi(t_1) - \psi(s))^{\eta_i+q-1} - (\psi(t_2) - \psi(s))^{\eta_i+q-1}\Big) g_i(s, u(s)) ds \right| \\
&+ \sum_{i=1}^{m} \frac{1}{\Gamma(\eta_i + q)} \left| \int_{t_2}^{t_1} \psi'(s)(\psi(t_1) - \psi(s))^{\eta_i+q-1} g_i(s, u(s)) ds \right| \\
&\leq \sum_{i=1}^{m} \frac{\varphi_i^*}{\Gamma(\eta_i + q + 1)} \Big(2|\psi(t_1) - \psi(t_2)|^{\eta_i+q} + \big|\Psi_0^{\eta_i+q}(t_2) - \Psi_0^{\eta_i+q}(t_1)\big| \Big).
\end{aligned}
$$

As $t_2 \to t_1$, so the right-hand side tends to zero. Thus B is equi-continuous. Therefore, it follows from the Arzelá–Ascoli theorem that B is a compact operator on B_r. We conclude that B is completely continuous.

Step 3. It remains to verify the condition (3) of Lemma 2.5. For any $v \in B_r$, we have

$$
\begin{aligned}
\|u(\cdot)\| &= \|Au(\cdot) + Bv(\cdot)\| \\
&\leq \|Au(\cdot)\| + \|Bv(\cdot)\| \\
&\leq \sup_{t\in J} \left\{ \left| I^{q;\psi}\Big(Du(s)C_p u(s)\Big)(t) + I^{q;\psi}\left(Du(s)\Big(\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\Big)\left(\frac{\upsilon_1}{b_1} - C_p u(s)\right)\right)(t) \right.\right. \\
&\left.\left. + \Omega_1\Big(\upsilon_2 - a_2 C_{p-\delta} u(\xi) - b_2 C_{p-\delta} u(T)\Big) I^{q;\psi}\left(Du(s)\Big(\Psi_0^2(s) - \Psi_0^1(T)\Psi_0^1(s)\Big)\right)(t) \right| \right\} \\
&+ \sup_{t\in J} \left\{ \sum_{i=1}^{m} I^{\eta_i+q;\psi}\big|g_i(s, v(s))\big|(t) \right\} \\
&\leq \chi^* \vartheta^* \frac{\Psi_0^{p+q}(T)}{\Gamma(p+1)\Gamma(q+1)} + \chi^*\left(\frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)}\right)\left(\frac{|\upsilon_1|}{|b_1|} + \frac{\Psi_0^p(T)}{\Gamma(p+1)}\vartheta^*\right) \\
&+ \chi^*|\Omega_1| \frac{(q+4)\Psi_0^{q+2}(T)}{\Gamma(q+3)}\left(|\upsilon_2| + \vartheta^* \frac{|a_2|\Psi_0^{p-\delta}(\xi) + |b_2|\Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)}\right) \\
&+ \sum_{i=1}^{n} \varphi_i^* \frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)}.
\end{aligned}
$$

Which implies, from the choice of r that $\|u\| \leq r$, and so $u \in B_r$. Hence all conditions of Lemma 2.5 are satisfied. Therefore, the operator equation $u(t) = Au(t) + Bu(t)$ has at least one solution in B_r. Consequently, the problem (1.1)–(1.2) has at least on solution on J. Thus the proof is completed. □

3.2 Uniqueness of solutions

In the next result, we apply the Banach fixed theorem to prove the uniqueness of solutions for the problem (1.1)–(1.2).

Theorem 3.2. *Assume that the hypotheses(H1)–(H3) together with the inequality*

$$\Lambda + \sum_{i=1}^{m} K_i^* \frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q)} < 1.$$

are satisfied, then the problem (1.1)–(1.2) *has an unique solution.*

Proof. According to Lemma 2.4, we define the operator $Q: E \to E$ by

$$Qu(t) = Au(t) + Bu(t).$$

First, we show that $Q(B_r) \subset B_r$. As in the previous proof (**step 3**) of Theorem 3.1, we can obtain for $u \in B_r$ and $t \in J$

$$\begin{aligned}\|Qu(\cdot)\| \le{}& \chi^*\vartheta^* \frac{\Psi_0^{p+q}(T)}{\Gamma(p+1)\Gamma(q+1)} + \chi^* \left(\frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)} \right) \left(\frac{|\upsilon_1|}{|b_1|} + \frac{\Psi_0^{p}(T)}{\Gamma(p+1)}\vartheta^* \right) \\ &+ \chi^*|\Omega_1| \frac{(q+4)\Psi_0^{q+2}(T)}{\Gamma(q+3)} \left(|\upsilon_2| + \vartheta^* \frac{|a_2|\Psi_0^{p-\delta}(\xi) + |b_2|\Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)} \right) \\ &+ \sum_{i=1}^{n} \varphi_i^* \frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)} \le r.\end{aligned}$$

This shows that $Q(B_r) \subset B_r$.
Next, we prove that the operator Q is a contractive operator. For $u, v \in B_r$

$$\|Qu(\cdot) - Qv(\cdot)\| \le \|Au(\cdot) - Av(\cdot)\| + \|Bu(\cdot) - Bv(\cdot)\|,$$

and

$$\begin{aligned}&\|Bu(\cdot) - Bv(\cdot)\| \\ &\le \sup_{t\in J} \left\{ \sum_{i=1}^{m} \frac{1}{\Gamma(\eta_i+q)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\eta_i+q-1} \big| g_i(s,u(s)) - g_i(s,v(s)) \big| ds \right\} \\ &\le \sum_{i=1}^{m} K_i^* \frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)} \|u(\cdot) - v(\cdot)\|.\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we get

$$\|Qu(\cdot) - Qv(\cdot)\| \le \left(\Lambda + \sum_{i=1}^{m} K_i^* \frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)} \right) \|u(\cdot) - v(\cdot)\|.$$

This implies that the operator Q is a contractive operator. Consequently, by Theorem 3.2, we conclude that Q has an unique fixed point, which is a solution of the problem (1.1)–(1.2). This completes the proof. ◻

4 Example

Consider the following fractional hybrid differential equation

$$\begin{cases} {}^cD^{\frac{5}{2};t}\left[\dfrac{{}^cD^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]=f\left(t,u(t),{}^cD^{\frac{5}{2};t}\left[\dfrac{{}^cD^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]\right), \\ u(0)=0, \quad {}^cD^{\frac{3}{4};t}u(0)=0, \\ 2\left[\dfrac{{}^cD^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]_{t=0}+\dfrac{2}{7}\left[\dfrac{{}^cD^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]_{t=1}=\dfrac{7}{2}, \\ \dfrac{7}{13}\,{}^cD^{\frac{4}{5};t}\left[\dfrac{{}^cD^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]_{t=\frac{4}{5}}+\dfrac{1}{2}\,{}^cD^{\frac{4}{5};t}\left[\dfrac{{}^cD^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]_{t=1}=2\,, \end{cases} \tag{4.1}$$

where

$$\begin{aligned}\sum_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))(s) &= I^{\frac{1}{3};t}\left(\frac{\sin^2 x(s)}{8(s+1)^2}\right)(t)+I^{\frac{3}{2};t}\left(\frac{1}{2\pi\sqrt{81+s^2}}\frac{|x(s)|}{2+|x(s)|}\right)(t)\\ &\quad + I^{\frac{7}{3};t}\left(\frac{\sin x(s)}{3\pi\sqrt{49+s^2}}\right)(t),\end{aligned}$$

$$h(t,u(t))=\frac{e^{-3t}\cos u(t)}{2t+40}+\frac{1}{80}(t^3+1),$$

and

$$\begin{aligned} &f\left(t,u(t),{}^cD^{\frac{5}{2};t}\left[\frac{{}^cD^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]\right)\\ &=\frac{1}{60\sqrt{t+81}}\left(\frac{|x(t)|}{3+|x(t)|}-\arctan\left({}^cD^{\frac{5}{2};t}\left[\frac{{}^cD^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]\right)\right).\end{aligned}$$

Here $T=1, p=\frac{5}{2}, q=\frac{3}{4}, m=3, \eta_1=\frac{1}{3}, \eta_2=\frac{3}{2}, \eta_3=\frac{7}{3}, \delta=\frac{4}{5}, a_1=2, a_2=\frac{7}{13}, b_1=\frac{2}{7}$, $b_2=\frac{1}{2}, \upsilon_1=\frac{7}{2}, \upsilon_2=2, \xi=\frac{4}{5}$, $g_1=\dfrac{\sin^2 x(t)}{8(t+1)^2}$, $g_2=\dfrac{1}{2\pi\sqrt{81+t^2}}\dfrac{|x(t)|}{2+|x(t)|}$, $g_3=\dfrac{\sin x(t)}{3\pi\sqrt{49+t^2}}$.

The hypotheses (H1), (H2) and (H4) are satisfied with the following positives functions: $L(t)=\dfrac{e^{-3}}{2t+40}$, $M(t)=\vartheta(t)=\dfrac{1}{60\sqrt{t+81}}$, $\varphi_1(t)=K_1(t)=\dfrac{1}{8(t+1)^2}$, $\varphi_2(t)=K_2(t)=\dfrac{1}{2\pi\sqrt{81+t^2}}$, $\varphi_3(t)=K_3(t)=\dfrac{1}{3\pi\sqrt{49+t^2}}$ and $\chi(t)=\dfrac{e^{-3}}{2t+40}+\dfrac{1}{80}(t^3+1)$, which gives us $L^*=\frac{1}{40}$, $M^*=\vartheta^*=\frac{1}{540}, \chi^*=\frac{3}{80}$, $\varphi_1^*=K_1^*=\frac{1}{8}, \varphi_2^*=K_2^*=\frac{1}{18\pi}$, $\varphi_3^*=K_3^*=\frac{1}{21\pi}$.

With the given data, we find that

$$\Omega_1 \simeq 1.81820508, \quad \Omega_2 \simeq 0.60797139, \quad \Omega_3 \simeq 1.60797139,$$

and

$$\Lambda \simeq 0.48820986 < 1.$$

By Theorem 3.1, the problem (4.1) has a solution on $[0,1]$.

Also, we have

$$\Lambda + \sum_{i=1}^{3} K_i^* \frac{\Psi_0^{\eta_i+q}(1)}{\Gamma(\eta_i+\frac{7}{4})} \simeq 0.61782704 < 1.$$

In view of Theorem 3.2 the problem (4.1) has an unique solution.

5 Conclusion

In this manuscript, we have successfully investigated the existence, uniqueness of the solutions for a new class of $\psi-$Caputo type hybrid fractional differential equations with hybrid conditions. The existence of solutions is provided by using a generalization of Krasnoselskii's fixed point theorem due to Dhage [5], whereas the uniqueness result is achieved by Banach's contraction mapping principle. Also, we have presented an illustrative example to support our main results. In future works, many results can be established when one takes a more generalized operator. Precisely, it will be of interest to study the current problem in this work for the fractional operator with variable order [22], and ψ-Hilfer fractional operator [19].

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Graded weakly 1-absorbing prime ideals

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ABSTRACT

In this paper, we introduce and study graded weakly 1-absorbing prime ideals in graded commutative rings. Let G be a group and R be a G-graded commutative ring with a nonzero identity $1 \neq 0$. A proper graded ideal P of R is called a graded weakly 1-absorbing prime ideal if for each nonunits $x, y, z \in h(R)$ with $0 \neq xyz \in P$, then either $xy \in P$ or $z \in P$. We give many properties and characterizations of graded weakly 1-absorbing prime ideals. Moreover, we investigate weakly 1-absorbing prime ideals under homomorphism, in factor ring, in rings of fractions, in idealization.

RESUMEN

En este artículo, introducimos y estudiamos ideales primos débilmente 1-absorbentes en anillos conmutativos gradados. Sea G un grupo y R un anillo conmutativo G-gradado con identidad no cero $1 \neq 0$. Un ideal gradado propio P de R se llama ideal primo gradado débilmente 1-absorbente si para cualquiera $x, y, z \in h(R)$ no-unidades con $0 \neq xyz \in P$, entonces o bien $xy \in P$ o $z \in P$. Entregamos muchas propiedades y caracterizaciones de ideales primos gradados débilmente 1-absorbentes. Más aún, investigamos ideales primos débilmente 1-absorbentes bajo homomorfismo, en anillos cociente, en anillos de fracciones, en idealización.

Keywords and Phrases: graded ideal, 1-absorbing prime ideal, weakly 1-absorbing prime ideal, graded weakly 1-absorbing prime ideal.

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1 Introduction

Throughout the paper, we focus only on graded commutative rings with a nonzero identity. R will always denote such a ring and G denotes a group with identity e. $u(R)$, $N(R)$ and $reg(R)$ denote the set of all unit elements, all nilpotent elements and all regular elements of R, respectively. Over the years, several types of ideals have been developed such as prime, maximal, primary, etc. The concept of prime ideals and its generalizations have a significant place in commutative algebra since they are used in understanding the structure of rings [6, 19, 11, 4]. R is said to be G-graded if $R = \bigoplus_{g\in G} R_g$ with $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$ where R_g is an additive subgroup of R for all $g \in G$. Sometimes we denote the G-graded ring R by $G(R)$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Also, we set $h(R) = \bigcup_{g\in G} R_g$. The support of $G(R)$ is defined as $supp(G(R)) = \{g \in G : R_g \neq \{0\}\}$. Moreover, as shown for example in [13] that R_e is a subring of R and $1 \in R_e$. Let P be an ideal of a graded ring R. Then P is said to be graded ideal if $P = \bigoplus_{g\in G}(P \cap R_g)$, *i.e.*, for $x \in P$, $x = \sum_{g\in G} x_g$ where $x_g \in P$ for all $g \in G$. It is known that an ideal of a graded ring need not be graded. Let R be a G-graded ring and P be a graded ideal of R. Then R/P is G-graded by $(R/P)_g = (R_g+P)/P$ for all $g \in G$. If R and S are G-graded rings, then $R\times S$ is a G-graded ring by $(R \times S)_g = R_g \times S_g$ for all $g \in G$.

Lemma 1.1 ([9, Lemma 2.1])**.** *Let R be a G-graded ring.*

(1) *If P and Q are graded ideals of R, then $P + Q$, PQ and $P \bigcap Q$ are graded ideals of R.*

(2) *If $x \in h(R)$, then $Rx = (x)$ is a graded ideal of R.*

Let P be a proper graded ideal of R. Then the graded radical of P is denoted by $\mathrm{Grad}(P)$ and it is defined as follows:

$$\mathrm{Grad}(P) = \left\{ x = \sum_{g\in G} x_g \in R : \text{ for all } g \in G, \text{ there exists } n_g \in \mathbb{N} \text{ such that } x_g^{n_g} \in P \right\}.$$

Note that $\mathrm{Grad}(P)$ is always a graded ideal of R (see [15]).

In [15], Refai *et al.* defined and studied graded prime ideals. A proper graded ideal P of a graded ring R is called graded prime ideal if whenever $xy \in P$ for some $x, y \in h(R)$ then either $x \in P$ or $y \in P$. Clearly, if P is a prime ideal of R and it is a graded ideal of R, then P is a graded prime ideal of R. On the other hand, the importance of graded prime ideals comes from the fact that graded prime ideals are not necessarily prime ideals, as we see in the next example.

Example 1.2. *Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$.*

Consider the graded ideal $I = 17R$ of R. We show that I is a graded prime ideal of R. Let $xy \in I$ for some $x, y \in h(R)$.

Case (1): *Assume that $x, y \in R_0$. In this case, $x, y \in \mathbb{Z}$ such that 17 divides xy, and then either 17 divides x or 17 divides y as 17 is a prime number, which implies that $x \in I$ or $y \in I$.*

Case (2): *Assume that $x, y \in R_1$. In this case, $x = ia$ and $y = ib$ for some $a, b \in \mathbb{Z}$ such that 17 divides $xy = -ab$, and then 17 divides a or 17 divides b in $\mathbb{Z}$, which implies that 17 divides $x = ia$ or 17 divides $y = ib$ in R. Then we have that $x \in I$ or $y \in I$.*

Case (3): *Assume that $x \in R_0$ and $y \in R_1$. In this case, $x \in \mathbb{Z}$ and $y = ib$ for some $b \in \mathbb{Z}$ such that 17 divides $xy = ixb$ in R, that is $ixb = 17(\alpha + i\beta)$ for some $\alpha, \beta \in \mathbb{Z}$. Then we obtain $xb = 17\beta$, that is 17 divides xb in $\mathbb{Z}$, and again 17 divides x or 17 divides b, which implies that 17 divides x or 17 divides $y = ib$ in R. Thus, $x \in I$ or $y \in I$.*

One can similarly show that $x \in I$ or $y \in I$ in other cases. So, I is a graded prime ideal of R. On the other hand, I is not a prime ideal of R since $(4 - i)(4 + i) \in I$, $(4 - i) \notin I$ and $(4 + i) \notin I$.

Several generalizations of graded prime ideals attracted attention by many authors. In [14], Refai and Al-Zoubi introduced graded primary ideals which is a generalization of graded prime ideals. A proper graded ideal P of a graded ring R is called graded primary ideal if $xy \in P$ for some $x, y \in h(R)$ implies that either $x \in P$ or $y \in \text{Grad}(P)$. They also studied graded primary G-decomposition related to graded primary ideals. Atani defined a generalization of graded prime ideals as graded weakly prime ideals in [5]. A proper graded ideal P of a graded ring R is said to be graded weakly prime ideal if whenever $x, y \in h(R)$ such that $0 \neq xy \in P$ then either $x \in P$ or $y \in P$. They gave some characterizations of graded weakly prime ideals and their homogeneous components. In [12], Naghani and Moghimi introduced 2-absorbing version of graded prime ideals and graded weakly prime ideals. A proper graded ideal P of a graded ring R is called graded 2-absorbing (graded weakly 2-absorbing) if whenever $x, y, z \in h(R)$ such that $xyz \in P$ ($0 \neq xyz \in P$) then $xy \in P$ or $yz \in P$ or $xz \in P$. They investigated some properties of this new class of graded ideals. Yassine *et al.* studied 1-absorbing prime ideals which is a generalization of prime ideals in [19]. A proper ideal P of R is said to be 1-absorbing prime ideal if for some nonunit elements $x, y, z \in R$ such that $xyz \in P$ implies that either $xy \in P$ or $z \in P$. Authors determined 1-absorbing prime ideals in some special rings such as principal ideal domains, valuation domains and Dedekind domains. Currently, Koç *et al.* defined weakly 1-absorbing prime ideals which is a generalization of 1-absorbing prime ideals in [11]. A proper ideal P of R is called weakly 1-absorbing prime ideal if $0 \neq xyz \in P$ for some nonunits $x, y, z \in R$ implies $xy \in P$ or $z \in P$. They gave certain properties of this new concept and characterized rings that every proper ideal is weakly 1-absorbing ideal. More recently, in [1], Dawwas *et al.* defined graded version of 1-absorbing prime ideals which is a generalization of both graded prime ideals and 1-absorbing prime ideals. A proper graded ideal P of a graded ring R is called graded 1-absorbing prime ideal if whenever for some nonunits x, y, z in $h(R)$ such that $xyz \in P$ then either $xy \in P$ or $z \in P$. Moreover, many studies have been made by researchers related to graded versions of known structures [3, 7, 8, 10, 16].

In this paper, we define graded weakly 1-absorbing prime ideal which is a generalization of graded 1-absorbing prime ideals. A proper graded ideal P of a graded ring R is said to be graded weakly 1-absorbing prime ideal if whenever for some nonunits x, y, z in $h(R)$ such that $0 \neq xyz \in P$ then either $xy \in P$ or $z \in P$. Every graded 1-absorbing prime ideal is a graded weakly 1- absorbing prime ideals but the converse is not true in general (see, Example 3.2). In addition to many properties of this new class of graded ideals, we also investigate behavior of graded weakly 1-absorbing ideals under homomorphism, in factor ring, in rings of fractions, in idealization (see, Theorem 3.15, Proposition 3.14, Theorem 3.16, Theorem 3.18 and Theorem 3.23).

2 Motivation

Graded prime ideals play an essential role in graded commutative ring theory. Indeed, graded prime ideals are interesting because they correspond to irreducible varieties and schemes in the graded case and because of their connection to factorization. Also, graded prime ideals are important because they have applications to combinatorics and they have structural significance in graded ring theory. Thus, this concept has been generalized and studied in several directions. The significance of some of these generalizations is same as the graded prime ideals. In a feeling of animate being, they determine how far an ideal is from being graded prime. Several generalizations of graded prime ideals attracted attention by many authors. For instance, graded weakly prime ideals, graded primary ideals, graded almost prime ideals, graded 2-absorbing ideals, graded 2-absorbing primary ideals and graded 1-absorbing prime ideals. In continuation of these generalizations, we present the concept of graded weakly 1-absorbing prime ideals, as a new generalization to graded prime ideals, in order to benefit from this new concept in more applications, and to make the study of graded prime ideals more flexible.

3 Graded weakly 1-absorbing prime ideals

Definition 3.1. *Let R be a G-graded ring and P be a proper graded ideal of R. Then, P is called graded weakly 1-absorbing prime ideal of R if whenever $0 \neq xyz \in P$ for some nonunit elements x, y, z in $h(R)$ then $xy \in P$ or $z \in P$.*

Example 3.2. *Every graded 1-absorbing prime ideal is a graded weakly 1-absorbing prime ideal. The converse may not be true. Let $R = \mathbb{Z}_{21}$ and consider the trivial grading on R. $P = (\bar{0})$ is graded* *weakly 1-absorbing prime ideal. But it is not graded 1-absorbing prime ideal since* $\bar{3}.\bar{3}.\bar{7} = \bar{0} \in P$, $\bar{3}.\bar{3} \notin P$ *and* $\bar{7} \notin P$.

Example 3.3. *Let* $R = \mathbb{Z}_8[i] = \mathbb{Z}_8 \oplus i\mathbb{Z}_8$. *Then note that* R *is a* $\mathbb{Z}_2$*-graded ring and* $h(R) = \mathbb{Z}_8 \cup i\mathbb{Z}_8$. *Now, put* $P = (\bar{4})$. *Since* $\bar{2}(\bar{1}+i)(\bar{1}-i) = \bar{4} \in P$ *but* $\bar{2}(\bar{1}+i) \notin P$ *and* $(\bar{1}-i) \notin P$, *it follows that* P *is not a weakly 1-absorbing prime ideal of* R. *However, the set of nonunit homogeneous elements of* R *is* $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{2}i, \bar{4}i, \bar{6}i\}$. *Let* $x, y, z \in h(R)$ *be nonunit elements. Then note that* $xyz = \bar{0} \in P$, *which implies that* P *is a graded weakly 1-absorbing prime ideal of* R.

$N(R)$ denotes the set of all nilpotent elements of R. Recall that a ring R is said to be reduced if $N(R) = 0$.

Theorem 3.4. *Let* R *be a* G*-graded reduced ring and* P *be a graded weakly 1-absorbing prime ideal of* R. *Then,* $\mathrm{Grad}(P)$ *is a graded weakly prime ideal of* R.

Proof. Suppose that $0 \neq xy \in \mathrm{Grad}(P)$ where $x, y \in h(R)$. Then there exists $n \in \mathbb{N}$ such that $(xy)^n \in P$. We have $0 \neq (xy)^n = x^k x^{n-k} y^n \in P$ for some positive integer $k < n$. If x or y is unit in $h(R)$, we are done. So, assume that x and y are nonunit elements in $h(R)$. As P is graded weakly 1-absorbing prime ideal, $x^n \in P$ or $y^n \in P$ showing that $x \in \mathrm{Grad}(P)$ or $y \in \mathrm{Grad}(P)$. □

Theorem 3.5. *Let* R *be a* G*-graded ring and* P *be a graded weakly 1-absorbing prime ideal of* R. *Then,* $(P : a)$ *is a graded weakly prime ideal of* R *where* a *is a regular nonunit element in* $h(R) - P$.

Proof. From [1, Lemma 2.4], $(P : a)$ is a graded ideal of R. Suppose $0 \neq xy \in (P : a)$ for some $x, y \in h(R)$. Then $0 \neq (xa)y \in P$ where $xa, y \in h(R)$. If x or y is unit, there is nothing to prove. So, we can assume that x and y are nonunit elements in $h(R)$. Since P is graded weakly 1-absorbing prime ideal of R, we get either $xa \in P$ or $y \in P$. It gives $x \in (P : a)$ or $y \in (P : a)$, as needed. □

Definition 3.6. *Let* R *be a* G*-graded ring and* P *be a graded ideal of* R. *Then,* P *is called* g*-weakly 1-absorbing prime ideal of* R *for* $g \in G$ *with* $P_g \neq R_g$ *if* $0 \neq xyz \in P$ *for some nonunit elements* x, y, z *in* R_g *implies that* $xy \in P$ *or* $z \in P$.

We say that a proper graded ideal P of a G-graded ring R is said to be a g-weakly prime for $g \in G$ if $P_g \neq R_g$ and whenever $0 \neq xy \in P$ for some $x, y \in R_g$ implies $x \in P$ or $y \in P$.

Proposition 3.7. *Let* R *be a* G*-graded reduced ring and* P *be a* g^n*-weakly 1-absorbing prime ideal of* R *for each* $n \in \mathbb{N}$. *Then, Grad*(P) *is a* g*-weakly prime ideal of* R.

Proof. It immediately follows from Theorem 3.4. □

Recall from [1] that a proper graded ideal P of a G-graded ring R is said to be a g-1-absorbing prime for $g \in G$ if $P_g \neq R_g$ and whenever $xyz \in P$ for some nonunits $x, y, z \in R_g$ implies $xy \in P$ or $z \in P$.

Proposition 3.8. *Let R be a G-graded ring. If R has a g-weakly-1-absorbing prime ideal that is not a g-weakly prime ideal of R and (0) is a g-1-absorbing prime ideal of R, then, for each unit element u in R_g and for each nonunit element v in R_g the sum $u + v$ is a unit element in R_g.*

Proof. Assume that P is a g-weakly 1-absorbing prime ideal of R that is not a g-weakly prime ideal of R. Then, there exist nonunit elements $x, y \in R_g$ such that $xy \in P$ but $x \notin P$ and $y \notin P$. Then we have $vxy \in P$ where v is a nonunit element in R_g. If $vxy = 0 \in (0)$, then $vx \in P$ since (0) is a g-1 absorbing prime ideal and $y \notin P$. If $0 \neq vxy \in P$, we have $vx \in P$ since P is a g-weakly 1-absorbing prime ideal of R. Now we will show that $u + v$ is a unit element in R_g where u is a unit element in R_g. Suppose to the contrary. If $(u + v)xy = 0 \in (0)$, we get $(u + v)x \in P$. This implies $ux \in P$ giving that $x \in P$ which is a contadiction. If we assume $0 \neq (u + v)xy \in P$, then again we get a contradiction by using the fact that P is a g-weakly 1-absorbing prime ideal and so it completes the proof. ◻

Theorem 3.9. *Let R be a G-graded ring and P be a proper graded ideal of R. Consider the following statements.*

(i) *P is a graded weakly 1-absorbing prime ideal of R.*

(ii) *If $xy \notin P$ for some nonunits $x, y \in h(R)$, then $(P : xy) = P \cup (0 : xy)$.*

(iii) *If $xy \notin P$ for some nonunits $x, y \in h(R)$, then either $(P : xy) = P$ or $(P : xy) = (0 : xy)$.*

(iv) *If $0 \neq xyK \subseteq P$ for some nonunits $x, y \in h(R)$ and proper graded ideal K of R, then either $xy \in P$ or $K \subseteq P$.*

(v) *If $0 \neq xJK \subseteq P$ for some nonunit $x \in h(R)$ and proper graded ideals J, K of R, then either $xJ \subseteq P$ or $K \subseteq P$.*

(vi) *If $0 \neq IJK \subseteq P$ for proper graded ideals I, J, K of R, then either $IJ \subseteq P$ or $K \subseteq P$.*

Then, $(vi) \Rightarrow (v) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$.

Proof. $(vi) \Rightarrow (v)$: Suppose that $0 \neq xJK \subseteq P$ for some nonunit $x \in h(R)$ and proper graded ideals J, K of R. Now, put $I = (x)$. Then I is a proper graded ideal of R and $0 \neq IJK \subseteq P$. By (vi), we have $xJ \subseteq IJ \subseteq P$ or $K \subseteq P$, which completes the proof.

$(v) \Rightarrow (iv)$: Suppose that $0 \neq xyK \subseteq P$ for some nonunits $x, y \in h(R)$ and proper graded ideal K of R. Now, consider the proper graded ideal $J = (y)$ of R and note that $0 \neq xJK \subseteq P$. So by (v), we get $xy \in xJ \subseteq P$ or $K \subseteq P$.

$(iv) \Rightarrow (iii):$ Let $x, y \in h(R)$ be nonunit elements such that $xy \notin P$. It is easy to see that $xy(P : xy) \subseteq P$.

Case 1: Assume that $xy(P : xy) = 0$. This gives $(P : xy) \subseteq (0 : xy) \subseteq (P : xy)$, that is, $(P : xy) = (0 : xy)$.

Case 2: Assume that $xy(P : xy) \neq 0$. Then by (iv), we have $(P : xy) \subseteq P$ which implies that $(P : xy) = P$.

$(iii) \Rightarrow (ii):$ It is straightforward.

$(ii) \Rightarrow (i):$ Let $x, y, z \in h(R)$ be nonunits such that $0 \neq xyz \in P$. If $xy \in P$, then we are done. So assume that $xy \notin P$. Since $z \in (P : xy) - (0 : xy)$ and $(P : xy) = (0 : xy) \cup P$, we have $z \in P$ which completes the proof. □

In the following example, we show that the condition "P is a graded weakly 1-absorbing prime ideal" does not ensure that the conditions (ii)-(vi) in Theorem 3.9 hold. In fact, we will show that $(i) \not\Rightarrow (ii)$.

