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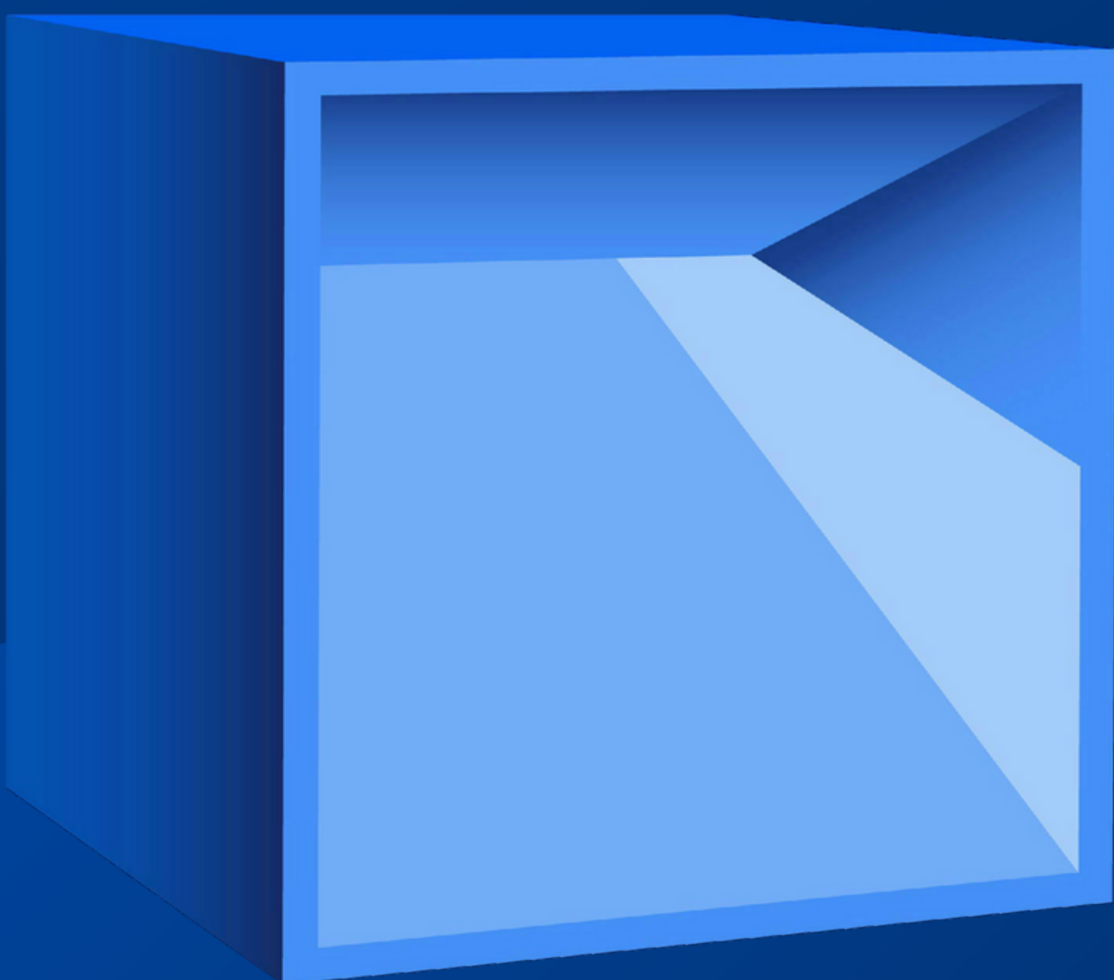

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SUMMARY

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Some observations on a clopen version of the Rothberger property

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ABSTRACT

In this paper, we prove that a clopen version $S_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ of the Rothberger property and Borel strong measure zeroness are independent. For a zero-dimensional metric space (X, d), X satisfies $S_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ if, and only if, X has Borel strong measure zero with respect to each metric which has the same topology as d has. In a zero-dimensional space, the game $G_1(\mathcal{O}, \mathcal{O})$ is equivalent to the game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ and the point-open game is equivalent to the point-clopen game. Using reflections, we obtain that the game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ and the point-clopen game are strategically and Markov dual. An example is given for a space on which the game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ is undetermined.

RESUMEN

En este artículo, probamos que una versión clopen $S_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ de la propiedad de Rothberger y la nulidad de la medida fuerte de Borel son independientes. Para un espacio métrico (X, d) cero-dimensional, X satisface $S_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ si, y sólo si, X tiene una medida Borel fuerte cero con respecto a cada métrica que tenga la misma topología que d tiene. En un espacio cero-dimensional, el juego $G_1(\mathcal{O}, \mathcal{O})$ es equivalente al juego $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ y el juego punto-abierto es equivalente al juego punto-cerrado. Usando reflexiones, obtenemos que el juego $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ y el juego punto-clopen son estratégicamente y Markov duales. Se entrega un ejemplo de un espacio para el cual el juego $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ es indeterminado

Keywords and Phrases: Strong measure zero, selection principles, point-clopen game, zero-dimensional space

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1 Introduction

In 1938, Rothberger [12] (see also [9]) introduced covering property in topological spaces. A space X is said to have *Rothberger property* if for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle V_n : n \in \omega \rangle$ such that for each n, V_n is an element of $\mathcal{U}_n$ and each $x \in X$ belongs to V_n for some n. This property is stronger than Lindelöf and preserved under continuous images.

Usually, each selection principle $S_1(\mathcal{A}, \mathcal{B})$ can be associated with some topological game $G_1(\mathcal{A}, \mathcal{B})$. So the Rothberger property $S_1(\mathcal{O}, \mathcal{O})$ is associated with the Rothberger game $G_1(\mathcal{O}, \mathcal{O})$.

Let X be a topological space. The Rothberger game $G_1(\mathcal{O}, \mathcal{O})$ played on X is a game with two players Alice and Bob.

1st round: Alice chooses an open cover $\mathcal{U}_1$ of X. Bob chooses a set $U_1 \in \mathcal{U}_1$.

2nd round: Alice chooses an open cover $\mathcal{U}_2$ of X. Bob chooses a set $U_2 \in \mathcal{U}_2$.

etc.

If the family $\{U_n : n \in \omega\}$ is a cover of the space X then Bob wins the game $G_1(\mathcal{O}, \mathcal{O})$. Otherwise, Alice wins.

A topological space is Rothberger if, and only if, Alice has no winning strategy in the game $G_1(\mathcal{O}, \mathcal{O})$ [11].

In [8] Galvin proved that for a first-countable space X Bob has a winning strategy in $G_1(\mathcal{O}, \mathcal{O})$ if, and only if, X is countable.

In this paper, we continue to study the mildly Rothberger-type properties, started in papers [2, 3, 4], and, we define a new game - the mildly Rothberger game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$. In a zero-dimensional space, the Rothberger game is equivalent to the mildly Rothberger game. Using reflections, we obtained that $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ and the point-clopen game are strategically and Markov dual.

2 Preliminaries

Let (X, τ) or X be a topological space. If a set is open and closed in a topological space, then it is called *clopen*. Let ω be the first infinite cardinal and ω_1 the first uncountable cardinal. For the terms and symbols that we do not define, follow [7].

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a topological space X.

The symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ there exists a sequence $\langle U_n : n \in \omega \rangle$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \omega\} \in \mathcal{B}$, [13].

In this paper $\mathcal{A}$ and $\mathcal{B}$ will be collections of the following open covers of a space X:

$\mathcal{O}$: the collection of all open covers of X.

$\mathcal{C}_{\mathcal{O}}$: the collection of all clopen covers of X.

Clearly, X has the Rothberger property if, and only if, X satisfies $S_1(\mathcal{O}, \mathcal{O})$.

A space X is said to have the *mildly Rothberger property* if it satisfies the selection principles $S_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$.

It can be noted that $S_1(\mathcal{O}, \mathcal{O}) \Rightarrow S_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$ and also every connected space must satisfy $S_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$. Then the set of real numbers with usual topology satisfies $S_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$ but it does not satisfy $S_1(\mathcal{O}, \mathcal{O})$.

Let (X, τ) be a topological space and $\mathcal{T}_X = \tau \setminus \{\emptyset\}$ be a topology without empty set.

- Let $\mathcal{T}_{X,x} = \{U \in \mathcal{T}_X : x \in U\}$ be the local point-base at $x \in X$.
- Let $\mathcal{P}_X = \{\mathcal{T}_{X,x} : x \in X\}$ be the collection of local point-bases of X.
- Let $\mathcal{C}_{\mathcal{T}_{X,x}} = \{U \in \mathcal{T}_X : U \text{ is a clopen set in } X,\ x \in U\}$.
- Let $\mathcal{C}_X = \{\mathcal{C}_{\mathcal{T}_{X,x}} : x \in X\}$.

3 Results on $S_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$

3.1 $S_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$ and Borel strong measure zeroness are independent

Recall that a set of reals X is *null* (or has measure zero) if for each positive ϵ there exists a cover $\{I_n\}_{n\in\omega}$ of X such that $\Sigma_n \operatorname{diam}(I_n) < \epsilon$.

To restrict the notion of measure zero or null set, in 1919, Borel [1] defined a notion stronger than measure zeroness. Now this notion is known as strong measure zeroness or strongly null set.

Borel strong measure zero: Y is *Borel strong measure zero* if there is for each sequence $\langle \epsilon_n : n \in \omega \rangle$ of positive real numbers a sequence $\langle J_n : n \in \omega \rangle$ of subsets of Y such that each J_n is of diameter $< \epsilon_n$, and Y is covered by $\{J_n : n \in \omega\}$.

But Borel was unable to construct a nontrivial (that is, an uncountable) example of a Borel strong measure zero set. He therefore conjectured that there exists no such examples.

In 1928, Sierpinski observed that every Luzin set is Borel strong measure zero, thus the Continuum Hypothesis implies that Borel's Conjecture is false.

Sierpinski asked whether the property of being Borel strong measure zero is preserved under taking homeomorphic (or even continuous) images.

In 1941, the answer given by Rothberger is negative under the Continuum Hypothesis. This lead Rothberger to introduce the following topological version of Borel strong measure zero (which is preserved under taking continuous images).

In 1988, Miller and Fremlin [10] proved that a space Y has the Rothberger property ($S_1(\mathcal{O},\mathcal{O})$) if, and only if, it has Borel strong measure zero with respect to each metric on Y which generates the topology of Y.

Recall that a space X is zero-dimensional if it has a base consisting clopen sets. Now we show that $S_1(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$ and Borel strong measure zeroness are independent to each other. Since the set of real numbers does not have measure zero, it does not have Borel strong measure zero but it satisfies $S_1(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$. Since every metric space with Borel strong measure zero must be zero-dimensional and separable, $S_1(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$ is equivalent to $S_1(\mathcal{O},\mathcal{O})$ (see below Theorem 3.1). So by Theorem 6(c) in [10], there is a subset of reals with Borel strong measure zero but it does not satisfy $S_1(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$.

The proof of the following result easily follows from replacing the open sets with sets of a clopen base of the topological space.

Theorem 3.1. *For a zero-dimensional space X, $S_1(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$ is equivalent to $S_1(\mathcal{O},\mathcal{O})$.*

From Theorem 1 in [10], we obtain the following corollary.

Corollary 3.2. *For a zero-dimensional metric space (X,d) the following statements are equivalent:*

(1) *X satisfies $S_1(\mathcal{O},\mathcal{O})$;*

(2) *X satisfies $S_1(\mathcal{C}_\mathcal{O},\mathcal{C}_\mathcal{O})$;*

(3) *X has Borel strong measure zero with respect to every metric which generates the original topology;*

(4) *every continuous image of X in Baire space ω^ω with usual metric has Borel strong measure zero.*

3.2 Dual selection games

The *selection game* $G_1(\mathcal{A},\mathcal{B})$ is an ω-length game for two players, Alice and Bob. During round n, Alice choose $A_n \in \mathcal{A}$, followed by Bob choosing $B_n \in A_n$. Player Bob wins in the case that $\{B_n : n < \omega\} \in \mathcal{B}$, and Player Alice wins otherwise.

We consider the following strategies:

- A *strategy for player Alice* in $G_1(\mathcal{A},\mathcal{B})$ is a function $\sigma : (\bigcup \mathcal{A})^{<\omega} \to \mathcal{A}$. A strategy σ for Alice is called *winning* if whenever $x_n \in \sigma\langle x_i : i < n\rangle$ for all $n < \omega$, $\{x_n : n \in \omega\} \notin \mathcal{B}$. If player Alice has a winning strategy, we write $Alice \uparrow G_1(\mathcal{A},\mathcal{B})$.

- A strategy for player Bob in $G_1(\mathcal{A},\mathcal{B})$ is a function $\tau : \mathcal{A}^{<\omega} \to \bigcup \mathcal{A}$. A strategy τ for Bob is *winning* if $A_n \in \mathcal{A}$ for all $n < \omega$, $\{\tau(A_0,\ldots,A_n) : n < \omega\} \in \mathcal{B}$.

- A *predetermined strategy* for Alice is a strategy which only considers the current turn number. Formally it is a function $\sigma : \omega \to \mathcal{A}$. If Alice has a winning predetermined strategy, we write $Alice \underset{pre}{\uparrow} G_1(\mathcal{A},\mathcal{B})$.

- A *Markov strategy* for Bob is a strategy which only considers the most recent move of player Alice and the current turn number. Formally it is a function $\tau : \mathcal{A} \times \omega \to \bigcup \mathcal{A}$. If Bob has a winning Markov strategy, we write $Bob \underset{mark}{\uparrow} G_1(\mathcal{A},\mathcal{B})$.

Note that, $Bob \underset{mark}{\uparrow} G_1(\mathcal{A},\mathcal{B}) \Rightarrow Bob \uparrow G_1(\mathcal{A},\mathcal{B}) \Rightarrow Alice \not\uparrow G_1(\mathcal{A},\mathcal{B}) \Rightarrow Alice \underset{pre}{\not\uparrow} G_1(\mathcal{A},\mathcal{B})$.

It is worth noting that $Alice \underset{pre}{\not\uparrow} G_1(\mathcal{A},\mathcal{B})$ is equivalent to the selection principle $S_1(\mathcal{A},\mathcal{B})$.

Two games G_1 and G_2 are said to be **strategically dual** provided that the following two hold:

- $Alice \uparrow G_1$ iff $Bob \uparrow G_2$

- $Alice \uparrow G_2$ iff $Bob \uparrow G_1$.

Two games G_1 and G_2 are said to be **Markov dual** provided that the following two hold:

- $Alice \underset{pre}{\uparrow} G_1$ iff $Bob \underset{mark}{\uparrow} G_2$

- $Alice \underset{pre}{\uparrow} G_2$ iff $Bob \underset{mark}{\uparrow} G_1$.

Two games G_1 and G_2 are said to be **dual** provided that they are both strategically dual and Markov dual.

For a set X, let $\mathcal{C}(X) = \{f \in (\bigcup X)^X : x \in X \Rightarrow f(x) \in x\}$ be the collection of all choice functions on X.

Write $X \preceq Y$ if X is coinitial in Y with respect to $\subseteq$; that is, $X \subseteq Y$, and for all $y \in Y$, there exists $x \in X$ such that $x \subseteq y$.

In the context of selection games, $\mathcal{A}'$ is a *selection basis* for $\mathcal{A}$ when $\mathcal{A}' \preceq \mathcal{A}$ [6].

Definition 3.3 ([6])**.** *The set $\mathcal{R}$ is said to be a* **reflection** *of the set $\mathcal{A}$ if $\{range(f) : f \in \mathcal{C}(\mathcal{R})\}$ is a selection basis for $\mathcal{A}$.*

Let $G_1(\mathcal{A},\neg\mathcal{B}) := G_1(\mathcal{A},\mathcal{P}(\bigcup \mathcal{A}) \setminus \mathcal{B})$.

Theorem 3.4 ([6], Corollary 26)**.** *If $\mathcal{R}$ is a reflection of $\mathcal{A}$, then $G_1(\mathcal{A},\mathcal{B})$ and $G_1(\mathcal{R},\neg\mathcal{B})$ are dual.*

The point-open game $PO(X)$ is a game where Alice chooses points of X, Bob chooses an open neighborhood of each chosen point, and Alice wins if Bob's choices are a cover.

Theorem 3.5 ([8])**.** *The game $G_1(\mathcal{O},\mathcal{O})$ is strategically dual to the point-open game on each topological space.*

Theorem 3.6 ([5])**.** *The game $G_1(\mathcal{O},\mathcal{O})$ is Markov dual to the point-open game on each topological space.*

Corollary 3.7. *The game $G_1(\mathcal{O},\mathcal{O})$ is dual to the point-open game on each topological space.*

Recall that two games G and G' are *equivalent* (isomorphic) if

(1) $Alice \uparrow G$ iff $Alice \uparrow G'$.

(2) $Bob \uparrow G$ iff $Bob \uparrow G'$.

Since $\mathcal{P}_X$ is a reflection of $\mathcal{O}$ [6, Proposition 28], the Rothberger game $G_1(\mathcal{O},\mathcal{O})$ and $G_1(\mathcal{P}_X,\neg\mathcal{O})$ are dual [6, Corollary 29]. It is well known that the game $G_1(\mathcal{P}_X,\neg\mathcal{O})$ is equivalent to the point-open game.

3.3 The point-clopen and quasi-component-clopen games

The *point-clopen game* $PC(X)$ on a space X is played according to the following rules:

In each inning $n \in \omega$, Alice picks a point $x_n \in X$, and then Bob chooses a clopen set $U_n \subseteq X$ with $x_n \in U_n$. At the end of the play

$$x_0, U_0, x_1, U_1, x_2, U_2, \ldots, x_n, U_n, \ldots,$$

the winner is Alice if $X = \bigcup_{n\in\omega} U_n$, and Bob otherwise.

We denote the collection of all non-empty clopen subsets of a space X by τ_c and the collection of all finite subsets of τ_c by $\tau_c^{<\omega}$.

A strategy for Alice in the point-clopen game on a space X is a function $\varphi : \tau_c^{<\omega} \to X$.

A strategy for Bob in the point-clopen game on a space X is a function $\psi : X^{<\omega} \to \tau_c$ such that, for all $\langle x_0, x_1, \ldots, x_n\rangle \in X^{<\omega} \setminus \{\langle\rangle\}$, we have $x_n \in \psi(\langle x_0, \ldots, x_n\rangle) = U_n$.

A strategy $\varphi : \tau_c^{<\omega} \to X$ for Alice in the point-clopen game on a space X is a winning strategy for Alice if, for every sequence $\langle U_n : n \in \omega\rangle$ of clopen subsets of a space X such that $\forall n \in \omega$,

($x_n = \varphi(\langle U_0, U_1, \ldots, U_{n-1}\rangle) \in U_n$), we have $X = \bigcup_{n\in\omega} U_n$. If Alice has a winning strategy in the point-clopen game on a space X, we write $Alice{\uparrow}PC(X)$.

A strategy $\psi : X^{<\omega} \to \tau_c$ for Bob in the point-clopen game on a space X is a winning strategy for Bob if, for every sequence $\langle x_n : n \in \omega\rangle$ of points of a space X, we have $X = \bigcup_{n\in\omega}\{U_n : U_n = \psi(\langle x_0, x_1, \ldots, x_n\rangle)\}$. If Bob has a winning strategy in the point-clopen game on a space X, we write $Bob{\uparrow}PC(X)$.

The game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ is a game for two players, Alice and Bob, with an inning per each natural number n. In each inning, Alice picks a clopen cover of the space and Bob selects one member from this cover. Bob wins if the sets he selected throughout the game cover the space. If this is not the case, Alice wins.

The intersection of all clopen sets containing a component is called a *quasi-component* of the space [7].

The *quasi-component-clopen game* $QC(X)$ on a space X is played according to the following rules:

In each inning $n \in \omega$, Alice picks a quasi-component A_n of X, and then Bob chooses a clopen set $U_n \subseteq X$ with $A_n \subseteq U_n$. At the end of the play

$$A_0, U_0, A_1, U_1, A_2, U_2, \ldots, A_n, U_n, \ldots,$$

the winner is Alice if $X = \bigcup_{n\in\omega} U_n$, and Bob otherwise.

We denote the collection of all quasi-components of a space X by Q_X and the collection of all finite subsets of Q_X by $Q_X^{<\omega}$.

A strategy for Alice in the quasi-component-clopen game on a space X is a function $\varphi : \tau_c^{<\omega} \to Q_X$.

A strategy for Bob in the quasi-component-clopen game on a space X is a function $\psi : Q_X^{<\omega} \to \tau_c$ such that, for all $\langle A_0, A_1, \ldots, A_n\rangle \in Q_X^{<\omega} \setminus \{\langle\rangle\}$, we have $A_n \subseteq \psi(\langle A_0, \ldots, A_n\rangle) = U_n$.

A strategy $\varphi : \tau_c^{<\omega} \to Q_X$ for Alice in the quasi-component-clopen game on a space X is a winning strategy for Alice if, for every sequence $\langle U_n : n \in \omega\rangle$ of clopen subsets of a space X such that $\forall n \in \omega, (A_n = \varphi(\langle U_0, U_1, \ldots, U_{n-1}\rangle) \subseteq U_n)$, we have $X = \bigcup_{n\in\omega} U_n$. If Alice has a winning strategy in the quasi-component-clopen game on a space X, we write $Alice{\uparrow}QC(X)$.

A strategy $\psi : Q_X^{<\omega} \to \tau_c$ for Bob in the quasi-component-clopen game on a space X is a winning strategy for Bob if, for every sequence $\langle A_n : n \in \omega\rangle$ of quasi-components of a space X, we have $X = \bigcup_{n\in\omega}\{U_n : U_n = \psi(\langle A_0, A_1, \ldots, A_n\rangle)\}$. If Bob has a winning strategy in the quasi-component-clopen game on a space X, we write $Bob{\uparrow}QC(X)$.

Proposition 3.8. *The point-clopen game is equivalent to the quasi-component-clopen game.*

Proof. Let $\varphi : \tau_c^{<\omega} \to X$ be a winning strategy for Alice in the point-clopen game on a space

X. Then the function $\psi : \tau_c^{<\omega} \to Q_X$ such that $\psi(\langle U_0, U_1, \dots, U_{n-1}\rangle) = Q[\varphi(\langle U_0, U_1, \dots, U_{n-1}\rangle)]$ ($Q[x]$ is the quasi-component of x) for every sequence $\langle U_n : n \in \omega\rangle$ of clopen subsets of a space X and $n \in \omega$, is a winning strategy for Alice in the quasi-component-clopen game. This follows from the fact that $x_n = \varphi(\langle U_0, U_1, \dots, U_{n-1}\rangle) \in Q[x_n] \subseteq U_n$.

Let $\varphi : \tau_c^{<\omega} \to Q_X$ be a winning strategy for Alice in the quasi-component-clopen game on a space X. Then the function $\psi : \tau_c^{<\omega} \to X$ such that $\psi(\langle U_0, U_1, \dots, U_{n-1}\rangle) \in \varphi(\langle U_0, U_1, \dots, U_{n-1}\rangle)$ for every sequence $\langle U_n : n \in \omega\rangle$ of clopen subsets of a space X and $n \in \omega$, is a winning strategy for Alice in the point-clopen game.

Let $\psi : X^{<\omega} \to \tau_c$ be a winning strategy for Bob in the point-clopen game on X. Then the function $\rho : Q_X^{<\omega} \to \tau_c$ such that $\rho(\langle A_0, A_1, \dots, A_n\rangle) = \psi(\langle x_0, x_1, \dots, x_n\rangle)$ for every sequence $\langle A_n : n \in \omega\rangle$ of quasi-components of a space X and some $x_0, \dots, x_n$ that $A_i = Q[x_i]$ for each $i = 0, \dots, n$, is a winning strategy for Bob in the quasi-component-clopen game.

Let $\psi : Q_X^{<\omega} \to \tau_c$ be a winning strategy for Bob in the quasi-component-clopen game on X. Then the function $\rho : X^{<\omega} \to \tau_c$ such that $\rho(\langle x_0, x_1, \dots, x_n\rangle) = \psi(\langle A_0, A_1, \dots, A_n\rangle)$ for every sequence $\langle x_n : n \in \omega\rangle$ of points of a space X where $A_i = Q[x_i]$ for each $i = 0, \dots, n$, is a winning strategy for Bob in the point-clopen-clopen game. □

Proposition 3.9. *$\mathcal{C}_X$ is a reflection of $\mathcal{C}_{\mathcal{O}}$.*

Proof. For every clopen cover $\mathcal{U}$, the corresponding choice function $f \in \mathcal{C}(\mathcal{C}_X)$ is simply the witness that $x \in f(\mathcal{C}_{\mathcal{T}_{X,x}}) \in \mathcal{U}$. □

By Theorem 3.4, we get the following result.

Corollary 3.10. *$G_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$ and $G_1(\mathcal{C}_X, \neg\mathcal{C}_{\mathcal{O}})$ are dual.*

Note that $PC(X)$ and $G_1(\mathcal{C}_X, \neg\mathcal{C}_{\mathcal{O}})$ are the same game.

By Proposition 3.8, $PC(X)$ and $QC(X)$ are equivalent, hence, we get the following result.

Proposition 3.11. *The game $G_1(\mathcal{C}_X, \neg\mathcal{C}_{\mathcal{O}})$ is equivalent to the quasi-component-clopen game.*

Corollary 3.12. *If a space X is a union of countable number of quasi-components, then Bob $\uparrow G_1(\mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{O}})$.*

The following chain of implications always holds:

$$X \text{ is a union of countable number of quasi-components}$$
$$\Downarrow$$
$$Bob \uparrow G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$$
$$\Downarrow$$
$$Alice \not\uparrow G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$$
$$\Updownarrow$$
$$X \text{ has mildly Rothberger property.}$$

The proof of the following result easily follows from replacing the open sets with sets of a clopen base of the topological space.

Theorem 3.13. *For a zero-dimensional space, the following statements hold:*

(1) *The game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ is equivalent to the game $G_1(\mathcal{O}, \mathcal{O})$.*

(2) *The point-clopen game is equivalent to the point-open game.*

From [11] and [**?**], we have the following result.

Theorem 3.14. *For a space X, the following statements hold:*

(1) *[11] X satisfies $S_1(\mathcal{O}, \mathcal{O})$ iff $Alice \not\uparrow G_1(\mathcal{O}, \mathcal{O})$.*

(2) *[**?**] X satisfies $S_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ iff $Alice \not\uparrow G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$.*

Corollary 3.15. *For a space X, the following statements are equivalent:*

(1) *X satisfies $S_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$;*

(2) *$Alice \underset{pre}{\not\uparrow} G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$;*

(3) *$Alice \not\uparrow G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$;*

(4) *$Bob \not\uparrow G_1(\mathcal{C}_X, \neg\mathcal{C}_\mathcal{O})$;*

(5) *$Bob \underset{mark}{\not\uparrow} G_1(\mathcal{C}_X, \neg\mathcal{C}_\mathcal{O})$;*

(6) *$Bob \not\uparrow PC(X)$;*

(7) *$Bob \not\uparrow QC(X)$;*

(8) *$Bob \underset{mark}{\not\uparrow} PC(X)$;*

(9) *$Bob \underset{mark}{\not\uparrow} QC(X)$.*

Corollary 3.16. *For a zero-dimensional space X, the following statements are equivalent:*

(1) *X satisfies $S_1(\mathcal{O}, \mathcal{O})$;*

(2) *X satisfies $S_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$;*

(3) *$Alice \underset{pre}{\not\uparrow} G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$;*

(4) *$Alice \underset{pre}{\not\uparrow} G_1(\mathcal{O}, \mathcal{O})$;*

(5) *$Alice \not\uparrow G_1(\mathcal{O}, \mathcal{O})$;*

(6) *$Alice \not\uparrow G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$;*

(7) *Bob* $\not\uparrow G_1(\mathcal{P}_X, \neg\mathcal{O})$;

(8) *Bob* $\not\uparrow G_1(\mathcal{C}_X, \neg\mathcal{C}_\mathcal{O})$;

(9) *Bob* $\not\uparrow PO(X)$;

(10) *Bob* $\not\uparrow PC(X)$;

(11) *Bob* $\not\uparrow QC(X)$;

(12) *Bob* $\underset{mark}{\not\uparrow} PO(X)$;

(13) *Bob* $\underset{mark}{\not\uparrow} PC(X)$;

(14) *Bob* $\underset{mark}{\not\uparrow} QC(X)$.

In [8], Galvin and Telgársky in [14, Theorem 6.3] prove: If X is a Lindelöf space in which each element is G_δ, then Bob has a winning strategy in $G_1(\mathcal{O}, \mathcal{O})$ if, and only if, X is countable.

Theorem 3.17. *Let X be a space in which each quasi-component is an intersection of countably many clopen sets, then Bob $\uparrow G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ if, and only if, X is a union of countably many quasi-components.*

Proof. Let Bob have a winning strategy in the game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ on X. Since the game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ and the point-clopen game are dual and, by Proposition 3.8, the point-clopen game and the quasi-component-clopen game are equivalent.

Let Alice have a winning strategy in the quasi-component-clopen game. Let φ be a winning strategy of Alice in the quasi-component-clopen game on X. For every quasi-component Q, there is a sequence $\langle V_k : k \in \omega\rangle$ of clopen sets such that $Q = \bigcap_{k\in\omega} V_k$.

So we restrict the move of Bob from $\{V_k : k \in \omega\}$ for Q played by Alice.

Let Alice start the play of the point-clopen game by quasi-component $\varphi(\langle\rangle) = Q_{\langle\rangle}$. Then Bob replies with a clopen set of the form $V_{k_0,\langle\rangle}$ for some $k_0 \in \omega$.

Alice's next move in the play is a quasi-component $\varphi(\langle V_{k_0,\langle\rangle}\rangle) = Q_{\langle k_0\rangle}$. Then Bob replies with a clopen set of the form $V_{k_1,\langle k_0\rangle}$ for some $k_1 \in \omega$.

Now Alice's next move in the play is a quasi-component $\varphi(\langle V_{k_0,\langle\rangle}, V_{k_1,\langle k_0\rangle}\rangle) = Q_{\langle k_0,k_1\rangle}$. Then Bob replies with a clopen set of the form $V_{k_2,\langle k_0,k_1\rangle}$ for some $k_2 \in \omega$ and so on.

Similarly we are defining $\langle Q_s : s \in \omega^{<\omega}\rangle$ by setting $Q_{\langle\rangle} = \varphi(\langle\rangle)$ and for each $s \in \omega^{<\omega}$ and for each $k \in \omega$, defining

$$Q_{s\frown\langle k\rangle} = \varphi(\langle V_{s(0),s\restriction 0}, V_{s(1),s\restriction 1}, \dots, V_{s(m-1),s\restriction(m-1)}, V_{k,s}\rangle),$$

where $m = dom(s)$. From this we construct a countable collection $\{Q_s : s \in \omega^{<\omega}\}$.

Now to show that $\bigcup\{Q_s : s \in \omega^{<\omega}\} = X$. If possible suppose that $\bigcup\{Q_s : s \in \omega^{<\omega}\} \neq X$, then there is $y \in X \setminus \{Q_s : s \in \omega^{<\omega}\}$. Then $y \notin Q_s$ for any $s \in \omega^{<\omega}$. For each $Q_n \in \{Q_s : s \in \omega^{<\omega}\}$, there is some k_n such that $y \notin V_{k_n,n}$. Then Alice loses the following play of the quasi-component-clopen game

$$\langle Q_0, V_{k_0,0}, Q_1, V_{k_1,1}, \ldots, Q_n, V_{k_n,n}, \ldots \rangle$$

in which Alice uses the strategy φ since $y \notin \bigcup_{n\in\omega} V_{k_n,n}$, a contradiction.

The converse follows from Corollary 3.12. □

3.4 Determinacy and $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ game

A game G played between two players Alice and Bob is determined if either Alice has a winning strategy in game G or Bob has a winning strategy in game G. Otherwise G is undetermined.

It can be observed that the game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ is determined for every countable space. But in a mildly Rothberger space in which each quasi-component is an intersection of countably many clopen sets with uncountable many quasi-components, none of the players Alice and Bob have a winning strategy. So $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ is undetermined for a mildly Rothberger space in which each quasi-component is an intersection of countably many clopen sets with uncountable many quasi-components. Thus every uncountable zero-dimensional mildly Rothberger metric space is undetermined.

Recall that an uncountable set L of reals is a *Luzin set* if for each meager set M, $L \cap M$ is countable. The Continuum Hypothesis implies the existence of a Luzin set. A Luzin set is an example of a space for which the game $G_1(\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{O})$ is undetermined.

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Existence of solutions for higher order $\phi-$Laplacian BVPs on the half-line using a one-sided Nagumo condition with nonordered upper and lower solutions

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ABSTRACT

In this paper, we consider the following $(n+1)$st order bvp on the half line with a $\phi-$Laplacian operator

$$\begin{cases} (\phi(u^{(n)}))'(t) = f(t, u(t), \dots, u^{(n)}(t)), & a.e.,\ t \in [0, +\infty), \\ & n \in \mathbb{N} \setminus \{0\}, \\ u^{(i)}(0) = A_i,\ i = 0, \dots, n-2, \\ u^{(n-1)}(0) + au^{(n)}(0) = B, \\ u^{(n)}(+\infty) = C. \end{cases}$$

The existence of solutions is obtained by applying Schaefer's fixed point theorem under a one-sided Nagumo condition with nonordered lower and upper solutions method where f is a L^1-Carathéodory function.

RESUMEN

En este artículo, consideramos el siguiente pvf en la semi-recta de orden $(n+1)$ con un operador $\phi-$Laplaciano

$$\begin{cases} (\phi(u^{(n)}))'(t) = f(t, u(t), \dots, u^{(n)}(t)), & a.e.,\ t \in [0, +\infty), \\ & n \in \mathbb{N} \setminus \{0\}, \\ u^{(i)}(0) = A_i,\ i = 0, \dots, n-2, \\ u^{(n-1)}(0) + au^{(n)}(0) = B, \\ u^{(n)}(+\infty) = C. \end{cases}$$

Se obtiene la existencia de soluciones aplicando el teorema de punto fijo de Schaefer bajo una condición unilateral de Nagumo con un método de soluciones inferiores y superiores no-ordenadas donde f es una función L^1-Carathéodory.

Keywords and Phrases: Boundary value problem, One-sided Nagumo condition, Lower and upper solutions, A priori estimates.

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1 Introduction

Differential equations of nth order were studied in many works, with different boundary value conditions by using different methods on bounded and unbounded domains, we quote [5, 6, 8, 9, 10] and references therein.

In this paper we consider the following $\phi-$Laplacian ordinary differential equation of order $n+1$ given by

$$(\phi(u^{(n)}))'(t) = f(t, u(t), \dots, u^{(n)}(t)), \ a.e., \ t \in [0, +\infty), \tag{1.1}$$

where $n \in \mathbb{N} \setminus \{0\}$, ϕ is an increasing homeomorphism satisfying $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$.

Concerning the nonlinearity, we suppose that $f : [0, +\infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a L^1-Carathéodory function.

This equation is subject to the following Sturm-Liouville type boundary conditions:

$$\begin{cases} u^{(i)}(0) = A_i, \ i = 0, \dots, n-2, \\ u^{(n-1)}(0) + au^{(n)}(0) = B, \\ u^{(n)}(+\infty) = C, \end{cases} \tag{1.2}$$

where $a < 0, B, C \in \mathbb{R}$, $A_i \in \mathbb{R}$, $i = 0, 1, \dots, n-2$ and $u^{(n)}(+\infty) = \lim_{t \to +\infty} u^{(n)}(t)$.

To prove the existence of solutions for this problem we use Scheafer's fixed point theorem combined with the upper and lower solutions method with a one-sided Nagumo condition.

The upper and lower solutions method have witnessed qualitative progress in recent years by providing various results, following some papers that use this method [2, 3, 4, 7, 11, 12, 14, 15, 16].

In [12] and [7], the authors study the existence of solutions to the following two problems using the Schauder fixed point theorem with upper and lower solutions method with a one-sided Nagumo condition. The first problem is given by

$$u'''(t) = f\left(t, u(t), u'(t), u''(t)\right), t \in [0, +\infty),$$

$u(0) = A$, $au'(0) + bu''(0) = B$, $u''(+\infty) = C$, with $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is a L^1- Carathéodory function, $a > 0$, $b < 0$, $A, B, C \in \mathbb{R}$. The second problem is

$$u^{(4)}(t) = f\left(t, u(t), u'(t), u''(t), u'''(t)\right), \quad t \in [0, +\infty),$$

where $f : [0, +\infty) \times \mathbb{R}^4 \to \mathbb{R}$ is a L^1-Carathéodory function, and the boundary conditions are of

Sturm-Liouville type,

$$u(0) = A, u'(0) = B, \quad u''(0) + au'''(0) = C, \quad u'''(+\infty) = D,$$

$A, B, C, D \in \mathbb{R}$, $a < 0$ and $u'''(+\infty) := \lim_{t\to+\infty} u'''(t)$.

In the present paper, we have obtained the same results as in [12] and [7], but for a more general problem, where we combine an nth order ordinary differential equation with a $\phi-$Laplacian operator on the half line using non-ordered upper and lower solutions and to compensate the lack of compactness of the interval $[0, +\infty)$ we invoke the Corduneanu lemma (see Lemma 2.6).

This problem has many applications with regards to higher order problems defined on unbounded intervals. We quote, *e.g.*, [14] for $n = 2$. In the case where $\phi(t) = t$, we cite [12] and [7] for the third and fourth order, respectively.

The paper is divided into four sections. Section 2 is devoted to some preliminary definitions and the proof of technical lemmas. In Section 3, we prove the main result and in Section 4, we propose an example where we show the applicability of the main result.

2 Definitions and preliminary results

Let

$$X = \left\{ u \in \mathcal{C}^n[0, +\infty) : \lim_{t\to+\infty} u^{(n)}(t) \text{ exists in } \mathbb{R} \right\}$$

and define the norm $\|u\|_X := \max\{\|u\|_0, \|u'\|_1, \|u''\|_2, \dots, \|u^{(n)}\|_n\}$, where

$$\|u^{(i)}\|_i = \sup_{0\le t<+\infty} \left| \frac{u^{(i)}(t)}{1 + t^{n-i}} \right|, \; i = 0, 1, 2, \dots, n.$$

Lemma 2.1. *For each fixed $n \in \mathbb{N} \setminus \{0\}$, let $u \in \mathcal{C}^n([0, +\infty))$. If $\lim_{t\to+\infty} u^{(n)}(t) = \ell$, then*

$$\lim_{t\to+\infty} u^{(n)}(t) = (n-i)! \lim_{t\to+\infty} \frac{u^{(i)}(t)}{1 + t^{n-i}}, \quad \textit{for} \quad i \in \{0, 1, \dots, n-1\}.$$

Proof. Let n be fixed in $\mathbb{N}^*$ and $u \in \mathcal{C}^n([0, +\infty))$ such that $\lim_{t\to+\infty} u^{(n)}(t) = \ell$. We have

$$\lim_{t\to+\infty} u^{(n)}(t) - \ell + 1 = 1.$$

So,

$$\lim_{t\to+\infty} u^{(n-1)}(t) - \ell t + t + d_{n-1} = +\infty$$

where d_{n-1} is a real constant. By using L'Hospital's rule, we deduce that

$$\lim_{t\to+\infty}\frac{u^{(n-1)}(t)-\ell t+t+d_{n-1}}{1+t}=\lim_{t\to+\infty}u^{(n)}(t)-\ell+1.$$

Hence,

$$\lim_{t\to+\infty}u^{(n)}(t)=\lim_{t\to+\infty}\frac{u^{(n-1)}(t)}{1+t}=(n-(n-1))!\lim_{t\to+\infty}\frac{u^{(n-1)}(t)}{1+t^{(n-(n-1))}}\cdot$$

In this case, $i=n-1$. To evaluate $\lim_{t\to+\infty}\frac{u^{(n-2)}}{1+t^2}$, we repeat the formula twice:

$$\lim_{t\to+\infty}u^{(n-1)}(t)-\ell t+t+d_{n-1}=+\infty.$$

Then,

$$\lim_{t\to+\infty}u^{(n-2)}(t)-\ell\frac{t^2}{2}+\frac{t^2}{2}+d_{n-1}t+d_{n-2}=+\infty.$$

Using L'Hospital's rule twice, we get

$$\begin{aligned}\lim_{t\to+\infty}\frac{u^{(n-2)}(t)-\frac{\ell}{2}t^2+\frac{1}{2}t^2+d_{n-1}t+d_{n-2}}{1+t^2}&=\lim_{t\to+\infty}\frac{u^{(n-1)}(t)-\ell t+t+d_{n-1}}{2t}\\&=\lim_{t\to+\infty}\frac{u^{(n)}(t)}{2}-\frac{\ell}{2}+\frac{1}{2}\cdot\end{aligned}$$

Then,

$$\lim_{t\to+\infty}u^{(n)}(t)=\lim_{t\to+\infty}2\frac{u^{(n-2)}(t)}{1+t^2}=(n-(n-2))!\lim_{t\to+\infty}\frac{u^{(n-2)}(t)}{1+t^{(n-(n-2))}}\cdot$$

Here $i=n-2$ and d_{n-1},d_{n-2} are real constants. At the order i,

$$\begin{aligned}&\lim_{t\to+\infty}\frac{u^{(i)}(t)}{1+t^{n-i}}-\frac{\ell}{(n-i)!}+\frac{1}{(n-i)!}\\&=\lim_{t\to+\infty}\frac{u^{(i)}(t)-\frac{\ell}{(n-i)!}t^{n-i}+\frac{1}{(n-i)!}t^{n-i}+\frac{1}{(n-i-1)!}t^{n-i-1}+\cdots+d_{i-1}t+d_i}{1+t^{n-i}}\\&=\\&\vdots\\&=\lim_{t\to+\infty}\frac{u^{(n)}(t)}{(n-i)!}-\frac{\ell}{(n-i)!}+\frac{1}{(n-i)!}\cdot\end{aligned}$$

In conclusion,

$$\lim_{t\to+\infty}u^{(n)}(t)=(n-i)!\lim_{t\to+\infty}\frac{u^{(i)}(t)}{1+t^{n-i}}.$$

□

By this Lemma, $(X,\|\cdot\|_X)$ is a Banach space.

The following definition establishes the assumptions assumed on the nonlinearity.

Definition 2.2. *A function $f : [0,+\infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ is called a L^1-Carathéodory function if it satisfies:*

(i) for each $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$, $t \mapsto f(t, x_0, x_1, \ldots, x_n)$ is measurable on $[0,+\infty)$;

(ii) for almost every $t \in [0,+\infty), (x_0, x_1, \ldots, x_n) \mapsto f(t, x_0, x_1, \ldots, x_n)$ is continuous in $\mathbb{R}^{n+1}$;

(iii) $\forall \rho > 0$, $\exists \varphi_\rho \in L^1[0,+\infty)$, $\forall x \in X$

$$\|x\|_X < \rho \Rightarrow |f(t, x(t), x'(t), \ldots, x^{(n)}(t))| \le \varphi_\rho(t), \quad a.e., \quad t \in [0,+\infty).$$

Lemma 2.3. *Let $\eta \in L^1[0,+\infty)$. The linear boundary value problem*

$$(\phi(u^{(n)}))'(t) + \eta(t) = 0, \quad a.e., \quad t \in [0,+\infty), \tag{2.1}$$

with boundary conditions (1.2), has a unique solution in X. Moreover, this solution can be expressed as

$$\begin{aligned} u(t) =& A_0 + A_1 t + \cdots + \frac{A_{n-2}}{(n-2)!} t^{n-2} + \frac{B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(s)\, ds\right)}{(n-1)!} t^{n-1} \\ &+ \int_0^t \left(\frac{(t-s)^{n-1}}{(n-1)!}\right) \phi^{-1}\left(\phi(C) + \int_s^{+\infty} \eta(\tau)\, d\tau\right) ds. \end{aligned} \tag{2.2}$$

Proof. We integrate (2.1) from t to $+\infty$,

$$\phi(u^{(n)}(t)) = \phi(C) + \int_t^{+\infty} \eta(\tau)\, d\tau$$

to get

$$u^{(n)}(t) = \phi^{-1}\left(\phi(C) + \int_t^{+\infty} \eta(\tau)\, d\tau\right). \tag{2.3}$$

So,

$$u^{(n)}(0) = \phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(\tau)\, d\tau\right). \tag{2.4}$$

By integrating (2.3) on $(0, t]$ and using (1.2) with (2.4),

$$u^{(n-1)}(t) = B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(s)\, ds\right) + \int_0^t \phi^{-1}\left(\phi(C) + \int_s^{+\infty} \eta(\tau)\, d\tau\right) ds. \tag{2.5}$$

Integrating (2.5) on $(0, t]$, we get

$$\begin{aligned} u^{(n-2)}(t) =& A_{n-2} + Bt - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(s)\, ds\right) t \\ &+ \int_0^t (t-s)\, \phi^{-1}\left(\phi(C) + \int_s^{+\infty} \eta(\tau)\, d\tau\right) ds. \end{aligned} \tag{2.6}$$

By integrating (2.6) on $(0,t]$,

$$u^{(n-3)}(t) = A_{n-3} + A_{n-2}t + \frac{B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(s)\,ds\right)}{2}t^2 + \int_0^t \frac{(t-s)^2}{2}\phi^{-1}\left(\phi(C) + \int_s^{+\infty} \eta(\tau)\,d\tau\right) ds.$$

Integrating again over $(0,t]$, we find for $i = 0, 1, \ldots, n-1$,

$$u^{(i)}(t) = \sum_{k=i}^{k=n-2} \frac{A_k}{(k-i)!}t^{k-i} + B\,\frac{t^{n-1-i}}{(n-1-i)!} - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(s)\,ds\right)\frac{t^{n-1-i}}{(n-1-i)!} + \int_0^t \frac{(t-s)^{n-1-i}}{(n-1-i)!}\phi^{-1}\left(\phi(C) + \int_s^{+\infty} \eta(\tau)\,d\tau\right) ds. \tag{2.7}$$

By (2.7),

$$u(t) = A_0 + A_1 t + \cdots + \frac{A_{n-2}}{(n-2)!}t^{n-2} + \frac{B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(s\,ds\right)}{(n-1)!}t^{n-1} + \int_0^t \left(\frac{(t-s)^{n-1}}{(n-1)!}\right)\phi^{-1}\left(\phi(C) + \int_s^{+\infty} \eta(\tau)\,d\tau\right) ds. \qquad \square$$

Now, we need to have an *a priori* estimate for $u^{(n)}$, for this let $\gamma_i, \Gamma_i \in C[0,+\infty)$, $\gamma_i(t) \le \Gamma_i(t)$, $i = 0, 1, 2, \ldots, n-1$, with $\sup_{t\ge 0} \frac{|\gamma_{n-1}(t)|}{1+t} < +\infty$ and $\sup_{t\ge 0} \frac{|\Gamma_{n-1}(t)|}{1+t} < +\infty$. Define the set

$$E = \left\{(t, x_0, x_1, \cdots, x_n) \in [0,+\infty) \times \mathbb{R}^{n+1} : \gamma_i(t) \le x_i \le \Gamma_i(t), i = 0, 1, 2, \ldots, n-1\right\}.$$

Definition 2.4. A function $f : E \to \mathbb{R}$ is said to satisfy the one-sided Nagumo type growth condition in E if it satisfies either

$$f(t, x_0, x_1, \ldots, x_n) \le \psi(t)h(|x_n|), \quad \forall (t, x_0, x_1, \ldots, x_n) \in E, \tag{2.8}$$

or

$$f(t, x_0, x_1, \ldots, x_n) \ge -\psi(t)h(|x_n|), \quad \forall (t, x_0, x_1, \ldots, x_n) \in E, \tag{2.9}$$

for some positive continuous functions ψ, h, and some $\nu > 1$, such that

$$\sup_{0\le t<+\infty} \psi(t)(1+t)^{\nu} < +\infty, \quad \int^{+\infty} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))}\,ds = +\infty, \quad \int^{+\infty} \frac{\phi^{-1}(-s)}{h(|\phi^{-1}(-s)|)}\,ds = -\infty. \tag{2.10}$$

Next lemma provides an *a priori* bound.

Lemma 2.5. *Let $f : [0,+\infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a L^1-Carathéodory function satisfying* (2.8) *with* (2.10)*, or* (2.9) *with* (2.10)*. Then there exists $R > 0$ such that every solution u of* (1.1)- (1.2) *satisfying*

$$\gamma_i(t) \leq u^{(i)}(t) \leq \Gamma_i(t), \quad i = 0, \ldots, n-1 \tag{2.11}$$

for $t \in [0,+\infty)$ is such that $\|u^{(n)}\|_n < R$ where R does not depend on the solution u.

Proof. Let u be a solution of (1.1)-(1.2) such that (2.11) holds. Consider $r > 0$ such that

$$r > \max\left\{\left|\frac{B - \Gamma_{n-1}(0)}{a}\right|, \left|\frac{B - \gamma_{n-1}(0)}{a}\right|, |C|\right\}. \tag{2.12}$$

With this inequality we cannot have $|u^{(n)}(t)| > r$ for all $t \in [0,+\infty)$, because

$$|u^{(n)}(0)| = \left|\frac{B - u^{(n-1)}(0)}{a}\right| \leq \max\left\{\left|\frac{B - \Gamma_{n-1}(0)}{a}\right|, \left|\frac{B - \gamma_{n-1}(0)}{a}\right|\right\} < r \tag{2.13}$$

and $|u^{(n)}(+\infty)| = |C| < r$.

In the case where $|u^{(n)}(t)| \leq r$ for all $t \in [0,+\infty)$, it is enough to consider $R > r/2$ to complete the proof:

$$\|u^{(n)}\|_n = \sup_{0\leq t<+\infty}\left|\frac{u^{(n)}(t)}{2}\right| \leq \frac{r}{2} < R.$$

If there exists $t \in (0,+\infty)$ such that $|u^{(n)}(t)| > r$, then by (2.10), we can take $R > r$ such that

$$\int_{\phi(r)}^{\phi(R)} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))}\, ds > M\left\{M_1 + \sup_{0\leq t<+\infty}\frac{|\Gamma_{n-1}(t)|}{1+t}\frac{\nu}{\nu-1}\right\}$$

and

$$\int_{\phi(-R)}^{\phi(-r)} \frac{\phi^{-1}(s)}{h(|\phi^{-1}(s)|)}\, ds < M\left\{-M_1 + \inf_{0\leq t<+\infty}\frac{-|\gamma_{n-1}(t)|}{1+t}\frac{\nu}{\nu-1}\right\}$$

with $M := \sup\limits_{0\leq t<+\infty}\psi(t)(1+t)^\nu$ and $M_1 := \sup\limits_{0\leq t<+\infty}\dfrac{\Gamma_{n-1}(t)}{(1+t)^\nu} - \inf\limits_{0\leq t<+\infty}\dfrac{\gamma_{n-1}(t)}{(1+t)^\nu}$.

Assume that the growth condition (2.8) holds. By (2.12), suppose that there exist $t_*,\, t_+ \in (0,+\infty)$ such that $u^{(n)}(t_*) = r$ and $u^{(n)}(t) > r$ for all $t \in (t_*, t_+]$. Then

$$\begin{aligned}
\int_{\phi(u^{(n)}(t_*))}^{\phi(u^{(n)}(t_+))} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))}\, ds &= \int_{t_*}^{t_+} \frac{u^{(n)}(s)}{h(u^{(n)}(s))}(\phi(u^{(n)}))'(s)\, ds \\
&= \int_{t_*}^{t_+} \frac{f(s, u(s), u'(s), u''(s), u'''(s), \ldots, u^{(n)}(s))}{h(u^{(n)}(s))} u^{(n)}(s)\, ds \\
&\leq \int_{t_*}^{t_+} \psi(s)\ u^{(n)}(s)\, ds \leq M \int_{t_*}^{t_+} \frac{u^{(n)}(s)}{(1+s)^\nu}\, ds
\end{aligned}$$

$$
\begin{aligned}
&= M\int_{t_*}^{t_+}\left(\left(\frac{u^{(n-1)}(s)}{(1+s)^{\nu}}\right)' + \frac{\nu u^{(n-1)}(s)}{(1+s)^{1+\nu}}\right) ds \\
&= M\left(\frac{u^{(n-1)}(t_+)}{(1+t_+)^{\nu}} - \frac{u^{(n-1)}(t_*)}{(1+t_*)^{\nu}} + \int_{t_*}^{t_+}\frac{\nu u^{(n-1)}(s)}{(1+s)^{1+\nu}}\, ds\right) \\
&\le M\left(M_1 + \sup_{0\le t<+\infty}\frac{|\Gamma_{n-1}(t)|}{1+t}\int_0^{+\infty}\frac{\nu}{(1+s)^{\nu}}\, ds\right) \\
&< \int_{\phi(r)}^{\phi(R)}\frac{\phi^{-1}(s)}{h(\phi^{-1}(s))}\, ds.
\end{aligned}
$$

So $u^{(n)}(t_+) < R$ and as t_*, t_+ are arbitrary in $(0,+\infty)$, we have $u^{(n)}(t) < R$ for all $t \in [0,+\infty)$.

By the same technique using (2.12), and considering t_- and t_* such that $u^{(n)}(t_*) = -r$, $u^{(n)}(t) < -r$ for all $t \in [t_-, t_*)$, it can be proved that $u^{(n)}(t) > -R$ for all $t \in [0,+\infty)$, therefore $\|u^{(n)}\|_n < R/2 < R$.

If f satisfies (2.9), following similar arguments we get the same conclusion. □

We also need a compactness criterion.

Lemma 2.6 ([1]). *A set $M \subset X$ is relatively compact if the following three conditions hold:*

(1) *all functions from M are uniformly bounded;*

(2) *all functions from M are equicontinuous on any compact interval of $[0,+\infty)$;*

(3) *all functions from M are equiconvergent at infinity, that is, for any given $\epsilon > 0$, there exists a $t_\epsilon > 0$ such that*

$$
\left|\frac{u^{(i)}(t)}{1+t^{n-i}} - \lim_{t\to+\infty}\frac{u^{(i)}(t)}{1+t^{n-i}}\right| < \epsilon, \text{ for all } t > t_\epsilon, u \in M \text{ and } i = 0,1,2,3,\dots,n.
$$

To end this section, we present the Schaefer Fixed Point Theorem with the definition of lower and upper solutions for our problem (1.1)-(1.2).

Theorem 2.7 ([13]). *Let E be a Banach space and $T : E \to E$ be a completely continuous operator. If the set*

$$
\{x \in E : \ x = \lambda T x \quad for\ \lambda \in (0,1)\}
$$

is bounded, then T has at least one fixed point.

Definition 2.8. A function $\alpha \in X$ is said to be a lower solution of problem (1.1)-(1.2) if $\phi(\alpha^{(n)}) \in AC[0,+\infty)$ such that

$$(\phi(\alpha^{(n)}))'(t) \geq f(t, \overline{\alpha}(t), \alpha'(t), \ldots, \alpha^{(n)}(t)),\ a.e.,\ t \in [0,+\infty),$$

and

$$\begin{cases} \alpha^{(i)}(0) \leq A_i,\ i = 1, \ldots, n-2, \\ \alpha^{(n-1)}(0) + a\alpha^{(n)}(0) \leq B, \\ \alpha^{(n)}(+\infty) < C, \end{cases} \tag{2.14}$$

where $a < 0$, $B, C \in \mathbb{R}$, $A_i \in \mathbb{R}$, $i = 0, \ldots, n-2$ and $\overline{\alpha}(t) := \alpha(t) - \alpha(0) + A_0$.

A function $\beta \in X$ where $\phi(\beta^{(n)}) \in AC[0,+\infty)$ is an upper solution if it satisfies the reversed inequalities with $\overline{\beta}(t) := \beta(t) - \beta(0) + A_0$.

3 Main existence result

Theorem 3.1. *Let $f : [0,+\infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a L^1-Carathéodory function, ϕ an increasing homeomorphism satisfying $\phi(0) = 0$, and α, β lower and upper solutions of* (1.1)-(1.2)*, respectively, such that*

$$\alpha^{(n-1)}(t) \leq \beta^{(n-1)}(t), \quad \forall t \in [0,+\infty). \tag{3.1}$$

If f satisfies the one-sided Nagumo condition (2.8)*, or* (2.9)*, on the set*

$$E_* = \Big\{(t, x_0, x_1, \ldots, x_n) \in [0,+\infty) \times \mathbb{R}^{n+1} : \overline{\alpha}(t) \leq x_0 \leq \overline{\beta}(t),\ \alpha'(t) \leq x_1 \leq \beta'(t), \ldots, \\ \alpha^{(n-1)}(t) \leq x_{n-1} \leq \beta^{(n-1)}(t)\Big\}$$

and

$$\begin{aligned} f(t, \overline{\alpha}(t), \alpha'(t), \ldots, \alpha^{(n-2)}(t), x_{n-1}, x_n) &\geq f(t, x_0, \ldots, x_n) \\ &\geq f(t, \overline{\beta}(t), \beta'(t), \ldots, \beta^{(n-2)}(t), x_{n-1}, x_n), \end{aligned} \tag{3.2}$$

for (t, x_{n-1}, x_n) fixed and $\overline{\alpha}(t) \leq x_0 \leq \overline{\beta}(t)$, $\alpha'(t) \leq x_1 \leq \beta'(t), \ldots, \alpha^{(n-2)}(t) \leq x_{n-2} \leq \beta^{(n-2)}(t)$, then problem (1.1)-(1.2) *has at least a solution $u \in X$ with $\phi(u^{(n)}) \in AC[0,+\infty)$ and there exists $R > 0$ such that*

$$\overline{\alpha}(t) \leq u(t) \leq \overline{\beta}(t), \alpha'(t) \leq u'(t) \leq \beta'(t), \ldots, \alpha^{(n-1)}(t) \leq u^{(n-1)}(t) \leq \beta^{(n-1)}(t), \\ -R < u^{(n)}(t) < R, \quad \forall t \in [0,+\infty).$$

Remark 3.2. *α and β are almost-ordered. In fact, α and β can be chosen such that $\alpha \not\leq \beta$ but we have necessary that $\overline{\alpha} \leq \overline{\beta}$.*

Indeed, from condition (3.1), for all $t \in [0,+\infty)$, we have $\alpha^{(n-1)}(t) \leq \beta^{(n-1)}(t)$. As $\alpha^{(n-2)}(0) \leq A_{n-2} \leq \beta^{(n-2)}(0)$, integrating on $[0,+\infty)$

$$\alpha^{(n-2)}(t) - \alpha^{(n-2)}(0) = \int_0^t \alpha^{(n-1)}(s)\,ds \leq \int_0^t \beta^{(n-1)}(s)\,ds = \beta^{(n-2)}(t) - \beta^{(n-2)}(0).$$

As

$$\alpha^{(n-2)}(t) - \alpha^{(n-2)}(0) + A_{n-2} \leq \beta^{(n-2)}(t) - \beta^{(n-2)}(0) + A_{n-2},$$

then

$$\alpha^{(n-2)}(t) \leq \beta^{(n-2)}(t).$$

By the same technique, one shows that $\alpha^{(i)} \leq \beta^{(i)}$, for $i = 1,2,\ldots,n-3$, then

$$\alpha(t) - \alpha(0) = \int_0^t \alpha'(s)\,ds \leq \int_0^t \beta'(s)\,ds = \beta(t) - \beta(0),$$

So,

$$\overline{\alpha}(t) \leq \overline{\beta}(t), \quad \forall t \in [0,+\infty).$$

Proof. Consider the j-modified equation for $j = 1,2$

$$\begin{aligned}(\phi(u^{(n)}))'(t) &= f(t, \delta_0(t,u(t)),\ldots,\delta_{n-1}(t,u^{(n-1)}(t)),\delta_{nj}(t,u^{(n)}(t))) \\ &\quad + \frac{1}{1+t^2}\frac{u^{(n-1)}(t) - \delta_{n-1}(t,u^{(n-1)}(t))}{1+|u^{(n-1)}(t) - \delta_{n-1}(t,u^{(n-1)}(t))|}, \quad a.e.,\ t \in [0,+\infty),\end{aligned} \tag{3.3}$$

where the functions $\delta_i, \delta_{nj} : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$, $i = 0,1,2,3,\ldots,n-1$ and $j = 1,2$ are given by

$$\delta_0(t,x) = \begin{cases} \overline{\beta}(t), & x > \overline{\beta}(t), \\ x, & \overline{\alpha}(t) \leq x \leq \overline{\beta}(t), \\ \overline{\alpha}(t), & x < \overline{\alpha}(t), \end{cases}$$

$$\delta_i(t,y_i) = \begin{cases} \beta^{(i)}(t), & y_i > \beta^{(i)}(t), \\ y_i, & \alpha^{(i)}(t) \leq y_i \leq \beta^{(i)}(t), \\ \alpha^{(i)}(t), & y_i < \alpha^{(i)}(t), \end{cases} \qquad i = 1,2,\ldots,n-1,$$

$$\delta_{n1}(t,w) = \begin{cases} N, & w > N, \\ w, & -N \leq w \leq N, \\ -N, & w < -N, \end{cases}$$

where $N > \max\left\{\sup_{0\leq t<+\infty}|\alpha^{(n)}(t)|, \sup_{0\leq t<+\infty}|\beta^{(n)}(t)|\right\}$, and

$$\delta_{n2}(t,w) = w.$$

For convenience, the proof is divided into three principal steps.

Step 1: Every solution of (3.3)-(1.2), satisfies $\alpha^{(n-1)}(t) \leq u_j^{(n-1)}(t) \leq \beta^{(n-1)}(t)$ for all $t \in [0,+\infty)$, $j = 1,2$. Let u_j be a solution of the j-modified problem (3.3)-(1.2), $j = 1,2$ and suppose, by contradiction, that there exists $t \in (0,+\infty)$ such that $\alpha^{(n-1)}(t) > u_j^{(n-1)}(t)$, $j = 1,2$. Therefore

$$\inf_{0\leq t<+\infty}\left(u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)\right) < 0,\ j = 1,2.$$

By (2.14) this infimum cannot be attained at $+\infty$. In fact,

$$\inf_{0\leq t<+\infty}\left(u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)\right) := u_j^{(n-1)}(+\infty) - \alpha^{(n-1)}(+\infty) < 0$$

and

$$u_j^{(n)}(+\infty) - \alpha^{(n)}(+\infty) \leq 0.$$

We reach the following contradiction

$$0 \geq u_j^{(n)}(+\infty) - \alpha^{(n)}(+\infty) > C - C = 0.$$

If

$$\inf_{0\leq t<+\infty}\left(u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)\right) := u_j^{(n-1)}(0^+) - \alpha^{(n-1)}(0^+) < 0, \quad j = 1,2.$$

Then we have the following contradiction for $j = 1,2$

$$0 \leq u_j^{(n)}(0^+) - \alpha^{(n)}(0^+) \leq \frac{B - u_j^{(n-1)}(0)}{a} + \frac{\alpha^{(n-1)}(0) - B}{a} = -\frac{1}{a}(u_j^{(n-1)}(0) - \alpha^{(n-1)}(0)) < 0.$$

If there is $t_* \in (0,+\infty)$, we can define for $j = 1,2$

$$\min_{0\leq t<+\infty}\left(u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)\right) := u_j^{(n-1)}(t_*) - \alpha^{(n-1)}(t_*) < 0,$$

with $u_j^{(n)}(t_*) = \alpha^{(n)}(t_*)$. Then there exists $\bar{t} > t_*$, such that

$$u_j^{(n-1)}(t) - \alpha^{(n-1)}(t) < 0, \quad u_j^{(n)}(t) - \alpha^{(n)}(t) \geq 0, \quad \text{for all} \quad t \in (t_*,\bar{t}).$$

Therefore by (3.2) and Definition 2.8, we get a contradiction for $j=1,2$

$$
\begin{aligned}
(\phi(u_j^{(n)}))'(t)-(\phi(\alpha^{(n)}))'(t) &= f(t,\delta_0(t,u_j(t)),\dots,\delta_{n-1}(t,u_j^{(n-1)}(t)),\delta_{nj}(t,u_j^{(n)}(t)))\\
&\quad+\frac{1}{1+t^2}\frac{u_j^{(n-1)}(t)-\delta_{n-1}(t,u_j^{(n-1)}(t))}{1+|u_j^{(n-1)}(t)-\delta_{n-1}(t,u_j^{(n-1)}(t))|}-(\phi(\alpha^{(n)}))'(t)\\
&= f(t,\delta_0(t,u_j(t)),\dots,\delta_{n-2}(t,u_j^{(n-2)}(t)),\alpha^{(n-1)}(t),\alpha^{(n)}(t))\\
&\quad+\frac{1}{1+t^2}\frac{u_j^{(n-1)}(t)-\alpha^{(n-1)}(t)}{1+|u_j^{(n-1)}(t)-\alpha^{(n-1)}(t)|}-(\phi(\alpha^{(n)}))'(t)\\
&\le \frac{1}{1+t^2}\frac{u_j^{(n-1)}(t)-\alpha^{(n-1)}(t)}{1+|u_j^{(n-1)}(t)-\alpha^{(n-1)}(t)|}<0, \quad a.e. \quad t\in(t_*,\bar{t}).
\end{aligned}
$$

So, the function $\phi(u_j^{(n)}(t))-\phi(\alpha^{(n)}(t))$ is decreasing for all $t\in(t_*,\bar{t})$. If $t\in(t_*,\ \bar{t})$,

$$0=\phi(u_j^{(n)}(t_*))-\phi(\alpha^{(n)}(t_*))>\phi(u_j^{(n)}(t))-\phi(\alpha^{(n)}(t))$$

and $u_j^{(n)}(t)-\alpha^{(n)}(t)<0$. Therefore $u_j^{(n-1)}(t)-\alpha^{(n-1)}(t)$ is decreasing in $(t_*,\bar{t})$, which is a contradiction. So $u_j^{(n-1)}(t)\ge\alpha^{(n-1)}(t)$, $\forall t\in[0,+\infty)$, $j=1,2$. In the same way, we show that $u_j^{(n-1)}(t)\le\beta^{(n-1)}(t)$, $\forall t\in[0,+\infty)$, $j=1,2$.

As $\alpha^{(n-2)}(0)\le A_{n-2}\le\beta^{(n-2)}(0)$ and $u_j^{(n-2)}(0)=A_{n-2}$, integrating on $[0,+\infty)$ for $j=1,2$,

$$
\begin{aligned}
\alpha^{(n-2)}(t)-\alpha^{(n-2)}(0) &= \int_0^t \alpha^{(n-1)}(s)\,ds \le \int_0^t u_j^{(n-1)}(s)\,ds = u_j^{(n-2)}(t)-A_{n-2}\\
&\le \int_0^t \beta^{(n-1)}(s)\,ds = \beta^{(n-2)}(t)-\beta^{(n-2)}(0).
\end{aligned}
$$

As

$$\alpha^{(n-2)}(t)-\alpha^{(n-2)}(0)+A_{n-2}\le u_j^{(n-2)}(t)\le\beta^{(n-2)}(t)-\beta^{(n-2)}(0)+A_{n-2},$$

then

$$\alpha^{(n-2)}(t)\le u_j^{(n-2)}(t)\le\beta^{(n-2)}(t).$$

By the same technique, one shows that $\alpha^{(i)}\le u_j^{(i)}\le\beta^{(i)}$, for $i=1,2,\dots,n-3$, $j=1,2$, then

$$\alpha(t)-\alpha(0)=\int_0^t\alpha'(s)\,ds\le\int_0^t u_j'(s)\,ds=u_j(t)-A_0\le\int_0^t\beta'(s)\,ds=\beta(t)-\beta(0),$$

$$\overline{\alpha}(t)\le u_j(t)\le\overline{\beta}(t),\quad j=1,2.$$

Step 2: By Lemma 2.5, if u is a solution of the 2-modified problem (3.3)-(1.2), then there exists $R_1 > 0$, not depending on u, such that

$$\|u^{(n)}\|_n < R_1.$$

Now, we need to consider $N = N_1$, where

$$N_1 > \max\left\{2R_1, \sup_{0\le t<+\infty} |\alpha^{(n)}(t)|, \sup_{0\le t<+\infty} |\beta^{(n)}(t)|\right\}.$$

If the 1-modified problem (3.3)-(1.2) has a solution u, then u is a solution of problem (1.1)-(1.2), where

$$\|u^{(n)}\|_n < R_1 < \frac{N_1}{2} < N_1.$$

Step 3: Problem (3.3)-(1.2) for $j = 1$ has at least one solution. Let us define the operator $T : X \to X$ by

$$\begin{aligned} Tu(t) =& A_0 + A_1 t + \cdots + \frac{A_{n-2}}{(n-2)!}t^{n-2} + \frac{B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} F(u(s))\,ds\right)}{(n-1)!}t^{n-1} \\ &+ \int_0^t \left(\frac{(t-s)^{n-1}}{(n-1)!}\right)\phi^{-1}\left(\phi(C) + \int_s^{+\infty} F(u(\tau))\,d\tau\right) ds. \end{aligned}$$

with

$$\begin{aligned} F(u(s)) :=& -f(s, \delta_0(s, u(s)), \ldots, \delta_{n-1}(s, u^{(n-1)}(s)), \delta_{n1}(s, u^{(n)}(s))) \\ &- \frac{1}{1+s^2}\frac{u^{(n-1)}(s) - \delta_{n-1}(s, u^{(n-1)}(s))}{1 + |u^{(n-1)}(s) - \delta_{n-1}(s, u^{(n-1)}(s))|}. \end{aligned}$$

From Lemma 2.3, one can see that the fixed points of T are solutions of the 1-modified (3.3)-(1.2) problem. So it is sufficient to prove that T has a fixed point in X. For this aim, it is enough to prove that the operator T satisfies the condition of the Schaefer fixed point theorem 2.7. The proof is split into three steps.

(1) $T : X \to X$ is well defined. Let $u \in X$. As f is a L^1-Carathéodory function, so, for

$$\rho > \max\{N_1, \|\overline{\alpha}\|_0, \|\overline{\beta}\|_0\} \cup \{\|\alpha^{(i)}\|_i, \|\beta^{(i)}\|_i,\ i = 1, 2, \ldots, n-1\},$$

we obtain

$$\begin{aligned} \int_0^{+\infty} |F(u(s))|\,ds &\le \int_0^{+\infty} \varphi_\rho(s) + \frac{1}{1+s^2}\frac{|u^{(n-1)}(s) - \delta_{n-1}(s, u^{(n-1)}(s))|}{1 + |u^{(n-1)}(s) - \delta_{n-1}(s, u^{(n-1)}(s))|}\,ds \\ &\le \int_0^{+\infty}\left(\varphi_\rho(s) + \frac{1}{1+s^2}\right) ds = M_\rho < +\infty, \end{aligned} \tag{3.4}$$

this means that F is also a L^1-Carathéodory function. Then,

$$\lim_{t\to+\infty}(Tu)^{(n)}(t) = \lim_{t\to+\infty}\phi^{-1}\left(\phi(C)+\int_t^{+\infty}F(u(\tau))\,d\tau\right) = C = \lim_{t\to+\infty}\frac{(Tu)^{(n-1)}(t)}{1+t}$$
$$= 2!\lim_{t\to+\infty}\frac{(Tu)^{(n-2)}(t)}{1+t^2} = 3!\lim_{t\to+\infty}\frac{(Tu)^{(n-3)}(t)}{1+t^3} = \cdots = n!\frac{(Tu)(t)}{1+t^n}\cdot$$

Therefore, $Tu \in X$.

(2) T is continuous. Let $(u_m)\subset X$, such that $u_m \to u$ in X. There exists $r>0$ such that $\|u_m\|_X < r$, $\forall m\in\mathbb{N}$. We have to prove that $\|Tu_m - Tu\|_X \overset{m\to+\infty}{\longrightarrow} 0$. To this end, we can see that

$$\|Tu_m - Tu\|_0 \overset{m\to+\infty}{\longrightarrow} 0, \|(Tu_m)'-(Tu)'\|_1 \overset{m\to+\infty}{\longrightarrow} 0, \|(Tu_m)''-(Tu)''\|_2 \overset{m\to+\infty}{\longrightarrow} 0,\ldots,$$
$$\|(Tu_m)^{(n)}-(Tu)^{(n)}\|_n \overset{m\to+\infty}{\longrightarrow} 0.$$

We have,

$$\sup_{0\le t<+\infty}|\phi((Tu_m)^{(n)})(t)-\phi((Tu)^{(n)})(t)| = \sup_{0\le t<+\infty}\left|\int_t^{+\infty}F(u_m(\mu))\,d\mu - \int_t^{+\infty}F(u(\mu))\,d\mu\right|$$
$$\le \int_0^{+\infty}\Big|F(u_m(\mu))-F(u(\mu))\Big|\,d\mu \le 2M_\rho < +\infty.$$

From Lebesgue Dominated Convergence Theorem, $F(u_m(t))$ converges to $F(u(t))$ a.e., $t\in[0,+\infty)$, as $m\to+\infty$, because F is L^1-Carathéodory function, so

$$\int_0^{+\infty}\Big|F(u_m(\mu))-F(u(\mu))\Big|d\mu \to 0,$$

as $m\to+\infty$, then,

$$\|(Tu_m)^{(n)}-(Tu)^{(n)}\|_n \to 0,$$

as $m\to+\infty$. Moreover, we have that for $i=0,1,2,\ldots,n-1$,

$$\sup_{0\le t<+\infty}\left|\frac{(Tu_m)^{(i)}(t)}{1+t^{n-i}}-\frac{(Tu)^{(i)}(t)}{1+t^{n-i}}\right| = \sup_{0\le t<+\infty}\left|-a\frac{\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u_m(s))ds\right)}{1+t^{n-i}}\frac{t^{n-i-1}}{(n-i-1)!}\right.$$
$$+\frac{\int_0^t(t-s)^{n-i-1}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u_m(\tau))d\tau\right)ds}{(1+t^{n-i})(n-i-1)!}$$
$$+a\frac{\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)}{1+t^{n-i}}\frac{t^{n-i-1}}{(n-i-1)!}$$
$$\left.-\frac{\int_0^t(t-s)^{n-i-1}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)ds}{(1+t^{n-i})(n-i-1)!}\right|$$

$$\begin{aligned}
&\leq \sup_{0\leq t<+\infty} \frac{t^{n-i-1}}{(1+t^{n-i})(n-i-1)!}\Big|a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right) \\
&\quad - a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u_m(s))ds\right)\Big| \\
&+\sup_{0\leq t<+\infty} \frac{1}{(1+t^{n-i})(n-i-1)!}\int_0^t (t-s)^{n-i-1} \\
&\quad \Big|\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u_m(\tau))d\tau\right) \\
&\quad - \phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)\Big| ds \\
&\leq \sup_{0\leq t<+\infty} \frac{2|a|t^{n-i-1}}{(1+t^{n-i})(n-i-1)!}\|(Tu_m)^{(n)}-(Tu)^{(n)}\|_n \\
&+\sup_{0\leq t<+\infty}\int_0^t \frac{2(t-s)^{n-i-1}}{(1+t^{n-i})(n-i-1)!}\|(Tu_m)^{(n)}-(Tu)^{(n)}\|_n ds \\
&\leq 2|a|\|(Tu_m)^{(n)}-(Tu)^{(n)}\|_n + 2\|(Tu_m)^{(n)}-(Tu)^{(n)}\|_n \to 0,
\end{aligned}$$

as $m\to+\infty$.

(3) T is compact. Let

$$L_\rho = \max\left\{\phi^{-1}(|\phi(C)|+M_\rho), \left|\phi^{-1}(-|\phi(C)|-M_\rho)\right|\right\}.$$

Let $U\subset X$ be any bounded subset, *i.e.*, there is $r>0$ such that $\|u\|_X<r$ for all $u\in U$. For each $u\in U$, one has for $i=0,1,\dots,n-1$.

$$\begin{aligned}
\|(Tu)^{(i)}\|_i &= \sup_{0\leq t<+\infty}\Bigg|\frac{\sum_{k=i}^{k=n-2}\frac{A_k}{(k-i)!}t^{k-i}+B\frac{t^{n-1-i}}{(n-1-i)!}-a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)\frac{t^{n-1-i}}{(n-1-i)!}}{1+t^{n-i}} \\
&\quad + \int_0^t \frac{(t-s)^{n-1-i}}{(n-1-i)!(1+t^{n-i})}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)ds\Bigg| \\
&\leq \sum_{k=i}^{k=n-2}\frac{|A_k|}{(k-i)!}+\frac{|B|+|a|L_\rho}{(n-1-i)!}+\frac{L_\rho}{(n-i)!}<+\infty,
\end{aligned}$$

and

$$\|(Tu)^{(n)}\|_n = \frac{1}{2}\sup_{0\leq t<+\infty}\left|\phi^{-1}\left(\phi(C)+\int_t^{+\infty}F(u(\mu))d\mu\right)\right|\leq L_\rho<+\infty.$$

So,

$$\|Tu\|_X \leq |A_0|+|A_1|+|A_2|+\cdots+|A_{n-2}|+|B|+(|a|+1)L_\rho<+\infty.$$

That is, TU is uniformly bounded.

In order to prove that TU is equicontinuous, let $L>0$ and $t_1,t_2\in[0,L]$ with $t_1<t_2$.

We have

$$\begin{aligned}\left|\phi((Tu)^{(n)})(t_2)-\phi((Tu)^{(n)})(t_1)\right| &= \left|\phi(C)+\int_{t_2}^{+\infty}F(u(\tau))d\tau-\phi(C)-\int_{t_1}^{+\infty}F(u(\tau))d\tau\right|\\ &= \Big|\int_{t_2}^{+\infty}F(u(\tau))d\tau-\int_{t_1}^{+\infty}F(u(\tau))d\tau\Big|\\ &= \Big|\int_{t_1}^{t_2}F(u(\tau))d\tau\Big|\to 0,\end{aligned}$$

as $t_1\to t_2$. Also,

$$\begin{aligned}\left|\frac{(Tu)^{(n-1)}(t_2)}{1+t_2}-\frac{(Tu)^{(n-1)}(t_1)}{1+t_1}\right| &= \left|\frac{B-a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)}{(1+t_2)}\right.\\ &\quad+\int_0^{t_2}\frac{1}{1+t_2}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)ds\\ &\quad-\frac{B-a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)}{(1+t_1)}\\ &\quad\left.-\int_0^{t_1}\frac{1}{1+t_1}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)ds\right|\\ &\le\left|\frac{B-a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)}{(1+t_2)}\right.\\ &\quad\left.-\frac{B-a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)}{(1+t_1)}\right|\\ &\quad+L_\rho\int_0^{t_1}\left|\frac{1}{1+t_2}-\frac{1}{1+t_1}\right|ds\\ &\quad+L_\rho\int_{t_1}^{t_2}\left|\frac{1}{1+t_2}\right|ds\to 0,\end{aligned}$$

as $t_1\to t_2$. Moreover, we have that for $i=0,1,\dots,n-2$

$$\begin{aligned}\left|\frac{(Tu)^{(i)}(t_2)}{1+t_2^{n-i}}-\frac{(Tu)^{(i)}(t_1)}{1+t_1^{n-i}}\right| &= \left|\frac{\sum\limits_{k=i}^{k=n-2}\frac{A_k}{(k-i)!}t_2^{k-i}-\left(a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)-B\right)\frac{t_2^{n-1-i}}{(n-1-i)!}}{1+t_2^{n-i}}\right.\\ &\quad-\frac{\sum\limits_{k=i}^{k=n-2}\frac{A_k}{(k-i)!}t_1^{k-i}-\left(a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)-B\right)\frac{t_1^{n-1-i}}{(n-1-i)!}}{1+t_1^{n-i}}\\ &\quad+\int_0^{t_2}\frac{(t_2-s)^{n-1-i}}{(n-1-i)!(1+t_2^{n-i})}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)ds\\ &\quad\left.-\int_0^{t_1}\frac{(t_1-s)^{n-1-i}}{(n-1-i)!(1+t_1^{n-i})}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)ds\right|\end{aligned}$$

$$\leq \left| \frac{\sum_{k=i}^{k=n-2} \frac{A_k}{(k-i)!} t_2^{k-i} - \left(a\phi^{-1}\left(\phi(C)+\int_0^{+\infty} F(u(s))ds\right)-B\right)\frac{t_2^{n-1-i}}{(n-1-i)!}}{1+t_2^{n-i}} - \frac{\sum_{k=i}^{k=n-2} \frac{A_k}{(k-i)!} t_1^{k-i} - \left(a\phi^{-1}\left(\phi(C)+\int_0^{+\infty} F(u(s))ds\right)-B\right)\frac{t_1^{n-1-i}}{(n-1-i)!}}{1+t_1^{n-i}} \right|$$
$$+ L_\rho \int_0^{t_1} \left| \frac{(t_2-s)^{n-1-i}}{(n-1-i)!(1+t_2^{n-i})} - \frac{(t_1-s)^{n-1-i}}{(n-1-i)!(1+t_1^{n-i})} \right| ds$$
$$+ L_\rho \int_{t_1}^{t_2} \left| \frac{(t_2-s)^{n-1-i}}{(n-1-i)!(1+t_2^{n-i})} \right| ds \to 0,$$

as $t_1 \to t_2$.

Furthermore, $TU \subset X$ is equiconvergent at infinity. We use that F is L^1-Carathéodory function and the continuity of ϕ^{-1}. From Lemma 2.1, we have that for all $u \in U$, $\lim_{t\to+\infty} (Tu)^{(n)}(t) = C$, then, $\lim_{t\to+\infty} \frac{(Tu)^{(n-i)}(t)}{1+t^i} = \frac{C}{i!}$ for $i \in \{1,\dots,n\}$. So,

$$\left|(Tu)^{(n)}(t) - C\right| = \left|\phi^{-1}\left(\phi(C) + \int_t^{+\infty} F(u(\mu))d\mu\right) - C\right| \to 0,$$

as $t \to +\infty$. Regarding the next derivative, we have

$$\begin{aligned}
\lim_{t\to+\infty}\left|\frac{(Tu)^{(n-1)}(t)}{1+t} - C\right| &= \lim_{t\to+\infty}\left|\frac{1}{1+t}\left(B - a\phi^{-1}\left(\phi(C)+\int_0^{+\infty} F(u(s))\,ds\right)\right.\right. \\
&\quad \left.\left. + \int_0^t \phi^{-1}\left(\phi(C) + \int_s^{+\infty} F(u(\tau))\,d\tau\right) ds\right) - C\right| \\
&\leq \lim_{t\to+\infty}\left|\frac{1}{1+t}\left(B - a\phi^{-1}\left(\phi(C)+\int_0^{+\infty} F(u(s))\,ds\right)\right)\right| \\
&\quad + \lim_{t\to+\infty}\left|\frac{1}{1+t}\int_0^t \phi^{-1}\left(\phi(C)+\int_s^{+\infty} F(u(\tau))\,d\tau\right) ds - C\right| \\
&= \lim_{t\to+\infty}\left|\frac{1}{1+t}\left(B - a\phi^{-1}\left(\phi(C)+\int_0^{+\infty} F(u(s))\,ds\right)\right)\right| \\
&\quad + \lim_{t\to+\infty}\left|\phi^{-1}\left(\phi(C)+\int_t^{+\infty} F(u(\tau))\,d\tau\right) - C\right| = 0.
\end{aligned}$$

By the same technique, one can show that

$$\left|\frac{(Tu)^{(n-i)}(t)}{1+t^i} - \frac{C}{i!}\right| \to 0,$$

as $t \to +\infty$, $i = 2, 3, \dots, n$. So, by Lemma 2.6, the set TU is relatively compact.

Moreover, the set

$$\{u \in X : \ u = \lambda Tu, \ \lambda \in (0,1)\}$$

is bounded, as

$$\|\lambda Tu\|_X \leq |A_0| + |A_1| + |A_2| + \cdots + |A_{n-2}| + |B| + (|a|+1)L_\rho < +\infty, \ \forall \lambda \in (0,1).$$

By Theorem 2.7, T has at least one fixed point $u \in X$ such that

$$\overline{\alpha}(t) \leq u(t) \leq \overline{\beta}(t), \alpha'(t) \leq u'(t) \leq \beta'(t), \ldots, \alpha^{(n-1)}(t) \leq u^{(n-1)}(t) \leq \beta^{(n-1)}(t),$$
$$-2R_1 < u^{(n)}(t) < 2R_1, \quad \forall t \in [0, +\infty).$$

□

Remark 3.3. *If $n = 1$, problem (1.1)-(1.2) is written as*

$$\begin{cases} (\phi(u'))'(t) = f(t, u(t), u'(t)), \ \textit{a.e. } t \in [0, +\infty), \\ u(0) + au'(0) = B, \\ u'(+\infty) = C. \end{cases}$$

In this case, we cannot consider the functions $\overline{\alpha}$ and $\overline{\beta}$. Moreover, in Theorem 3.1, we do not need to suppose the condition (3.2) and the upper and lower solutions are automatically ordered.

4 Example

Consider the $(n+1)$st order differential equation for a fixed $n \in \mathbb{N} \setminus \{0, 1\}$

$$((u^{(n)})^3)'(t) = f(t, u(t), u'(t), \ldots, u^{(n)}(t)), \ \text{a.e.}, \ t \geq 0, \tag{4.1}$$

with the boundary conditions

$$\begin{cases} u(0) = 2, \\ u^{(i)}(0) = 0, \ i = 1, \ldots, n-2, \\ u^{(n-1)}(0) - \dfrac{1}{(n+1)!} u^{(n)}(0) = \dfrac{1}{3}, \\ u^{(n)}(+\infty) = \dfrac{n!}{2}, \end{cases} \tag{4.2}$$

where

$$f(t, x_0, x_1, \ldots, x_n) = \frac{\Big|x_{n-1} - n!t - (n-1)!\Big|(-x_0 + 2) - (x_1 + x_2 + \cdots + x_{n-2})\Big|x_n - n!\Big|}{(1+t^3)(2+t^n)^2}. \tag{4.3}$$

Moreover, the functions $\alpha(t) \equiv 2$ and $\beta(t) = t^n + t^{n-1} + \cdots + t + 1$ are respectively, non-ordered lower and upper solutions for (4.1)-(4.2), with $\overline{\alpha}(t) = 2$ and $\overline{\beta}(t) = t^n + t^{n-1} + \cdots + t + 2$. As,

$\alpha^{(i)} \leq 0 \leq \beta^{(i)}(t)$, $i = 1, \ldots, n-2$, $0 = \alpha^{(n)}(+\infty) < \frac{n!}{2} < \beta^{(n)}(+\infty) = n!$ and

$$\beta^{(n-1)}(0) - \frac{\beta^{(n)}(0)}{(n+1)!} = (n-1)! - \frac{n!}{(n+1)!} \geq \frac{1}{3} \geq 0 = \alpha^{(n-1)}(0) - \frac{\alpha^{(n)}(0)}{(n+1)!}.$$

Also, $\beta^{(n-1)}(t) = n!t + (n-1)!$, $\beta^{(n)}(t) = n!$, $((\alpha^{(n)})^3)'(t) = 0$ and $((\beta^{(n)})^3)'(t) = 0$, then, $f(t, \overline{\alpha}(t), \alpha'(t), \ldots, \alpha^{(n)}(t)) = 0$ and $f(t, \overline{\beta}(t), \beta'(t), \ldots, \beta^{(n)}(t)) = 0$ for all $t \geq 0$. The nonlinearity f satisfies the one-sided Nagumo condition (2.8) with

$$\psi(t) = \frac{1}{1+t^3}, \quad 1 < \nu < 3, \quad h(w) = 2$$

on the set

$$E_0 = \Big\{(t, x_0, x_1, \ldots, x_n) \in [0, +\infty) \times \mathbb{R}^{n+1} : \overline{\alpha}(t) \leq x_0 \leq \overline{\beta}(t), \alpha'(t) \leq x_1 \leq \beta'(t),$$
$$\alpha''(t) \leq x_2 \leq \beta''(t), \ldots, \alpha^{n-1}(t) \leq x_{n-1} \leq \beta^{(n-1)}(t)\Big\},$$

and satisfies the assumptions of Theorem 3.1.

Therefore, there is at least a nontrivial solution u of (4.1)-(4.2), and $R > 0$, such that

$$\overline{\alpha}(t) \leq u(t) \leq \overline{\beta}(t), \alpha'(t) \leq u'(t) \leq \beta'(t), \ldots, \alpha^{(n-1)}(t) \leq u^{(n-1)}(t) \leq \beta^{(n-1)}(t),$$
$$-R < u^{(n)}(t) < R, \quad \forall t \in [0, +\infty).$$

From this, we see that u is a nonnegative function and its derivatives $u^{(i)}$ are nonnegative for $i \in \{1, \ldots, n-1\}$ and nondecreasing for $i \in \{1, \ldots, n-2\}$.

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Several inequalities for an integral transform of positive operators in Hilbert spaces with applications

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ABSTRACT

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a postive operator on a complex Hilbert space H.
We show among others that, if $\beta \ge A \ge \alpha > 0$, $B > 0$ with $M \ge B - A \ge m > 0$ for some constants α, β, m, M, then

$$\begin{aligned} 0 &\le \frac{m^2}{M^2}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right] \\ &\le \frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right](B-A)^{-1} \\ &\le \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\ &\le \frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right](B-A)^{-1} \\ &\le \frac{M^2}{m^2}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right]. \end{aligned}$$

Some examples for operator monotone and operator convex functions as well as for integral transforms $\mathcal{D}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

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RESUMEN

Para una función contínua y positiva $w(\lambda)$, $\lambda > 0$ y μ una medida positiva sobre $(0, \infty)$ consideramos la siguiente *transformada integral*

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

donde se asume que la integral existe para un operador positivo T, sobre el espacio complejo de Hilbert H.
Mostramos, entre otras cosas, que si $\beta \geq A \geq \alpha > 0$, $B > 0$ con $M \geq B - A \geq m > 0$ para algunas constantes α, β, m, M, entonces

$$\begin{aligned} 0 &\leq \frac{m^2}{M^2}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right] \\ &\leq \frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right](B-A)^{-1} \\ &\leq \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\ &\leq \frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right](B-A)^{-1} \\ &\leq \frac{M^2}{m^2}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right]. \end{aligned}$$

También se proporcionan algunos ejemplos para las funciones operador monótono y operador convexo, así como de transformadas integrales $\mathcal{D}(\cdot,\cdot)$ relacionadas con las funciones exponencial y logarítmica.

Keywords and Phrases: Operator monotone functions, Operator convex functions, Operator inequalities, Löwner-Heinz inequality, Logarithmic operator inequalities.

2020 AMS Mathematics Subject Classification: 47A63, 47A60.

1 Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [6], see for instance [1, p. 144–145]:

Theorem 1.1. *A function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} \, d\mu(\lambda), \tag{1.1}$$

where $b \geq 0$ and a positive measure μ on $[0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} \, d\mu(\lambda) < \infty. \tag{1.2}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1 - \lambda) A + \lambda B) \leq (\geq) (1 - \lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex. We have the following representation of operator convex functions [1, p. 147]:

Theorem 1.2. *A function $f : [0, \infty) \to \mathbb{R}$ is operator convex in $[0, \infty)$ with $f'_+(0) \in \mathbb{R}$ if and only if it has the representation*

$$f(t) = f(0) + f'_+(0) t + ct^2 + \int_0^\infty \frac{t^2\lambda}{t + \lambda} \, d\mu(\lambda), \tag{1.3}$$

where $c \geq 0$ and a positive measure μ on $[0, \infty)$ such that (1.2) holds.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} \, d\lambda. \tag{1.4}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right), \quad \text{for all } u > 0$$

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)}, \quad \text{for all } t > 0. \tag{1.5}$$

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda), \quad t > 0, \tag{1.6}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.6) exists for all $t > 0$. For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda, \quad t > 0. \tag{1.7}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \quad t > 0. \tag{1.8}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1)\mathcal{D}(w_{\ln})(t), \quad t > 0. \tag{1.9}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda), \tag{1.10}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \quad \text{for } T > 0. \tag{1.11}$$

From (1.8) we have the representation

$$T^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(T) \tag{1.12}$$

where $T > 0$ and from (1.9)

$$(T-1)^{-1} \ln T = \mathcal{D}(w_{\ln})(T) \tag{1.13}$$

provided $T > 0$ and $T - 1$ is invertible.

In what follows, if A is an operator and a is a real number, then by $A \geq a$ we understand $A \geq aI$, where I is the identity operator.

In this paper we show among others that, if $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right] \\
&\leq \frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right](B-A)^{-1} \\
&\leq \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\
&\leq \frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right].
\end{aligned}$$

Some examples for operator monotone and operator convex functions as well as for integral transforms $\mathcal{D}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2 Main results

In the following, whenever we write $\mathcal{D}(w,\mu)$ we mean that the integral from (1.6) exists and is finite for all $t > 0$.

Theorem 2.1. *For all A, $B > 0$ with $B - A \geq 0$ we have the representation*

$$\begin{aligned}
0 &\leq (B-A)^{1/2}\left[\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B)\right](B-A)^{1/2} \qquad (2.1)\\
&= \int_0^\infty \left(\int_0^1 \left[(B-A)^{1/2}(\lambda + sB + (1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right) \times w(\lambda)\, d\mu(\lambda).
\end{aligned}$$

Proof. Observe that, for all A, $B > 0$

$$\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) = \int_0^\infty w(\lambda)\left[(\lambda+B)^{-1} - (\lambda+A)^{-1}\right] d\mu(\lambda). \qquad (2.2)$$

Let $T, S > 0$. The function $f(t) = -t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t\to 0}\left[\frac{f(T+tS) - f(T)}{t}\right] = T^{-1}ST^{-1}, \quad \text{for } T, S > 0 \qquad (2.3)$$

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D] : \{(1-t)C + tD,\ t \in [0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t) := f((1-t)C + tD),\ t \in [0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t))\, dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C)\, dt. \qquad (2.4)$$

If we write this equality for the function $f(t) = -t^{-1}$ and $C, D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1} (D - C) ((1-t)C + tD)^{-1} dt. \tag{2.5}$$

Now, if we take in (2.5) $C = \lambda + B$, $D = \lambda + A$, then

$$\begin{aligned} &(\lambda + B)^{-1} - (\lambda + A)^{-1} \\ &= \int_0^1 ((1-t)(\lambda + B) + t(\lambda + A))^{-1} (A - B) \times ((1-t)(\lambda + B) + t(\lambda + A))^{-1} dt \\ &= \int_0^1 (\lambda + (1-t)B + tA)^{-1} (A - B) (\lambda + (1-t)B + tA)^{-1} dt \end{aligned} \tag{2.6}$$

and by (2.2) we derive

$$\begin{aligned} &\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\ &= \int_0^\infty w(\lambda) \left(\int_0^1 (\lambda + (1-t)B + tA)^{-1} (B - A) \times (\lambda + (1-t)B + tA)^{-1} dt \right) d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \left(\int_0^1 (\lambda + sB + (1-s)A)^{-1} (B - A) \times (\lambda + sB + (1-s)A)^{-1} ds \right) d\mu(\lambda) \end{aligned} \tag{2.7}$$

for all $A, B > 0$, where for the last equality we used the change of variable $s = 1 - t$, $t \in [0,1]$. Now, since $B - A \geq 0$, hence by multiplying both sides with $(B - A)^{1/2}$ we get

$$\begin{aligned} &(B - A)^{1/2} [\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B)] (B - A)^{1/2} \\ &= \int_0^\infty w(\lambda) \left(\int_0^1 (B - A)^{1/2} (\lambda + sB + (1-s)A)^{-1} (B - A) \right. \\ &\left. \times (\lambda + sB + (1-s)A)^{-1} (B - A)^{1/2} ds \right) d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \left(\int_0^1 (B - A)^{1/2} (\lambda + sB + (1-s)A)^{-1} (B - A)^{1/2} \right. \\ &\left. \times (B - A)^{1/2} (\lambda + sB + (1-s)A)^{-1} (B - A)^{1/2} ds \right) d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \times \left(\int_0^1 \left[(B - A)^{1/2} (\lambda + sB + (1-s)A)^{-1} (B - A)^{1/2} \right]^2 ds \right) d\mu(\lambda), \end{aligned} \tag{2.8}$$

which proves the identity in (2.1). Since

$$\left[(B - A)^{1/2} (\lambda + sB + (1-s)A)^{-1} (B - A)^{1/2} \right]^2 \geq 0$$

then by integrating over s on $[0,1]$, multiplying by $w(\lambda) \geq 0$ and integrating over $d\mu(\lambda)$, we deduce the inequality in (2.1). □

The case of operator monotone functions is as follows:

Corollary 2.2. *Assume that f is operator monotone on $[0,\infty)$, then all A, $B>0$ with $B-A\geq 0$ we have the equality*

$$\begin{aligned} 0 &\leq (B-A)^{1/2}\left[f(A)A^{-1}-f(B)B^{-1}\right](B-A)^{1/2} \\ &- f(0)(B-A)^{1/2}\left(A^{-1}-B^{-1}\right)(B-A)^{1/2} \\ &= \int_0^\infty \left(\int_0^1 \left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)\lambda\, d\mu(\lambda) \end{aligned} \tag{2.9}$$

for some positive measure $\mu(\lambda)$. If $f(0)=0$, then

$$\begin{aligned} 0 &\leq (B-A)^{1/2}\left[f(A)A^{-1}-f(B)B^{-1}\right](B-A)^{1/2} \\ &= \int_0^\infty \left(\int_0^1 \left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)\times\lambda\, d\mu(\lambda). \end{aligned} \tag{2.10}$$

Proof. From (1.1) we have the representation

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t), \tag{2.11}$$

with $\ell(\lambda)=\lambda$, for some positive measure $\mu(\lambda)$ and nonnegative number b. Since

$$\begin{aligned} \mathcal{D}(\ell,\mu)(A)-\mathcal{D}(\ell,\mu)(B) &= [f(A)-f(0)]A^{-1}-[f(B)-f(0)]B^{-1} \\ &= f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right), \end{aligned}$$

hence by (2.1) we get (2.9). □

The case of operator convex functions is as follows:

Corollary 2.3. *Assume that f is operator convex on $[0,\infty)$, then all A, $B>0$ with $B-A\geq 0$ we have that*

$$\begin{aligned} 0 &\leq (B-A)^{1/2}\left[f(A)A^{-2}-f(B)B^{-2}\right](B-A)^{1/2} \\ &- f'_+(0)(B-A)^{1/2}\left(A^{-1}-B^{-1}\right)(B-A)^{1/2} \\ &- f(0)(B-A)^{1/2}\left(A^{-2}-B^{-2}\right)(B-A)^{1/2} \\ &= \int_0^\infty \left(\int_0^1 \left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)\times\lambda\, d\mu(\lambda), \end{aligned} \tag{2.12}$$

for some positive measure $\mu(\lambda)$. If $f(0)=0$, then

$$\begin{aligned} 0 &\leq (B-A)^{1/2}\left[f(A)A^{-2}-f(B)B^{-2}\right](B-A)^{1/2} \\ &- f'_+(0)(B-A)^{1/2}\left(A^{-1}-B^{-1}\right)(B-A)^{1/2} \\ &= \int_0^\infty \left(\int_0^1 \left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)\times\lambda\, d\mu(\lambda). \end{aligned} \tag{2.13}$$

Proof. From (1.3) we have that

$$\frac{f(t) - f(0) - f'_+(0)\,t}{t^2} - c = \mathcal{D}(\ell, \mu)(t),$$

for $t > 0$. Then for A, $B > 0$,

$$\begin{aligned}\mathcal{D}(\ell,\mu)(A) - \mathcal{D}(\ell,\mu)(B) &= f(A)\,A^{-2} - f'_+(0)\,A^{-1} - f(0)\,A^{-2} - f(A)\,B^{-2} + f'_+(0)\,B^{-1} + f(0)\,B^{-2} \\ &= f(A)\,A^{-2} - f(B)\,B^{-2} - f'_+(0)\left(A^{-1} - B^{-1}\right) - f(0)\left(A^{-2} - B^{-2}\right)\end{aligned}$$

and by (2.1) we derive (2.13). □

When more conditions are imposed on the operators A and B we have the following refinements and reverses of the inequality

$$0 \le \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B)$$

that hold for $B - A \ge 0$.

Theorem 2.4. *If $\beta \ge A \ge \alpha > 0$, $B > 0$ with $M \ge B - A \ge m > 0$ for some constants α, β, m, M, then*

$$\begin{aligned} 0 &\le \frac{m^2}{M^2}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right] \\ &\le \frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right](B-A)^{-1} \\ &\le \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\ &\le \frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right](B-A)^{-1} \\ &\le \frac{M^2}{m^2}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right]. \end{aligned} \tag{2.14}$$

Proof. For $s \in [0,1]$ we have

$$\lambda + sB + (1-s)\,A = \lambda + s\,(B-A) + A.$$

We have

$$\lambda + s\,(B-A) + A \ge \lambda + sm + A \ge \lambda + sm + \alpha = \lambda + (1-s)\,\alpha + s\,(m+\alpha),$$

$s \in [0,1]$ and $\lambda \ge 0$, which implies that

$$(\lambda + sB + (1-s)\,A)^{-1} \le [\lambda + (1-s)\,\alpha + s\,(m+\alpha)]^{-1}$$

and, by multiplying both sides by $(B-A)^{1/2} \ge 0$,

$$\begin{aligned}(B-A)^{1/2}(\lambda + sB + (1-s)\,A)^{-1}(B-A)^{1/2} &\le [\lambda + (1-s)\,\alpha + s\,(m+\alpha)]^{-1}(B-A) \\ &\le M\,[\lambda + (1-s)\,\alpha + s\,(m+\alpha)]^{-1}.\end{aligned}$$

Furthermore,

$$\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 \leq M^2\left[\lambda+(1-s)\alpha+s(m+\alpha)\right]^{-2},$$

for $s\in[0,1]$ and $\lambda\geq 0$, which implies by integration that

$$\begin{aligned}
&\int_0^\infty w(\lambda)\left(\int_0^1 \left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)d\mu(\lambda)\\
&\leq M^2\int_0^\infty w(\lambda)\left(\int_0^1 \left[\lambda+(1-s)\alpha+s(m+\alpha)\right]^{-2}ds\right)d\mu(\lambda)\\
&=\frac{M^2}{m}\int_0^\infty w(\lambda)\left(\int_0^1 \left[\lambda+(1-s)\alpha+s(m+\alpha)\right]^{-1}(m+\alpha-\alpha)\right.\\
&\left.\times\left[\lambda+(1-s)\alpha+s(m+\alpha)\right]^{-1}ds\right)d\mu(\lambda) \quad \text{(and by (2.7))}\\
&=\frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}(w,\mu)(m+\alpha)\right].
\end{aligned}$$

Using (2.8) we get

$$(B-A)^{1/2}\left[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\right](B-A)^{1/2} \leq \frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}(w,\mu)(m+\alpha)\right].$$

Multiplying both sides with $(B-A)^{-1/2}$ we deduce the fourth inequality in (2.14). We also have

$$\lambda+s(B-A)+A\leq\lambda+sM+A\leq\lambda+sM+\beta=\lambda+(1-s)\beta+s(M+\beta),$$

which implies that

$$(\lambda+sB+(1-s)A)^{-1}\geq\left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-1}$$

and, by multiplying both sides by $(B-A)^{1/2}\geq 0$,

$$\begin{aligned}
(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2} &\geq\left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-1}(B-A)\\
&\geq m\left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-1},
\end{aligned}$$

for $s\in[0,1]$ and $\lambda\geq 0$. By taking the square, we get

$$\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 \geq m^2\left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-2},$$

for $s\in[0,1]$ and $\lambda\geq 0$. By taking the integrals in this inequality we obtain

$$\begin{aligned}
&\int_0^\infty w(\lambda)\left(\int_0^1 \left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)d\mu(\lambda)\\
&\geq m^2\int_0^\infty w(\lambda)\left(\int_0^1 \left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-2}ds\right)d\mu(\lambda)\\
&=\frac{m^2}{M}\int_0^\infty w(\lambda)\left(\int_0^1 \left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-1}(M+\beta-\beta)\right.\\
&\left.\times\left[\lambda+(1-s)\beta+s(M+\beta)\right]^{-1}ds\right)d\mu(\lambda) \quad \text{(and by (2.7))}\\
&=\frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta)-\mathcal{D}(w,\mu)(M+\beta)\right].
\end{aligned}$$

Using (2.8) we get

$$(B-A)^{1/2}\left[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\right](B-A)^{1/2}\geq\frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta)-\mathcal{D}(w,\mu)(M+\beta)\right].$$

Multiplying both sides with $(B-A)^{-1/2}$ we deduce the second inequality in (2.14). The rest of the inequalities are obvious. □

It is well known that, if $P\geq 0$, then

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle,$$

for all x, $y\in H$. Therefore, if $T>0$, then

$$0\leq\langle x,x\rangle^2=\left\langle T^{-1}Tx,x\right\rangle^2=\left\langle Tx,T^{-1}x\right\rangle^2\leq\langle Tx,x\rangle\left\langle TT^{-1}x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle,$$

for all $x\in H$. If $x\in H$, $\|x\|=1$, then

$$1\leq\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle\leq\langle Tx,x\rangle\sup_{\|x\|=1}\left\langle x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1}\leq T. \tag{2.15}$$

Remark 2.5. *If $A>0$ and $B-A>0$, then obviously $\|A\|\geq A\geq\left\|A^{-1}\right\|^{-1}$ and $\|B-A\|\geq B-A\geq\left\|(B-A)^{-1}\right\|^{-1}$. So, if we take $\beta=\|A\|$, $\alpha=\left\|A^{-1}\right\|^{-1}$, $M=\|B-A\|$ and $m=\left\|(B-A)^{-1}\right\|^{-1}$ in (2.14), then we get*

$$\begin{aligned}
0&\leq\frac{\mathcal{D}(w,\mu)(\|A\|)-\mathcal{D}(w,\mu)(\|B-A\|+\|A\|)}{\|B-A\|^2\left\|(B-A)^{-1}\right\|^2}\\
&\leq\frac{\mathcal{D}(w,\mu)(\|A\|)-\mathcal{D}(w,\mu)(\|B-A\|+\|A\|)}{\|B-A\|\left\|(B-A)^{-1}\right\|^2}(B-A)^{-1}\\
&\leq\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\\
&\leq\|B-A\|^2\left\|(B-A)^{-1}\right\|\\
&\times\left[\mathcal{D}(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right)-\mathcal{D}(w,\mu)\left(\left\|(B-A)^{-1}\right\|^{-1}+\left\|A^{-1}\right\|^{-1}\right)\right]\times(B-A)^{-1}\\
&\leq\|B-A\|^2\left\|(B-A)^{-1}\right\|^2\\
&\times\left[\mathcal{D}(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right)-\mathcal{D}(w,\mu)\left(\left\|(B-A)^{-1}\right\|^{-1}+\left\|A^{-1}\right\|^{-1}\right)\right].
\end{aligned}\tag{2.16}$$

Corollary 2.6. *Assume that f is operator monotone on $[0,\infty)$. If $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then*

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\frac{f(\beta)}{\beta} - \frac{f(M+\beta)}{M+\beta} - \frac{M}{\beta(M+\beta)}f(0)\right] \\
&\leq \frac{m^2}{M}\left[\frac{f(\beta)}{\beta} - \frac{f(M+\beta)}{M+\beta} - \frac{M}{\beta(M+\beta)}f(0)\right](B-A)^{-1} \\
&\leq f(A)A^{-1} - f(B)B^{-1} - f(0)\left(A^{-1} - B^{-1}\right) \\
&\leq \frac{M^2}{m}\left[\frac{f(\alpha)}{\alpha} - \frac{f(m+\alpha)}{m+\alpha} - \frac{m}{\alpha(m+\alpha)}f(0)\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\frac{f(\alpha)}{\alpha} - \frac{f(m+\alpha)}{m+\alpha} - \frac{m}{\alpha(m+\alpha)}f(0)\right].
\end{aligned} \tag{2.17}$$

If $f(0) = 0$, then

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\frac{f(\beta)}{\beta} - \frac{f(M+\beta)}{M+\beta}\right] \leq \frac{m^2}{M}\left[\frac{f(\beta)}{\beta} - \frac{f(M+\beta)}{M+\beta}\right](B-A)^{-1} \\
&\leq f(A)A^{-1} - f(B)B^{-1} \leq \frac{M^2}{m}\left[\frac{f(\alpha)}{\alpha} - \frac{f(m+\alpha)}{m+\alpha}\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\frac{f(\alpha)}{\alpha} - \frac{f(m+\alpha)}{m+\alpha}\right].
\end{aligned} \tag{2.18}$$

The proof follows by (2.14) and the representation (2.11).

Remark 2.7. *If $A > 0$ and $B - A > 0$, then for f an operator monotone function on $[0,\infty)$ with $f(0) = 0$, we obtain from (2.18) some similar inequalities to the ones in Remark 2.5. We omit the details.*

The case of operator convex functions is as follows:

Corollary 2.8. *Assume that f is operator convex on $[0,\infty)$. If $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then*

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\frac{f(\beta)}{\beta^2} - \frac{f(M+\beta)}{(M+\beta)^2} - f'_+(0)\frac{M}{\beta(M+\beta)} - f(0)\frac{M(M+2\beta)}{\beta^2(M+\beta)^2}\right] \\
&\leq \frac{m^2}{M}\left[\frac{f(\beta)}{\beta^2} - \frac{f(M+\beta)}{(M+\beta)^2} - f'_+(0)\frac{M}{\beta(M+\beta)} - f(0)\frac{M(M+2\beta)}{\beta^2(M+\beta)^2}\right] \times (B-A)^{-1} \\
&\leq f(A)A^{-2} - f(B)B^{-2} - f'_+(0)\left(A^{-1} - B^{-1}\right) - f(0)\left(A^{-2} - B^{-2}\right) \\
&\leq \frac{M^2}{m}\left[\frac{f(\alpha)}{\alpha^2} - \frac{f(m+\alpha)}{(m+\alpha)^2} - f'_+(0)\frac{m}{\alpha(m+\alpha)} - f(0)\frac{m(m+2\alpha)}{\alpha^2(m+\alpha)^2}\right] \times (B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\frac{f(\alpha)}{\alpha^2} - \frac{f(m+\alpha)}{(m+\alpha)^2} - f'_+(0)\frac{m}{\alpha(m+\alpha)} - f(0)\frac{m(m+2\alpha)}{\alpha^2(m+\alpha)^2}\right].
\end{aligned} \tag{2.19}$$

If $f(0)=0$, *then*

$$\begin{aligned}0 &\le \frac{m^2}{M^2}\left[\frac{f(\beta)}{\beta^2}-\frac{f(M+\beta)}{(M+\beta)^2}-f'_+(0)\frac{M}{\beta(M+\beta)}\right]\\ &\le \frac{m^2}{M}\left[\frac{f(\beta)}{\beta^2}-\frac{f(M+\beta)}{(M+\beta)^2}-f'_+(0)\frac{M}{\beta(M+\beta)}\right](B-A)^{-1}\\ &\le f(A)A^{-2}-f(B)B^{-2}-f'_+(0)\left(A^{-1}-B^{-1}\right) \qquad (2.20)\\ &\le \frac{M^2}{m}\left[\frac{f(\alpha)}{\alpha^2}-\frac{f(m+\alpha)}{(m+\alpha)^2}-f'_+(0)\frac{m}{\alpha(m+\alpha)}\right](B-A)^{-1}\\ &\le \frac{M^2}{m^2}\left[\frac{f(\alpha)}{\alpha^2}-\frac{f(m+\alpha)}{(m+\alpha)^2}-f'_+(0)\frac{m}{\alpha(m+\alpha)}\right].\end{aligned}$$

Remark 2.9. *If* $A>0$ *and* $B-A>0$, *then for* f *an operator convex function on* $[0,\infty)$ *with* $f(0)=0$, *we obtain from (2.20) some similar inequalities to the ones in Remark 2.5. We omit the details.*

3 Some examples

The function $f(t)=t^r$, $r\in(0,1]$ is operator monotone on $[0,\infty)$ and by (2.18) we obtain the power inequalities

$$\begin{aligned}0 &\le \frac{m^2}{M^2}\left[\beta^{r-1}-(M+\beta)^{r-1}\right] \le \frac{m^2}{M}\left[\beta^{r-1}-(M+\beta)^{r-1}\right](B-A)^{-1}\\ &\le A^{r-1}-B^{r-1} \le \frac{M^2}{m}\left[\alpha^{r-1}-(m+\alpha)^{r-1}\right](B-A)^{-1} \qquad (3.1)\\ &\le \frac{M^2}{m^2}\left[\alpha^{r-1}-(m+\alpha)^{r-1}\right],\end{aligned}$$

provided that $\beta\ge A\ge\alpha>0$, $B>0$ with $M\ge B-A\ge m>0$ for some constants α, β, m, M.