Example 3.10. *Let $R = \mathbb{Z}_{12}[X]$, where X is an indeterminate over $\mathbb{Z}_{12}$. Then $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is a $\mathbb{Z}$-graded ring, where $R_0 = \mathbb{Z}_{12}$ and $R_n = \mathbb{Z}_{12}X^n$ if $n > 0$, otherwise $R_n = 0$. Then note that $h(R) = \bigcup_{n\geq 0} \mathbb{Z}_{12}X^n$ and the set of nonunits homogeneous elements of R is $nh(R) = \{\overline{2k}, \overline{3k}, \overline{a}X^n : k, a \in \mathbb{Z}$ and $n \in \mathbb{N}\}$. Consider the graded ideal $P = (X, \overline{4})$ of R. Let $f, g, h \in nh(R)$ such that $0 \neq fgh \in P$. If at least one of the f, g, h is of the form $\overline{a}X^n$, then we are done. So assume that $f, g, h \in \{\overline{2k}, \overline{3k} : k \in \mathbb{Z}\}$. Since $0 \neq fgh \in P = (X, \overline{4})$, we have $0 \neq fgh = \overline{4k}$ for some $k \in \mathbb{Z}$ with $\gcd(k, 3) = 1$. Since $4|fgh$ and $3 \nmid fgh$, we conclude that $f, g, h \in \{\overline{2}, \overline{4}, \overline{8}, \overline{10}\}$. This implies that $fg \in P$, that is, P is a graded weakly 1-absorbing prime ideal of R. Now, we will show that P does not satisfy the condition (ii) in Theorem 3.9. Take nonunits homogeneous elements $c = \overline{2}, d = \overline{3}$ of R. Then note that $cd \notin P$. On the other hand, it is clear that $\overline{2} \in (0 : cd) - P$ and $X \in P - (0 : cd)$. This gives $z = \overline{2} + X \in (P : cd) - ((0 : cd) \cup P)$. Thus, we have $(P : cd) \supsetneq (0 : cd) \cup P$, i.e., P does not satisfy the condition (ii) in Theorem 3.9.*

Definition 3.11. *Let P be a graded weakly 1-absorbing prime ideal of R and $x_{g_1}, y_{g_2}, z_{g_3}$ be nonunits in $h(R)$. Then, $(x_{g_1}, y_{g_2}, z_{g_3})$ is called graded 1-triple zero if $x_{g_1}y_{g_2}z_{g_3} = 0$, $x_{g_1}y_{g_2} \notin P$ and $z_{g_3} \notin P$, where $g_1, g_2, g_3 \in G$.*

Theorem 3.12. *Let $P = \bigoplus_{g\in G} P_g$ be a graded weakly 1-absorbing prime ideal that is not graded 1-absorbing prime ideal and $(x_{g_1}, y_{g_2}, z_{g_3})$ be a graded 1-triple zero of P, where $g_1, g_2, g_3 \in G$. Then,*

(i) $x_{g_1}y_{g_2}P_{g_3} = 0$.

(ii) $x_{g_1}z_{g_3} \notin P_{g_1g_3}$ *and* $y_{g_2}z_{g_3} \notin P_{g_2g_3}$ *imply that* $x_{g_1}z_{g_3}P_{g_2} = y_{g_2}z_{g_3}P_{g_1} = x_{g_1}P_{g_2}P_{g_3} = y_{g_2}P_{g_1}P_{g_3} = z_{g_3}P_{g_1}P_{g_2} = 0$. *In particular,* $P_{g_1}P_{g_2}P_{g_3} = 0$.

Proof. $(i):$ Let $P = \bigoplus_{g\in G} P_g$ be a graded weakly 1-absorbing prime ideal that is not graded 1-absorbing prime ideal and $(x_{g_1}, y_{g_2}, z_{g_3})$ be a graded 1-triple zero of P. Assume that $x_{g_1}y_{g_2}P_{g_3} \neq 0$. Then, there exists $a \in P_{g_3} = P \cap R_{g_3}$ such that $0 \neq x_{g_1}y_{g_2}a$. So, we have $0 \neq x_{g_1}y_{g_2}a = x_{g_1}y_{g_2}(z_{g_3} + a) \in P$. If $z_{g_3} + a$ is unit, then $x_{g_1}y_{g_2} \in P$ which gives a contradiction. Since P is graded weakly 1-absorbing prime ideal and $x_{g_1}y_{g_2} \notin P$, $z_{g_3}+a \in P$. This shows $z_{g_3} \in P$, a contradiction.

$(ii):$ Let $x_{g_1}z_{g_3} \notin P_{g_1g_3}$ and $y_{g_2}z_{g_3} \notin P_{g_2g_3}$. Then, $x_{g_1}z_{g_3}, y_{g_2}z_{g_3} \notin P$. Now choose $a \in P_{g_2}$. So, we have $x_{g_1}(y_{g_2} + a)z_{g_3} = x_{g_1}az_{g_3} \in P$ since $x_{g_1}y_{g_2}z_{g_3} = 0$. If $y_{g_2} + a$ is unit, then we obtain $x_{g_1}z_{g_3} \in P$, which is a contradiction. Thus, $y_{g_2} + a$ is not unit. If $x_{g_1}az_{g_3} \neq 0$, then $0 \neq x_{g_1}(y_{g_2} + a)z_{g_3} \in P$. Thus, $x_{g_1}(y_{g_2} + a) \in P$ or $z_{g_3} \in P$ implying that $x_{g_1}y_{g_2} \in P$ or $z_{g_3} \in P$, a contradiction. This shows $x_{g_1}az_{g_3} = 0$ and so $x_{g_1}z_{g_3}P_{g_2} = 0$. Similarly, $y_{g_2}z_{g_3}P_{g_1} = 0$.

Now assume that $x_{g_1}P_{g_2}P_{g_3} \neq 0$. Then there exist $a_{g_2} \in P_{g_2}, b_{g_3} \in P_{g_3}$ such that $x_{g_1}a_{g_2}b_{g_3} \neq 0$. This gives $0 \neq x_{g_1}(y_{g_2} + a_{g_2})(z_{g_3} + b_{g_3}) = x_{g_1}y_{g_2}z_{g_3} + x_{g_1}y_{g_2}b_{g_3} + x_{g_1}a_{g_2}z_{g_3} + x_{g_1}a_{g_2}b_{g_3} = x_{g_1}a_{g_2}b_{g_3} \in P$. If $(y_{g_2} + a_{g_2})$ is unit, $x_{g_1}(z_{g_3} + b_{g_3}) \in P$. It means that $x_{g_1}z_{g_3} \in P$, which is a contradiction. Hence, $(y_{g_2} + a_{g_2})$ is nonunit. Similar argument shows that $(z_{g_3} + b_{g_3})$ is nonunit. Since P is graded weakly 1-absorbing prime ideal, $x_{g_1}(y_{g_2}+a_{g_2}) \in P$ or $z_{g_3}+b_{g_3} \in P$. This proves $x_{g_1}y_{g_2} \in P$ or $z_{g_3} \in P$ which is a contradiction. So, $x_{g_1}P_{g_2}P_{g_3} = 0$. Similarly we have $y_{g_2}P_{g_1}P_{g_3} = z_{g_3}P_{g_1}P_{g_2} = 0$.

Suppose $P_{g_1}P_{g_2}P_{g_3} \neq 0$. Then there exist $a_{g_1} \in P_{g_1}, b_{g_2} \in P_{g_2}, c_{g_3} \in P_{g_3}$ such that $a_{g_1}b_{g_2}c_{g_3} \neq 0$. So, we have $0 \neq (a_{g_1} + x_{g_1})(b_{g_2} + y_{g_2})(c_{g_3} + z_{g_3}) = a_{g_1}b_{g_2}c_{g_3} \in P$ since $x_{g_1}z_{g_3}P_{g_2} = y_{g_2}z_{g_3}P_{g_1} = x_{g_1}P_{g_2}P_{g_3} = y_{g_2}P_{g_1}P_{g_3} = z_{g_3}P_{g_1}P_{g_2} = 0$ and $x_{g_1}y_{g_2}z_{g_3} = 0$. If $a_{g_1} + x_{g_1}$ is unit, $(b_{g_2} + y_{g_2})(c_{g_3} + z_{g_3}) \in P$ and it implies $y_{g_2}z_{g_3} \in P$, a contradiction. So, $a_{g_1} + x_{g_1}$ is not unit. Similar argument shows that $b_{g_2} + y_{g_2}, c_{g_3} + z_{g_3}$ are nonunits. Since P is graded weakly 1-absorbing prime ideal, we have either $(a_{g_1} + x_{g_1})(b_{g_2} + y_{g_2}) \in P$ or $c_{g_3} + z_{g_3} \in P$. Thus, we conclude that $x_{g_1}y_{g_2} \in P$ or $z_{g_3} \in P$ giving a contradiction. Therefore, $P_{g_1}P_{g_2}P_{g_3} = 0$. □

Let R be a G-graded ring. It is clear that for each $g \in G$, R_g is an R_e-module and P_g is an R_e-submodule of R_g.

Theorem 3.13. *Let $P = \bigoplus_{g\in G} P_g$ be a graded 1-absorbing prime ideal of $G(R)$ and $g \in G$. If $x, y \in R_g$ are nonunits such that $xy \notin P$, then $(P_{g^2} :_{R_e} xy) = P_e$.*

Proof. Let $z \in (P_{g^2} :_{R_e} xy)$, where $x, y \in R_g$ are nonunits. Then, $xyz \in P_{g^2} \subseteq P$. If z is a unit, $xy \in P$ which gives a contradiction. So, z is not unit. As P is graded 1-absorbing prime ideal and $xy \notin P$ we get $z \in P$. Thus, $z \in P \cap R_e = P_e$. This shows $(P_{g^2} :_{R_e} xy) \subseteq P_e$.

On the other hand, suppose $z \in P_e \subseteq P$. Then, $xyz \in P \cap R_{g^2} = P_{g^2}$ proving $z \in (P_{g^2} :_{R_e} xy)$, as desired. □

Proposition 3.14. *Let R be a G-graded ring and $J \subseteq I$ be proper graded ideals of R. Then the followings statements are satisfied.*

(i) *If I is graded weakly 1-absorbing prime ideal, then I/J is graded weakly 1-absorbing prime ideal of R/J.*

(ii) *Suppose that J consists of all nilpotent elements of R. If J is a graded weakly 1-absorbing prime ideal of R and I/J is a graded weakly 1-absorbing prime ideal of R/J, then I is a graded weakly 1-absorbing prime ideal of R.*

(iii) *If (0) is graded 1-absorbing prime ideal of R and I is graded weakly 1-absorbing prime ideal of R, then I is graded 1-absorbing prime ideal of R.*

Proof. (i) : Let $0+J \neq (x+J)(y+J)(z+J) \in I/J$ for some nonunits $x+J, y+J, z+J \in h(R/J)$. Then, $0 \neq xyz + J \in I/J$ and so $0 \neq xyz \in I$ where x, y, z are nonunits in $h(R)$. As I is a graded weakly 1-absorbing prime ideal, either $xy \in I$ or $z \in I$. Hence, $xy + J \in I/J$ or $z + J \in I/J$, as desired.

(ii) : Suppose $0 \neq xyz \in I$ for some nonunits $x, y, z \in h(R)$. Then, $xyz+J = (x+J)(y+J)(z+J) \in I/J$. If $xyz \in J$, then $xy \in J \subseteq I$ or $z \in J$ since $J \subseteq I$ is graded weakly 1-absorbing prime ideal. So we can assume $xyz \notin J$. Then we have $0 + J \neq (x + J)(y + J)(z + J) \in I/J$. As I/J is graded weakly 1-absorbing prime ideal of R/J, $(x + J)(y + J) \in I/J$ or $z + J \in I/J$. It implies either $xy \in I$ or $z \in I$.

(iii) : Suppose that $xyz \in I$ for some nonunits $x, y, z \in h(R)$. If $xyz \neq 0$, then we are done. So, we can assume $xyz = 0 \in (0)$. Then, we get either $xy = 0 \in I$ or $z = 0 \in I$ since (0) is graded 1-absorbing prime ideal. Therefore, we conclude that $xy \in I$ or $z \in I$. □

Let R and S be two G-graded rings. A ring homomorphism $f : R \to S$ is said to be graded homomorphism if $f(R_g) \subseteq S_g$ for all $g \in G$.

Theorem 3.15. *Let R_1 and R_2 be two G-graded rings and $f : R_1 \longrightarrow R_2$ be a graded homomorphism such that $f(1_{R_1}) = 1_{R_2}$. The following statements are satisfied.*

(i) *If f is injective, J is a graded weakly 1-absorbing prime ideal of R_2 and $f(x)$ is a nonunit element of R_2 for all nonunit elements $x \in h(R_1)$, then $f^{-1}(J)$ is a graded weakly 1-absorbing prime ideal of R_1.*

(ii) *If f is surjective and I is a graded weakly 1-absorbing prime ideal of R_1 with $\ker(f) \subseteq I$, then $f(I)$ is a graded weakly 1-absorbing prime ideal of R_2.*

Proof. $(i):$ It is clear that $f^{-1}(J)$ is a graded ideal of R_1. Let $0 \neq xyz \in f^{-1}(J)$ for some nonunits x, y, z in $h(R_1)$. So, $f(x), f(y)$ and $f(z)$ are nonunits in $h(R_2)$ by the assumption. Since f is injective and $xyz \neq 0$, we have $f(xyz) \neq 0$. Then we get $0 \neq f(x)f(y)f(z) = f(xyz) \in J$. As J is a graded weakly 1-absorbing prime ideal of R_2, $f(x)f(y) \in J$ or $f(z) \in J$. It implies that we have either $xy \in f^{-1}(J)$ or $z \in f^{-1}(J)$.

$(ii):$ Suppose that $0 \neq abc \in f(I)$ for some nonunits $a, b, c \in h(R_2)$. Then, there exist nonunits $x, y, z \in h(R_1)$ such that $f(x) = a, f(y) = b$ and $f(z) = c$. It gives that $0 \neq f(x)f(y)f(z) = abc \in f(I)$. So, there exists $i \in I$ such that $f(xyz) = f(i)$. This means $xyz - i \in \ker(f) \subseteq I$ giving $xyz \in I$. Since I is a graded weakly 1-absorbing prime ideal and $0 \neq xyz \in I$, we conclude that $xy \in I$ or $z \in I$. It shows $f(x)f(y) = ab \in f(I)$ or $f(z) = c \in f(I)$, as needed. □

Let $S \subseteq h(R)$ be a multiplicative set and R be a G-graded ring. Then $S^{-1}R$ is a G-graded ring with $(S^{-1}R)_g = \left\{\frac{a}{s} : a \in R_h, s \in S \cap R_{hg^{-1}}\right\}$. Let I be a graded ideal of R. Then we denote the set $\{a \in R : ab \in I \text{ for some } b \in R - I\}$ by $Z_I(R)$.

Theorem 3.16. *Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset. The following statements are satisfied.*

(i) *If I is a graded weakly 1-absorbing prime ideal of R with $I \cap S = \emptyset$, then $S^{-1}I$ is a graded weakly 1-absorbing prime ideal of $S^{-1}R$.*

(ii) *If $S^{-1}I$ is a graded weakly 1-absorbing prime ideal of $S^{-1}R$, $u(S^{-1}R) = \{\frac{x}{s} : x \in u(R), s \in S\}$, $S \subseteq reg(R)$ and $S \cap Z_I(R) = \emptyset$, then I is a graded weakly 1-absorbing prime ideal of R.*

Proof. $(i):$ Suppose that $0 \neq \frac{x}{s}\frac{y}{t}\frac{z}{u} \in S^{-1}I$ for some nonunits $\frac{x}{s}, \frac{y}{t}, \frac{z}{u} \in h(S^{-1}R)$. Then $0 \neq a(xyz) = (ax)yz \in I$ for some $a \in S$. Here, ax, y, z are nonunits in $h(R)$. Otherwise, we would have $\frac{x}{s}, \frac{y}{t}, \frac{z}{u}$ are units in $S^{-1}R$, a contradiction. As I is a graded weakly 1-absorbing prime ideal of R, we have either $axy \in I$ or $z \in I$. This implies that $\frac{x}{s}\frac{y}{t} = \frac{axy}{ast} \in S^{-1}I$ or $\frac{z}{u} \in S^{-1}I$. Thus, $S^{-1}I$ is a graded weakly 1-absorbing prime ideal of $S^{-1}R$.

$(ii):$ Let $0 \neq xyz \in I$ for some nonunits $x, y, z \in h(R)$. Since $S \subseteq reg(R)$, we conclude that $0 \neq \frac{x}{1}\frac{y}{1}\frac{z}{1} \in S^{-1}I$. Here, $\frac{x}{1}, \frac{y}{1}, \frac{z}{1}$ are nonunits in $h(S^{-1}R)$. Since $S^{-1}I$ is a graded weakly 1-absorbing prime ideal of $S^{-1}R$, we conclude either $\frac{x}{1}\frac{y}{1} = \frac{xy}{1} \in S^{-1}I$ or $\frac{z}{1} \in S^{-1}I$. Then there exists $s \in S$ such that $sxy \in I$ or $sz \in I$. We can assume that $sxy \in I$. If $xy \notin I$, then we have $s \in Z_I(R) \cap S$ which is a contradiction. Thus we have $xy \in I$. In other case, similarly, we get $z \in I$. Therefore, I is a graded weakly 1-absorbing prime ideal of R. □

Theorem 3.17. *Let $P = \bigoplus_{g \in G} P_g$ be a graded weakly 1-absorbing prime ideal of R and $g \in G$. Then, $(P_{g^2} :_{R_e} xy) = P_e \cup (0 :_{R_e} xy)$ where $x, y \in R_g$ are nonunits such that $xy \notin P$.*

Proof. Clearly $(0 :_{R_e} xy) \subseteq (P_{g^2} :_{R_e} xy)$. Let $z \in P_e \subseteq P$. This implies that $xyz \in P \cap R_{g^2} = P_{g^2}$ and so $z \in (P_{g^2} :_{R_e} xy)$. Hence, $P_e \cup (0 :_{R_e} xy) \subseteq (P_{g^2} :_{R_e} xy)$. Now, we will show that $(P_{g^2} :_{R_e} xy) \subseteq (0 :_{R_e} xy) \cup P_e$. Let $z \in (P_{g^2} :_{R_e} xy)$. Then, we have $xyz \in P_{g^2} \subseteq P$. If z is a unit, then we have $xy \in P$, a contradiction. Suppose that z is a nonunit of R. If $xyz \neq 0$, then $z \in P \cap R_e = P_e$. So assume that $xyz = 0$. It gives $z \in (0 :_{R_e} xy)$. Thus we have $z \in P_e \cup (0 :_{R_e} xy)$. Therefore, $(P_{g^2} :_{R_e} xy) = P_e \cup (0 :_{R_e} xy)$. □

Let $R = \bigoplus_{g \in G} R_g$ be a graded ring. Recall from [18] that R is said to be a graded field if every nonzero homogenous element is a unit in R.

Theorem 3.18. *Suppose that R_1, R_2 be two G-graded commutative rings that are not graded fields and $R = R_1 \times R_2$. Let P be a nonzero proper graded ideal of R. The following statements are equivalent.*

- (i) *P is a graded weakly 1-absorbing prime ideal of R.*
- (ii) *$P = P_1 \times R_2$ for some graded prime ideal P_1 of R_1 or $P = R_1 \times P_2$ for some graded prime ideal P_2 of R_2.*
- (iii) *P is a graded prime ideal of R.*
- (iv) *P is a graded weakly prime ideal of R.*
- (v) *P is a graded 1-absorbing prime ideal of R.*

Proof. $(i) \Rightarrow (ii):$ Let P be a nonzero proper graded ideal of R. Then we can write $P = P_1 \times P_2$ for some graded ideals P_1 of R_1 and P_2 of R_2. Since P is nonzero, $P_1 \neq 0$ or $P_2 \neq 0$. Without loss of generality, we may assume that $P_1 \neq 0$. Then there exists a homogeneous element $0 \neq x \in P_1$. Since P is a graded weakly 1-absorbing prime ideal and $(0,0) \neq (1,0)(1,0)(x,1) \in P$, we conclude either $(1,0) \in P$ or $(x,1) \in P$. Then we have either $P_1 = R_1$ or $P_2 = R_2$. Assume that $P_1 = R_1$. Now we will show that P_2 is a graded prime ideal of R_2. Let $yz \in P_2$ for some $y, z \in h(R_2)$. If y or z is a unit, then we have either $y \in P_2$ or $z \in P_2$. So assume that y, z are nonunits in $h(R_2)$. Since R_1 is not a graded field, there exists a nonzero nonunit $t \in h(R_1)$. This implies that $(0,0) \neq (t,1)(1,y)(1,z) = (t, yz) \in P$. As P is a graded weakly 1-absorbing prime ideal of R, we conclude either $(t,1)(1,y) = (t,y) \in P$ or $(1,z) \in P$. Thus we get $y \in P_2$ or $z \in P_2$ and so P_2 is a graded prime ideal of R_2. In other case, one can similarly show that $P = P_1 \times R_2$ and P_1 is a graded prime ideal of R_1.

$(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i):$ It is obvious.

$(iii) \Rightarrow (v):$ It is clear.

$(v) \Rightarrow (i):$ It is straightforward. □

Definition 3.19. *Let R be a ring and M be an R-module. A proper submodule N of M is called a 1-absorbing R-submodule if whenever $xym \in N$ where $x, y \in R$ are nonunits, $m \in M$, then either $xy \in (N :_R M)$ or $m \in N$.*

Theorem 3.20. *Let $P = \bigoplus_{g\in G} P_g$ be a graded 1-absorbing prime ideal of $G(R)$. If $P_g \neq R_g$, then P_g is a 1-absorbing R_e-submodule of R_g.*

Proof. Let $xyr \in P_g \subseteq P$ for some nonunits $x, y \in R_e$ and $r \in R_g$. As P is graded 1-absorbing prime ideal, $xy \in P$ or $r \in P$. This implies that $xy \in (P_g :_{R_e} R_g)$ since $xyR_g \subseteq PR_g \subseteq P \cap R_g = P_g$ or $r \in P \cap R_g = P_g$. □

Definition 3.21. *Let $P = \bigoplus_{g\in G} P_g$ be a graded ideal of $G(R)$. A graded component P_g of P is called 1-absorbing prime subgroup of R_g if $xyz \in P_g$ for some nonunits $x, y, x \in h(R)$ implies either $xy \in P_g$ or $z \in P_g$.*

Proposition 3.22. *Let $P = \bigoplus_{g\in G} P_g$ be a graded ideal of $G(R)$. If P_g is a 1-absorbing prime subgroup of R_g for all $g \in G$, then P is a graded 1-absorbing prime ideal of R.*

Proof. Suppose $xyz \in P$ for some nonunits $x, y, z \in h(R)$. Then, $xyz \in P_g$ for some $g \in G$. Since P_g is 1-absorbing prime subgroup of R_g, $xy \in P_g$ or $z \in P_g$. This gives $xy \in P$ or $z \in P$, as needed. □

Let M be an R-module. The idealization $R \ltimes M = \{(r, m) : r \in R \text{ and } m \in M\}$ of M is a commutative ring with componentwise addition and multiplication: $(x, m_1)+(y, m_2) = (x+y, m_1+m_2)$ and $(x, m_1)(y, m_2) = (xy, xm_2 + ym_1)$ for each $x, y \in R$ and $m_1, m_2 \in M$. Let G be an Abelian group and M be a G-graded R-module. Then $X = R \ltimes M$ is a G-graded ring by $X_g = R_g \ltimes M_g = R_g \oplus M_g$ for all $g \in G$. Note that, X_g is an additive subgroup of X for all $g \in G$. Also, for $g, h \in G$, $X_g X_h = (R_g \ltimes M_g)(R_h \ltimes M_h) = R_g R_h \ltimes (R_g M_h + R_h M_g) \subseteq R_{gh} \ltimes (M_{gh} + M_{hg}) \subseteq R_{gh} \ltimes M_{gh} = X_{gh}$ as G is Abelian (see [2, 17]).

Theorem 3.23. *Let G be an Abelian group, M be a G-graded R-module and P be an ideal of R. Then, the following statements are equivalent.*

(i) *$P \ltimes M$ is a graded weakly 1-absorbing prime ideal of $R \ltimes M$.*

(ii) *P is a graded weakly 1-absorbing prime ideal of R and if $x_{g_1} y_{g_2} z_{g_3} = 0$ such that $x_{g_1} y_{g_2} \notin P$ and $z_{g_3} \notin P$ for some nonunit elements $x_{g_1}, y_{g_2}, z_{g_3}$ in $h(R)$, where $g_1, g_2, g_3 \in G$, then $x_{g_1} y_{g_2} M_{g_3} = x_{g_1} z_{g_3} M_{g_2} = y_{g_2} z_{g_3} M_{g_1} = 0$.*

Proof. $(i) \Rightarrow (ii)$: By [17, Theorem 3.3], P is a graded ideal of R. Suppose that $0 \neq abc \in P$ where a, b, c are nonunits in $h(R)$. Since $(0, 0) \neq (a, 0)(b, 0)(c, 0) \in P \ltimes M$ and $(a, 0), (b, 0), (c, 0)$

are nonunits in $h(R \ltimes M)$, we get $(a,0)(b,0) \in P \ltimes M$ or $(c,0) \in P \ltimes M$. Thus, we conclude that $ab \in P$ or $c \in P$, as needed. Now suppose $x_{g_1}y_{g_2}z_{g_3} = 0$ such that $x_{g_1}y_{g_2} \notin P$ and $z_{g_3} \notin P$ for some nonunit elements $x_{g_1}, y_{g_2}, z_{g_3}$ in $h(R)$, where $g_1, g_2, g_3 \in G$. Let $x_{g_1}y_{g_2}M_{g_3} \neq 0$. Then there exists $m_{g_3} \in M_{g_3}$ such that $x_{g_1}y_{g_2}m_{g_3} \neq 0$. This gives $(0,0) \neq (x_{g_1},0)(y_{g_2},0)(z_{g_3},m_{g_3}) = (0, x_{g_1}y_{g_2}m_{g_3}) \in P \ltimes M$ for some nonunits $(x_{g_1},0)$, $(y_{g_2},0)$, $(z_{g_3},m_{g_3}) \in h(R \ltimes M)$ and $P \ltimes M$ is a graded weakly 1-absorbing prime ideal, we have $(x_{g_1},0)(y_{g_2},0) = (x_{g_1}y_{g_2},0) \in P \ltimes M$ or $(z_{g_3},m_{g_3}) \in P \ltimes M$. This gives $x_{g_1}y_{g_2} \in P$ or $z_{g_3} \in P$, a contradiction. Hence, $x_{g_1}y_{g_2}M_{g_3} = 0$. Similar argument shows that $x_{g_1}z_{g_3}M_{g_2} = y_{g_2}z_{g_3}M_{g_1} = 0$.

$(ii) \Rightarrow (i):$ By [17, Theorem 3.3], $P \ltimes M$ is a graded ideal of $R \ltimes M$. Assume that $(0,0) \neq (x_{g_1},m_{g_1})(y_{g_2},m_{g_2})(z_{g_3},m_{g_3}) = (x_{g_1}y_{g_2}z_{g_3}, x_{g_1}y_{g_2}m_{g_3} + x_{g_1}z_{g_3}m_{g_2} + y_{g_2}z_{g_3}m_{g_1}) \in P \ltimes M$ for some nonunits $(x_{g_1},m_{g_1}),(y_{g_2},m_{g_2}),(z_{g_3},m_{g_3})$ in $h(R \ltimes M)$. Then we get $x_{g_1}y_{g_2}z_{g_3} \in P$ for some nonunits $x_{g_1}, y_{g_2}, z_{g_3} \in h(R)$.

Case 1: Assume that $x_{g_1}y_{g_2}z_{g_3} = 0$. If $x_{g_1}y_{g_2} \notin P$ and $z_{g_3} \notin P$, we have $x_{g_1}y_{g_2}M_{g_3} = x_{g_1}z_{g_3}M_{g_2} = y_{g_2}z_{g_3}M_{g_1} = 0$. This implies that $x_{g_1}y_{g_2}m_{g_3} + x_{g_1}z_{g_3}m_{g_2} + y_{g_2}z_{g_3}m_{g_1} = 0$ and so $(x_{g_1},m_{g_1})(y_{g_2},m_{g_2})(z_{g_3},m_{g_3}) = (0,0)$ giving a contradiction. Hence, we must have $x_{g_1}y_{g_2} \in P$ or $z_{g_3} \in P$. This gives $(x_{g_1},m_{g_1})(y_{g_2},m_{g_2}) \in P \ltimes M$ or $(z_{g_3},m_{g_3}) \in P \ltimes M$.

Case 2: Now, assume that $x_{g_1}y_{g_2}z_{g_3} \neq 0$. This gives $x_{g_1}y_{g_2} \in P$ or $z_{g_3} \in P$ since P is graded weakly 1-absorbing prime ideal. Then we conclude that $(x_{g_1},m_{g_1})(y_{g_2},m_{g_2}) \in P \ltimes M$ or $(z_{g_3},m_{g_3}) \in P \ltimes M$ which completes the proof. □

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On Severi varieties as intersections of a minimum number of quadrics

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ABSTRACT

Let $\mathscr{V}$ be a variety related to the second row of the Freudenthal-Tits Magic square in N-dimensional projective space over an arbitrary field. We show that there exist $M \leq N$ quadrics intersecting precisely in $\mathscr{V}$ if and only if there exists a subspace of projective dimension $N-M$ in the secant variety disjoint from the Severi variety. We present some examples of such subspaces of relatively large dimension. In particular, over the real numbers we show that the Cartan variety (related to the exceptional group $\mathsf{E}_6(\mathbb{R})$) is the set-theoretic intersection of 15 quadrics.

RESUMEN

Sea $\mathscr{V}$ una variedad relacionada a la segunda fila del cuadrado Mágico de Freudenthal-Tits en el espacio proyectivo N-dimensional sobre un cuerpo arbitrario. Mostramos que existen $M \leq N$ cuádricas intersectandose precisamente en $\mathscr{V}$ si y solo si existe un subespacio de dimensión proyectiva $N-M$ en la variedad secante disjunta de la variedad de Severi. Presentamos algunos ejemplos de tales subespacios de dimensión relativamente grande. En particular, sobre los números reales, mostramos que la variedad de Cartan (relacionada al grupo excepcional $\mathsf{E}_6(\mathbb{R})$) es la intersección conjuntista de 15 cuádricas.

Keywords and Phrases: Cartan variety, quadrics, exceptional geometry, Severi variety, quaternion veronesian.

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1 Introduction

It is well known that the Grassmannians of the (split) spherical buildings related to semi-simple algebraic groups over algebraically closed fields can be described as the intersection of a number of quadrics, see [7] for the complex case, and [3] and [10] for the more general case. In this paper, we consider the Grassmannians (or "varieties") related to the second row of the Freudenthal-Tits Magic square. Over the complex numbers, these are the so-called "Severi varieties". However, these can be considered over any field $\mathbb{K}$ (not necessarily algebraically closed anymore), and these geometries will be also called Severi varieties. A Severi variety lives in a projective space of dimension $N = 5$, 8, 14 or 26 and is the set-theoretic and scheme-theoretic intersection of $N + 1$ quadrics, the equations of which carry a particularly elegant combinatorics, see [11]. The question we'd like to put forward in this paper is whether we can describe the Severi varieties set-theoretically with fewer quadrics, and ultimately try to find the minimum number of quadrics the intersection of which is precisely the given Severi variety. Our motivation is entirely curiosity and beauty; the latter under the form of a rather unexpected connection we found.

We will show that the $N + 1$ quadrics referred to above are linearly independent from each other. Also, every quadric containing the given Severi variety is a linear combination of these $N + 1$ quadrics. These two facts point, in our opinion, to the conjecture that no set of N quadrics can intersect precisely in the Severi variety. However, the quadric Veronese surface (the case $N = 5$ Severi variety) over fields of characteristic 2 is the set-theoretic intersection of three quadrics, see Lemma 4.20 in [6]. Moreover, it was stated in [2], however without proof, that in the case $N = 8$, the Severi variety is the set-theoretic intersection of only 6 quadrics. Hence the above conjecture is false. In general, we will show the following equivalence:

Main Result. *There exist $M \leq N$ quadrics intersecting precisely in the given Severi variety $\Longleftrightarrow$ there exists a subspace of projective dimension $N - M$ in the secant variety disjoint from the Severi variety.*

A more detailed and precise statement will be provided in Section 3. In fact, that statement and its proof allow one, in principle, to describe all equivalence classes of systems of $M \leq N$ quadratic equations exactly describing a given Severi variety. As an application, we will do this explicitly in the simplest case, $N = 5$. For the other cases we content ourselves with giving examples for relatively small M. In particular we will exhibit the real Cartan variety (the Grassmanian of type $\mathsf{E}_{6,1}$ in 26-dimensional real projective space) as the intersection of only 15 quadrics (whereas initially, we had 27 of them). It would require additional methods and ideas to pin down the minimal M for each case and each field, so we consider that to be out of the scope of this paper.

About the method of our proof: Usually, the equations of the $N + 1$ initial quadrics are partial derivatives of a cubic form (which has to be taken for granted). In the present paper, we start with the combinatorics of the equations of the quadrics and derive the cubic form from that. This enables us to make a few geometric observations and interpretations which lead to a proof of the Main Result.

Since the secant variety of a Severi variety always contains at least one point outside the variety, we recover in our special case of Severi variety already the general result of Kronecker saying that any projective variety in $\mathbb{P}^N_{\mathbb{K}}$ is a set theoretic intersection of (at most) N hypersurfaces (in our case quadrics), see Corollary 2 in [5]. One could also ask the equivalent question for the scheme-theoretic intersection of quadrics, but we did not consider that. It seems to us that the answer we give in the present paper for the Segre variety is also valid in the scheme-theoretic sense, but the minimal examples for the line Grassmannian and the Cartan variety are not.

2 Preliminaries

2.1 The varieties

The main objects in this paper are the *quadric Veronese surface* $\mathscr{V}_2(\mathbb{K})$ over any field $\mathbb{K}$, the *Segre variety* $\mathscr{S}_{2,2}(\mathbb{K})$ corresponding to the product of two projective planes over $\mathbb{K}$, the *line Grassmannian* $\mathscr{G}_{2,6}(\mathbb{K})$ of projective 5-space over $\mathbb{K}$, and the *Cartan variety* $\mathscr{E}_6(\mathbb{K})$ associated to the 27-dimensional module of the (split) exceptional group of Lie type E_6 over the field $\mathbb{K}$. These varieties can be defined as intersections of quadrics (and we will do so in Subsection 4.1 below), but it might be insightful to also have the classical definition, which we now present. In what follows, $\mathbb{K}$ is an arbitrary field and $\mathbb{P}^N_{\mathbb{K}}$ or $\mathbb{P}^N$ denotes the N-dimensional projective space over $\mathbb{K}$, which we suppose to be coordinatized with homogeneous coordinates from $\mathbb{K}$ after an arbitrary choice of a basis.

The quadric Veronese surface $\mathscr{V}_2(\mathbb{K})$—This is the image of the *Veronese map* $\nu : \mathbb{P}^2 \to \mathbb{P}^5 : (x,y,z) \mapsto (x^2,y^2,z^2,yz,zx,xy)$.

The Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$—This is the image of the *Segre map* $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8 : (x,y,z;u,v,w) \mapsto (xu,yu,zu,xv,yv,zv,xw,yw,zw)$.

We may view the set of 3×3 matrices over $\mathbb{K}$ as a 9-dimensional vector space, and the set of symmetric 3×3 matrices as a 6-dimensional subspace. Then we may consider the corresponding projective spaces of (projective) dimension 8 and 5, respectively, in the classical way by considering the 1-spaces as the points. In this way, the Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$ corresponds exactly with the rank 1 matrices; explicitly

$$\mathbb{K}(xu,yu,zu,xv,yv,zv,xw,yw,zw) \leftrightarrow \mathbb{K}\begin{pmatrix} xu & yu & zu \\ xv & yv & zv \\ xw & yw & zw \end{pmatrix}.$$

Similarly, the quadric Veronese surface $\mathscr{V}_2(\mathbb{K})$ corresponds exactly with the rank 1 symmetric matrices; explicitly

$$\mathbb{K}(x^2, y^2, z^2, yz, zx, xy) \leftrightarrow \mathbb{K}\begin{pmatrix} x^2 & yx & zx \\ xy & y^2 & zy \\ xz & yz & z^2 \end{pmatrix}.$$

In particular, $\mathscr{V}_2(\mathbb{K})$ is a subvariety of $\mathscr{S}_{2,2}(\mathbb{K})$ obtained by intersecting with a 5-dimensional subspace.

There exist other Segre varieties; in general $\mathscr{S}_{n,m}(\mathbb{K})$ is defined as the image in $\mathbb{P}^{nm-1}$ of the map $(x_i, y_j)_{1\leq i\leq n, 1\leq j\leq m} \mapsto (x_iy_j)_{1\leq i\leq n,1\leq j\leq m}$. The images of the marginal maps defined by either fixing the x_i, $1 \leq i \leq n$, or the y_j, $1 \leq j \leq m$, are called the *generators* of the variety (in case of $\mathscr{S}_{2,2}(\mathbb{K})$ the generators are 2-dimensional projective subspaces).

The line Grassmannian $\mathscr{G}_{2,6}(\mathbb{K})$—Denote the set of lines of $\mathbb{P}^5$, or equivalently, the set of 2-spaces of $\mathbb{K}^6$ by $\binom{\mathbb{K}^6}{\mathbb{K}^2}$. Then $\mathscr{G}_{2,6}(\mathbb{K})$ is the image of the Plücker map

$$\binom{\mathbb{K}^6}{\mathbb{K}^2} \to \mathbb{P}^{14} : \langle (x_1, x_2, \ldots, x_6).(y_1, y_2, \ldots, y_6)\rangle \mapsto (x_iy_j - x_jy_i)_{1\leq i<j\leq 6}.$$

Denote the coordinate of $\mathbb{P}^{14}$ corresponding to the entry $x_iy_j - x_jy_i$ by p_{ij}, $1 \leq i < j \leq 6$. By restricting to $y_1 = y_2 = y_3 = x_4 = x_5 = x_6 = 0$, we see that $\mathscr{S}_{2,2}(\mathbb{K})$ is a subvariety of $\mathscr{G}_{2,6}(\mathbb{K})$ obtained by intersecting with an 8-dimensional projective subspace with equation $p_{12} = p_{13} = p_{23} = p_{45} = p_{46} = p_{56} = 0$.

The Cartan variety $\mathscr{E}_6(\mathbb{K})$—This variety is traditionally defined using a trilinear or cubic form, and we postpone this to Subsection 4.1. It is an exceptional variety in the sense that it cannot be defined, using classical notions like Plücker or Grassmann coordinates, from a projective space.

The above varieties share the following properties, see [9]. Set $N = 2+3M$, with $M = 1, 2, 4, 8$. Let $\mathscr{V}$ be one of the varieties $\mathscr{V}_2(\mathbb{K})$, $\mathscr{S}_{2,2}(\mathbb{K})$, $\mathscr{G}_{2,6}(\mathbb{K})$ or $\mathscr{E}_6(\mathbb{K})$, in $\mathbb{P}^N$, with $M = 1, 2, 4, 8$, respectively. Then there exists a unique set $\mathscr{H}$ of $(M + 1)$-dimensional subspaces, called *host spaces*, satisfying

(1) every pair of points of $\mathscr{V}$ is contained in at least one host space;

(2) the intersection of $\mathscr{V}$ with any host space is a non-degenerate quadric of maximal Witt index in the host space.

Borrowing some terminology from the theory of parapolar spaces, we shall refer to the quadrics in (2) as *symps*. Also, we shall call two points of the variety *collinear* when all points of the joining projective line belong to the variety.

If we specialize $\mathbb{K} = \mathbb{C}$, then $\mathscr{V}$ is sometimes called a *Severi variety*; these are the only complex varieties with the property that their secant varieties are not the whole projective space, but the secant variety of every variety of the same dimension in a lower dimensional projective space coincides with the ambient space. So we will also refer to these varieties over an arbitrary field as the *Severi varieties*.

2.2 A generalized quadrangle

The introduction of an appropriate cubic form and explicit descriptions using coordinates will be greatly facilitated by using the language of finite generalized quadrangles. A finite *generalized quadrangle (of order* (s,t)*)* is an incidence system $\Gamma = (\mathscr{P}, \mathscr{L})$ of finitely many points ($\mathscr{P}$) and lines ($\mathscr{L}$), where each line is a subset of $\mathscr{P}$, such that each line contains $s+1$ points, through each point pass $t+1$ lines, and for each point p and each line L with $p \notin L$, there exists a unique point-line pair (q, M) such that $p \in M$ and $q \in L \cap M$. We are only interested in generalized quadrangles of order $(2,t)$, and then, by 1.2.2 and 1.2.3 of [8], necessarily $t \in \{1,2,4\}$. Moreover, by 5.2.3 and 5.3.2 of [8], for each $t \in \{1,2,4\}$, there is a unique generalized quadrangle $\mathsf{GQ}(2,t)$ of order $(2,t)$ and $\mathsf{GQ}(2,1)$ is contained in $\mathsf{GQ}(2,2)$ as a subgeometry, and $\mathsf{GQ}(2,2)$ is contained in $\mathsf{GQ}(2,4)$ as a subgeometry.

In the rest of this paper, we denote by $\Gamma = (\mathscr{P}, \mathscr{L})$ the generalized quadrangle $\mathsf{GQ}(2,4)$. An explicit construction of Γ runs as follows, see Section 6.1 of [8]. Let $\mathscr{P}'$ be the set of all 2-subsets of the 6-set $\{1,2,3,4,5,6,\}$, and define

$$\mathscr{P} = \mathscr{P}' \cup \{1,2,3,4,5,6\} \cup \{1',2',3',4',5',6'\}.$$

Denote briefly the 2-subset $\{i,j\}$ by ij, for all appropriate i,j. Let $\mathscr{L}'$ be the set of partitions of $\{1,2,3,4,5,6\}$ into 2-subsets and define

$$\mathscr{L} = \mathscr{L}' \cup \{\{i, j', ij\} \mid i,j \in \{1,2,3,4,5,6\}, i \neq j\}.$$

Then $\Gamma = (\mathscr{P}, \mathscr{L})$ is a model of $\mathsf{GQ}(2,4)$. The subgeometry $\Gamma' = (\mathscr{P}', \mathscr{L}')$ is a model of $\mathsf{GQ}(2,2)$. Further restriction to

$$\mathscr{P}'' = \{ij \mid i \in \{1,2,3\}, j \in \{4,5,6\}\},$$

with induced line set

$$\mathscr{L}'' = \{\{14,25,36\},\{15,26,34\},\{16,24,35\},\{14,26,35\},\{15,24,36\},\{16,25,34\}\},$$

produces a model $\Gamma'' = (\mathscr{P}'', \mathscr{L}'')$ of $\mathsf{GQ}(2,1)$, which we sometimes refer to as a 3×3 grid.

The sets $\{1,2,3,4,5,6\}$ and $\{1',2',3',4',5',6'\}$ have the property that they both do not contain any pair of collinear points, and that non-collinearity is a paring between the two sets. Such a pair of 6-sets is usually called a *double six*.

Finally, we need the notion of a *partial spread*, which is just a set of disjoint lines. A *spread* is a partial spread that partitions the point set. Every generalized quadrangle of order $(2,t)$, $t = 1,2,4$, satisfies the following property (again, see Section 6.1 of [8]):

(∗) *Every pair of disjoint lines is contained in a unique generalized subquadrangle of order* $(2,1)$

Three mutually disjoint lines of a subquadrangle of order $(2,1)$ will be called a *regulus*. Property (∗) can be reformulated as "every pair of disjoint lines is contained in a unique regulus". A partial spread which is closed under taking reguli of pairs of its members is called *regular*. The $\mathsf{GQ}(2,4)$ contains regular spreads; a maximum regular partial spread of $\mathsf{GQ}(2,2)$ has size 3, and obviously the $\mathsf{GQ}(2,1)$ contains exactly two regular spreads. In this paper, we will fix the following regular spread $\mathscr{S}$ of Γ, which induces maximum regular partial spreads in Γ' and Γ'':

$$\begin{aligned}\mathscr{S} &= \{\{14,25,36\},\{15,26,34\},\{16,24,35\},\{12,2,1'\},\{23,3,2'\},\{13,1,3'\},\\ &\quad \{45,4,5'\},\{56,5,6'\},\{46,6,4'\}\}.\end{aligned}$$

The lines $\{14,25,36\},\{15,26,34\},\{16,24,35\}$ form a maximum regular partial spread in both Γ' and Γ''.

3 Main result

In this paper, we prove the following connection between the minimum number of quadrics needed to describe a Severi variety and the largest dimension of a projective subspace in the secant variety disjoint from the variety itself.

Theorem 3.1. *Let $\mathscr{V}$ be either the quadratic Veronese surface $\mathscr{V}_2(\mathbb{K})$, the Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$, the line Grassmannian $\mathscr{G}_{2,6}(\mathbb{K})$, or the Cartan variety $\mathscr{E}_6(\mathbb{K})$, in N-dimensional projective space $\mathbb{P}^N$ over $\mathbb{K}$, with $N=5,8,14,26$, respectively. Then $\mathscr{V}$ is the intersection of $N-d$ quadrics and no less, where d is the dimension of a maximum dimensional projective subspace of $\mathbb{P}^N$ entirely consisting of points lying on a secant of $\mathscr{V}$, or in the nucleus plane if $\mathscr{V}=\mathscr{V}_2(\mathbb{K})$ with $\operatorname{char}\mathbb{K}=2$, but not on $\mathscr{V}$. More precisely, the equivalence classes of the systems of $N-d$ linearly independent quadrics intersecting precisely in $\mathscr{V}$ are in natural bijective correspondence with the d-dimensional projective subspaces of $\mathbb{P}^N$ entirely consisting of points lying on a secant of $\mathscr{V}$, or in the nucleus plane if $\mathscr{V}=\mathscr{V}_2(\mathbb{K})$ with $\operatorname{char}\mathbb{K}=2$, but not on $\mathscr{V}$.*

To fix the ideas, we provide a full proof for the variety $\mathscr{E}_6(\mathbb{K})$. The other cases are completely similar. We comment on them along the way, if differences arise.

4 Proof of Theorem 3.1

4.1 A cubic form

The Cartan variety $\mathscr{E}_6(\mathbb{K})$ is the intersection of 27 well chosen degenerate quadrics. The equations of these quadrics can be described as follows. Let $\mathbb{K}^{27}$ be the vector space underlying $\mathbb{P}^{26}$ and denote by $\langle v\rangle$ the point of $\mathbb{P}^{26}$ corresponding to the nonzero vector $v \in \mathbb{K}^{27}$. Recall that $\Gamma = (\mathscr{P}, \mathscr{L})$ is the generalized quadrangle of order $(2,4)$ and $\mathscr{S}$ is a regular spread of Γ. Label the standard basis vectors of $\mathbb{K}^{27}$ with the points of Γ; so the standard basis is $\{e_p : p \in \mathscr{P}\}$. Each point $p \in \mathscr{P}$ defines a unique quadratic form Q_p given in coordinates by

$$Q_p(v) = X_{q_1}X_{q_2} - \sum_{\{p,r_1,r_2\}\in\mathscr{L}\setminus\mathscr{S}} X_{r_1}X_{r_2},$$

where $\{p, q_1, q_2\} \in \mathscr{S}$. Now define the map $\phi : \mathbb{K}^{27} \to \mathbb{K}^{27} : v \mapsto (Q_p(v))_{p\in\mathscr{P}}$. Our basic observation is the following identity.

Observation 4.1. *For all $v \in \mathbb{K}^{27}$ we have $\phi(\phi(v)) = C(v)v$, where*

$$C(v) = \sum_{\{p,q,r\}\in\mathscr{S}} X_pX_qX_r - \sum_{\{p,q,r\}\in\mathscr{L}\setminus\mathscr{S}} X_pX_qX_r.$$

Also, $\phi(v) = \nabla C(v)$ (the gradient in the classical sense).

Proof. The last assertion is obvious. We show the first one. We have to prove the following identity for each point $p \in \mathscr{P}$:

$$Q_{q_1}(v)Q_{q_2}(v) - \sum_{\{p,r_1,r_2\}\in\mathscr{L}\setminus\mathscr{S}} Q_{r_1}(v)Q_{r_2}(v) \quad = \quad C(v)X_p, \tag{4.1}$$

where $\{p, q_1, q_2\} \in \mathscr{S}$ and $v = (X_q)_{q\in\mathscr{P}}$. Since each $Q_q(v)$, $q \in \mathscr{P}$, has five terms of degree 2 in the coordinates of v, the above sum has 125 terms of degree 4. Since each $Q_q(v)$ has a unique term containing X_p, there are five terms containing X_p^2 and another 40 containing X_p but not X_p^2. The terms with X_p^2 are easily seen to be

$$X_p^2X_{q_1}X_{q_2} \quad - \sum_{\{p,r_1,r_2\}\in\mathscr{L}\setminus\mathscr{S}} X_p^2X_{r_1}X_{r_2}. \tag{4.2}$$

For each line $\{q_1, s, s'\} \in \mathscr{L}$, we have the combined terms $X_pX_{q_1}$ of $Q_{q_2}(v)$ and $-X_sX_{s'}$ of $Q_{q_1}(v)$, resulting in a term $-X_pX_{q_1}X_sX_{s'}$ in the left hand side of Equation (4.1). Note that $\{q_1, s, s'\} \notin \mathscr{S}$. Similarly for the lines through q_2. We conclude that the terms of $Q_{q_1}Q_{q_2}$ containing X_p but not X_p^2 are given by

$$- \sum_{i=1}^{2} \sum_{\{q_i,s,s'\}\in\mathscr{L}\setminus\mathscr{S}} X_pX_{q_i}X_sX_{s'}. \tag{4.3}$$

Now let $r \in \mathscr{P}$ be collinear to p but distinct from q_1 and q_2. Let $\{r, s, s'\} \in \mathscr{L}$, with $p \notin \{s, s'\}$. First suppose that $\{r, s, s'\} \in \mathscr{S}$. Let $r' \in \mathscr{P}$ be such that $\{p, r, r'\} \in \mathscr{L} \setminus \mathscr{S}$. Then have the combined terms $-X_pX_r$ of $Q_{r'}(v)$ and $X_sX_{s'}$ of $Q_{q_1}(v)$, resulting in a term $-X_pX_rX_sX_{s'}$ in the left hand side of Equation (4.1). If $\{r, s, s'\} \in \mathscr{L} \setminus \mathscr{S}$, then we obtain the same term, but with the opposite sign. These terms, together with those of Expressions (4.2) and (4.3) already provide the full right hand side of Equality (4.1). The remaining $125-5-40=80$ terms in the left hand side of Equality (4.1) should now cancel pairwise. Disregarding the signs, they are all of the form $X_{s_1}X_{s_1'}X_{s_2}X_{s_2'}$, where $\{r_i,s_i,s_i'\}\in\mathscr{L}$, $i=1,2$, with $\{p,r_1,r_2\}\in\mathscr{L}$. These three lines are contained in a unique grid

$$\begin{pmatrix} p & r_1 & r_2\\ q & s_1 & s_2\\ q' & s_1' & s_2'\end{pmatrix},\quad \text{for some } q,q'\in\mathscr{P},$$

where the rows and columns correspond to lines of Γ. Hence in the term $Q_q(v)Q_{q'}(v)$ also appears a term $X_{s_1}X_{s_2}X_{s_1'}X_{s_2'}$, up to sign. We now have to see that the signs are opposite. If $\{p,r_1,r_2\}\in\mathscr{S}$, then both signs are $+$, but the terms nevertheless cancel since $Q_q(v)Q_{q'}(v)$ appears with a minus sign in Equality (4.1). Note that it does not make any difference whether $\{q,s_1,s_2\}\in\mathscr{S}$ or not, since, by the regularity property of $\mathscr{S}$ we have $\{q,s_1,s_2\}\in\mathscr{S}$ if and only if $\{q',s_1',s_2'\}\in\mathscr{S}$.

Now suppose $\{p,r_1,r_2\}\in\mathscr{L}\setminus\mathscr{S}$. We may also assume that $\{p,q,q'\}\in\mathscr{L}\setminus\mathscr{S}$, as otherwise we are back in the previous case by interchanging the roles of $\{r_1,r_2\}$ and $\{q,q'\}$. If exactly one of the other lines of the grid belongs to the spread $\mathscr{S}$, then the signs are opposite. The regularity of $\mathscr{S}$ implies that at most one other line belongs to $\mathscr{S}$; we now claim that every 3×3 grid of Γ contains at least one spread line. Indeed, we count 12 grids with three spread lines and $9\cdot 12=108$ grids with a unique spread line. In total there are 45 lines, each in 16 grids, but each also counted 6 times. Hence there are 120 3×3 grids in total, which shows our claim and the observation. □

Comments on the other cases.

(*i*) The Grassmannian variety $\mathscr{G}_{2,6}(\mathbb{K})$ arises from the Cartan variety above by setting $X_p=0$ for all points p in a double six. Indeed, the analogue of the construction above considers Γ' in place of Γ and a maximal regular partial spread $\mathscr{S}'$ in place of $\mathscr{S}$ ($\mathscr{S}'$ consists just of three disjoint lines of a grid). That this works can be seen through the model of Γ,Γ' and $\mathscr{S}$ given in Subsection 2.2. Since $\mathscr{G}_{2,6}(\mathbb{K})$ is the intersection of all quadrics with equation $p_{ij}p_{k\ell}+p_{ik}p_{\ell j}+p_{i\ell}p_{jk}=0$ (as follows from Theorem 3.8 in [6]), it suffices to make a choice between each p_{ij} and p_{ji} in order to get the signs lined up with the above rule and the choice of $\mathscr{S}'$. But this can simply be done by retaining p_{ij} for $i\in\{1,2,3\}$ and $j\in\{4,5,6\}$, and $(ij)\in\{(12),(23),(31),(45),(56),(64)\}$, as an elementary calculation shows.

(*ii*) The Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$ arises from the Cartan variety by setting $X_p=0$ for all points outside a regulus of spread lines. This can easily be seen through the construction in Subsection 2.1, denoting the point of the grid associated to the entry (i,j) in the 3×3 matrix by q_{ij} and the corresponding coordinate by x_{ij}, we let the grid be defined by the lines $\{q_{ij},q_{k\ell},q_{mn}\}$ with $\{i,k,m\}=\{j,\ell,n\}=\{1,2,3\}$. If we choose the spread lines as

$\{q_{11}, q_{22}, q_{33}\}$, $\{q_{12}, q_{23}, q_{31}\}$ and $\{q_{13}, q_{32}, q_{21}\}$, then we see that Q_p is exactly the co-factor of the entry corresponding to p in the matrix $(x_{ij})_{1\leq i,j\leq 3}$. This indeed defines $\mathscr{S}_{2,2}(\mathbb{K})$ as can be deduced from Theorem 4.94 in [6], or from [2].

(*iii*) The quadric Veronese variety $\mathscr{V}_2(\mathbb{K})$ arises from the Cartan variety by setting $X_p = 0$ for all points outside a regulus $\{L_1, L_2, L_3\}$ of spread lines and $X_{p_1} = X_{p_2}$ for collinear points $p_i \in L_i$, $i = 1, 2$. Indeed, in the previous paragraph, choosing $L_3 = \{q_{11}, q_{22}, q_{33}\}$, collinear points outside this line correspond to symmetric entries of the matrix. Here, the gradient is not identical to ϕ; the last three coordinates of the gradient are twice the last three coordinates of ϕ, hence there is special behaviour in characteristic 2.

Denoting by $v.w$ the ordinary dot product of v and w in $\mathbb{K}^{27}$, we observe the following.

Observation 4.2. *For arbitrary $v, w \in \mathbb{K}^{27}$ and $t \in \mathbb{K}$, we have*

$$C(v + tw) = C(v) + t\phi(v).w + t^2 v.\phi(w) + t^3 C(w). \tag{4.4}$$

Proof. It is clear that the coefficient of t^0 and t^3 are $C(v)$ and $C(w)$, respectively. It remains to explain the coefficient of t, as the one of t^2 is obtained by switching the roles of v and w. Now, obviously, the coefficient of t is linear in w, hence if suffices to set $w = e_p$ for $p \in \mathscr{P}$. Then we see that the coefficient of t in $C(v + te_p)$ is equal to $\frac{\partial C(v)}{\partial X_p} e_p = Q_p(v)e_p$. Now Identity (4.4) follows. □

Hence we deduce that the *adjoint square* $v^\sharp$ in the sense of Aschbacher [1], is, up to reordering the coordinates, exactly equal to $\phi(v)$. Hence $C(v)$ is the cubic form related to $\mathscr{E}_6(\mathbb{K})$ and the Chevalley group $\mathsf{E}_6(\mathbb{K})$ acts on $\mathbb{P}^{26}$ with three orbits, which are easily seen to be defined as

(*i*) the points of the variety $\mathscr{E}_6(\mathbb{K})$, namely those corresponding to the vectors v with $\phi(v) = \vec{o}$. These points are the *white points*;

(*ii*) the points off the variety $\mathscr{E}_6(\mathbb{K})$ corresponding to the vectors v with $C(v) = 0$. These points are the *grey points*;

(*iii*) the points corresponding to vectors v with $C(v) \neq 0$. These points are the *black points*.

We have taken the notions of white, grey and black from Aschbacher [1]. See also Cohen [4] for a very comprehensive introduction.

Comments on the other cases. For the quadric Veronese variety $\mathscr{V}_2(\mathbb{K})$ the group has more than three orbits; in this case, and if $\operatorname{char}\mathbb{K} = 2$, the grey points also comprise all points of the *nucleus plane*.

It now follows from (*i*), (*ii*) and Identity (4.4) that the projective null set of the cubic form C is exactly the secant variety of $\mathscr{E}_6(\mathbb{K})$.

Observation 4.3. *Let v be a nonzero vector of $\mathbb{K}^{27}$.*

(i) *The point $\langle v\rangle$ is a white point if and only if $\phi(v)=\vec{o}$;*

(ii) *the point $\langle v\rangle$ is grey if and only if $\phi(v)\neq\vec{o}$ and the point $\langle\phi(v)\rangle$ is white;*

(iii) *the point $\langle v\rangle$ is black if and only if $\langle\phi(v)\rangle$ is a black point.*

Proof. (i) This follows immediately from the definition of white points above.

(ii) By definition, the point $\langle v\rangle$ is grey if and only if $\phi(v)\neq\vec{o}$ and $C(v)=0$. The latter is equivalent to $\phi(\phi(v))=\vec{o}$, which is equivalent to $\phi(v)$ being white by (i).

(iii) Suppose $\langle v\rangle$ is black. If $\langle\phi(v)\rangle$ is white or grey, then $C(\phi(v))=0$, implying $\phi(\phi(\phi(v)))=\vec{o}$. But the left hand side is equal to $\phi(C(v)v)=C(v)^2\phi(v)\neq\vec{o}$, a contradiction. Now suppose $\phi(v)$ is black. Then $\phi(\phi(\phi(v)))$ is a non-zero multiple of $\phi(v)$, and so $\phi(\phi(v))$ cannot be equal to $\vec{o}$, implying $C(v)\neq 0$ and $\langle v\rangle$ is black. □

It follows from the previous observation that $\langle\phi(v)\rangle$ is never a grey point. We record this for further reference.

Corollary 4.4. *For each $v\in\mathbb{K}^{27}$, $\langle\phi(v)\rangle$ is never a grey point.*

We also observe that transitivity of the automorphism group of $\mathscr{E}_6(\mathbb{K})$ implies the following.