The function $f(t)=\ln(t+1)$ is operator monotone on $[0,\infty)$ and by (2.18) we get

$$\begin{aligned}0 &\le \frac{m^2}{M^2}\left[\frac{\ln(\beta+1)}{\beta}-\frac{\ln(M+\beta+1)}{M+\beta}\right] \le \frac{m^2}{M}\left[\frac{\ln(\beta+1)}{\beta}-\frac{\ln(M+\beta+1)}{M+\beta}\right](B-A)^{-1}\\ &\le A^{-1}\ln(A+1)-B^{-1}\ln(B+1) \le \frac{M^2}{m}\left[\frac{\ln(\alpha+1)}{\alpha}-\frac{\ln(m+\alpha+1)}{m+\alpha}\right](B-A)^{-1} \qquad (3.2)\\ &\le \frac{M^2}{m^2}\left[\frac{\ln(\alpha+1)}{\alpha}-\frac{\ln(m+\alpha+1)}{m+\alpha}\right],\end{aligned}$$

provided that $\beta\ge A\ge\alpha>0$, $B>0$ with $M\ge B-A\ge m>0$ for some constants α, β, m, M.

The function $f(t) = -\ln(t+1)$ is operator convex, and by (2.20) we obtain

$$\begin{aligned} 0 &\le \frac{m^2}{M^2}\left[\frac{\ln(M+\beta+1)}{(M+\beta)^2} - \frac{\ln(\beta+1)}{\beta^2} + \frac{M}{\beta(M+\beta)}\right] \\ &\le \frac{m^2}{M}\left[\frac{\ln(M+\beta+1)}{(M+\beta)^2} - \frac{\ln(\beta+1)}{\beta^2} + \frac{M}{\beta(M+\beta)}\right](B-A)^{-1} \\ &\le B^{-2}\ln(B+1) - A^{-2}\ln(A+1) + A^{-1} - B^{-1} \\ &\le \frac{M^2}{m}\left[\frac{\ln(m+\alpha+1)}{(m+\alpha)^2} - \frac{\ln(\alpha+1)}{\alpha^2} + \frac{m}{\alpha(m+\alpha)}\right](B-A)^{-1} \\ &\le \frac{M^2}{m^2}\left[\frac{\ln(m+\alpha+1)}{(m+\alpha)^2} - \frac{\ln(\alpha+1)}{\alpha^2} + \frac{m}{\alpha(m+\alpha)}\right], \end{aligned} \tag{3.3}$$

provided that $\beta \ge A \ge \alpha > 0$, $B > 0$ with $M \ge B - A \ge m > 0$ for some constants α, β, m, M.
Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \ge 0$ and $a > 0$. Then

$$D(e_{-a})(t) := \int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda}\, d\lambda = E_1(at)\exp(at), \quad t \ge 0,$$

where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}\, du, \quad t \ge 0. \tag{3.4}$$

For $a = 1$ we have

$$D(e_{-1})(t) := \int_0^\infty \frac{\exp(-\lambda)}{t+\lambda}\, d\lambda = E_1(t)\exp(t), \quad t \ge 0.$$

Let $\beta \ge A \ge \alpha > 0$, $B > 0$ with $M \ge B - A \ge m > 0$ for some constants α, β, m, M. Then by (2.14) we have

$$\begin{aligned} 0 &\le \frac{m^2}{M^2}\left[E_1(a\beta)\exp(a\beta) - E_1(a(M+\beta))\exp(a(M+\beta))\right] \\ &\le \frac{m^2}{M}\left[E_1(a\beta)\exp(a\beta) - E_1(a(M+\beta))\exp(a(M+\beta))\right](B-A)^{-1} \\ &\le E_1(aA)\exp(aA) - E_1(aB)\exp(aB) \\ &\le \frac{M^2}{m}\left[E_1(a\alpha)\exp(a\alpha) - E_1(a(m+\alpha))\exp(a(m+\alpha))\right](B-A)^{-1} \\ &\le \frac{M^2}{m^2}\left[E_1(a\alpha)\exp(a\alpha) - E_1(a(m+\alpha))\exp(a(m+\alpha))\right], \end{aligned} \tag{3.5}$$

for $a > 0$. For $a = 1$ we have

$$\begin{aligned} 0 &\le \frac{m^2}{M^2}\left[E_1(\beta)\exp(\beta) - E_1(M+\beta)\exp(M+\beta)\right] \\ &\le \frac{m^2}{M}\left[E_1(\beta)\exp(\beta) - E_1(M+\beta)\exp(M+\beta)\right](B-A)^{-1} \\ &\le E_1(A)\exp(A) - E_1(B)\exp(B) \\ &\le \frac{M^2}{m}\left[E_1(\alpha)\exp(\alpha) - E_1(m+\alpha)\exp(m+\alpha)\right](B-A)^{-1} \\ &\le \frac{M^2}{m^2}\left[E_1(\alpha)\exp(\alpha) - E_1(m+\alpha)\exp(m+\alpha)\right]. \end{aligned} \tag{3.6}$$

More examples of such transforms are

$$D(w_{1/(\ell^2+a^2)})\,(t) := \int_0^\infty \frac{1}{(t+\lambda)\,(\lambda^2+a^2)}\,d\lambda = \frac{\pi t - 2a\ln(t/a)}{2a\,(t^2+a^2)}, \quad t \geq 0$$

and

$$D(w_{\ell/(\ell^2+a^2)})\,(t) := \int_0^\infty \frac{\lambda}{(t+\lambda)\,(\lambda^2+a^2)}\,d\lambda = \frac{\pi a + 2t\ln(t/a)}{2a\,(t^2+a^2)}, \quad t \geq 0,$$

for $a > 0$. The interested reader may state other similar results by employing the examples of monotone operator functions provided in [2, 3, 4, 7] and [8].

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On stability of nonlocal neutral stochastic integro differential equations with random impulses and Poisson jumps

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ABSTRACT

This article aims to examine the existence and Hyers-Ulam stability of non-local random impulsive neutral stochastic integrodifferential delayed equations with Poisson jumps. Initially, we prove the existence of mild solutions to the equations by using the Banach fixed point theorem. Then, we investigate stability via the continuous dependence of solutions on the initial value. Next, we study the Hyers-Ulam stability results under the Lipschitz condition on a bounded and closed interval. Finally, we give an illustrative example of our main result.

RESUMEN

Este artículo examina la existencia y estabilidad de Hyers-Ulam de ecuaciones integrodiferenciales con retardo no-locales aleatorias impulsivas neutrales estocásticas con saltos de Poisson. Inicialmente probamos la existencia de soluciones mild de las ecuaciones usando el teorema del punto fijo de Banach. Luego, investigamos la estabilidad a través de la dependencia continua de las soluciones respecto del valor inicial. A continuación, estudiamos resultados acerca de la estabilidad de Hyers-Ulam bajo la condición de Lipschitz en un intervalo cerrado y acotado. Finalmente, damos un ejemplo ilustrativo de nuestro resultado principal.

Keywords and Phrases: Existence of mild solutions, Hyers-Ulam (HU) stability, random impulsive, stochastic integro differential equations, time delays.

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1 Introduction

A model to represent the system with the occurrence of a sudden change in state at some time points is provided by impulsive differential equations. Differential equations (DEs) with fixed time impulses have been studied by many authors [7, 15, 22]. However, in the real world, impulses frequently occur at unpredictable times. Wu and Meng [21] introduced the generic DEs with random impulses, where the impulsive moments are random variables and any solution of the equations is a stochastic process, to better depict this phenomenon in reality. Examples of integer-order DEs with random impulses that have moderate solutions have been mentioned in [9, 18, 19]. The stochastic differential equations (SDEs) with random impulse involving fractional derivatives also have been studied in [10, 20, 24].

Poisson jumps are now a common modelling element in the fields of physics, biology, medicine, economics, and finance. A jump term must naturally be included in the SDEs. Furthermore, many real-world systems (such those that experience abrupt price changes or jumps as a result of stock market crashes, earthquakes, epidemics, etc.) could experience some jump-type stochastic disturbances. Since these system's sample pathways are not continuous, stochastic processes with jumps are a better fit for describing these models. These jump models typically come from Poisson random measurements. Such system's sample pathways (abbreviated c'adl'ag) are right continuous and have left limits. For more details, see the monographs [1, 23] and references therein.

On the other hand, impulsive differential equations also caught the interest of researchers see [2, 11, 12, 13]. Differential equations with fixed moments of impulses have become a natural framework for modeling processes in economics, physics, and population dynamics. The impulses usually exist at deterministic or random points. The properties of fixed-type random impulses are investigated in many articles [18, 19]. A. Anguraj *et al.* [4] established the existence and HU stability of random impulsive stochastic functional integrodifferential equations with finite delays. Moreover, Lang, Wenxuan, *et al.* [16] investigated the existence and HU stability of solutions for SDEs with random impulses. D. Chalishajar *et al.* [6] studied the existence, uniqueness, and stability of non-local random impulsive neutral stochastic differential equations with Poisson jumps. Recently, D. Baleanu, *et al.* [5] discussed the existence and stability results of mild solutions for random impulsive stochastic integro-differential equations (RISIDEs) with noncompact semigroups and resolvent operators in Hilbert spaces. R. Kasinathan *et al.* [14] investigated the existence and stability results of mild solutions for RISIDEs with noncompact semigroups via resolvent operators.

In A. Anguraj *et al.* [3] have been studied the existence and UH stability of SDEs with random impulse driven by Poisson jumps of the type

$$\begin{aligned} d(z(t)) &= f(t,z_t)+g(t,z_t)dW(t)+\int_{\mathfrak{U}} h(t,z_t,z)\tilde{K}(ds,dz), \quad t\geq t_0, \quad t\neq t_q,\\ z(\sigma_q) &= b_q(\delta_q)z(\sigma_q^-),\ q=1,2,\ldots\\ z_{t_0} &= \sigma=\{\sigma(\theta):-\delta\leq\theta\leq 0\}. \end{aligned}$$

Motivated by the above works, this paper aims to fill this gap by investigating the existence, stability and HU stability of non-local random impulsive neutral stochastic integrodifferential delayed equations (NRINSIDEs) and Poisson jumps.

The considered following NRINSIDEs with Poisson jumps of the type

$$d[z(t)+h(t,z_t)] = \left[f(t,z_t)+\int_0^t k(t,s,z_s)\,ds\right]dt+g(t,z_t)\,dW(t) + \int_{\mathfrak{U}} P(t,z_t,z)\tilde{K}(ds,dz), \quad t\geq t_0, \quad t\neq t_q, \tag{1.1}$$

$$z(\sigma_q)=b_q(\delta_q)z(\sigma_q^-), \quad q=1,2,\ldots, \tag{1.2}$$

$$z_{t_0}+r(z)=z_0=\sigma=\{\sigma(\theta):-\delta\leq\theta\leq 0\}, \tag{1.3}$$

where δ_q is a random variable defined from Ω to $\mathcal{D}_q \overset{def}{=} (0,d_q)$ for $q=1,2,\ldots$, where $0<d_q<\infty$. Moreover, suppose that $\delta_\imath$ and $\delta_\jmath$ are independent of each other as $\imath\neq\jmath$ for $\imath,\jmath=1,2\ldots$ Here $f:[t_0,\mathcal{T}]\times\mathfrak{C}\to\mathbb{R}^d$, $h:[t_0,\mathcal{T}]\times\mathfrak{C}\to\mathbb{R}^d$, $g:[t_0,\mathcal{T}]\times\mathfrak{C}\times\to\mathbb{R}^{d\times m}$, $k:[t_0,\mathcal{T}]\times[t_0,\mathcal{T}]\times\mathfrak{C}\to\mathbb{R}^d$, $r:\mathfrak{C}\to\mathfrak{C}$ and $b_q:\mathcal{D}_q\to\mathbb{R}^{d\times d}$ are Borel measurable functions, and z_t is $\mathbb{R}^d$-valued stochastic process such that

$$z_t=\{z(t+\theta):-\delta\leq\theta\leq 0\}, \quad z_t\in\mathbb{R}^d.$$

We assume that $\sigma_0=t_0$ and $\sigma_q=\sigma_{q-1}+\delta_q$ for $q=1,2,\ldots$ Obviously, $\{\sigma_q\}$ is a process with independent increments. The impulsive moments σ_q from a strictly increasing sequence, *i.e.* $\sigma=\sigma_0<\sigma_1<\sigma_2<\cdots<\lim_{k\to\infty}\sigma_q=\infty$, and $z(\sigma_q^-)=\lim_{t\to\sigma_q-0}z(t)$. Denote by $\{\mathbb{G}(t),t\geq 0\}$ the simple counting process generated by $\{\sigma_q\}$, and $\{\mathbb{K}(t),t\geq 0\}$ is a given m-dimensional Wiener process, and denote $\mathfrak{F}_t^{(1)}$ the σ-algebra generated by $\{\mathbb{G}_t,t\geq 0\}$, and denote $\mathfrak{F}_t^{(2)}$ the σ-algebra generated by $\{\mathbb{K}_t,t\geq 0\}$. We assume that $\mathfrak{F}_\infty^{(2)},\mathfrak{F}_\infty^{(2)}$ and σ are mutually independent. In (1.1)-(1.3), $\tilde{K}(dt,dz)=K(dt,dz)-dt\,\nu(du)$ denotes the compensated Poisson measure independent of $W(t)$ and $\tilde{K}(dt,dz)$ represents the Poisson counting measure associated with a characteristic measure ν.

Highlights:

(1) This work extends the work of A. Anguraj *et al.* [3].

(2) Time delay of NRINSIDEs and Poisson jumps is taken care of by the prescribed phase space $\mathcal{B}$.

The structure of this article is as follows: In section 2, we mention some concepts and principles. Section 3 is devoted to studying the existence of mild solutions of the system (1.1)-(1.3). In section 4, the stability of the mild solution of the equations (1.1)-(1.3) is studied. In section 5, we investigate the HU stability of the system (1.1)-(1.3). An example is given to illustrate the theory in section 6. At the end, the last section deals with the conclusion and acknowledgement.

2 Preliminaries

Suppose that $(\Omega, \mathfrak{F}_t, \mathcal{P})$ is a probability space with filtration $\{\mathfrak{F}_t\},\ t \geq 0$ fulfilling $\mathfrak{F}_t = \mathfrak{F}_t^{(1)} \cup \mathfrak{F}_t^{(2)}$. Let $\pounds^p = (\Omega, \mathbb{R}^d)$ be the collection of all strongly measurable, p^{th} integrable, $\mathfrak{F}_t$ measurable, $\mathbb{R}^d$-random variables in z with the norm $\|z\|_{\pounds_p} = (\mathbb{E}\|z\|_t^p)^{1/p}$. Let $\delta > 0$ and denote the Banach space of all piecewise continuous $\mathbb{R}^d$-valued stochastic process $\{\sigma(t),\ t \in [-\delta, 0]\}$ by $\mathfrak{C}([-\delta, 0], \pounds(\Omega, \mathbb{R}^d))$ equipped with the norm

$$\|\psi\|_{\mathfrak{C}} = \left(\sup_{-\delta \leq \theta \leq 0} \mathbb{E}\|\psi(\theta)\|_t^p \right)^{1/p}.$$

The initial data

$$z_{t_0} + r(z) = z_0 = \sigma = \{\sigma(\theta) : -\delta \leq \theta \leq 0\}, \tag{2.1}$$

is an $\mathfrak{F}_{t_0}$ measurable, $[-\delta, 0]$ to $\mathbb{R}^d$-valued random variable such that $\mathbb{E}\|\sigma\|^p < \infty$.

2.1 Poisson jump process

Let $(p(t))_{t \geq 0}$ be an $\mathcal{H}$-valued, σ-finite stationary $\mathfrak{F}_t$-adapted Poisson point process on $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathcal{P})$. The counting random measure K defined by

$$K((t_1, t_2] \times \mathfrak{U})(w) = \sum_{t_1 < s \leq t_2} I_{\mathfrak{U}}(p(s)(w)),$$

for any $\mathfrak{U} \in \mathcal{B}_\sigma(\mathcal{H})$ is called the Poisson random measure associated to the Poisson point process p. This measure ν is said to be a Levy measure. Then the measure $\hat{K}$ is defined by

$$\hat{K}((0, t] \times \mathfrak{U}) = K((0, t] \times \mathfrak{U}) - t\nu(\mathfrak{U}).$$

This measure $\hat{K}(dt, du)$ is called the compensated Poisson random measure, and $dt\,\nu(\mathfrak{U})$ is called the compensator.

Definition 2.1. *For a given $\mathcal{T} \in (t_0, \infty)$, a $\mathbb{R}^d$-valued stochastic process $z(t)$ on $t_0 - \delta \leq t \leq \mathcal{T}$ is called the solution to equation (1.1)-(1.3) with the initial data (2.1), if for each $t_0 \leq t \leq \mathcal{T}$, $z_{t_0} = \sigma$, $\{z_{t_0}\}_{t_0 \leq t \leq \mathcal{T}}$ is $\mathfrak{F}_t$-adapted and*

$$\begin{aligned} z(t) &= \sum_{q=0}^{\infty}\Bigg[\prod_{\imath=1}^{q} b_\imath(\delta_\imath)\sigma(0) - r(z) + h(0,\sigma) - \prod_{\imath=1}^{q} b_\imath(\delta_\imath)h(t,z_t) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s,z_s)\,ds + \int_{\sigma_q}^{t} f(s,z_s)\,ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma ds + \int_{\sigma_q}^{t}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s,z_s)\,dW(s) + \int_{\sigma_q}^{t} g(s,z_s)\,dW(s) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du) + \int_{\sigma_q}^{t}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du)\Bigg] I_{[\sigma_q,\sigma_{q+1})}(t) \end{aligned}$$

where $\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) = b_q(\delta_q)b_{q-1}(\delta_{q-1})\cdots b_\imath(\delta_\imath)$, *and $I_{\mathcal{L}}(.)$ is the index function, i.e.,*

$$I_{\mathcal{L}}(t) = \begin{cases} 1 & \textit{if} \quad t \in \mathcal{L}, \\ 0 & \textit{if} \quad t \notin \mathcal{L}. \end{cases}$$

Definition 2.2 (HU stability)**.** *Suppose that $w(t)$ is a $\mathbb{R}^d$-valued stochastic process. If there exists a real number $N > 0$, such that for arbitrary $\epsilon \geq 0$, satisfying*

$$\begin{aligned} &\mathbb{E}\Bigg\| w(t) - \sum_{q=0}^{\infty}\Bigg[\prod_{\imath=1}^{q} b_\imath(\delta_\imath)\sigma(0) - r(z) + h(0,\sigma) - \prod_{\imath=1}^{q} b_\imath(\delta_\imath)h(t,z_t) + \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s,z_s)\,ds \\ &+ \int_{\sigma_q}^{t} f(s,z_s)\,ds + \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma\,ds + \int_{\sigma_q}^{\zeta}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma\,ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s,z_s)\,dW(s) + \int_{\sigma_q}^{t} g(s,z_s)\,dW(s) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du) + \int_{\sigma_q}^{t}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du)\Bigg] I_{[\sigma_q,\sigma_{q+1})}(t)\Bigg\|^p \leq \epsilon. \end{aligned}$$

For each solution $z(t)$ with the initial value $z_{t_0} = w_{t_0} = \sigma$, if there exists a solution $z(t)$ of equations (1.1)-(1.3) with

$$\mathbb{E}\|w(t) - z(t)\| \leq N\epsilon, \quad \forall\ t\ (t_0 - \tau, \mathcal{T}).$$

Then equation (1.1)-(1.3) has the HU stability.

Lemma 2.3 ([8])**.** *Let $\vartheta, \psi \in \mathfrak{C}([a,b], \mathbb{R}^d)$ be two functions. We suppose that $\vartheta(t)$ is nondecreasing. If $z(t) \in \mathfrak{C}([a,b], \mathbb{R}^d)$ is a solution of the following inequality*

$$z(t) \leq \vartheta(t) + \int_a^t \psi(s) z(s)\, ds, \quad t \in [a,b],$$

then $z(t) \leq \vartheta(t) \exp\left(\int_a^t \psi(s)\, ds\right)$.

Lemma 2.4 ([17])**.** *For any $p \geq 1$ and for any predictable process $z \in \pounds^p_{d\times m} = [0, \mathcal{T}]$ the inequality holds,*

$$\sup \mathbb{E}\|z(t)\, dW(t)\|^p \leq (p/2(p-1))^{p/2} \left(\int_0^t \mathbb{E}\|z(s)\|^p)^{2/p} ds\right)^{p/2}, \quad t \in [0, \mathcal{T}].$$

3 Main results

In order to derive the existence and uniqueness of the system (1.1)-(1.3), we shall impose the following assumptions:

(A1): The functions $h : [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $f : [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, and $g : [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^{d\times m}$. There exist positive constant $L_h > 0$, $L_f > 0$ and $L_g > 0$ such that,

$$\begin{aligned}
\mathbb{E}\|h(t,\psi_1) - h(t,\psi_2)\|^p &\leq L_h \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}}, \\
\mathbb{E}\|h(t,\psi)\|^p &\leq L_h \mathbb{E}\|\psi\|^p_{\mathfrak{C}}. \\
\mathbb{E}\|f(t,\psi_1) - f(t,\psi_2)\|^p &\leq L_f \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}}, \\
\mathbb{E}\|f(t,\psi)\|^p &\leq L_f \mathbb{E}\|\psi\|^p_{\mathfrak{C}}. \\
\mathbb{E}\|g(t,\psi_1) - g(\zeta,\psi_2)\|^p &\leq L_g \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}}, \\
\mathbb{E}\|g(t,\psi)\|^p &\leq L_g \mathbb{E}\|\psi\|^p_{\mathfrak{C}},
\end{aligned}$$

for all $t \in [t_0, \mathcal{T}]$ and ψ_1, ψ_2 and $\psi \in \mathfrak{C}$.

(A2): The function $k : [t_0, \mathcal{T}] \times [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, there exists a positive constant $L_k > 0$ such that,

$$\begin{aligned}
\int_0^t \mathbb{E}\|k(t,s,\psi_1) - k(t,s,\psi_2)\|^p &\leq L_k \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}}, \\
\int_0^t \mathbb{E}\|k(t,s,\psi)\|^p &\leq L_k \mathbb{E}\|\psi\|^p_{\mathfrak{C}},
\end{aligned}$$

for all $t \in [t_0, \mathcal{T}]$ and ψ_1, ψ_2 and $\psi \in \mathfrak{C}$.

(A3): The condition $\max_{\imath,q}\left\{\prod_{\jmath=\imath}^{q}\|b_\jmath(\tau_\jmath)\|\right\}<\infty$. That is to say, there exists a constant $C>0$ such that

$$\mathbb{E}\left(\max_{\imath,q}\left\{\prod_{\jmath=\imath}^{q}\|b_\imath(\tau_\jmath)\|\right\}\right)^p\leq C.$$

(A4): The function $P:[t_0,\mathcal{T}]\times\mathfrak{C}\times\mathfrak{U}\to\mathbb{R}^d$, there exists a positive constant $L_P>0$ such that,

$$\int_{\mathfrak{U}}\mathbb{E}\|P(t,\psi_1,u)-P(t,\psi_2,u)\|^p\nu\,dz\leq L_P\mathbb{E}\|\psi_1-\psi_2\|^p_{\mathfrak{C}},$$
$$\int_{\mathfrak{U}}\mathbb{E}\|P(t,s,\psi)\|^p\nu\,dz\leq L_P\mathbb{E}\|\psi\|^p_{\mathfrak{C}},$$

for all $t\in[t_0,\mathcal{T}]$ and ψ_1,ψ_2 and $\psi\in\mathfrak{C}$.

(A5): The function $r:\mathfrak{C}\to\mathfrak{C}$ is continuous and there exists some constant $L_r>0$ such that,

$$\mathbb{E}\|r(t,\psi_1)-r(t,\psi_2)\|^p\leq L_r\mathbb{E}\|\psi_1-\psi_2\|^p_{\mathfrak{C}},$$
$$\mathbb{E}\|r(t,\psi)\|^p\leq L_r\mathbb{E}\|\psi\|^p_{\mathfrak{C}},$$

for all $t\in[t_0,\mathcal{T}]$ and ψ_1,ψ_2 and $\psi\in\mathfrak{C}$.

Theorem 3.1. *Assume that the assumptions (A1)–(A5) are satisfied. Then the system* (1.1)-(1.3) *has a unique solution in $\mathcal{B}$.*

Proof. Let $\mathcal{B}$ be the phase space $\mathcal{B}=\mathfrak{C}([t_0-\delta,\mathcal{T}],\mathcal{L}^p(\Omega,\mathbb{R}^d))$ endowed with the norm

$$\|z\|^p_{\mathcal{B}}=\sup_{t\in[t_0,\mathcal{T}]}\|z_t\|^p_{\mathfrak{C}},$$

where $\|z_t\|_{\mathfrak{C}}=\sup_{-\delta\leq s\leq t}\mathbb{E}\|z_t\|^p$. Denote $B_m=\{z\in\mathcal{B},\|z\|^p_{\mathcal{B}}\leq m\}$, which is the closed ball with center z and radius $m>0$. For any initial value $(t_0,z_0,)$ with $t_0\geq 0$ and $z_0\in B_m$, we define the operator $S:\mathcal{B}\to\mathcal{B}$ by

$$(Sz)(t)=\begin{cases}\sigma(t)-r(t), \qquad t\in(\infty,t_0]\\ \sum_{q=0}^{\infty}\Big[\prod_{\imath=1}^{q}b_\imath(\delta_\imath)\sigma(0)-r(t)+h(0,\sigma)-\prod_{\imath=1}^{q}b_\imath[(\delta_\imath)h(t,z_t)\\ +\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\tau_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}f(s,z_s)ds+\int_{\sigma_q}^{t}f(s,z_s)ds+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma,z_\varsigma)d\varsigma ds\\ +\int_{\sigma_q}^{t}\int_0^s k(s,\varsigma,z_\varsigma)d\varsigma+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}g(s,z_s)dW(s)+\int_{\sigma_q}^{t}g(s,z_s)dW(s)\\ +\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}}P(s,z_s,u)\tilde{K}(ds,du)\\ +\int_{\sigma_q}^{t}\int_{\mathfrak{U}}P(s,z_s,u)\tilde{K}(ds,du)\Big]I_{[\sigma_q,\sigma_{q+1})}(t), \qquad t\in[t_0,\mathcal{T}].\end{cases}$$

Now we have to prove that S maps $\mathcal{B}$ into itself.

$$\begin{aligned}\|(Sz)(t)\|^p = \Bigg\| \sum_{q=0}^{\infty} \Bigg[\prod_{\imath=1}^{q} b_\imath(\delta_\imath)[\sigma(0) - r(t) + h(0,\sigma)] - \prod_{\imath=1}^{q} b_\imath(\delta_\imath) h(t, z_t) \\ + \Bigg[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s, z_s)\, ds + \int_{\sigma_q}^{\sigma} f(s, z_s)\, ds \Bigg] \\ + \Bigg[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_0^s k(s, \varsigma, z_\varsigma)\, d\varsigma\, ds + \int_{\sigma_q}^{\sigma} \int_0^s k(s, \varsigma, z_\varsigma)\, d\varsigma \Bigg] \\ + \Bigg[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s, z_s)\, dW(s) + \int_{\sigma_q}^{\sigma} g(s, z_s)\, dW(s) \Bigg] \\ + \Bigg[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_{\mathfrak{U}} P(s, z_s, u) \tilde{K}(ds, du) + \int_{\sigma_q}^{t} \int_{\mathfrak{U}} P(s, z_s, u) \tilde{K}(ds, du) \Bigg] \Bigg] I_{[\sigma_q, \sigma_{q+1})}(t) \Bigg\|^p\end{aligned}$$

$$\begin{aligned}\mathbb{E}\|Sz(t)\|^p &\le 4^{p-1} \mathbb{E}\Bigg[\max_{\imath,q} \Bigg\{ \prod_{\imath=\jmath}^{q} \|b_\jmath(\delta_\jmath)\| \Bigg\} \Bigg]^p [\|\sigma(0) - r(z) + h(0,\sigma)\|^p] \\ &+ 4^{p-1} \mathbb{E}\Bigg[\max_{\imath,q} \Bigg\{ \prod_{\imath=\jmath}^{q} \|b_\jmath(\delta_\jmath)\| \Bigg\} \Bigg]^p \|h(t, z_t)\|^p \\ &+ 4^{p-1} \mathbb{E}\Bigg[\max_{\imath,q} \Bigg\{ 1, \prod_{\imath=\jmath}^{q} \|b_\jmath(\delta_\jmath)\| \Bigg\} \Bigg]^p \Bigg[\int_{t_0}^{t} \|f(s, z_s)\|\, ds\, I_{[\sigma_q, \sigma_{q+1})}(t) \Bigg]^p \\ &+ 4^{p-1} \mathbb{E}\Bigg[\max_{\imath,q} \Bigg\{ 1, \prod_{\imath=\jmath}^{q} \|b_\jmath(\delta_\jmath)\| \Bigg\} \Bigg]^p \Bigg[\int_{t_0}^{t} \int_0^s \|k(s, \varsigma, z_\varsigma)\, d\varsigma\, ds\, I_{[\sigma_q, \sigma_{q+1})}(t) \Bigg]^p \\ &+ 4^{p-1} \mathbb{E}\Bigg[\max_{\imath,q} \Bigg\{ 1, \prod_{\imath=\jmath}^{q} \|b_\jmath(\delta_\jmath)\| \Bigg\} \Bigg]^p \Bigg[\int_{t_0}^{t} \|g(s, z_s) dW(s)\|\, ds\, I_{[\sigma_q, \sigma_{q+1})}(t) \Bigg]^p \\ &+ 4^{p-1} \mathbb{E}\Bigg[\max_{\imath,q} \Bigg\{ 1, \prod_{\imath=\jmath}^{q} \|b_\jmath(\delta_\jmath)\| \Bigg\} \Bigg]^p \Bigg[\int_{t_0}^{t} \int_{\mathfrak{U}} \|P(s, z_s, u) \tilde{K}(ds, du)\|\, ds\, I_{[\sigma_q, \sigma_{q+1})}(t) \Bigg]^p \\ &\le 4^{p-1} C[\mathbb{E}\|\sigma(0)\|^p + L_r \mathbb{E}\|z\|^p] + 4^{p-1} C L_h \mathbb{E}\|\sigma\|^p + 4^{p-1} C L_h \mathbb{E}\|z_t\|_{\mathfrak{C}}^p \\ &+ 4^{p-1} \max\{1, C\}(t - t_0)^{p-1} L_f \int_{t_0}^{t} \mathbb{E}\|z_s\|_{\mathfrak{C}}^p\, ds + 4^{p-1} \max\{1, C\}(t - t_0)^{p-1} L_k \int_{t_0}^{t} \mathbb{E}\|z_s\|_{\mathfrak{C}}^p\, ds \\ &+ 4^{p-1} \max\{1, C\}(t - t_0)^{p/2-1} L_g L_p \int_{t_0}^{t} \mathbb{E}\|z_s\|_{\mathfrak{C}}^p\, ds + 4^{p-1} \max\{1, C\}(t - t_0)^{p/2} L_P c_p \int_{t_0}^{t} \mathbb{E}\|z_s\|_{\mathfrak{C}}^p\, ds.\end{aligned}$$

Thus

$$\begin{aligned}\sup_{s\in[t-\tau,t]} \mathbb{E}\|Sz(t)\|^p \le 4^{p-1} C[\mathbb{E}\|\sigma(0)\|^p + L_h \mathbb{E}\|\sigma\|^p] + \big\{ 4^{p-1} C (L_r + L_h)(t - t_0)^{-1} \\ + 4^{p-1} \max\{1, C\} \big[(t - t_0)^{p-1} L_f + (t - t_0)^{p-1} L_k + (t - t_0)^{p/2-1} L_g L_p \\ + (t - t_0)^{p-1} c_p (L_P + L_P^{P/2}) \big] \big\} (t - t_0) \sup_{s\in[t-\delta,t]} \mathbb{E}\|z_s\|_{\mathfrak{C}}^p.\end{aligned}$$

Therefore S maps $\mathcal{B}$ into itself.

Now, we have to prove that S is a contraction mapping.

$$
\begin{aligned}
\mathbb{E}\|(Sz)(t)-(Sw)(t)\|^p \le\ & 3^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\|r(z)-r(w)\|I_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p\\
&+3^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\|h(t,z_t)-h(t,w_t)\|I_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p\\
&+3^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{1,\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\int_{t_0}^{t}\|f(s,z_s)-f(s,w_s)\|dsI_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p\\
&+3^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{1,\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\int_{t_0}^{t}\int_0^s\|k(s,\varsigma,z_\varsigma)-k(s,\varsigma,z_\varsigma)\|d\varsigma dsI_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p\\
&+3^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{1,\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\int_{t_0}^{t}\|g(s,z_s)-g(s,w_s)\|dW(s)dsI_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p\\
&+3^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{1,\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\int_{t_0}^{t}\int_{\mathfrak{U}}\|P(s,z_s,u)-P(s,z_s,u)\|\tilde{K}(ds,du)I_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p\\
&\le 3^{p-1}C\mathbb{E}\|r(z)-r(w)\|^p+3^{p-1}C\mathbb{E}\|h(t,z_t)-h(t,w_t)\|^p\\
&+3^{p-1}\max\{1,C\}(t-t_0)^pL_f\times\int_{t_0}^{t}\mathbb{E}\|f(s,z_s)-f(s,w_s)\|^pds\\
&+3^{p-1}\max\{1,C\}(t-t_0)^pL_k\int_{t_0}^{t}\int_0^s\mathbb{E}\|k(s,\varsigma,z_\varsigma)-k(s,\varsigma,w_\varsigma)\|^pd\varsigma ds\\
&+3^{p-1}\max\{1,C\}(t-t_0)^{p/2}L_pL_g\int_{t_0}^{t}\mathbb{E}\|g(s,z_s)-g(s,w_s)\|^pdW(s)\\
&+3^{p-1}\max\{1,C\}(t-t_0)^pc_pL_P\int_{t_0}^{t}\mathbb{E}\|P(s,z_s,u)-P(s,z_s,u)\|^pds\\
&\le 3^{p-1}CL_r\mathbb{E}\|z-w\|^p_{\mathfrak{C}}+3^{p-1}CL_h\mathbb{E}\|z-w\|^p_{\mathfrak{C}}\\
&+3^{p-1}\max\{1,C\}(t-t_0)^pL_f\mathbb{E}\|z_s-w_s\|^p_{\mathfrak{C}}ds\\
&+3^{p-1}\max\{1,C\}(t-t_0)^pL_k\mathbb{E}\|z_s-w_s\|^p_{\mathfrak{C}}ds\\
&+3^{p-1}\max\{1,C\}(t-t_0)^{p/2}L_pL_g\mathbb{E}\|z_s-w_s\|^p_{\mathfrak{C}}ds\\
&+3^{p-1}\max\{1,C\}(t-t_0)^pc_p(L_P+L_P^{p/2})\mathbb{E}\|z_s-w_s\|^p_{\mathfrak{C}}ds\\
&\le\big\{3^{p-1}C(L_r+L_h)+3^{p-1}\max\{1,C\}\big[\big(t-t_0)^pL_f+(t-t_0)^pL_k\\
&+(t-t_0)^{p/2}L_pL_g+(t-t_0)^pc_p(L_P+L_P^{p/2})\big]\big\}\sup_{\theta\in[-\delta,0]}\mathbb{E}\|z(t+\theta)-w(t+\theta)\|^p_{\mathfrak{C}}\\
&\le\big\{3^{p-1}C(L_r+L_h)+3^{p-1}\max\{1,C\}\big[\big(t-t_0)^pL_f+(t-t_0)^pL_k\\
&+(t-t_0)^{p/2}L_pL_g+(t-t_0)^pc_p(L_P+L_P^{p/2})\big]\big\}\sup_{s\in[t-\delta,t]}\mathbb{E}\|z(s)-w(s)\|^p_{\mathfrak{C}}.
\end{aligned}
$$

Taking the supremum over t, we get

$$\|(Sz)(t)-(Sw)(t)\|^p_{\mathcal{B}}\le\mathfrak{A}(\mathcal{T})\mathbb{E}\|z-w\|^p_{\mathcal{B}},$$

with

$$\mathfrak{A}(\mathcal{T})=3^{p-1}C(L_r+L_h)+3^{p-1}\max\{1,C\}\big[\big(t-t_0)^p(L_f+L_k+c_p(L_P+L_P^{p/2}))+(t-t_0)^{p/2}L_pL_g\big].$$

By taking a suitable $0 < \mathcal{T}_1 < \mathcal{T}$ sufficient small such that $\mathfrak{A}(\mathcal{T}) < 1$. Hence S is a contraction on $\mathcal{B}_{\mathcal{T}_1}$. $Sz = z$ is a unique solution of equation (1.1)-(1.3) by the Banach fixed point theorem. □

4 Stability

The stability through continuous dependence of solutions on the initial condition is investigated.

Definition 4.1 ([4])**.** *A mild solution $z(t)$ of the system* (1.1) *and* (1.2) *with initial condition σ satisfying* (2.1) *is said to be stable in the mean square if for all $\epsilon > 0$, there exist, $\beta > 0$ such that,*

$$\begin{aligned} \mathbb{E}\|z - w\|_t^p \leq \epsilon, &\quad \textit{whenever}, \\ \mathbb{E}\|\sigma_1 - \sigma_2\|^p < \beta, &\quad \textit{for all} \quad t \in [t_0, T], \end{aligned}$$

where $w(t)$ is another mild solution of the system (1.1) *and* (1.2) *with initial value σ defined in* (1.3)*.*

Theorem 4.2. *Let $z(t)$ and $w(t)$ be mild solutions of the system* (1.1)-(1.3) *with initial values σ_1 and σ_2 respectively. If the hypotheses of theorem 3.1 are fulfilled, the mean solution of the system* (1.1)- (1.3) *is stable in the mean square.*

Proof. Under assumptions, $z(t)$ and $w(t)$ be two mild solutions of the system (1.1)-(1.3) with initial values σ_1 and σ_2 respectively.

$$\begin{aligned} z(t) - w(t) = \sum_{q=0}^{\infty} \Bigg[&\prod_{\imath=1}^{q} b_\imath(\delta_\imath)[\sigma_1 - \sigma_2] + [r(z) - r(w)] + [h(0,\sigma_1) - h(0,\sigma_2)] + \prod_{\imath=1}^{q} b_\imath(\delta_\imath)[h(t,z_t) - h(t,w_t)] \\ &+ \Bigg[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} [f(s,z_s) - f(s,w_s)]\, ds + \int_{\sigma_q}^{t} [f(s,z_s) - f(s,w_s)]\, ds \Bigg] \\ &+ \Bigg[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_0^s [k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)]\, d\varsigma\, ds + \int_{\sigma_q}^{t} \int_0^s [k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)]\, d\varsigma \Bigg] \\ &+ \Bigg[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} [g(s,z_s) - g(s,w_s)]\, dW(s) + \int_{\sigma_q}^{t} [g(s,z_s) - g(s,w_s)]\, dW(s) \Bigg] \\ &+ \Bigg[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_{\mathfrak{U}} [P(s,z_s,u) - P(s,w_s,u)] \tilde{K}(ds,du) \\ &+ \int_{\sigma_q}^{t} \int_{\mathfrak{U}} [P(s,z_s,u) - P(s,w_s,u)] \tilde{K}(ds,du) \Bigg] \Bigg] I_{[\sigma_q,\sigma_{q+1})}(t). \end{aligned}$$

Then,

$$\begin{aligned}\mathbb{E}\|z(t)-w(t)\|^p \leq{}& 4^{p-1}C(1+L_h)\mathbb{E}\|\sigma_1-\sigma_2\|^p + 4^{p-1}CL_r\mathbb{E}\|z-w\|^p + 4^{p-1}C^pL_h\mathbb{E}\|z(t)-w(t)\|^p\\ &+ 4^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_f\int_{t_0}^t \mathbb{E}\|z(s)-w(s)\|^p ds\\ &+ 4^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_k\int_{t_0}^t \mathbb{E}\|z(s)-w(s)\|^p ds\\ &+ 4^{p-1}\max\{1,C\}(t-t_0)^{p/2-1}L_gL_p\int_{t_0}^t \mathbb{E}\|z(s)-w(s)\|^p ds\\ &+ 4^{p-1}\max\{1,C\}(t-t_0)^{p/2}L_Pc_p\int_{t_0}^t \mathbb{E}\|z(s)-w(s)\|^p ds.\end{aligned}$$

Furthermore,

$$\begin{aligned}\sup_{s\in[t-\tau,t]}\mathbb{E}\|z(t)-w(t)\|^p \leq{}& 4^{p-1}C(1+L_h)\mathbb{E}\|\sigma_1-\sigma_2\|^p + 4^{p-1}C(L_r+L_h)\sup_{t\in[t-\tau,t]}\mathbb{E}\|z(t)-w(t)\|^p\\ &+ 4^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_f\int_{t_0}^t \sup_{s\in[t-\tau,t]}\mathbb{E}\|z(s)-w(s)\|^p ds\\ &+ 4^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_k\int_{t_0}^t \sup_{s\in[t-\tau,t]}\mathbb{E}\|z(s)-w(s)\|^p ds\\ &+ 4^{p-1}\max\{1,C\}(t-t_0)^{p/2-1}L_gL_p\int_{t_0}^t \sup_{s\in[t-\tau,t]}\mathbb{E}\|z(s)-w(s)\|^p ds\\ &+ 4^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_Pc_p\int_{t_0}^t \sup_{s\in[t-\tau,t]}\mathbb{E}\|z(s)-w(s)\|^p ds.\end{aligned}$$

Thus,

$$\sup_{s\in[t-\tau,t]}\mathbb{E}\|z(t)-w(t)\|^p \leq \gamma\mathbb{E}\|\sigma_1-\sigma_2\|^p,$$

where,

$$\gamma = \frac{4^{p-1}C(1+L_h)}{1-\left[4^{p-1}C(L_r+L_h)+4^{p-1}\max\{1,C\}(t-t_0)^p[(L_f+L_k)+(t-t_0)^{-p/2}L_gL_p+c_p(L_P+L_P^{p/2})]\right]}.$$

Given $\epsilon>0$, choose $\beta=\frac{\epsilon}{\gamma}$ such that $\mathbb{E}\|\sigma_1-\sigma_2\|^p<\beta$. Then,

$$\|z-w\|_{\mathcal{B}}^p \leq \epsilon.$$

This completes the proof. □

5 HU stability

In this section, we investigate the HU stability of equations (1.1)-(1.3) under the assumptions (A1)-(A5). We have the following HU stability theorem.

Theorem 5.1. *Under the assumptions (A1)-(A5). Then equations* (1.1)-(1.3) *has the HU stability.*

Proof.

$$\begin{aligned} z(t) = \sum_{q=0}^{\infty} \Big[&\prod_{\imath=1}^{q} b_\imath(\delta_\imath)\sigma(0) - r(z) + h(0,\sigma) - \prod_{\imath=1}^{q} b_\imath(\tau_\imath)h(t,z_t) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s,z_s)\,ds + \int_{\sigma_q}^{t} f(s,z_s)\,ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma\,ds + \int_{\sigma_q}^{\zeta}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma\,ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s,z_s)\,dW(s) + \int_{\sigma_q}^{t} g(s,z_s)\,dW(s) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du) + \int_{\sigma_q}^{t}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du)\Big] I_{[\sigma_q,\sigma_{q+1})}(\zeta). \end{aligned}$$

It follows from the condition that

$$\begin{aligned} \mathbb{E}\Big\| w(s) - \sum_{q=0}^{\infty} \Big[&\prod_{\imath=1}^{q} b_\imath(\delta_\imath)\sigma(0) - r(z) + h(0,\sigma) - \prod_{\imath=1}^{q} b_\imath(\tau_\imath)h(t,z_t) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s,z_s)\,ds + \int_{\sigma_q}^{t} f(s,z_s)\,ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma\,ds + \int_{\sigma_q}^{t}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma\,ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s,z_s)\,dW(s) + \int_{\sigma_q}^{t} g(s,z_s)\,dW(s) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du) + \int_{\sigma_q}^{t}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du)\Big] I_{[\sigma_q,\sigma_{q+1})}(t)\Big\|^p \le \epsilon. \end{aligned}$$

When $t \in [t_0-\delta, t_0]$, we get $\mathbb{E}\|w(t)-z(t)\|^p = 0$. And when $t \in [0,\mathcal{T}]$, we get

$$\begin{aligned} \mathbb{E}\|w(t)-z(t)\|^p \le 2^{p-1}\mathbb{E}\Big\| w(s) - \sum_{q=0}^{\infty}\Big[&\prod_{\imath=1}^{q} b_\imath(\delta_\imath)\sigma(0) - r(z) + h(0,\sigma) - \prod_{\imath=1}^{q} b_\imath(\delta_\imath)h(t,z_t) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\tau_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s,z_s)\,ds + \int_{\sigma_q}^{t} f(s,z_s)\,ds + \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma\,ds \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\sigma_q}^{t} \int_0^s k(s,\varsigma,z_\varsigma)\, d\varsigma ds + \sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s,z_s)\, dW(s) + \int_{\sigma_q}^{t} g(s,z_s)\, dW(s) \\
&+ \sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_{\mathfrak{U}} P(s,z_s,u) \tilde{K}(ds,du) + \int_{\sigma_q}^{t} \int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du) \Big] I_{[\sigma_q,\sigma_{q+1})}(t) \\
&+ 2^{p-1}\mathbb{E}\Big\| \sum_{q=0}^{\infty} \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} [f(s,z_s) - f(s,w_s)]\, ds + \int_{\sigma_q}^{t} [f(s,z_s) - f(s,w_s)]\, ds \Big] \\
&+ \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_0^s [k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)]\, d\varsigma\, ds + \int_{\sigma_q}^{t} \int_0^s [k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)]\, d\varsigma \Big] \\
&+ \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} [g(s,z_s) - g(s,w_s)]\, d\mathcal{W}(s) + \int_{\sigma_q}^{t} [g(s,z_s) - g(s,w_s)]\, d\mathcal{W}(s) \Big] \\
&+ \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_{\mathfrak{U}} [P(s,z_s,u) - P(s,w_s,u)] \tilde{K}(ds,du) \\
&+ \int_{\sigma_q}^{t} \int_{\mathfrak{U}} [P(s,z_s,u) - P(s,w_s,u)] \tilde{K}(ds,du) \Big] I_{[\sigma_q,\sigma_{q+1})}(t) \Big\|^p \\
&\leq 2^{p-1}\epsilon + 2^{p-1}N,
\end{aligned}
$$

where

$$
\begin{aligned}
N &= 4^{p-1}\mathbb{E}\Big\| \sum_{q=0}^{\infty} \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} [f(s,z_s) - f(s,w_s)] ds + \int_{\sigma_q}^{t} [f(s,z_s) - f(s,w_s)] ds \Big] \\
&+ \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_0^s [k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)] d\varsigma ds + \int_{\sigma_q}^{t} \int_0^s [k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)] d\varsigma \Big] \\
&+ \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} [g(s,z_s) - g(s,w_s)] dW(s) + \int_{\sigma_q}^{t} [g(s,z_s) - g(s,w_s)] dW(s) \Big] \\
&+ \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_{\mathfrak{U}} [P(s,z_s,u) - P(s,w_s,u)] \tilde{K}(ds,du) \\
&+ \int_{\sigma_q}^{t} \int_{\mathfrak{U}} [P(s,z_s,u) - P(s,w_s,u)] \tilde{K}(ds,du) \Big] \Big] I_{[\sigma_q,\sigma_{q+1})}(t) \Big\|^p \\
&\leq 4^{p-1}(\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}).
\end{aligned}
$$

Take

$$
\begin{aligned}
\mathcal{A} &= \mathbb{E}\Big\| \sum_{q=0}^{\infty} \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} [f(s,z_s) - f(s,w_s)] ds + \int_{\sigma_q}^{t} [f(s,z_s) - f(s,w_s)] ds \Big] I_{[\sigma_q,\sigma_{q+1})}(t) \Big\|^p \\
&\leq (C^p + 1)(\mathcal{T} - t_0)^{p-1} \int_{t_0}^{t} \mathbb{E}\big\| f(s,z_s) - f(s,w_s) \big\|^p ds \\
&\leq (C^p + 1)(\mathcal{T} - t_0)^{p-1} \int_{t_0}^{t} \mathbb{E}\big\| f(s,z_s) - f(s,w_s) \big\|^p ds \\
&\leq (C^p + 1)L_f(\mathcal{T} - t_0)^{p-1} \int_{t_0}^{t} \big\| z_s - w_s \big\|_{\mathfrak{C}}^p ds.
\end{aligned}
$$

By (A2), we have

$$\begin{aligned}\mathcal{B} &= \mathbb{E}\Big\| \sum_{q=0}^{\infty} \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \big[k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)\big] d\varsigma\, ds \\ &+ \int_{\sigma_q}^{t} \int_0^s \big[k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)\big]\, d\varsigma\, ds \Big] I_{[\sigma_q,\sigma_{q+1})}(t) \Big\|^p \\ &\le (C^p+1)(\mathcal{T}-t_0)^{p-1} \int_{t_0}^{t} \mathbb{E}\big\|k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)\big] d\varsigma ds \\ &\le (C^p+1)L_k(\mathcal{T}-t_0)^{p-1} \int_{t_0}^{t} \big\|z_s - w_s\big\|_{\mathfrak{C}}^p ds.\end{aligned}$$

Using Lemma 2.4, we have

$$\begin{aligned}\mathcal{C} &= \mathbb{E}\Big\| \sum_{q=0}^{\infty} \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \big[g(s,z_s) - g(s,w_s)\big] dW(s) \\ &+ \int_{\sigma_q}^{t} \big[g(s,z_s) - g(s,w_s)\big] dW(s) \Big] I_{[\sigma_q,\sigma_{q+1})}(t) \Big\|^p \\ &\le (C^p+1)(p(p-1)/2)(\mathcal{T}-t_0)^{p-2/2} \int_{t_0}^{t} \mathbb{E}\big\|g(s,z_s) - g(s,w_s)\big\|^p ds \\ &\le (C^p+1)L_g(p(p-1)/2)(\mathcal{T}-t_0)^{p-2/2} \int_{t_0}^{t} \big\|z_s - w_s\big\|_{\mathfrak{C}}^p ds.\end{aligned}$$

By (A4), we have

$$\begin{aligned}\mathcal{D} &= \mathbb{E}\Big\| \sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_{\mathfrak{U}} [P(s,z_s,u) - P(s,w_s,u)] \tilde{K}(ds,du) \\ &+ \int_{\sigma_q}^{t} \int_{\mathfrak{U}} [P(s,z_s,u) - P(s,w_s,u)] \tilde{K}(ds,du) \Big] I_{[\sigma_q,\sigma_{q+1})}(t) \Big\|^p \\ &\le (C^p+1)c_p(\mathcal{T}-t_0)^{p-1} \Big[\int_{t_0}^{t} \int_{\mathfrak{U}} \mathbb{E}\|P(s,z_s,u) - P(s,w_s,u)\|^p \nu(dz)\, ds \\ &+ \Big(\int_{t_0}^{t} \int_{\mathfrak{U}} \mathbb{E}\|P(s,z_s,u) - P(s,w_s,u)\|^{p/2} \nu(dz)\, ds \Big)^{1/2} \Big] \\ &\le (C^p+1)c_p(L_P + L_P^{p/2})(\mathcal{T}-t_0)^{p-1} \int_{t_0}^{t} \big\|z_s - w_s\big\|_{\mathfrak{C}}^p ds.\end{aligned}$$

Therefore,

$$F = H \int_{t_0}^{t} \big\|z(s) - w(s)\big\|_{\mathfrak{C}}^p ds, \quad \text{with}$$

$H = 4^{p-1}(C^p+1)(\mathcal{T}-\zeta_0)^{p/2-1}\big[L_f(\mathcal{T}-t_0)^{p/2} + L_k(\mathcal{T}-t_0)^{p/2} + L_g(p(p-1)/2)^{p/2} + c_p(L_P + L_P^{p/2})(\mathcal{T}-t_0)^{p/2}\big].$

Then, we get that

$$\mathbb{E}\|z(t) - w(t)\|^p \le 2^{p-1}\epsilon + 2^{p-1} H \int_{t_0}^{t} \|w(s) - z(s)\|_{\mathfrak{C}}^p ds.$$

Considering,

$$\begin{aligned}\int_{t_0}^{t} \left\|w(s)-z(s)\right\|_{\mathfrak{C}}^{p} ds &= \int_{t_0}^{t} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|w(s+\theta)-z(s+\theta)\|^p ds \\ &= \sup_{\theta\in[-\tau,0]} \int_{\zeta_0}^{t} \mathbb{E}\|w(s+\theta)-z(s+\theta)\|^p ds \\ &= \sup_{\theta\in[-\tau,0]} \int_{t_0+\theta}^{t+\theta} \mathbb{E}\|w(I)-z(I)\|^p dI.\end{aligned}$$

Notice that, when $t\in[t_0-\tau,t_0]$,

$$\mathbb{E}\|w(I)-z(I)\|^p dI = 0.$$

Therefore,

$$\int_{t_0}^{t} \left\|w_s-z_s\right\|_{\mathfrak{C}}^{p} ds = \sup_{\theta\in[-\tau,0]} \int_{t_0}^{t+\theta} \mathbb{E}\|w(I)-z(I)\|^p dI = \int_{t_0}^{t} \mathbb{E}\|w(I)-z(I)\|^p dI.$$

So, we get

$$\mathbb{E}\|w(t)-z(t)\|^p \le 2^{p-1}\epsilon + 2^{p-1}H\int_{t_0}^{t} \mathbb{E}\|w(I)-z(I)\|^p dI.$$

By Lemma 2.3, we have

$$\mathbb{E}\|w(t)-z(t)\|^p \le 2^{p-1}\epsilon + 2^{p-1}\exp(2^{p-1}H).$$

Therefore, there exists $N = 2^{p-1}\exp(2^{p-1}K)$ such that

$$\mathbb{E}\|w(t)-z(t)\|^p \le N\epsilon.$$

Thus the proof gets completed. □

6 An application

The considered NRINSIDEs with Poisson jumps is of the form

$$\begin{aligned} d\left[(z(\zeta)+\int_{-\alpha}^{0} u_1(\theta)z(\zeta+\theta)\right] &= \left[\int_{-\alpha}^{0} u_2(\theta)z(\zeta+\theta) + \int_{-\alpha}^{0}\int_{0}^{\zeta} u_3(\theta)z(\zeta+\theta)\right]d\zeta \\ &\quad + \left[\int_{-\alpha}^{0} u_4(\theta)z(\zeta+\theta)\right]dW(\zeta) \\ &\quad + \left[\int_{-\alpha}^{0}\int_{\mathfrak{U}} u_5(\theta)z(\zeta+\theta)\right]\tilde{K}(ds,du), \quad t\ge t_0, \quad t\ne\zeta_q\end{aligned}$$

$$z(\sigma_q) = b_q(\delta_q) z(\sigma_q^-), \quad q = 1, 2, \ldots,$$

$$z(0) + \sum_{\imath}^{m} c_\imath(r_{\imath,z}) = z_0, \quad 0 \leq r_1 \leq r_2 \leq \cdots \leq r_p \leq \mathcal{T}.$$

Let $\alpha > 0$, z is $\mathbb{R}$-valued stochastic process, $\sigma \in \mathfrak{C}([-\delta, 0], \mathcal{L}^2(\Omega, \mathbb{R}))$. δ_q is defined from Ω to $\mathcal{D}_q \stackrel{\text{def}}{=} (0, d_q)$ for all $q = 1, 2, \ldots$ Suppose that τ_q follow Erlang distribution and let $\delta_\imath$ and $\delta_\jmath$ are independent of every other as $\imath \neq \jmath$ for $\imath, \jmath = 1, 2, \ldots$, $\zeta_0 = \sigma_0 < \sigma_1 < \sigma_2 < \cdots$ and $\sigma_q = \sigma_{q-1} + \tau_q$ for $q = 1, 2, \ldots$ Let $W(t) \in \mathbb{R}$ be a one-dimensional Brownian motion, where b is a function of q. $u_1, u_2, u_3 : [-\delta, 0] \to \mathbb{R}$ are continuous functions. Define $h : [\zeta_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $f : [\zeta_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $g : [\zeta_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^{d \times m}$, $r : \mathfrak{C} \to \mathfrak{C}$, $k : [\zeta_0, \mathcal{T}] \times [\zeta_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $P : [\zeta_0, \mathcal{T}] \times \mathfrak{C} \times \mathfrak{U} \to \mathbb{R}^d$, and $b_q : \mathcal{D}_q \to \mathbb{R}^{d \times d}$ by

$$h(\zeta, z(\zeta))(.) = \int_{-\alpha}^{0} u_1(\theta) z(\zeta + \theta)\, d\theta(.), \quad f(\zeta, z(\zeta))(.) = \int_{-\alpha}^{0} u_2(\theta) z(\zeta + \theta)\, d\theta(.),$$

$$k(\zeta, z(\zeta))(.) = \int_{-\alpha}^{0} u_3(\theta) z(\zeta + \theta)\, d\theta(.), \quad g(\zeta, z(\zeta))(.) = \int_{-\alpha}^{0} u_4(\theta) z(\zeta + \theta)\, d\theta(.),$$

$$P(\zeta, z(\zeta))(.) = \int_{-\alpha}^{0} u_3(\theta) z(\zeta + \theta)\, d\theta(.).$$

For $z(t + \theta) \in \mathfrak{C}$, we suppose that the following conditions hold:

(1) $\max\limits_{\imath, q} \left\{ \prod\limits_{\jmath = \imath}^{q} \mathbb{E} \| b_\imath(\delta_\imath) \|^2 \right\} < \infty,$

(2) $\int_{-\alpha}^{0} u_1(\theta)^2 d\theta, \int_{-\alpha}^{0} u_2(\theta)^2 d\theta, \int_{-\alpha}^{0} u_3(\theta)^2 d\theta < \int_{-\alpha}^{0} u_4(\theta)^2 d\theta < \int_{-\alpha}^{0} u_5(\theta)^2 d\theta < \infty.$

Suppose the conditions (1) and (2) are fulfilled. Then the assumptions (A1)-(A5) holds. The system (1.1)-(1.3) has a unique mild solution z and is HU stable.