Observation 4.5. *Let v be a nonzero vector of $\mathbb{K}^{27}$. Then $\langle v\rangle$ is a white point if and only if there exists a grey point $\langle w\rangle$ with $\langle\phi(w)\rangle=\langle v\rangle$.*

Proof. If $\langle w\rangle$ is grey, then by Observation 4.3 (ii), $\langle\phi(w)\rangle$ is white. Now let $\langle v\rangle$ be a white point. Let $\langle w_0\rangle$ be a grey point (for instance the point $\langle e_p+e_q\rangle$ with p and q collinear points of Γ). Then by Observation 4.3 (ii), $\langle\phi(w_0)\rangle$ is white. Let g be an automorphism of $\mathscr{E}_6(\mathbb{K})$ mapping $\langle\phi(w_0)\rangle$ to $\langle v\rangle$. Then $\langle w_0^g\rangle$ is grey and $\langle\phi(w_0^g)\rangle=\langle\phi(w_0)^g=\langle v\rangle$. □

Observation 4.6. *For every white point $\langle v\rangle$, the set $\{\langle w\rangle\mid\vec{o}\neq\phi(w)\in\langle v\rangle\}$ is the set of grey points of a (9-dimensional) host space of $\mathbb{P}^{26}$ (hence generated by the points of some fixed symp).*

Proof. Let $p\in\mathscr{P}$ be arbitrary. Let $\langle w\rangle$ be a grey point belonging to the host space $U_p:=\langle e_q\mid p\perp q\in\mathscr{P}\rangle$. Then clearly $\phi(w)$ is a nonzero multiple of e_p. By transitivity of the automorphism group, we thus see that for every white point $\langle v\rangle$, the set $\{\langle w\rangle\mid\vec{o}\neq\phi(w)\in\langle v\rangle\}$ is the set of grey points of a union of host spaces of $\mathbb{P}^{26}$. Suppose that we have the union of at least two host spaces. By transitivity, we may assume that two of these host spaces are U_p and U_q, with $p,q\in\mathscr{P}$. But we already know that these map to $\langle e_p\rangle$ and $\langle e_q\rangle$, respectively, which are distinct. The assertion now follows. □

4.2 A lemma

Lemma 4.7. *Let Q be a quadratic form whose null set contains $\mathscr{E}_6(\mathbb{K})$. Then Q is a linear combination (with constant coefficients in $\mathbb{K}$) of the Q_p, $p \in \mathscr{P}$.*

Proof. Let Q be given by the polynomial

$$Q(v) = \sum_{\{p,q\} \subseteq \mathscr{P}} a_{\{p,q\}} X_p X_q,$$

with $a_{\{p,q\}} \in \mathbb{K}$. Since all points corresponding to the standard basis vectors belong to $\mathscr{E}_6(\mathbb{K})$, we have $a_{\{p\}} = 0$, for all $p \in \mathscr{P}$. Now let $p, q \in \mathscr{P}$ be distinct but non-collinear in Γ. Then one easily checks that $\langle e_p + e_q \rangle \in \mathscr{E}_6(\mathbb{K})$. Hence the coefficient $a_{\{p,q\}}$ of $X_p X_q$ in $Q(v)$ is also 0.

Now consider a line $L \in \mathscr{S}$ and a line $M \in \mathscr{L} \setminus \mathscr{S}$ with $L \cap M = \{p\}$, $p \in \mathscr{P}$. Let $L = \{p, q_1, q_2\}$ and $M = \{p, r_1, r_2\}$. Then clearly the point $\langle e_{q_1} + e_{q_2} + e_{r_1} + e_{r_2} \rangle$ belongs to $\mathscr{E}_6(\mathbb{K})$. This implies that $a_{\{q_1,q_2\}} = -a_{\{r_1,r_2\}} =: a_p$. Now it is clear that $Q(v) = \sum_{p \in \mathscr{P}} a_p Q_p(v)$, proving the lemma. □

Noting that, for collinear points $q_1, q_2 \in \mathscr{P}$, the vector $e_{q_1} + e_{q_2}$ belongs to the null set of each quadratic form Q_p, $p \in \mathscr{P}$, except for the unique point p with $\{p, q_1, q_2\} \in \mathscr{L}$, we see that

Observation 4.8. *The set $\{Q_p : p \in \mathscr{P}\}$ is a linearly independent set of quadratic forms and no proper subset of it intersects precisely in $\mathscr{E}_6(\mathbb{K})$.*

Comments on the other cases. Care has to be taken for the case $\mathscr{V}_2(\mathbb{K})$, not only since the automorphism group can have more than three orbits on the points (and on the hyperplanes) of the surrounding projective space, but also since this case behaves in an exceptional way for small fields. Let us provide some quick details. With respect to the representation given as definition in Subsection 2.1, we have

$$\phi(x_1, x_2, x_3, x_{23}, x_{31}, x_{12}) = (x_2x_3 - x_{23}^2, x_3x_1 - x_{31}^2, x_1x_2 - x_{12}^2, x_{31}x_{12} - x_1x_{23}, x_{12}x_{23} - x_2x_{31}, x_{23}x_{31} - x_3x_{12}),$$

and

$$C(x_1, x_2, x_3, x_{23}, x_{31}, x_{12}) = x_1x_2x_3 + 2x_{12}x_{23}x_{31} - x_1x_{23}^2 - x_2x_{31}^2 - x_3x_{12}^2.$$

Observations 4.5 and 4.6 need an alternative proof, since the group does not act transitively on the grey points. However, one calculates easily that

$$\phi(0,0,0,k,-\ell,0) = (-k^2, -\ell^2, 0, 0, 0, -k\ell) = -\nu(k, \ell, 0)$$

and

$$\phi(1, 1, a^2 + b^2, -b, -a, 0) = (a^2, b^2, 1, b, a, ab) = \nu(a, b, 1),$$

which covers all points of $\mathscr{V}_2(\mathbb{K})$. This shows the nontrivial direction of Observation 4.5. Observation 4.6 follows from a similar calculation and is left to the reader. Lemma 4.7 only holds for fields with at least four elements. To prove this, one just expresses that a generic point $\nu(x, y, z)$ satisfies a quadratic equation, and one argues that, if the field has at least 4 elements, then the corresponding quadratic form is a linear combination of the $X_iX_j - X_{ij}^2$, $ij \in \{12, 23, 31\}$, and the $X_iX_{jk} - X_{ki}X_{ij}$, $ijk \in \{123, 231, 312\}$. For $|\mathbb{K}| \leq 3$, all points of $\mathscr{V}_2(\mathbb{K})$ satisfy $X_1X_{23} - X_2X_{23} = 0$ since $x = x^3$ for all $x \in \mathbb{K}$, and this is not a linear combination of the basic quadratic equations. Finally, Observation 4.8 is false, see Subsection 5.2 below.

4.3 Reducing the number of quadrics—End of the proof

Lemma 4.7 and Observation 4.8 indicate that we need all 27 quadratic forms to describe $\mathscr{E}_6(\mathbb{K})$ as the intersection of quadrics. However, making suitable linear combinations, we can actually reduce the number of quadrics. To do this, let U be a subspace of $\mathbb{K}^{27}$ such that all its non-zero vectors correspond to grey points, and we use the same notation U for the corresponding subspace of $\mathbb{P}^{26}$. Let $\{H_i : i \in I\}$ be a minimal set of hyperplanes of $\mathbb{K}^{27}$ whose intersection is exactly U (then $|I| + \dim U = 27$, where $\dim U$ is the vector dimension of U). For each

$$H_i \leftrightarrow \sum_{p \in \mathscr{P}} a_p^{(i)} X_p = 0, \quad i \in I,$$

define the quadratic form Q_i given by

$$Q_i(v) = \sum_{p \in \mathscr{P}} a_p^{(i)} Q_p(v).$$

Note that for a vector $v \in \mathbb{K}^{27}$ we have $Q_i(v) = 0$ if and only if $\phi(v) \in H_i$.

Clearly, the null set of each Q_i contains the vectors corresponding to $\mathscr{E}_6(\mathbb{K})$. Conversely, suppose some nonzero vector v belongs to the null set of each Q_i, $i \in I$. Then, by construction, $\langle \phi(v) \rangle \in U$. If $\phi(v) \neq \vec{o}$, this would mean that $\langle \phi(v) \rangle$ is a grey point, contradicting Corollary 4.4.

Conversely, suppose $\mathscr{E}_6(\mathbb{K})$ is the intersection of the null sets of a number of quadratic forms Q_i, $i \in I$. By Lemma 4.7, each quadratic form Q_i is a linear combination of the Q_p, $p \in \mathscr{P}$, say

$$Q_i(v) = \sum_{p \in \mathscr{P}} a_p^{(i)} Q_p(v), \quad a_p^{(i)} \in \mathbb{K}.$$

For $i \in I$, let the hyperplane H_i be given by the equation

$$H_i \leftrightarrow \sum_{p \in \mathscr{P}} a_p^{(i)} X_p = 0.$$

Suppose there is a white or black point $\langle v \rangle$ contained in each hyperplane H_i, $i \in I$. Then Lemma 4.5 and the definition of $C(v)$ implies that there exists $w \in \mathbb{K}^{27}$ with $\phi(w) = v$ and with $\langle w \rangle$ grey or black. It follows that w is in the null set of each Q_i, $i \in I$, contradicting the fact that $\langle w \rangle$ is not white. This actually shows the last claim of Theorem 3.1, and the first claim also follows.

Hence the minimum number of quadrics completely describing $\mathscr{E}_6(\mathbb{K})$ as their intersection is equal to $27 - d'$, where $d' = d + 1$ is the dimension of a maximum dimensional subspace containing no vectors corresponding to white or black points.

5 Examples and applications

In this section, we determine the exact value of d for some specific cases. Our results will show that d strongly depends on the field $\mathbb{K}$ and therefore the determination of d for every field $\mathbb{K}$ is beyond the scope of this paper.

We begin with some general observations.

5.1 General observations

To ease notation, we will identify the projective version of ϕ with ϕ, *i.e.*, we will write $\langle \phi(v) \rangle$ as $\phi(\langle v \rangle)$. This projective version is then not defined on the points of $\mathscr{E}_6(\mathbb{K})$, and it induces an involutive bijection from the set of black points onto itself.

In this section, let U be a subspace of $\mathbb{P}^{26}$ entirely consisting of grey points; we will briefly call this a *grey subspace*. Then $\phi(U)$ corresponds to a set of points of $\mathscr{E}_6(\mathbb{K})$. We prove some properties of $\phi(U)$.

Lemma 5.1. *Let $p, q \in U$, $p \neq q$. Denote the line joining p and q by L, and note that $L \subseteq U$. Then*

(i) If $\phi(p) = \phi(q)$, then $\phi(L) = \phi(p)$;

(ii) if $\phi(p)$ and $\phi(q)$ are collinear on $\mathscr{E}_6(\mathbb{K})$, then ϕ is bijective on L and $\phi(L)$ is a conic on $\mathscr{E}_6(\mathbb{K})$ which is contained in a singular plane of $\mathscr{E}_6(\mathbb{K})$;

(iii) if $\phi(p)$ and $\phi(q)$ are not collinear on $\mathscr{E}_6(\mathbb{K})$, then ϕ is bijective on L and $\phi(L)$ is a conic on $\mathscr{E}_6(\mathbb{K})$ which is not contained in a singular plane of $\mathscr{E}_6(\mathbb{K})$.

Proof. Define the cross product $v \times w$ as the linearization of ϕ, *i.e.*, $v \times w = \phi(v+w) - \phi(v) - \phi(w)$. Denote the projective version also by $\times$, *i.e.*, $\langle v \rangle \times \langle w \rangle = \langle v \times w \rangle$. Then one calculates that, for all $\lambda, \mu \in \mathbb{K}$,

$$\phi(\lambda v + \mu w) = \lambda^2 \phi(v) + \lambda\mu(v \times w) + \mu^2 \phi(w). \tag{5.1}$$

If $v \times w$ is linearly dependent on $\phi(v)$ and $\phi(w)$, then also $\phi(v+w)$ is a linear combination of $\phi(v)$ and $\phi(w)$. Suppose first that $\phi(v)$ and $\phi(w)$ are not collinear on $\mathscr{E}_6(\mathbb{K})$. Then, since the only points of $\mathscr{E}_6(\mathbb{K})$ on the line $\langle v, w\rangle$ are $\langle v\rangle$ and $\langle w\rangle$, and since $\langle \phi(v+w)\rangle$ is by assumption a point of $\mathscr{E}_6(\mathbb{K})$, we deduce without loss of generality $\phi(\langle v+w\rangle) = \phi(\langle v\rangle)$. Hence $\langle v\rangle$ and $\langle v+w\rangle$ are contained in the same host space, implying $\langle w\rangle$ is also, and we are in Situation (i), a contradiction. Hence Equation (5.1) defines a conic.

Now suppose that $\phi(v)$ and $\phi(w)$ are collinear on $\mathscr{E}_6(\mathbb{K})$. Let ξ and ζ be the symplecta the host spaces of which contain $\langle v\rangle$ and $\langle w\rangle$, respectively. Let $U = \xi \cap \zeta$. Select maximal singular subspaces $V \subseteq \xi$ and $W \subseteq \zeta$ disjoint from U. Then simple dimension arguments show that every point of V is collinear to a unique point of W. Moreover $\langle V, W\rangle \cap \mathscr{E}_6(\mathbb{K})$ is a Segre variety $\mathscr{S}$ isomorphic to $\mathscr{S}_{1,4}(\mathbb{K})$, and every 4-dimensional generator of that Segre variety is contained in a unique symp also containing U. This follows from the similar but easy to check fact for $\mathscr{S}_{2,2}(\mathbb{K})$ and the fact that $\mathscr{S}_{2,2}(\mathbb{K})$ is amply contained in $\mathscr{E}_6(\mathbb{K})$ (by [11]). Now $v = v_1 + v_2$, with $\langle v_1\rangle \in U$ and $v_2 \in V$, and $w = w_1 + w_2$, with $\langle w_1\rangle \in U$ and $\langle w_2\rangle \in W$. Notice that $p \times q = \phi(p+q)$ for points $p, q \in \mathscr{E}_6(\mathbb{K})$. If $\langle v_2\rangle$ and $\langle w_2\rangle$ are not contained in the same 1-dimensional generator of $\mathscr{S}$, then $\langle v_2 + w_2\rangle$ is not contained in $\mathscr{S}$ and hence $\phi(v_2 + w_2)$ is not contained in a symplecton through U (as each host space through U intersects $\langle \mathscr{S}\rangle$ in a 4-dimensional generator of $\mathscr{S}$). Consequently in that case,

$$v \times w = v_1 \times w_2 + v_2 \times w_1 + v_2 \times w_2 = \phi(v_1 + w_2) + \phi(v_2 + w_1) + \phi(v_2 + w_2)$$

is linearly independent from $\phi(v)$ and $\phi(w)$ (since $\phi(v_2 + w_1)$ is a (possibly trivial) multiple of $\phi(\langle v\rangle)$ and $\phi(v_1 + w_2)$ a multiple of $\phi(w)$). So in this case, (ii) holds.

So we may assume that $\langle v_2\rangle$ and $\langle w_2\rangle$ are collinear on $\mathscr{E}_6(\mathbb{K})$, *i.e.*, $v_2 \times w_2 = \vec{o}$. It then follows that there exists a unique point p on the line through $\langle v_1\rangle$ and $\langle w_1\rangle$ collinear with both $\langle v_2\rangle$ and $\langle w_2\rangle$ (if some point q on that line were collinear to $\langle v_2\rangle$ but not to $\langle w_2\rangle$, then ζ would be determined by $\langle w_2\rangle$ and q and would contain $\langle v_2\rangle$). Hence there exists $\ell \in \mathbb{K}^\times$ with $v_2 \times (v_1 + \ell w_1) = \vec{o} = w_2 \times (v_1 + \ell w_1)$. Then, using the bilinearity of the cross-product, we calculate

$$v \times w = v_1 \times w_2 + v_2 \times w_1 = -\ell w_1 \times w_2 - \ell^{-1} v_1 \times v_2 = -\ell\phi(w) - \ell^{-1}\phi(v).$$

Hence, substituting this in Equation (5.1), we obtain

$$\phi(\lambda v + \mu w) = (\lambda^2 - \ell^{-1}\lambda\mu)\phi(v) + (\mu^2 - \ell\lambda\mu)\phi(w). \tag{5.2}$$

which becomes $\vec{o}$ for $\mu = \ell\lambda$, a contradiction.

The lemma now follows. □

We call lines of type (i) *short*, lines of type (ii) *flat* and lines of type (iii) *conical*. We now have the following result.

Proposition 5.2. *Set $d = \dim(U)$, where we use projective dimensions.*

(1) *If all lines of U are short, then $\phi(U)$ is a point.*

(2) *If only all lines in a hyperplane of U are short, then either all other lines are flat, or all other lines are conical. In both cases $\phi(U)$ is a quadric of Witt index 1 spanning a $(d+1)$-dimensional space in $\mathbb{P}^{26}$, which is singular in the flat case, and in the conical case the quadric is contained in a symp as a subquadric.*

(3) *In all cases $\phi(U)$ is the quotient (or projection) of a Veronese variety $\mathscr{V}_d(\mathbb{K})$, where the image of a conic is either a conic, or a single point. If U does not contain short lines, then $\dim\langle\phi(U)\rangle \geq d$.*

(4) *If U contains two disjoint planes containing only short lines, then every line intersecting both planes is conical.*

For $\mathscr{G}_{2,6}(\mathbb{K})$, the last statement becomes:

(4′) *If U contains two disjoint short lines, then every line intersecting both lines is conical.*

Proof. We start with noting that (1) is obvious: all points of U are contained in the same host space.

Let $e_0, \ldots, e_d$ be a (vector) basis of U. Then, using the definition of the cross product and the bilinearity of it, we calculate that $\phi(U)$ is the image of the map

$$(\lambda_0, \ldots, \lambda_d) \mapsto \sum_{i=0}^{d} \lambda_i^2 \phi(e_i) + \sum_{i=0}^{d-1} \sum_{j=i+1}^{d} \lambda_i \lambda_j (e_i \times e_j), \tag{5.3}$$

which is a Veronese variety $\mathscr{V}_d(\mathbb{K})$ if all $\phi(e_i)$ and $e_i \times e_j$ are linearly independent. But if not, then this is just an obvious quotient of $\mathscr{V}_d(\mathbb{K})$. If $\phi(U)$ does not contain short lines, then no point of the subspace from which one projects lies on a tangent, and since tangents at one point fill the whole tangent space, the latter are isomorphically projected. Hence (3).

To show (2), we may assume that all lines of the subspace $H := \langle e_1, \ldots, e_d\rangle$ are short. Hence there exist constants $k_1, \ldots, k_{d-1}$ such that $\phi(e_i) = k_i \phi(e_d)$, $k_i \in \mathbb{K}$, $i = 1, \ldots, d-1$. Then $\phi(e_i + e_j)$ is a multiple of $\phi(e_d)$, $i, j \in \{1, \ldots, d\}$, $i \neq j$, and so we may write $e_i \times e_j = \ell_{ij}\phi(e_d)$, $i, j \in \{1, \ldots, d\}$, $i < j$, for some $\ell \in \mathbb{K}$. The mapping (5.3) becomes

$$(\lambda_0, \ldots, \lambda_d) \mapsto \lambda_0^2 \phi(e_0) + \left(\sum_{i=1}^{d} k_i \lambda_i^2 + \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \ell_{ij} \lambda_i \lambda_j \right) \phi(e_d) + \sum_{i=1}^{d} \lambda_0 \lambda_i (e_0 \times e_i). \tag{5.4}$$

If $\phi(e_0), \phi(e_d)$ and all $e_0 \times e_i$, $i = 1, \ldots, d$, are linearly independent from each other, then, with respect to that basis, and denoting the coordinate corresponding to $e_0 \times e_i$ by X_i, the one corresponding to $\phi(e_0)$ by X_0 and the one corresponding to $\phi(e_d)$ by X_{d+1}, it is an elementary exercise to calculate that a point is in the image of the map (5.4) if and only if its coordinates $(X_0,\dots,X_{d+1})$ satisfy

$$X_0X_{d+1}=\sum_{i=1}^{d}k_iX_i^2+\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\ell_{ij}X_iX_j. \tag{5.5}$$

Note that the right hand side of Equation (5.5) is an anisotropic quadratic form; indeed, suppose there exist $x_i\in\mathbb{K}$, $i=1,\dots,d$, such that $\sum_{i=1}^{d}k_ix_i^2+\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\ell_{ij}x_ix_j=0$. Then setting $\lambda_0=0$ and $\lambda_i=x_i$, we see that the right hand side of the map in (5.4) becomes $\vec{o}$, a contradiction, as this would yield a white point in U.

Hence Equation (5.5) defines a quadric Q of Witt index 1. If $\phi(e_0),\phi(e_d)$ and all $e_0\times e_i$, $i=1,\dots,d$, are not linearly independent from each other, then $\phi(S)$ is a projection of Q. However, considering a point p in $\langle Q\rangle$ in the subspace from which we project, we can select a plane α through p containing two points of $\phi(U)$, and then α contains a conic, which is either not projected bijectively, or projected into a line, both of which are contradictions to Lemma 5.1. Hence $\phi(U)$ spans a space of dimension $d+1$.

If some line L of U is flat, then, for each point $p\in L\setminus H$, $\phi(p)$ and $\phi(L\cap H)$ are collinear on $\mathscr{E}_6(\mathbb{K})$. But $\phi(L\cap H)=\phi(H)=\phi(q)$, for each $q\in H$. Hence all lines of U intersecting L in some point not in H are flat. Replacing L with each such a line, we obtain that all lines of U not contained in H are flat.

This completes the proof of (2). We now address (4). Suppose that α and β are two disjoint planes all of whose lines are short, and suppose for a contradiction that there is a flat line L intersecting α and β in some point $\langle v\rangle$ and $\langle w\rangle$, respectively. Then $\phi(\alpha)=\phi(\langle v\rangle)$ and $\phi(\beta)=\phi(\langle w\rangle)$ are collinear on $\mathscr{E}_6(\mathbb{K})$. We now use the same notation as in the proof of Lemma 5.1 (*ii*). So ξ and ζ are the symplecta with $\alpha\subseteq\langle\xi\rangle$ and $\beta\subseteq\langle\zeta\rangle$, and $U=\xi\cap\zeta$. Also, V and W are maximal singular subspaces of ξ and ζ, respectively, disjoint from U. Let α_2 and β_2 be the projection of α and β, respectively, from U onto V and W, respectively. Since $\langle V,W\rangle\cap\mathscr{E}_6(\mathbb{K})$ is a Segre variety, a dimension argument implies that some point $\langle v_2\rangle$ of α_2 is collinear on $\mathscr{E}_6(\mathbb{K})$ with some point $\langle w_2\rangle$ of β_2. But, as one can read in the last part of the proof of Lemma 5.1, this leads to a contradiction. □

Corollary 5.3. *With the above notation, if U intersects the space spanned by a symp ξ in a subspace of dimension 1, 2 or 4 in the cases $\mathscr{V}=\mathscr{S}_{2,2}(\mathbb{K})$, $\mathscr{G}_{2,6}(\mathbb{K})$ or $\mathscr{E}_6(\mathbb{K})$, respectively, then either U is contained in $\langle\xi\rangle$, or $\mathscr{V}=\mathscr{G}_{2,6}(\mathbb{K})$ and all lines of U that intersect $\langle\xi\rangle$ are flat.*

Proof. Set $d=\dim U$. Without loss of generality, we may assume that $U\cap\langle\xi\rangle$ is a hyperplane of U. Then Proposition 5.2 implies that $\phi(U)$ is contained in a subspace W of $\mathbb{P}^{6d-4}$ of dimension $d+1=3,4,6$ for the respective cases. So W can only be a singular subspace of $\mathscr{V}$ if $\mathscr{V}=\mathscr{G}_{2,6}(\mathbb{K})$. If W is not singular, then $\phi(U)$ is quadric of Witt index 1 arising as the intersection of a symp with a subspace of dimension 3, 4 and 6, respectively. But such subspaces always have lines in common with the symp, since they intersect each maximal singular subspace of the symp in a line, by an obvious dimension argument, a contradiction. □

Now we consider the separate varieties in turn. Notice first that, since $\mathscr{V}_2(\mathbb{K}) \subseteq \mathscr{S}_{2,2}(\mathbb{K}) \subseteq \mathscr{G}_{2,6}(\mathbb{K}) \subseteq \mathscr{E}_6(\mathbb{K})$, each example for a certain variety carries over to the next variety, as ordered in the inclusions just given.

5.2 The quadric Veronesean $\mathscr{V}_2(\mathbb{K})$

Recall that $\mathscr{V}_2(\mathbb{K})$ is given by the image of the Veronese map

$$\mathbb{P}^2 \to \mathbb{P}^5 : (x, y, z) \mapsto (x^2, y^2, z^2, yz, zx, xy).$$

The line given by the points $(0, 0, 0, k, \ell, 0)$ entirely consists of grey points, hence in general, $6 - 2 = 4$ quadrics suffice to describe $\mathscr{V}_2(\mathbb{K})$. After a little calculation, ordering the coordinates like $(X_1, X_2, X_3, X_{23}, X_{31}, X_{12})$, these turn out to be $X_1X_2 = X_{12}^2$, $X_3X_1 = X_{31}^2$, $X_2X_3 = X_{23}^2$, and any one of $X_1X_{23} = X_{31}X_{12}$, $X_2X_{31} = X_{12}X_{23}$ or $X_3X_{12} = X_{23}X_{31}$. In characteristic 2, the whole nucleus plane consists of grey points and hence the first three equations suffice (see also Lemma 4.20 in Hirschfeld & Thas [6]). This somehow reflects the property of the gradient being identically zero in the last three coordinates.

We now determine all grey planes, showing in particular that in characteristic not equal to 2 there do not exist such planes, and in characteristic 2 only the nucleus plane is a grey plane, except if the underling field is $\mathbb{F}_2$. We are grateful to J. Thas for hinting the use of conic bundles in the below argument (our original proof consisted merely of boring calculations).

So suppose π is a grey plane containing at least one point p contained in a secant L. Obviously there are no flat lines. Then Corollary 5.3 implies that π only contains conical lines. Proposition 5.2(3) now implies that ϕ is bijective from π onto $\mathscr{V}_2(\mathbb{K})$. Hence the map ρ mapping each point $p \in \pi$ to the unique conic C on $\mathscr{V}_2(\mathbb{K})$ with $p \in \langle C \rangle$ is a bijection. Let $L \cap \mathscr{V}_2(\mathbb{K}) = \{x, y\}$. Consider the bundle $\mathscr{B}$ of conics of $\mathbb{P}^2$ defined by intersecting $\mathscr{V}_2(\mathbb{K})$ with all hyperplanes containing the solid $\langle \pi, L \rangle$. By the bijectivity of ρ, each conic D on $\mathscr{V}_2(\mathbb{K})$ containing x generates, together with π and L, a hyperplane H_D. Hence $H_D \cap \mathscr{V}_2(\mathbb{K})$ is a degenerate conic in $\mathbb{P}^2$, which also contains y. So if $y \notin D$, then $H_D \cap \mathscr{V}_2(\mathbb{K})$ contains a conic of $\mathscr{V}_2(\mathbb{K})$ through y. It follows that $\mathscr{B}$ consists solely of degenerate conics. But an arbitrary pair of members of $\mathscr{B}$ not containing the line of $\mathbb{P}^2$ corresponding to the conic of $\mathscr{V}_2(\mathbb{K})$ containing x and y generates a bundle containing exactly three degenerate members. Hence $|\mathbb{K}| = 2$. In this case one can easily check that π is the unique plane in a solid spanned by the complement in $\mathscr{V}_2(\mathbb{F}_2)$ of a conic (a conic corresponding to a line of $\mathbb{P}^2_{\mathbb{F}_2}$). Hence there are seven such planes. Each such plane intersects the nucleus plane in a unique point, namely the unique point of the solid not lying in a plane spanned by any three of the four points of $\mathscr{V}_2(\mathbb{F}_2)$ it contains.

5.3 The Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$

This is the only case with a uniform answer for arbitrary fields. Indeed, we will show that there always exists a grey plane π, and never a grey solid.

First, if we represent $\mathscr{S}_{2,2}(\mathbb{K})$ as the 3×3 rank 1 matrices, up to a scalar, then we can define π as the plane containing all skew-symmetric matrices (with 0 on every diagonal entry). It is easy to see that a skew-symmetric matrix, which always has determinant 0, has rank 1 if and only if it is the 0-matrix. Here every line of π is conical and $\phi(\pi)$ is a Veronesean isomorphic to $\mathscr{V}_2(\mathbb{K})$ embedded in $\mathscr{S}_{2,2}(\mathbb{K})$. It follows that the following system of equations in the unknowns $X_{00},\ldots,X_{22}$ defines $\mathscr{S}_{2,2}(\mathbb{K})$:

$$\begin{cases} X_{11}X_{22} = X_{12}X_{21} \\ X_{00}X_{22} = X_{02}X_{20} \\ X_{00}X_{11} = X_{01}X_{10} \\ X_{10}X_{02} + X_{01}X_{20} = (X_{12}+X_{21})X_{00} \\ X_{01}X_{12} + X_{10}X_{21} = (X_{02}+X_{20})X_{11} \\ X_{02}X_{21} + X_{20}X_{12} = (X_{01}+X_{10})X_{22} \end{cases}$$

In characteristic 2, the plane π is the nucleus plane of the Veronese surface contained in $\mathscr{S}_{2,2}(\mathbb{K})$ obtained by restricting $\mathscr{S}_{2,2}(\mathbb{K})$ to the symmetric (rank 1) 3×3 matrices.

Now suppose there exists a grey solid S. If S contains a short line, then considering any plane in S containing that short line, Corollary 5.3 leads to a contradiction. If S contains only conical lines, then let L_1 and L_2 be two non-intersecting lines of S. Let ξ_i be the symp containing $\phi(L_i)$, $i=1,2$. Clearly $\xi_1\neq\xi_2$ as otherwise every point of $\phi(L_1)$ is collinear to two points of $\phi(L_2)$, yielding flat lines. Hence ξ_1 and ξ_2 intersect nontrivially and since $\phi(L_1)$ is an ovoid of ξ_1, some point $x_1\in\phi(L_1)$ is collinear to a point of the intersection $\xi_1\cap\xi_2$. Then x_1 is collinear to a line of ξ_2, and since $\phi(L_2)$ is an ovoid of ξ_2, x_1 is collinear to some point $x_2\in\phi(L_2)$, a contradiction (as $\langle x_1,x_2\rangle$ is then the image under ϕ of a flat line of S). Hence there is at least one flat line $L\subseteq S$. Let π be the plane spanned by $\phi(L)$. If $\phi(S)\subseteq\pi$, then S only contains flat lines. By Lemma 5.2 (3) the dimension of π is at least 3, a contradiction. Hence there is some point $p\in S$ with $\phi(p)\notin\pi$. Since there is a unique point in π collinear to $\phi(p)$, we can pick two points $x_1,x_2\in L$ such that $\phi(x_i)$ is not collinear to $\phi(p)$, $i=1,2$. Let ξ_i be the symp determined by $\phi(x_i)$ and $\phi(p)$, $i=1,2$. Then $\xi_1\cap\xi_2$ is obviously equal to the line through $\phi(p)$ intersecting π. The argument above shows that for each point q_1 on the line $\langle x_1,p\rangle$, the point $\phi(q_1)$ is collinear in $\mathscr{S}_{2,2}(\mathbb{K})$ to a unique point $\phi(q_2)$, with $q_2\in\langle x_2,p\rangle$. But clearly $\phi(\langle q_1,q_2\rangle)$ is contained in a plane disjoint from π, contradicting the fact that $\langle p,L\rangle$ is a projective plane in S. So we ruled out all possibilities for S to exist.

Nevertheless one can sometimes find other grey planes. For instance, if $\mathbb{P}^2$ admits a linear collineation without fixed points, then one can find such a plane in the span of two disjoint singular planes of $\mathscr{S}_{2,2}(\mathbb{K})$. Such a plane only has flat lines. As an example suppose $\mathbb{K}$ is a field admitting a cubic extension $\mathbb{L}$; let the corresponding cubic polynomial be given by $x^3 - Tx^2 + Qx - N$, with $T, Q, N \in \mathbb{K}$. The plane containing the points

$$\begin{pmatrix} m & k - Qm & -\ell \\ \ell - Tm & Nm & k \\ 0 & 0 & 0 \end{pmatrix}, k, \ell, m \in \mathbb{K},$$

is grey, as one can calculate (in the calculations one might need the fact that also the polynomial $x^3 + Qx^2 + TNx + N^2$ is irreducible; its roots in the cubic extension $\mathbb{L}$ are the opposites of the pairwise products of the roots of the original polynomial). Applying ϕ we obtain that the mapping

$$(k, l, m) \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k^2 + N\ell m - Qkm & \ell^2 - T\ell m + km & (N - QT)m^2 - l\ell + Tkm + Q\ell m \end{pmatrix},$$

$k, \ell, m \in \mathbb{K}$, induces a bijection from $\mathbb{P}^2$ onto a singular plane of $\mathscr{S}_{2,2}(\mathbb{K})$, where each line is mapped to a conic. In fact, these conics form the net of all conics passing through three given conjugate points in the plane over the cubic extension $\mathbb{L}$.

Remark 5.4. *One might wonder how the net of conics in $\mathbb{P}^2$ of the last example can be a projection of the quadric Veronese surface, as required by Proposition 5.2 (3). To see this directly, one considers the above net of conics in $\mathbb{P}^2$, take its image under the Veronese map, and project the Veronese surface from the intersection of the hyperplanes spanned by the image of three linearly independent members of the net. This intersection is a plane consisting merely of black points.*

5.4 The line Grassmannian $\mathscr{G}_{2,6}(\mathbb{K})$

By the previous subsection, there always exists a grey plane. But we can do better for certain fields, in particular, if the field $\mathbb{K}$ admits a quadratic extension (separable or not). We will see that in this case we can find a grey 5-dimensional subspace of $\mathbb{P}^{14}$. But we start with a curious example in the case that $\mathbb{P}^3$ admits a linear collineation without fixed elements.

Example 5.5 (Dimension 3)**.** *Therefore, we consider a point $p \in \mathscr{G}_{2,6}(\mathbb{K})$ and the subspace $U_p \subseteq \mathbb{P}^{14}$ generated by all singular lines on p. Then $\dim U_p = 8$ and $\mathscr{G}_{2,6}(\mathbb{K}) \cap U_p$ is a cone with vertex p and base $\mathscr{S}_{1,3}(\mathbb{K})$ (the latter is indeed the residue at p). Consider any base space W; that is, a 7-dimensional subspace of U_p not containing p. Then $\mathscr{S} := W \cap \mathscr{G}_{2,6}(\mathbb{K}) \cong \mathscr{S}_{1,3}(\mathbb{K})$. Take two singular solids S_1, S_2 of $\mathscr{S}$. The mapping $\theta : S_1 \to S_2 : x_1 \mapsto x_2$ defined by $\langle x_1, x_2 \rangle \subseteq \mathscr{S}$ is a (linear) collineation from S_1 to S_2. Now let φ be a linear collineation of S_2 without fixed elements.*

Then let $\mathscr{S}' \cong \mathscr{S}_{1,3}(\mathbb{K})$ *be the Segre variety with as set of maximal singular* 1*-dimensional subspaces the set of lines* $\{\langle p, p^{\theta\varphi}\rangle\}$*, and select a* 3*-dimensional singular subspace* S' *of* $\mathscr{S}'$ *distinct from* S_1 *and* S_2*. We claim that* $S' \cap \mathscr{S} = \emptyset$*. Indeed, each point in* S' *is on a unique line intersecting both* S_1 *and* S_2*, and if that line would belong to* $\mathscr{S}$ *and intersect* S_i *in* x_i*,* $i = 1, 2$*, then* $x_1^{\theta} = x_2 = x_1^{\theta\varphi}$*, implying* x_2 *is a fixed point of* φ*, a contradiction.*

Suppose now that two points $p', q' \in S'$ *are contained in a common host space of some symplecton* ξ *of* $\mathscr{G}_{2,6}(\mathbb{K})$*. Then it is easy to see that* $\xi \cap S_i = L_i$ *is a line,* $i = 1, 2$*. In the solid* $\langle L_1, L_2\rangle$ *there is a unique line* L' *containing* p' *and intersecting* L_i *in some point* p_i*; then* $p_1^{\theta\varphi} = p_2$*. Likewise, there is a unique line* M' *containing* q' *and intersecting* L_i *in some point* q_i*; then* $q_1^{\theta\varphi} = q_2$*. Hence* $L_1^{\theta\varphi} = \langle p_1, q_1\rangle^{\theta\varphi} = \langle p_2, q_2\rangle = L_2$*. But the latter also coincides with* L_1^{θ} *(as* L_1, L_2 *is contained in a hyperbolic quadric completely contained in* $\mathscr{S}$*). Hence* φ *fixes* L_2*, a contradiction.*

We conclude that $\phi(S')$ *is the bijective projection of a Veronese variety* $\mathscr{V}_3(\mathbb{K})$ *into a hyperbolic quadric in some* 5*-dimensional projective space (that quadric corresponds to the point* p*; it consists of the images under* ϕ *of the symplecta passing through* p*). This is a rather remarkable situation. But that inclusion can abstractly be seen directly by sending a point* x *of* S_2 *to the image of the line* $\langle x, x^{\varphi}\rangle$ *under the Klein correspondence. We deduce that every plane of the Klein quadric contains a unique conic of that image.*

If, in the above, φ *has no fixed points, but does admit fixed lines, then we can still find* S' *and it is still a grey solid. But* $\phi(S')$ *is the union of elliptic quadratic surfaces (in* 3*-dimensional subspaces). An extreme situation is that the fixed lines of* φ *form a spread of* S'*, in which case* $\phi(S')$ *coincides with one such elliptic quadric. It is clear that this situation arises if and only if* $\mathbb{K}$ *admits a quadratic extension. But in this case we can extend* S' *to a* 5*-dimensional grey subspace, as evidenced by the next example.*

Example 5.6 (Dimension 5)**.** *Let* $x^2 - Tx + N$ *be an irreducible quadratic polynomial over* $\mathbb{K}$ *(with coefficients in* $\mathbb{K}$*), defining the quadratic extension* $\mathbb{L}$ *of* $\mathbb{K}$*. Let* p_1, p_2, p_3 *be three points on a line of the quadrangle* Γ' *of order* $(2, 2)$*, and suppose* $\{p_1, p_2, p_3\}$ *is a spread line. Let* $\{p_i, q_i, r_i\}$ *and* $\{p_i, s_i, t_i\}$ *be the other two lines passing through* p_i*,* $i = 1, 2, 3$*. We may choose this notation such that* $\{q_1, q_2, q_3\}$ *and* $\{r_1, r_2, r_3\}$ *are the other two spread lines in* Γ'*, and the other six lines of* Γ' *are* $\{s_1, q_2, t_3\}$*,* $\{s_1, t_2, r_3\}$ *and cyclic permutations of the indices. (For an explicit realization inside the model given in Subsection 2.2, see Example 5.8.) Define the following subspace:*

$$\begin{cases} 0 = X_{p_i}, & i = 1, 2, 3, \\ 0 = X_{r_i} + X_{q_i}, & i = 1, 2, 3, \\ 0 = X_{t_i} + NX_{s_i} + TX_{q_i}, & i = 1, 2, 3. \end{cases}$$

Since we have nine linearly independent equations, this defines a 5*-dimensional projective subspace* U*. In order to apply* ϕ *we write a generic point of* U *with coordinates* $X_{p_i} = 0$*,* $i = 1, 2, 3$*, the coordinates* X_{q_i} *and* X_{s_i} *are considered as running parameters,* $i = 1, 2, 3$*, and the coordinates*

X_{r_i} and X_{t_i} linearly depend on these parameters as given above, namely $X_{r_i} = -X_{q_i}$ and $X_{t_i} = -NX_{s_i} - TX_{q_i}$. We denote the coordinate vector of such a generic point by $v_{X_{q_1},X_{q_2},X_{q_3};X_{s_1},X_{s_2},X_{s_3}}$, or simply v in the sequel. Calculating $\phi(v)$ we obtain a vector with p_i-coordinate equal to

$$X_{q_i}^2 + TX_{q_1}X_{s_i} + NX_{s_i}^2, \quad i = 1, 2, 3.$$

Clearly, such coordinate is 0 if and only if $X_{q_i} = X_{s_i} = 0$, showing that no point of U is white. Calculating $C(v)$, we simply obtain 0, showing that U is a grey space.

One now sees that the short lines in U form a regular spread; they are the point set of a projective plane $\mathbb{P}^2_{\mathbb{L}}$ the lines of which are the 3-dimensional subspaces of U generated by two distinct short lines. This is the spread representation of $\mathbb{P}^2_{\mathbb{L}}$. We now claim that ϕ transforms this representation into the corresponding Hermitian Veronesean of $\mathbb{P}^2_{\mathbb{L}}$. Indeed, let δ be one of the roots in $\mathbb{L}$ of the polynomial $x^2 - Tx + N$, and let $x \mapsto \overline{x}$ be the corresponding Galois involution of $\mathbb{L}$. Note that $\overline{a + b\delta} = a + Tb - b\delta$, $a, b \in \mathbb{K}$. Denoting the p-coordinate of $\phi(v)$ by Y_p, $p \in \mathscr{P}$, a straightforward calculation reveals:

$$\begin{aligned} (\overline{X_{q_2} + X_{s_2}\delta})(X_{q_3} + X_{s_3}\delta) &= Y_{r_1} + Y_{t_1}\delta, \\ (\overline{X_{q_1} + X_{s_1}\delta})(X_{q_1} + X_{s_1}\delta) &= Y_{p_1}, \\ Y_{s_1} &= NY_{t_1}, \\ Y_{q_1} &= Y_{r_1} - TY_{t_1}, \end{aligned}$$

and the same equation for cyclic permutations of the indices, which shows that $\phi(U)$ is projectively equivalent to the point set

$$\{(\overline{X}_1X_1, \overline{X}_2X_2, \overline{X}_3X_3, \overline{X}_2X_3, \overline{X}_3X_1, \overline{X}_1X_2) \mid X_1, X_2, X_3 \in \mathbb{L}\},$$

where the first three coordinates are considered to belong to $\mathbb{K}$, and the last three to $\mathbb{K} \times \mathbb{K}$ via the obvious identification $a + b\delta \to (a, b)$. This shows our claim.

5.5 The Cartan variety $\mathscr{E}_6(\mathbb{K})$

By the previous subsections, there always exists a grey plane, and if $\mathbb{K}$ admits a quadratic extension, there is always a grey 5-space. We can slightly generalise the latter, and we can also give an example of a grey 11-dimensional space if $\mathbb{K}$ is the centre of a quaternion division algebra, or $\operatorname{char}\mathbb{K} = 2$ and $\mathbb{K}$ admits a degree 4 inseparable field extension. Also, we will show that there always exists a grey 4-space, whatever the field.

Example 5.7 (Dimensions 4 and 5)**.** *Let $\Gamma' = (\mathscr{P}', \mathscr{L}')$ be a subquadrangle of $\Gamma = (\mathscr{P}, \mathscr{L})$ of order $(2, 2)$. Let W be the 12-dimensional vector subspace of $\mathbb{K}^{27}$ generated by the e_p not belonging to Γ'. The points outside $\mathscr{P}'$ form a double six $\{p_1, \ldots, p_6, q_1, \ldots, q_6\}$, where $\{p_1, \ldots, p_6\}$ and $\{q_1, \ldots, q_6\}$*

are cocliques and p_i *is collinear to* q_j *if and only if* $i \neq j$*, for all* $i, j \in \{1, \ldots, 6\}$*. Let* $w \in W$ *have coordinates* $(x_p)_{p \in \mathscr{P}}$ *(with* $x_p = 0$ *if* $p \in \mathscr{P}'$*). Then* $\phi(w) = 0$ *if and only if* $x_{p_i} x_{q_j} = \pm x_{p_j} x_{q_i}$*, for all* $i, j \in \{1, \ldots, 6\}$ *with* $i \neq j$*, and where each sign depends on the position of the spread* $\mathscr{S}$*. However, changing the sign of the coordinates related to the points of one single six collinear to the points of one single three with respect to the grid in* Γ' *defined by intersecting* $\mathscr{L}'$ *with* $\mathscr{S}$*, we see that all signs become positive. This means that, denoting the projective subspace defined by* W *also by* W*, the intersection* $W \cap \mathscr{E}_6(\mathbb{K})$ *is a Segre variety* $\mathscr{S}_{1,5}(\mathbb{K})$*. As before, given a fixed point free linear collineation of* $\mathbb{P}^5$*, one can select a* 5*-dimensional subspace* U *of* W *disjoint from* $\mathscr{E}_6(\mathbb{K})$*, which is automatically a grey subspace. If* $\mathbb{K}$ *admits a separable quadratic extension, then we may choose* W *such that it contains a regular spread of short lines, and* $\phi(U)$ *is a Hermitian Veronesean variety on* $\mathscr{E}_6(\mathbb{K})$*, as in Example 5.6. However, note that in the inseparable case, the corresponding spread is elementwise fixed only by the identity. We hence conjecture that also in the separable case, the current* 5*-space is not projectively equivalent to the one of Example 5.6 (meaning the current subspace* U *is not contained in the space spanned by any subvariety of* $\mathscr{E}_6(\mathbb{K})$ *isomorphic to* $\mathscr{G}_{2,6}(\mathbb{K})$*).*

Now let $\mathbb{K}$ *be arbitrary and let* M *be a* 6×6 *upper triangular matrix with entries in* $\mathbb{K}$*, with* 1*s on the diagonal and such that* $M - I$ *(with* I *the identity matrix) has rank 5. Then the corresponding linear collineation* θ *of* $\mathbb{P}^5$ *has a unique fixed point. Let* U' *be a* 5*-space in* W *constructed as above from* θ*; then* $U' \cap \mathscr{E}_6(\mathbb{K})$ *is a point* p *corresponding to the unique fixed point of* θ*. Hence any hyperplane of* U' *not containing* p *is a grey* 4*-space.*

Example 5.8 (Dimension 11)**.** *Let* $x_1^2 - Tx_1x_2 + Nx_2^2 - \ell x_3^2 + \ell T x_3 x_4 - \ell N x_4^2$ *be the norm form of a quaternion division algebra* $\mathbb{H}$ *over* $\mathbb{K}$*, with* $\ell, T, N \in \mathbb{K}$*, or with* $T = 0$ *and* $\operatorname{char} \mathbb{K} = 2$*, and then we assume it is just an inseparable field extension of degree 4.*

It is convenient to work with the explicit description of Γ *and* $\mathscr{S}$ *given in Subsection 2.2. The current example will extend Example 5.6 with*

$$\begin{aligned}
(p_1, p_2, p_3) &= (25, 14, 36),\\
(q_1, q_2, q_3) &= (34, 26, 15),\\
(r_1, r_2, r_3) &= (16, 35, 24),\\
(s_1, s_2, s_3) &= (46, 56, 45),\\
(t_1, t_2, t_3) &= (13, 23, 12).
\end{aligned}$$

The subspace U *we want to define can be described by a system of fifteen equations, nine of which are given in Example 5.6 (using the above identification). The other six read (denoting the coordinate corresponding to the point* i *by* X_i *and the one corresponding to* i' *by* X'_i*,* $i = 1, \ldots, 6$*):*

$$\begin{cases} 0 = X'_i - \ell X_j, & (i,j) = (2,5), (1,4), (3,6),\\ 0 = X'_i - \ell N X_j - \ell T X_i, & (i,j) = (5,2), (4,1), (6,3). \end{cases}$$

In completely the same way as in Example 5.6, one checks that U *is a grey* 11*-dimensional subspace, and that its image under* ϕ *is the corresponding quaternion Veronesean of* $\mathbb{P}^2_{\mathbb{H}}$*.*

As a corollary of the last example, we obtain that every quaternion Veronese variety of the plane $\mathbb{P}^2_{\mathbb{H}}$, with $\mathbb{H}$ a quaternion division algebra over the field $\mathbb{K}$, or a degree 4 inseparable field extension in characteristic 2, is a projection of the Veronese variety $\mathscr{V}_{11}(\mathbb{K})$.

Over the real numbers, we can choose $-\ell = N = 1$ and $T = 0$. It follows that in this case $\mathscr{E}_6(\mathbb{R})$ has a particularly nice description as the intersection of fifteen quadrics, whose forms can be given as follows. Choose a fixed spread line L of $\mathscr{S}$. Three of the forms are Q_p, with $p \in L$. The other twelve forms are all of shape $Q_a + Q_b$, where $\{a, b, p\} \in \mathscr{L}$ is a line of Γ with $p \in L$.

5.6 Conclusion

We conclude by noting that we gave a full answer for the minimality of the number of quadrics describing a Severi variety in the cases of $\mathscr{V}_2(\mathbb{K})$ and $\mathscr{S}_{2,2}(\mathbb{K})$. For the two other case, we were only able to give some examples (yielding bounds) over fields with certain properties. Since we think that some of the dimensions we obtained are pretty high, we conjecture that

(C1) If $\mathbb{K}$ admits a quadratic extension, then the maximum projective dimension of a grey subspace for $\mathscr{G}_{2,6}(\mathbb{K})$ is 5.

(C2) If $\mathbb{K}$ admits a quaternion division algebra, or a degree 4 inseparable field extension in characteristic 2, then the maximum projective dimension of a grey subspace for $\mathscr{E}_6(\mathbb{K})$ is 11.

Remark 5.9. *We note that the minimum number of quadrics found in the present paper for a certain variety, is exactly equal to the dimension of the vector space related to the variety of the previous case, ranking the cases in increasing dimension, and adding a trivial variety in the beginning consisting of three spanning points in a projective plane (three* 1*-spaces generating a* 3*-dimensional vector space; this is the line-residue of the long root geometry of type* D_4 *which is sometimes added as zeroth column in the fourth row of the Freudenthal-Tits magic square; the Severi varieties are the line-residues of the other varieties of the fourth column). We do not think this is a coincidence; further research should give evidence for this.*

Finally, one could wonder which quadrics one can obtain by linearly combining the 27 basic quadrics in the case $\mathscr{E}_6(\mathbb{K})$*, or 9 and 15 basic quadrics in the cases* $\mathscr{S}_{2,2}(\mathbb{K})$ *and* $\mathscr{G}_{2,6}(\mathbb{K})$*, respectively. It is proved in a yet unpublished manuscript of A. De Schepper and M. Victoor that there are exactly three possibilities (corresponding to the "duals" of the white, grey and black points): For* $\mathscr{E}_6(\mathbb{K})$*, these are non-degenerate parabolic quadrics (hence of maximal Witt index) and degenerate quadrics with an* 8*- or* 16*-dimensional radical (projective dimension) and hyperbolic base. Similarly,* *for* $\mathscr{G}_{2,6}(\mathbb{K})$*, we have non-degenerate parabolic quadrics and degenerate quadrics with a* 4- *or* 8-*dimensional radical and hyperbolic base; for* $\mathscr{S}_{2,2}(\mathbb{K})$*, we have non-degenerate parabolic quadrics and degenerate quadrics with a* 2- *or* 4-*dimensional radical and hyperbolic base.*

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Ideal based graph structures for commutative rings

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ABSTRACT

We introduce a graph structure $\Gamma_2^*(R)$ for commutative rings with unity. We study some of the properties of the graph $\Gamma_2^*(R)$. Also we study some parameters of $\Gamma_2^*(R)$ and find rings for which $\Gamma_2^*(R)$ is split.

RESUMEN

Introducimos una estructura de grafo $\Gamma_2^*(R)$ para anillos conmutativos con unidad. Estudiamos algunas de las propiedades del grafo $\Gamma_2^*(R)$. También estudiamos algunos parámetros de $\Gamma_2^*(R)$ y encontramos anillos para los cuales $\Gamma_2^*(R)$ se escinde.

Keywords and Phrases: Maximal ideal, idempotent, clique number, domination number, split graph.

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1 Introduction

The idea of relating a commutative ring to a graph was introduced by Istvan Beck [3]. He introduced a graph, $\Gamma(R)$, whose vertices are the elements of R and two distinct vertices x and y are adjacent if and only if $xy = 0$. In [1], Anderson and Livingston modified the definition of Beck to introduce the *zero-divisor graph*, $\Gamma^*(R)$, and investigated many of its properties. $\Gamma^*(R)$ is the subgraph of $\Gamma(R)$ induced by the set of non-zero zero-divisors of R. Cherian Thomas introduced many graph structures for R in [10] and obtained many interesting results.

Throughout the paper, the word 'ring' shall mean a commutative ring with $1 \neq 0$ which is not a field. We denote the Jacobson radical of a ring R by $\mathfrak{J}(R)$ and the set of all maximal ideals by $\max R$.

For the basic concepts from graph theory refer [4, 9]; for commutative ring theory, see [2].

We give two ideal based graphs, $\Gamma_1(R)$ and $\Gamma_2(R)$, introduced in [10].

The graph $\Gamma_1(R)$ has all ideals of R as vertices and two distinct vertices $\mathfrak{a}$ and $\mathfrak{b}$ are adjacent if and only if $\mathfrak{a}\mathfrak{b} = 0$.

The graph $\Gamma_2(R)$ has the same vertex set as that of $\Gamma_1(R)$ and two distinct vertices $\mathfrak{a}$ and $\mathfrak{b}$ are adjacent if and only if $\mathfrak{a} + \mathfrak{b} = R$.

In [5], the authors have studied the subgraph $\Gamma_1^*(R)$ of $\Gamma_1(R)$ induced by all the non-zero proper ideals of R.

We state the following result:

Theorem 1.1 ([5])**.** *Let R be an Artin ring. $\Gamma_1^*(R)$ is complete if and only if one of the following holds:*

(i) $R \cong F_1 \oplus F_2$ *where* F_1 *and* F_2 *are fields.*

(ii) R *is local with maximal ideal* $\mathfrak{m}$ *having index of nilpotency* 2.

(iii) R *is local with principal maximal ideal* $\mathfrak{m}$ *having index of nilpotency* 3.

In [8], S. C. Mathew has introduced and studied some basic properties of $\Gamma_2^*(R)$ which is the subgraph of $\Gamma_2(R)$ induced by the set of all non-zero proper ideals of R. In this paper we include those results, for the sake of completeness. We compare the graphs $\Gamma_1^*(R)$ and $\Gamma_2^*(R)$ and find the clique number and domination number of $\Gamma_2^*(R)$. Also we investigate the properties of rings for which $\Gamma_2^*(R)$ is split.

2 The graph $\Gamma_2^*(R)$ and its properties

In this section we define the graph $\Gamma_2^*(R)$ and investigate some properties of the graph.

Definition 2.1. Let R be a ring. We associate a graph $\Gamma_2^*(R)$ to R whose vertex set is the set of all non-zero proper ideals of R and for distinct ideals $\mathfrak{a}$ and $\mathfrak{b}$, the corresponding vertices are adjacent if and only if $\mathfrak{a}+\mathfrak{b}=R$.

Remark 2.2. $\Gamma_2^*(R)$ is totally disconnected if and only if R is local.

Remark 2.3. $\Gamma_2^*(R)=K_1$ if and only if $(R,\mathfrak{m})$ is local with $\mathfrak{m}$ principal and $\mathfrak{m}^2=0$.

Theorem 2.4. *Let R be a non-local ring. Then $\Gamma_2^*(R)$ is connected if and only if $\mathfrak{J}(R)=0$.*

Proof. **(⇒):** Assume $\Gamma_2^*(R)$ is connected. If $\mathfrak{J}(R)\neq 0$, then $\mathfrak{J}(R)$ is an isolated vertex in $\Gamma_2^*(R)$.

(⇐): Assume that $\mathfrak{J}(R)=0$. Now, $\max R$ induces a complete subgraph in $\Gamma_2^*(R)$. Let $\mathfrak{a}$ be any proper non-zero non maximal ideal. Since $\mathfrak{J}(R)=0$, there exists a maximal ideal $\mathfrak{m}$ such that $\mathfrak{a}\nsubseteq\mathfrak{m}$. Thus $\mathfrak{a}$ is adjacent to $\mathfrak{m}$ and hence $\Gamma_2^*(R)$ is connected. □

Corollary 2.5. *If $\Gamma_2^*(R)$ is connected,* $\operatorname{diam}\Gamma_2^*(R)\leq 3$.

Remark 2.6. $\mathfrak{a}$ is an isolated vertex of $\Gamma_2^*(R)$ if and only if $\mathfrak{a}\subseteq\mathfrak{J}(R)$.

Next result follows from the proof of Theorem 2.4 and Remark 2.6.

Theorem 2.7. *$\Gamma_2^*(R)$ is connected except for isolated vertices. That is, $\Gamma_2^*(R)$ has at most one component different from K_1.*

Theorem 2.8. *$\Gamma_2^*(R)\cong K_2$ if and only if R is a direct sum of two fields.*

Proof. **(⇒):** Let $R\cong F_1\oplus F_2$ where F_1 and F_2 are fields. Then the ideals of R are $F_1\oplus 0$, $0\oplus F_2$, $0\oplus 0$ and $F_1\oplus F_2$. Then, $\Gamma_2^*(R)\cong K_2$.

(⇐): Suppose $\Gamma_2^*(R)\cong K_2$. Then R is non-local. Also, R cannot have more than two maximal ideals. Therefore R has exactly two maximal ideals, say $\mathfrak{m}_1$ and $\mathfrak{m}_2$ with $\mathfrak{m}_1\cap\mathfrak{m}_2=0$. This implies $R\cong\dfrac{R}{\mathfrak{m}_1}\oplus\dfrac{R}{\mathfrak{m}_2}$, a direct sum of two fields. □

Theorem 2.9. *The only triangle free connected graphs that can be realized as $\Gamma_2^*(R)$ are K_1 and K_2.*

Proof. Let G be a triangle free connected graph. Since G is triangle free R can have at most two maximal ideals. Also since G is connected the result follows. □

Theorem 2.10. *$\Gamma_2^*(R)$ is complete if and only if either R is a direct sum of two fields or R is local with principal maximal ideal having index of nilpotency* 2.

Proof. **(⇒):** If $\Gamma_2^*(R)$ is complete, R can have at most two maximal ideals. For, assume R has 3 maximal ideals say, $\mathfrak{m}_1, \mathfrak{m}_2$ and $\mathfrak{m}_3$. Then $\mathfrak{m}_1\mathfrak{m}_2 = 0$; otherwise $\mathfrak{m}_1\mathfrak{m}_2$ is a vertex of $\Gamma_2^*(R)$ and will not be adjacent to $\mathfrak{m}_1$ and $\mathfrak{m}_2$. For the same reason, $\mathfrak{m}_1\mathfrak{m}_3 = 0$. Then $\mathfrak{m}_1(\mathfrak{m}_2 + \mathfrak{m}_3) = 0$. This implies $\mathfrak{m}_1 = 0$ which is not possible. Now assume that R has exactly 2 maximal ideals say, $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Then $\mathfrak{J}(R) = \mathfrak{m}_1\mathfrak{m}_2 = \{0\}$. Thus R is a direct sum of 2 fields. Now, if R is local with maximal ideal $\mathfrak{m}$, since $\Gamma_2^*(R)$ is complete, $\mathfrak{m}$ must be principal with index of nilpotency 2.

(⇐): If R is a direct sum of two fields, $\Gamma_2^*(R) \cong K_2$ and if R is local with principal maximal ideal having index of nilpotency 2, $\Gamma_2^*(R) \cong K_1$. □

The following corollary is immediate.

Corollary 2.11. *The only complete graphs that can be realized as $\Gamma_2^*(R)$ are K_1 and K_2.*

3 Comparison between $\Gamma_1^*(R)$ and $\Gamma_2^*(R)$

Theorem 3.1. *Assume* $\operatorname{diam}\Gamma_2^*(R) = 2$. *Then any two vertices in $\Gamma_2^*(R)$ which are not adjacent are also not adjacent in $\Gamma_1^*(R)$. That is, $\Gamma_1^*(R)$ is a subgraph of $\Gamma_2^*(R)$.*

Proof. Let $\operatorname{diam}\Gamma_2^*(R) = 2$. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are not adjacent in $\Gamma_2^*(R)$. Then, there exists a maximal ideal $\mathfrak{m}$ such that $\mathfrak{a} + \mathfrak{m} = R = \mathfrak{b} + \mathfrak{m}$. Therefore, $(\mathfrak{a} + \mathfrak{m})(\mathfrak{b} + \mathfrak{m}) = R$. That is, $\mathfrak{ab} + \mathfrak{am} + \mathfrak{bm} + \mathfrak{m}^2 = R$.

But, $\mathfrak{ab} + \mathfrak{am} + \mathfrak{bm} + \mathfrak{m}^2 \subseteq \mathfrak{ab} + \mathfrak{m}$. Therefore, $\mathfrak{ab} + \mathfrak{m} = R$. This implies, in particular, $\mathfrak{ab} \neq 0$. Thus, $\mathfrak{a}$ and $\mathfrak{b}$ are not adjacent in $\Gamma_1^*(R)$. □

Remark 3.2. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are adjacent in $\Gamma_2^*(R)$. Then, $\mathfrak{a} + \mathfrak{b} = R$. This implies $\mathfrak{ab} = \mathfrak{a} \cap \mathfrak{b}$. Hence $\mathfrak{a}$ is adjacent to $\mathfrak{b}$ in $\Gamma_1^*(R)$ if and only if $\mathfrak{a} \cap \mathfrak{b} = 0$. This must hold for every pair of comaximal ideals $\mathfrak{a}$ and $\mathfrak{b}$.