Lemma 6.1. *If $P = 0$ in* (1.1)-(1.3)*, then the system behaves as NRINSIDEs of the form:*

$$d[z(t) + h(t, z_t)] = \left[f(t, z_t) + \int_0^t k(t, s, z_s)\, ds \right] dt + g(t, z_t)\, dW(t), \quad t \geq t_0, \quad t \neq t_q,$$

$$z(\sigma_q) = b_q(\delta_q) z(\sigma_q^-), \quad q = 1, 2, \ldots,$$

$$z_{t_0} + r(z) = z_0 = \sigma = \{\sigma(\theta) : -\delta \leq \theta \leq 0\}$$

By applying Theorem 3.1 under the assumptions (A1)-(A5), then the above guarantees the existence of the mild solution.

7 Conclusion

This article is devoted to discuss the existence and HU stability. First, we used the Banach fixed point theorem to demonstrate the existence of mild solutions to the equations (1.1)-(1.3). Then, we examined the stability via the continuous dependence of solutions on the initial value. Next, we investigated the HU stability results under the Lipschitz condition on a bounded and closed interval. In addition, this result could be extended to investigate the controllability of random impulsive neutral stochastic differential equations finite/infinite state-dependent delay in the future. The fractional order of NRINSDEs with Poisson jumps would be quite interesting. This will be the focus of future research.

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Existence and controllability of integrodifferential equations with non-instantaneous impulses in Fréchet spaces

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ABSTRACT

In this paper, we investigate existence of mild solutions to a non-instantaneous integrodifferential equation via resolvent operators in the sense of Grimmer in Fréchet spaces. Utilizing the technique of measures of noncompactness in conjunction with the Darbo's fixed point theorem, we present sufficient criteria ensuring the controllability of the given problem. An illustrative example is also discussed.

RESUMEN

En este artículo, investigamos la existencia de soluciones mild de una ecuación integrodiferencial no-instantánea vía operadores resolventes en el sentido de Grimmer en espacios de Fréchet. Usando la técnica de medidas de nocompacidad junto con el teorema de punto fijo de Darbo, presentamos criterios suficientes para asegurar la controlabilidad del problema dado. Se discute, además, un ejemplo ilustrativo.

Keywords and Phrases: Integrodifferential equation, mild solution, measures of noncompactness, resolvent operator controllability, fixed point theorem, Fréchet space.

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1 Introduction

In recent years, the theory of fractional differential equations has been extensively developed by many authors. For a detailed account of the subject, we refer the reader to [1, 3, 4, 33]. Hernández and O'Regan initiated the theory of abstract impulsive differential equations with non-instantaneous impulses in [21]. Later, the authors studied instantaneous and non-instantaneous impulsive integrodifferential equations in Banach spaces in [2].

The controllability of linear and nonlinear differential systems in finite dimensional spaces received considerable attention, for example, see [8, 9, 10], while some interesting results on the controllability of such systems in infinite-dimensional Banach spaces with unbounded operators can be found in the monographs [10, 12, 23, 31]. For more details on the subject, see the papers [5, 13, 19, 20, 32] and the references cited therein. Lasiecka and Triggiani [22] discussed the exact controllability of semilinear abstract systems with application to waves and plates boundary control problems. For some results on evolution equations, for instance, see [1, 11, 28, 29].

Recently, in [15], the authors used Schauder's fixed point theorem to study the existence of mild solutions by considering two cases of the resolvent operators for the following integrodifferential problem:

$$\begin{cases} \xi'(t) = \Psi_1\xi(t) + \int_0^t \Psi_2(t-\theta)\xi(\theta)\,d\theta + \wp\left(t,\xi(t),(H\xi)(t)\right); & \text{if } t\in[0,a],\\ \xi(0) = g(\xi)+\xi_0. \end{cases}$$

Motivated by the works [2, 15], we will investigate the existence and controllability of mild solutions to the following impulsive integrodifferential equations via resolvent operators:

$$\begin{cases} \xi'(t) = \Psi_1\xi(t) + \wp\left(t,\xi(t),(H\xi)(t)\right) + \int_0^t \Psi_2(t-\theta)\xi(\theta)\,d\theta; & \text{if } t\in\Theta_j; \quad j=0,1,\ldots,\\ \xi(t) = \varpi_j\left(t,\xi\left(t_j^-\right)\right); \quad \text{if } t\in\tilde{\Theta}_j, \quad j=1,2,\ldots,\\ \xi(0)=\xi_0, \end{cases} \tag{1.1}$$

where $\Theta_0 = [0,\ t_1], \Theta_j := (\theta_j, t_{j+1}]$ and $\tilde{\Theta}_j = (t_j,\theta_j]$ with $0=\theta_0 < t_1 \le \theta_1 \le t_2 < \cdots < \theta_{\ell-1} \le t_\ell \le \theta_\ell \le t_{\ell+1} \le \cdots \le +\infty$, $\Psi_1 : D(\Psi_1)\subset \Xi \to \Xi$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\ge 0}$, $\Psi_2(t)$ is a closed linear operator with domain $D(\Psi_1)\subset D(\Psi_2(t))$, the operator H is defined by

$$(H\xi)(t) = \int_0^a \hbar(t,\theta,\xi(\theta))\,d\theta,$$

for $a>0,\ D_\hbar = \{(t,\theta)\in\mathbb{R}^2\ ;\ 0\le\theta\le t\le a\}$ and $\hbar : D_\hbar\times\Xi\to\Xi$. The nonlinear term

$\wp : \Theta_j \times \Xi \times \Xi \to \Xi;\ j = 0, \ldots,\ \varpi_j : \tilde{\Theta}_j \times \Xi \to \Xi;\ j = 1, 2, \ldots,$ are given functions, where $\Theta = [0, +\infty)$, and $(\Xi, \|\cdot\|)$ is a Banach space, $\xi_0 \in \Xi$.

We emphasize that the novelty of our work includes the investigation of problem (1.1) under a diverse set of conditions. Specifically, we incorporated non-instantaneous impulses in the integrodifferential system on an unbounded domain to broaden its scope, in contrast to previous research efforts. The controllability of the given integrodifferential problem with non-instantaneous impulses is also studied. Our results generalize the ones presented in the articles [2, 15].

The rest of this paper is organized as follows. In Section 2, we recall some preliminary results and definitions related to our study. In Section 3, we will present the existence result by using the technique of measures of noncompactness in conjunction with the Darbo's fixed point theorem. We will also study the controllability for the given problem. An example is given to illustrate the applicability of the abstract results.

2 Preliminaries

Let us begin this section with some preliminary concepts related to the study of the problem at hand. Let $C(\Theta, \Xi)$ be the space of continuous functions from $\Theta := [0; +\infty)$ into Ξ and $B(\Xi)$ denotes the space of all bounded linear operators from Ξ into Ξ equipped with the norm

$$\|T\|_{B(\Xi)} = \sup\{\|T(\xi)\| \ :\ \|\xi\| = 1\}.$$

A measurable function $\xi : [0; +\infty) \to \Xi$ is Bochner integrable if and only if $\|\xi\|$ is Lebesgue integrable. For the properties of the Bochner integral, for instance, see [30].

Let $L^1([0; +\infty), \Xi)$ denote the Banach space of measurable functions $\xi : [0; +\infty) \to \Xi$ which are Bochner integrable, with the norm

$$\|\xi\|_{L^1} = \int_0^{+\infty} \|\xi(t)\|\, dt.$$

We consider the following Cauchy problem

$$\begin{cases} \xi'(t) = \Psi_1 \xi(t) + \displaystyle\int_0^t \Psi_2(t-\theta)\xi(\theta)\, d\theta, & \text{for } t \geq 0, \\ \xi(0) = \xi_0 \in \Xi. \end{cases} \tag{2.1}$$

The existence and properties of the resolvent operator have been discussed in [18]. In what follows, we suppose the following assumptions:

$(R1)$ Ψ_1 is the infinitesimal generator of a uniformly continuous semigroup $\{T(t)\}_{t>0}$;

$(R2)$ For all $t \geq 0, \Psi_2(t)$ is a closed linear operator from $\mathcal{D}(\Psi_1)$ to Ξ and $\Psi_2(t) \in \Psi_2(D(\Psi_1), \Xi)$. For any $\xi \in D(\Psi_1)$, the map $t \to \Psi_2(t)\xi$ is bounded, differentiable and the derivative $t \to \Psi_2{}'(t)\xi$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 2.1 ([18]). *If the assumptions $(R1)$ and $(R2)$ are satisfied, then the problem (2.1) has a unique resolvent operator.*

Let $\{t_i\}_{i=0}^{\infty}$ be the sequence of real numbers such that

$$0 = t_0 < t_1 < t_2 < \cdots, \text{ and } \lim_{i \to +\infty} t_i = +\infty.$$

Let $PC(\mathbb{R}^+, \Xi)$ be the Banach space defined by

$$PC(\mathbb{R}^+, \Xi) = \Big\{ \xi : \mathbb{R}^+ \to \Xi : \xi|_{\tilde{\Theta}_j} = \varpi_j; j = 1, \ldots, \ell,\ \xi|_{\Theta_j}; j = 0, \ldots, \ell, \text{ are continuous } \xi\left(\theta_j^-\right),\ \xi\left(\theta_j^+\right),\ \xi\left(t_j^-\right) \text{ and } \xi\left(t_j^+\right) \text{ exist with } \xi\left(t_j^-\right) = \xi\left(t_j\right) \Big\},$$

endowed with the family of seminorms:

$$\|x\|_n = \sup\{\|x(t)\| \ : \ t \in [0, t_n]\},\ n = 1, 2, \ldots$$

Define by $\mathfrak{F} = C(\Theta, \Xi)$ the Fréchet space of continuous functions $\Im$ from $\mathbb{R}_+$ into Ξ, with the norm

$$\|\Im\|_n = \sup_{t \in \tilde{\Theta}_n} \|\Im(t)\|, \quad \tilde{\Theta}_n := [0, n],\ n \in \mathbb{N},$$

and the distance

$$d(\xi, \Im) = \sum_{n=1}^{\infty} \frac{2^{-n}\|\xi - \Im\|_n}{1 + \|\xi - \Im\|_n}; \quad \xi, \Im \in C(\mathbb{R}_+, \Xi).$$

Let χ represent the Kuratowski measure of noncompactness in Ξ. The properties of χ can be found in [6].

Definition 2.2 ([16]). *Let $\daleth_{\mathfrak{F}}$ be the family of all nonempty and bounded subsets of a Fréchet space $\mathfrak{F}$. A family of functions $\{\chi_n\}_{n \in \mathbb{N}}$, where $\chi_n : \daleth_{\mathfrak{F}} \to [0, \infty)$ is a family of measures of noncompactness in the real Fréchet space $\mathfrak{F}$, if for all Ω, Ω_1 and $\Omega_2 \in \daleth_{\mathfrak{F}}$, the following conditions are satisfied:*

(C_1) *$\{\chi_n\}_{n \in \mathbb{N}}$ is full, that is $\chi_n(\Omega) = 0$ for $n \in \mathbb{N}$ if and only if Ω is precompact;*

(C_2) *$\chi_n(\Omega_1) < \chi_n(\Omega_2)$, for $\Omega_1 \subset \Omega_2$ and $n \in \mathbb{N}$;*

(C_3) *$\chi_n(\operatorname{Conv} \Omega) = \chi_n(\Omega)$, for $n \in \mathbb{N}$;*

(C_4) *If* $\{\Omega_i\}$ *is a sequence of closed sets from* $\beth_{\mathfrak{F}}$*, such that* $\Omega_{i+1} \subset \Omega_i, i = 1, \ldots,$ *and* $\lim\limits_{i\to\infty} \chi_n(\Omega_i) = 0$*, for each* $n \in \mathbb{N}$*, then the intersection set* $\Omega_\infty = \bigcap_{i=1}^{\infty} \Omega_i$ *is nonempty.*

Example 2.3. *For* $\Omega \in \beth_{\mathfrak{F}},\ x \in \Omega,\ n \in \mathbb{N}$ *and* $\epsilon > 0$*, let us denote by* $\beta^n(x, \epsilon)$ *for* $n \in \mathbb{N}$*, the modulus of continuity of the function* x *on the interval* $\tilde{\Theta}_n$ *defined by*

$$\beta^n(x, \epsilon) = \sup\{|x(t) - x(\theta)| \ ; \ t, \theta \in \tilde{\Theta}_n \quad |t - \theta| < \epsilon\}.$$

Further, let us set

$$\beta^n(\Omega, \epsilon) = \sup\{\beta^n(x, \epsilon) \ ; \ x \in \Omega\}, \quad \beta_0^n(\Omega) = \lim_{\epsilon\to 0^+} \beta^n(\Omega, \epsilon)$$

and

$$\alpha_n(\Omega) = \beta_0^n(\Omega) + \sup_{t\in\tilde{\Theta}_n} \chi\big(\Omega(t)\big).$$

If the family of mappings $\{\alpha_n\}_{n\in\mathbb{N}}$*, where* $\alpha_n : \beth_{\mathfrak{F}} \to \Theta$*, satisfies the conditions* (C_1)*–*(C_4)*, then the family of maps* $\{\alpha_n\}_{n\in\mathbb{N}}$ *defined above is a family of measures of noncompactness in the Fréchet space* $\mathfrak{F}$*.*

Definition 2.4 ([27])**.** *A nonempty subset* $\Omega \subset \mathfrak{F}$ *is bounded if, for* $n \in \mathbb{N}$*, there exists* $\beth_n > 0$*, such that*

$$\|\xi\|_n \leq \beth_n, \ \text{for each } \xi \in \Omega.$$

Lemma 2.5 ([16])**.** *If* $\mathfrak{M}$ *is a bounded subset of a Banach space* Ξ*, then for each* $\epsilon > 0$*, there is a sequence* $\{\xi_j\}_{j=1}^{\infty} \subset \mathfrak{M}$ *such that*

$$\chi(\mathfrak{M}) \leq 2\chi\left(\{\xi_j\}_{j=1}^{\infty}\right) + \epsilon.$$

Lemma 2.6 ([24])**.** *If* $\{\xi_j\}_{j=0}^{\infty} \subset L^1$ *is uniformly integrable, then the function* $t \to \alpha(\{\xi_j(t)\}_{j=0}^{\infty})$ *is measurable and*

$$\chi\left(\left\{\int_0^t \xi_j(\theta)d\theta\right\}_{j=0}^{\infty}\right) \leq 2\int_0^t \chi\left(\{\xi_j(\theta)\}_{j=0}^{\infty}\right)d\theta, \ \text{for } t \in \tilde{\Theta}_n, \ n \in \mathbb{N}.$$

For more details about measures of noncompactness, see [7, 16, 17].

3 Main results

In this subsection, we discuss the existence of mild solutions for the problem (1.1).

3.1 Existence of mild solutions

Definition 3.1. *A function $\xi \in PC(\mathbb{R}^+, \Xi)$ is called a mild solution to the problem (1.1) if it satisfies*

$$\xi(t) = \begin{cases} R(t)\xi_0 + \int_0^t R(t-\theta)\wp(\theta, \xi(\theta), (H\xi)(\theta))\, d\theta; & \text{if } t \in \Theta_0, \\ R(t-\theta_j)\left[\varpi_j(\theta, \xi(\theta_j^-))\right] + \int_{\theta_j}^t R(t-\theta)\wp(\theta, \xi(\theta), (H\xi)(\theta))\, d\theta; & \text{if } t \in \Theta_j, \\ \varpi_j(t, \xi(t_j^-)); & \text{if } t \in \tilde{\Theta}_j, \end{cases}$$

where $j = 1, 2, \dots$

In the sequel, we need the following hypotheses.

$(A1)$ (i) $\wp : \Theta \times \Xi \times \Xi \to \Xi$ is a Carathéodory function and there exist a function $p \in L^1(\Theta, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : \Theta \to (0, +\infty)$, such that

$$||\wp(t, \xi, \bar{\xi})|| \le p(t)\psi(\|\xi\| + \|\bar{\xi}\|), \quad \text{for } \xi,\ \bar{\xi} \in \Xi.$$

(ii) There exists a function $l_\wp \in L^1(\Theta, \mathbb{R}^+)$ such that for any bounded set $B \subset \Xi$ and $t \in \Theta$, we have

$$\chi(\wp(t, B, H(B))) \le l_\wp(t)\chi(B).$$

$(A2)$ The function $\hbar : D_\hbar \times \Xi \times \Xi \to \Xi$ is continuous and there exists $c_1 > 0$ such that,

$$\|\hbar(t, \theta, \xi) - \hbar(t, \theta, \bar{\xi})\| \le c_1\|\xi - \bar{\xi}\|, \quad \text{for each } (t, \theta) \in D_\hbar \text{ and } \xi,\ \bar{\xi} \in \Xi,$$

with

$$\hbar^* = \sup\{\|\hbar(t, \theta, 0)\|\ ,\ (t, \theta) \in D_\hbar\} < \infty.$$

$(A3)$ $\varpi_j : \tilde{\Theta}_j \times \Xi \to \Xi$ are continuous and there exist positive constants $L_{\varpi_j}, j \in \mathbb{N}$ and $\tau > 1$

such that

$$\|\varpi_j(.,\xi) - \varpi_j(.,\Im)\| \leq \frac{L_{\varpi_j}}{\tau}\|\xi - \Im\|, \quad \text{for all } \xi, \Im \in \Xi,\ j = 1, 2, \ldots$$

$(A4)$ Assume that $(R1) - (R2)$ hold, and there exist $\gimel_R \geq 1$ and $b \geq 0$, such that

$$\|R(t)\|_{B(\Xi)} \leq \gimel_R e^{-bt}.$$

Using the methods employed in [25, 26], we can verify that the following example contains a family of measures of noncompactness in $PC(\mathbb{R}^+, \Xi)$:

$$\chi_n(\Pi) = \max_{i=0,\ldots,\ell} \beta_0(\gamma_i^p, \Pi) + \sup_{t\in\tilde{\Theta}_n} \left\{ e^{-\tau\widetilde{\zeta}(t)} \chi(\Pi(t)) \right\}; \quad p = 0, 1, 2 \text{ and } \ell = 0, 1, \ldots,$$

with γ_i^p a partition of $\mathbb{R}^+$. In particular,

$$\gamma_i^p = \begin{cases} \Theta_0; & \text{if } p = 0,\ \ell = 0, \\ \Theta_\ell; & \text{if } p = 1,\ \ell = 1, 2, \ldots, \\ \tilde{\Theta}_\ell; & \text{if } p = 2,\ \ell = 1, 2, \ldots, \end{cases}$$

and $\widetilde{\zeta}(t) = \int_0^t \zeta(\theta)\, d\theta,\ \zeta(t) = 4\gimel_R l(t),\ \tau > 1$, where $\Pi(t) = \{\pi(t) \in \mathfrak{F}\ ;\ \pi \in \Pi\}$, $t \in \tilde{\Theta}_n$. Moreover, if the set Π is equicontinuous, then $\beta_0(\gamma_i^p, \Pi) = 0$.

Theorem 3.2. *If the conditions $(A1) - (A4)$ are satisfied and*

$$\gimel_R L_{\varpi_j} < \tau,$$

then the system (1.1) has at least one mild solution.

Proof. Transform the problem (1.1) into a fixed point problem by introducing an operator $\aleph : PC(\mathbb{R}^+, \Xi) \to PC(\mathbb{R}^+, \Xi)$ as

$$\aleph\xi(t) = \begin{cases} R(t)\xi_0 + \int_0^t R(t-\theta)\wp(\theta, \xi(\theta), (H\xi)(\theta))\, d\theta; & \text{if } t \in \Theta_0, \\ R(t-\theta_j)\left[\varpi_j(\theta, \xi(\theta_j^-))\right] + \int_{\theta_j}^t R(t-\theta)\wp(\theta, \xi(\theta), (H\xi)(\theta))\, d\theta; & \text{if } t \in \Theta_j, \\ \varpi_j(t, \xi(t_j^-)); & \text{if } t \in \tilde{\Theta}_j, \end{cases}$$

where $j = 1, 2, \ldots$ Clearly, the fixed points of the operator $\aleph$ are mild solutions to the problem (1.1). Next, we verify that the operator $\aleph$ satisfies the hypothesis of Darbo's fixed point theorem [16].

Let $\delta_n > 0$ and $D_{\delta_n} = \{\xi \in PC(\mathbb{R}^+, \Xi);\ \|\xi\|_n \leq \delta_n\}$, where

$$\max\left\{\beth_R(\|\xi_0\| + \psi(K^*_{\delta_n})), \frac{\beth_R(\varpi_0 + \psi(K^*_{\delta_n}))}{1 - \frac{\beth_R L_{\varpi_j}}{\tau}}\right\} \leq \delta_n,$$

and

$$K^*_{\delta_n} = \big((c_1+1)\delta_n + a\hbar^*\big)\|p\|_{L^1}.$$

Notice that the set D_{δ_n} is bounded, closed and convex.

Step 1: $\aleph(D_{\delta_n}) \subset D_{\delta_n}$.

- **Case 1:** For any $n \in \mathbb{N}$, $\xi \in D_{\delta_n}$, $t \in \Theta_0 \cap \tilde{\Theta}_n$, it follows by $(A1)$ that

$$\begin{aligned}\|\aleph\xi(t)\| &\leq \beth_R\|\xi_0\| + \beth_R\int_0^t \psi(\|\xi(\theta)\| + \|H\xi(\theta)\|)p(\theta)\,d\theta \\ &\leq \beth_R\|\xi_0\| + \beth_R\psi((c_1+1)\delta_n + a\hbar^*)\|p\|_{L^1}.\end{aligned}$$

  Then we have

$$\|\aleph\xi\|_n \leq \beth_R\Big[\|\xi_0\| + \psi((c_1+1)\delta_n + a\hbar^*)\|p\|_{L^1}\Big].$$

- **Case 2:** For $t \in \Theta_j \cap \tilde{\Theta}_n$ and for each $\xi \in D_{\delta_n}$, by $(A1), (A2)$ and $(A3)$, we have

$$\|\varpi_j(\cdot, \xi(\cdot))\| \leq \frac{L_{\varpi_j}}{\tau}\|\xi(\cdot)\| + \varpi_0,$$

  and

$$\|\aleph\xi\|_n \leq \beth_R\left[\frac{L_{\varpi_j}}{\tau}\delta_n + \varpi_0 + \psi((c_1+1)\delta_n + a\hbar^*)\|p\|_{L^1}\right].$$

- **Case 3:** For $t \in \tilde{\Theta}_j \cap \tilde{\Theta}_n$, and for each $\xi \in D_{\delta_n}$, by $(A3)$, we have

$$\|\aleph\xi\|_n \leq \frac{L_{\varpi_j}}{\tau}\delta_n + \varpi_0.$$

Thus,

$$\|\aleph\xi\|_n \leq \delta_n.$$

Step 2: $\aleph$ is continuous.

Let ξ_ℓ be a sequence such that $\xi_\ell \to \xi_*$ in Ξ. We complete the proof in several steps.

- **Case 1:** For $t \in \Theta_0 \cap \tilde{\Theta}_n$, we have

$$\|(\aleph\xi_\ell)(t) - (\aleph\xi_*)(t)\| \leq \daleth_R \int_{\theta_j}^{t} \|\wp(\theta, \xi_\ell(\theta), H\xi_\ell(\theta)) - \wp(\theta, \xi_*(\theta), H\xi_*(\theta))\| \, d\theta.$$

It follows by continuity of $\hbar$ and $\wp$ that

$$\hbar(t, \theta, \xi_\ell(\theta)) \to \hbar(t, \theta, \xi_*(\theta)) \quad \text{as } \ell \to +\infty,$$

and

$$\|\hbar(t, \theta, \xi_\ell(\theta)) - \hbar(t, \theta, \xi_*(\theta))\| \leq c_1 \|\xi_\ell - \xi_*\|.$$

By the Lebesgue dominated convergence theorem, we obtain

$$\int_0^t \hbar(t, \theta, \xi_\ell(\theta)) \, d\theta \to \int_0^t \hbar(t, \theta, \xi_*(\theta)) \, d\theta, \quad \text{as } \ell \to +\infty.$$

Then, by $(A1)$, we get

$$\wp(\theta, \xi_\ell(\theta), H\xi_\ell(\theta)) \to \wp(\theta, \xi_*(\theta), H\xi_*(\theta)), \quad \text{as } \ell \to +\infty,$$

which implies that

$$\|(\aleph\xi_\ell) - (\aleph\xi_*)\|_n \to 0, \quad \text{as } \ell \to +\infty.$$

- **Case 2:** Let $t \in \Theta_j \cap \tilde{\Theta}_n$. Then we have

$$\begin{aligned} \|(\aleph\xi_\ell)(t) - (\aleph\xi_*)(t)\| \leq\ & \daleth_R \|\varpi_j(\theta_j, \xi_\ell(\theta_j^-)) - \varpi_j((\theta_j, \xi_*(\theta_j^-)))\| \\ & + \daleth_R \int_0^t \|\wp(\theta, \xi_\ell(\theta), H\xi_\ell(\theta)) - \wp(\theta, \xi_*(\theta), H\xi_*(\theta))\| \, d\theta. \end{aligned}$$

As argued in case 1, by the continuity of $\hbar, \wp$ and ϖ_j, we get

$$\|(\aleph\xi_\ell) - (\aleph\xi_*)\|_n \to 0, \quad \text{as } \ell \to +\infty.$$

- **Case 3:** For $t \in \tilde{\Theta}_j \cap \tilde{\Theta}_n$, we obtain

$$\|(\aleph\xi_\ell)(t) - (\aleph\xi_*)(t)\| \leq \|\varpi_j(t_j, \xi_\ell(t_j^-)) - \varpi_j((t_j, \xi_*(t_j^-)))\|,$$

which, in view of the continuity of ϖ_j, implies that

$$\|(\aleph\xi_\ell) - (\aleph\xi_*)\|_n \to 0, \quad \text{as } \ell \to +\infty.$$

Thus, $\aleph$ is continuous.

Step 3 Since we have $\aleph(D_{\delta_n}) \subset D_{\delta_n}$, therefore, $\aleph(D_{\delta_n})$ is bounded.

Step 4 Let Π be a bounded equicontinuous subset of D_{δ_n}, then $\{\aleph(\Pi)\}$ is equicontinuous, which implies that $\beta_0(\gamma_i^p, \aleph(\Pi)) = 0$. Now, for any $\varrho > 0$, there exists a sequence $\{\xi_\ell\}_{j=0}^{\infty} \subset \Pi$ and we complete the proof of this part in certain steps.

- **Case 1:** Let $t \in \Theta_0 \cap \tilde{\Theta}_n$. Setting $O_{fv(\theta)} = \wp(\theta, \xi(\theta), H\xi(\theta))$, we have

$$\begin{aligned}
\chi\Big\{\int_0^t R(t-\theta)O_{fv(\theta)}d\theta \ ; \ \xi \in \Pi\Big\} &\leq 2\chi\left\{\int_0^t R(t-\theta)O_{f\xi_\ell(\theta)}d\theta \ ;\xi \in \Pi\right\} + \varrho \\
&\leq 4\int_0^t \gimel_R l_\wp(\theta)\chi(\{\Pi(\theta)\})d\theta + \varrho \\
&\leq \int_0^t \zeta(\theta)\chi(\Pi(\theta))d\theta + \varrho \\
&\leq \int_0^t e^{\tau\widetilde{\zeta}(\theta)}e^{-\tau\widetilde{\zeta}(\theta)}\zeta(\theta)\chi(\Pi(\theta))d\theta + \varrho \\
&\leq \int_0^t \zeta(\theta)e^{\tau\widetilde{\zeta}(\theta)} \sup_{\theta\in[0,t]} e^{-\tau\widetilde{\zeta}(\theta)}\chi(\Pi(\theta))d\theta + \varrho \\
&\leq \chi_n(\Pi)\int_0^t \left(\frac{e^{\tau\widetilde{\zeta}(\theta)}}{\tau}\right)' d\theta + \varrho \\
&\leq \frac{e^{\tau\widetilde{\zeta}(t)}}{\tau}\chi_n(\Pi) + \varrho,
\end{aligned}$$

  which implies that

$$\chi(\aleph(\Pi)(t)) \leq \frac{e^{\tau\widetilde{\zeta}(t)}}{\tau}\chi_n(\Pi) + \varrho.$$

  Since ϱ is arbitrary, so

$$\chi(\aleph(\Pi)(t)) \leq \frac{e^{\tau\widetilde{\zeta}(t)}}{\tau}\chi_n(\Pi),$$

  and hence

$$\chi_n(\aleph(\Pi)) \leq \frac{1}{\tau}\chi_n(\Pi).$$

- **Case 2:** For $t \in \Theta_j \cap \widetilde{\Theta}_n$, we proceed as in the proof of Case 1 to obtain

$$\begin{aligned}
\chi(\aleph(\Pi)(t)) &\leq \gimel_R \ \chi(\{\varpi_j(\theta, \xi_\ell(\theta_j^-)); \xi \in \Pi\}) + \frac{e^{\tau\widetilde{\zeta}(t)}}{\tau}\chi_n(\Pi) + \varrho \\
&\leq \frac{e^{\tau\widetilde{\zeta}(t)}(\gimel_R L_{\varpi_j} + 1)}{\tau}\chi_n(\Pi) + \varrho.
\end{aligned}$$

Therefore, we get

$$\chi_n(\aleph(\Pi)) \quad \leq \quad \frac{(\daleth_R L_{\varpi_j}+1)}{\tau}\chi_n(\Pi).$$

- **Case 3:** Let $t \in \tilde{\Theta}_j \cap \tilde{\Theta}_n$. By $(A3)$, the set $\left\{\varpi_j(t,\xi_j^-)\right\}_{j=1}^n$ is equicontinuous, and that $\beta_0\left(\gamma_i^p, G(\Pi)\right)=0$, with $\{Gv(t)\}=\left\{\varpi_j(t,\xi_j^-)\right\}$.
  On the other hand, we have

  $$\|\varpi_j(t,\xi(.))-\varpi_j(t,\overline{\xi}(.))\| \leq \frac{L_{\varpi_j}}{\tau}\|\xi(.)-\overline{\xi}(.)\|,$$

  which yields

  $$e^{-\tau\widetilde{\zeta}(t)}\|\varpi_j(t,\xi(t_j^-))-\varpi_j(t,\overline{\xi}(t_j^-))\| \leq \frac{L_{\varpi_j}}{\tau}e^{-\tau\widetilde{\zeta}(t)}\|\xi(t_j^-)-\overline{\xi}(t_j^-)\|.$$

  Hence, we get

  $$\chi_n(\aleph(\Pi)) \quad \leq \quad \frac{L_{\varpi_j}}{\tau}\chi_n(\Pi),$$

which shows that $\aleph$ is contraction (in terms of a measure of noncompactness), since $\daleth_R L_{\varpi_j}+1<\tau$. Therefore, by Darbo's fixed point theorem [16], we deduce that $\aleph$ has at least one fixed point which is a mild solution to the problem (1.1). □

3.2 Controllability of the system

In this subsection, we discuss the controllability for the system:

$$\begin{cases} \xi'(t) &= \Psi_1(t)\xi(t)+\wp\left(t,\xi(t),(H\xi)(t)\right) \\ & \quad +\displaystyle\int_0^t \Psi_2(t-\theta)\xi(\theta)\,d\theta + Cu(t); \quad \text{if } t\in\Theta_j, \quad j=0,1,\dots, \\ \xi(t) &= \varpi_j\left(t,\xi\left(t_j^-\right)\right); \quad \text{if } t\in\tilde{\Theta}_j, \quad j=1,2,\dots, \\ \xi(0) &= \xi_0, \end{cases} \tag{3.1}$$

where $u\in L^2(\Theta,\mathfrak{S})$ is the control function, $\mathfrak{S}$ is the Banach space of admissible control functions and C is a bounded linear operator from $\mathfrak{S}$ into Ξ. Before proceeding further, we define the solution for the problem (3.1).

Definition 3.3. *The system (3.1) is said to be controllable on the interval Θ, if for every initial function $\xi_0=\xi(0)\in\Xi$ and $\hat{\hat{\xi}}\in\Xi$, there is some control $u\in L^2([0;n];\Xi)$ for some $n>0$, such that the mild solution $\xi(\cdot)$ of the system (3.1) satisfies the terminal condition $\xi(n)=\hat{\hat{\xi}}$.*

To obtain the controllability of mild solutions of (3.1), we assume the following conditions.

$(A5)$ There exists a positive constant ρ_n, such that

$$\max\left\{\varphi_1^\rho; \varphi_2^\rho; \frac{\varpi_0}{1-\frac{L_{\varpi_j}}{\tau}}\right\} \leq \rho_n,$$

where

$$\varphi_1^\rho = \left\{\beth_R\left[\|\xi_0\| + \psi(K^*_{\rho_n})\|p\|_{L^1} + c_5c_6\left(\frac{\rho_n}{\beth_R} + \|\xi_0\| + \psi(K^*_{\rho_n})\|p\|_{L^1}\right)\right]\right\},$$

$$\varphi_2^\rho = \left\{\beth_R\left[\frac{L_{\varpi_j}}{\tau}\rho_n + \varpi_0 + \psi(K^*_{\rho_n})\|p\|_{L^1} + c_5c_6\left(\frac{\rho_n}{\beth_R} + \|\xi_0\| + \psi(K^*_{\rho_n})\|p\|_{L^1}\right)\right]\right\},$$

and

$$K^*_{\rho_n} = \big((c_1+1)\rho_n + a\hbar^*\big)\|p\|_{L^1}.$$

$(A6)$ (i) For each n, the linear operator $W : L^2(\tilde{\Theta}_n, \mathfrak{S}) \to \mathfrak{F}$, defined by

$$Wu = \int_0^n R(n-\theta)Cu(\theta)\,d\theta,$$

has a pseudo inverse operator W^{-1}, which takes values in $L^2(\tilde{\Theta}_n, \mathfrak{S})\setminus \ker(W)$.

(ii) There exist positive constants c_5, c_6, such that

$$\|C\| \leq c_5 \quad \text{and} \quad \|W^{-1}\| \leq c_6.$$

(iii) There exist $p_w \in L^1(\Theta, \mathbb{R}^+)$, $k_C \geq 0$, and for any bounded sets $V_1 \subset \Xi$, $V_2 \subset \mathfrak{S}$,

$$\chi((W^{-1}V_1)(t)) \leq p_w(t)\chi(V_1), \qquad \chi((CV_2)(t)) \leq k_C\chi(V_2(t)).$$

Theorem 3.4. *Suppose that the hypotheses* $(A1)-(A5)$ *hold. Then the problem (3.1) is controllable.*

Proof. For $n \in \mathbb{N}$, we define a family of measures of non compactness in $PC(\Theta, \mathfrak{F})$ as

$$\widetilde{\chi}_n(\Pi) = \max_{i=0,\ldots,\ell} \beta_0\left(\gamma_i^p, \Pi\right) + \sup_{t\in\tilde{\Theta}_n}\left\{e^{-\tau\widetilde{\varkappa}(t)}\chi(\Pi(t))\right\}, \quad p = 0,1,2 \quad \text{and } \ell = 0,1,\ldots,$$

where $\widetilde{\varkappa}(t) = \displaystyle\int_0^t \varkappa(\theta)\,d\theta$, $\varkappa(t) = 4\beth_R(l_\wp(t) + k_C(\beth_R\|l_\wp\|^1)p_w(t))$, $\tau > 1$. Using $(A5)$, we define the control:

$$u_\xi(t) = \begin{cases} W^{-1}\Big[\xi(n) - R(n)\xi_0 - \displaystyle\int_0^n R(n-\theta)\wp(\theta,\xi(\theta),(H\xi)(\theta))\,d\theta\Big]; & \text{if } t\in\Theta_0,\\ W^{-1}\Big[\xi(n) - R(n-\theta_j)\left[\varpi_j(\theta,\xi(\theta_j^-))\right] & \\ -\displaystyle\int_{\theta_j}^n R(t-\theta)\wp(\theta,\xi(\theta),(H\xi)(\theta))\,d\theta\Big]; & \text{if } t\in\Theta_j, \quad j=1,2,\dots \end{cases}$$

Using the above control, it will be shown that the operator defined by

$$\Upsilon\xi(t) = \begin{cases} R(t)\xi_0 + \displaystyle\int_0^t R(t-\theta)\wp(\theta,\xi(\theta),(H\xi)(\theta))\,d\theta + \int_0^t R(t-\theta)Cu_\xi(\theta)\,d\theta; \quad \text{if } t\in\Theta_0,\\ R(t-\theta_j)\left[\varpi_j(\theta,\xi(\theta_j^-))\right] + \displaystyle\int_{\theta_j}^t R(t-\theta)\wp(\theta,\xi(\theta),(H\xi)(\theta))\,d\theta \\ +\displaystyle\int_{\theta_j}^t R(t-\theta)Cu_\xi(\theta)\,d\theta; \quad \text{if } t\in\Theta_j, \quad j=1,2,\dots,\\ \varpi_j(t,\xi(t_j^-)); \quad \text{if } t\in\tilde\Theta_j, \quad j=1,2,\dots, \end{cases}$$

has a fixed point which is a mild solution to the system (3.1), and hence the system is controllable. By $(A4)$, we define a closed, bounded and convex subset B_{ρ_n} for any $n\in\mathbb{N}$ as follows: $B_{\rho_n} = B(0,\rho_n) = \{x\in PC \ : \ \|x\|_n \le \rho_n\}$. We establish the proof in several steps.

Step 1: $\aleph(B_{\rho_n}) \subset B_{\rho_n}$. For any $\xi\in B_{\rho_n}$, we accomplish the following cases by using the assumptions $(A1),(A4)$ and $(A5)$.

- **Case 1:** Let $t\in\Theta_0\cap\tilde\Theta_n$. For any $n\in\mathbb{N}$, $\xi\in B_{\rho_n}$, $t\in\Theta_0\cap\tilde\Theta_n$, it follows by $(A1)$ that

$$\begin{aligned}\|\Upsilon\xi(t)\| &\le \beth_R\Big(\|\xi_0\| + \psi((c_1+1)\rho_n + a\hbar^*)\|p\|_{L^1} + c_5c_6\big(\frac{\rho_n}{\beth_R} + \|\xi_0\| + \psi(K^*_{\rho_n})\|p\|_{L^1}\big)\Big)\\ &\le \rho_n.\end{aligned}$$

- **Case 2:** For $t\in\Theta_j\cap\tilde\Theta_n$, and for each $\xi\in B_{\rho_n}$, by $(A1),(A2)$ and $(A3)$, we obtain

$$\begin{aligned}\|\Upsilon\xi(t)\| &\le \beth_R\Big[\frac{L_{\varpi_j}}{\tau}\rho_n + \varpi_0 + \psi((c_1+1)\delta_n + a\hbar^*)\|p\|_{L^1}\\ &\quad + c_5c_6\left(\frac{\rho_n}{\beth_R} + \|\xi_0\| + \psi(K^*_{\rho_n})\|p\|_{L^1}\right)\Big]\\ &\le \rho_n.\end{aligned}$$

- **Case 3:** Let $t\in\tilde\Theta_j\cap\tilde\Theta_n$. Then, for each $\xi\in B_{\rho_n}$, it follows by $(A3)$ that

$$\|\Upsilon\xi(t)\| \le \frac{L_{\varpi_j}}{\tau}\rho_n + \varpi_0 \le \rho_n.$$

Thus, we get

$$\|\Upsilon\xi\|_n \leq \rho_n,$$

which implies that $\Upsilon(B_{\rho_n}) \subset B_{\rho_n}$ and $\Upsilon(B_{\rho_n})$ is bounded.

Step 2: Υ is continuous on B_{ρ_n}. Let ξ_n be a sequence such that $\xi_n \to \xi_*$ in B_{ρ_n}. Since $\wp, \hbar, \varpi_j, C$ are continuous, therefore, it follows by the Lebesgue dominated convergence theorem that

$$\int_0^t R(t-\theta)Cu_{\xi_n}(\theta)\,d\theta \to \int_0^t R(t-\theta)Cu_{\xi_*}(\theta)\,d\theta,$$

which yields

$$\|(\Upsilon\xi_n) - (\Upsilon\xi_*)\|_n \to 0, \quad \text{as } n \to +\infty.$$

Thus, we deduce that Υ is continuous.

Step 3: Let Π be a bounded equicontinuous subset of B_{ρ_n}, then $\{\Upsilon(\Pi)\}$ is equicontinuous, and that $\beta_0(\gamma_i^p, \Upsilon(\Pi)) = 0$. Now, for any $\varrho > 0$, there exists a sequence $\{\xi_j\}_{j=0}^{\infty} \subset \Pi$. Then we complete the proof for several cases.

- **Case 1:** For $t \in \Theta_0 \cap \tilde{\Theta}_n$, we have

$$\begin{aligned}
\chi(\Upsilon(\Pi)(t)) &\leq 2\chi\left(\left\{\int_0^t R(t-\theta)(\wp(\theta,\xi_j(\theta),H\xi_j(\theta)) + u_{\xi_j}(\theta))\,d\theta\ ;\ \xi \in \Pi\right\}\right) + \varrho \\
&\leq 4\int_0^t \daleth_R(l_\wp(\theta) + k_C(\daleth_R\|l_\wp\|_L^1)p_w(\theta))\chi(\{\Pi(\theta)\})\,d\theta + \varrho \\
&\leq \frac{e^{\tau\widetilde{\varkappa}(t)}}{\tau}\widetilde{\chi}_n(\Pi) + \varrho.
\end{aligned}$$

  Since ϱ is arbitrary, we have

$$\chi(\Upsilon(\Pi)(t)) \leq \frac{e^{\tau L(t)}}{\tau}\widetilde{\chi}_n(\Pi),$$

  and hence

$$\widetilde{\chi}_n(\Upsilon(\Pi)) \leq \frac{1}{\tau}\widetilde{\chi}_n(\Pi).$$

- **Case 2:** Let $t \in \Theta_j \cap \tilde{\Theta}_n$. Then, as in the proof of Case 1, we get

$$\begin{aligned}
\chi(\Upsilon(\Pi)(t)) &\leq 4\int_0^t \daleth_R(l_\wp(\theta) + k_C(\daleth_R\|l_\wp\|_L^1)p_w(\theta))\chi(\{\Pi(\theta)\})d\theta + \varrho + \frac{\daleth_R L_{\varpi_j}}{\tau}\chi(\{\Pi(t)\}) \\
&\leq \frac{e^{\tau\widetilde{\varkappa}(t)}(\daleth_R L_{\varpi_j} + 1)}{\tau}\widetilde{\chi}_n(\Pi) + \varrho.
\end{aligned}$$

  Since ϱ is arbitrary, we obtain

$$\widetilde{\chi}_n(\Upsilon(\Pi)) \leq \frac{\daleth_R L_{\varpi_j} + 1}{\tau}\widetilde{\chi}_n(\Pi).$$

- **Case 3:** Let $t \in \tilde{\Theta}_j \cap \tilde{\Theta}_n$. Then, by $(A3)$, the set $\left\{\varpi_j(t, z_j^-)\right\}_{j=1}^n$ is equicontinuous, and that $\beta_0\left(\gamma_i^p, G(\Pi)\right) = 0$, with $\{Gz(t)\} = \left\{\varpi_j(t, z_j^-)\right\}$. On the other hand, we have

$$\|\varpi_j(t, z(.)) - \varpi_j(t, \overline{z}(.))\| \leq \frac{L_{\varpi_j}}{\tau}\|z(.) - \overline{z}(.)\|,$$

which implies that

$$e^{-\tau\widetilde{\zeta}(t)}\|\varpi_j(t, z(t_j^-)) - \varpi_j(t, \overline{z}(t_j^-))\| \leq \frac{L_{\varpi_j}}{\tau}e^{-\tau\widetilde{\zeta}(t)}\|z(t_j^-) - \overline{z}(t_j^-)\|.$$

Therefore, we have

$$\widetilde{\chi}_n(\Upsilon(\Pi)) \quad \leq \quad \frac{L_{\varpi_j}}{\tau}\widetilde{\chi}_n(\Pi),$$

which shows that Υ is contraction in view of the assumption

$$\daleth_R L_{\varpi_j} + 1 < \tau.$$

Hence, by Darbo's fixed point theorem [16], the operator Υ has a fixed point, which implies that the given system is controllable. □

4 An example

Consider the following impulsive integro-differential equations:

$$\begin{cases} \dfrac{\partial}{\partial t}\gamma(t,x) = -\dfrac{\partial}{\partial x}\gamma(t,x) - \pi\gamma(t,x) - \displaystyle\int_0^t \Gamma(t-\theta)\left(\dfrac{\partial}{\partial x}\gamma(\theta,x) + \pi\gamma(\theta,x)\right)d\theta \\ \qquad\qquad + \dfrac{\|\gamma(t,x)\|_{L^2}}{1+t^3\sin^2(t)} + \left(1+t^3\sin^2(t)\right)^{-1}\sin\left[\displaystyle\int_0^a \cos^2(\theta t)|\gamma(\theta,x)|\,d\theta\right] \\ \qquad\qquad + Cu(t,x), \quad \text{if } t \in \Theta_j,\ x \in (0,1), \\ \gamma(t,x) = \dfrac{\|\gamma(2j^- - 1, x)\|_{L^2}}{1 + 17(\|\gamma(2j^- - 1, x)\|_{L^2} + 1)}, \quad \text{if } t \in \tilde{\Theta}_j, \quad x \in (0,1), \\ \gamma(t,0) = \gamma(t,1) = 0, \quad t \in \mathbb{R}^+, \\ \gamma(0,x) = e^x, \quad x \in (0,1), \end{cases} \tag{4.1}$$

where $\Theta_0 = (0,1]$, $\Theta_j = (2j; 2j+1], j = 0,1,\ldots,$ $\tilde{\Theta}_j = (2j-1; 2j], j = 1,2,\ldots$ Set $\mathfrak{F} = L^2(0,1)$ and let Ψ_1 be defined by

$$(\Psi_1\varphi)(x) = -\left(\frac{d}{dx}\varphi(x) + \pi\varphi(x)\right),$$

and

$$D(\Psi_1) = \{\varphi \in L^2(0,1) \ / \ \varphi, \Psi_1\varphi \in L^2(0,1) \quad ; \quad \varphi(0) = \varphi(1) = 0\}.$$

The operator Ψ_1 is the infinitesimal generator of a C_0-semigroup on $\mathfrak{F}$ with domain $D(\Psi_1)$. Now, we define the operator $\Psi_2(t) : \mathfrak{F} \mapsto \mathfrak{F}$ as follows:

$$\Psi_2(t)z = \Gamma(t)\Psi_1 z, \quad \text{for } t \geq 0, \quad z \in D(\Psi_1).$$

As argued in [14], for some $r_2 > r_1 > 0$, it follows that $\|\Gamma(t)\| \leq \dfrac{e^{-r_2 t}}{r_1}$, and $\|\Gamma'(t)\| \leq \dfrac{e^{-r_2 t}}{r_1^2}$. From [18], we have that the resolvent operator $(R(t))_{t\geq 0}$ exists on $\mathfrak{F}$ which is norm continuous and $\|R(t)\| \leq e^{(r_1^{-1}-1)t}$. Therefore, the assumption $(A4)$ holds with $\beth_R = 1$ and $b = 1 - r_1^{-1}$. Now, we define

$$\gamma(t)(x) = \gamma(t,x),$$

$$\wp(t,\gamma(t),H\gamma(t))(x) = \frac{\|\gamma(t,x)\|_{L^2}}{1+t^3\sin^2(t)} + \left(1+t^3\sin^2(t)\right)^{-1}\sin\left[\int_0^a \cos^2(\theta t)|\gamma(\theta,x)|\,d\theta\right],$$

$$H\gamma(t)(x) = \int_0^a \cos^2(\theta t)|\gamma(\theta,x)|\,d\theta,$$

and

$$\varpi_j(t,\gamma(t_{j^-},x)) = \frac{\|\gamma(2j^- - 1,x)\|_{L^2}}{1+17(\|\gamma(2j^- - 1,x)\|_{L^2}+1)}.$$

Case 01: $Cu = 0$. With the above setting, the system (4.1) can be expressed in the following abstract form:

$$\begin{cases} \gamma'(t) = \Psi_1\gamma(t) + \wp\left(t,\gamma(t),(H\gamma)(t)\right) + \displaystyle\int_0^t \Psi_2(t-\theta)\gamma(\theta)\,d\theta, & \text{if } t \in \Theta_j, \\ \gamma(t) = \varpi_j\left(t,\gamma\left(t_j^-\right)\right), & \text{if } t \in \tilde{\Theta}_j, \\ \gamma(0) = \gamma_0. \end{cases} \tag{4.2}$$

On the other hand, we have

$$|\wp(t,\gamma_1(t),\gamma_2(t))| \leq (1+t^3\sin^2(t))^{-1}\Big(|\gamma_1(t)| + |\gamma_2(t)| + 1\Big).$$

Also, for any bounded set $\Sigma \subset \mathfrak{F}$, we have

$$\chi(\wp(t,\Sigma,H(\Sigma))) \leq (1+t^3\sin^2(t))^{-1}\chi(\Sigma).$$

So

$$p(t) = (1+t^3\sin^2(t))^{-1}, \text{ which certainly belongs to } L^1(\Theta,\mathbb{R}^+),$$

and $\psi(t) = 1+t$ is a continuous nondecreasing function from Θ to $[1,+\infty)$. Moreover, we have the estimates:

$$\|\hbar(t,\theta,\gamma_1) - \hbar(t,\theta,\gamma_2)\|_{\mathfrak{F}} \leq a\|\gamma_1 - \gamma_2\|_{\mathfrak{F}},$$

and

$$\|\varpi_j(\gamma_1) - \varpi_j(\gamma_2)\|_{\mathfrak{F}} \leq \frac{1}{18}\|\gamma_1 - \gamma_2\|_{\mathfrak{F}}.$$

For $\daleth_R < 3$, all the assumptions of Theorem 3.2 are satisfied. Hence, the problem (4.1) has at least one mild solution defined on $\mathbb{R}^+$.

Case 02 : $Cu = \varkappa u(t,\gamma)$ for $\varkappa > 0$. Let the operator $C : L^2(0,1) \to L^2(0,1)$ be defined by $Cu = \varkappa u(t,\gamma)$. Then, the system (4.1) takes the form:

$$\begin{cases} \gamma'(t) = \Psi_1\gamma(t) + \wp\left(t, \gamma(t), (H\gamma)(t)\right) + \displaystyle\int_0^t \Psi_2(t-\theta)\gamma(\theta)\, d\theta + Cu(t), & \text{if } t \in \Theta_j, \\ \gamma(t) = \varpi_j\left(t, \gamma\left(t_j^-\right)\right), \quad \text{if } t \in \tilde{\Theta}_j, \\ \gamma(0) = \gamma_0. \end{cases} \tag{4.3}$$

As argued in Case 01, we can easily verify the assumptions $(A1) - (A5)$. If we assume that the operator W given by $Wu = \displaystyle\int_0^n R(n-\theta)\varkappa u(\theta)\, d\theta$, satisfies $(A6)$, then all the assumptions given in Theorem 3.4 are verified. Therefore, the problem (4.1) is controllable.

5 Conclusions

In this research, we investigated existence of mild solutions for a non-instantaneous integrodifferential equation via resolvent operators in the sense of Grimmer in a Fréchet space. We applied Darbo's fixed point theorem in conjunction with the technique of measures of noncompactness to establish the desired results. The controllability of the given problem is also discussed. An example is presented for illustrating the application of our key findings. Our results are novel in the given configuration and contribute significantly to the literature on the topic. We believe that the present study can lead to new avenues for research, such as coupled systems, problems with infinite delays, and their fractional counterparts. Thus, this article will serve as a starting point for future endeavors in aforementioned areas.

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Maximum, anti-maximum principles and monotone methods for boundary value problems for Riemann-Liouville fractional differential equations in neighborhoods of simple eigenvalues

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ABSTRACT

It has been shown that, under suitable hypotheses, boundary value problems of the form, $Ly + \lambda y = f$, $BCy = 0$ where L is a linear ordinary or partial differential operator and BC denotes a linear boundary operator, then there exists $\Lambda > 0$ such that $f \geq 0$ implies $\lambda y \geq 0$ for $\lambda \in [-\Lambda, \Lambda] \setminus \{0\}$, where y is the unique solution of $Ly + \lambda y = f$, $BCy = 0$. So, the boundary value problem satisfies a maximum principle for $\lambda \in [-\Lambda, 0)$ and the boundary value problem satisfies an anti-maximum principle for $\lambda \in (0, \Lambda]$. In an abstract result, we shall provide suitable hypotheses such that boundary value problems of the form, $D_0^\alpha y + \beta D_0^{\alpha-1} y = f$, $BCy = 0$ where D_0^α is a Riemann-Liouville fractional differentiable operator of order α, $1 < \alpha \leq 2$, and BC denotes a linear boundary operator, then there exists $\mathcal{B} > 0$ such that $f \geq 0$ implies $\beta D_0^{\alpha-1} y \geq 0$ for $\beta \in [-\mathcal{B}, \mathcal{B}] \setminus \{0\}$, where y is the unique solution of $D_0^\alpha y + \beta D_0^{\alpha-1} y = f$, $BCy = 0$. Two examples are provided in which the hypotheses of the abstract theorem are satisfied to obtain the sign property of $\beta D_0^{\alpha-1} y$. The boundary conditions are chosen so that with further analysis a sign property of βy is also obtained. One application of monotone methods is developed to illustrate the utility of the abstract result.

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RESUMEN

Se ha demostrado que, bajo hipótesis apropiadas, problemas de valor en la frontera de la forma $Ly + \lambda y = f$, $BCy = 0$, donde L es un operador diferencial lineal ordinario o parcial y BC denota un operador lineal de frontera, entonces existe $\Lambda > 0$ tal que $f \geq 0$ implica $\lambda y \geq 0$ para $\lambda \in [-\Lambda, \Lambda] \setminus \{0\}$, donde y es la única solución de $Ly + \lambda y = f$, $BCy = 0$. Así, el problema de valor en la frontera satisface un principio del máximo para $\lambda \in [-\Lambda, 0)$ y el problema de valor en la frontera satisface un anti-principio del máximo para $\lambda \in (0, \Lambda]$. En un resultado abstracto, entregaremos hipótesis apropiadas tales que los problemas de valor en la frontera de la forma $D_0^\alpha y + \beta D_0^{\alpha-1} y = f$, $BCy = 0$ donde D_0^α es un operador diferencial fraccionario de Riemann-Liouville de orden α, $1 < \alpha \leq 2$, y BC denota un operador lineal de frontera, entonces existe $\mathcal{B} > 0$ tal que $f \geq 0$ implica $\beta D_0^{\alpha-1} y \geq 0$ para $\beta \in [-\mathcal{B}, \mathcal{B}] \setminus \{0\}$, donde y es la única solución de $D_0^\alpha y + \beta D_0^{\alpha-1} y = f$, $BCy = 0$. Se entregan dos ejemplos en los cuales las hipótesis del teorema abstracto se satisfacen para obtener la propiedad de signo de $\beta D_0^{\alpha-1} y$. Las condiciones de frontera se eligen de tal forma de obtener también una propiedad de signo para βy con un análisis adicional. Se desarrolla una aplicación de métodos monótonos para ilustrar la utilidad del resultado abstracto.

Keywords and Phrases: Maximum principle, anti-maximum principle, Riemann-Liouville fractional differential equation, boundary value problem, monotone methods, upper and lower solution.

2020 AMS Mathematics Subject Classification: 34B08, 34B18, 34B27, 34L15.

1 Introduction

For $\gamma > 0$, $y \in \mathcal{L}[0,1]$, the space of Lebesgue integrable functions, the expression

$$I_0^\gamma y(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} y(s)\, ds, \quad 0 \le t \le 1,$$

denotes a Riemann-Liouville fractional integral of y of order γ, where Γ denotes the special gamma function. For $\gamma = 0$, I_0^0 is defined to be the identity operator.

Let n denote a positive integer and assume $n-1 < \alpha \le n$. Then $D_0^\alpha y(t) = D^n I_0^{n-\alpha} y(t)$, where $D^n = \dfrac{d^n}{dt^n}$ and if this expression exists, denotes a Riemann-Liouville fractional derivative of y of order α. So, if $1 < \alpha < 2$, $D_0^\alpha y(t) = \dfrac{d^2}{dt^2} \displaystyle\int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} y(s)\, ds$ if the right hand side exists. In the case α is a positive integer, we may write $D_0^\alpha y(t) = D^\alpha y(t)$ or $I_0^\alpha y(t) = I^\alpha y(t)$ since the Riemann-Liouville derivative or integral agrees with the classical derivative or integral if α is a positive integer.

For authoritative accounts on the development of fractional calculus, we refer to the monographs [11, 16, 20]. For the sake of self-containment, we state properties that we shall employ in this study. It is well-know that the Riemann-Liouville fractional integrals commute; that is if $\gamma_1, \gamma_2 > 0$, and $y \in \mathcal{L}[0,1]$, then

$$I_0^{\gamma_1} I_0^{\gamma_2} y(t) = I_0^{\gamma_1+\gamma_2} y(t) = I_0^{\gamma_2} I_0^{\gamma_1} y(t).$$

A power rule is valid for the Riemann-Liouville fractional integral; if $\delta > -1$ and $\gamma \ge 0$, then

$$I_0^\gamma t^\delta = I_0^\gamma (t-0)^\delta = \frac{\Gamma(\delta+1)}{\Gamma(\delta+1+\gamma)} t^{\delta+\gamma}.$$

A power rule is valid for the Riemann-Liouville fractional derivative; if $\delta > -1$ and $\gamma \ge 0$ then

$$D_0^\gamma t^\delta = \frac{\Gamma(\delta+1)}{\Gamma(\delta+1-\gamma)} t^{\delta-\gamma}.$$

Since the gamma function is unbounded at 0, it is the convention that if $\delta + 1 - \gamma = 0$, then $D_0^\gamma t^\delta = 0$. Note that if $1 < \alpha \le 2$, and if $D_0^\alpha y(t)$ exists, then $D_0^{\alpha-1} y(t)$ exists and

$$D_0^\alpha y(t) = D^2 I_0^{2-\alpha} y(t) = D D I_0^{1-(\alpha-1)} y(t) = D D_0^{\alpha-1} y(t).$$

In [12], a boundary value problem,

$$D_0^\alpha y(t) = f(t, y(t)), \quad 0 < t \le 1, \tag{1.1}$$

$$y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1), \tag{1.2}$$

where $1 < \alpha \leq 2$, was studied. This is an example of a boundary value problem at resonance since $< t^{\alpha-1} >$, the linear span of $t^{\alpha-1}$, denotes the solution space of the homogeneous problem, $D_0^\alpha y = 0$, with homogeneous boundary conditions, (1.2). In [12], the purpose of that article was to consider an equivalent shifted equation, $D_0^\alpha y(t) - K^2 y(t) = f(t, y(t)) - K^2 y(t)$, $0 < t \leq 1$, and apply the method of quasilinearization to the shifted boundary value problem. The method of quasilinearization is ideally suited when the boundary value problem, in this case the shifted boundary value problem, satisfies a maximum principle [19]. In particular, in [12], a nonpositive Green's function for the shifted boundary value problem was explicitly constructed. Not surprisingly, Mittag-Leffler functions were key to the construction and signing of the Green's function. The case, $D_0^\alpha y(t) + K^2 y(t) = f(t, y(t)) + K^2 y(t)$, $0 < t \leq 1$, was not addressed in [12].

The maximum principle is well-known and is an important tool in the qualitative study of differential equations; we refer the reader to the well-known monograph [19] for many applications. In recent years, the maximum principle has become an important tool in the study of boundary value problems for fractional differential equations. Early applications appear in [24] and [3] where explicit Green's functions, expressed in terms of power functions, where constructed and sign properties were analyzed so that fixed point theorems could be applied. Many authors have continued the strategy to construct and analyze explicit Green's functions and apply fixed point theory to nonlinear boundary value problems for fractional differential equations.

In the example, $y'' + \lambda y = f$, with Neumann boundary conditions, $y'(0) = 0$, $y'(1) = 0$, if $\lambda < 0$, then this boundary value problem satisfies a maximum principle. In particular, for $f \in \mathcal{L}[0, 1]$, the boundary value problem is uniquely solvable and f nonnegative implies y is nonpositive where y is the unique solution associated with f. Clément and Peletier [9] were the first to discover an anti-maximum principle. They were primarily interested in partial differential equations, but they illustrated the anti-maximum principle with the the same boundary value problem, $y'' + \lambda y = f$, $y'(0) = 0$, $y'(1) = 0$, but now, $0 < \lambda < \frac{\pi^2}{4}$. For this particular boundary value problem, if $0 < \lambda < \frac{\pi^2}{4}$ and if $f \in \mathcal{L}[0, 1]$, then the boundary value problem is uniquely solvable and f nonnegative implies y is nonnegative where y is the unique solution associated with f. At $\lambda = 0$, the boundary value problem is at resonance, and more precisely, $\lambda = 0$ denotes a simple eigenvalue of the linear problem. So there has been a change in the sign property, maximum principle or anti-maximum principle, through the simple eigenvalue $\lambda = 0$. Since the publication of [9] there have been many studies of boundary value problems with parameter and the change of behavior from maximum to anti-maximum principles as a function of the parameter. In the case of partial differential equations, we refer to [1, 2, 8, 10, 14, 17, 18, 21]. In the case of ordinary differential equations we refer to [4, 5, 6, 7, 13, 22].

In this article, we intend to study this change in behavior for a boundary value problem for a Riemann-Liouville fractional differential equation. We shall modify the methods developed in [8], where in [8], those authors began with an ordinary differential equation

$$y''(t) + \lambda y(t) = f(t), \quad t \in [0, 1], \tag{1.3}$$

and considered either periodic boundary conditions or Neumann boundary conditions. Key to their argument is that for $f = 0$, at $\lambda = 0$, the boundary value problem, (1.3) with periodic or Neumann boundary conditions, is at resonance since constant functions are nontrivial solutions. That is, $\lambda = 0$ is a simple eigenvalue for the problem, (1.3) with periodic or Neumann boundary conditions, and the eigenspace is $< 1 >$, where $< 1 >$ denotes the linear span of the 1 function. Rewriting the boundary value problem as an abstract equation and employing the resolvent, the inverse of $(D^2 + \lambda \mathcal{I})$ for $\lambda \neq 0$, under the imposed boundary conditions, if it exists, and the partial resolvent for $\lambda = 0$, then under the assumption that $f \geq 0$ (with $f \in \mathcal{L}[0, 1]$), the authors in [8] exhibited sufficient conditions for the existence of $\Lambda > 0$, and a constant $K > 0$, independent of f, such that

$$\lambda y(t) \geq K|f|_1, \quad \lambda \in [-\Lambda, \Lambda] \setminus \{0\}, \quad 0 \leq t \leq 1,$$

where y is the unique solution of the boundary value problem associated with (1.3) and $|f|_1 = \int_0^1 |f(s)|\, ds$. With this one inequality the authors showed that for $-\Lambda \leq \lambda < 0$ the boundary value problem, (1.3) with periodic or Neumann boundary conditions, satisfies a maximum principle and for $0 < \lambda \leq \Lambda$, the boundary value problem (1.3) with periodic or Neumann boundary conditions, satisfies an anti-maximum principle. They referred to this principle as a maximum principle (we shall take the liberty to refer to it as a signed maximum principle in y) and then proceeded to produce many nice examples.

Recently, [13], the arguments developed in [8] were adapted to study boundary value problems for the ordinary differential equation

$$y''(t) + \beta y'(t) = f(t), \quad t \in [0, 1]; \tag{1.4}$$

sufficient conditions for a signed maximum principle in Dy, where $Dy = y'$, were obtained. That is, under the assumption that $f \geq 0$ (with $f \in \mathcal{L}[0, 1]$), sufficient conditions were exhibited to imply the existence of $\mathcal{B} > 0$, and a constant $K > 0$, independent of f such that

$$\beta Dy(t) \geq K|f|_1, \quad \lambda \in [-\mathcal{B}, \mathcal{B}] \setminus \{0\}, \quad 0 \leq t \leq 1.$$

Two examples of boundary value problems were presented in which if a solution y of the boundary value problem is such that $Dy = y'$ has constant sign on $[0, 1]$, then y has constant sign on $[0, 1]$. For one of the examples, an appropriate partial order in $C^1[0, 1]$, depending on the sign of β, was defined and the method of upper and lower solutions, coupled with monotone methods, was employed to obtain sufficient conditions for the existence of solutions of the boundary value problem for a nonlinear differential equation,

$$y''(t) = f(t, y(t), y'(t)), \quad t \in [0, 1].$$

Motivated by the work in [13], we shall adapt the methods developed in [8] and exhibit sufficient conditions to obtain a signed maximum principle in $D_0^{\alpha-1}y$, for the boundary value problem $D_0^{\alpha}y(t) + \beta D_0^{\alpha-1}y(t) = f(t)$, with boundary conditions

$$BCy = 0, \quad D_0^{\alpha-1}y(0) = D_0^{\alpha-1}y(1), \tag{1.5}$$

where BC denotes a linear boundary operator mapping a function y to the reals. In particular, we shall exhibit sufficient conditions that imply the existence of $\mathcal{B} > 0$, and a constant $K > 0$, independent of f, such that

$$\beta D_0^{\alpha-1}y(t) \geq K|f|_1, \quad \beta \in [-\mathcal{B}, \mathcal{B}] \setminus \{0\}, \quad 0 \leq t \leq 1. \tag{1.6}$$

In two examples, the boundary condition BC will be such that if y satisfies the boundary conditions (1.5) and $\beta D_0^{\alpha-1}y(t) > 0$ on $[0, 1]$, then $\beta y(t) \geq 0$ on $(0, 1]$. In one of the examples, an appropriate partial order in a Banach space is defined and the method of upper and lower solutions, coupled with monotone methods, is applied to obtain sufficient conditions for the existence of solutions of the nonlinear differential equation

$$D_0^{\alpha}y(t) = f(t, y(t), \beta D_0^{\alpha-1}y(t)), \quad t \in (0, 1],$$

satisfying the boundary conditions, (1.5).

In Section 2, following the lead of [8], we shall first define the concept of a signed maximum principle in $D_0^{\alpha-1}y$. Then analogous to Lemma 1, Lemma 2 and Lemma 3 in [8], we shall prove the main theorem and obtain sufficient conditions for (1.6) and hence, obtain sufficient conditions for adherence to a signed maximum principle in $D_0^{\alpha-1}y$. In Section 3, we shall exhibit two examples that adhere to a strong signed maximum principle in $D_0^{\alpha-1}y$ and furthermore (1.6) implies $\beta y(t) \geq 0$ on $(0, 1]$. We shall close in Section 4 with an application of a monotone method applied to a nonlinear problem related to one of the examples produced in Section 4. At $\beta = 0$, the problem is at resonance. The problem is shifted [15] by $\beta D_0^{\alpha-1}y$ and $\beta > 0$ or $\beta < 0$ is chosen as a function of the monotonicity properties of the nonlinear term $f(t, y(t), \beta D_0^{\alpha-1}y(t))$.

2 The main theorem

As is standard, let $C[0,1]$ denote the Banach space of continuous functions defined on $[0,1]$ with the supremum norm, $|\cdot|_0$, and let $\mathcal{L}[0,1]$ denote the space of Lebesgue integrable functions with the usual $\mathcal{L}_1$ norm. Employing notation introduced in [23], assume $1 < \alpha \leq 2$ and define

$$C_{\alpha-2}[0,1] = \left\{ y : (0,1] \to \mathbb{R} : y(t) \text{ is continuous for } t \in (0,1], \text{ and } \lim_{t\to 0^+} t^{2-\alpha} y(t) \text{ exists} \right\}.$$

It is clear that $y \in C_{\alpha-2}[0,1]$ if, and only if, there exists $z \in C[0,1]$ such that $y(t) = t^{\alpha-2} z(t)$ for $t \in (0,1]$. Define $|y|_{\alpha-2} = |z|_0$ and $C_{\alpha-2}[0,1]$ with norm $|\cdot|_{\alpha-2}$ is a Banach space.

Let $\mathcal{X}_{\alpha-2}$ denote the Banach space

$$\mathcal{X}_{\alpha-2} = \{y : (0,1] \to \mathbb{R} : y \in C_{\alpha-2}[0,1], D_0^{\alpha-1} y \in C[0,1]\},$$

with

$$||y|| = \max\{|y|_{\alpha-2}, |D_0^{\alpha-1} y|_0\}.$$

The following definition is motivated by Definition 1 found in [8].

Definition 2.1. *Assume $\mathcal{A}$ is a linear operator with Dom$(\mathcal{A}) \subset \mathcal{X}_{\alpha-2}$ and Im$(\mathcal{A}) \subset \mathcal{L}[0,1]$. For $\beta \in \mathbb{R} \setminus \{0\}$, the operator $\mathcal{A} + \beta D_0^{\alpha-1}$ satisfies a **signed maximum principle in** $D_0^{\alpha-1} y$ if for each $f \in \mathcal{L}[0,1]$, the equation*

$$(\mathcal{A} + \beta D_0^{\alpha-1}) y = f, \quad y \in Dom(\mathcal{A}),$$

*has unique solution, y, and $f(t) \geq 0$, $0 \leq t \leq 1$, implies $\beta D_0^{\alpha-1} y(t) \geq 0$, $0 \leq t \leq 1$. The operator $\mathcal{A} + \beta D_0^{\alpha-1}$ satisfies a **strong signed maximum principle in** $D_0^{\alpha-1} y$ if $f(t) \geq 0$, $0 \leq t \leq 1$, and $f(t) \not\equiv 0$ a.e., implies $\beta D_0^{\alpha-1} y(t) > 0$, $0 \leq t \leq 1$.*

Remark 2.2. *Throughout this article, the phrases "maximum principle" or "anti-maximum principle" may be used loosely. If so, we mean the following. If $f \geq 0$ implies $D_0^{\alpha-1} y \leq 0$ the phrase maximum principle may be used. This is precisely the case for the classical second order ordinary differential equation with Dirichlet boundary conditions. If $f \geq 0$ implies $D_0^{\alpha-1} y \geq 0$ the phrase anti-maximum principle may be used. This is the case observed in [9] for $\alpha = 2$, where the phrase anti-maximum principle was coined.*

For $f \in \mathcal{L}[0,1]$ (or $f \in C[0,1]$), let $|f|_1 = \int_0^1 |f(s)|\, ds$ and define $\overline{f} = \int_0^1 f(t)\, dt$. Define

$$\tilde{\mathcal{C}} \subset C[0,1] = \{f \in C[0,1] : \overline{f} = 0\}, \quad \tilde{\mathcal{L}} \subset \mathcal{L}[0,1] = \{f \in \mathcal{L}[0,1] : \overline{f} = 0\}.$$

Assume $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \to \mathcal{L}[0,1]$ denotes a linear operator satisfying

$$\mathrm{Dom}(\mathcal{A}) \subset \mathcal{X}_{\alpha-2}, \quad \ker(\mathcal{A}) = < t^{\alpha-1} + ct^{\alpha-2} >, \quad \mathrm{Im}(\mathcal{A}) = \tilde{\mathcal{L}}, \tag{2.1}$$

for some real constant c, where $< t^{\alpha-1} + ct^{\alpha-2} >$ denotes the linear span of $t^{\alpha-1} + ct^{\alpha-2}$. Assume further that for $\tilde{f} \in \tilde{\mathcal{L}}$, the problem $\mathcal{A}y = \tilde{f}$ is uniquely solvable with solution $y \in \mathrm{Dom}(\mathcal{A})$ and such that $\int_0^1 D_0^{\alpha-1} y(t)\,dt = \overline{(D_0^{\alpha-1} y)} = 0$. In particular, define

$$\mathrm{Dom}(\tilde{\mathcal{A}}) = \left\{ y \in \mathrm{Dom}(\mathcal{A}) : \overline{(D_0^{\alpha-1} y)} = 0 \right\},$$

and then

$$\mathcal{A}|_{\mathrm{Dom}(\tilde{\mathcal{A}})} : \mathrm{Dom}(\tilde{\mathcal{A}}) \to \tilde{\mathcal{L}}$$

is one to one and onto. Moreover, if $\mathcal{A}\tilde{y} = \tilde{f}$ for $\tilde{f} \in \tilde{\mathcal{L}}$, $\tilde{y} \in \mathrm{Dom}(\tilde{\mathcal{A}})$, assume there exists a constant $M > 0$ depending only on $\mathcal{A}$ such that

$$|D_0^{\alpha-1} \tilde{y}|_0 \le M |\tilde{f}|_1. \tag{2.2}$$

For $f \in \mathcal{L}$, define

$$\tilde{f} = f - \overline{f},$$

which implies $\tilde{f} \in \tilde{\mathcal{L}}$, and for $y \in \mathrm{Dom}(\mathcal{A})$ define

$$\tilde{y} = y - \overline{(D_0^{\alpha-1} y)} \frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

which implies $\tilde{y} \in \mathrm{Dom}(\tilde{\mathcal{A}})$ since

$$D_0^{\alpha-1}(\tilde{y}) = D_0^{\alpha-1} y - \overline{(D_0^{\alpha-1} y)}.$$

Finally assume there exists $\mathcal{A}' : \mathrm{Dom}(\mathcal{A}') \to \mathcal{L}$ such that $\mathcal{A} = \mathcal{A}' D_0^{\alpha-1}$. In this context, we rewrite

$$\mathcal{A}y + \beta D_0^{\alpha-1} y = f, \quad y \in \mathrm{Dom}(\mathcal{A}), \tag{2.3}$$

as

$$(\mathcal{A}' + \beta \mathcal{I}) D_0^{\alpha-1} y = f, \quad D_0^{\alpha-1} y \in \mathrm{Dom}(\mathcal{A}'). \tag{2.4}$$

Define $\mathrm{Dom}(\tilde{\mathcal{A}}') = \{ v \in \mathrm{Dom}(\mathcal{A}') : \overline{v} = 0 \} \subset C[0,1]$ and it follows that

$$\mathcal{A}'|_{\mathrm{Dom}(\tilde{\mathcal{A}}')} : \mathrm{Dom}(\tilde{\mathcal{A}}') \to \tilde{\mathcal{L}}$$

is one to one and onto. With the decompositions $\tilde{f} = f - \overline{f}$ and $\tilde{y} = y - \overline{D_0^{\alpha-1} y}\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, it follows that $\tilde{f} \in \tilde{\mathcal{L}}$ and $\tilde{y} \in \mathrm{Dom}(\tilde{\mathcal{A}})$, or more appropriately, $D_0^{\alpha-1}\tilde{y} \in \mathrm{Dom}(\tilde{\mathcal{A}}')$. So, equation (2.3) or equation (2.4) decouples as follows:

$$\mathcal{A}' D_0^{\alpha-1}\tilde{y} + \beta D_0^{\alpha-1}\tilde{y} = (\mathcal{A}' + \beta\mathcal{I}) D_0^{\alpha-1}\tilde{y} = \tilde{f}, \tag{2.5}$$

$$\beta D_0^{\alpha-1}\left(\overline{D_0^{\alpha-1} y}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right) = \beta\overline{D_0^{\alpha-1} y} = \overline{f}. \tag{2.6}$$

Denote the inverse of $(\mathcal{A}' + \beta\mathcal{I})$, if it exists, by $\mathcal{R}_\beta$ and denote the inverse of $\mathcal{A}'|_{\mathrm{Dom}(\tilde{\mathcal{A}}')}$ by $\mathcal{R}_0$. So, $\mathcal{R}_0 : \tilde{\mathcal{L}} \to C[0,1]$ and

$$D_0^{\alpha-1}\tilde{y} = \mathcal{R}_0\tilde{f} \quad \text{if, and only if,} \quad \mathcal{A}'(D_0^{\alpha-1}\tilde{y}) = \tilde{f}. \tag{2.7}$$

Note that (2.7) implies that since $D_0^{\alpha-1}\tilde{y} \in \mathrm{Dom}(\tilde{\mathcal{A}}')$, then

$$D_0^{\alpha-1}\tilde{y} = \mathcal{R}_0\mathcal{A}' D_0^{\alpha-1}\tilde{y}. \tag{2.8}$$

Since $\tilde{\mathcal{C}} \subset \tilde{\mathcal{L}}$, we can also consider $\mathcal{R}_0 : \tilde{\mathcal{C}} \to C[0,1]$. Let

$$||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}} = \sup_{|v|_0=1} |\mathcal{R}_0 v|_0, \quad v, \mathcal{R}_0 v \in C[0,1],$$

and

$$||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}} = \sup_{|v|_1=1} |\mathcal{R}_0 v|_0, \quad v \in \mathcal{L}[0,1], \quad \mathcal{R}_0 v \in C[0,1].$$

Since $D_0^{\alpha-1}\tilde{y} \in \tilde{\mathcal{C}}$ then $|\mathcal{R}_0 D_0^{\alpha-1}\tilde{y}|_0 \le ||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}} |D_0^{\alpha-1}\tilde{y}|_0$. Similarly, $\tilde{f} \in \tilde{\mathcal{L}}$ implies $|\mathcal{R}_0\tilde{f}|_0 \le ||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}} |\tilde{f}|_1$.

The following theorem is proved in [13] for the case $\alpha = 2$ and closely models the motivating lemmas and proofs found in [8]. We supply the proof again for $1 < \alpha \le 2$, for the sake of self-containment.

Theorem 2.3. *Assume $\mathcal{A} : Dom(\mathcal{A}) \to \mathcal{L}[0,1]$ denotes a linear operator satisfying (2.1) and (2.2), and assume that for $\tilde{f} \in \tilde{\mathcal{L}}$, the problem $\mathcal{A}y = \tilde{f}$ is uniquely solvable with solution $y \in Dom(\mathcal{A})$ such that $\overline{D_0^{\alpha-1} y} = 0$. Further, assume there exists $\mathcal{A}' : Dom(\mathcal{A}') \to \mathcal{L}[0,1]$ such that $\mathcal{A} = \mathcal{A}' D_0^{\alpha-1}$. Assume $\tilde{\mathcal{A}}'|_{Dom(\tilde{\mathcal{A}}')} : Dom(\tilde{\mathcal{A}}') \to \tilde{\mathcal{L}}$ is one to one and onto. Then there exists $B_1 > 0$ such if $0 < |\beta| \le B_1$, then $\mathcal{R}_\beta$, the inverse of $(\mathcal{A}' + \beta\mathcal{I})$, exists. Moreover, if $\tilde{f} \in \tilde{L}$, if $B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}} < 1$, where $\mathcal{R}_0$ denotes the inverse of $\mathcal{A}'|_{Dom(\tilde{\mathcal{A}}')}$, and if $0 < |\beta| \le B_1$, then*

$$|\mathcal{R}_\beta\tilde{f}|_0 \le \frac{||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}} |\tilde{f}|_1. \tag{2.9}$$

Further, there exists $\mathcal{B} \in (0, B_1)$ such that if $0 < |\beta| \le \mathcal{B}$, then the operator $(\mathcal{A} + \beta D_0^{\alpha-1})$ satisfies *a strong signed maximum principle in* $D_0^{\alpha-1}y$.

Proof. Employ (2.8) and apply $\mathcal{R}_0$ to (2.5) to obtain

$$D_0^{\alpha-1}\tilde{y} + \beta\mathcal{R}_0 D_0^{\alpha-1}\tilde{y} = \mathcal{R}_0\tilde{f}.$$

Note that (2.2) implies that $\mathcal{R}_0 : \tilde{\mathcal{L}} \to \tilde{\mathcal{C}}$ is continuous and hence, bounded. Assume $|\beta|||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}} < 1$. Then $(\mathcal{I} + \beta\mathcal{R}_0) : \tilde{C} \to \tilde{C}$ is invertible and

$$D_0^{\alpha-1}\tilde{y} = (\mathcal{I} + \beta\mathcal{R}_0)^{-1}\mathcal{R}_0\tilde{f}.$$

So, assume $0 < B_1 < \dfrac{1}{||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}$ and assume $|\beta| \leq B_1$. Then $\mathcal{R}_\beta = (\mathcal{I} + \beta\mathcal{R}_0)^{-1}\mathcal{R}_0$ exists. Moreover,

$$\begin{aligned}|D_0^{\alpha-1}\tilde{y}|_0 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}|D_0^{\alpha-1}\tilde{y}|_0 &\leq |D_0^{\alpha-1}\tilde{y}|_0 - |\beta|||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}|D_0^{\alpha-1}\tilde{y}|_0 \\ &\leq |(\mathcal{I} + \beta\mathcal{R}_0)D\tilde{y}|_0 = |\mathcal{R}_0\tilde{f}|_0 \leq ||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}|\tilde{f}|_1\end{aligned}$$

and (2.9) is proved since $\mathcal{R}_\beta\tilde{f} = D_0^{\alpha-1}\tilde{y} \subset C[0,1]$.

Now assume $f \in \mathcal{L}$ and assume $f \geq 0$ a.e. Then $\bar{f} = |f|_1$. Let $0 < |\beta| \leq B_1 < \dfrac{1}{||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}$, write $f = \bar{f} + \tilde{f}$ and consider

$$\beta D_0^{\alpha-1}y = \beta\mathcal{R}_\beta f = \beta\mathcal{R}_\beta(\bar{f} + \tilde{f}).$$

Note that $\beta\mathcal{R}_\beta\bar{f} = \bar{f}$ since $(\mathcal{A}' + \beta\mathcal{I})\bar{f} = \beta\bar{f}$. So,

$$\begin{aligned}\beta D_0^{\alpha-1}y = \beta\mathcal{R}_\beta f &= \beta\mathcal{R}_\beta(\bar{f} + \tilde{f}) \\ &= \bar{f} + \beta\mathcal{R}_\beta\tilde{f} \geq |f|_1 - |\beta||\mathcal{R}_\beta\tilde{f}|_0.\end{aligned}$$

Continuing to assume that $0 < |\beta| \leq B_1$, it now follows from (2.9) that

$$\beta D_0^{\alpha-1}y \geq |f|_1 - |\beta|\Big(\frac{||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}\Big)|\tilde{f}|_1.$$

Since $\tilde{f} = f - \bar{f}$, and $|\tilde{f}|_1 \leq |f|_1 + \bar{f} = 2|f|_1$, the theorem is proved with

$$\mathcal{B} < \min\Big\{B_1, \Big(\frac{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}{2||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}\Big)\Big\}.$$

In particular, if $0 < |\beta| \leq \mathcal{B}$, then

$$\beta D_0^{\alpha-1}y(t) \geq K|f|_1 = \Big(1 - \mathcal{B}\Big(\frac{2||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}\Big)\Big)|f|_1. \quad \square$$

3 Two examples

This article is modeled after [13] and in [13] the conclusion of the theorem analogous to Theorem 2.3 is that the operator $(\mathcal{A} + \beta D)$ satisfies a strong signed maximum principle in Dy. So the elementary observation that $\beta Dy > 0$ on an interval implies that βy is monotone increasing on that interval is employed to consider boundary value problems for which $\beta Dy > 0$ on $[0, 1]$ implies that βy has constant sign on $(0, 1)$. In the following lemma, we state and prove a modest extension of this principle to the fractional Riemann-Liouville derivative of order $\gamma = \alpha - 1$, $0 < \gamma \le 1$.

Lemma 3.1. *Assume* $0 < \gamma \le 1$. *Assume* $\beta \ne 0$. *Assume* $y \in C_{\gamma-1}[0, 1]$ *and assume* $D_0^\gamma y(t) \in C[0, 1]$. *Assume* $\beta D_0^\gamma y(t) > 0$, $0 \le t \le 1$, *and assume* $\beta \lim_{t\to 0^+} t^{1-\gamma} y(t) \ge 0$. *Then* $\beta y(t) > 0$, $0 < t \le 1$.

Proof. If $\gamma = 1$, then y can be extended continuously to $[0, 1]$ and $\beta y(0) \ge 0$. Then βy is increasing on $[0, 1]$ and the result is true.

So, assume $0 < \gamma < 1$ and define $a = \lim\limits_{t\to 0^+} t^{1-\gamma} y(t)$. Thus, $\beta a \ge 0$. Then [11, Theorem 2.23] or [23, Proposition 6.1],

$$y(t) = at^{\gamma-1} + I_0^\gamma D_0^\gamma y(t), \quad 0 < t \le 1,$$

and

$$\beta y(t) = \beta a t^{\gamma-1} + I_0^\gamma \beta D_0^\gamma y(t), \quad 0 < t \le 1.$$

If $a = 0$, then $I_0^\gamma \beta D_0^\gamma y(t) > 0$ if $0 < t \le 1$ and the statement is proved. If $\beta a > 0$, then both terms $\beta a t^{\gamma-1}$ and $I_0^\gamma \beta D_0^\gamma y(t)$ are positive for $t \in (0, 1]$, and the statement is proved. □

Example 3.2. Let $1 < \alpha \le 2$, and consider the linear boundary value problem

$$D_0^\alpha y(t) + \beta D_0^{\alpha-1} y(t) = f(t), \quad 0 \le t \le 1, \tag{3.1}$$

$$y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1). \tag{3.2}$$

For the boundary value problem (3.1), (3.2), $\mathcal{A} = D_0^\alpha$, $\mathcal{A}' = D = \frac{d}{dt}$, $\ker(\mathcal{A}) =< t^{\alpha-1} >$. We show that the operators $\mathcal{A}$ and $\mathcal{A}'$ satisfy the hypotheses of Theorem 2.3.

One can show directly that $\mathrm{Im}(\mathcal{A}) = \tilde{\mathcal{L}}$. If $f \in \mathrm{Im}(\mathcal{A})$ then there exists a solution y of

$$D_0^\alpha y(t) = f(t), \quad 0 \le t \le 1, \quad y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1),$$

which implies

$$0 = D_0^{\alpha-1} y(1) - D_0^{\alpha-1} y(0) = \int_0^1 D_0^\alpha y(t)\, dt = \int_0^1 f(t)\, dt,$$

and $f \in \tilde{\mathcal{L}}$. Likewise, if $f \in \tilde{\mathcal{L}}$, then

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds = I_0^\alpha f(t) \in \mathrm{Dom}(\tilde{A}) \tag{3.3}$$

is a solution of

$$D_0^\alpha y(t) = f(t), \quad 0 \le t \le 1, \quad y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1).$$

To see that $y \in \mathrm{Dom}(\tilde{A})$, note that

$$D_0^{\alpha-1} y(t) = D_0^{\alpha-1} I_0^\alpha f(t) = D_0^{\alpha-1} I_0^{\alpha-1} I^1 f(t) = \int_0^t f(s)\, ds.$$

So, $\overline{D_0^{\alpha-1} y} = \overline{f} = 0$. To see that the boundary conditions are satisfied, $y(0) = I_0^\alpha y|_{t=0}$, and the condition $y(0) = 0$ is clear. Moreover, $D_0^{\alpha-1} I_0^\alpha f(t) = \int_0^t f(s)\, ds$, which implies $D_0^{\alpha-1} I_0^\alpha f|_{t=0} = D_0^{\alpha-1} I_0^\alpha f|_{t=1} = 0$ since $f \in \tilde{\mathcal{L}}$.

To argue that $\mathcal{A} y = \tilde{f}$ is uniquely solvable with solution $y \in \mathrm{Dom}(\tilde{\mathcal{A}})$, (3.3) implies the solvability. For uniqueness, if y_1 and y_2 are two such solutions, then $(y_1 - y_2)(t) = ct^{\alpha-1}$ and $y_1 - y_2 \in \mathrm{Dom}(\tilde{\mathcal{A}})$ implies $c = 0$.

Finally, (3.3) implies (2.2) is satisfied with $M = 1$ since

$$|D_0^{\alpha-1} y(t)| = \left| \int_0^t f(s)\, ds \right| \le |f|_1.$$

Theorem 2.3 applies and there exists $\mathcal{B} > 0$ such that if $0 < |\beta| \le \mathcal{B}$ then $(\mathcal{A} + \beta D_0^{\alpha-1} y)$ has the strong maximum principle in $D_0^{\alpha-1} y$. Thus, $f \ge 0$ implies $\beta D_0^{\alpha-1} y \ge 0$. To apply Lemma 3.1, recall [11, Theorem 2.23] or [23, Theorem 6.8], that

$$\begin{aligned} y(t) &= at^{\alpha-2} + \frac{D_0^{\alpha-1} y|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + I_0^\alpha D_0^\alpha y(t) \\ &= at^{(\alpha-1)-1} + \frac{D_0^{\alpha-1} y|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + I_0^{\alpha-1} I D D_0^{\alpha-1} y(t) \\ &= at^{(\alpha-1)-1} + \frac{D_0^{\alpha-1} y|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + I_0^{\alpha-1} (D_0^{\alpha-1} y(t) - D_0^{\alpha-1} y|_{t=0}) \\ &= at^{(\alpha-1)-1} + I_0^{\alpha-1} D_0^{\alpha-1} y(t). \end{aligned}$$

where $a = \lim_{t \to 0^+} t^{2-\alpha} y(t) = \lim_{t \to 0^+} t^{1-(\alpha-1)} y(t)$. Since $y(0) = 0$ implies $a = 0$, Lemma 3.1 applies with $\gamma = \alpha - 1$ and $\beta a = 0$. Thus, $\beta y(t) \ge 0$, for $0 < t \le 1$, and if $|f|_1 > 0$, then $\beta y(t) > 0$, for $0 < t \le 1$.

Hence, a natural partial order on $\mathcal{X}_{\alpha-2}$ in which to apply the method of upper and lower solutions and monotone methods to a nonlinear boundary value problem is

$$y \in \mathcal{X}_{\alpha-2} \succeq 0 \Longleftrightarrow \beta y(t) \geq 0, \quad 0 < t \leq 1, \quad \text{and} \quad \beta D_0^{\alpha-1} y(t) \geq 0, \quad 0 \leq t \leq 1, \tag{3.4}$$

and

$$y \in \mathcal{X}_{\alpha-2} \succ 0 \Longleftrightarrow \beta y(t) > 0, \quad 0 < t \leq 1, \quad \text{and} \quad \beta D_0^{\alpha-1} y(t) > 0, \quad 0 \leq t \leq 1.$$

In Section 4, we shall employ monotone methods with respect to this partial order and obtain sufficient conditions for existence of maximal and minimal solutions of a nonlinear boundary value problem

$$D_0^\alpha y(t) = f(t, y(t), D_0^{\alpha-1} y(t)), \quad t \in (0,1],$$

associated with the boundary conditions (3.2).

Example 3.3. For the second example, let $0 < h < 1$, and consider a family of boundary conditions

$$\lim_{t \to 0^+} t^{2-\alpha} y(t) = hy(1), \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1). \tag{3.5}$$

Remark 3.4. *Note that the boundary condition $y(1)$ can be expressed as $t^{2-\alpha} y(t)|_{t=1}$ and so, if $h = 1$ in (3.5), we intend that these boundary conditions represent a Riemann-Liouville fractional analogue of periodic boundary conditions. In this example however, we require that $0 < h < 1$.*

For the boundary value problem (3.1), (3.5), $\mathcal{A} = D_0^\alpha$, $\mathcal{A}' = D = \frac{d}{dt}$ and

$$\ker(\mathcal{A}) = \left\langle t^{\alpha-1} + \frac{h}{1-h} t^{\alpha-2} \right\rangle.$$

Precisely as in Example (3.2), $\mathrm{Im}(\mathcal{A}) = \tilde{\mathcal{L}}$. Again, $f \in \tilde{\mathcal{L}}$ implies

$$\mathrm{Dom}(\tilde{\mathcal{A}}) = \left\{ y \in \mathcal{X}_{\alpha-2} : \overline{D_0^{\alpha-1} y} = 0 \right\}.$$

Again, M in (2.2) can be computed since if $f \in \tilde{\mathcal{L}}$, then

$$\tilde{y}(t) = I_0^\alpha f(t) + \frac{h}{1-h} I_0^\alpha f(1) t^{\alpha-2}$$

denotes the unique solution $y \in \mathrm{Dom}(\tilde{\mathcal{A}})$ of the boundary value problem $D_0^\alpha y = f$, (3.5). Thus, Theorem 2.3 applies and there exists $\mathcal{B} > 0$ such that if $0 < |\beta| \leq \mathcal{B}$ then $(\mathcal{A} + \beta D_0^{\alpha-1})$ satisfies a strong maximum principle in $D_0^{\alpha-1} y$.

To determine a sign condition on βy we appeal to Lemma 3.1. Let $a = \lim_{t \to 0^+} t^{2-\alpha} y(t)$. We first rule out the case $a = 0$. Assume $0 < |\beta| \leq \mathcal{B}$, and $0 < h < 1$. If $a = 0$, then $y(t) = I_0^{\alpha-1} D_0^{\alpha-1} y(t)$ and

$\beta y(1) > 0$. In particular, $y(1) \neq 0$. Since $\lim_{t\to 0^+} t^{2-\alpha}y(t) = a = 0$, y does not satisfy the boundary condition, $\lim_{t\to 0^+} t^{2-\alpha}y(t) = hy(1)$. Thus, $a \neq 0$.

Now, continue to assume $0 < |\beta| \leq \mathcal{B}$, and assume $0 < h < 1$. If $\beta D_0^{\alpha-1}y(t) > 0$, $0 \leq t \leq 1$, we rule out the case $\beta a < 0$. The condition $0 < h < 1$, the boundary condition, $\lim_{t\to 0^+} t^{2-\alpha}y(t) = hy(1)$ and the identity $y(t) = at^{\alpha-2} + I_0^{\alpha-1}D_0^{\alpha-1}y(t)$ imply that with $a = \lim_{t\to 0^+} t^{2-\alpha}y(t) = \lim_{t\to 0^+} t^{1-(\alpha-1)}y(t)$, then

$$0 < \frac{a}{a + I_0^{\alpha-1}D_0^{\alpha-1}y|_{t=1}} < 1,$$

or

$$0 < \frac{\beta a}{\beta a + I_0^{\alpha-1}\beta D_0^{\alpha-1}y|_{t=1}} < 1.$$

If $\beta a < 0$, then $\beta a < \beta a + I_0^{\alpha-1}\beta D_0^{\alpha-1}y|_{t=1} < 0$ and $|\beta a| > \left|\beta a + I_0^{\alpha-1}\beta D_0^{\alpha-1}y|_{t=1}\right|$, which implies

$$\frac{\beta a}{\beta a + I_0^{\alpha-1}\beta D_0^{\alpha-1}y|_{t=1}} > 1,$$

and so the condition $0 < h < 1$ is contradicted. So, $\beta a > 0$ and Lemma 3.1 applies with $\gamma = \alpha - 1$. Thus, if $0 < |\beta| \leq \mathcal{B}$ and $0 < h < 1$, then a natural partial order in which to apply the method of upper and lower solutions and monotone methods to a nonlinear problem is

$$y \in \mathcal{X}_{\alpha-2} \succeq 0 \iff \beta y(t) \geq 0, \quad 0 < t \leq 1, \quad \text{and} \quad \beta D_0^{\alpha-1}y(t) \geq 0, \quad 0 \leq t \leq 1,$$

and

$$y \in \mathcal{X}_{\alpha-2} \succ 0 \iff \beta y(t) > 0, \quad 0 < t \leq 1, \quad \text{and} \quad \beta D_0^{\alpha-1}y(t) > 0, \quad 0 \leq t \leq 1.$$

In particular, there is a transition from a maximum principle to an anti-maximum principle at $\beta = 0$.

Remark 3.5. *The work in this article extends the work produced in [13], where* $\alpha = 2$. *In [13], it is shown if* $1 < h$, *then* $\beta D_0^{\alpha-1}y(t) = \beta D^1 y(t) \geq 0$, $0 \leq t \leq 1$, *implies* $\beta y(t) \leq 0$, $0 \leq t \leq 1$. *In [13], the sign of the derivative implies monotonicity of the function. For the fractional case,* $1 < \alpha < 2$, *the case* $1 < h$ *remains open.*

4 A Monotone Method

Assume $1 < \alpha \leq 2$. Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Consider the boundary value problem

$$D_0^{\alpha}y(t) = f(t, y(t), D_0^{\alpha-1}y(t)), \quad 0 \leq t \leq 1, \tag{4.1}$$

$$y(0) = 0, \quad D_0^{\alpha-1}y(0) = D_0^{\alpha-1}y(1). \tag{4.2}$$

Assume that f satisfies the following monotonicity properties.

$$\begin{aligned} f(t,y,z_1) &< f(t,y,z_2) \quad \text{for} \quad (t,y)\in[0,1]\times\mathbb{R}, \quad z_1>z_2, \\ f(t,y_1,z) &< f(t,y_2,z) \quad \text{for} \quad (t,z)\in[0,1]\times\mathbb{R}, \quad y_1>y_2; \end{aligned} \tag{4.3}$$

that is, f is monotone decreasing in each of the third component and second component.

Note for $y \in C_{\alpha-2}$, one should initially consider the differential equation $D_0^{\alpha}y(t) = f(t,y(t),D_0^{\alpha-1}y(t))$ on $(0,1]$. The boundary condition $y(0)=0$ implies the functions produced in the following iterative schemes exist on $[0,1]$ and so, we assume (4.1) on $[0,1]$.

Apply a shift [15] to (4.1) and consider the equivalent boundary value problem,

$$D_0^{\alpha}y(t)+\beta D_0^{\alpha-1}y(t) = f(t,y(t),D_0^{\alpha-1}y(t))+\beta D_0^{\alpha-1}y(t), \quad 0\le t\le 1,$$

with boundary conditions (4.2), where $-\mathcal{B}\le\beta<0$ and $\mathcal{B}>0$ is shown to exist in Theorem 2.3. Note that if $g(t,y,z)=f(t,y,z)+\beta z$ and f satisfies (4.3), then g satisfies (4.3) if $\beta<0$.

Assume the existence of solutions, $w_1, v_1 \in \mathcal{X}_{\alpha-2}$, of the following boundary value problems for fractional differential inequalities

$$\begin{aligned} &D_0^{\alpha}w_1(t)\ge f(t,w_1(t),D_0^{\alpha-1}w_1(t)), \quad D_0^{\alpha}v_1(t)\le f(t,v_1(t),D_0^{\alpha-1}v_1(t)), \quad 0\le t\le 1, \\ &w_1(0)=0, \quad D_0^{\alpha-1}w_1(0)=D_0^{\alpha-1}w_1(1), \qquad v_1(0)=0, \quad D_0^{\alpha-1}v_1(0)=D_0^{\alpha-1}v_1(1). \end{aligned} \tag{4.4}$$

Assume further that

$$(v_1(t)-w_1(t))\ge 0, \quad 0\le t\le 1, \quad (D_0^{\alpha-1}v_1(t)-D_0^{\alpha-1}w_1(t))\ge 0, \quad 0\le t\le 1. \tag{4.5}$$

Motivated by (3.4) and noting that $\beta<0$, define a partial order $\succeq_{\beta<0}$ on $\mathcal{X}_{\alpha-2}$ by

$$u\in\mathcal{X}_{\alpha-2}\succeq_{\beta<0} 0 \iff u(t)<0, \quad 0<t\le 1, \quad \text{and} \quad D_0^{\alpha-1}u(t)\le 0, \quad 0\le t\le 1.$$

Then the assumption (4.5) implies $w_1 \succeq_{\beta<0} v_1$.

Define iteratively, the sequences $\{v_k\}_{k=1}^{\infty}$, $\{w_k\}_{k=1}^{\infty}$, where

$$\begin{aligned} D_0^{\alpha}v_{k+1}(t)+\beta D_0^{\alpha-1}v_{k+1}(t) &= f(t,v_k(t),D_0^{\alpha-1}v_k(t))+\beta D_0^{\alpha-1}v_k(t), \quad 0\le t\le 1, \\ v_{k+1}(0)=0, \quad D_0^{\alpha-1}v_{k+1}(0) &= D_0^{\alpha-1}v_{k+1}(1), \end{aligned} \tag{4.6}$$

and

$$D_0^\alpha w_{k+1}(t) + \beta D_0^{\alpha-1} w_{k+1}(t) = f(t, w_k(t), D_0^{\alpha-1} w_k(t)) + \beta D_0^{\alpha-1} w_k(t), \quad 0 \le t \le 1, \tag{4.7}$$
$$w_{k+1}(0) = 0, \quad D_0^{\alpha-1} w_{k+1}(0) = D_0^{\alpha-1} w_{k+1}(1).$$

Theorem 2.3 implies the existence of each v_{k+1}, w_{k+1} since if $|\beta| \le \mathcal{B}$, the inverse of $(\mathcal{A} + \beta D)$ exists.

Theorem 4.1. *Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and assume f satisfies the monotonicity properties* (4.3). *Assume the existence of functions $v_1, w_1 \in \mathcal{X}_{\alpha-2}$ satisfying* (4.4) *and* (4.5). *Define the sequences of iterates $\{v_k\}_{k=1}^\infty$, $\{w_k\}_{k=1}^\infty$ by* (4.6) *and* (4.7) *respectively. Then, for each $k \in \mathbb{N}_1$,*

$$w_k \succeq_{\beta<0} w_{k+1} \succeq_{\beta<0} v_{k+1} \succeq_{\beta<0} v_k. \tag{4.8}$$

Moreover, $\{v_k\}_{k=1}^\infty$ converges in $\mathcal{X}_{\alpha-2}$ to a solution v of the boundary value problem (4.1), (4.2) *and $\{w_k\}_{k=1}^\infty$ converges in $\mathcal{X}_{\alpha-2}$ to a solution w of the boundary value problem* (4.1), (4.2) *satisfying*

$$w_k \succeq_{\beta<0} w_{k+1} \succeq_{\beta<0} w \succeq_{\beta<0} v \succeq_{\beta<0} v_{k+1} \succeq_{\beta<0} v_k. \tag{4.9}$$

Proof. Since v_1 satisfies a differential inequality given in (4.5), then for $0 \le t \le 1$,

$$D_0^\alpha v_2(t) + \beta D_0^{\alpha-1} v_2(t) = f(t, v_1(t), D_0^{\alpha-1} v_1(t)) + \beta D_0^{\alpha-1} v_1(t) \ge D_0^\alpha v_1(t) + \beta D_0^{\alpha-1} v_1(t).$$

Set $u = v_2 - v_1$ and u satisfies a boundary value problem for a differential inequality,

$$D_0^\alpha u(t) + \beta D_0^{\alpha-1} u(t) \ge 0, \quad 0 \le t \le 1, \quad u(0) = 0, \quad D_0^{\alpha-1} u(0) = D_0^{\alpha-1} u(1).$$

The signed maximum principle applies and $u \succeq_{\beta<0} 0$; in particular, $v_2 \succeq_{\beta<0} v_1$. Similarly, $w_1 \succeq_{\beta<0} w_2$. Now set $u = w_2 - v_2$ and

$$\begin{aligned} D_0^\alpha u(t) + \beta D_0^{\alpha-1} u(t) &= \Big(f(t, w_1(t), D_0^{\alpha-1} w_1(t)) - f(t, v_1(t), D_0^{\alpha-1} v_1(t)) \Big) \\ &\quad + \beta (D_0^{\alpha-1} w_1(t) - D_0^{\alpha-1} v_1(t)), \quad 0 \le t \le 1, \\ u(0) = 0, &\quad D_0^{\alpha-1} u(0) = D_0^{\alpha-1} u(1). \end{aligned}$$

Since f satisfies (4.3) and $w_1 \succeq_{\beta<0} v_1$, then

$$D_0^\alpha u(t) + \beta D_0^{\alpha-1} u(t) \ge 0, \quad 0 \le t \le 1,$$

and again the signed maximum principle applies and $u \succeq_{\beta<0} 0$. In particular, $w_2 \succeq_{\beta<0} v_2$. Thus, (4.8) is proved for $k = 1$. A straightforward induction implies that (4.8) is valid using the arguments presented in this paragraph.

To obtain the existence of limiting solutions v and w satisfying (4.9), note that the sequence $\{D_0^{\alpha-1}v_k\}$ is monotone decreasing and bounded below by $\{D_0^{\alpha-1}w_1\}$. So the sequence $\{D_0^{\alpha-1}v_k\}$ is converging pointwise on $[0,1]$ to some function g defined on $[0,1]$. Moreover, $D_0^\alpha v_k = DD_0^{\alpha-1}v_k$ is uniformly bounded on

$$\Omega = \{(t,y,z) : w_1(t) \le y \le v_1(t),\ D_0^{\alpha-1}w_1(t) \le z \le D_0^{\alpha-1}v_1(t),\ 0 \le t \le 1\},$$

and so the pointwise limit g is continuous on $[0,1]$. Dini's theorem applies and $\{D_0^{\alpha-1}v_k\}$ is converging uniformly to g on $[0,1]$. Note $a = 0$, and so, we can define $v_k(0) = 0$ and extend v_k to a continuous function on $[0,1]$. The sequence $\{v_k\}$ is monotone decreasing and bounded below, and so there exists v such that $\{v_k\}$ is converging pointwise to v on $[0,1]$. Note that since $v_k(0) = 0$, then $v_k = I_0^{\alpha-1}D_0^{\alpha-1}v_k$ which converges uniformly $I_0^{\alpha-1}g$. So $v = I_0^{\alpha-1}g$ which implies $D_0^{\alpha-1}v = g$. To summarize, v_k is converging to v in $C_{\alpha-2}$ and $\{D_0^{\alpha-1}v_k\}$ is converging to $\{D_0^{\alpha-1}v\}$ in $C[0,1]$.

Finally, using $D_0^\alpha v_{k+1}(t) = f(t, v_k(t), D_0^{\alpha-1}v_k(t)) + \beta(D_0^{\alpha-1}v_k(t) - D_0^{\alpha-1}v_{k+1}(t))$, it now follows that the sequence $\{D_0^\alpha v_k\}$ converges uniformly on $[0,1]$ to $f(t, v(t), D_0^{\alpha-1}v(t))$. Since $D_0^\alpha v_k = D^1 D_0^{\alpha-1}v_k$, we conclude that $\lim\limits_{k\to\infty} D_0^\alpha v_k = D_0^\alpha v$.

Similar details apply to $\{w_k\}$ and the theorem is proved. □

Suppose now f satisfies the "anti"-inequalities to (4.3); that is suppose f satisfies

$$\begin{aligned} &f(t,y,z_1) > f(t,y,z_2) \quad \text{for} \quad (t,y) \in [0,1]\times\mathbb{R}, \quad z_1 > z_2, \\ &f(t,y_1,z) > f(t,y_2,z) \quad \text{for} \quad (t,z) \in [0,1]\times\mathbb{R}, \quad y_1 > y_2. \end{aligned} \tag{4.10}$$

One can appeal to the signed maximum principle and apply a shift to (4.1) and consider the equivalent boundary value problem, $D_0^\alpha y(t) + \beta D_0^{\alpha-1}y(t) = f(t, y(t), D_0^{\alpha-1}y(t)) + \beta D_0^{\alpha-1}y(t), 0 \le t \le 1$, where $0 < \beta < \mathcal{B}$. Note, if f satisfies (4.10) and $\beta > 0$, then $g(t,y,z) = f(t,y,z) + \beta z$ satisfies (4.10).

Now, assume the existence of solutions, $w_1, v_1 \in \mathcal{X}_{\alpha-2}$, of the following differential inequalities

$$\begin{gathered} D_0^\alpha w_1(t) \le f(t, w_1(t), D_0^{\alpha-1}w_1(t)), \quad D_0^\alpha v_1(t) \ge f(t, v_1(t), D_0^{\alpha-1}v_1(t)), \quad 0 \le t \le 1, \\ w_1(0) = 0, \quad D_0^{\alpha-1}w_1(0) = D_0^{\alpha-1}w_1(1), \quad v_1(0) = 0, \quad D_0^{\alpha-1}v_1(0) = D_0^{\alpha-1}v_1(1). \end{gathered} \tag{4.11}$$

Assume further that

$$(v_1(t) - w_1(t)) \ge 0, \quad 0 < t \le 1, \quad (D_0^{\alpha-1}v_1(t) - D_0^{\alpha-1}w_1(t)) \ge 0, \quad 0 \le t \le 1. \tag{4.12}$$

Noting that $\beta > 0$ define a partial order $\succeq_{\beta>0}$ on $\mathcal{X}_{\alpha-2}$ by

$$u \in \mathcal{X}_{\alpha-2} \succeq_{\beta>0} 0 \Longleftrightarrow u(t) \geq 0, \quad 0 < t \leq 1, \quad \text{and} \quad D_0^{\alpha-1}u(t) \geq 0, \quad 0 \leq t \leq 1.$$

In particular, in (4.12), assume $v_1 \succeq_{\beta>0} w_1$.

Theorem 4.2. *Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and assume f satisfies the monotonicity properties,* (4.10). *Assume the existence of $w_1, v_1 \in \mathcal{X}_{\alpha-2}$ satisfying* (4.11) *and* (4.12). *Define the sequences of iterates $\{v_k\}_{k=1}^{\infty}$, $\{w_k\}_{k=1}^{\infty}$ by* (4.6) *and* (4.7) *respectively. Then, for each $k \in \mathbb{N}_1$,*

$$v_k \succeq_{\beta>0} v_{k+1} \succeq_{\beta>0} w_{k+1} \succeq_{\beta>0} w_k.$$

Moreover, $\{v_k\}_{k=1}^{\infty}$ converges in $\mathcal{X}_{\alpha-2}$ to a solution v of (4.1) *and $\{w_k\}_{k=1}^{\infty}$ converges in $\mathcal{X}_{\alpha-2}$ to a solution w of* (4.1) *satisfying*

$$v_k \succeq_{\beta>0} v_{k+1} \succeq_{\beta>0} v \succeq_{\beta>0} w \succeq_{\beta>0} w_{k+1} \succeq_{\beta>0} w_k.$$

We close the article with two corollaries of Theorem 4.2 in which upper and lower solutions, v_1 and w_1 are explicitly produced.