Theorem 3.3. *Let R be a non-local ring. Then, $\Gamma_2^*(R)$ is a subgraph of $\Gamma_1^*(R)$ if and only if R is a direct sum of two fields; and hence $\Gamma_1^*(R) = \Gamma_2^*(R)$ only when R is a direct sum of two fields.*

Proof. **(⇒):** $\Gamma_2^*(R)$ is a subgraph of $\Gamma_1^*(R)$ if and only if for any pair of comaximal ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R, $\mathfrak{ab} = 0$. So, if $\Gamma_2^*(R)$ is a subgraph of $\Gamma_1^*(R)$, in particular, $\mathfrak{m}_1\mathfrak{m}_2 = 0$ where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are two maximal ideals of R. Hence, $R \cong \frac{R}{\mathfrak{m}_1} \oplus \frac{R}{\mathfrak{m}_2}$.

(⇐): If R is a direct sum of two fields, $\Gamma_1^*(R) = \Gamma_2^*(R) = K_2$. □

Theorem 3.4. *If R is a finite direct sum of fields, $\Gamma_1^*(R) \cong \Gamma_2^*(R)$.*

Proof. Let $R = F_1 \oplus F_2 \oplus \cdots \oplus F_n$ where F_i's are fields. Thus, an ideal $\mathfrak{a}$ of R is of the form, $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_n$ where, $\mathfrak{a}_i = 0$ or F_i.

Define $\varphi : V(\Gamma_1^*(R)) \to V(\Gamma_2^*(R))$ by $\varphi(\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_n) = \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \cdots \oplus \mathfrak{b}_n$ where

$$\mathfrak{b}_i = \begin{cases} F_i, & \text{if } \mathfrak{a}_i = (0) \\ 0, & \text{if } \mathfrak{a}_i = F_i. \end{cases}$$

Clearly, φ is a bijection.

Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are adjacent in $\Gamma_1^*(R)$. Thus $\mathfrak{b}$ must contain 0 at the positions in which $\mathfrak{a}$ contains F_i's. Therefore, $\varphi(\mathfrak{b})$ contains F_i's at the positions where $\varphi(\mathfrak{a})$ contains 0. Then, $\varphi(\mathfrak{a})$ is adjacent to $\varphi(\mathfrak{b})$.

Similarly, if $\varphi(\mathfrak{a})$ and $\varphi(\mathfrak{b})$ are adjacent in $\Gamma_2^*(R)$ then, $\mathfrak{a}$ and $\mathfrak{b}$ are adjacent in $\Gamma_1^*(R)$. Thus, φ is a graph isomorphism. That is, $\Gamma_1^*(R) \cong \Gamma_2^*(R)$. □

Remark 3.5. *In the context of Theorem 3.4, we can explicitly determine $\Gamma_1^*(R)$ and $\Gamma_2^*(R)$ by identifying the vertex set with the power set $P(X) \setminus \{X, \varnothing\}$ where $X = \{1, 2, \ldots, n\}$ and $A \subset X$ with $\bigoplus_{i \in A} F_i$. Then A and B are adjacent in $\Gamma_1^*(R)$ if and only if $A \cap B = \varnothing$ and A and B are adjacent in $\Gamma_2^*(R)$ if and only if $A \cup B = X$.*

Theorem 3.6. *$\Gamma_1^*(R)$ and $\Gamma_2^*(R)$ are edge disjoint if and only if R has no non-trivial idempotents.*

Proof. **(⇒):** Suppose that R contains a non-trivial idempotent e. Then, $R = Re \oplus R(1-e)$. This implies, $Re + R(1-e) = R$ and $Re \cap R(1-e) = ReR(1-e) = 0$. That is, $\Gamma_1^*(R)$ and $\Gamma_2^*(R)$ are not edge disjoint.

(⇐): Assume that $\Gamma_1^*(R)$ and $\Gamma_2^*(R)$ are not edge disjoint and then there exist two ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a} + \mathfrak{b} = R$ and $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b} = 0$. Then, $R = \mathfrak{a} \oplus \mathfrak{b}$ and hence, $\mathfrak{a} = Re$ and $\mathfrak{b} = R(1-e)$ for some idempotent e. Since $\mathfrak{a}$ and $\mathfrak{b}$ are non-zero proper ideals, e must be non-trivial. □

Theorem 3.7. *Let R be a non-local ring. If $\Gamma_1^*(R) = \overline{\Gamma_2^*(R)}$, R is not semi-local.*

Proof. Assume that R is semi-local with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$. Then, there are the following possibilities.

Case (I): $\Gamma_2^*(R)$ is connected.

This assumption implies $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n = 0$, by Theorem 2.4. Therefore, $(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}), \mathfrak{m}_n$ are adjacent in $\Gamma_1^*(R)$ as well as in $\Gamma_2^*(R)$, which means $\Gamma_1^*(R) \neq \overline{\Gamma_2^*(R)}$.

Case (II): $\Gamma_2^*(R)$ is disconnected.

This implies $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n = \mathfrak{J}(R) \neq 0$. We subdivide this case into two.

Case (II)(a): $\mathfrak{J}(R)$ is nilpotent.

Then there exist least positive integers $k_1, k_2, \ldots, k_n$ such that $\mathfrak{m}_1^{k_1}\mathfrak{m}_2^{k_2}\cdots\mathfrak{m}_n^{k_n} = 0$ with at least one $k_j > 1$ for $1 \leq j \leq n$, say $k_n > 1$.

If $k_n > 2$, we have $(\mathfrak{m}_1\cdots\mathfrak{m}_n) + \mathfrak{m}_n \neq R$ and $(\mathfrak{m}_1\cdots\mathfrak{m}_n)\mathfrak{m}_n \neq 0$. That is, $\Gamma_1^*(R) \neq \overline{\Gamma_2^*(R)}$.

Now consider the case when $k_n = 2$. If $k_i > 1$ for some $i \neq n$, $(\mathfrak{m}_1\cdots\mathfrak{m}_n)+\mathfrak{m}_n \neq R$ and $(\mathfrak{m}_1\cdots\mathfrak{m}_n)\mathfrak{m}_n \neq 0$. If $k_i = 1$ $\forall i \neq n$, $(\mathfrak{m}_1\cdots\mathfrak{m}_{l-1}\mathfrak{m}_{l+1}\cdots\mathfrak{m}_n^2) + \mathfrak{m}_l = R$ where $l \neq n$. But, $(\mathfrak{m}_1\cdots\mathfrak{m}_{l-1}\mathfrak{m}_{l+1}\cdots\mathfrak{m}_n^2)\mathfrak{m}_l = 0$. So, $\Gamma_1^*(R) \neq \overline{\Gamma_2^*(R)}$.

Case (II)(b): $\mathfrak{J}(R)$ is not nilpotent.

In this case we have $(\mathfrak{m}_1\cdots\mathfrak{m}_n) + \mathfrak{m}_1 \neq R$ and $(\mathfrak{m}_1\cdots\mathfrak{m}_n)\mathfrak{m}_1 \neq 0$.

Thus, if $\Gamma_1^*(R) = \overline{\Gamma_2^*(R)}$, R cannot be semi-local. □

Theorem 3.8. *Let $(R, \mathfrak{m})$ be an Artin local ring. Then, $\Gamma_1^*(R) = \overline{\Gamma_2^*(R)}$ if and only if either $\mathfrak{m}$ has index of nilpotency 2 or $\mathfrak{m}$ is principal with index of nilpotency 3.*

Proof. Follows from Remark 2.2 and Theorem 1.1. □

4 Some parameters of $\Gamma_2^*(R)$

In this section we find the clique number and the domination number of $\Gamma_2^*(R)$.

Theorem 4.1. $cl(\Gamma_2^*(R)) = |\max R|$.

Proof. Clearly $\max R$ induces a complete subgraph. Let $\mathfrak{a}$ be any non-zero non-maximal proper ideal of R. Then $\mathfrak{a}$ is contained in a maximal ideal. That is, there exists a maximal ideal $\mathfrak{m}$ such that $\mathfrak{a}$ is not adjacent to $\mathfrak{m}$. Thus, $\max R$ induces a maximal complete subgraph.

Now suppose $S = \{\mathfrak{a}_i : i \in \Lambda\}$, where Λ is an index set, induces a complete subgraph in $\Gamma_2^*(R)$. Then one maximal ideal can contain at most one $\mathfrak{a}_i \in S$. That is, there exists an injective map from S to $\max R$. This implies, $|S| \leq |\max R|$. Thus, $cl(\Gamma_2^*(R)) = |\max R|$. □

Theorem 4.2. *Let R be a semi local ring with $|\max R| = n > 2$. Then, $\gamma(\Gamma_2^*(R)) = |\max R| +$ Number of isolated vertices in $\Gamma_2^*(R)$.*

Proof. Let $\Gamma_2^{**}(R)$ be the connected component of $\Gamma_2^*(R)$ induced by the non-isolated vertices of $\Gamma_2^*(R)$. Now, by Theorem 2.7, it is enough to show that $\gamma(\Gamma_2^{**}(R)) = |\max R|$.

Let $\max R = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$. Clearly $\max R$ is a dominating set for $\Gamma_2^{**}(R)$. Now consider, $S = \{\mathfrak{m}_2\cdots\mathfrak{m}_n, \mathfrak{m}_1\mathfrak{m}_3\cdots\mathfrak{m}_n, \ldots, \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_{n-1}\}$, which is an independent set in $\Gamma_2^{**}(R)$. Note that any ideal $\mathfrak{a} \notin S$ can be adjacent only to at most one element of S. So every dominating set in $\Gamma_2^{**}(R)$ must contain at least n elements. Thus, $\gamma(\Gamma_2^{**}(R)) = n = |\max R|$. Hence the result follows. $\square$

Remark 4.3. If R is a semi-local ring with $|\max R| = 2$ then, the above result is not true. For example, if R is a direct sum of two fields, $\gamma(\Gamma_2^{*}(R)) = \gamma(K_2) = 1$ but $|\max R| = 2$.

5 Splitness

A graph (V, E) is said to be a *split graph* if V is the disjoint union of two sets K and S where K induces a complete subgraph and S is an independent set. Then, we can assume either K is a clique or S is a maximal independent set. In [6] & [7], the authors have carried out a detailed study on splitness of some graphs associated with a ring. In this section we continue the study in the case of $\Gamma_2^{*}(R)$.

Lemma 5.1. *Let $R = R_1 \times R_2 \times R_3$ be a ring. If $\Gamma_2^{*}(R)$ is split, each R_i must be a field.*

Proof. Suppose R_1 is not a field. Then there exists a proper non-zero ideal I of R_1. Then, $\{I \times R_2 \times R_3, R_1 \times R_2 \times 0, 0 \times R_2 \times R_3, R_1 \times 0 \times 0\}$ induces a C_4 in $\Gamma_2^{*}(R)$, a contradiction. $\square$

Lemma 5.2. *If F_i $(1 \leq i \leq 3)$ are fields and $R = F_1 \times F_2 \times F_3$ then $\Gamma_2^{*}(R)$ is split.*

Proof. $V(\Gamma_2^{*}(R))$ can be partitioned into $K = \{F_1 \times F_2 \times 0, F_1 \times 0 \times F_3, 0 \times F_2 \times F_3\}$ and $S = \{F_1 \times 0 \times 0, 0 \times F_2 \times 0, 0 \times 0 \times F_3\}$ where K induces a complete subgraph and S is an independent set. $\square$

Lemma 5.3. *Let F be a field and R_1 a local ring. Let $R = R_1 \times F$. Then $\Gamma_2^{*}(R)$ is split.*

Proof. Let $\{I_j : j \in J\}$ be the collection of non-zero proper ideals of R_1. Then $\{I_j \times F : j \in J\} \cup \{I_j \times 0 : j \in J\}$ is an independent set and $\{0 \times F, R_1 \times 0\}$ is a K_2. This forms a partition of $V(\Gamma_2^{*}(R))$. Thus, $\Gamma_2^{*}(R)$ is split. $\square$

Lemma 5.4. *Suppose R has exactly n maximal ideals $\mathfrak{m}_i$ $(1 \leq i \leq n)$ with each $\mathfrak{m}_i$ being generated by an idempotent e_i. Then $R \cong \prod_{i=1}^{n} F_i$ where each $F_i \cong R/\mathfrak{m}_i$, a field.*

Proof. Let $e = \prod_{i=1}^{n} e_i$. Then $e \in \mathfrak{J}(R)$. Therefore, $1 - e$ is a unit (and an idempotent). So, $1 - e = 1 \Rightarrow e = 0$. Then by the Chinese Remainder Theorem,

$$R \cong \frac{R}{\prod_{i=1}^{n} Re_i} \cong \frac{R}{\bigcap_{i=1}^{n} Re_i} \cong \prod_{i=1}^{n} \frac{R}{Re_i}.$$

$\square$

Theorem 5.5. *Let R be a ring. $\Gamma_2^*(R)$ is a split graph if and only if one of the following conditions holds:*

(i) R *is local.*

(ii) $R \cong R_1 \times F$ *where* R_1 *is a local ring and* F *is a field.*

(iii) $R \cong F_1 \times F_2 \times F_3$ *where* F_i*'s are fields.*

Proof. First we note that $\Gamma_2^*(R)$ is split if and only if $\Gamma_2(R)$ is split. Also, if R is local, $\Gamma_2^*(R)$ is split. Sufficiency of other conditions follows from the lemmas. To prove the necessity of the conditions, we assume that R is not local and $V(\Gamma_2(R))$ is the disjoint union of two sets K and S where K induces a complete subgraph and S is an independent set. We assume that K and S are non-empty. Also, S can contain at most one maximal ideal.

Case (I): S contains a maximal ideal, say $\mathfrak{m}_1$.

In this case, R can have only one maximal ideal other than $\mathfrak{m}_1$. For, if $\mathfrak{m}_2$ and $\mathfrak{m}_3$ are distinct maximal ideals other than $\mathfrak{m}_1$, then $\mathfrak{m}_2$ and $\mathfrak{m}_3$ are in K. Then, $\mathfrak{m}_2\mathfrak{m}_3 \in S$, $\mathfrak{m}_1 \in S$. Clearly, $\mathfrak{m}_1 + \mathfrak{m}_2\mathfrak{m}_3 = R$, a contradiction. Thus, R contains only one maximal ideal other than $\mathfrak{m}_1$, say $\mathfrak{m}_2$ which belongs to K. Let $x_i \in \mathfrak{m}_i$ $(i = 1, 2)$ with $x_1 + x_2 = 1$. As $\mathfrak{m}_2^2 + \mathfrak{m}_1 = R$, $\mathfrak{m}_2^2 \in K$ which implies $\mathfrak{m}_2^2 = \mathfrak{m}_2$. Similarly, as $Rx_2 + \mathfrak{m}_1 = R$, $Rx_2 \in K$ which implies $\mathfrak{m}_2 = Rx_2$. Then, $\mathfrak{m}_2$ is a finitely generated maximal ideal which is idempotent. Hence, $\mathfrak{m}_2$ is generated by an idempotent. So, $R \cong R_1 \times F$ where F is a field and $\mathfrak{m}_2$ is isomorphic to the ideal $R_1 \times \{0\}$. Further, R_1 must be local.

Case (II): S contains no maximal ideal.

In this case, R can have at most three maximal ideals, for, if $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ and $\mathfrak{m}_4$ are distinct maximal ideals, $\mathfrak{m}_1\mathfrak{m}_2$ and $\mathfrak{m}_3\mathfrak{m}_4$ are in S which leads to a contradiction. If R has only two maximal ideals, say, $\mathfrak{m}_1$ and $\mathfrak{m}_2$, then $\mathfrak{m}_1, \mathfrak{m}_2 \in K$. Since, $\mathfrak{m}_i^2 + \mathfrak{m}_i \neq R$ $(i = 1, 2)$, we have $\mathfrak{m}_1^2, \mathfrak{m}_2^2 \in S$. But $\mathfrak{m}_1^2 + \mathfrak{m}_2^2 = R$. So, to avoid a contradiction we have to assume $\mathfrak{m}_1^2 = \mathfrak{m}_1$ or $\mathfrak{m}_2^2 = \mathfrak{m}_2$. That is, $R \cong R_1 \times F$ where F is a field and R_1 is a local ring. So, let us assume R has exactly 3 maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$ and $\mathfrak{m}_3$. Note that $\mathfrak{m}_i \in K$ $(i = 1, 2, 3)$. Then, as $\mathfrak{m}_1 + \mathfrak{m}_2\mathfrak{m}_3 = R$, there exists $x_1 \in \mathfrak{m}_1$ such that $Rx_1 + \mathfrak{m}_2\mathfrak{m}_3 = R$ which implies $Rx_1 \in K$ and hence, $Rx_1 = \mathfrak{m}_1$. Similarly arguing with $\mathfrak{m}_1^2 + \mathfrak{m}_2\mathfrak{m}_3 = R$, we get $\mathfrak{m}_1 = \mathfrak{m}_1^2$. Then $\mathfrak{m}_1$ is generated by an idempotent. Similarly each $\mathfrak{m}_j$ $(j = 2, 3)$ is generated by an idempotent. Then by the Lemma 5.4, $R \cong F_1 \times F_2 \times F_3$ where F_i $(1 \leq i \leq 3)$ are fields. □

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Fixed point results of (ϕ, ψ)-weak contractions in ordered b-metric spaces

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ABSTRACT

The purpose of this paper is to prove some results on fixed point, coincidence point, coupled coincidence point and coupled common fixed point for the mappings satisfying generalized (ϕ, ψ)-contraction conditions in complete partially ordered b-metric spaces. Our results generalize, extend and unify most of the fundamental metrical fixed point theorems in the existing literature. A few examples are illustrated to support our findings.

RESUMEN

El propósito de este artículo es demostrar algunos resultados sobre puntos fijos, puntos de coincidencia, puntos de coincidencia acoplados y puntos de coincidencia acoplados comunes para aplicaciones que satisfacen condiciones de (ϕ, ψ)-contracción generalizadas en b-espacios métricos completos parcialmente ordenados. Nuestros resultados generalizan, extienden y unifican la mayoría de los teoremas de punto fijo métricos fundamentales en la literatura existente. Se ilustran algunos ejemplos para apoyar nuestros resultados.

Keywords and Phrases: Fixed point, coupled coincidence point, coupled common fixed point, partially ordered b-metric space, compatible, mixed f-monotone.

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1 Introduction

The usual metric space has been generalized and enhanced in many different directions, one of such generalizations is a b-metric space which was first coined by Czerwik in [16] and is also known as metric type space (Khamsi and Hussain [35] used recently the term "metric type space")'. Indeed, in some papers it is considered that this concept has been introduced by Bourbaki [14] in 1974, or that it has been introduced by Bakhtin [12] in 1989, or by Czerwik [16] in 1993 or even by Czerwik [17] in 1998. After extensive searches in zbMATH and Mathematical Reviews, it appears that the first fixed point theorem in a quasimetric space (b-metric spaces) has been established in 1981 by Vulpe *et al.* [55], who transposed the Picard-Banach contraction mapping principle from metric spaces to the framework of a quasimetric space. Some important information on the introduction of a b-metric spaces can be found from the article "The early developments in fixed point theory on b-metric spaces: a brief survey and some important related aspects" by Berinde and Pacurar [13]. Later, a series of papers have been dedicated to the improvement of fixed point results for single valued and multi-valued operators on b-metric spaces by following various topological properties, some of such are from [1, 3, 6, 5, 9, 20, 22, 28, 29, 30, 32, 34, 36, 39, 40, 41, 43, 53].

The concept of coupled fixed points for certain mappings in ordered spaces was first introduced by Bhaskar *et al.* [23] and applied their results to study the existence and uniqueness of the solutions for boundary valued problems. While the concept of coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings with monotone property in complete partially ordered metric spaces was first introduced by Lakshmikantham *et al.* [37]. Since then, several authors have carried out further generalizations and improvements in various spaces (see [10, 18, 21, 24, 44, 48]). Aghajani *et al.* [2] proved some coupled coincidence and coupled fixed point results for mappings satisfying generalized (ψ, ϕ, θ)-contractive conditions in partially ordered complete b-metric spaces. Later, the results of [2] have been improved and generalized by Huaping Huang *et al.* [27] in the same space. More works on coupled coincidence and coupled fixed point results for generalized contraction mappings in ordered spaces can be seen from [4, 7, 8, 11, 15, 19, 25, 26, 31, 38, 42, 45, 46, 47, 49, 50, 51, 52]. Recently, some results on fixed point, coincidence point and coupled coincidence points for the mappings satisfying generalized weak contraction contractions in partially ordered b-metric spaces have been discussed by Belay Mituku *et al.* [39], Seshagiri Rao *et al.* [53, 54] and Kalyani *et al.* [33].

The aim of this work is to provide some results on fixed point and coincidence point, coupled coincidence point for the mappings satisfying generalized (ϕ, ψ)-contractive conditions in an ordered b-metric space. Our results are the variations and the generalizations of the results of [25, 26, 31, 38, 42, 45, 52] and several comparable results in the existing literature. A few numerical examples are illustrated to support the findings.

2 Mathematical Preliminaries

The following definitions and results will be needed in what follows.

Definition 2.1 ([39, 53]). *A mapping $d : P \times P \to [0, +\infty)$, where P is a non-empty set is said to be a b-metric, if it satisfies the properties given below for any $\upsilon, \xi, \mu \in P$ and for some real number $s \geq 1$,*

(a) $d(\upsilon, \xi) = 0$ *if and only if* $\upsilon = \xi$,

(b) $d(\upsilon, \xi) = d(\xi, \upsilon)$,

(c) $d(\upsilon, \xi) \leq s\,(d(\upsilon, \mu) + d(\mu, \xi))$.

Then (P, d, s) is known as a b-metric space. If $(P, \preceq)$ is still a partially ordered set, then $(P, d, s, \preceq)$ is called a partially ordered b-metric space.

Definition 2.2 ([39, 53]). *Let (P, d, s) be a b-metric space. Then*

(1) a sequence $\{\upsilon_n\}$ is said to converge to υ, if $\lim_{n\to+\infty} d(\upsilon_n, \upsilon) = 0$ and written as $\lim_{n\to+\infty} \upsilon_n = \upsilon$.

(2) $\{\upsilon_n\}$ is said to be a Cauchy sequence in P, if $\lim_{n,m\to+\infty} d(\upsilon_n, \upsilon_m) = 0$.

(3) (P, d) is said to be complete, if every Cauchy sequence in it is convergent.

Definition 2.3. *If the metric d is complete then $(P, d, s, \preceq)$ is called complete partially ordered b-metric space.*

Definition 2.4 ([39]). *Let $(P, \preceq)$ be a partially ordered set and let $\mathscr{f}, \mathscr{g} : P \to P$ be two mappings. Then*

(1) $\mathscr{g}$ is called monotone non-decreasing, if $\mathscr{g}\upsilon \preceq \mathscr{g}\xi$ for all $\upsilon, \xi \in P$ with $\upsilon \preceq \xi$.

(2) an element $\upsilon \in P$ is called a coincidence (common fixed) point of $\mathscr{f}$ and $\mathscr{g}$, if $\mathscr{f}\upsilon = \mathscr{g}\upsilon$ ($\mathscr{f}\upsilon = \mathscr{g}\upsilon = \upsilon$).

(3) $\mathscr{f}$ and $\mathscr{g}$ are called commuting, if $\mathscr{f}\mathscr{g}\upsilon = \mathscr{g}\mathscr{f}\upsilon$, for all $\upsilon \in P$.

(4) $\mathscr{f}$ and $\mathscr{g}$ are called compatible, if any sequence $\{\upsilon_n\}$ with $\lim_{n\to+\infty} \mathscr{f}\upsilon_n = \lim_{n\to+\infty} \mathscr{g}\upsilon_n = \mu$, for $\mu \in P$ then $\lim_{n\to+\infty} d(\mathscr{g}\mathscr{f}\upsilon_n, \mathscr{f}\mathscr{g}\upsilon_n) = 0$.

(5) a pair of self maps $(\mathscr{f}, \mathscr{g})$ is called weakly compatible, if $\mathscr{f}\mathscr{g}\upsilon = \mathscr{g}\mathscr{f}\upsilon$, when $\mathscr{g}\upsilon = \mathscr{f}\upsilon$ for some $\upsilon \in P$.

(6) $\mathscr{g}$ is called monotone $\mathscr{f}$-non-decreasing, if

$$\mathscr{f}\upsilon \preceq \mathscr{f}\xi \quad \textit{implies} \quad \mathscr{g}\upsilon \preceq \mathscr{g}\xi, \ \textit{for any } \upsilon, \xi \in P.$$

(7) a non empty set P is called well ordered set, if every two elements of it are comparable i.e., $\upsilon \preceq \xi$ or $\xi \preceq \upsilon$, for $\upsilon, \xi \in P$.

Definition 2.5 ([2, 37]). *Let $(P, \preceq)$ be a partially ordered set and, let $\hbar : P \times P \to P$ and $\ell : P \to P$ be two mappings. Then*

(1) $\hbar$ has the mixed ℓ-monotone property, if $\hbar$ is non-decreasing ℓ-monotone in its first argument and is non-increasing ℓ-monotone in its second argument, that is for any $\upsilon, \xi \in P$

$$\upsilon_1, \upsilon_2 \in P, \quad \ell\upsilon_1 \preceq \ell\upsilon_2 \quad \text{implies} \quad \hbar(\upsilon_1, \xi) \preceq \hbar(\upsilon_2, \xi) \text{ and}$$
$$\xi_1, \xi_2 \in P, \quad \ell\xi_1 \preceq \ell\xi_2 \quad \text{implies} \quad \hbar(\upsilon, \xi_1) \succeq \hbar(\upsilon, \xi_2).$$

Suppose, if ℓ is the identity mapping then $\hbar$ is said to have the mixed monotone property.

(2) an element $(\upsilon, \xi) \in P \times P$ is called a coupled coincidence point of $\hbar$ and ℓ, if $\hbar(\upsilon, \xi) = \ell\upsilon$ and $\hbar(\xi, \upsilon) = \ell\xi$. Note that, if ℓ is the identity mapping then (υ, ξ) is said to be a coupled fixed point of $\hbar$.

(3) an element $\upsilon \in P$ is called a common fixed point of $\hbar$ and ℓ, if $\hbar(\upsilon, \upsilon) = \ell\upsilon = \upsilon$.

(4) $\hbar$ and ℓ are commutative, if for all $\upsilon, \xi \in P$, $\hbar(\ell\upsilon, \ell\xi) = \ell(\hbar\upsilon, \hbar\xi)$.

(5) $\hbar$ and ℓ are said to be compatible, if

$$\lim_{n\to+\infty} d(\ell(\hbar(\upsilon_n, \xi_n)), \hbar(\ell\upsilon_n, \ell\xi_n)) = 0 \text{ and } \lim_{n\to+\infty} d(\ell(\hbar(\xi_n, \upsilon_n)), \hbar(\ell\xi_n, \ell\upsilon_n)) = 0,$$

whenever $\{\upsilon_n\}$ and $\{\xi_n\}$ are any two sequences in P such that $\lim_{n\to+\infty} \hbar(\upsilon_n, \xi_n) = \lim_{n\to+\infty} \ell\upsilon_n = \upsilon$ and $\lim_{n\to+\infty} \hbar(\xi_n, \upsilon_n) = \lim_{n\to+\infty} \ell\xi_n = \xi$, for any $\upsilon, \xi \in P$.

We know that a b-metric is not continuous and then we use frequently the following lemma in the proof of our results for the convergence of sequences in b-metric spaces.

Lemma 2.6 ([2]). *Let $(P, d, s, \preceq)$ be a b-metric space with $s > 1$ and suppose that $\{\upsilon_n\}$ and $\{\xi_n\}$ are b-convergent to υ and ξ respectively. Then we have*

$$\frac{1}{s^2} d(\upsilon, \xi) \leq \lim_{n\to+\infty} \inf d(\upsilon_n, \xi_n) \leq \lim_{n\to+\infty} \sup d(\upsilon_n, \xi_n) \leq s^2 d(\upsilon, \xi).$$

In particular, if $\upsilon = \xi$, then $\lim_{n\to+\infty} d(\upsilon_n, \xi_n) = 0$. Moreover, for each $\tau \in P$, we have

$$\frac{1}{s} d(\upsilon, \tau) \leq \lim_{n\to+\infty} \inf d(\upsilon_n, \tau) \leq \lim_{n\to+\infty} \sup d(\upsilon_n, \tau) \leq s d(\upsilon, \tau).$$

3 Main Results

The following distance functions are used throughout the paper.

A self mapping ϕ defined on $[0, +\infty)$ is said to be an altering distance function, if it satisfies the following conditions:

(i) ϕ is non-decreasing and continuous function,

(iii) $\phi(t) = 0$ if and only if $t = 0$.

Let us denote the set of all altering distance functions on $[0, +\infty)$ by Φ.

Similarly, Ψ denotes the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(i) ψ is lower semi-continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Let $(P, d, s, \preceq)$ be a partially ordered b-metric space with parameter $s > 1$ and, let $g : P \to P$ be a mapping. Set

$$M(\upsilon, \xi) = \max\left\{\frac{d(\xi, g\xi)\,[1 + d(\upsilon, g\upsilon)]}{1 + d(\upsilon, \xi)}, \frac{d(\upsilon, g\upsilon)\; d(\upsilon, g\xi)}{1 + d(\upsilon, g\xi) + d(\xi, g\upsilon)}, d(\upsilon, \xi)\right\}, \tag{3.1}$$

and

$$N(\upsilon, \xi) = \max\left\{\frac{d(\xi, g\xi)\,[1 + d(\upsilon, g\upsilon)]}{1 + d(\upsilon, \xi)}, d(\upsilon, \xi)\right\}. \tag{3.2}$$

Let $\phi \in \Phi$ and $\psi \in \Psi$. The mapping g is a generalized (ϕ, ψ)-contraction mapping if it satisfies the following condition

$$\phi(sd(g\upsilon, g\xi)) \leq \phi(M(\upsilon, \xi)) - \psi(N(\upsilon, \xi)), \tag{3.3}$$

for any $\upsilon, \xi \in P$ with $\upsilon \preceq \xi$ and M, N are same as above.