Corollary 4.3. *Let $\mathcal{B}$ be given by Theorem 2.3. Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, assume there exists $\beta \in (0, \mathcal{B}]$ such that $f(t,y,z) + \beta z$ is bounded on $[0,1] \times \mathbb{R}^2$, and assume $g(t,y,z) = f(t,y,z) + \beta z$ satisfies the monotonicity conditions* (4.10). *Then $v_1(t) = \dfrac{M}{\beta\Gamma(\alpha)} t^{\alpha-1} \in \mathcal{X}_{\alpha-2}$ and $w_1(t) = -v_1(t) \in \mathcal{X}_{\alpha-2}$ satisfy* (4.11) *and* (4.12) *where $M = \sup_{[0,1]\times\mathbb{R}^2} |f(t,y,z) + \beta z|$; in particular, there exists a solution $y \in \mathcal{X}_{\alpha-2}$ of the boundary value problem* (4.1), (4.2) *satisfying*

$$v_1 \succeq_{\beta>0} y \succeq_{\beta>0} w_1.$$

Remark 4.4. *Remove the hypothesis that g satisfies* (4.10), *and the Schauder fixed point theorem implies the existence of a solution of the boundary value problem* (4.1), (4.2) *in the case g is bounded.*

Corollary 4.5. *Let $\mathcal{B}$ be given by Theorem 2.3. Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, assume there exists $\beta \in (0, \mathcal{B}]$ such that $g(t,y,z) = f(t,y,z) + \beta z$ satisfies the monotonicity conditions* (4.10). *Assume there exist $\sigma \in C[0,1]$ and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|g(t,y,z)| \leq \sigma(t)\psi(|y|), \quad (t,y,z) \in [0,1] \times \mathbb{R}^2.$$

Moreover, assume there exists $M > 0$ such that

$$\frac{\beta M}{|\sigma|_0 \psi\left(\frac{M}{\Gamma(\alpha)}\right)} > 1.$$

Then there exists a solution of the boundary value problem (4.1), (4.2).

Proof. Set $v_1(t) = \dfrac{M}{\Gamma(\alpha)} t^{\alpha-1} \in \mathcal{X}_{\alpha-2}$. Then

$$D_0^\alpha v_1(t) + \beta D_0^{\alpha-1} v_1(t) = \beta M > |\sigma|_0 \psi\left(\frac{M}{\Gamma(\alpha)}\right) \geq g(t, v_1, D_0^{\alpha-1} v_1(t)).$$

Set $w_1(t) = -v_1(t)$ and $v_1(t), w_1(t) \in \mathcal{X}_{\alpha-2}$ satisfy (4.11) and (4.12). □

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Stability of ternary antiderivation in ternary Banach algebras via fixed point theorem

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ABSTRACT

In this paper, we introduce the concept of ternary antiderivation on ternary Banach algebras and investigate the stability of ternary antiderivation in ternary Banach algebras, associated to the (α, β)-functional inequality:

$$\begin{aligned}&\|\mathcal{F}(x+y+z)-\mathcal{F}(x+z)-\mathcal{F}(y-x+z)-\mathcal{F}(x-z)\|\\&\leq\|\alpha(\mathcal{F}(x+y-z)+\mathcal{F}(x-z)-\mathcal{F}(y))\|+\|\beta(\mathcal{F}(x-z)\\&+\mathcal{F}(x)-\mathcal{F}(z))\|\end{aligned}$$

where α and β are fixed nonzero complex numbers with $|\alpha|+|\beta|<2$ by using the fixed point method.

RESUMEN

En este arículo, introducimos el concepto de antiderivación ternaria en álgebras de Banach ternarias e investigamos la estabilidad de las antiderivaciones ternaria en álgebras de Banach ternarias, asociadas a la (α, β)- desigualdad funcional:

$$\begin{aligned}&\|\mathcal{F}(x+y+z)-\mathcal{F}(x+z)-\mathcal{F}(y-x+z)-\mathcal{F}(x-z)\|\\&\leq\|\alpha(\mathcal{F}(x+y-z)+\mathcal{F}(x-z)-\mathcal{F}(y))\|+\|\beta(\mathcal{F}(x-z)\\&+\mathcal{F}(x)-\mathcal{F}(z))\|\end{aligned}$$

donde α y β son números complejos no cero fijos, con $|\alpha|+|\beta|<2$ usando el método de punto fijo.

Keywords and Phrases: Hyers-Ulam stability; stability; fixed point method; ternary antiderivation; ternary Banach algebra; additive functional inequality.

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1 Introduction

A ternary Banach algebra is a complex Banach space $\mathcal{A}$, endowed with a ternary product $(x, y, z) \to [x, y, z]$ of $\mathcal{A}^3$ into $\mathcal{A}$, which is $\mathbb{C}-$linear in each variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$ for all $x, y, z, w, u \in \mathcal{A}$.

If a ternary Banach algebra $(\mathcal{A}, [\cdot, \cdot, \cdot])$ has an unit, *i.e.*, an element $e \in \mathcal{A}$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in \mathcal{A}$, then it is routine to verify that $\mathcal{A}$, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital algebra. Conversely, if $(\mathcal{A}, \circ)$ is a unital algebra, then $[x, y, z] := x \circ y^* \circ z$ makes $\mathcal{A}$ into a ternary Banach algebra.

A $\mathbb{C}-$linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a ternary homomorphism if $H([x, y, z]) = [H(x), H(y), H(z)]$ for all $x, y, z \in \mathcal{A}$. A $\mathbb{C}-$linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a ternary derivation if

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in \mathcal{A}$.

We say that an equation is stable if any function satisfying the equation approximately is near to an exact solution of the equation.

The stability problem of functional equations started from a question of Ulam, in 1940, on the stability of group homomorphisms. In 1941, Hyers [17] gave an answer to the question of Ulam in the context of Banach spaces in the case of additive mappings, that was an major step toward further solutions in this field.

During the last two decades, a number of articles and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, k-additive mappings, multiplicative mappings, bounded nth differences, Euler-Lagrange functional equations, differential equations, and Navier-Stokes equations (see [1, 2, 4, 5, 19, 22, 25, 26, 27, 28, 29]).

Also, approximate generalized Lie derivations have been already established in [6, 7].

Ternary algebraic structures appear in various domains of theoretical and mathematical physics, such as the quark model and Nambu mechanics [18, 21]. Today, many physical systems can be modeled as a linear system. The principle of additivity has various applications in physics especially in calculating the internal energy in thermodynamic and also the meaning of the superposition principle.

In recent years, the Hyers-Ulam stability of various (among others functional, differential and integral) equations and other objects (for example in groups, Banach algebra, ternary Banach algebras and C^*-ternary algebras) has been intensively studied (see [8, 9, 10, 11, 15, 16, 30]).

Fixed-point theory has been studied by various methods. The study on fixed point theory provides essential tools for solving problems arising in various fields of functional analysis, such as dynamical systems, equilibrium problems and differential equations (see for instance [3, 14, 24]).

We recall a fundamental result in fixed point theory.

Definition 1.1 ([12]). *Let $\mathcal{X}$ be a non-empty set and $d : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ a mapping such that*

(1) $d(x, y) = 0$ *if and only if* $x = y$,

(2) $d(x, y) = d(y, x)$ *for all* $x, y \in \mathcal{X}$,

(3) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in \mathcal{X}$.

Then d is called a generalized metric and $(\mathcal{X}, d)$ is a generalized metric space.

Theorem 1.2 ([12]). *Let $(\mathcal{X}, d)$ be a complete generalized metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping, that is,*

$$d(\mathcal{T}x, \mathcal{T}y) \leq Ld(x, y)$$

for some $L < 1$ and all $x, y \in \mathcal{X}$. Then for each given element $x \in \mathcal{X}$, either

$$d(\mathcal{T}^n x, \mathcal{T}^{n+1} x) = +\infty$$

for all $n \geq 0$ or

$$d(\mathcal{T}^n x, \mathcal{T}^{n+1} x) < +\infty, \qquad \forall n \geq n_0,$$

for some positive integer n_0. Moreover, if the second alternative holds, then

(i) the sequence $\{\mathcal{T}^n x\}$ is convergent to a fixed point y^* of $\mathcal{T}$;

(ii) y^* is the unique fixed point of $\mathcal{T}$ in the set $Y := \{y \in \mathcal{X}, d(\mathcal{T}^{n_0} x, y) < +\infty\}$ and $d(y, y^*) \leq \dfrac{1}{1-L} d(y, \mathcal{T}y)$ for all $y \in Y$.

In this paper, we consider the following functional inequality

$$\begin{aligned} &\|\mathcal{F}(x+y+z) - \mathcal{F}(x+z) - \mathcal{F}(y-x+z) - \mathcal{F}(x-z)\| \\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| \end{aligned} \tag{1.1}$$

for all $x, y, z \in \mathcal{A}$, where α and β are fixed nonzero complex numbers with $|\alpha| + |\beta| < 2$.

Throughout this paper, assume that $\mathcal{A}$ is a ternary Banach algebra and α and β are fixed nonzero complex numbers with $|\alpha| + |\beta| < 2$.

The aim of the present paper is to establish the stability problem of ternary antiderivations in complex ternary Banach algebras by using the fixed point method.

2 Stability of (α, β)-functional inequality (1.1)

In this section, we prove the Hyers-Ulam stability of the additive (α, β)-functional inequality (1.1) by using the fixed point method.

Lemma 2.1. *Let $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying*

$$\begin{aligned} &\|\mathcal{F}(x+y+z) - \mathcal{F}(x+z) - \mathcal{F}(y-x+z) - \mathcal{F}(x-z)\| \\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) - \mathcal{F}(x) + \mathcal{F}(z))\| \end{aligned} \tag{2.1}$$

for all $x, y, z \in \mathcal{A}$. Then the mapping $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is additive.

Proof. Assume that $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ satisfies (2.1).

Putting $x = y = z = 0$ in (2.1), we have

$$2\|\mathcal{F}(0)\| \leq (|\alpha| + |\beta|)\|\mathcal{F}(0)\|$$

and thus $\mathcal{F}(0) = 0$, since $|\alpha| + |\beta| < 2$.

Letting $z = x$ in (2.1), we obtain

$$\|\mathcal{F}(2x+y) - \mathcal{F}(2x) - \mathcal{F}(y)\| \leq 0$$

and so $\mathcal{F}(2x+y) = \mathcal{F}(2x) + \mathcal{F}(y)$ for all $x, y \in \mathcal{A}$. Therefore $\mathcal{F}$ is additive. □

Theorem 2.2. *Suppose that $\Lambda : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ is a function such that there exists an $L < 1$ with*

$$\Lambda(x, y, z) \leq \frac{L}{2}\Lambda(2x, 2y, 2z) \tag{2.2}$$

for all $x, y, z \in \mathcal{A}$. Let $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying

$$\begin{aligned} &\|\mathcal{F}(x+y+z) - \mathcal{F}(x+z) - \mathcal{F}(y-x+z) - \mathcal{F}(x-z)\| \\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \Lambda(x, y, z) \end{aligned} \tag{2.3}$$

for all $x, y, z \in \mathcal{A}$. *Then there exists a unique additive mapping* $\Delta : \mathcal{A} \to \mathcal{A}$ *such that*

$$\|\mathcal{F}(x) - \Delta(x)\| \leq \frac{L}{2(1-L)} \Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right) \tag{2.4}$$

for all $x \in \mathcal{A}$.

Proof. Setting $x = y = z = 0$ in (2.3), we have

$$2\|\mathcal{F}(0)\| \leq (|\alpha| + |\beta|)\|\mathcal{F}(0)\| + \Lambda(0,0,0)$$

and thus $\mathcal{F}(0) = 0$, since $|\alpha| + |\beta| < 2$ and by (2.2) $\Lambda(0,0,0) = 0$.

Letting $x = z = \frac{t}{2}$ and $y = t$ in (2.3), we get

$$\|\mathcal{F}(2t) - 2\mathcal{F}(t)\| \leq \Lambda\left(\frac{t}{2}, t, \frac{t}{2}\right) \tag{2.5}$$

for all $t \in \mathcal{A}$.

Now, consider the set $\Omega = \{\omega : \mathcal{A} \to \mathcal{A} : \omega(0) = 0\}$ and the mapping d defined on $\Omega \times \Omega$ by

$$d(\delta, \omega) = \inf\left\{k \in \mathbb{R}_+ : \|\delta(x) - \omega(x)\| \leq k\Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right), \forall x \in \mathcal{A}\right\},$$

where as usual, $\inf \emptyset = +\infty$. d is a complete generalized metric on Ω (see [20]).

Now, let us consider the linear mapping $\mathcal{T} : \Omega \to \Omega$ such that

$$\mathcal{T}\delta(x) := 2\delta\left(\frac{x}{2}\right)$$

for all $x \in \mathcal{A}$. Thus $d(\delta, \omega) = \varepsilon$ implies that

$$\|\delta(x) - \omega(x)\| \leq \epsilon\Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right)$$

for all $x \in \mathcal{A}$. Hence

$$\|\mathcal{T}\delta(x) - \mathcal{T}\omega(x)\| = \left\|2\delta\left(\frac{x}{2}\right) - 2\omega\left(\frac{x}{2}\right)\right\| \leq 2\varepsilon\Lambda\left(\frac{x}{4}, \frac{x}{2}, \frac{x}{4}\right) \leq L\varepsilon\Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right)$$

for all $x \in \mathcal{A}$, that is $d(\delta, \omega) = \varepsilon$ implies that $d(\mathcal{T}\delta(x), \mathcal{T}\omega(x)) \leq L\varepsilon$. This means that

$$d(\mathcal{T}\delta(x), \mathcal{T}\omega(x)) \leq Ld(\delta, \omega)$$

for all $\delta, \omega \in \Omega$.

Next, from (2.5), we get

$$\left\|\mathcal{F}(x)-2\mathcal{F}\left(\frac{x}{2}\right)\right\| \leq \Lambda\left(\frac{x}{4},\frac{x}{2},\frac{x}{4}\right) \leq \frac{L}{2}\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x \in \mathcal{A}$, it follows that $d(\mathcal{F},\mathcal{T}\mathcal{F}) \leq \frac{L}{2}$.

Using the fixed point alternative we deduce the existence of a unique fixed point of $\mathcal{T}$, that is, the existence of a mapping $\Delta : \mathcal{A} \to \mathcal{A}$ such that

$$\Delta(x) = 2\Delta\left(\frac{x}{2}\right)$$

with the following property: there exists a $k \in (0,\infty)$ satisfying

$$\|\mathcal{F}(x)-\Delta(x)\| \leq k\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x \in \mathcal{A}$.

Since $\lim_{n\to\infty} d(\mathcal{T}^n\mathcal{F},\Delta) = 0$,

$$\lim_{n\to\infty} 2^n\mathcal{F}\left(\frac{x}{2^n}\right) = \Delta(x)$$

for all $x \in \mathcal{A}$.

Also, $d(\mathcal{F},\Delta) \leq \dfrac{1}{1-L}d(\mathcal{F},\mathcal{T}\mathcal{F})$ which implies

$$\|\mathcal{F}(x)-\Delta(x)\| \leq \frac{L}{2(1-L)}\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x \in \mathcal{A}$. It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\|\Delta(x+y+z)-\Delta(x+z)-\Delta(y-x+z)-\Delta(x-z)\| \\
&= \lim_{n\to\infty} 2^n\left\|\mathcal{F}\left(\frac{x+y+z}{2^n}\right)-\mathcal{F}\left(\frac{x+z}{2^n}\right)-\mathcal{F}\left(\frac{y-x+z}{2^n}\right)-\mathcal{F}\left(\frac{x-z}{2^n}\right)\right\| \\
&\leq \lim_{n\to\infty} 2^n\left\|\alpha\left(\mathcal{F}\left(\frac{x+y-z}{2^n}\right)+\mathcal{F}\left(\frac{x-z}{2^n}\right)-\mathcal{F}\left(\frac{y}{2^n}\right)\right)\right\| \\
&\quad + \lim_{n\to\infty} 2^n\left\|\beta\left(\mathcal{F}\left(\frac{x-z}{2^n}\right)+\mathcal{F}\left(\frac{x}{2^n}\right)-\mathcal{F}\left(\frac{z}{2^n}\right)\right)\right\| + \lim_{n\to\infty} 2^n\Lambda\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) \\
&= \|\alpha(\Delta(x+y-z)+\Delta(x-z)-\Delta(y))\| + \|\beta(\Delta(x-z)+\Delta(x)-\Delta(z))\|
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Therefore, by Lemma 2.1, the mapping Δ is additive. □

Corollary 2.3. *Let $\mathcal{F}:\mathcal{A}\to\mathcal{A}$ be a mapping satisfying*

$$\begin{aligned}&\|\mathcal{F}(x+y+z)-\mathcal{F}(x+z)-\mathcal{F}(y-x+z)-\mathcal{F}(x-z)\|\\&\le\|\alpha(\mathcal{F}(x+y-z)+\mathcal{F}(x-z)-\mathcal{F}(y))\|+\|\beta(\mathcal{F}(x-z)+\mathcal{F}(x)-\mathcal{F}(z))\|+\|[x,y,z]\|\end{aligned}$$

for all $x,y,z\in\mathcal{A}$. Then there exists a unique additive mapping $\Delta:\mathcal{A}\to\mathcal{A}$ such that

$$\|\mathcal{F}(x)-\Delta(x)\|\le\|[x,x,x]\|$$

for all $x\in\mathcal{A}$.

Proof. The proof follows from Theorem 2.2 by taking $L=\frac{8}{9}$ and $\Lambda(x,y,z)=\|[x,y,z]\|$ for all $x,y,z\in\mathcal{A}$. □

3 Stability of ternary antiderivations in ternary algebras

In this section we introduce the concept of ternary antiderivation in ternary Banach algebras and prove the stability of ternary antiderivations associated to (1.1) in ternary Banach algebras.

Definition 3.1. *Let $\mathcal{A}$ be a ternary Banach algebra. A $\mathbb{C}-$linear mapping $\mathcal{I}:\mathcal{A}\to\mathcal{A}$ is called a ternary antiderivation if it satisfies*

$$[\mathcal{I}(x),\mathcal{I}(y),\mathcal{I}(z)]=\mathcal{I}[\mathcal{I}(x),y,z]+\mathcal{I}[x,\mathcal{I}(y),z]+\mathcal{I}[x,y,\mathcal{I}(z)]$$

for all $x,y,z\in\mathcal{A}$.

Example 3.2. *The complex number set $\mathbb{C}$ with a ternary product $[x,y,z]=xyz$ for all $x,y,z\in\mathbb{C}$, is a ternary Banach algebra.*

Define $\mathcal{I}:\mathbb{C}\to\mathbb{C}$ by

$$\mathcal{I}(x)=3x$$

for all $x\in\mathbb{C}$. Then $\mathcal{I}$ is a ternary antiderivation.

Definition 3.3 ([13])**.** *Let $\mathcal{A}$ be a ternary Banach algebra. A double sequence $\{a_{n,m}\}$ in $\mathcal{A}$ converges to $L\in\mathcal{A}$ and we write $\lim\limits_{n,m\to\infty}a_{n,m}=L$ if for every $\epsilon>0$ there is an integer N such that for all $n,m\ge N$,*

$$|a_{n,m}-L|<\epsilon.$$

If no such number L exists, we say that $\{a_{n,m}\}$ diverges.

Lemma 3.4 ([23])**.** *Let $\mathcal{A}$ be complex Banach algebra and let $f : \mathcal{A} \to \mathcal{A}$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x \in \mathcal{A}$, then f is $\mathbb{C}-$linear.*

Theorem 3.5. *Let $\Lambda : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a function, and let there exists an $L < 1$ with satisfying*

$$\Lambda(x, y, z) \leq \frac{L}{8}\Lambda(2x, 2y, 2z) \tag{3.1}$$

for all $x, y, z \in \mathcal{A}$. Assume that $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is a mapping such that

$$\begin{aligned} &\|\mathcal{F}(\mu(x+y+z)) - \mu\mathcal{F}(x+z) - \mu\mathcal{F}(y-x+z) - \mu\mathcal{F}(x-z)\| \\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \Lambda(x, y, z) \end{aligned} \tag{3.2}$$

and

$$\|[\mathcal{F}(x), \mathcal{F}(y), \mathcal{F}(z)] - \mathcal{F}[\mathcal{F}(x), y, z] - \mathcal{F}[x, \mathcal{F}(y), z] - \mathcal{F}[x, y, \mathcal{F}(z)]\| \leq \Lambda(x, y, z) \tag{3.3}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{A}$. If $\mathcal{F}$ is continuous and in addition, $\mathcal{F}_n(x) := 2^n\mathcal{F}\left(\frac{x}{2^n}\right)$ converges uniformly for all $x \in \mathcal{A}$, double sequences $\{2^{n+m}\mathcal{F}\left(\mathcal{F}\left(\frac{x}{2^n}\right)\frac{y}{2^m}\right)\}$ and $\{2^{n+m}\mathcal{F}\left(\frac{x}{2^n}\mathcal{F}\left(\frac{y}{2^m}\right)\right)\}$ are convergent for all $x, y \in \mathcal{A}$, then there exists a unique continuous ternary antiderivation $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ such that

$$\|\mathcal{F}(x) - \mathcal{I}(x)\| \leq \frac{L}{2(4-L)}\Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right)$$

for all $x \in \mathcal{A}$.

Proof. Assume that $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ satisfies (3.2).

Putting $\mu = 1$ and $x = y = z = 0$ in (3.2), we obtain

$$2\|\mathcal{F}(0)\| \leq (|\alpha| + |\beta|)\|\mathcal{F}(0)\| + \Lambda(0, 0, 0)$$

and thus $\mathcal{F}(0) = 0$, since $|\alpha| + |\beta| < 2$ and by (3.1), $\Lambda(0, 0, 0) = 0$.

Letting $x = z = \frac{t}{2}$ and $y = t$ in (3.2), we have

$$\|\mathcal{F}(2\mu t) - 2\mu\mathcal{F}(t)\| \leq \Lambda\left(\frac{t}{2}, t, \frac{t}{2}\right) \tag{3.4}$$

for all $\mu \in \mathbb{T}^1$ and all $t \in \mathcal{A}$.

Next, consider the set

$$\Omega := \{\omega : \mathcal{A} \to \mathcal{A} : \omega(0) = 0\}$$

and define the generalized metric on Ω

$$d(\theta,\omega)=\inf\left\{k\in\mathbb{R}_{\geq 0}:\|\theta(x)-\omega(x)\|\leq k\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right),\forall x\in A\right\},$$

where as usual, $\inf\emptyset=+\infty$. By [20, Lemma 1.2], (Ω,d) is a complete generalized metric space.

Now we define the linear mapping $\mathcal{T}:\Omega\to\Omega$ such that

$$\mathcal{T}\theta(x)=2\theta\left(\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$.

Let $\theta,\omega\in\Omega$ be given such that $d(\theta,\omega)=\varepsilon$. Then

$$\|\theta(x)-\omega(x)\|\leq\varepsilon\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$. Hence

$$\|\mathcal{T}\theta(x)-\mathcal{T}\omega(x)\|=\left\|2\theta\left(\frac{x}{2}\right)-2\omega\left(\frac{x}{2}\right)\right\|\leq 2\varepsilon\Lambda\left(\frac{x}{4},\frac{x}{2},\frac{x}{4}\right)\leq\frac{L}{4}\varepsilon\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$. So $d(\theta,\omega)=\varepsilon$ implies that $d(\mathcal{T}\theta(x),\mathcal{T}\omega(x))\leq\frac{L}{4}\varepsilon$. Hence

$$d(\mathcal{T}\theta(x),\mathcal{T}\omega(x))\leq\frac{L}{4}d(\theta,\omega)$$

for all $\theta,\omega\in\Omega$. It follows from (3.4) that

$$\left\|\mathcal{F}(x)-2\mathcal{F}\left(\frac{x}{2}\right)\right\|\leq\Lambda\left(\frac{x}{4},\frac{x}{2},\frac{x}{4}\right)\leq\frac{L}{8}\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$ and so $d(\mathcal{F},\mathcal{T}\mathcal{F})\leq\frac{L}{8}$.

Using the fixed point alternative we deduce the existence of a unique fixed point of $\mathcal{T}$, that is, the existence of a mapping $\mathcal{I}:\mathcal{A}\to\mathcal{A}$ such that

$$\mathcal{I}(x)=2\mathcal{I}\left(\frac{x}{2}\right)$$

with the following property: there exists a $k\in(0,\infty)$ satisfying

$$\|\mathcal{F}(x)-\mathcal{I}(x)\|\leq k\Lambda\left(\frac{x}{2},x,\frac{x}{2}\right)$$

for all $x\in\mathcal{A}$.

Since $\lim_{n\to\infty}d(\mathcal{T}^n\mathcal{F},\mathcal{I})=0$,

$$\lim_{n\to\infty}2^n\mathcal{F}\left(\frac{x}{2^n}\right)=\mathcal{I}(x)$$

for all $x \in \mathcal{A}$. Also, $d(\mathcal{F},\mathcal{I}) \leq \frac{1}{1-\frac{L}{4}} d(\mathcal{F},\mathcal{T}\mathcal{F})$ which implies

$$\|\mathcal{F}(x) - \mathcal{I}(x)\| \leq \frac{L}{2(4-L)} \Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right)$$

for all $x \in \mathcal{A}$. It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\|\mathcal{I}(x+y+z) - \mathcal{I}(x+z) - \mathcal{I}(y-x+z) - \mathcal{I}(x-z)\| \\
&= \lim_{n\to\infty} \left\| 2^n \left(\mathcal{F}\left(\frac{x+y+z}{2^n}\right) - \mathcal{F}\left(\frac{x+z}{2^n}\right) - \mathcal{F}\left(\frac{y-x+z}{2^n}\right) - \mathcal{F}\left(\frac{x-z}{2^n}\right) \right) \right\| \\
&\leq \lim_{n\to\infty} 2^n \left\| \alpha \left(\mathcal{F}\left(\frac{x+y-z}{2^n}\right) + \mathcal{F}\left(\frac{x-z}{2^n}\right) - \mathcal{F}\left(\frac{y}{2^n}\right) \right) \right\| \\
&\quad + \lim_{n\to\infty} 2^n \left\| \beta \left(\mathcal{F}\left(\frac{x-z}{2^n}\right) + \mathcal{F}\left(\frac{x}{2^n}\right) - \mathcal{F}\left(\frac{z}{2^n}\right) \right) \right\| + \lim_{n\to\infty} 2^n \Lambda\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&= \|\alpha(\mathcal{I}(x+y-z) + \mathcal{I}(x-z) - \mathcal{I}(y))\| + \|\beta(\mathcal{I}(x-z) + \mathcal{I}(x) - \mathcal{I}(z))\|
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. By Lemma 2.1, the mapping $\mathcal{I}$ is additive.

Letting $x = z = \frac{t}{2}$ and $y = 0$ in (3.2), we get

$$\|\mathcal{F}(\mu t) - \mu\mathcal{F}(t)\| \leq \Lambda\left(\frac{t}{2}, 0, \frac{t}{2}\right)$$

for all $\mu \in \mathbb{T}^1$ and all $t \in \mathcal{A}$. Thus

$$\begin{aligned}
\|\mathcal{I}(\mu x) - \mu\mathcal{I}(x)\| &= \lim_{n\to\infty} 2^n \left\| \mathcal{F}\left(\mu\frac{x}{2^n}\right) - \mu\mathcal{F}\left(\frac{x}{2^n}\right) \right\| \\
&\leq \lim_{n\to\infty} 2^n \Lambda\left(\frac{x}{2^{n+1}}, 0, \frac{x}{2^{n+1}}\right) \leq \lim_{n\to\infty} \left(\frac{L}{4}\right)^n \Lambda\left(\frac{x}{2}, 0, \frac{x}{2}\right),
\end{aligned}$$

which tends to zero as $n \to \infty$ and so $\mathcal{I}(\mu x) = \mu\mathcal{I}(x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Therefore, by Lemma 3.4, the mapping $\mathcal{I}$ is $\mathbb{C}-$linear.

Since $\mathcal{F}$ is continuous and $\mathcal{F}_n$ converges uniformly, $\mathcal{I}$ is continuous. It follows from (3.1) and (3.3) that

$$\begin{aligned}
&\|[\mathcal{I}(x), \mathcal{I}(y), \mathcal{I}(z)] - \mathcal{I}[\mathcal{I}(x), y, z] - \mathcal{I}[x, \mathcal{I}(y), z] - \mathcal{I}[x, y, \mathcal{I}(z)]\| \\
&= \lim_{n\to\infty} \left\| 2^{3n} \left[\mathcal{F}\left(\frac{x}{2^n}\right), \mathcal{F}\left(\frac{y}{2^n}\right), \mathcal{F}\left(\frac{z}{2^n}\right) \right] - 2^n\mathcal{I}\left[\mathcal{F}\left(\frac{x}{2^n}\right), y, z \right] \right. \\
&\qquad \left. - 2^n\mathcal{I}\left[x, \mathcal{F}\left(\frac{y}{2^n}\right), z \right] - 2^n\mathcal{I}\left[x, y, \mathcal{F}\left(\frac{z}{2^n}\right) \right] \right\| \\
&= \lim_{n\to\infty} 2^{3n} \left\| \left[\mathcal{F}\left(\frac{x}{2^n}\right), \mathcal{F}\left(\frac{y}{2^n}\right), \mathcal{F}\left(\frac{z}{2^n}\right) \right] - \mathcal{F}\left[\mathcal{F}\left(\frac{x}{2^n}\right), \frac{y}{2^n}, \frac{z}{2^n} \right] \right. \\
&\qquad \left. - \mathcal{F}\left[\frac{x}{2^n}, \mathcal{F}\left(\frac{y}{2^n}\right), \frac{z}{2^n} \right] - \mathcal{F}\left[\frac{x}{2^n}, \frac{y}{2^n}, \mathcal{F}\left(\frac{z}{2^n}\right) \right] \right\| \\
&\leq \lim_{n\to\infty} 2^{3n} \Lambda\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \leq \lim_{n\to\infty} L^n \Lambda(x, y, z)
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{A}$. Since $L < 1$, the $\mathbb{C}-$linear mapping $\mathcal{I}$ is a ternary antiderivation. $\square$

Corollary 3.6. *Let $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying*

$$\begin{aligned}&\|\mathcal{F}(\mu(x+y+z)) - \mu\mathcal{F}(x+z) - \mu\mathcal{F}(y-x+z) - \mu\mathcal{F}(x-z)\| \\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \|[[x,x,y],y,z]\|,\end{aligned}$$

$$\|[\mathcal{F}(x), \mathcal{F}(y), \mathcal{F}(z)] - \mathcal{F}[\mathcal{F}(x), y, z] + \mathcal{F}[x, \mathcal{F}(y), z] + \mathcal{F}[x, y, \mathcal{F}(z)]\| \leq \|[[x,x,y],y,z]\|$$

for all $x, y, z \in \mathcal{A}$. If $\mathcal{F}$ is continuous and in addition, $\mathcal{F}_n(x) := 2^n\mathcal{F}\left(\frac{x}{2^n}\right)$ converges uniformly for all $x \in \mathcal{A}$, double sequences $\{2^{n+m}\mathcal{F}\left(\mathcal{F}\left(\frac{x}{2^n}\right)\frac{y}{2^m}\right)\}$ and $\{2^{n+m}\mathcal{F}\left(\frac{x}{2^n}\mathcal{F}\left(\frac{y}{2^m}\right)\right)\}$ are convergent for all $x, y \in \mathcal{A}$, then there exists a unique continuous ternary antiderivation $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ such that

$$\|\mathcal{F}(x) - \mathcal{I}(x)\| \leq \frac{1}{50}\|x\|^5$$

for all $x \in \mathcal{A}$.

Proof. The proof follows from Theorem 3.5 by taking $L = \frac{32}{33}$ and $\Lambda(x, y, z) = \|[[x,x,y],y,z]\|$ for all $x, y, z \in \mathcal{A}$. $\square$

4 Stability of continuous ternary antiderivations in ternary Banach algebras

In this section, we prove the stability of continuous ternary antiderivations in ternary Banach algebras.

Theorem 4.1. *Let $\Lambda : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a function. If there exists an $L < 1$ with satisfying*

$$\Lambda(x, y, z) \leq \frac{L}{8}\Lambda(2x, 2y, 2z) \tag{4.1}$$

for all $x, y, z \in \mathcal{A}$. Assume that $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying

$$\begin{aligned}&\|\mathcal{F}(\mu(x+y+z)) - \mu\mathcal{F}(x+z) - \mu\mathcal{F}(y-x+z) - \mu\mathcal{F}(x-z)\| \\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \Lambda(x,y,z)\end{aligned} \tag{4.2}$$

and (3.3) for all μ with $|\mu| < 1$ (resp. $|\mu| > 1$). If $\mathcal{F}$ is continuous and in addition, $\mathcal{F}_n(x) := 2^n\mathcal{F}\left(\frac{x}{2^n}\right)$ converges uniformly for all $x \in \mathcal{A}$, double sequences $\{2^{n+m}\mathcal{F}\left(\mathcal{F}\left(\frac{x}{2^n}\right)\frac{y}{2^m}\right)\}$ and

$\{2^{n+m}\mathcal{F}\left(\frac{x}{2^n}\mathcal{F}\left(\frac{y}{2^m}\right)\right)\}$ *are convergent for all* $x, y \in \mathcal{A}$*, then there exists a unique continuous ternary antiderivation* $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ *such that*

$$\|\mathcal{F}(x) - \mathcal{I}(x)\| \leq \frac{L}{2(4-L)}\Lambda\left(\frac{x}{2}, x, \frac{x}{2}\right) \tag{4.3}$$

for all $x \in \mathcal{A}$*.*

Proof. Let $\mu \in \mathbb{T}^1$. Then there exists a sequence $\{\mu_n\}_{n=1}^{\infty}$ with $|\mu_n| < 1$ (resp. $|\mu_n| > 1$) such that

$$\lim_{n\to\infty} \mu_n = \mu.$$

By (4.2) we get

$$\begin{aligned}&\|\mathcal{F}(\mu_n(x+y+z)) - \mu_n\mathcal{F}(x+z) - \mu_n\mathcal{F}(y-x+z) - \mu_n\mathcal{F}(x-z)\|\\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \Lambda(x,y,z)\end{aligned}$$

for all positive integers n, all μ_n with $|\mu_n| < 1$ (resp. $|\mu_n| > 1$) and all $x, y, z \in \mathcal{A}$.

Passing to the limit as $n \to \infty$, and using the continuity of $\mathcal{F}$ and $\|\cdot\|$, we obtain

$$\begin{aligned}&\|\mathcal{F}(\mu(x+y+z)) - \mu\mathcal{F}(x+z) - \mu\mathcal{F}(y-x+z) - \mu\mathcal{F}(x-z)\|\\ &\leq \|\alpha(\mathcal{F}(x+y-z) + \mathcal{F}(x-z) - \mathcal{F}(y))\| + \|\beta(\mathcal{F}(x-z) + \mathcal{F}(x) - \mathcal{F}(z))\| + \Lambda(x,y,z)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{A}$

Therefore, by the same reasoning as in the proof of Theorem 3.5, there exists a unique ternary antiderivation $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ satisfying (4.3). □

Declarations

Availablity of data and materials

Not applicable.

Human and animal rights

We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest

The authors declare that they have no competing interests.

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Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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On generalized Hardy spaces associated with singular partial differential operators

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ABSTRACT

We define and study the Hardy spaces associated with singular partial differential operators. Also, a characterization by mean of atomic decomposition is investigated.

RESUMEN

Definimos y estudiamos los espacios de Hardy asociados a operadores diferenciales parciales singulares. También investigamos una caracterización por medio de la descomposición atómica.

Keywords and Phrases: Riemann-Liouville operator, Hardy spaces, Poisson maximal function, atomic decomposition.

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1 Introduction

The foundations of the real Hardy space $H^p(\mathbb{R}^n)$, $p \in [1,+\infty[$, were started with the works of C. Fefferman and E. M. Stein [10]. Hardy spaces were deeply developed later by R. Coifman and G. Weiss [8]. The theory of Hardy spaces $H^p(\mathbb{R}^n)$, plays a very important role in harmonic analysis and operator theory and it is shown that it has many interesting applications, for more details we refer the reader to [20]. In the euclidean case, there are many equivalent definitions of the Hardy spaces $H^p(\mathbb{R}^n)$ either by using the Poisson maximal function or by using the atomic decomposition. Uchiyama [19] characterized also the Hardy spaces $H^p(\mathbb{R}^n)$ by means of Littlewood-Paley g-function.

In [5], Baccar, Ben Hamadi and Rachdi have considered the following singular partial differential operators

$$\begin{cases} \Delta_1 = \dfrac{\partial}{\partial x}, \\ \Delta_2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2}; \quad (r,x) \in]0,+\infty[\times\mathbb{R}; \quad \alpha \geqslant 0, \end{cases}$$

and they associated to Δ_1 and Δ_2 the so called Riemann-Liouville operator $\mathscr{R}_\alpha$ defined on $\mathscr{C}_e(\mathbb{R}^2)$ (The space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable), by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}\,dt\,ds, & \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)\dfrac{dt}{\sqrt{(1-t^2)}}, & \text{if } \alpha = 0. \end{cases}$$

The Riemann-Liouville operator $\mathscr{R}_\alpha$ generalizes the spherical mean operator given by

$$\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\sin\theta, x + r\cos\theta)\,d\theta,$$

which plays an important role in image processing of the so-called synthetic aperture radar (SAR) data, and in the linearized inverse scattering problem in acoustics, as well as in the interpretation of many physical phenomena in quantum mechanics, see [9, 11, 12].

According to [5], the Fourier transform $\mathscr{F}_\alpha$ associated with the Riemann-Liouville operator is defined for every $(s,y) \in \Upsilon$, by

$$\mathscr{F}_\alpha(f)(s,y) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\mathscr{R}_\alpha\left(\cos(s.)e^{-iy.}\right)(r,x)\frac{r^{2\alpha+1}dr\,dx}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}},$$

for a suitable integrable function, where Υ is a set that will be defined later.

Many harmonic analysis results have been already proved by Baccar, Ben Hamadi, Rachdi, Rouz and Omri for the Riemann-Liouville operator and its Fourier transform [3, 4, 5, 6, 7, 18]. Hleili, Mejjaoli, Omri and Rachdi have also established several uncertainty principles for the same Fourier transform $\mathscr{F}_\alpha$ [13, 15, 16, 17] .

Our purpose in this work is to define and study the Hardy's spaces $\mathcal{H}_\alpha^p$ related to the Riemann-Liouville operator and to characterize theses spaces for $p \in [1,+\infty[$ by using Poisson maximal operator associated to $\mathscr{R}_\alpha$ and by using atomic decomposition as well.

The paper is organized as follows. In the second section, we give some classical harmonic analysis results related to the Riemann-Liouville operator, the third section is devoted to the characterization of the Hardy spaces related to $\mathscr{R}_\alpha$ by using its Poisson maximal function. In the last section, we introduce the atomic decomposition which allows us to characterize $\mathcal{H}_\alpha^1$.

2 Riemann-Liouville operator

In this section we give and develop some harmonic analysis results related to the Riemann-Liouville operator that we will use later. For the proofs of these results we refer the reader to [5] and [7]. In [5] Baccar, Ben Hamadi and Rachdi considered the following system

$$\begin{cases} \triangle_1 u = -i\lambda u(r,x) \\ \triangle_2 u = -\mu^2 u(r,x) \\ u(0,0) = 1, \quad \frac{\partial u}{\partial x}(0,x), \quad \forall x \in \mathbb{R} \end{cases}$$

and showed that for all $(\mu,\lambda) \in \mathbb{C}^2$, this system admits a unique infinitely differentiable solution given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)e^{-i\lambda x},$$

where j_α is the modified Bessel function of the first kind and index α, see [14, 21].

The function $\varphi_{\mu,\lambda}$ is bounded on $[0,+\infty[\times\mathbb{R}$ if and only if (μ,λ) belongs to the set

$$\Upsilon = \mathbb{R}^2 \cup \big\{(ir,x),\ (r,x) \in \times\mathbb{R}^2,\ |r| \leqslant |x|\big\}.$$

In this case, we have

$$\sup_{(r,x)\in\mathbb{R}^2} |\varphi_{\mu,\lambda}(r,x)| = 1.$$

In the following we denote by

- ν_α the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$d\nu_\alpha(r,x) = \frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\,dr\,dx,$$

- $L^p(d\nu_\alpha)$, $p\in[1,+\infty]$, is the Lebesgue space of all measurable functions f on $[0,+\infty[\times\mathbb{R}$ such that $\|f\|_{p,\nu_\alpha}<+\infty$, where

$$\|f\|_{p,\nu_\alpha} = \begin{cases} \left(\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|^p d\nu_\alpha(r,x)\right)^{\frac{1}{p}}, & \text{if } p\in[1,+\infty[\\ \underset{(r,x)\in[0,+\infty[\times\mathbb{R}}{\text{ess sup}} |f(r,x)|, & \text{if } p=+\infty. \end{cases}$$

- $L^1_{loc}(d\nu_\alpha)$ the space of measurable functions on $[0,+\infty[\times\mathbb{R}$ that are locally integrable on $[0,+\infty[\times\mathbb{R}$ with respect to the measure ν_α.

  According to [2], the eigenfunction $\varphi_{\mu,\lambda}$ satisfies the following product formula

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \left(\varphi_{\mu,\lambda}(\sqrt{r^2+s^2+2rs\cos\theta}, x+y)\right)\sin^{2\alpha}\theta\,d\theta.$$

- $\langle .|.\rangle_\alpha$is the inner product on the Hilbert space $L^2(d\nu_\alpha)$ defined by

$$\langle f|g\rangle_\alpha = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\overline{g(r,x)}\,d\nu_\alpha(r,x).$$

This allows us to define the translation operators as follows.

Definition 2.1. *For every $(r,x)\in[0,+\infty[\times\mathbb{R}$, the translation operator $\mathcal{T}_{(r,x)}$ associated with the operator $\mathscr{R}_\alpha$ is defined on $L^1(d\nu_\alpha)$ by, for every $(s,y)\in[0,+\infty[\times\mathbb{R}$*

$$\mathcal{T}_{(r,x)}(f)(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)(\sin\theta)^{2\alpha}d\theta,$$

whenever the integral on the right hand side is well defined.

Proposition 2.2. *Let f be in $L^1(d\nu_\alpha)$, then for every $(r,x)\in[0,+\infty[\times\mathbb{R}$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}} \mathcal{T}_{(r,x)}(f)(s,y)\,d\nu_\alpha(s,y) = \int_0^{+\infty}\int_{\mathbb{R}} f(s,y)\,d\nu_\alpha(s,y).$$

Proposition 2.3. *For every $f\in L^p(d\nu_\alpha)$, $1\leqslant p\leqslant+\infty$, and for every $(r,x)\in[0,+\infty[\times\mathbb{R}$, the function $\mathcal{T}_{(r,x)}(f)$ belongs to $L^p(d\nu_\alpha)$ and we have*

$$\|\mathcal{T}_{(r,x)}(f)\|_{p,\nu_\alpha}\leqslant\|f\|_{p,\nu_\alpha}. \tag{2.1}$$

Definition 2.4. *The convolution product of two measurable functions f and g on $[0,+\infty[\times\mathbb{R}$ is defined on $[0,+\infty[\times\mathbb{R}$, by*

$$(f*g)(r,x)=\int_0^{+\infty}\int_{\mathbb{R}}\mathcal{T}_{(r,-x)}(\check{f})(s,y)g(s,y)\,d\nu_\alpha(s,y),$$

where $\check{f}(s,y)=f(s,-y)$, whenever the integral an the right hand side is well defined.

Theorem 2.5. *If $p, q, r \in [1, +\infty]$ are such that $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ then, for every function $f \in L^p(d\nu_\alpha)$ and $g \in L^q(d\nu_\alpha)$, $f * g$ belongs to $L^r(d\nu_\alpha)$ and we have the Young's inequality*

$$\|f * g\|_{r,d\nu_\alpha} \leqslant \|f\|_{p,\nu_\alpha} \|g\|_{q,\nu_\alpha}.$$

Definition 2.6. *The Fourier transform $\mathscr{F}_\alpha$ associated with the operator $\mathscr{R}_\alpha$ is defined for every integrable function f on $[0, +\infty[\times\mathbb{R}$ with respect to the measure ν_α, by*

$$\forall(\mu, \lambda) \in \Upsilon, \quad \mathscr{F}_\alpha(f)(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x)\varphi_{\mu,\lambda}(r, x)\, d\nu_\alpha(r, x).$$

Proposition 2.7.

(i) Let $f \in L^1(d\nu_\alpha)$ and $(r, x) \in [0, +\infty[\times\mathbb{R}$ we have

$$\forall(\mu, \lambda) \in \Upsilon, \quad \mathscr{F}_\alpha(\mathcal{T}_{(r,-x)}(f))(\mu, \lambda) = \varphi_{\mu,\lambda}(r, x)\mathscr{F}_\alpha(f)(\mu, \lambda).$$

(ii) Let $f, g \in L^1(d\nu_\alpha)$, then we have

$$\forall(\mu, \lambda) \in \Upsilon, \quad \mathscr{F}_\alpha(f * g)(\mu, \lambda) = \mathscr{F}_\alpha(f)(\mu, \lambda)\mathscr{F}_\alpha(g)(\mu, \lambda).$$

In the following, we denote by

- Υ_+ the subspace of Υ given by

$$\Upsilon_+ = [0, +\infty[\times\mathbb{R} \cup \big\{(ir, x),\ (r, x) \in [0, +\infty[\times\mathbb{R},\ 0 \leqslant r \leqslant |x|\big\}.$$

- B_{Υ_+} the $\sigma-$algebra defined on Υ_+ by

$$B_{\Upsilon_+} = \{\theta^{-1}(B),\ B \in Bor([0, +\infty[\times\mathbb{R})\}$$

  where $Bor([0, +\infty[\times\mathbb{R})$ is the usual Borel $\sigma-$algebra on $[0, +\infty[\times\mathbb{R}$ and θ is bijective function defined by

$$\begin{array}{rccc} \theta: & \Upsilon_+ & \longrightarrow & [0, +\infty[\times\mathbb{R} \\ & (\mu, \lambda) & \longmapsto & \left(\sqrt{\mu^2 + \lambda^2}, \lambda\right). \end{array}$$

- γ_α the mesure defined on B_{Υ_+} by

$$\gamma_\alpha(A) = \nu_\alpha(\theta(A)), \quad \forall A \in B_{\Upsilon_+}.$$

- $L^p(d\gamma_\alpha)$, $p \in [1, +\infty]$ is the Lebesgue space of measurable functions f defined on Υ_+ satisfying $\|f\|_{p,\gamma_\alpha} < +\infty$, where

$$\|f\|_{p,\gamma_\alpha} = \begin{cases} \left(\displaystyle\int_{\Upsilon_+} |f(\mu,\lambda)|^p d\gamma_\alpha(\mu,\lambda) \right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty[\\ \operatorname*{ess\,sup}_{(\mu,\lambda)\in\Upsilon_+} |f(\mu,\lambda)|, & \text{if } p = +\infty. \end{cases}$$

- $\mathscr{S}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreassing together with all their derivatives, even with respect the first variable.

  The space $\mathscr{S}_e(\mathbb{R}^2)$ is equipped with the topology associated to the countable family of norms

$$\forall m \in \mathbb{N}, \quad \rho_m(\varphi) = \sup_{\substack{(r,x)\in[0,+\infty[\times\mathbb{R} \\ k+|\beta|\leq m}} (1 + r^2 + x^2)^k |D^\beta(\varphi)(r,x)|.$$

- $\mathscr{D}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$ with compact support, even with respect the first variable.

Proposition 2.8. *Let $f \in L^1(d\nu_\alpha)$. For every $(\mu,\lambda) \in \Upsilon$, we have*

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = \widetilde{\mathscr{F}}_\alpha(f) \circ \theta(\mu,\lambda),$$

where

$$\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r,x) j_\alpha(r\mu) e^{-i\lambda x} d\nu_\alpha(r,x).$$

Theorem 2.9. *$\widetilde{\mathscr{F}}_\alpha$ is an isomorphism from $\mathscr{S}_e(\mathbb{R}^2)$ onto itself.*

Proposition 2.10. *For every $f \in L^1(d\nu_\alpha)$ and for all $(r,x),(\mu,\lambda) \in [0,+\infty[\times\mathbb{R}$, we have*

$$\widetilde{\mathscr{F}}_\alpha(\mathcal{T}_{(r,x)} f)(\mu,\lambda) = j_\alpha(r\mu) e^{-ix\lambda} \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda).$$

Theorem 2.11 (Inversion formula for $\mathscr{F}_\alpha$)**.** *Let $f \in L^1(d\nu_\alpha)$ such that $\mathscr{F}_\alpha(f)$ belongs to $L^1(d\gamma_\alpha)$, then for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}$,*

$$f(r,x) = \int_{\Upsilon_+} \mathscr{F}_\alpha(f)(\mu,\lambda) \overline{\varphi_{\mu,\lambda}(r,x)}\, d\gamma_\alpha(\mu,\lambda).$$

Theorem 2.12 (Plancherel's theorem)**.** *The Fourier transform $\mathscr{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$.*

3 Hardy space associated with the Riemann-Liouville operator

Definition 3.1. *For every $t > 0$, the Poisson kernel p_t, associated with the Riemann-Liouville operator $\mathscr{R}_\alpha$ is defined on $\mathbb{R}^2$ by*

$$p_t(r,x) = \int_{\Upsilon_+} e^{-t\sqrt{s^2+2y^2}}\overline{\varphi_{s,y}(r,x)}\, d\gamma_\alpha(s,y) = \mathscr{F}_\alpha^{-1}\left(e^{-t\sqrt{.^2+2.^2}}\right)(r,x).$$

Lemma 3.2. *For every $t > 0$, the Poisson kernel p_t is given by*

$$\forall (r,x) \in \mathbb{R}^2, \quad p_t(r,x) = 2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)\frac{t}{(t^2+r^2+x^2)^{\alpha+2}}.$$

Proof. See [2]. □

Definition 3.3 (Bounded distribution)**.** *Let $v \in \mathscr{S}_e'(\mathbb{R}^2)$, we say that v is a bounded tempered distribution, if*

$$\forall \varphi \in \mathscr{S}_e(\mathbb{R}^2), \quad \varphi * v \in L^\infty(d\nu_\alpha)$$

and if the operator

$$\begin{array}{rccc} \phi_v: & \mathscr{S}_e(\mathbb{R}^2) & \longrightarrow & L^\infty(d\nu_\alpha) \\ & \varphi & \longmapsto & \varphi * v \end{array}$$

is bounded.

Proposition 3.4. *Let $v \in \mathscr{S}_e'(\mathbb{R}^2)$ be a bounded tempered distribution and $f \in L^1(d\nu_\alpha)$. Then, for every $\varphi \in \mathscr{S}_e(\mathbb{R}^2)$*

$$\langle f * v, \varphi\rangle = \int_0^{+\infty}\int_{\mathbb{R}} \breve{\varphi} * v(r,x)\check{f}(r,x)\, d\nu_\alpha(r,x).$$

where $\breve{\varphi}(r,x) = \varphi(-r,-x)$, is well defined. Moreover, $f * v$ is a tempered distribution.

Proof. Let v be a bounded tempered distribution. For all $f \in L^1(d\nu_\alpha)$ and $\varphi \in \mathscr{S}_e(\mathbb{R}^2)$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}} |\breve{\varphi} * v(r,x)||\check{f}(r,x)|\, d\nu_\alpha(r,x) \leq \|\breve{\varphi} * v\|_{\infty,\nu_\alpha}\|f\|_{1,\nu_\alpha} < +\infty$$

and consequently, the integral

$$\langle f * v, \varphi\rangle = \int_0^{+\infty}\int_{\mathbb{R}} \breve{\varphi} * v(r,x)\check{f}(r,x)\, d\nu_\alpha(r,x)$$

is well defined.