Now, we prove some results for the existence of fixed point, coincidence point, coupled coincidence point and coupled common fixed point of the mappings satisfying a generalized (ϕ, ψ)-contraction condition in the context of partially ordered b-metric space. We begin with the following fixed point theorem in this paper.

Theorem 3.1. *Suppose that $(P, d, s, \preceq)$ is a complete partially ordered b-metric space with parameter $s > 1$. Let $g : P \to P$ be a generalized (ϕ, ψ)-contractive mapping, and be continuous, non-decreasing mapping with respect to $\preceq$. If there exists $\upsilon_0 \in P$ with $\upsilon_0 \preceq g\upsilon_0$, then g has a fixed point in P.*

Proof. For some $\upsilon_0 \in P$ such that $g\upsilon_0 = \upsilon_0$, then we have the result. Assume that $\upsilon_0 \prec g\upsilon_0$, then construct a sequence $\{\upsilon_n\} \subset P$ by $\upsilon_{n+1} = g\upsilon_n$, for $n \geq 0$. Since g is non-decreasing, then by induction we obtain that

$$\upsilon_0 \prec g\upsilon_0 = \upsilon_1 \preceq \cdots \preceq \upsilon_n \preceq g\upsilon_n = \upsilon_{n+1} \preceq \cdots . \tag{3.4}$$

If for some $n_0 \in \mathbb{N}$ such that $\upsilon_{n_0} = \upsilon_{n_0+1}$ then from (3.4), υ_{n_0} is a fixed point of g and we have nothing to prove. Suppose that $\upsilon_n \neq \upsilon_{n+1}$, for all $n \geq 1$. Since $\upsilon_n > \upsilon_{n-1}$ for all $n \geq 1$ and then by condition (3.3), we have

$$\begin{aligned}\phi(d(\upsilon_n, \upsilon_{n+1})) = \phi(d(g\upsilon_{n-1}, g\upsilon_n)) &\leq \phi(sd(g\upsilon_{n-1}, g\upsilon_n)) \\ &\leq \phi(M(\upsilon_{n-1}, \upsilon_n)) - \psi(N(\upsilon_{n-1}, \upsilon_n)).\end{aligned} \tag{3.5}$$

From (3.5), we get

$$d(\upsilon_n, \upsilon_{n+1}) = d(g\upsilon_{n-1}, g\upsilon_n) \leq \frac{1}{s} M(\upsilon_{n-1}, \upsilon_n), \tag{3.6}$$

where

$$\begin{aligned}M(\upsilon_{n-1}, \upsilon_n) &= \max\left\{\frac{d(\upsilon_n, g\upsilon_n)\,[1 + d(\upsilon_{n-1}, g\upsilon_{n-1})]}{1 + d(\upsilon_{n-1}, \upsilon_n)}, \frac{d(\upsilon_{n-1}, g\upsilon_{n-1})\; d(\upsilon_{n-1}, g\upsilon_n)}{1 + d(\upsilon_{n-1}, g\upsilon_n) + d(\upsilon_n, g\upsilon_{n-1})},\right. \\ &\qquad \left. d(\upsilon_{n-1}, \upsilon_n)\right\} \\ &= \max\left\{d(\upsilon_n, \upsilon_{n+1}), \frac{d(\upsilon_{n-1}, \upsilon_n)\; d(\upsilon_{n-1}, \upsilon_{n+1})}{1 + d(\upsilon_{n-1}, \upsilon_{n+1})}, d(\upsilon_{n-1}, \upsilon_n)\right\} \\ &\leq \max\{d(\upsilon_n, \upsilon_{n+1}), d(\upsilon_{n-1}, \upsilon_n)\}.\end{aligned} \tag{3.7}$$

If $\max\{d(\upsilon_n, \upsilon_{n+1}), d(\upsilon_{n-1}, \upsilon_n)\} = d(\upsilon_n, \upsilon_{n+1})$ for some $n \geq 1$, then from (3.6) follows

$$d(\upsilon_n, \upsilon_{n+1}) \leq \frac{1}{s} d(\upsilon_n, \upsilon_{n+1}), \tag{3.8}$$

which is a contradiction. This means that $\max\{d(\upsilon_n, \upsilon_{n+1}), d(\upsilon_{n-1}, \upsilon_n)\} = d(\upsilon_{n-1}, \upsilon_n)$ for $n \geq 1$. Hence, we obtain from (3.6) that

$$d(\upsilon_n, \upsilon_{n+1}) \leq \frac{1}{s} d(\upsilon_{n-1}, \upsilon_n). \tag{3.9}$$

Since, $\frac{1}{s} \in (0, 1)$ then the sequence $\{\upsilon_n\}$ is a Cauchy sequence by [1, 6, 41, 22]. But P is complete, then there exists $\mu \in P$ such that $\upsilon_n \to \mu$.

Also, the continuity of g implies that

$$g\mu = g(\lim_{n\to+\infty} \upsilon_n) = \lim_{n\to+\infty} g\upsilon_n = \lim_{n\to+\infty} \upsilon_{n+1} = \mu. \tag{3.10}$$

Therefore, μ is a fixed point of g in P. □

Last result is still valid for g not necessarily continuous, assuming an additional hypothesis on P.

Theorem 3.2. *In Theorem 3.1 assume that P satisfies,*

$$\text{if a non-decreasing sequence } \{\upsilon_n\} \to \mu \text{ in } P, \text{ then } \upsilon_n \preceq \mu \text{ for all } n \in \mathbb{N}, \text{i.e., } \mu = \sup \upsilon_n.$$

Then a non-decreasing mapping g has a fixed point in P.

Proof. From Theorem 3.1, we take the same sequence $\{\upsilon_n\}$ in P such that $\upsilon_0 \preceq \upsilon_1 \preceq \cdots \preceq \upsilon_n \preceq \upsilon_{n+1} \preceq \cdots$, that is, $\{\upsilon_n\}$ is non-decreasing and converges to some $\mu \in P$. Thus from the hypotheses, we have $\upsilon_n \preceq \mu$, for any $n \in \mathbb{N}$, implies that $\mu = \sup \upsilon_n$.

Next, we prove that μ is a fixed point of g in P, that is $g\mu = \mu$. Suppose that $g\mu \neq \mu$. Let

$$M(\upsilon_n, \mu) = \max\left\{\frac{d(\mu, g\mu)\,[1 + d(\upsilon_n, g\upsilon_n)]}{1 + d(\upsilon_n, \mu)}, \frac{d(\upsilon_n, g\upsilon_n)\ d(\upsilon_n, g\mu)}{1 + d(\upsilon_n, g\mu) + d(\mu, g\upsilon_n)}, d(\upsilon_n, \mu)\right\}, \tag{3.11}$$

and

$$N(\upsilon_n, \mu) = \max\left\{\frac{d(\mu, g\mu)\,[1 + d(\upsilon_n, g\upsilon_n)]}{1 + d(\upsilon_n, \mu)}, d(\upsilon_n, \mu)\right\}. \tag{3.12}$$

Letting $n \to +\infty$ and from the fact that $\lim_{n\to+\infty} \upsilon_n = \mu$, we get

$$\lim_{n\to+\infty} M(\upsilon_n, \mu) = \max\{d(\mu, g\mu), 0, 0\} = d(\mu, g\mu), \tag{3.13}$$

and

$$\lim_{n\to+\infty} N(\upsilon_n, \mu) = \max\{d(\mu, g\mu), 0\} = d(\mu, g\mu). \tag{3.14}$$

We know that $\upsilon_n \preceq \mu$ for all n, then from contraction condition (3.3), we get

$$\phi(d(\upsilon_{n+1}, g\mu)) = \phi(d(g\upsilon_n, g\mu) \leq \phi(sd(g\upsilon_n, g\mu) \leq \phi(M(\upsilon_n, \mu)) - \psi(N(\upsilon_n, \mu)). \tag{3.15}$$

Letting $n \to +\infty$ and use of (3.13) and (3.14), we get

$$\phi(d(\mu, g\mu)) \leq \phi(d(\mu, g\mu)) - \psi(d(\mu, g\mu)) < \phi(d(\mu, g\mu)), \tag{3.16}$$

which is a contradiction under (3.16). Thus, $g\mu = \mu$, that is g has a fixed point μ in P. □

Now we give a sufficient condition for the uniqueness of the fixed point that exists in Theorem 3.1 and Theorem 3.2.

$$\text{every pair of elements has a lower bound or an upper bound.} \tag{3.17}$$

This condition is equivalent to,

$$\text{for every } \upsilon, \xi \in P, \text{ there exists } w \in P \text{ which is comparable to } \upsilon \text{ and } \xi.$$

Theorem 3.3. *In addition to the hypotheses of Theorem 3.1 (or Theorem 3.2), condition* (3.17) *provides the uniqueness of a fixed point of g in P.*

Proof. From Theorem 3.1 (or Theorem 3.2), we conclude that g has a nonempty set of fixed points. Suppose that υ^* and ξ^* be two fixed points of g then, we claim that $\upsilon^* = \xi^*$. Suppose that $\upsilon^* \neq \xi^*$, then from the hypotheses we have

$$\phi(d(g\upsilon^*, g\xi^*)) \leq \phi(sd(g\upsilon^*, g\xi^*)) \leq \phi(M(\upsilon^*, \xi^*)) - \psi(N(\upsilon^*, \xi^*)). \tag{3.18}$$

Consequently, we get

$$d(\upsilon^*, \xi^*) = d(g\upsilon^*, g\xi^*) \leq \frac{1}{s} M(\upsilon^*, \xi^*), \tag{3.19}$$

where

$$\begin{aligned} M(\upsilon^*, \xi^*) &= \max\left\{ \frac{d(\xi^*, g\xi^*)\,[1 + d(\upsilon^*, g\upsilon^*)]}{1 + d(\upsilon^*, \xi^*)}, \frac{d(\upsilon^*, g\upsilon^*)\ d(\upsilon^*, g\xi^*)}{1 + d(\upsilon^*, g\xi^*) + d(\xi^*, g\upsilon^*)}, d(g\upsilon^*, g\xi^*) \right\} \\ &= \max\left\{ \frac{d(\xi^*, \xi^*)\,[1 + d(\upsilon^*, \upsilon^*)]}{1 + d(\upsilon^*, \xi^*)}, \frac{d(\upsilon^*, \upsilon^*)\ d(\upsilon^*, \xi^*)}{1 + d(\upsilon^*, \xi^*) + d(\xi^*, \upsilon^*)}, d(\upsilon^*, \xi^*) \right\} \\ &= \max\{0, 0, d(\upsilon^*, \xi^*)\} \\ &= d(\upsilon^*, \xi^*). \end{aligned} \tag{3.20}$$

From (3.19), we obtain that

$$d(\upsilon^*, \xi^*) \leq \frac{1}{s} d(\upsilon^*, \xi^*) < d(\upsilon^*, \xi^*), \tag{3.21}$$

which is a contradiction. Hence, $\upsilon^* = \xi^*$. This completes the proof. □

Let $(P, d, s, \preceq)$ be a partially ordered b-metric space with parameter $s > 1$, and let $g, f : P \to P$ be two mappings. Set

$$M_f(\upsilon, \xi) = \max\left\{ \frac{d(f\xi, g\xi)\,[1 + d(f\upsilon, g\upsilon)]}{1 + d(f\upsilon, f\xi)}, \frac{d(f\upsilon, g\upsilon)\ d(f\upsilon, g\xi)}{1 + d(f\upsilon, g\xi) + d(f\xi, g\upsilon)}, d(f\upsilon, f\xi) \right\}, \tag{3.22}$$

and

$$N_f(\upsilon, \xi) = \max\left\{ \frac{d(f\xi, g\xi)\,[1 + d(f\upsilon, g\upsilon)]}{1 + d(f\upsilon, f\xi)}, d(f\upsilon, f\xi) \right\}. \tag{3.23}$$

Now, we introduce the following definition.

Definition 3.4. *Let $(P,d,s,\preceq)$ be a partially ordered b-metric space with $s>1$. The mapping $\mathscr{g}:P\to P$ is called a generalized (ϕ,ψ)-contraction mapping with respect to $\mathscr{f}:P\to P$ for some $\phi\in\Phi$ and $\psi\in\Psi$, if*

$$\phi(sd(\mathscr{g}\upsilon,\mathscr{g}\xi))\le\phi(M_{\mathscr{f}}(\upsilon,\xi))-\psi(N_{\mathscr{f}}(\upsilon,\xi)), \tag{3.24}$$

for any $\upsilon,\xi\in P$ with $\mathscr{f}\upsilon\preceq\mathscr{f}\xi$, where $M_{\mathscr{f}}(\upsilon,\xi)$ and $N_{\mathscr{f}}(\upsilon,\xi)$ are given by (3.22) *and* (3.23) *respectively.*

Theorem 3.5. *Suppose that $(P,d,s,\preceq)$ is a complete partially ordered b-metric space with $s>1$. Let $\mathscr{g}:P\to P$ be a generalized (ϕ,ψ)-contractive mapping with respect to $\mathscr{f}:P\to P$ and, $\mathscr{g}$ and $\mathscr{f}$ are continuous such that $\mathscr{g}$ is a monotone $\mathscr{f}$-non-decreasing mapping, compatible with $\mathscr{f}$ and $\mathscr{g}P\subseteq\mathscr{f}P$. If for some $\upsilon_0\in P$ such that $\mathscr{f}\upsilon_0\preceq\mathscr{g}\upsilon_0$, then $\mathscr{g}$ and $\mathscr{f}$ have a coincidence point in P.*

Proof. By following the proof of Theorem 2.2 in [8], we construct two sequences $\{\upsilon_n\}$ and $\{\xi_n\}$ in P such that

$$\xi_n=\mathscr{g}\upsilon_n=\mathscr{f}\upsilon_{n+1}\quad\text{for all } n\ge 0, \tag{3.25}$$

for which

$$\mathscr{f}\upsilon_0\preceq\mathscr{f}\upsilon_1\preceq\cdots\preceq\mathscr{f}\upsilon_n\preceq\mathscr{f}\upsilon_{n+1}\preceq\cdots. \tag{3.26}$$

Again from [8], we have to show that

$$d(\xi_n,\xi_{n+1})\le\lambda d(\xi_{n-1},\xi_n), \tag{3.27}$$

for all $n\ge1$ and where $\lambda\in[0,\frac{1}{s})$. Now from (3.24) and using (3.25) and (3.26), we get

$$\begin{aligned}\phi(sd(\xi_n,\xi_{n+1}))&=\phi(sd(\mathscr{g}\upsilon_n,\mathscr{g}\upsilon_{n+1}))\\&\le\phi(M_{\mathscr{f}}(\upsilon_n,\upsilon_{n+1}))-\psi(N_{\mathscr{f}}(\upsilon_n,\upsilon_{n+1})),\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}M_{\mathscr{f}}(\upsilon_n,\upsilon_{n+1})&=\max\left\{\frac{d(\mathscr{f}\upsilon_{n+1},\mathscr{g}\upsilon_{n+1})\,[1+d(\mathscr{f}\upsilon_n,\mathscr{g}\upsilon_n)]}{1+d(\mathscr{f}\upsilon_n,\mathscr{f}\upsilon_{n+1})},\frac{d(\mathscr{f}\upsilon_n,\mathscr{g}\upsilon_n)\;d(\mathscr{f}\upsilon_n,\mathscr{g}\upsilon_{n+1})}{1+d(\mathscr{f}\upsilon_n,\mathscr{g}\upsilon_{n+1})+d(\mathscr{f}\upsilon_{n+1},\mathscr{g}\upsilon_n)},\right.\\&\qquad\left.d(\mathscr{f}\upsilon_n,\mathscr{f}\upsilon_{n+1})\right\}\\&=\max\left\{\frac{d(\xi_n,\xi_{n+1})\,[1+d(\xi_{n-1},\xi_n)]}{1+d(\xi_{n-1},\xi_n)},\frac{d(\xi_{n-1},\xi_n)\;d(\xi_{n-1},\xi_{n+1})}{1+d(\xi_{n-1},\xi_{n+1})+d(\xi_n,\xi_n)},d(\xi_{n-1},\xi_n)\right\}\\&=\max\{d(\xi_{n-1},\xi_n),d(\xi_n,\xi_{n+1})\}\end{aligned}$$

and

$$\begin{aligned}N_{\mathcal{f}}(\upsilon_n,\upsilon_{n+1}) &= \max\left\{\frac{d(\mathcal{f}\upsilon_{n+1},\mathcal{g}\upsilon_{n+1})\,[1+d(\mathcal{f}\upsilon_n,\mathcal{g}\upsilon_n)]}{1+d(\mathcal{f}\upsilon_n,\mathcal{f}\upsilon_{n+1})}, d(\mathcal{f}\upsilon_n,\mathcal{f}\upsilon_{n+1})\right\}\\ &= \max\left\{\frac{d(\xi_n,\xi_{n+1})\,[1+d(\xi_{n-1},\xi_n)]}{1+d(\xi_{n-1},\xi_n)}, d(\xi_{n-1},\xi_n)\right\}\\ &= \max\{d(\xi_{n-1},\xi_n), d(\xi_n,\xi_{n+1})\}.\end{aligned}$$

Therefore from equation (3.28), we get

$$\phi(sd(\xi_n,\xi_{n+1})) \leq \phi(\max\{d(\xi_{n-1},\xi_n), d(\xi_n,\xi_{n+1})\}) - \psi(\max\{d(\xi_{n-1},\xi_n), d(\xi_n,\xi_{n+1})\}). \quad (3.29)$$

If $0 < d(\xi_{n-1},\xi_n) \leq d(\xi_n,\xi_{n+1})$ for some $n \in \mathbb{N}$, then from (3.29) we get

$$\phi(sd(\xi_n,\xi_{n+1})) \leq \phi(d(\xi_n,\xi_{n+1})) - \psi(d(\xi_n,\xi_{n+1})) < \phi(d(\xi_n,\xi_{n+1})), \quad (3.30)$$

or equivalently

$$sd(\xi_n,\xi_{n+1}) \leq d(\xi_n,\xi_{n+1}). \quad (3.31)$$

This is a contradiction. Hence from (3.29) we obtain that

$$sd(\xi_n,\xi_{n+1}) \leq d(\xi_{n-1},\xi_n). \quad (3.32)$$

Thus equation (3.27) holds, where $\lambda \in [0, \frac{1}{s})$. Therefore from (3.27) and Lemma 3.1 of [32], we conclude that $\{\xi_n\} = \{\mathcal{g}\upsilon_n\} = \{\mathcal{f}\upsilon_{n+1}\}$ is a Cauchy sequence in P and then converges to some $\mu \in P$ as P is complete such that

$$\lim_{n\to+\infty} \mathcal{g}\upsilon_n = \lim_{n\to+\infty} \mathcal{f}\upsilon_{n+1} = \mu.$$

Thus by the compatibility of $\mathcal{g}$ and $\mathcal{f}$, we obtain that

$$\lim_{n\to+\infty} d(\mathcal{f}(\mathcal{g}\upsilon_n), \mathcal{g}(\mathcal{f}\upsilon_n)) = 0, \quad (3.33)$$

and from the continuity of $\mathcal{g}$ and $\mathcal{f}$, we have

$$\lim_{n\to+\infty} \mathcal{f}(\mathcal{g}\upsilon_n) = \mathcal{f}\mu, \qquad \lim_{n\to+\infty} \mathcal{g}(\mathcal{f}\upsilon_n) = \mathcal{g}\mu. \quad (3.34)$$

Further, from the triangular inequality of a b-metric and, from equations (3.33) and (3.34) , we get

$$\frac{1}{s}d(\mathcal{g}\mu,\mathcal{f}\mu) \leq d(\mathcal{g}\mu,\mathcal{g}(\mathcal{f}\upsilon_n)) + sd(\mathcal{g}(\mathcal{f}\upsilon_n),\mathcal{f}(\mathcal{g}\upsilon_n)) + sd(\mathcal{f}(\mathcal{g}\upsilon_n),\mathcal{f}\mu). \quad (3.35)$$

Finally, we arrive at $d(\mathcal{g}v,\mathcal{f}v) = 0$ as $n \to +\infty$ in (3.35). Therefore, v is a coincidence point of $\mathcal{g}$

and $\mathcal{f}$ in P. □

Relaxing the continuity of the mappings $\mathcal{f}$ and $\mathcal{g}$ in Theorem 3.5, we obtain the following result.

Theorem 3.6. *In Theorem 3.5, assume that P satisfies*

> *for any non-decreasing sequence $\{\mathcal{f}\upsilon_n\} \subset P$ with $\lim_{n\to+\infty} \mathcal{f}\upsilon_n = \mathcal{f}\upsilon$ in $\mathcal{f}P$, where $\mathcal{f}P$ is a closed subset of P implies that $\mathcal{f}\upsilon_n \preceq \mathcal{f}\upsilon, \mathcal{f}\upsilon \preceq \mathcal{f}(\mathcal{f}\upsilon)$ for $n \in \mathbb{N}$.*

If there exists $\upsilon_0 \in P$ such that $\mathcal{f}\upsilon_0 \preceq \mathcal{g}\upsilon_0$, then the weakly compatible mappings $\mathcal{g}$ and $\mathcal{f}$ have a coincidence point in P. Furthermore, $\mathcal{g}$ and $\mathcal{f}$ have a common fixed point, if $\mathcal{g}$ and $\mathcal{f}$ commute at their coincidence points.

Proof. The sequence, $\{\xi_n\} = \{\mathcal{g}\upsilon_n\} = \{\mathcal{f}\upsilon_{n+1}\}$ is a Cauchy sequence from the proof of Theorem 3.5. Since $\mathcal{f}P$ is closed, then there is some $\mu \in P$ such that

$$\lim_{n\to+\infty} \mathcal{g}\upsilon_n = \lim_{n\to+\infty} \mathcal{f}\upsilon_{n+1} = \mathcal{f}\mu.$$

Thus from the hypotheses, we have $\mathcal{f}\upsilon_n \preceq \mathcal{f}\mu$ for all $n \in \mathbb{N}$. Now, we have to prove that μ is a coincidence point of $\mathcal{g}$ and $\mathcal{f}$.

From equation (3.24), we have

$$\phi(sd(\mathcal{g}\upsilon_n, \mathcal{g}\upsilon)) \le \phi(M_{\mathcal{f}}(\upsilon_n, \upsilon)) - \psi(N_{\mathcal{f}}(\upsilon_n, \upsilon)), \tag{3.36}$$

where

$$\begin{aligned} M_{\mathcal{f}}(\upsilon_n, \mu) &= \max\left\{\frac{d(\mathcal{f}\mu, \mathcal{g}\mu)\,[1 + d(\mathcal{f}\upsilon_n, \mathcal{g}\upsilon_n)]}{1 + d(\mathcal{f}\upsilon_n, \mathcal{f}\mu)}, \frac{d(\mathcal{f}\upsilon_n, \mathcal{g}\upsilon_n)\; d(\mathcal{f}\upsilon_n, \mathcal{g}\mu)}{1 + d(\mathcal{f}\upsilon_n, \mathcal{g}\mu) + d(\mathcal{f}\mu, \mathcal{g}\upsilon_n)}, d(\mathcal{f}\upsilon_n, \mathcal{f}\mu)\right\} \\ &\to \max\{d(\mathcal{f}\mu, \mathcal{g}\mu), 0, 0\} \\ &= d(\mathcal{f}\mu, \mathcal{g}\mu) \quad \text{as } n \to +\infty, \end{aligned}$$

and

$$\begin{aligned} N_{\mathcal{f}}(\upsilon_n, \mu) &= \max\left\{\frac{d(\mathcal{f}\mu, \mathcal{g}\mu)\,[1 + d(\mathcal{f}\upsilon_n, \mathcal{g}\upsilon_n)]}{1 + d(\mathcal{f}\upsilon_n, \mathcal{f}\mu)}, d(\mathcal{f}\upsilon_n, \mathcal{f}\mu)\right\} \\ &\to \max\{d(\mathcal{f}\mu, \mathcal{g}\mu), 0\} \\ &= d(\mathcal{f}\mu, \mathcal{g}\mu) \quad \text{as } n \to +\infty. \end{aligned}$$

Therefore equation (3.36) becomes

$$\phi(s \lim_{n\to+\infty} d(\mathcal{g}\upsilon_n, \mathcal{g}\upsilon)) \le \phi(d(\mathcal{f}\mu, \mathcal{g}\mu)) - \psi(d(\mathcal{f}\mu, \mathcal{g}\mu)) < \phi(d(\mathcal{f}\mu, \mathcal{g}\mu)). \tag{3.37}$$

Consequently, we get

$$\lim_{n\to+\infty} d(g\upsilon_n, g\upsilon) < \frac{1}{s} d(f\mu, g\mu). \tag{3.38}$$

Further by triangular inequality, we have

$$\frac{1}{s} d(f\mu, g\mu) \leq d(f\mu, g\upsilon_n) + d(g\upsilon_n, g\mu), \tag{3.39}$$

then (3.38) and (3.39) lead to contradiction, if $f\mu \neq g\mu$. Hence, $f\mu = g\mu$. Let $f\mu = g\mu = \rho$, that is g and f commute at ρ, then $g\rho = g(f\mu) = f(g\mu) = f\rho$. Since $f\mu = f(f\mu) = f\rho$, then by equation (3.36) with $f\mu = g\mu$ and $f\rho = g\rho$, we get

$$\phi(sd(g\mu, g\rho)) \leq \phi(M_f(\mu, \rho)) - \psi(N_f(\mu, \rho)) < \phi(d(g\mu, g\rho)), \tag{3.40}$$

or equivalently,

$$sd(g\mu, g\rho) \leq d(g\mu, g\rho),$$

which is a contradiction, if $g\mu \neq g\rho$. Thus, $g\mu = g\rho = \rho$. Hence, $g\mu = f\rho = \rho$, that is ρ is a common fixed point of g and f. □

Definition 3.7. *Let $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$, $\phi \in \Phi$ and $\psi \in \Psi$. A mapping $h : P \times P \to P$ is said to be a generalized (ϕ, ψ)-contractive mapping with respect to $f : P \to P$ such that*

$$\phi(s^k d(h(\upsilon, \xi), h(\rho, \tau))) \leq \phi(M_f(\upsilon, \xi, \rho, \tau)) - \psi(N_f(\upsilon, \xi, \rho, \tau)), \tag{3.41}$$

for all $\upsilon, \xi, \rho, \tau \in P$ with $f\upsilon \preceq f\rho$ and $f\xi \succeq f\tau$, $k > 2$ where

$$M_f(\upsilon, \xi, \rho, \tau) = \max\left\{ \frac{d(f\rho, h(\rho, \tau))\,[1 + d(f\upsilon, h(\upsilon, \xi))]}{1 + d(f\upsilon, f\rho)}, \frac{d(f\upsilon, h(\upsilon, \xi))\ d(f\upsilon, h(\rho, \tau))}{1 + d(f\upsilon, h(\rho, \tau)) + d(f\rho, h(\upsilon, \xi))}, d(f\upsilon, f\rho) \right\},$$

and

$$N_f(\upsilon, \xi, \rho, \tau) = \max\left\{ \frac{d(f\rho, h(\rho, \tau))\,[1 + d(f\upsilon, h(\upsilon, \xi))]}{1 + d(f\upsilon, f\rho)}, d(f\upsilon, f\rho) \right\}.$$

Theorem 3.8. *Let $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$. Suppose that $h : P \times P \to P$ be a generalized (ϕ, ψ)- contractive mapping with respect to $f : P \to P$ and, h and f are continuous functions such that h has the mixed f-monotone property and commutes with f. Also assume that $h(P \times P) \subseteq f(P)$. Then h and f have a coupled coincidence point in P, if there exists $(\upsilon_0, \xi_0) \in P \times P$ such that $f\upsilon_0 \preceq h(\upsilon_0, \xi_0)$ and $f\xi_0 \succeq h(\xi_0, \upsilon_0)$.*

Proof. From the hypotheses and following the proof of Theorem 2.2 of [8], we construct two sequences $\{\upsilon_n\}$ and $\{\xi_n\}$ in P such that

$$\ell\upsilon_{n+1} = \hbar(\upsilon_n, \xi_n), \quad \ell\xi_{n+1} = \hbar(\xi_n, \upsilon_n), \quad \text{for all } n \geq 0.$$

In particular, $\{\ell\upsilon_n\}$ is non-decreasing and $\{\ell\xi_n\}$ is non-increasing sequences in P. Now from (3.41) by replacing $\upsilon = \upsilon_n, \xi = \xi_n, \rho = \upsilon_{n+1}, \tau = \xi_{n+1}$, we get

$$\begin{aligned}\phi(s^k d(\ell\upsilon_{n+1}, \ell\upsilon_{n+2})) &= \phi(s^k d(\hbar(\upsilon_n, \xi_n), \hbar(\upsilon_{n+1}, \xi_{n+1}))) \\ &\leq \phi(M_\ell(\upsilon_n, \xi_n, \upsilon_{n+1}, \xi_{n+1})) - \psi(N_\ell(\upsilon_n, \xi_n, \upsilon_{n+1}, \xi_{n+1})),\end{aligned} \tag{3.42}$$

where

$$M_\ell(\upsilon_n, \xi_n, \upsilon_{n+1}, \xi_{n+1}) \leq \max\{d(\ell\upsilon_n, \ell\upsilon_{n+1}), d(\ell\upsilon_{n+1}, \ell\upsilon_{n+2})\} \tag{3.43}$$

and

$$N_\ell(\upsilon_n, \xi_n, \upsilon_{n+1}, \xi_{n+1}) = \max\{d(\ell\upsilon_n, \ell\upsilon_{n+1}), d(\ell\upsilon_{n+1}, \ell\upsilon_{n+2})\}. \tag{3.44}$$

Therefore from (3.42), we have

$$\begin{aligned}\phi(s^k d(\ell\upsilon_{n+1}, \ell\upsilon_{n+2})) &\leq \phi(\max\{d(\ell\upsilon_n, \ell\upsilon_{n+1}), d(\ell\upsilon_{n+1}, \ell\upsilon_{n+2})\}) \\ &\quad - \psi(\max\{d(\ell\upsilon_n, \ell\upsilon_{n+1}), d(\ell\upsilon_{n+1}, \ell\upsilon_{n+2})\}).\end{aligned} \tag{3.45}$$

Similarly by taking $\upsilon = \xi_{n+1}, \xi = \upsilon_{n+1}, \rho = \upsilon_n, \tau = \upsilon_n$ in (3.41), we get

$$\begin{aligned}\phi(s^k d(\ell\xi_{n+1}, \ell\xi_{n+2})) &\leq \phi(\max\{d(\ell\xi_n, \ell\xi_{n+1}), d(\ell\xi_{n+1}, \ell\xi_{n+2})\}) \\ &\quad - \psi(\max\{d(\ell\xi_n, \ell\xi_{n+1}), d(\ell\xi_{n+1}, \ell\xi_{n+2})\}).\end{aligned} \tag{3.46}$$

From the fact that $\max\{\phi(c), \phi(d)\} = \phi\{\max\{c, d\}\}$ for all $c, d \in [0, +\infty)$. Then combining (3.45) and (3.46), we get

$$\begin{aligned}\phi(s^k\delta_n) &\leq \phi(\max\{d(\ell\upsilon_n, \ell\upsilon_{n+1}), d(\ell\upsilon_{n+1}, \ell\upsilon_{n+2}), d(\ell\xi_n, \ell\xi_{n+1}), d(\ell\xi_{n+1}, \ell\xi_{n+2})\}) \\ &\quad - \psi(\max\{d(\ell\upsilon_n, \ell\upsilon_{n+1}), d(\ell\upsilon_{n+1}, \ell\upsilon_{n+2}), d(\ell\xi_n, \ell\xi_{n+1}), d(\ell\xi_{n+1}, \ell\xi_{n+2})\}),\end{aligned} \tag{3.47}$$

where

$$\delta_n = \max\{d(\ell\upsilon_{n+1}, \ell\upsilon_{n+2}), d(\ell\xi_{n+1}, \ell\xi_{n+2})\}. \tag{3.48}$$

Let us denote,

$$\Delta_n = \max\{d(\ell\upsilon_n, \ell\upsilon_{n+1}), d(\ell\upsilon_{n+1}, \ell\upsilon_{n+2}), d(\ell\xi_n, \ell\xi_{n+1}), d(\ell\xi_{n+1}, \ell\xi_{n+2})\}. \tag{3.49}$$

Hence from equations (3.45)-(3.48), we obtain

$$s^k \delta_n \leq \Delta_n. \tag{3.50}$$

Next, we prove that

$$\delta_n \leq \lambda \delta_{n-1}, \tag{3.51}$$

for all $n \geq 1$ and where $\lambda = \frac{1}{s^k} \in [0,1)$.