It is clear that $f * v$ is a linear operator. Now, for every $\varphi \in \mathscr{S}_e(\mathbb{R}^2)$, we have

$$\begin{aligned}|\langle f * v, \varphi\rangle| &\leqslant \int_0^{+\infty}\int_{\mathbb{R}} |\check{\varphi} * v(r,x)||\check{f}(r,x)|\, d\nu_\alpha(r,x)\\ &\leqslant \|\check{\varphi} * v\|_{\infty,d\nu_\alpha}\|\check{f}\|_{1,\nu_\alpha} \leqslant C\rho_m(\check{\varphi})\|f\|_{1,\nu_\alpha}\\ &\leqslant C\|f\|_{1,d\nu_\alpha}\rho_m(\varphi).\end{aligned}$$

Then, $f * v$ is a tempered distribution. □

Proposition 3.5. *For every bounded tempered distribution $v \in \mathscr{S}'_e(\mathbb{R}^2)$ and for every $t > 0$, $p_t * v \in L^\infty(d\nu_\alpha)$.*

Proof. By Urysohn's lemma, we know that there exits a function $f \in \mathcal{D}_e(\mathbb{R}^2)$ such that

$$\begin{cases} f \equiv 1, & \text{on } \quad \overline{B(0,1/2)}\\ f \equiv 0, & \text{on } \quad B^c(0,1)\\ 0 \leqslant f \leqslant 1. \end{cases}$$

Let $\varphi = \widetilde{\mathscr{F}}_\alpha^{-1}(f)$ then $\varphi \in \mathscr{S}_e(\mathbb{R}^2)$ and $\widetilde{\mathscr{F}}_\alpha(\varphi) = f = 1$ on $\overline{B(0,1/2)}$, hence for $\eta = 1 - \widetilde{\mathscr{F}}_\alpha(\varphi)$, we deduce that $\eta \in \mathscr{C}_e^\infty(\mathbb{R}^2)$ and $\eta = 0$ on $\overline{B(0,1/2)}$. Finally, let g the function defined by $g(r,x) = e^{-|(r,x)|}\eta(r,x)$ and $\psi = \widetilde{\mathscr{F}}_\alpha^{-1}(g)$, then for all $t > 0$ and for all $(r,x) \in \mathbb{R}^2$, we have

$$\begin{aligned}\widetilde{\mathscr{F}}_\alpha(p_t)(r,x) &= e^{-t\sqrt{r^2+x^2}}\\ &= e^{-t\sqrt{r^2+x^2}}\left(\widetilde{\mathscr{F}}_\alpha(\varphi)(tr,tx) + \eta(tr,tx)\right)\\ &= e^{-t\sqrt{r^2+x^2}}\widetilde{\mathscr{F}}_\alpha(\varphi_t)(r,x) + \widetilde{\mathscr{F}}_\alpha(\psi_t)(r,x)\\ &= \widetilde{\mathscr{F}}_\alpha(p_t)(r,x)\widetilde{\mathscr{F}}_\alpha(\varphi_t)(r,x) + \widetilde{\mathscr{F}}_\alpha(\psi_t)(r,x)\\ &= \widetilde{\mathscr{F}}_\alpha(p_t * \varphi_t + \psi_t)(r,x)\end{aligned}$$

Consequently, by the fact that $\widetilde{\mathscr{F}}_\alpha$ is injective, we get

$$p_t = p_t * \varphi_t + \psi_t$$

and therefore

$$p_t * v = p_t * \varphi_t * v + \psi_t * v.$$

Since φ_t and ψ_t belongs to $\mathscr{S}_e(\mathbb{R}^2)$, $\varphi_t * v$ and $\psi_t * v$ are bounded on $\mathbb{R}^2$ and $p_t \in L^1(\nu_\alpha)$, then $p_t * \varphi_t * v$ is a bounded function and the same holds for $p_t * v$. □

Definition 3.6. *Let $f \in \mathcal{S}'_e(\mathbb{R}^2)$ be a bounded tempered distribution. The Poisson maximal function $\mathcal{P}_f^\alpha$ associated with the Riemann-Liouville operator $\mathcal{R}_\alpha$ is defined on $\mathbb{R}^2$ by*

$$\mathcal{P}_f^\alpha(r,x) = \sup_{t>0} |p_t * f(r,x)|.$$

Definition 3.7 (Hardy space)**.** *For every $p \in [1,+\infty[$, the Hardy space $\mathcal{H}_\alpha^p$ associated with the Riemann-Liouville operator is the space of all the bounded tempered distributions f on $\mathbb{R}^2$ satisfying*

$$\mathcal{P}_f^\alpha \in L^p(d\nu_\alpha).$$

We set

$$\|f\|_{\mathcal{H}_\alpha^p} = \|\mathcal{P}_f^\alpha\|_{p,\nu_\alpha}. \tag{3.1}$$

Proposition 3.8. *Let $f \in \mathcal{S}'_e(\mathbb{R}^2)$ be a bounded tempered distribution. Then,*

$$\lim_{t\to 0} p_t * f = f \quad in \quad \mathcal{S}'_e(\mathbb{R}^2).$$

Proof. Let $\eta \in \mathcal{S}_e(\mathbb{R}^2)$. First, we will show that $\lim_{t\to 0} p_t * \eta * f = \eta * f$ in $\mathcal{S}'_e(\mathbb{R}^2)$, thus by using Fubini's theorem, we deduce that for every $\psi \in \mathcal{S}_e(\mathbb{R}^2)$, we have

$$\begin{aligned}
\langle p_t * \eta * f, \psi\rangle_\alpha &= \int_0^{+\infty}\int_{\mathbb{R}} (p_t * \eta * f)(r,x)\psi(r,x)\, d\nu_\alpha(r,x) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \left(\int_0^{+\infty}\int_{\mathbb{R}} \mathcal{T}_{(r,-x)}(\check{p}_t)(s,u)\eta * f(s,u) d\nu_\alpha(s,u)\right)\psi(r,x)\, d\nu_\alpha(r,x) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \left(\int_0^{+\infty}\int_{\mathbb{R}} \mathcal{T}_{(r,-x)}(\check{p}_t)(s,u)\psi(r,x)\, d\nu_\alpha(r,x)\right)\eta * f(s,u)\, d\nu_\alpha(s,u) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \left(\int_0^{+\infty}\int_{\mathbb{R}} \mathcal{T}_{(s,-u)}(\check{p}_t)(r,x))\psi(r,x)\, d\nu_\alpha(r,x)\right)\eta * f(s,u)\, d\nu_\alpha(s,u) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} p_t * \psi(s,t)\eta * f(s,t)\, d\nu_\alpha(s,t).
\end{aligned}$$

Using the dominated convergence theorem, we have

$$\begin{aligned}
\lim_{t\to 0}\langle p_t * \eta * f, \psi\rangle_\alpha &= \lim_{t\to 0}\int_0^{+\infty}\int_{\mathbb{R}} p_t * \psi(s,t)\eta * f(s,t)\, d\nu_\alpha(s,t) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \lim_{t\to 0} p_t * \psi(s,t)\eta * f(s,t)\, d\nu_\alpha(s,t) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \psi(s,t)\ \eta * f(s,t)\, d\nu_\alpha(s,t) \\
&= \langle \eta * f, \psi\rangle_\alpha.
\end{aligned}$$

Then,

$$\lim_{t\to 0} p_t * \eta * f = \eta * f \quad \text{in} \quad \mathscr{S}'_e(\mathbb{R}^2). \tag{3.2}$$

Now, we want to show that

$$\lim_{t\to 0} \mathscr{F}_\alpha(p_t)(1-\mathscr{F}_\alpha(\eta))\mathscr{F}_\alpha(f) = (1-\mathscr{F}_\alpha(\eta))\mathscr{F}_\alpha(f) \quad \text{in} \quad \mathscr{S}'_e(\mathbb{R}^2).$$

$\mathscr{F}_\alpha(p_t)(1-\mathscr{F}_\alpha(\eta))$ is a infinitely differentiable function on $\mathbb{R}^2$, then for any $\psi \in \mathscr{S}_e(\mathbb{R}^2)$, we have

$$\begin{aligned}
\lim_{t\to 0}\langle \mathscr{F}_\alpha(p_t)(1-\mathscr{F}_\alpha(\eta))\mathscr{F}_\alpha(f), \psi\rangle_\alpha &= \lim_{t\to 0}\langle \mathscr{F}_\alpha(f), \mathscr{F}_\alpha(p_t)(1-\mathscr{F}_\alpha(\eta))\psi\rangle_\alpha \\
&= \left\langle \mathscr{F}_\alpha(f), \lim_{t\to 0} \mathscr{F}_\alpha(p_t)(1-\mathscr{F}_\alpha(\eta))\psi \right\rangle_\alpha \\
&= \langle \mathscr{F}_\alpha(f), (1-\mathscr{F}_\alpha(\eta))\psi\rangle_\alpha \\
&= \langle (1-\mathscr{F}_\alpha(\eta))\mathscr{F}_\alpha(f), \psi\rangle_\alpha.
\end{aligned}$$

Hence,

$$\lim_{t\to 0} \mathscr{F}_\alpha(p_t)(1-\mathscr{F}_\alpha(\eta))\mathscr{F}_\alpha(f) = (1-\mathscr{F}_\alpha(\eta))\mathscr{F}_\alpha(f) \quad \text{in} \quad \mathscr{S}'_e(\mathbb{R}^2). \tag{3.3}$$

Consequently,

$$\lim_{t\to 0} \mathscr{F}_\alpha(p_t * f - p_t * \eta * f) = \mathscr{F}_\alpha(f - \eta * f) \quad \text{in} \quad \mathscr{S}'_e(\mathbb{R}^2),$$

which implies that

$$\lim_{t\to 0} p_t * f - p_t * \eta * f = f - \eta * f.$$

Then,

$$\lim_{t\to 0} p_t * f - \lim_{t\to 0} p_t * \eta * f = f - \eta * f.$$

From the Relation (3.2), we have

$$\lim_{t\to 0} p_t * f - \eta * f = f - \eta * f.$$

Then,

$$\lim_{t\to 0} p_t * f = f - \eta * f + \eta * f = f,$$

which achieves the proof. □

Definition 3.9 (Hardy-Littlewood maximal function)**.** *Let $f \in L^1_{loc}(d\nu_\alpha)$. The Hardy-Littlewood maximal function $M_\alpha(f)$ associated with the Riemann-Liouville operator $\mathcal{R}_\alpha$ is defined on* $[0, +\infty[\times\mathbb{R}$*, by*

$$M_\alpha(f)(r,x) = \sup_{\eta>0} \frac{1}{\nu_\alpha(B((0,0),\eta))} \int_{B((0,0),\eta)} \mathcal{T}_{(r,-x)}(|\check{f}|)(s,y)\, d\nu_\alpha(s,y).$$

Theorem 3.10 (The boundedness of M_α)**.** *For every $p \in]1,+\infty]$, the maximal operator M_α is of strong type (p,p) from $L^p(d\nu_\alpha)$ into itself, that is for every $p \in]1,+\infty[$ there exists $C_p > 0$ such that for every $f \in L^p(d\nu_\alpha)$*

$$\|M_\alpha(f)\|_{p,\nu_\alpha} \leqslant C_p \|f\|_{p,\nu_\alpha}$$

Proof. See [1]. □

Proposition 3.11. *Let k be a nonnegative decreasing funtion on $[0,+\infty[$ which is continuous except possibly at finite number of points. We define the function K on $[0,+\infty[\times\mathbb{R}$ by*

$$K(r,x) = k\left(\sqrt{r^2+x^2}\right).$$

Then, for every locally integrable function f on $[0,+\infty[\times\mathbb{R}$ we have

$$\sup_{\epsilon>0}(K_\epsilon * |f|(r,x)) \leqslant \|K\|_{1,\nu_\alpha} M_\alpha(f)(r,x), \tag{3.4}$$

where $K_\epsilon(r,x) = \dfrac{1}{\epsilon^{2\alpha+3}} K\left(\dfrac{r}{\epsilon}, \dfrac{x}{\epsilon}\right)$.

Proof. First, we prove the relation (3.4), when K is continuous with compact support such that $supp(K) \subset B(0,R)$, where $R > 0$ and $f \in L^1_{loc}(d\nu_\alpha)$. We will prove that

$$\sup_{\epsilon>0}(K_\epsilon * |f|(0,0)) \leqslant \frac{1}{(2\pi)^{\frac{1}{2}} 2^\alpha \Gamma(\alpha+1)} M_\alpha(f)(0,0)\|K\|_{1,\nu_\alpha}. \tag{3.5}$$

$$\begin{aligned}
K_\epsilon * |f|(0,0) &= \int_0^{+\infty}\int_{\mathbb{R}} |f(s,x)| K_\epsilon(s,x)\, d\nu_\alpha(s,x) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} |f(s,x)| K_\epsilon(s,x) \frac{s^{2\alpha+1}}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)} ds\, dx \\
&= \int_0^{+\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(r\cos\theta, r\sin\theta)| K_\epsilon(r,0) r \frac{(r\cos(\theta))^{2\alpha+1}}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)} dr\, d\theta.
\end{aligned}$$

Let F and G be the functions defined on $[0,+\infty[$ by

$$\begin{aligned}
F(r) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(r\cos\theta, r\sin\theta))| \cos^{2\alpha+1}\theta \frac{d\theta}{\sqrt{2\pi}} \\
G(r) &= \int_0^r F(y) y^{2\alpha+2} \frac{dy}{2^\alpha \Gamma(\alpha+1)}.
\end{aligned}$$

By integration by parts, we obtain

$$\begin{aligned}
K_\epsilon * |f|(0,0) &= \int_0^{+\infty} F(r)K_\epsilon(r,0)r^{2\alpha+2}\frac{dr}{2^\alpha\Gamma(\alpha+1)} \\
&= \int_0^{\epsilon R} F(r)K_\epsilon(r,0)r^{2\alpha+2}\frac{dr}{2^\alpha\Gamma(\alpha+1)} \\
&= G(\epsilon R)K_\epsilon(\epsilon R,0) - G(0)K_\epsilon(0,0) - \int_0^{\epsilon R} G(r)dK_\epsilon(r,0) \\
&= -\int_0^{\epsilon R} G(r)dK_\epsilon(r,0) \\
&= \int_0^{+\infty} G(r)d\left(-K_\epsilon\right)(r,0),
\end{aligned}$$

where the last integrals are understood in the Lebesgue-Stieltjes sense.

On the other hand,

$$\begin{aligned}
G(r) &= \int_0^r F(y)\frac{y^{2\alpha+2}}{2^\alpha\Gamma(\alpha+1)}dy \\
&= \int_0^r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(r\cos\theta, r\sin\theta))|\frac{y^{2\alpha+2}\cos^{2\alpha+1}\theta}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)}d\theta\, dy \\
&= \int_{\left\{(s,x)\in[0,+\infty[\times\mathbb{R}:\ \sqrt{s^2+x^2}\leqslant r\right\}} |f(s,x)|\, d\nu_\alpha(s,x) \\
&\leqslant M_\alpha(f)(0,0)\nu_\alpha\left(\left\{(s,x)\in[0,+\infty[\times\mathbb{R} : \sqrt{s^2+x^2}\leqslant r\right\}\right) \\
&= M_\alpha(f)(0,0)\nu_\alpha\left(\left\{(s,x)\in[0,+\infty[\times\mathbb{R} : \sqrt{s^2+x^2}\leqslant 1\right\}\right) r^{2\alpha+3}.
\end{aligned}$$

Consequently, we use the integration by parts, we obtain

$$\begin{aligned}
&\int_0^{+\infty} G(r)d\left(-K_\epsilon\right)(r,0) \\
&\leqslant M_\alpha(f)(0,0)\nu_\alpha\left(\left\{(s,x)\in[0,+\infty[\times\mathbb{R} : \sqrt{s^2+x^2}\leqslant 1\right\}\right)\left(\int_0^{+\infty} r^{2\alpha+3}d\left(-K_\epsilon\right)(r,0)\right) \\
&= (2\alpha+3)M_\alpha(f)(0,0)\nu_\alpha\left(\left\{(s,x)\in[0,+\infty[\times\mathbb{R} : \sqrt{s^2+x^2}\leqslant 1\right\}\right)\int_0^{+\infty} r^{2\alpha+2}K_\epsilon(r,0)\, dr
\end{aligned}$$

Since,

$$\begin{aligned}
\nu_\alpha\left(\left\{(s,x)\in[0,+\infty[\times\mathbb{R} : \sqrt{s^2+x^2}\leqslant 1\right\}\right) &= \int_{\left\{(r,x)\in[0,+\infty[\times\mathbb{R}:\sqrt{s^2+x^2}\leqslant 1\right\}} \frac{s^{2\alpha+1}}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)}ds\, dx \\
&= \frac{1}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)}\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (r\cos\theta)^{2\alpha+1} r\, dr\, d\theta \\
&= \frac{1}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2\alpha+3}\cos^{2\alpha+1}\theta\, d\theta.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_0^{+\infty} G(r)d(-K_\epsilon)(r,0) \\
&\leqslant \frac{2\alpha+3}{(2\pi)^{\frac{1}{2}}2^\alpha\Gamma(\alpha+1)} M_\alpha(f)(0,0)\int_0^{+\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2\alpha+3}(r\cos\theta)^{2\alpha+1} rK_\epsilon(r,0)\,dr\,d\theta \\
&= M_\alpha(f)(0,0)\|K_\epsilon\|_{1,\nu_\alpha} \\
&= M_\alpha(f)(0,0)\|K\|_{1,\nu_\alpha}.
\end{aligned}$$

For the general case, let us consider an integrable function K on $[0,+\infty[\times\mathbb{R}$. We know that $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$. Then, for every $K \in L^1(d\nu_\alpha)$, there exists a sequence $(K_j)_{j\in\mathbb{N}}$ of radial, compactly supported, continuous functions increase to K such that

$$\lim_{j\to+\infty} K_j = K$$

From the Relation (3.5), we have

$$\lim_{j\to+\infty}\sup_{\epsilon>0}\left(K_{j,\epsilon} * |f|(0,0)\right) \leqslant \lim_{j\to+\infty} M_\alpha(f)(0,0)\|K_j\|_{1,\nu_\alpha}.$$

Then,

$$\sup_{\epsilon>0}\left(K_\epsilon * |f|(0,0)\right) \leqslant M_\alpha(f)(0,0)\|K\|_{1,\nu_\alpha}.$$

Let $f \in L^1_{loc}(d\nu_\alpha)$ and $(\mu,\lambda) \in [0,+\infty[\times\mathbb{R}$, we denote by

$$g(x,y) = \mathcal{T}_{(\mu,-\lambda)}(|\check{f}|)(x,y), \quad \forall (x,y) \in [0,+\infty[\times\mathbb{R}.$$

$$\begin{aligned}
M_\alpha(g)(0,0) &= \sup_{\eta>0}\frac{1}{\nu_\alpha(B((0,0),\eta))}\int_{B((0,0),\eta)} \mathcal{T}_{(0,0)}(|\check{g}|)(s,y)\,d\nu_\alpha(s,y) \\
&= \sup_{\eta>0}\frac{1}{\nu_\alpha(B((0,0),\eta))}\int_{B((0,0),\eta)} |\check{g}|(s,y)\,d\nu_\alpha(s,y) \\
&= \sup_{\eta>0}\frac{1}{\nu_\alpha(B((0,0),\eta))}\int_{B((0,0),\eta)} \mathcal{T}_{(\mu,-\lambda)}(|\check{f}|)(s,y)\,d\nu_\alpha(s,y) \\
&= M_\alpha(f)(\mu,\lambda).
\end{aligned}$$

Moreover, for all $\epsilon > 0$ we have

$$\begin{aligned}
K_\epsilon * |g|(0,0) &= \int_0^{+\infty}\int_{\mathbb{R}} \mathcal{T}_{(0,0)}(|\check{g}|)(s,y)K_\epsilon(s,x)\,d\nu_\alpha(s,x) = \int_0^{+\infty}\int_{\mathbb{R}} |\check{g}|(s,y)K_\epsilon(s,x)\,d\nu_\alpha(s,x) \\
&= K_\epsilon * |f|(r,x).
\end{aligned}$$

Using the Relation (3.5)

$$\sup_{\epsilon>0}(K_\epsilon * |f|(r,x)) \leqslant M_\alpha(f)(r,x)\|K\|_{1,\nu_\alpha}. \qquad \square$$

Theorem 3.12. *For every $p \in]1,+\infty[$, $\mathcal{H}^p_\alpha$ coincides with $L^p(d\nu_\alpha)$. Moreover, there exists a constant $C_p>0$ such that for every $f \in \mathcal{H}^p_\alpha$, we have*

$$\|f\|_{p,\nu_\alpha} \leqslant \|f\|_{\mathcal{H}^p_\alpha} \leqslant C_p\|f\|_{p,\nu_\alpha}.$$

Proof. Let $f \in \mathcal{H}^p_\alpha$. Using the Relation (3.1),

$$|p_t * f(r,x)| \leqslant \mathcal{P}^\alpha_f(r,x), \quad \forall (r,x) \in \mathbb{R}^2.$$

This implies that

$$\|p_t * f\|_{p,\nu_\alpha} \leqslant \|\mathcal{P}^\alpha_f\|_{p,\nu_\alpha} = \|f\|_{\mathcal{H}^p_\alpha} < +\infty.$$

We deduce that the set $\{p_t * f, t>0\}$ lies in the closed ball $\overline{B(0,\|f\|_{\mathcal{H}^p_\alpha})}$ of $L^p(d\nu_\alpha)$. Moreover, $L^p(d\nu_\alpha)$ is the dual space of $L^q(d\nu_\alpha)$, where q is the conjugate exponent of p.

We define

$$\begin{array}{rccc} \Phi: & L^p(d\nu_\alpha) & \longrightarrow & (L^q(d\nu_\alpha))^* \\ & f & \longmapsto & \Phi_f \end{array}$$

where,

$$\begin{array}{rccc} \Phi_f: & L^q(d\nu_\alpha) & \longrightarrow & \mathbb{C} \\ & g & \longmapsto & \displaystyle\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)g(r,x)\,d\nu_d(r,x). \end{array}$$

We know that for every $f \in L^p(d\nu_\alpha)$,

$$\|\Phi_f\|_{(L^q(d\nu_\alpha))^*} = \|f\|_{p,\nu_\alpha}.$$

Then,

$$\|\Phi_{p_t*f}\|_{(L^q(d\nu_\alpha))^*} = \|p_t * f\|_{p,\nu_\alpha} \leqslant \|f\|_{\mathcal{H}^p_\alpha} < +\infty.$$

We deduce that the set $\{\Phi_{p_t*f}, t>0\}$ lies in the closed ball $\overline{B(0,\|f\|_{\mathcal{H}^p_\alpha})}$ of $L^p(d\nu_\alpha)$. Hence, by Banach-Alaoglu theorem, there exist a sequence $(t_j)_{j\in\mathbb{N}}$ and $f_0 \in L^p(d\nu_\alpha)$ such that

$$\lim_{t_j\to 0}\Phi_{p_{t_j}*f} = \Phi_{f_0}$$

in the the weak* topology of $L^p(d\nu_\alpha)$. Then,

$$\lim_{t_j\to 0} p_{t_j} * f = f_0 \quad \text{in} \quad L^p(d\nu_\alpha).$$

By Proposition 3.8, we obtain that for every $f \in S'_e(\mathbb{R}^2)$ bounded tempered distribution

$$\lim_{t\to 0} p_t * f = f \quad \text{in} \quad S'_e(\mathbb{R}^2).$$

Thus, f and f_0 coincides. We have (see [2])

$$\lim_{t\to 0} \|p_t * f - f\|_{p,\nu_\alpha} = 0.$$

Moreover, we have

$$\begin{aligned}\|f\|_{p,\nu_\alpha} &\leqslant \|p_t * f - f\|_{p,\nu_\alpha} + \|p_t * f\|_{p,\nu_\alpha} \\ &\leqslant \|p_t * f - f\|_{p,\nu_\alpha} + \|\mathcal{P}^\alpha_f\|_{p,\nu_\alpha}.\end{aligned}$$

Then,

$$\|f\|_{p,\nu_\alpha} \leqslant \|\mathcal{P}^\alpha_f\|_{p,\nu_\alpha} = \|f\|_{\mathcal{H}^p_\alpha}.$$

Using Proposition 3.11 for the function p_t, we have

$$\sup_{t>0} |p_t * f| \leqslant M_\alpha(f).$$

Then,

$$\|\mathcal{P}^\alpha_f\|_{p,\nu_\alpha} \leqslant \|M_\alpha(f)\|_{p,\alpha}.$$

Thus,

$$\|f\|_{\mathcal{H}^p_\alpha} \leqslant \|M_\alpha(f)\|_{p,\alpha}.$$

Now, from Theorem 3.10 we know that M_α is of strong type (p,p), $p \in]1,+\infty]$, we deduce that there exists a constant $C_p > 0$ such that

$$\|f\|_{\mathcal{H}^p_\alpha} \leqslant \|M_\alpha(f)\|_{p,\nu_\alpha} \leqslant C_p \|f\|_{p,\nu_\alpha},$$

which achieves the proof. □

Throughout this paper C denotes a positive constant that can change frome one line to next.

4 Atomic Decomposition of Hardy Spaces

Definition 4.1 (Cube). *A cube of* $[0,+\infty[\times\mathbb{R}$ *is a subset of* $\mathbb{R}^2$ *such that*

$$Q=[a_0,b_0]\times[a_1,b_1],$$

where $b_0-a_0=b_1-a_1=L>0$.

Definition 4.2 (Atomic Decomposition). *A measurable function* f *on* $\mathbb{R}^2$ *even with respect to the first variable is called an* L^∞*-atom for* $\mathcal{H}^1_\alpha$*, if there exists a cube* Q *satisfying*

(i) $Supp(f)\subset Q$.

(ii) $\|f\|_{\infty,\nu_\alpha}\leqslant \dfrac{1}{\nu_\alpha(Q)}$.

(iii) $\displaystyle\int_Q f(r,x)d\nu_\alpha(r,x)=0$.

In the next we define the atomic space $\mathcal{H}^\alpha_{atomic}$

Definition 4.3. *The space* $\mathcal{H}^\alpha_{atomic}$ *is defined as the vector space of all functions* $f\in L^1(d\nu_\alpha)$ *for which there exists a sequence* $\{f_i\}_{i\in\mathbb{N}}$ *of* L^∞*-atoms of* $\mathcal{H}^1_\alpha$ *and a sequence* $(\lambda_i)_{i\in\mathbb{N}}\in\ell^1(\mathbb{N})$*, such that*

$$f=\sum_{i=1}^{+\infty}\lambda_i f_i.$$

We set

$$\|f\|_{\mathcal{H}^\alpha_{atomic}}=\inf\left\{\sum_{i=1}^{+\infty}|\lambda_i|\mid f=\sum_{i=1}^{+\infty}\lambda_i f_i\right\}.$$

Now we introduce the following notations

- $Z^{[\alpha]}$ the set of functions $\varphi\in\mathcal{C}^1([0,+\infty[\times\mathbb{R},\mathbb{C})$ satisfying $\varphi(0,0)>0$ and for every $(x,y)\in[0,+\infty[\times\mathbb{R}$.
  - $0\leqslant\varphi(x,y)\leqslant\dfrac{C}{(1+x^2+y^2)^{\alpha+2}}$.
  - $0\leqslant\dfrac{\partial\varphi}{\partial x}(x,y)\leqslant\dfrac{Cx}{(1+x^2+y^2)^{\alpha+3}}$.
  - $0\leqslant\dfrac{\partial\varphi}{\partial y}(x,y)\leqslant\dfrac{Cy}{(1+x^2+y^2)^{\alpha+3}}$.

  Where C a positive constant depending on φ.
- We define the function h on $]0,+\infty[\times]0,+\infty[$ by

$$h(r,\gamma)=\begin{cases}\gamma r^{-2\alpha-1} & \text{if} \quad \gamma<r^{2\alpha+2},\\ \gamma^{\frac{1}{2\alpha+2}} & \text{if} \quad \gamma\geqslant r^{2\alpha+2}.\end{cases}$$

- For every $\gamma > 0$ and for every $\varphi \in Z^{[\alpha]}$, $(r,x) \in]0,+\infty[\times\mathbb{R}$ and $(s,y) \in [0,+\infty[\times\mathbb{R}$, we set

$$\Phi^\gamma((r,x),(s,y)) = \gamma\mathcal{T}_{(r,x)}(\varphi_{h(r,\gamma)})(-s,-y),$$

  where $\varphi_{h(r,\gamma)}(r,x) = \frac{1}{(h(r,\gamma))^{2\alpha+2}}\varphi\left(\frac{r}{h(r,\gamma)}, \frac{x}{h(r,\gamma)}\right)$.

- $d_\alpha((r,x),(s,y)) = \max\left(\left|\int_r^s t^{2\alpha+1}dt\right|, |x-y|\right)$, where $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$.
- $p((r,x),(s,y)) = \max(|r-s|, |x-y|)$, where $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$.
- $p'((r,x),(y,y')) = \max(|r-s|^{\frac{1}{2\alpha+2}}, |x-y|)$, where $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$.

Our goal now, is to prove that $\|.\|_{\mathcal{H}^1_\alpha}$ and $\|.\|_{\mathcal{H}^\alpha_{atomic}}$ are equivalent. To do this, we need some preparation.

Proposition 4.4. *Let $f \in L^1(d\nu_\alpha)$. For every $\lambda > 0$ and $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$, we have*

$$\mathcal{T}_{(\lambda r,\lambda x)}(f)(\lambda s,\lambda y) = \lambda^{-2\alpha-2}\mathcal{T}_{(r,x)}(f_{\lambda^{-1}})(s,y).$$

Proof.

$$\begin{aligned}
\mathcal{T}_{(\lambda r,\lambda x)}(f)(\lambda s,\lambda y) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f\left(\sqrt{(\lambda r)^2+(\lambda s)^2+2\lambda^2 rs\cos\theta}, \lambda x+\lambda y\right)\sin^{2\alpha}\theta\,d\theta \\
&= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f\left(\lambda\sqrt{r^2+s^2+2rs\cos\theta}, \lambda(x+y)\right)\sin^{2\alpha}\theta\,d\theta \\
&= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \lambda^{-2\alpha-2}f_{\lambda^{-1}}\left(\lambda\sqrt{r^2+s^2+2rs\cos\theta}, \lambda(x+y)\right)\sin^{2\alpha}\theta\,d\theta \\
&= \lambda^{-2\alpha-2}\mathcal{T}_{(r,x)}(f_{\lambda^{-1}})(s,y).
\end{aligned}$$

□

Proposition 4.5. *For every $\gamma, \lambda > 0$ and $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$, we have*

(i) $h(\lambda x, \lambda^{2\alpha+2}\gamma) = \lambda h(r,\gamma)$.

(ii) $d_\alpha((\lambda r, \lambda^{2\alpha+2}x),(\lambda s,\lambda^{2\alpha+2}y)) = \lambda^{2\alpha+2}d_\alpha((r,x),(s,y))$.

(iii) $\Phi^\gamma((\lambda r,\lambda x),(\lambda s,\lambda y)) = \Phi^{\lambda^{-2\alpha-2}\gamma}((r,x),(s,y))$.

Proof. (i)

$$\begin{aligned}
h(\lambda x, \lambda^{2\alpha+2}\gamma) &= \begin{cases} \lambda^{2\alpha+2}\gamma(\lambda x)^{-2\alpha-1}; & \text{if} \quad \lambda^{2\alpha+2}\gamma < (\lambda x)^{2\alpha+2}, \\ \lambda\gamma^{\frac{1}{2\alpha+2}}; & \text{if} \quad \gamma \geqslant x^{2\alpha+2}. \end{cases} \\
&= \begin{cases} \lambda\gamma x^{-2\alpha-1}; & \text{if} \quad \gamma < x^{2\alpha+2}, \\ \lambda\gamma^{\frac{1}{2\alpha+2}}; & \text{if} \quad \gamma \geqslant x^{2\alpha+2}. \end{cases} \\
&= \lambda h(r,\gamma).
\end{aligned}$$

(ii)

$$\begin{aligned} d_\alpha((\lambda r, \lambda^{2\alpha+2}x), (\lambda s, \lambda^{2\alpha+2}y)) &= \max\left(\left|\int_{\lambda s}^{\lambda r} t^{2\alpha+1}dt\right|, |\lambda^{2\alpha+2}x - \lambda^{2\alpha+2}y|\right) \\ &= \max\left(\lambda^{2\alpha+2}\left|\int_s^r t^{2\alpha+1}dt\right|, \lambda^{2\alpha+2}|x-y|\right) \\ &= \lambda^{2\alpha+2}d_\alpha((r,x),(s,y)). \end{aligned}$$

(iii) Using Proposition 4.4, we get

$$\begin{aligned} \Phi^\gamma((\lambda r, \lambda x), (\lambda s, \lambda y)) &= \gamma \mathcal{T}_{(\lambda r, \lambda x)}(\varphi_{h(\lambda r, \gamma)})(-\lambda s, -\lambda y) \\ &= \gamma \mathcal{T}_{(r,x)}(\varphi_{\lambda h(r, \lambda^{-2\alpha-2}\gamma)})(-\lambda s, -\lambda y) \\ &= \lambda^{-2\alpha-2}\gamma \mathcal{T}_{(r,x)}(\varphi_{h(r,\lambda^{-2\alpha-2}\gamma)})(-s,-y) \\ &= \Phi^{\lambda^{-2\alpha-2}\gamma}((r,x),(s,y)). \end{aligned}$$

□

Lemma 4.6. *There exist constants $C > 0$, $\beta > 0$ such that for every $(r,x),(s,y),(t,z) \in [0,+\infty[\times\mathbb{R}$ and $\gamma > 0$ we have*

$$|\Phi^\gamma((r,x),(s,y)) - \Phi^\gamma((r,x),(t,z))| \leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^\beta. \tag{4.1}$$

Proof. It is sufficient to prove the Relation (4.1) for $d_\alpha((s,y),(t,z)) < \frac{\gamma}{C}$, where C is a fixed constant large enough.

First, we will show that

$$L = |\Phi^\gamma((1,x),(s,y)) - \Phi^\gamma((1,x),(t,z))| \leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{r}\right)^\beta. \tag{4.2}$$

$$\begin{aligned} L &= C\gamma\left|\int_0^\pi \varphi_{h(1,\gamma)}\left(\sqrt{1+s^2-2s\cos\theta}, x-y\right)\sin^{2\alpha}\theta\, d\theta\right. \\ &\quad \left. - \int_0^\pi \varphi_{h(1,\gamma)}\left(\sqrt{1+t^2-2t\cos\theta}, x-z\right)\sin^{2\alpha}\theta\, d\theta\right| \\ &= C\frac{\gamma}{(h(1,\gamma))^{2\alpha+2}}\left|\int_0^\pi \left(\varphi\left(\frac{\sqrt{1+s^2-2s\cos\theta}}{h(1,\gamma)}, \frac{x-y}{t(1,\gamma)}\right)\right.\right. \\ &\quad \left.\left. - \varphi\left(\frac{\sqrt{1+t^2-2t\cos\theta}}{h(1,\gamma)}, \frac{x-z}{h(1,\gamma)}\right)\right)\sin^{2\alpha}\theta\, d\theta\right|. \end{aligned}$$

Let f be a function defined by

$$\begin{aligned} g:\quad [0,+\infty[\times\mathbb{R} &\longrightarrow \mathbb{R} \\ (s,t) &\longmapsto (g_1(s,t), g_2(s,t)) = \left(\frac{\sqrt{1+s^2-2s\cos\theta}}{h(1,\gamma)}, \frac{x-t}{h(1,\gamma)}\right) \end{aligned}$$

$$\begin{aligned}\frac{\partial(\varphi\circ g)}{\partial s}(s,t) &= \frac{\partial\varphi}{\partial s}(g(s,t))\frac{\partial g_1}{\partial s}(s,t)+\frac{\partial\varphi}{\partial t}(g(s,t))\frac{\partial g_2}{\partial s}(s,t)\\ &= \frac{1}{h(1,\gamma))}\left(\frac{s-\cos\theta}{\sqrt{1+s^2-2s\cos\theta}}\frac{\partial\varphi}{\partial s}(g(s,t))-\frac{\partial\varphi}{\partial t}(g(s,t))\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial(\varphi\circ g)}{\partial t}(s,t) &= \frac{\partial\varphi}{\partial s}(g(s,t))\frac{\partial g_1}{\partial t}(s,t)+\frac{\partial\varphi}{\partial t}(g(s,t))\frac{\partial g_2}{\partial t}(s,t)\\ &= -\frac{1}{h(1,\gamma)}\frac{\partial\varphi}{\partial t}(g(s,t)).\end{aligned}$$

Since $\varphi\in Z^{[\alpha]}$, we use the mean value theorem , there exist $(u,u')\in[(s,y),(t,z)]$ such that

$$\begin{aligned}|\varphi\circ g(s,y)-\varphi\circ g(t,z)| &\leqslant \|(s,y)-(t,z)\|_\infty \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\|d(\varphi\circ g)(u,u')\|\\ &= p((s,y),(t,z))\sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\|d(\varphi\circ g)(u,u')\|\end{aligned}$$

Then,

$$\begin{aligned}L &\leqslant C\frac{\gamma}{(h(1,\gamma))^{2\alpha+2}}p((s,y),(t,z))\left|\int_0^\pi \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\|d(\varphi\circ g)(u,u')\|\sin^{2\alpha}\theta\,d\theta\right|\\
&\leqslant C\frac{\gamma}{(h(1,\gamma))^{2\alpha+3}}p((s,y),(t,z))\\
&\times\sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\int_0^\pi\left|\frac{u-\cos\theta}{\sqrt{1+u^2-2u\cos\theta}}\frac{\partial\varphi}{\partial s}\left(\frac{\sqrt{1+u^2-2u\cos\theta}}{h(1,\gamma)},\frac{x-u'}{h(1,\gamma)}\right)\right|\\
&+2\left|\frac{\partial\varphi}{\partial t}\left(\frac{\sqrt{1+u^2-2u\cos\theta}}{h(1,\gamma)},\frac{x-u'}{h(1,\gamma)}\right)\right|\sin^{2\alpha}\theta\,d\theta\\
&\leqslant C\gamma(h(1,\gamma))^2p((s,y),(t,z))\\
&\times\sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\int_0^\pi\frac{(|u-\cos\theta|+2|x-u'|)}{((h(1,\gamma))^2+1+u^2-2u\cos\theta+(x-u')^2)^{\alpha+3}}\sin^{2\alpha}\theta\,d\theta\\
&\leqslant C\gamma(h(1,\gamma))^2p((s,y),(t,z))\\
&\times\sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\int_0^\pi\frac{(1-\cos\theta)+|u-1|+2|x-u'|)}{((h(1,\gamma))^2+1+u^2-2u\cos\theta+(x-u')^2)^{\alpha+3}}\sin^{2\alpha}\theta\,d\theta\\
&\leqslant C\gamma(h(1,\gamma))^2p((s,y),(t,z))\\
&\times\sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\int_0^\pi\frac{(1-\cos\theta)+3p((1,x),(u,u'))}{((h(1,\gamma))^2+1+u^2-2u\cos\theta+(x-u')^2)^{\alpha+3}}\sin^{2\alpha}\theta\,d\theta\\
&\leqslant C\gamma(h(1,\gamma))^2p((s,y),(t,z))\\
&\sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\int_0^\pi\frac{(1-\cos\theta)+3p((1,x),(u,u'))}{((h(1,\gamma))^2+(p((1,x),(u,u')))^2+2u(1-\cos\theta))^{\alpha+3}}\sin^{2\alpha}\theta\,d\theta\end{aligned}$$

Set

$$E(\gamma,u,u',\theta)=\frac{(1-\cos\theta)+3p((1,x),(u,u'))}{((h(1,\gamma))^2+(p((1,x),(u,u')))^2+2u(1-\cos\theta))^{\alpha+3}}.$$

Case 1 $\gamma\geqslant 1$**:** we have $h(1,\gamma)=\gamma^{\frac{1}{2\alpha+2}}$

- If $p((1,x),(u,u'))\leqslant\gamma^{\frac{1}{2\alpha+2}}$. Then,

$$E(\gamma,u,u',\theta)\leqslant(2+3\gamma^{\frac{1}{2\alpha+2}})\gamma^{-\frac{2\alpha+6}{2\alpha+2}}=3\gamma^{-\frac{2\alpha+5}{2\alpha+2}}(\gamma^{-\frac{1}{2\alpha+2}}+1)\leqslant 6\gamma^{-\frac{2\alpha+5}{2\alpha+2}}.$$

We have

$$p((s,y),(t,z))\leqslant(d_\alpha((s,y),(t,z)))^{\frac{1}{2\alpha+2}}.$$

Then,

$$L\leqslant C\gamma\gamma^{\frac{2}{2\alpha+2}}\gamma^{-\frac{2\alpha+5}{2\alpha+2}}p((y,y'),(z,z'))\leqslant C\frac{p((y,y'),(z,z'))}{\gamma^{\frac{1}{2\alpha+2}}}\leqslant C\left(\frac{d_\alpha((y,y'),(z,z'))}{\gamma}\right)^{\frac{1}{2\alpha+2}}$$

- If $p((1,x),(u,u'))\geqslant\gamma^{\frac{1}{2\alpha+2}}$. Then,

$$E(r,u,u',\theta)\leqslant 3\frac{1+p((1,x),(u,u'))}{(p((1,x),(u,u')))^{2\alpha+6}}\leqslant 6\frac{p((1,x),(u,u'))}{(p((1,x),(u,u')))^{2\alpha+6}}=6(p((1,x),(u,u')))^{-2\alpha-5}$$

Thus,

$$\begin{aligned}L&\leqslant C\gamma p((s,y),(t,z))\gamma^{\frac{2}{2\alpha+2}}\sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}(p((1,x),(u,u')))^{-2\alpha-5}\\&\leqslant C\frac{p((s,y),(t,z))}{\gamma^{\frac{1}{2\alpha+2}}}\\&\leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^{\frac{1}{2\alpha+2}}.\end{aligned}$$

Case 2 $\gamma<1$**:** $h(1,\gamma)=\gamma$.

- $p((1,x),(u,u'))\geqslant\frac{1}{4}$

$$\begin{aligned}E(\gamma,u,u',\theta)&=\frac{(1-\cos\theta)+3p((1,x),(u,u'))}{(\gamma^2+(p((1,x),(u,u'))^2+2u(1-\cos\theta))^{\alpha+3}}\\&\leqslant\frac{2+3p((1,x),(u,u'))}{(p((1,x),(u,u')))^{2\alpha+6}}\\&\leqslant 11(p((1,x),(u,u')))^{-2\alpha-5}\\&\leqslant 4^{2\alpha+5}11.\end{aligned}$$

Then,

$$L\leqslant C\gamma^2p((s,y),(t,z))\leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^{\frac{1}{2\alpha+2}}.$$

- If $p((1,x),(u,u')) < \frac{1}{4}$ and $p((1,x),(u,u')) > \frac{\gamma}{4}$.

$$\begin{aligned}
L &\leqslant C\gamma^3 p((s,y),(t,z)) \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}} \int_0^{p((1,x),(u,u'))} \frac{\theta^{2\alpha}(\frac{\theta^2}{2}+3p((1,x),(u,u')))}{(p((1,x),(u,u')))^{2\alpha+6}}\,d\theta \\
&+ \int_{p((1,x),(u,u'))}^{\pi} \frac{\theta^{2\alpha}(\frac{\theta^2}{2}+3p((1,x),(u,u')))}{\theta^{2\alpha+6}}d\theta \\
&\leqslant C\gamma^3 p((s,y),(t,z)) \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}} \frac{p((1,x),(u,u'))+1}{(p((1,x),(u,u')))^4} \\
&+ \int_{p((1,x),(u,u'))}^{\pi} \frac{1}{2\theta^4} + \frac{3p((1,x),(u,u'))}{\theta^5}\,d\theta \\
&\leqslant C\gamma^3 p((s,y),(t,z)) \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}} \frac{5}{4(p((1,x),(u,u')))^4} + \frac{2}{(p((1,x),(u,u')))^3} \\
&\leqslant C\gamma^3 p((s,y),(t,z)) \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}} \frac{1}{(p((1,x),(u,u')))^4} \\
&\leqslant C\frac{p((s,y),(t,z))}{\gamma}.
\end{aligned}$$

- If $p((1,x),(u,u')) < \frac{\gamma}{4}$

$$\begin{aligned}
L &\leqslant C\gamma^3 p((s,y),(t,z)) \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}} \int_0^{\gamma} \frac{\theta^{2\alpha}(\frac{\theta^2}{2}+2p((1,x),(u,u')))}{\gamma^{2\alpha+6}}\,d\theta \\
&+ \int_{\gamma}^{\frac{\pi}{2}} \frac{\theta^{2\alpha}(\frac{\theta^2}{4}+2p((1,x),(u,u')))}{\theta^{2\alpha+6}}d\theta + \int_{\frac{\pi}{2}}^{\pi} \frac{\theta^{2\alpha}(\frac{\theta^2}{4}+2p((1,x),(u,u')))}{\theta^{2\alpha+6}}d\theta \\
&\leqslant C\gamma^3 p((s,y),(t,z)) \\
&\times \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}} \left(\frac{1}{4\gamma^3} + \frac{2p((1,x),(u,u'))}{\gamma^5} + \frac{1}{8\gamma^3} + \frac{p((1,x),(u,u'))}{2\gamma^5} + 2p((1,x),(u,u'))\right) \\
&\leqslant C\gamma^3 p((s,y),(t,z)) \left(\frac{3}{8\gamma^3} + \frac{1}{\gamma^4} + \frac{\gamma}{2}\right) \\
&\leqslant C p((s,y),(t,z)) \left(\frac{3}{8} + \frac{1}{\gamma} + \frac{\gamma^3}{2}\right) \\
&\leqslant C\frac{p((s,y),(t,z))}{\gamma}.
\end{aligned}$$

Using (ii) and (iii) of Proposition 4.5 and the Relation (4.2), we get

$$|\Phi^{\gamma}((r,x),(s,y)) - \Phi^{\gamma}((r,x),(t,z))| \leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^{\beta}. \qquad \square$$

Proposition 4.7. *There exist constants $A > 0$ and $\beta > 0$ such that*

(i) $\Phi^{\gamma}((r,x),(r,x)) > \frac{1}{A}$, $\gamma > 0$ *and* $(r,x) \in [0,+\infty[\times\mathbb{R}$.

(ii) $0 \leqslant \Phi^{\gamma}((r,x),(s,y)) \leqslant A\left(1 + \frac{d_\alpha((r,x),(s,y))}{\gamma}\right)^{-1-\beta}$, $\gamma > 0$ *and* $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$.

(iii) For every $\gamma > 0$ *and* $(r,x),(s,y),(t,z) \in [0,+\infty[\times\mathbb{R}$*, such that*

$$d_\alpha((s,y),(t,z)) \leqslant \frac{1}{4A}(\gamma + d_\alpha((r,x),(s,y))),$$

we have

$$|\Phi^\gamma((r,x),(s,y)) - \Phi^\gamma((r,x),(t,z))| \leqslant A\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^\beta \left(1+\frac{d_\alpha((r,x),(s,y))}{\gamma}\right)^{-1-2\beta}.$$

Proof. Let $\varphi \in Z^{[\alpha]}$

(i) First, we will show that there exists a constant $A > 0$ such that

$$\forall x \in \mathbb{R}, \quad \Phi^\gamma((1,x),(1,x)) > \frac{1}{A}.$$

We know that $\varphi(0,0) > 0$. Then, there exist constants $a > 0$ and $b > 0$ such that for every $0 < r < b$ we have

$$\varphi(r,0) > a. \tag{4.3}$$

- If $\gamma < 1$, then $h(1,\gamma) = \gamma$.

$$\begin{aligned}\Phi^\gamma((1,x),(1,x)) &= \gamma \mathcal{T}_{(1,x)}(\varphi_\gamma)(-1,-x) \\ &= \gamma \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \varphi_\gamma\left(\sqrt{2(1-\cos\theta)},0\right)\sin^{2\alpha}\theta\,d\theta. \\ &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \frac{1}{\gamma^{2\alpha+1}}\varphi\left(\frac{\sqrt{2(1-\cos\theta)}}{\gamma},0\right)\sin^{2\alpha}\theta\,d\theta.\end{aligned}$$

By the Relation (4.3), there exists b' such that for every $0 < \theta < \gamma b'$, we have

$$\varphi\left(\frac{\sqrt{2(1-\cos\theta)}}{\gamma},0\right) > a.$$

Then,

$$\begin{aligned}\Phi^\gamma((1,x),(1,x)) &\geqslant \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^{\gamma b'} \frac{a}{\gamma^{2\alpha+1}}\sin^{2\alpha}\theta\,d\theta. \\ &\geqslant \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^{\gamma b'} a\sin^{2\alpha}\theta\,d\theta.\end{aligned}$$

- If $\gamma \geqslant 1$, then $h(1,\gamma) = \gamma^{\frac{1}{2\alpha+2}}$.

$$\begin{aligned}\Phi^\gamma((1,x),(1,x)) &= \gamma \mathcal{T}_{(1,x)}\left(\varphi_{\gamma^{\frac{1}{2\alpha+2}}}\right)(-1,-x) \\ &= \gamma \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \varphi_{\gamma^{\frac{1}{2\alpha+2}}}\left(\sqrt{2(1-\cos\theta)},0\right)\sin^{2\alpha}\theta\,d\theta \\ &= \gamma \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \frac{1}{\gamma}\varphi\left(\frac{\sqrt{2(1-\cos\theta)}}{\gamma^{\frac{1}{2\alpha+2}}},0\right)\sin^{2\alpha}\theta\,d\theta\end{aligned}$$

$$= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_0^{\pi} \varphi\left(\frac{\sqrt{2(1-\cos\theta)}}{\gamma^{\frac{1}{2\alpha+2}}}, 0\right) \sin^{2\alpha}\theta\, d\theta$$
$$\geqslant \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_0^{b'} a \sin^{2\alpha}\theta\, d\theta.$$

Thus,

$$\Phi^{\gamma}((1,x),(1,x)) > \frac{1}{A}, \quad \forall x \in \mathbb{R}.$$

Let $(r,x) \in]0,+\infty[\times\mathbb{R}$. Using (iii) of Proposition 4.5 we have

$$\Phi^{\gamma}((r,x),(r,x)) = \Phi^{\gamma r^{-2\alpha-3}}\left(\left(1,\frac{x}{r}\right),\left(1,\frac{x}{r}\right)\right) \geqslant \frac{1}{A}.$$

For $r = 0$ is obvious .

(ii) First, we have to show that

$$0 \leqslant \Phi^{\gamma}((1,x),(s,y)) \leqslant C\left(1 + \frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-1-\beta}. \tag{4.4}$$

Case 1 $\gamma < 1$**:** $h(1,\gamma) = \gamma$.

$$\begin{aligned}
\Phi^{\gamma}((1,x),(s,y)) &= \gamma \mathcal{T}_{(1,x)}(\varphi_\gamma)(-s,-y)\\
&= \gamma \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_0^{\pi} \varphi_\gamma\left(\sqrt{1+s^2-2s\cos\theta}, \frac{x-y}{\gamma}\right) \sin^{2\alpha}\theta\, d\theta\\
&= \gamma \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_0^{\pi} \frac{1}{\gamma^{2\alpha+2}} \varphi\left(\frac{\sqrt{1+s^2-2s\cos\theta}}{\gamma}, \frac{x-y}{\gamma}\right) \sin^{2\alpha}\theta\, d\theta\\
&\leqslant \gamma \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_0^{\pi} \frac{1}{\gamma^{2\alpha+2}} \frac{\gamma^{2\alpha+4}}{(\gamma^2+1+s^2-2s\cos\theta+(x-y)^2)^{\alpha+2}} \sin^{2\alpha}\theta\, d\theta\\
&\leqslant \gamma^3 \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_0^{\pi} \frac{1}{(\gamma^2+(1-s)^2+2s-2s\cos\theta+(x-y)^2)^{\alpha+2}} \sin^{2\alpha}\theta\, d\theta.
\end{aligned}$$

- If $\frac{\gamma}{2} \leqslant |1-s|$ and $\frac{1}{2} \leqslant s \leqslant 2$, then

$$d_\alpha((1,x),(s,y)) \sim p\left((1,x),(s,y)\right).$$

In fact,

$$\frac{1}{2^{2\alpha+1}}|1-s| \leqslant \left|\int_1^s t^{2\alpha+1} dt\right| \leqslant 2^{2\alpha+1}|1-s|.$$

Then,

$$\frac{1}{2^{2\alpha+1}} p\left((1,x),(s,y)\right) \leqslant d_\alpha\left((1,x),(s,y)\right) \leqslant 2^{2\alpha+1} p\left((1,x),(s,y)\right).$$

$$\begin{aligned}
&\Phi^\gamma((1,x),(s,y))\\
&\leqslant \gamma^3 \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \frac{1}{\left(\gamma^2+\left(p\left((1,x),(s,y)\right)\right)^2+2s(1-\cos\theta)\right)^{\alpha+2}}\sin^{2\alpha}\theta\, d\theta\\
&\leqslant C\gamma^3\left(\int_0^{p((1,x),(s,y))}\frac{\sin^{2\alpha}(\theta)}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\,d\theta+\int_{p((1,x),(s,y))}^\pi \frac{\sin^{2\alpha}\theta}{\theta^{2\alpha+4}}\,d\theta\right)\\
&\leqslant C\gamma^3\left(\int_0^{p((1,x),(s,y))}\frac{\theta^{2\alpha}}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\,d\theta+\int_{p((1,x),(s,y))}^\pi \frac{\theta^{2\alpha}}{\theta^{2\alpha+4}}\,d\theta\right)\\
&\leqslant C\gamma^3\left(\frac{1}{(p\left((1,x),(s,y)\right))^3}-\frac{1}{3\pi^2}+\frac{1}{3(p\left((1,x),(s,y)\right))^3}\right)\\
&\leqslant C\gamma^3\frac{1}{(p\left((1,x),(s,y)\right))^3}.
\end{aligned}$$

Since, $\frac{\gamma}{2}\leqslant |1-s|\leqslant p\left((1,x),(s,y)\right)$ then

$$\gamma+p\left((1,x),(s,y)\right)\leqslant 3p\left((1,x),(s,y)\right).$$

Then,

$$\begin{aligned}
\Phi^\gamma((1,x),(s,y))&\leqslant C\gamma^3\frac{1}{(\gamma+p\left((1,x),(s,y)\right))^2}\\
&=C\gamma^3\left(1+\frac{p\left((1,x),(s,y)\right)}{\gamma}\right)^{-3}\\
&\leqslant C\left(1+\frac{d_\alpha\left((1,x),(s,y)\right)}{\gamma}\right)^{-3}.
\end{aligned}$$

- If $\frac{\gamma}{2}\leqslant|1-s|$, $|x-y|>1$ and $|1-s|>\frac{1}{2}$, then

$$d_\alpha((1,x),(s,y))\leqslant 2^{2\alpha+1}\left(p\left((1,x),(s,y)\right)\right)^{2\alpha+2}.$$

In fact, we have

$$\frac{1}{2^{2\alpha+1}}\left|\int_1^s t^{2\alpha+1}dt\right|\leqslant|1-s||1-s|^{2\alpha+1}\leqslant\left(p\left((1,x),(s,y)\right)\right)^{2\alpha+2},$$

and

$$|x-y|\leqslant|x-y|^{2\alpha+2}.$$

$$\begin{aligned}
\Phi^\gamma((1,x),(s,y))&\leqslant\gamma^3C\int_0^\pi\frac{\sin^{2\alpha}\theta}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\,d\theta\\
&\leqslant\gamma^3C\frac{1}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\\
&\leqslant\gamma^3C\frac{1}{(d_\alpha((1,x),(s,y)))^{\frac{2\alpha+4}{2\alpha+2}}}
\end{aligned}$$

$$
\begin{aligned}
&= \gamma^3 C \frac{1}{(d_\alpha((1,x),(s,y)))^{1+\frac{1}{\alpha+1}}} \\
&\leqslant \gamma^3 C \frac{1}{(r + d_\alpha((1,x),(s,y)))^{1+\frac{1}{\alpha+1}}} \\
&\leqslant \gamma^{1+\frac{1}{\alpha+1}} C \frac{1}{(\gamma + d_\alpha((1,x),(s,y)))^{1+\frac{1}{\alpha+1}}} \\
&= C\left(1 + \frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-1-\frac{1}{\alpha+1}}
\end{aligned}
$$

- If $\frac{\gamma}{2} \leqslant |1-s|$, $|x-y| \leqslant 1$ and $|1-s| > \frac{1}{2}$, then

$$\frac{1}{2^{2\alpha+1}} d_\alpha((1,x),(s,y)) \leqslant \left(p'\left((1,x),(s,y)\right)\right)^{2\alpha+2}.$$

In fact, we have

$$\frac{1}{2^{2\alpha+1}} \left|\int_1^s t^{2\alpha+1} dt\right| \leqslant |1-s||1-s|^{2\alpha+1} \leqslant |1-s|^{2\alpha+2},$$

and

$$|x-y| \leqslant \left(|x-y|^{\frac{1}{2\alpha+2}}\right)^{2\alpha+2}.$$

$$
\begin{aligned}
\Phi^\gamma((1,x),(s,y)) &\leqslant \gamma^3 C \int_0^\pi \frac{\sin^{2\alpha}\theta}{(p'\left((1,x),(s,y)\right))^{2\alpha+4}}\, d\theta \\
&\leqslant \gamma^3 C \frac{1}{(p\left((1,x),(s,y)\right))^{2\alpha+4}} \\
&\leqslant \gamma^3 C \frac{1}{(d_\alpha((1,x),(s,y)))^{\frac{2\alpha+4}{2\alpha+2}}} \\
&= \gamma^3 C \frac{1}{(d_\alpha((1,x),(s,y)))^{1+\frac{1}{\alpha+1}}} \\
&\leqslant \gamma^3 C \frac{1}{(\gamma + d_\alpha((1,x),(s,y)))^{1+\frac{1}{\alpha+1}}} \\
&\leqslant \gamma^{1+\frac{1}{\alpha+1}} C \frac{1}{(\gamma + d_\alpha((1,x),(s,y)))^{1+\frac{1}{\alpha+1}}} \\
&= C\left(1 + \frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-1-\frac{1}{\alpha+1}}
\end{aligned}
$$

- If $\frac{\gamma}{2} > |1-s|$ and $|x-y| < \frac{\gamma}{2}$ then $\frac{1}{2} < s < \frac{3}{2}$ and $p\left((1,x),(s,y)\right) \sim d_\alpha\left((1,x),(s,y)\right)$.
In fact, we have

$$\left(\frac{1}{2}\right)^{2\alpha+1} |1-s| \leqslant \left|\int_1^s t^{2\alpha+1} dt\right| \leqslant \left(\frac{3}{2}\right)^{2\alpha+1} |1-s|.$$

Then,

$$\left(\frac{1}{2}\right)^{2\alpha+1} p\left((1,x),(s,y)\right) \leqslant d_\alpha\left((1,x),(s,y)\right) \leqslant \left(\frac{3}{2}\right)^{2\alpha+1} p\left((1,x),(s,y)\right).$$

$$\begin{aligned}
\Phi^\gamma((1,x),(y,y')) &\leqslant \gamma^3 C\left(\int_0^\gamma \frac{\sin^{2\alpha}\theta}{\gamma^{2\alpha+4}}\,d\theta + \int_\gamma^\pi \frac{\sin^{2\alpha}\theta}{\theta^{2\alpha+4}}\,d\theta\right)\\
&\leqslant \gamma^3 C\left(\int_0^\gamma \frac{\theta^{2\alpha}}{\gamma^{2\alpha+4}}\,d\theta + \int_\gamma^\pi \frac{\theta^{2\alpha}}{\theta^{2\alpha+4}}\,d\theta\right)\\
&\leqslant \gamma^3 C\left(\frac{\gamma^{2\alpha+1}}{\gamma^{2\alpha+4}} + \int_\gamma^\pi \frac{1}{\theta^4}\,d\theta\right)\\
&\leqslant C.
\end{aligned}$$