Suppose that if $\Delta_n = \delta_n$ then from (3.50), we get $s^k \delta_n \leq \delta_n$ which leads to $\delta_n = 0$ as $s > 1$ and hence (3.51) holds. If $\Delta_n = \max\{d(\not{f}\upsilon_n, \not{f}\upsilon_{n+1}), d(\not{f}\xi_n, \not{f}\xi_{n+1})\}$, *i.e.*, $\Delta_n = \delta_{n-1}$ then (3.50) follows (3.51).

Now from (3.50), we obtain that $\delta_n \leq \lambda^n \delta_0$ and hence,

$$d(\not{f}\upsilon_{n+1}, \not{f}\upsilon_{n+2}) \leq \lambda^n \delta_0 \ \text{ and } \ d(\not{f}\xi_{n+1}, \not{f}\xi_{n+2}) \leq \lambda^n \delta_0. \tag{3.52}$$

Therefore from Lemma 3.1 of [32], the sequences $\{\not{f}\upsilon_n\}$ and $\{\not{f}\xi_n\}$ are Cauchy sequences in P. Hence, by following the remaining proof of Theorem 2.2 of [2], we can show that $\hbar$ and $\not{f}$ have a coincidence point in P. □

Corollary 3.9. *Let $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$, and $\hbar : P \times P \to P$ be a continuous mapping such that $\hbar$ has a mixed monotone property. Suppose there exists $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\phi(s^k d(\hbar(\upsilon, \xi), \hbar(\rho, \tau))) \leq \phi(M_{\not{f}}(\upsilon, \xi, \rho, \tau)) - \psi(N_{\not{f}}(\upsilon, \xi, \rho, \tau)),$$

for all $\upsilon, \xi, \rho, \tau \in P$ with $\upsilon \preceq \rho$ and $\xi \succeq \tau$, $k > 2$ where

$$M_{\not{f}}(\upsilon, \xi, \rho, \tau) = \max\left\{\frac{d(\rho, \hbar(\rho, \tau))\,[1 + d(\upsilon, \hbar(\upsilon, \xi))]}{1 + d(\upsilon, \rho)}, \frac{d(\upsilon, \hbar(\upsilon, \xi))\ d(\upsilon, \hbar(\rho, \tau))}{1 + d(\upsilon, \hbar(\rho, \tau)) + d(\rho, \hbar(\upsilon, \xi))}, d(\upsilon, \rho)\right\},$$

and

$$N_{\not{f}}(\upsilon, \xi, \rho, \tau) = \max\left\{\frac{d(\rho, \hbar(\rho, \tau))\,[1 + d(\upsilon, \hbar(\upsilon, \xi))]}{1 + d(\upsilon, \rho)}, d(\upsilon, \rho)\right\}.$$

Then $\hbar$ has a coupled fixed point in P, if there exists $(\upsilon_0, \xi_0) \in P \times P$ such that $\upsilon_0 \preceq \hbar(\upsilon_0, \xi_0)$ and $\xi_0 \succeq \hbar(\xi_0, \upsilon_0)$.

Proof. Set $\not{f} = I_P$ in Theorem 3.8. □

Corollary 3.10. *Let $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$, and $\hbar : P \times P \to P$ be a continuous mapping such that $\hbar$ has a mixed monotone property. Suppose* *there exists* $\psi \in \Psi$ *such that*

$$d(\hbar(\upsilon,\xi),\hbar(\rho,\tau)) \leq \frac{1}{s^k} M_{\ell}(\upsilon,\xi,\rho,\tau) - \frac{1}{s^k}\psi(N_{\ell}(\upsilon,\xi,\rho,\tau)),$$

for all $\upsilon,\xi,\rho,\tau \in P$ *with* $\upsilon \preceq \rho$ *and* $\xi \succeq \tau$, $k > 2$ *where*

$$M_{\ell}(\upsilon,\xi,\rho,\tau) = \max\left\{\frac{d(\rho,\hbar(\rho,\tau))\,[1+d(\upsilon,\hbar(\upsilon,\xi))]}{1+d(\upsilon,\rho)}, \frac{d(\upsilon,\hbar(\upsilon,\xi))\; d(\upsilon,\hbar(\rho,\tau))}{1+d(\upsilon,\hbar(\rho,\tau))+d(\rho,\hbar(\upsilon,\xi))}, d(\upsilon,\rho)\right\},$$

and

$$N_{\ell}(\upsilon,\xi,\rho,\tau) = \max\left\{\frac{d(\rho,\hbar(\rho,\tau))\,[1+d(\upsilon,\hbar(\upsilon,\xi))]}{1+d(\upsilon,\rho)}, d(\upsilon,\rho)\right\}.$$

If there exists $(\upsilon_0,\xi_0) \in P \times P$ *such that* $\upsilon_0 \preceq \hbar(\upsilon_0,\xi_0)$ *and* $\xi_0 \succeq \hbar(\xi_0,\upsilon_0)$, *then* $\hbar$ *has a coupled fixed point in* P.

Theorem 3.11. *In addition to Theorem 3.8, if for all* $(\upsilon,\xi),(r,s) \in P\times P$, *there exists* $(c^*,d^*) \in P \times P$ *such that* $(\hbar(c^*,d^*),\hbar(d^*,c^*))$ *is comparable to* $(\hbar(\upsilon,\xi),\hbar(\xi,\upsilon))$ *and to* $(\hbar(r,s),\hbar(s,r))$, *then* $\hbar$ *and* ℓ *have a unique coupled common fixed point in* $P \times P$.

Proof. From Theorem 3.8, we know that there exists at least one coupled coincidence point in P for $\hbar$ and ℓ. Assume that (υ,ξ) and (r,s) are two coupled coincidence points of $\hbar$ and ℓ, *i.e.*, $\hbar(\upsilon,\xi) = \ell\upsilon$, $\hbar(\xi,\upsilon,) = \ell\xi$ and $\hbar(r,s) = \ell r$, $\hbar(s,r) = \ell s$. Now, we have to prove that $\ell\upsilon = \ell r$ and $\ell\xi = \ell s$.

From the hypotheses, there exists $(c^*,d^*) \in P \times P$ such that $(\hbar(c^*,d^*),\hbar(d^*,c^*))$ is comparable to $(\hbar(\upsilon,\xi),\hbar(\xi,\upsilon))$ and to $(\hbar(r,s),\hbar(s,r))$. Suppose that

$$(\hbar(\upsilon,\xi),\hbar(\xi,\upsilon)) \leq (\hbar(c^*,d^*),\hbar(d^*,c^*)) \text{ and } (\hbar(r,s),\hbar(s,r)) \leq (\hbar(c^*,d^*),\hbar(d^*,c^*)).$$

Let $c_0^* = c^*$ and $d_0^* = d^*$ and then choose $(c_1^*,d_1^*) \in P \times P$ as

$$\ell c_1^* = \hbar(c_0^*,d_0^*),\ \ell d_1^* = \hbar(d_0^*,c_0^*) \quad (n \geq 1).$$

By repeating the same procedure above, we can obtain two sequences $\{\ell c_n^*\}$ and $\{\ell d_n^*\}$ in P such that

$$\ell c_{n+1}^* = \hbar(c_n^*,d_n^*),\ \ell d_{n+1}^* = \hbar(d_n^*,c_n^*) \quad (n \geq 0).$$

Similarly, define the sequences $\{\ell\upsilon_n\}$, $\{\ell\xi_n\}$ and $\{\ell r_n\}$, $\{\ell s_n\}$ as above in P by setting $\upsilon_0 = \upsilon$, $\xi_0 = \xi$ and $r_0 = r$, $s_0 = s$. Further, we have that

$$\ell\upsilon_n \to \hbar(\upsilon,\xi),\ \ell\xi_n \to \hbar(\xi,\upsilon),\ \ell r_n \to \hbar(r,s),\ \ell s_n \to \hbar(s,r) \quad (n \geq 1). \tag{3.53}$$

Since, $(\hbar(\upsilon,\xi),\hbar(\xi,\upsilon)) = (\ell\upsilon,\ell\xi) = (\ell\upsilon_1,\ell\xi_1)$ is comparable to $(\hbar(c^*,d^*),\hbar(d^*,c^*)) = (\ell c^*,\ell d^*) =$

$(\not{f} c_1^*, \not{f} d_1^*)$ and hence we get $(\not{f} \upsilon_1, \not{f} \xi_1) \leq (\not{f} c_1^*, \not{f} d_1^*)$. Thus, by induction we obtain that

$$(\not{f} \upsilon_n, \not{f} \xi_n) \leq (\not{f} c_n^*, \not{f} d_n^*) \quad (n \geq 0). \tag{3.54}$$

Therefore from (3.41), we have

$$\begin{aligned} \phi(d(\not{f} \upsilon, \not{f} c_{n+1}^*)) \leq \phi(s^k d(\not{f} \upsilon, \not{f} c_{n+1}^*)) &= \phi(s^k d(\hbar(\upsilon, \xi), \hbar(c_n^*, d_n^*))) \\ &\leq \phi(M_{\not{f}}(\upsilon, \xi, c_n^*, d_n^*)) - \psi(N_{\not{f}}(\upsilon, \xi, c_n^*, d_n^*)), \end{aligned} \tag{3.55}$$

where

$$\begin{aligned} M_{\not{f}}(\upsilon, \xi, c_n^*, d_n^*) &= \max\left\{ \frac{d(\not{f} c_n^*, \hbar(c_n^*, d_n^*)) \left[1 + d(\not{f} \upsilon, \hbar(\upsilon, \xi))\right]}{1 + d(\not{f} \upsilon, \not{f} c_n^*)} \right., \\ &\qquad \left. \frac{d(\not{f} \upsilon, \hbar(\upsilon, \xi))\ d(\not{f} \upsilon, \hbar(c_n^*, d_n^*))}{1 + d(\not{f} \upsilon, \hbar(c_n^*, d_n^*)) + d(\not{f} c_n^*, \hbar(\upsilon, \xi))}, d(\not{f} \upsilon, \not{f} c_n^*) \right\} \\ &= \max\{0, 0, d(\not{f} \upsilon, \not{f} c_n^*)\} \\ &= d(\not{f} \upsilon, \not{f} c_n^*) \end{aligned}$$

and

$$\begin{aligned} N_{\not{f}}(\upsilon, \xi, c_n^*, d_n^*) &= \max\left\{ \frac{d(\not{f} c_n^*, \hbar(c_n^*, d_n^*)) \left[1 + d(\hbar \upsilon, \hbar(\upsilon, \xi))\right]}{1 + d(\not{f} \upsilon, \not{f} c_n^*)}, d(\not{f} \upsilon, \not{f} c_n^*) \right\} \\ &= d(\not{f} \upsilon, \not{f} c_n^*). \end{aligned}$$

Thus from (3.55),

$$\phi(d(\not{f} \upsilon, \not{f} c_{n+1}^*)) \leq \phi(d(\not{f} \upsilon, \not{f} c_n^*)) - \psi(d(\not{f} \upsilon, \not{f} c_n^*)). \tag{3.56}$$

As by the similar process, we can prove that

$$\phi(d(\not{f} \xi, \not{f} d_{n+1}^*)) \leq \phi(d(\not{f} \xi, \not{f} d_n^*)) - \psi(d(\not{f} \xi, \not{f} d_n^*)). \tag{3.57}$$

From (3.56) and (3.57), we have

$$\begin{aligned} \phi(\max\{d(\not{f} \upsilon, \not{f} c_{n+1}^*), d(\not{f} \xi, \not{f} d_{n+1}^*)\}) &\leq \phi(\max\{d(\not{f} \upsilon, \not{f} c_n^*), d(\not{f} \xi, \not{f} d_n^*)\}) \\ &\quad - \psi(\max\{d(\not{f} \upsilon, \not{f} c_n^*), d(\not{f} \xi, \not{f} d_n^*)\}) \\ &\quad < \phi(\max\{d(\not{f} \upsilon, \not{f} c_n^*), d(\not{f} \xi, \not{f} d_n^*)\}). \end{aligned} \tag{3.58}$$

Hence by the property of ϕ, we get

$$\max\{d(\not{f} \upsilon, \not{f} c_{n+1}^*), d(\not{f} \xi, \not{f} d_{n+1}^*)\} < \max\{d(\not{f} \upsilon, \not{f} c_n^*), d(\not{f} \xi, \not{f} d_n^*)\},$$

which shows that $\max\{d(\not{f} \upsilon, \not{f} c_n^*), d(\not{f} \xi, \not{f} d_n^*)\}$ is a decreasing sequence and by a result there exists

$\gamma \geq 0$ such that

$$\lim_{n\to+\infty} \max\{d(fv, fc_n^*), d(f\xi, fd_n^*)\} = \gamma.$$

From (3.58) taking upper limit as $n \to +\infty$, we get

$$\phi(\gamma) \leq \phi(\gamma) - \psi(\gamma), \tag{3.59}$$

from which we get $\psi(\gamma) = 0$, implies that $\gamma = 0$. Thus,

$$\lim_{n\to+\infty} \max\{d(fv, fc_n^*), d(f\xi, fd_n^*)\} = 0.$$

Consequently, we get

$$\lim_{n\to+\infty} d(fv, fc_n^*) = 0 \quad \text{and} \quad \lim_{n\to+\infty} d(f\xi, fd_n^*) = 0. \tag{3.60}$$

By similar argument, we get

$$\lim_{n\to+\infty} d(fr, fc_n^*) = 0 \quad \text{and} \quad \lim_{n\to+\infty} d(fs, fd_n^*) = 0. \tag{3.61}$$

Therefore from (3.60) and (3.61), we get $fv = fr$ and $f\xi = fs$. Since $fv = \hbar(v, \xi)$ and $f\xi = \hbar(\xi, v)$, then by the commutativity of $\hbar$ and f, we have

$$f(fv) = f(\hbar(v, \xi)) = \hbar(fv, f\xi) \quad \text{and} \quad f(f\xi) = f(\hbar(\xi, v)) = \hbar(f\xi, fv). \tag{3.62}$$

Let $fv = a^*$ and $f\xi = b^*$ then (3.62) becomes

$$f(a^*) = \hbar(a^*, b^*) \quad \text{and} \quad f(b^*) = \hbar(b^*, a^*), \tag{3.63}$$

which shows that (a^*, b^*) is a coupled coincidence point of $\hbar$ and f. It follows that $f(a^*) = fr$ and $f(b^*) = fs$ that is $f(a^*) = a^*$ and $f(b^*) = b^*$. Thus from (3.63), we get $a^* = f(a^*) = \hbar(a^*, b^*)$ and $b^* = f(b^*) = \hbar(b^*, a^*)$. Therefore, (a^*, b^*) is a coupled common fixed point of $\hbar$ and f.

For the uniqueness, let (u^*, v^*) be another coupled common fixed point of $\hbar$ and f, then we have $u^* = fu^* = \hbar(u^*, v^*)$ and $v^* = fv^* = \hbar(v^*, u^*)$. Since (u^*, v^*) is a coupled common fixed point of $\hbar$ and f, then we get $fu^* = fv = a^*$ and $fv^* = f\xi = b^*$. Thus, $u^* = fu^* = fa^* = a^*$ and $v^* = fv^* = fb^* = b^*$. Hence the result. □

Theorem 3.12. *In addition to the hypotheses of Theorem 3.11, if fv_0 and $f\xi_0$ are comparable, then $\hbar$ and f have a unique common fixed point in P.*

Proof. From Theorem 3.11, $\hbar$ and ℓ have a unique coupled common fixed point $(\upsilon, \xi) \in P$. Now, it is enough to prove that $\upsilon = \xi$. From the hypotheses, we have $\ell\upsilon_0$ and $\ell\xi_0$ are comparable then we assume that $\ell\upsilon_0 \preceq \ell\xi_0$. Hence by induction we get $\ell\upsilon_n \preceq \ell\xi_n$ for all $n \geq 0$, where $\{\ell\upsilon_n\}$ and $\{\ell\xi_n\}$ are from Theorem 3.8.

Now by use of Lemma 2.6, we get

$$\begin{aligned}\phi(s^{k-2}d(\upsilon,\xi)) = \phi\left(s^k\frac{1}{s^2}d(\upsilon,\xi)\right) &\leq \lim_{n\to+\infty}\sup\phi(s^k d(\upsilon_{n+1},\xi_{n+1}))\\ &= \lim_{n\to+\infty}\sup\phi(s^k d(\hbar(\upsilon_n,\xi_n),\hbar(\xi_n,\upsilon_n)))\\ &\leq \lim_{n\to+\infty}\sup\phi(M_\ell(\upsilon_n,\xi_n,\xi_n,\upsilon_n)) - \lim_{n\to+\infty}\inf\psi(N_\ell(\upsilon_n,\xi_n,\xi_n,\upsilon_n))\\ &\leq \phi(d(\upsilon,\xi)) - \lim_{n\to+\infty}\inf\psi(N_\ell(\upsilon_n,\xi_n,\xi_n,\upsilon_n))\\ &< \phi(d(\upsilon,\xi)),\end{aligned}$$

which is a contradiction. Thus, $\upsilon = \xi$, *i.e.*, $\hbar$ and ℓ have a common fixed point in P. □

Remark 3.13. *It is well known that b-metric space is a metric space when $s = 1$. So, from the result of Jachymski [31], the condition*

$$\phi(d(\hbar(\upsilon,\xi),\hbar(\rho,\tau))) \leq \phi(\max\{d(\ell\upsilon,\ell\rho),d(\ell\xi,\ell\tau)\}) - \psi(\max\{d(\ell\upsilon,\ell\rho),d(\ell\xi,\ell\tau)\})$$

is equivalent to,

$$d(\hbar(\upsilon,\xi),\hbar(\rho,\tau)) \leq \varphi(\max\{d(\ell\upsilon,\ell\rho),d(\ell\xi,\ell\tau)\}),$$

where $\phi \in \Phi$, $\psi \in \Psi$ and $\varphi : [0,+\infty) \to [0,+\infty)$ is continuous, $\varphi(t) < t$ for all $t > 0$ and $\varphi(t) = 0$ if and only if $t = 0$. So, in view of above our results generalize and extend the results of [15, 23, 25, 31, 37, 38] and several other comparable results.

Corollary 3.14. *Suppose $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with parameter $s > 1$. Let $g : P \to P$ be a continuous, non-decreasing mapping with regards to $\preceq$ such that there exists $\upsilon_0 \in P$ with $\upsilon_0 \preceq g\upsilon_0$. Suppose that*

$$\phi(sd(g\upsilon,g\xi)) \leq \phi(M(\upsilon,\xi)) - \psi(M(\upsilon,\xi)), \tag{3.64}$$

where $M(\upsilon,\xi)$ and the conditions upon ϕ, ψ are same as in Theorem 3.1. Then g has a fixed point in P.

Proof. Set $N(\upsilon,\xi) = M(\upsilon,\xi)$ in a contraction condition (3.3) and apply Theorem 3.1, we have the required proof. □

Note 1. *Similarly by removing the continuity of a non-decreasing mapping* g *and taking a non-decreasing sequence* $\{\upsilon_n\}$ *as above in Theorem 3.2, we can obtain a fixed point for* g *in* P*. Also one can obtain the uniqueness of a fixed point of* g *by using condition* (3.17) *in* P *as by following the proof of Theorem 3.3.*

Note 2. *By following the proofs of Theorems 3.5 - 3.6, we can find the coincidence point for the mappings* g *and* f *in* P*. Similarly, from Theorem 3.8, Theorem 3.11 and Theorem 3.12, one can obtain a coupled coincidence point and its uniqueness, and a unique common fixed point for the mappings* $\hbar$ *and* f *in* $P \times P$ *and on* P *satisfying an almost generalized contraction condition* (3.64)*, where* $M(\upsilon,\xi)$*,* $M_f(\upsilon,\xi)$*,* $M_f(\upsilon,\xi,\rho,\tau)$ *and the conditions upon* ϕ,ψ *are same as above defined.*

Corollary 3.15. *Suppose that* $(P,d,s,\preceq)$ *be a complete partially ordered* b*-metric space with* $s>1$*. Let* $g: P\to P$ *be a continuous, non-decreasing mapping with regards to* $\preceq$*. If there exists* $k\in[0,1)$ *and for any* $\upsilon,\xi\in P$ *with* $\upsilon\preceq\xi$ *such that*

$$d(g\upsilon, g\xi) \le \frac{k}{s}\max\left\{\frac{d(\xi,g\xi)\,[1+d(\upsilon,g\upsilon)]}{1+d(\upsilon,\xi)}, \frac{d(\upsilon,g\upsilon)\; d(\upsilon,g\xi)}{1+d(\upsilon,g\xi)+d(\xi,g\upsilon)}, d(\upsilon,\xi)\right\}. \tag{3.65}$$

If there exists $\upsilon_0\in P$ *with* $\upsilon_0\preceq g\upsilon_0$*, then* g *has a fixed point in* P*.*

Proof. Set $\phi(t)=t$ and $\psi(t)=(1-k)t$, for all $t\in(0,+\infty)$ in Corollary 3.14. □

Note 3. *Relaxing the continuity of a map* g *in Corollary 3.15, one can obtains a fixed point for* g *on taking a non-decreasing sequence* $\{\upsilon_n\}$ *in* P *by following the proof of Theorem 3.2.*

Example 3.16. *Define a metric* $d: P\times P\to P$ *as below and* $\le$ *is an usual order on* P*, where* $P=\{1,2,3,4,5,6\}$

$$\begin{aligned}
&d(\upsilon,\xi)=d(\xi,\upsilon)=0,\ \textit{if } \upsilon,\xi=1,2,3,4,5,6 \textit{ and } \upsilon=\xi,\\
&d(\upsilon,\xi)=d(\xi,\upsilon)=3,\ \textit{if } \upsilon,\xi=1,2,3,4,5 \textit{ and } \upsilon\neq\xi,\\
&d(\upsilon,\xi)=d(\xi,\upsilon)=12,\ \textit{if } \upsilon=1,2,3,4 \textit{ and } \xi=6,\\
&d(\upsilon,\xi)=d(\xi,\upsilon)=20,\ \textit{if } \upsilon=5 \textit{ and } \xi=6.
\end{aligned}$$

Define a map $g: P\to P$ *by* $g1=g2=g3=g4=g5=1, g6=2$ *and let* $\phi(t)=\frac{t}{2}$*,* $\psi(t)=\frac{t}{4}$ *for* $t\in[0,+\infty)$*. Then* g *has a fixed point in* P*.*

Proof. It is apparent that, $(P,d,s,\preceq)$ is a complete partially ordered b-metric space for $s=2$. Consider the possible cases for υ, ξ in P:

Case 1 Suppose $\upsilon,\xi\in\{1,2,3,4,5\}$, $\upsilon<\xi$ then $d(g\upsilon,g\xi)=d(1,1)=0$. Hence,

$$\phi(2d(g\upsilon,g\xi))=0\le\phi(M(\upsilon,\xi))-\psi(M(\upsilon,\xi)).$$

Case 2 Suppose that $\upsilon \in \{1,2,3,4,5\}$ and $\xi = 6$, then $d(g\upsilon, g\xi) = d(1,2) = 3$, $M(6,5) = 20$ and $M(\upsilon, 6) = 12$, for $\upsilon \in \{1,2,3,4\}$. Therefore, we have the following inequality,

$$\phi(2d(g\upsilon, g\xi)) \leq \frac{M(\upsilon,\xi)}{4} = \phi(M(\upsilon,\xi)) - \psi(M(\upsilon,\xi)).$$

Thus, condition (3.64) of Corollary 3.14 holds. Furthermore, the remaining assumptions in Corollary 3.14 are fulfilled. Hence, g has a fixed point in P as Corollary 3.14 is appropriate to g, ϕ, ψ and $(P, d, s, \preceq)$. □

Example 3.17. *A metric* $d : P \times P \to P$, *where* $P = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \frac{1}{n}, \dots\}$ *with usual order* $\leq$ *is defined as follows*

$$d(\upsilon,\xi) = \begin{cases} 0, & \text{if } \upsilon = \xi \\ 1, & \text{if } \upsilon \neq \xi \in \{0,1\} \\ |\upsilon - \xi|, & \text{if } \upsilon, \xi \in \left\{0, \frac{1}{2n}, \frac{1}{2m} : n \neq m \geq 1\right\} \\ 3, & \text{otherwise.} \end{cases}$$

A map $g : P \to P$ *be such that* $g0 = 0, g\frac{1}{n} = \frac{1}{12n}$ *for all* $n \geq 1$ *and let* $\phi(t) = t$, $\psi(t) = \frac{4t}{5}$ *for* $t \in [0, +\infty)$. *Then,* g *has a fixed point in* P.

Proof. It is obvious that for $s = \frac{12}{5}$, $(P, d, s, \preceq)$ is a complete partially ordered b-metric space and also by definition, d is discontinuous b-metric space. Now for $\upsilon, \xi \in P$ with $\upsilon < \xi$, we have the following cases:

Case 1 If $\upsilon = 0$ and $\xi = \frac{1}{n}$, $n \geq 1$, then $d(g\upsilon, g\xi) = d(0, \frac{1}{12n}) = \frac{1}{12n}$ and $M(\upsilon,\xi) = \frac{1}{n}$ or $M(\upsilon,\xi) = \{1,3\}$. Therefore, we have

$$\phi\left(\frac{12}{5} d(g\upsilon, g\xi)\right) \leq \frac{M(\upsilon,\xi)}{5} = \phi(M(\upsilon,\xi)) - \psi(M(\upsilon,\xi)).$$

Case 2 If $\upsilon = \frac{1}{m}$ and $\xi = \frac{1}{n}$ with $m > n \geq 1$, then

$$d(g\upsilon, g\xi) = d\left(\frac{1}{12m}, \frac{1}{12n}\right) \text{ and } M(\upsilon,\xi) \geq \frac{1}{n} - \frac{1}{m} \text{ or } M(\upsilon,\xi) = 3.$$

Therefore,

$$\phi\left(\frac{12}{5} d(g\upsilon, g\xi)\right) \leq \frac{M(\upsilon,\xi)}{5} = \phi(M(\upsilon,\xi)) - \psi(M(\upsilon,\xi)).$$

Hence, condition (3.64) of Corollary 3.14 and remaining assumptions are satisfied. Thus, g has a fixed point in P. □

Example 3.18. *Let $P = C[a,b]$ be the set of all continuous functions. Let us define a b-metric d on P by*

$$d(\theta_1, \theta_2) = \sup_{t \in C[a,b]} \{|\theta_1(t) - \theta_2(t)|^2\}$$

for all $\theta_1, \theta_2 \in P$ with partial order $\preceq$ defined by $\theta_1 \preceq \theta_2$ if $a \leq \theta_1(t) \leq \theta_2(t) \leq b$, for all $t \in [a,b]$, $0 \leq a < b$. Let $\mathcal{g} : P \to P$ be a mapping defined by $\mathcal{g}\theta = \frac{\theta}{5}, \theta \in P$ and the two altering distance functions by $\phi(t) = t$, $\psi(t) = \frac{t}{3}$, for any $t \in [0, +\infty]$. Then $\mathcal{g}$ has a unique fixed point in P.

Proof. From the hypotheses, it is clear that $(P, d, s, \preceq)$ is a complete partially ordered b-metric space with parameter $s = 2$ and fulfill all the conditions of Corollary 3.14 and Note 1. Furthermore for any $\theta_1, \theta_2 \in P$, the function $\min(\theta_1, \theta_2)(t) = \min\{\theta_1(t), \theta_2(t)\}$ is also continuous and the conditions of Corollary 3.14 and Note 1 are satisfied. Hence, $\mathcal{g}$ has a unique fixed point $\theta = 0$ in P. □

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