Since $\frac{\gamma}{2} > p\left((1,x),(s,y)\right)$ then

$$1 + \frac{p\left((1,x),(s,y)\right)}{\gamma} < \frac{3}{2}.$$

Thus,

$$\frac{8}{27}\left(1 + \frac{p\left((1,x),(s,y)\right)}{\gamma}\right)^{-3} > 1.$$

Then

$$\begin{aligned}
\Phi^\gamma((1,x),(s,y)) &\leqslant C\left(1 + \frac{p\left((1,x),(s,y)\right)}{r}\right)^{-3}\\
&\leqslant C\left(1 + \frac{d_\alpha\left((1,x),(s,y)\right)}{\gamma}\right)^{-3}
\end{aligned}$$

- If $\frac{\gamma}{2} > |1-s|$ and $|x-y| \geqslant \frac{\gamma}{2}$ then, we have

$$\frac{1}{2} < s < \frac{3}{2} \ \text{ and } \ p\left((1,x),(s,y)\right) \leqslant d_\alpha\left((1,x),(s,y)\right).$$

$$\Phi^\gamma((1,x),(s,y)) \leqslant \gamma^3 \frac{1}{\left(p\left((1,x),(s,y)\right)\right)^3}.$$

Since, $\frac{\gamma}{2} \leqslant p\left((1,x),(s,y)\right)$ then, we have

$$\gamma + p\left((1,x),(s,y)\right) \leqslant 3p\left((1,x),(s,y)\right).$$

Then,

$$\begin{aligned}\Phi^{\gamma}((1,x),(s,y)) &\leqslant C\gamma^2 \frac{1}{(\gamma + p\left((1,x),(s,y)\right))^3} \\ &= C\left(1+\frac{p\left((1,x),(s,y)\right)}{\gamma}\right)^{-3} \\ &\leqslant C\left(1+\frac{d_\alpha\left((1,x),(s,y)\right)}{\gamma}\right)^{-3}.\end{aligned}$$

Case 2 $\gamma \geqslant 1$**:** $h(1,\gamma) = \gamma^{\frac{1}{2\alpha+2}}$.

$$\begin{aligned}\Phi^{\gamma}((1,x),(s,y)) &= \gamma \mathcal{T}_{(1,x)}\left(\varphi_{\gamma^{\frac{1}{2\alpha+2}}}\right)(-s,-y) \\ &\leqslant \gamma^{\frac{2\alpha+4}{2\alpha+2}} C \int_0^{\pi} \frac{\sin^{2\alpha}\theta}{(\gamma^{\frac{2}{2\alpha+2}} + (p\left((1,x),(s,y)\right))^2 + 2s(1-\cos\theta))^{\alpha+2}}\, d\theta\end{aligned}$$

- If $\gamma^{\frac{1}{2\alpha+2}} \leqslant |1-s|$ and $|x-y| > \gamma^{\frac{1}{2\alpha+2}}$, then $s \geqslant 2$ and

$$d_\alpha((1,x),(s,y))^{\frac{1}{2\alpha+2}} \sim p\left((1,x),(s,y)\right).$$

In fact, we have

$$|1-s|^{2\alpha+2} \leqslant \left|\int_1^s t^{2\alpha+1} dt\right|$$

and

$$|x-y|^{2\alpha+2} \leqslant |x-y|.$$

Then,

$$p\left((1,x),(s,y)\right)^{2\alpha+2} \leqslant d_\alpha((1,x),(s,y)).$$

We use the fact that $f(y) = \dfrac{y^{2\alpha+2}-1}{(y-1)^{2\alpha+2}}$ is a bounded function in $[2,+\infty[$. Then,

$$\left|\int_1^s t^{2\alpha+1} dt\right| \leqslant C|1-s|^{2\alpha+2} \tag{4.5}$$

$$|1-x| \leqslant |1-x|^{2\alpha+2}.$$

Then,

$$d_\alpha((1,x),(s,y)) \leqslant Cp\left((1,x),(s,y)\right)^{2\alpha+2}.$$

$$\begin{aligned}\Phi^{\gamma}((1,x),(s,y)) &\leqslant \gamma^{\frac{2\alpha+4}{2\alpha+2}} C \int_0^{\pi} \frac{\sin^{2\alpha}\theta}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\, d\theta \\ &\leqslant \gamma^{\frac{2\alpha+4}{2\alpha+2}} C \frac{1}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\end{aligned}$$

$$\leqslant C\frac{1}{\left(\frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{\frac{2\alpha+4}{2\alpha+2}}}$$
$$\leqslant C\left(1+\frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-1-\frac{1}{\alpha+1}}.$$

- If $\gamma^{\frac{1}{2\alpha+2}} \leqslant |1-s|$ and $|x-y| \leqslant \gamma^{\frac{1}{2\alpha+2}}$, then $s \geqslant 2$ and $|x-y| \leqslant |1-s|$. Using the Relation (4.5), we get

$$\left|\int_1^s t^{2\alpha+1}dt\right| \leqslant C|1-s|^{2\alpha+2} \leqslant Cp\left((1,x),(s,y)\right)^{2\alpha+2}$$

$$|x-y| \leqslant |1-s| \leqslant |1-s|^{2\alpha+2} \leqslant p\left((1,x),(s,y)\right)^{2\alpha+2}$$

$$d_\alpha((1,x),(s,y))^{\frac{1}{2\alpha+2}} \leqslant p\left((1,x),(s,y)\right).$$

Thus,
$$\Phi^\gamma((1,x),(s,y)) \leqslant C\left(1+\frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-1-\frac{1}{\alpha+1}}.$$

- If $\gamma^{\frac{1}{2\alpha+2}} > |1-s|$ and $\gamma^{\frac{1}{2\alpha+2}} > |x-y|$, then

$$d_\alpha((1,x),(s,y)) \leqslant 2^{2\alpha+2}\gamma.$$

In fact,

$$\left|\int_1^s t^{2\alpha+1}dt\right| \leqslant (\gamma^{\frac{1}{2\alpha+2}}+1)|1-s| \leqslant (\gamma^{\frac{1}{2\alpha+2}}+1)\gamma^{\frac{1}{2\alpha+2}} \leqslant (\gamma^{\frac{1}{2\alpha+2}}+1)\gamma \leqslant 2^{2\alpha+2}\gamma,$$

and
$$|x-y| \leqslant \gamma^{\frac{1}{2\alpha+2}} \leqslant \gamma$$
$$\Phi^\gamma((1,x),(s,y)) \leqslant \gamma^{\frac{2\alpha+4}{2\alpha+2}}C\int_0^\pi \frac{\sin^{2\alpha}\theta}{\gamma^{\frac{2\alpha+4}{2\alpha+2}}}\,d\theta \leqslant C \leqslant C\left(1+\frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-3}.$$

- If $\gamma^{\frac{1}{2\alpha+2}} > |1-s|$ and $\gamma^{\frac{1}{2\alpha+2}} \leqslant |x-y|$, then

$$d_\alpha((1,x),(s,y)) \leqslant (p((1,x),(s,y)))^{2\alpha+2}.$$

In fact, we have
$$|1-s| < |x-y|.$$

This implies that
$$p((1,x),(s,y)) = |x-y|$$
$$\left|\int_1^s t^{2\alpha+1}dt\right| \leqslant (\gamma^{\frac{1}{2\alpha+2}}+1)^{2\alpha+1} \leqslant 2^{2\alpha}(\gamma+1) \leqslant 2^{2\alpha+1}\gamma \leqslant 2^{2\alpha+1}|x-y|^{2\alpha+2},$$

and

$$|x-y| < |x-y|^{2\alpha+2}$$

$$\tilde{\phi}_r((1,x),(s,y)) \leqslant C\left(1+\frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-1-\frac{1}{\alpha+1}}.$$

From (4.4), we get

$$0 \leqslant \Phi^\gamma((r,x),(s,y)) \leqslant C\left(1+\frac{d_\alpha((r,x),(s,y))}{\gamma}\right)^{-1-\beta}.$$

Now, we will prove (iii) of the Proposition 4.7. Assume that for every $\gamma > 0$ and $(r,x),(s,y),(t,z) \in [0,+\infty[\times\mathbb{R}$, we have

$$d_\alpha((s,y),(t,z)) \leqslant \frac{r+d_\alpha((r,x),(s,y))}{4C}.$$

Then for every $\gamma' > 0$, we have

$$\left(1+\frac{d_\alpha((r,x),(t,z))}{\gamma}\right)^{-1-\gamma'} \leqslant \left(1+\frac{d_\alpha((r,x),(s,y))}{\gamma}\right)^{-1-\gamma'}. \tag{4.6}$$

Using (ii) and the Relation (4.6) , we have

$$|\Phi^\gamma((r,x),(s,y)) - \Phi^\gamma((r,x),(t,z))| \leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^{-1-\beta}. \tag{4.7}$$

Finally, using Lemma 4.1 and the Relation (4.8) we have

$$|\Phi^\gamma((r,x),(s,y)) - \Phi^\gamma((r,x),(t,z)))| \leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^{\beta}\left(1+\frac{d_\alpha((r,x),(s,y))}{\gamma}\right)^{-1-2\beta}. \quad \square$$

Proposition 4.8. *There exists a constant $C > 0$ such that for every $f \in L^1(d\nu_\alpha)$ we have*

$$\frac{1}{C}\|f\|_{\mathcal{H}^1_\alpha} \leqslant \|f\|_{\mathcal{H}^\alpha_{atomic}} \leqslant C\|f\|_{\mathcal{H}^1_\alpha}. \tag{4.8}$$

Proof. It is clear that $p_t \in Z^{[\alpha]}$. Using Proposition 4.7, Corollary 1 of [19]. Thus, we have show that (4.6) holds. $\square$

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Laeng-Morpurgo-type uncertainty inequalities for the Weinstein transform

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ABSTRACT

In this work, by combining Carlson-type and Nash-type inequalities for the Weinstein transform $\mathscr{F}_W$ on $\mathbb{K} = \mathbb{R}^{d-1} \times [0, \infty)$, we show Laeng-Morpurgo-type uncertainty inequalities. We establish also local-type uncertainty inequalities for the Weinstein transform $\mathscr{F}_W$, and we deduce a Heisenberg-Pauli-Weyl-type inequality for this transform.

RESUMEN

En este trabajo, combinando desigualdades de tipo Carlson y de tipo Nash para la transformada de Weinstein $\mathscr{F}_W$ en $\mathbb{K} = \mathbb{R}^{d-1} \times [0, \infty)$, demostramos desigualdades de incertidumbre de tipo Laeng-Morpurgo. Establecemos también desigualdades de incertidumbre de tipo local para la transformada de Weinstein $\mathscr{F}_W$, y deducimos una desigualdad de tipo Heisenberg-Pauli-Weyl para esta transformada.

Keywords and Phrases: Laeng-Morpurgo-type inequality; local-type inequality; Heisenberg-Pauli-Weyl-type inequality.

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1 Introduction

Uncertainty principles are mathematical arguments that give limitations on the simultaneous concentration of a function and its Fourier transform. They have implications in quantum physics and signal analysis. They also play an important role in harmonic analysis, many of them have already been studied from several points of view for the Fourier transform, Heisenberg-Pauli-Weyl inequality and local uncertainty [9, 10]. Laeng-Morpurgo and Morpurgo [4, 7] obtained Heisenberg inequality involving a combination of L^1 and L^2 norms.

In this paper, we consider the Weinstein transform $\mathscr{F}_W$ [2, 5, 6] defined on $L^1(\mathbb{K}, \nu_k)$ by

$$\mathscr{F}_W(f)(\xi) := \int_{\mathbb{K}} f(x)\Psi_\xi(x)\mathrm{d}\nu_k(x), \quad \xi = (\xi', \xi_d) \in \mathbb{K},$$

where $\mathbb{K} := \mathbb{R}^{d-1} \times [0, \infty)$, $\mathrm{d}\nu_k(x) := \frac{x_d^{2k+1}}{\pi^{(d-1)/2}2^{k+(d-1)/2}\Gamma(k+1)}\,\mathrm{d}x'\mathrm{d}x_d$ and

$$\Psi_\xi(x) = e^{-i\langle x', \xi'\rangle} j_k(x_d\xi_d), \quad x = (x', x_d) \in \mathbb{K}.$$

Here j_k is the spherical Bessel function.

Many uncertainty principles have already been proved for the Weinstein transform $\mathscr{F}_W$ on $\mathbb{K}$, namely Mejjaoli and Salhi are the first that describe the uncertainty principles for the Weinstein transform [6]. Next, Ben Salem and Nasr obtained Heisenberg-type inequalities [3] for the Weinstein transform $\mathscr{F}_W$. Saoudi [11] proved a variation of L^p uncertainty principles for the Weinstein transform $\mathscr{F}_W$. In this work, by using Carlson-type inequality and Nash-type inequality [2, 8] for the Weinstein transform $\mathscr{F}_W$ on $L^1 \cap L^2(\mathbb{K}, \nu_k)$; we deduce uncertainty inequalities of Heisenberg-type for the Weinstein transform $\mathscr{F}_W$ on $L^1 \cap L^2(\mathbb{K}, \nu_k)$. Next, due to a local uncertainty inequality for the Weinstein transform $\mathscr{F}_W$ on $L^2(\mathbb{K}, \nu_k)$, we show uncertainty inequality of Heisenberg-Pauli-Weyl-type for the transform $\mathscr{F}_W$ on $L^2(\mathbb{K}, \nu_k)$.

The analog uncertainty inequalities are also proved, for the Dunkl transform $\mathscr{F}_k$ on $\mathbb{R}^d$ by Soltani [12, 13].

This paper is organized as follows. In Section 2, we recall some results about the Weinstein transform $\mathscr{F}_W$ on $\mathbb{K}$. In Section 3, we prove uncertainty inequalities of Heisenberg-type for the Weinstein transform $\mathscr{F}_W$ on $L^1 \cap L^2(\mathbb{K}, \nu_k)$. We show also uncertainty inequality of Heisenberg-Pauli-Weyl-type for the transform $\mathscr{F}_W$ on $L^2(\mathbb{K}, \nu_k)$. In the last section, we summarize the obtained results and describe the future work.

2 Weinstein transform

In this section we recall some basic results related to the Weinstein analysis.

We consider the Weinstein operator Δ_W [1, 3, 8] defined on $\mathbb{R}^{d-1} \times (0, \infty)$ by

$$\Delta_W := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \frac{2k+1}{x_d} \frac{\partial}{\partial x_d} = \Delta_{d-1} + L_k, \quad d \geq 2, \quad k > -1/2,$$

where Δ_{d-1} is the Laplacian operator in $\mathbb{R}^{d-1}$ and L_k is the Bessel operator with respect to the variable x_d defined on $(0, \infty)$ by

$$L_k := \frac{\partial^2}{\partial x_d^2} + \frac{2k+1}{x_d} \frac{\partial}{\partial x_d}.$$

The Weinstein operator (also called Laplace-Bessel operator) has several applications in pure and applied mathematics. The harmonic analysis associated to this operator is studied in [1, 2, 3, 5, 6, 8] and references therein.

Throughout this subsection, let $k > -1/2$ and $\mathbb{K} := \mathbb{R}^{d-1} \times [0, \infty)$. We denote by $L^p(\mathbb{K}, \nu_k)$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{K}$, such that

$$\begin{aligned} \|f\|_{L^p(\mathbb{K},\nu_k)} &:= \left(\int_{\mathbb{K}} |f(x', x_d)|^p \mathrm{d}\nu_k(x', x_d) \right)^{1/p} < \infty, \quad p \in [1, \infty), \\ \|f\|_{L^\infty(\mathbb{K},\nu_k)} &:= \operatorname*{ess\,sup}_{(x',x_d)\in\mathbb{K}} |f(x', x_d)| < \infty, \end{aligned}$$

where

$$\mathrm{d}\nu_k(x) := \mathrm{d}\nu_k(x', x_d) = \frac{x_d^{2k+1}}{\pi^{(d-1)/2} 2^{k+(d-1)/2} \Gamma(k+1)} \mathrm{d}x' \mathrm{d}x_d,$$

and $\mathrm{d}x' = \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{d-1}$.

Let $r > 0$, the measure ν_k satisfies [3]:

$$\nu_k(|x| < r) = cr^\alpha, \tag{2.1}$$

where

$$c = \frac{1}{2^{\frac{\alpha}{2}} \Gamma(\frac{\alpha}{2} + 1)} \quad \text{and} \quad \alpha = 2k + d + 1. \tag{2.2}$$

For all $\xi \in \mathbb{K}$, the system

$$L_k u(x) = -\xi_d^2 u(x), \quad \frac{\partial^2 u}{\partial x_j^2}(x) = -\xi_j^2 u(x), \quad j = 1, \dots, d-1,$$
$$u(0) = 1, \quad \frac{\partial u}{\partial x_d}(0) = 0, \quad \frac{\partial u}{\partial x_j}(0) = -i\xi_j, \quad j = 1, \dots, d-1,$$

admits a unique solution $\Psi_\xi(x)$, given by

$$\Psi_\xi(x) = e^{-i\langle x', \xi' \rangle} j_k(x_d \xi_d), \quad x \in \mathbb{K},$$

where j_k is the spherical Bessel function given by

$$j_k(x) := \Gamma(k+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{2n}.$$

For all $x, \xi \in \mathbb{K}$, the Weinstein kernel $\Psi_\xi(x)$ satisfies

$$|\Psi_\xi(x)| \leq 1.$$

The Weinstein (or Laplace-Bessel) transform $\mathscr{F}_W$ [2, 5, 6] is defined for $f \in L^1(\mathbb{K}, \nu_k)$ by

$$\mathscr{F}_W(f)(\xi) := \int_{\mathbb{K}} f(x) \Psi_\xi(x) \, \mathrm{d}\nu_k(x), \quad \xi \in \mathbb{K}.$$

The transform $\mathscr{F}_W$ initially defined on $L^1 \cap L^2(\mathbb{K}, \nu_k)$ extends uniquely to an isometric isomorphism on $L^2(\mathbb{K}, \nu_k)$, that is,

$$\|\mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} = \|f\|_{L^2(\mathbb{K},\nu_k)}, \quad f \in L^2(\mathbb{K}, \nu_k). \tag{2.3}$$

Moreover if $f \in L^1(\mathbb{K}, \nu_k)$, then

$$\|\mathscr{F}_W(f)\|_{L^\infty(\mathbb{K},\nu_k)} \leq \|f\|_{L^1(\mathbb{K},\nu_k)}. \tag{2.4}$$

Finally, if f and $\mathscr{F}_W(f)$ are both in $L^1(\mathbb{K}, \nu_k)$, the inverse Weinstein transform is defined by

$$f(x) = \int_{\mathbb{K}} \mathscr{F}_W(f)(\xi) \Psi_{-\xi}(x) \, \mathrm{d}\nu_k(\xi), \quad \text{a.e} \quad x \in \mathbb{K}.$$

3 Heisenberg-type uncertainty principles

Similar results have been appeared in the literature by Soltani [13], he proved a Laeng-Morpurgo-type uncertainty inequalities for the Dunkl transform $\mathscr{F}_k$ on $\mathbb{R}^d$. In the following, we will give Laeng-Morpurgo-type uncertainty inequalities for the Weinstein transform $\mathscr{F}_W$ on $\mathbb{K}$.

Proposition 3.1 ([2, 8]).

(i) (Carlson-type inequality). Let $a>0$. There exists a constant $A(a,\alpha)>0$ such that for every $f\in L^1\cap L^2(\mathbb{K},\nu_k)$, we have

$$\|f\|_{L^1(\mathbb{K},\nu_k)} \le A(a,\alpha)\|f\|^{\frac{2a}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)}\|\,|x|^a f\|^{\frac{\alpha}{\alpha+2a}}_{L^1(\mathbb{K},\nu_k)}. \tag{3.1}$$

(ii) (Nash-type inequality). Let $b>0$. There exists a constant $B(b,\alpha)>0$ such that for every $f\in L^1\cap L^2(\mathbb{K},\nu_k)$, we have

$$\|f\|_{L^2(\mathbb{K},\nu_k)} \le B(b,\alpha)\|f\|^{\frac{2b}{\alpha+2b}}_{L^1(\mathbb{K},\nu_k)}\|\,|\xi|^b \mathscr{F}_W(f)\|^{\frac{\alpha}{\alpha+2b}}_{L^2(\mathbb{K},\nu_k)}. \tag{3.2}$$

Thanks to the above proposition, by combining and multiplying the two relations (3.1) and (3.2) we obtain the following uncertainty inequalities of Laeng-Morpurgo-type [4, 7] for the Weinstein transform $\mathscr{F}_W$ on $L^1\cap L^2(\mathbb{K},\nu_k)$.

Theorem 3.2. *Let $a,b>0$. There exist three constants $C(a,b,\alpha)>0$, $N(a,b,\alpha)>0$ and $D(a,b,\alpha)>0$ such that for every $f\in L^1\cap L^2(\mathbb{K},\nu_k)$, we have*

(i) $\|f\|^{\alpha+2a+2b}_{L^2(\mathbb{K},\nu_k)} \le C(a,b,\alpha)\|\,|x|^a f\|^{2b}_{L^1(\mathbb{K},\nu_k)}\|\,|\xi|^b\mathscr{F}_W(f)\|^{\alpha+2a}_{L^2(\mathbb{K},\nu_k)}$,

(ii) $\|f\|^{\alpha+2a+2b}_{L^1(\mathbb{K},\nu_k)} \le N(a,b,\alpha)\|\,|x|^a f\|^{\alpha+2b}_{L^1(\mathbb{K},\nu_k)}\|\,|\xi|^b\mathscr{F}_W(f)\|^{2a}_{L^2(\mathbb{K},\nu_k)}$,

(iii) $\|f\|^{\alpha+2a}_{L^1(\mathbb{K},\nu_k)}\|f\|^{\alpha+2b}_{L^2(\mathbb{K},\nu_k)} \le D(a,b,\alpha)\|\,|x|^a f\|^{\alpha+2b}_{L^1(\mathbb{K},\nu_k)}\|\,|\xi|^b\mathscr{F}_W(f)\|^{\alpha+2a}_{L^2(\mathbb{K},\nu_k)}$.

By application of the two relations (3.1) and (3.2) we deduce also the following results which are a local-type uncertainty inequalities for the Weinstein transform $\mathscr{F}_W$ on $L^1\cap L^2(\mathbb{K},\nu_k)$.

Theorem 3.3. *Let E be a measurable subset of $\mathbb{K}$ such that $0<\nu_k(E)<\infty$, and let $a,b>0$. If $f\in L^1\cap L^2(\mathbb{K},\nu_k)$, then*

(i) $\|\mathbf{1}_E\mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} \le A(a,\alpha)(\nu_k(E))^{1/2}\|f\|^{\frac{2a}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)}\|\,|x|^a f\|^{\frac{\alpha}{\alpha+2a}}_{L^1(\mathbb{K},\nu_k)}$, *where $A(a,\alpha)$ is the constant given by Proposition 3.1 (i).*

(ii) $\|\mathbf{1}_E\mathscr{F}_W(f)\|_{L^1(\mathbb{K},\nu_k)} \le B(b,\alpha)(\nu_k(E))^{1/2}\|f\|^{\frac{2b}{\alpha+2b}}_{L^1(\mathbb{K},\nu_k)}\|\,|\xi|^b\mathscr{F}_W(f)\|^{\frac{\alpha}{\alpha+2b}}_{L^2(\mathbb{K},\nu_k)}$, *where $B(b,\alpha)$ is the constant given by Proposition 3.1 (ii).*

Being $\mathbf{1}_E$ the characteristic function of the set E.

Proof. Let $f \in L^1 \cap L^2(\mathbb{K}, \nu_k)$ and $a, b > 0$.

(i) From (2.4) we have

$$\|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} \leq (\nu_k(E))^{1/2} \|\mathscr{F}_W(f)\|_{L^\infty(\mathbb{K},\nu_k)} \leq (\nu_k(E))^{1/2} \|f\|_{L^1(\mathbb{K},\nu_k)}.$$

The desired result follows from Proposition 3.1 (i).

(ii) From (2.3) we have

$$\|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^1(\mathbb{K},\nu_k)} \leq (\nu_k(E))^{1/2} \|\mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} \leq (\nu_k(E))^{1/2} \|f\|_{L^2(\mathbb{K},\nu_k)}.$$

The desired result follows from Proposition 3.1 (ii). □

Soltani [12] proved a Heisenberg-Pauli-Weyl uncertainty principle for the Dunkl transform $\mathscr{F}_k$ on $\mathbb{R}^d$. In the following, we will give Heisenberg-Pauli-Weyl uncertainty principle for the Weinstein transform $\mathscr{F}_W$ on $L^2(\mathbb{K}, \nu_k)$.

Proposition 3.4. *(local-type inequality). Let $a > 0$ and let $f \in L^2(\mathbb{K}, \nu_k)$. If E be a measurable subset of $\mathbb{K}$ such that $0 < \nu_k(E) < \infty$, then*

$$\|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} \leq A(a,\alpha)(\nu_k(E))^{\frac{a}{\alpha+2a}} \|f\|_{L^2(\mathbb{K},\nu_k)}^{\frac{2a}{\alpha+2a}} \|\,|x|^a f\|_{L^2(\mathbb{K},\nu_k)}^{\frac{\alpha}{\alpha+2a}}, \tag{3.3}$$

where $A(a,\alpha)$ is the constant given by Proposition 3.1 (i).

Proof. Let $f \in L^2(\mathbb{K}, \nu_k)$ and $a > 0$. The inequality holds if $\|\,|x|^a f\|_{L^2(\mathbb{K},\nu_k)} = \infty$. Assume that $\|\,|x|^a f\|_{L^2(\mathbb{K},\nu_k)} < \infty$. For all $r > 0$, we have

$$\begin{aligned} \|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} &\leq \|\mathbf{1}_E \mathscr{F}_W(\mathbf{1}_{B_r} f)\|_{L^2(\mathbb{K},\nu_k)} + \|\mathbf{1}_E \mathscr{F}_W((1-\mathbf{1}_{B_r})f)\|_{L^2(\mathbb{K},\nu_k)} \\ &\leq (\nu_k(E))^{1/2} \|\mathscr{F}_W(\mathbf{1}_{B_r} f)\|_{L^\infty(\mathbb{K},\nu_k)} + \|\mathscr{F}_W((1-\mathbf{1}_{B_r})f)\|_{L^2(\mathbb{K},\nu_k)}. \end{aligned}$$

Hence it follows from (2.3) and (2.4) that

$$\|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} \leq (\nu_k(E))^{1/2} \|\mathbf{1}_{B_r} f\|_{L^1(\mathbb{K},\nu_k)} + \|(1-\mathbf{1}_{B_r})f\|_{L^2(\mathbb{K},\nu_k)}. \tag{3.4}$$

On the other hand, by Hölder's inequality and (2.1), we obtain

$$\|\mathbf{1}_{B_r} f\|_{L^1(\mathbb{K},\nu_k)} \leq (c r^\alpha)^{1/2} \|f\|_{L^2(\mathbb{K},\nu_k)}, \tag{3.5}$$

where c is the constant given by (2.2).

Moreover,

$$\|(1-\mathbf{1}_{B_r})f\|_{L^2(\mathbb{K},\nu_k)} \leq r^{-a}\| \, |x|^a f\|_{L^2(\mathbb{K},\nu_k)}. \tag{3.6}$$

Combining the relations (3.4), (3.5) and (3.6), we deduce that

$$\|\mathbf{1}_E \mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_k)} \leq (\nu_k(E))^{1/2}(cr^\alpha)^{1/2}\|f\|_{L^2(\mathbb{K},\nu_k)} + r^{-a}\| \, |x|^a f\|_{L^2(\mathbb{K},\nu_k)}.$$

By choosing

$$r = \left(\frac{2a\| \, |x|^a f\|_{L^2(\mathbb{K},\nu_k)}}{\alpha c^{1/2}\|f\|_{L^2(\mathbb{K},\nu_k)}}\right)^{\frac{2}{\alpha+2a}} (\nu_k(E))^{-\frac{1}{\alpha+2a}},$$

we obtain the desired inequality. □

We shall use the local uncertainty principle to obtain uncertainty principle of Heisenberg-Pauli-Weyl-type for the Weinstein transform $\mathscr{F}_W$ on $L^2(\mathbb{K},\nu_k)$. We note that the following theorem is given in [3] but in the proof, the approach is not the same.

Theorem 3.5. *Let $a, b > 0$. There exists a constant $K(a,b,\alpha) > 0$ such that for every $f \in L^2(\mathbb{K},\nu_k)$, we have*

$$\|f\|^{a+b}_{L^2(\mathbb{K},\nu_k)} \leq K(a,b,\alpha)\| \, |x|^a f\|^b_{L^2(\mathbb{K},\nu_k)}\| \, |\xi|^b \mathscr{F}_W(f)\|^a_{L^2(\mathbb{K},\nu_k)}.$$

Proof. Let $a, b > 0$ and let $r > 0$. Then

$$\|f\|^2_{L^2(\mathbb{K},\nu_k)} = \|\mathbf{1}_{B_r}\mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)} + \|(1-\mathbf{1}_{B_r})\mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)}. \tag{3.7}$$

Firstly,

$$\|(1-\mathbf{1}_{B_r})\mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)} \leq r^{-2b}\| \, |\xi|^b \mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)}. \tag{3.8}$$

From (2.1) and (3.3), we get

$$\|\mathbf{1}_{B_r}\mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)} \leq (A(a,\alpha))^2(cr^\alpha)^{\frac{2a}{\alpha+2a}}\|f\|^{\frac{4a}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)}\| \, |x|^a f\|^{\frac{2\alpha}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)}, \tag{3.9}$$

where c is the constant given by (2.2).

Combining the relations (3.7), (3.8) and (3.9), we obtain

$$\|f\|^2_{L^2(\mathbb{K},\nu_k)} \leq (A(a,\alpha))^2(cr^\alpha)^{\frac{2a}{\alpha+2a}}\|f\|^{\frac{4a}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)}\| \, |x|^a f\|^{\frac{2\alpha}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)} + r^{-2b}\| \, |\xi|^b \mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)}.$$

By setting

$$r = \left(\frac{b(\alpha+2a)\| \, |\xi|^b \mathscr{F}_W(f)\|^2_{L^2(\mathbb{K},\nu_k)}}{a\alpha(A(a,\alpha))^2 c^{\frac{2a}{\alpha+2a}}\|f\|^{\frac{4a}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)}\| \, |x|^a f\|^{\frac{2\alpha}{\alpha+2a}}_{L^2(\mathbb{K},\nu_k)}}\right)^{\frac{\alpha+2a}{2a\alpha+2b(\alpha+2a)}},$$

we get the inequality with

$$K(a,b,\alpha) = (A(a,\alpha))^{2b(\alpha+2a)} c^{2ab} \left(\frac{b(\alpha+2a)}{a\alpha}\right)^{a\alpha} \left(1 + \frac{a\alpha}{b(\alpha+2a)}\right)^{a\alpha+b(\alpha+2a)}.$$

This completes the proof of the theorem. □

4 Conclusion and perspective

The manuscript deals with some uncertainty inequalities associated with the Weinstein transform $\mathscr{F}_W$. Especially, we studied Laeng-Morpurgo type uncertainty inequalities for this transform. As it is well known, uncertainty inequalities are of great interest in harmonic analysis, in applied mathematics and in several areas of mathematical physics. The results given in Section 3 are complements to those given in references [3, 6, 8] and others. They also represent our contribution in the study of local-type uncertainty inequalities and the Heisenberg type inequality for the Weinstein transform $\mathscr{F}_W$. Finally, in a future paper, we have the idea to study the Weinstein-Stockwell transform $\mathscr{S}_g$, $g \in L^2(\mathbb{K}, \nu_\alpha)$, in which we will prove some uncertainty inequalities for this transform analogous to those proven for the Weinstein transform $\mathscr{F}_W$ in this paper.

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Osculating varieties and their joins: $\mathbb{P}^1 \times \mathbb{P}^1$

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ABSTRACT

Let $X \subset \mathbb{P}^r$ be an integral projective variety. We study the dimensions of the joins of several copies of the osculating varieties $J(X, m)$ of X. Our methods are general, but we give a full description in all cases only if X is a linearly normal embedding of $\mathbb{P}^1 \times \mathbb{P}^1$. For these embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ we give several examples and then study the joins of one copy of $J(X, m)$ and an arbitrary number of copies of X.

RESUMEN

Sea $X \subset \mathbb{P}^r$ una variedad proyectiva entera. Estudiamos la dimensión de las adjunciones de varias copias de las variedades osculantes $J(X, m)$ de X. Nuestros métodos son generales, pero damos una descripción completa en todos los casos solo si X es un embebimiento linealmente normal de $\mathbb{P}^1 \times \mathbb{P}^1$. Para estos embebimientos de $\mathbb{P}^1 \times \mathbb{P}^1$ damos varios ejemplos y luego estudiamos las adjunciones de una copia de $J(X, m)$ y un número arbitrario de copias de X.

Keywords and Phrases: Osculating space; joins of projective varieties; secant varieties; quadric surface.

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1 Introduction

Let $X \subset \mathbb{P}^r$ be an integral projective variety defined over a fixed algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. We consider the classical problem about the dimension of joins of varieties related to X. Let $J(X, m) \subseteq \mathbb{P}^r$, $m \geq 0$, denote the m-osculating variety of X, *i.e.* the closure in $\mathbb{P}^r$ of the union of all m-osculating spaces to the smooth points of X. With our convention of m-osculating linear spaces we have $J(X, 0) = X$, while $J(X, 1)$ is the tangential variety of X, *i.e.* the with the convention and the dimension of the joins of several $J(X, m_i)$, i varying in a finite set. Our notation calls $J(X, 1)$ the tangential variety $\tau(X) \subseteq \mathbb{P}^r$ of X, *i.e.* the closure in $\mathbb{P}^r$ of the union in $\mathbb{P}^r$ of the tangent spaces T_pX of X at all $p \in X_{\mathrm{reg}}$. For us $J(X, m)$ is the closure in $\mathbb{P}^r$ of the union of the m-osculating spaces at all p in a non-empty open subset of X_{reg} at which these m-osculating spaces have constant dimension.

Take integral varieties $T, Y \subset \mathbb{P}^r$. The join $J(T, Y)$ of T and Y is defined in the following way. If $T = Y$ and Y is a point, p, then $J(\{p\}, \{p\}) = \{p\}$. In all other cases $J(T, Y)$ is the closure of the union of all lines spanned by a point of T and a different point of Y. The algebraic set $J(T, Y)$ is always an irreducible variety and $\dim(T, Y) \leq \min\{r, \dim T + \dim Y + 1\}$ if $\dim T > 0$. The integer $\min\{r, \dim T + \dim Y + 1\}$ is the *expected dimension* of $J(T, Y)$. One defines inductively the join $J(T_1, \dots, T_s)$ of $s \geq 3$ integral varieties $T_i \subset \mathbb{P}^r$ by the formula $J(T_1, \dots, T_s) := J(J(T_1, \dots, T_{s-1}), T_s)$ ([1]). If $\dim T_1 > 0$ we have $\dim J(T_1, \dots, T_s) \leq \min\{r, \dim T_1 + \cdots + \dim T_s + s - 1\}$. If $\dim J(T_1, \dots, T_s) = \min\{r, \dim T_1 + \cdots + \dim T_s + s - 1\}$ we say that $J(T_1, \dots, T_s)$ has the *expected dimension*. The most famous and useful join is the case $T_i = T_1$ for all i, *i.e.*, the s-secant variety of T_1. However, other cases appear. For instance when X is the Veronese variety the join of the tangential variety $J(X, 1)$ of X and $s - 1$ copies of X is related to a certain additive decomposition of forms ([4]).

By the Terracini lemma for joins ([1, Corollary 1.11]) to compute the dimension of the join of s varieties $J(X, m_i)$, $1 \leq i \leq s$, it is sufficient to compute the dimension of the linear span of the tangent spaces $T_{Q_i}J(X, m_i)$ at a general $Q_i \in J(X, m_i)$. Obviously, we first need to compute $\dim T_{Q_i}J(X, m_i)$, but in all our examples these integers are known and hence the only problem is to see how linearly independent are these linear spaces $T_{Q_i}J(X, m_i)$. Fix a general $Q_i \in J(X, m_i)$ and let $p_i \in X_{\mathrm{reg}}$ the point of X_{reg} corresponding to Q_i. A key property of the osculating spaces $T_{Q_i}J(X, m_i)$, is that even for $m > 1$ there is a zero-dimensional scheme $Z_i \subset X$ such that $(Z_i) = \{p_i\}$ and $T_{Q_i}J(X, m_i)$ is the linear span of Z_i (Remark 2.1). If $m > 0$ the scheme is not unique, it is associated to the choice of a line of $T_{p_i}X$ containing p_i (Remark 3.4). Fix a general $(p_1, \dots, p_s) \in X_{\mathrm{reg}}^s$. For each i with $m_i > 0$ choose a "general" Z_i. As always in this type of problems ([3, 6, 7, 8, 9, 10, 11, 12]) it is sufficient to find the schemes $Z_i \subset X$, $1 \leq i \leq s$, and then to prove that the dimension of the linear span of $Z_1 \cup \cdots \cup Z_s$ is the expected one, $\sum_{i=1}^{s} \deg(Z_i) - 1$.

Set $n := \dim X$. For the joins of several copies of X it is sufficient to take as schemes the first infinitesimal neighborhood $2p$, $p \in X_{\mathrm{reg}}$, *i.e.* the closed subscheme of X with $(\mathcal{I}_p)^2$ as its ideal sheaf (this is the classical Terracini lemma for secant varieties [1, Corollary 1.11]); in this case the scheme has degree $n+1$. We call it the case $m = 0$. For the tangential variety the scheme Z_1 has degree $2n+1$ and it was used in several papers ([3, 6, 7, 8, 10, 11, 12]), some of them also considering the general case with any $m_i > 0$. Contrary to the case $m = 0$ the schemes $Z_i \subset X_{\mathrm{reg}}$ are not uniquely determined by the point $p \in X_{\mathrm{reg}}$ such that $(Z_i) = \{p\}$. For any $m > 0$ the scheme $W(m,p)$ associated to $J(X,m)$ at p has degree $n + \binom{n+m}{n}$ and it is implicitly computed in [6] (and by the classical algebraic geometers quoted in [5, 6]) and given in full generality in [7, 8, 10] at least for the Veronese embeddings of projective spaces. It depends on the choice of some $p \in X_{\mathrm{reg}}$ and a line through p of the embedded tangent space of X at p (Remark 2.1).

Of course, to define the osculating spaces we also need to fix an embedding of X in a projective space or, more generally, a line bundle $\mathcal{L}$ on X and a linear subspace $V \subseteq H^0(\mathcal{L})$. This set-up was described in a modern language by R. Piene ([18]), first defining the bundles of principal parts $\mathbb{P}^m_X(\mathcal{L})$ of $\mathcal{L}$ and then considering an evaluation map $\mathcal{O}_X \otimes V \longrightarrow \mathbb{P}^m_X(L)$. Thus for a fixed $m \geq 0$ and a general $p \in X_{\mathrm{reg}}$ we may choose an irreducible family of zero-dimensional schemes $\mathcal{Z}(m,p)$ such that for each $Z \in \mathcal{Z}(m,p)$ we have $Z = \{p\}$ and $\deg(Z) = n + \binom{n+m}{m}$. Moreover, for any $s > 0$ and any $m_i \geq 0$, the join of $J(X,m_1), \dots, J(X,m_s)$ has dimension $\langle W(m_1,p_1) \cup \cdots \cup W(m_s,p_s) \rangle$, where $\langle \ \rangle$ denote the linear span and $(p_1, \dots, p_s)$ is general in X^s.

The freedom in the choice to define $W(m,p)$ for $m > 0$ will be used several times in our proofs.

We only consider the case $X = \mathbb{P}^1 \times \mathbb{P}^1$ with all its Segre-Veronese embedding. We prove the following result.

Theorem 1.1. *Fix integers $c \geq 0$, $m \geq 0$ and $a \geq b \geq m+3$. Let $W \subset X$ be a general union of one element of $\mathcal{Z}(m)$ and c 2-points. Then either $h^0(\mathcal{I}_W(a,b)) = 0$ or $h^1(\mathcal{I}_W(a,b)) = 0$.*

The following result may also be proved using the tools in [9, 12].

Proposition 1.2. *Fix integer $c \geq 0$, $m \geq 2$ and $a \geq b \geq m$. Let $W \subset X$ be a general union of one m-point and c 2-points. Then either $h^0(\mathcal{I}_W(a,b)) = 0$ or $h^1(\mathcal{I}_W(a,b)) = 0$, except in the case $m = 2$, $b = 2$, a even and $c = a/2$.*

In section 3 (again with $X = \mathbb{P}^1 \times \mathbb{P}^1$) we give several examples of our tools and tricks to compute the dimensions of several joins.

2 General tools

In this section we collect the necessary tools lifted from the literature and add some remarks which greatly improve their use to compute the dimensions of joins.

For all $p \in X$ and all integers $m > 0$ let mp denote the closed subscheme of X with $(\mathcal{I}_p)^m$ as its ideal sheaf. For any $Y \subseteq X$ and any $p \in Y_{\mathrm{reg}} \cap X_{\mathrm{reg}}$ set $(mp, Y) := mp \cap Y$. Since p is a smooth point of both X and Y, (mp, Y) is the closed subscheme of Y whose ideal sheaf is $(\mathcal{I}_{p,Y})^m \subset \mathcal{O}_Y$.

Let X be an integral projective variety, $\mathcal{L}$ a line bundle on X and $V \subseteq H^0(\mathcal{L})$ a linear subspace. Let $Z \subseteq W \subset X$ be a zero-dimensional scheme. Obviously if $V \cap H^0(\mathcal{I}_Z \otimes \mathcal{L}) = 0$, then $V \cap H^0(\mathcal{I}_W \otimes \mathcal{L}) = 0$. Since W is zero-dimensional, the restriction map $H^0(\mathcal{O}_W \otimes \mathcal{L}) \longrightarrow H^0(\mathcal{O}_Z \otimes \mathcal{L})$ is surjective. Thus $h^1(\mathcal{I}_Z \otimes \mathcal{L}) \le h^1(\mathcal{I}_W \otimes \mathcal{L})$.

Remark 2.1. *Let X be an integral projective variety. Set $n := \dim(X)$. The schemes $Z \in \mathcal{Z}(X, m)$, $m \ge 0$, used to detect the tangent space $T_Q J(X, n)$ at a general $Q \in J(X, m)$ are all schemes obtained in the following way. Set $\mathcal{Z}(X, 0) := \{2p\}_{p \in X_{\mathrm{reg}}}$. Now assume $m > 0$. Set $\mathcal{Z}(X, m) := \cup_{p \in X_{\mathrm{reg}}} \mathcal{Z}(X, p, m)$, where each $\mathcal{Z}(X, p, m)$ is defined in the following way. Fix $p \in X_{\mathrm{reg}}$. Any zero-dimensional scheme $Z \in \mathcal{Z}(X, p, m)$ will have $Z = \{p\}$ and hence to define each element Z of $\mathcal{Z}(X, p, m)$ it is sufficient to define the ideal $\mathcal{J}$ of the local ring $\mathcal{O}_{X,p}$ such that $\mathcal{O}_Z = \mathcal{O}_{X,p}/\mathcal{J}$. Let μ be the maximal ideal of $\mathcal{O}_{X,p}$. The ideal $\mathcal{J}$ is constructed taking a germ at p of a smooth curve contained in a neighborhood of p in X and containing p, i.e. taking a regular system of parameters $t_1, \dots, t_n$ of the local ring $\mathcal{O}_{X,p}$, i.e. any system of n generators $t_1, \dots, t_n$ of the maximal ideal μ of $\mathcal{O}_{X,p}$ and taking any germ of curve with $(t_2, \dots, t_n)$ as its ideal in $\mathcal{O}_{X,p}$. As ideal of Z we take $\mu^{m+2} + t_1^{m+1}\mu$. With obvious conventions (i.e. taking as L_Z the germ of X at p) this ideal gives the ideal μ^{m+2} if $n = 1$, i.e. for $n = 1$ it gives the correct answer $\mathcal{J} = \mu^{m+2}$. The scheme Z is uniquely determined by the choice of a one-dimensional linear subspace of the n-dimensional vector space μ/μ^2, i.e. by the choice of a non-zero element of μ/μ^2. We will say that Z depends on the choice of a tangent vector L_Z of X at p. Each $Z \in \mathcal{Z}(X, m)$ has as its reduction a unique $p \in X_{\mathrm{reg}}$. We have*

$$(m+1)p \subset Z \subset (m+2)p, \quad \deg(Z) = n + \deg((m+1)p) = n + \binom{m+n}{n}.$$

We say that Z is defined by p and the tangent vector L_Z, because L_Z is uniquely determined by a connected degree 2 scheme $E \subset X$ such that $E = \{p\}$.

We often write $\mathcal{Z}(m)$ (resp. $\mathcal{Z}(p, m)$) instead of $\mathcal{Z}(X, m)$ (resp. $\mathcal{Z}(X, p, m)$).

Remark 2.2. *Let X be an integral projective variety and D an effective Cartier divisor of X. For any zero-dimensional scheme $Z \subset X$ the residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed zero-dimensional subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal scheme. We have $\mathrm{Res}_D(Z) \subseteq Z$,*

$\deg(Z) = \deg(\mathrm{Res}_D(Z)) + \deg(Z \cap D)$ *and for every line bundle* $\mathcal{L}$ *on* X *there is an exact sequence*

$$0 \longrightarrow \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D) \longrightarrow \mathcal{I}_Z \otimes \mathcal{L} \longrightarrow \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D} \longrightarrow 0 \tag{2.1}$$

For any Z *and* $\mathcal{L}$ *we will say that* (2.1) *is the residual exact sequence of* D. *Fix* $Z \in \mathcal{Z}(X,m)$, $m > 0$, *and set* $\{p\} := Z$. *Let* L_Z *be the tangent vector of* X *at* p *defining* Z *and call* $t_1,\dots,t_n$ *a regular system of generators of the maximal ideal of* μ *of* $\mathcal{O}_{X,p}$ *such that* L_V *is defined by* $t_2 = \cdots = t_n = 0$, $t_1^2 = 0$. *Now assume* $p \in D_{\mathrm{reg}}$.

(a) *Assume that* L_Z *is not contained in the tangent space of* D *at* p. *Then* $D \cap Z = ((m+1)p, D)$ *and hence* $\deg(\mathrm{Res}_D(Z)) = n + \binom{m+n}{n} - \binom{m+n-1}{n-1} = n + \binom{m+n-1}{n}$. *Moreover,* $\mathrm{Res}_D(Z) \in \mathcal{Z}(X, m-1)$ *and if* $m \geq 2$ *the scheme* $\mathrm{Res}_D(Z)$ *is defined by the same tangent vector* L_Z.

(b) *Assume that* L_Z *is contained in the tangent space of* D *at* p. *Then* $D \cap Z \in \mathcal{Z}(D,m)$ *and hence* $\deg(\mathrm{Res}_D(Z)) = n + \binom{m+n}{n} - n + 1 - \binom{m+n-1}{n-1} = 1 + \binom{m+n-1}{n}$. *We have* $\mathrm{Res}_D(Z) \supset mp$ *and* $\deg(\mathrm{Res}_D(Z)) = \deg(mp) + 1$. *The scheme* $\mathrm{Res}_D(Z)$ *is the union of* mp *and the scheme* $t_1^{m+2} = t_2 = \cdots = t_n = 0$.

In both cases the scheme Z *is vertically graded with respect to* D *in the sense of [2] and hence we may apply the Differential Horace Horace Lemma to* Z *([2]).*

For any line bundle $\mathcal{L}$ on X, any closed subscheme B of X and any vector space $V \subseteq H^0(\mathcal{L})$ set $V(-B) := V \cap H^0(\mathcal{I}_B \otimes \mathcal{L})$.

We describe the case of 2-points of the Differential Horace Lemma ([2]). The reader will find in that paper explicitly the case of points with higher multiplicities and the case (vertically graded subschemes) sufficient to handle all $Z \in \mathcal{Z}(m)$. Let X be an integral projective n-dimensional variety, D an effective Cartier divisor of X, $\mathcal{L}$ a line bundle on X, $V \subseteq H^0(\mathcal{L})$ a linear subspace. Let V_D be the image of V by the restriction map $\rho : H^0(\mathcal{L}) \longrightarrow H^0(D, \mathcal{L}_{|D})$. Set $n := \dim X$. Let $V(-D)$ be the set of all $f \in H^0(\mathcal{L}(-D))$ such that $zf \in V$, where $z \in H^0(\mathcal{O}_X(D))$ is the equation of D. Take a general $p \in X_{\mathrm{reg}} \cap D_{\mathrm{reg}}$. To prove that $\dim V(-Z-2q) = \max\{0, \dim V(-Z) - n - 1\}$ for a general $q \in X_{\mathrm{reg}}$ it is sufficient to prove that one of the following sets of conditions is satisfied:

(a) $\dim V_D(-Z \cap D) \leq 1$ and

$$\dim V(-D)(-\mathrm{Res}_D(Z) - (2p, D)) = \max\{0, \dim W(-\mathrm{Res}_D(Z)) - n\};$$

(b) $\dim V_D(-Z \cap D) > 0$ and

$$\dim V(-D)(-\mathrm{Res}_D(Z) - (2p, D)) = \dim V(-D)(-\mathrm{Res}_D(Z)) - n.$$

Remark 2.3. *Take any projective variety X, any line bundle $\mathcal{L}$ on X and any vector space $V \subseteq H^0(\mathcal{L})$. Fix $(u,v) \in \mathbb{N}^2$. Let $B \subset X$ be a general union of u tangent vectors of X_{reg} and v points of X. By [14] we have* $\dim V(-B) = \max\{0, \dim V - 2u - v\}$.

Remark 2.3 is useful because it applies to non-complete linear systems, too. We will use this key feature in the proof of the next lemma.

Lemma 2.4. *Take a projective variety X, a line bundle $\mathcal{L}$ on X and an integral Cartier divisor $D \subset X$. Assume $h^1(\mathcal{L}) = h^1(\mathcal{L}(-D)) = h^1(D, \mathcal{L}_{|D}) = 0$. Fix $(u,v) \in \mathbb{N}^2$. Let $Z \subset X$ be a zero-dimensional scheme. Let $B \subset D$ be a general union of u tangent vectors of D_{reg} and v points of X. Assume $h^1(\mathcal{I}_Z \otimes \mathcal{L}) = 0$, $h^1(D, \mathcal{I}_{D\cap Z,D} \otimes \mathcal{L}_{|D}) = 0$ and $h^0(D, \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D}) \geq 2u + v$. Then $h^0(\mathcal{I}_{Z\cup B} \otimes \mathcal{L}) = \max\{0, h^0(\mathcal{I}_Z \otimes \mathcal{L}) - 2u - v\}$.*

Proof. Remark 2.3 applied to D, $\mathcal{L}_{|D}$, $H^0X, (\mathcal{I}_Z \otimes \mathcal{L})$ and $H^0(D, \mathcal{I}_{Z\cap D} \otimes \mathcal{L}_{|D})$ gives

$$h^0(D, \mathcal{I}_{(Z\cap D)\cup B,D} \otimes \mathcal{L}_{|D}) = h^0(D, \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D}) - 2u - v.$$

Use twice the residual exact sequence of D, first with $\mathcal{I}_Z \otimes \mathcal{L}$ in the middle and then with $\mathcal{I}_{Z\cup B} \otimes \mathcal{L}$ in the middle. Use that $\mathrm{Res}_D(Z \cup B) = \mathrm{Res}_D(Z)$, because $B \subset D$ (as schemes). □

If we take the set-up and assumptions of Lemma 2.4 except the inequality on $h^0(D, \mathcal{I}_{Z\cup D,D} \otimes \mathcal{L}_{|D})$ and we have $h^0(D, \mathcal{I}_{Z\cup D,D} \otimes \mathcal{L}_{|D}) \leq 2u + v$, then Remark 2.3 gives $h^0(D, \mathcal{I}_{(Z\cap D)\cup B,D} \otimes \mathcal{L}_{|D}) = 0$. Thus the residual exact sequence of D gives $h^0(\mathcal{I}_{Z\cup B} \otimes \mathcal{L}) = h^0(\mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D))$.

Remark 2.5. *Fix a line bundle $\mathcal{L}$ on an integral projective variety X. Let $Z_1 \subseteq Z_2$ be zero-dimensional schemes. Note that $h^1(\mathcal{I}_{Z_1} \otimes \mathcal{L}) \leq h^1(\mathcal{I}_{Z_2} \otimes \mathcal{L})$. If $h^0(\mathcal{I}_{Z_2} \otimes \mathcal{L}) = h^0(\mathcal{L}) - \deg(Z_2)$, then $h^0(\mathcal{I}_{Z_1} \otimes \mathcal{L}) = h^0(\mathcal{L}) - \deg(Z_1)$. Set $n := \dim X$. Let $\mathcal{U}$ (resp. $\mathcal{V}$) be the set of all triples $(e,f,g) \in \mathbb{N}^3$ such that $h^0(\mathcal{L}) \leq e(n+\binom{n+2}{n})+f(2n+1)+g(n+1)$ (resp. $h^0(\mathcal{L}) \geq e(n+\binom{n+2}{n})+f(2n+1)+g(n+1)$). Fix $(e,f,g) \in \mathcal{U}$. Let $Z \subset X$ be a general union of e elements of $\mathcal{Z}(2)$, f elements of $\mathcal{Z}(1)$ and g 2-points. Suppose you want to prove that $h^0(\mathcal{I}_Z \otimes \mathcal{L}) = h^0(\mathcal{L}) - e(n+\binom{n+2}{n}) - f(2n+1) - g(n+1)$. It is sufficient to show that $h^0(\mathcal{I}_Z \otimes \mathcal{L}) = h^0(\mathcal{L}) - e(n + \binom{n+2}{n}) - f(2n+1) - g'(n+1)$ for some integer $g' \geq g$, where Z' is the union of Z and $g' - g$ general 2-points. Thus to check for all $(e,f,g) \in \mathcal{U}$ that a general union of e elements of $\mathcal{Z}(2)$, f element of $\mathcal{Z}(1)$ and g 2-points imposes independent conditions to $h^0(\mathcal{L})$ it is sufficient to check all $(e,f,g) \in \mathbb{N}^3$ such that*

$$h^0(\mathcal{L}) - n \leq e\left(n + \binom{n+2}{n}\right) + f(2n+1) + g(n+1) \tag{2.2}$$

Suppose you want to prove that $h^0(\mathcal{I}_W \otimes \mathcal{L}) = 0$ for all $(u,v,w) \in \mathcal{V}$, where W is a general union of u elements of $\mathcal{Z}(2)$, v elements of $\mathcal{Z}(1)$ and w 2-points. Decreasing if necessary the zero-dimensional scheme, it is sufficient to check all $(u,v,w) \in \mathbb{N}^3$ satisfying one of the following sets *of conditions:*

$$h^0(\mathcal{L}) \le e\left(n + \binom{n+2}{n}\right) + f(2n+1) + g(n+1) \le h^0(\mathcal{L}) + n \tag{2.3}$$

$$g = 0,\ h^0(\mathcal{L}) \le e\left(n + \binom{n+2}{n}\right) + f(2n+1) \le h^0(\mathcal{L}) + 2n \tag{2.4}$$

$$f = g = 0,\ h^0(\mathcal{L}) \le e\left(n + \binom{n+2}{n}\right) + f(2n+1) \le h^0(\mathcal{L}) + n - 1 + \binom{n+2}{n} \tag{2.5}$$

With minimal modifications the interested reader may state similar statements for general unions of prescribed numbers of osculating spaces and multiple points with arbitrary multiplicities and for m-points instead of just 2*-points (see Remark 3.2).*

Fix a linear subspace $V \subseteq H^0(\mathcal{L})$*. Suppose* $\dim V = 1$*. Thus* $V(-p) = 0$ *for a general* $p \in X$*. Hence* $V(-2p) = 0$ *for a general* $p \in X_{\mathrm{reg}}$*. Suppose* $\dim V = 2$*. By [14]* $V(-A) = 0$ *for a general tangent vector* A *of* X_{reg}*. Thus* $V(-2p) = 0$ *for a general* $p \in X_{\mathrm{reg}}$*. Thus in* (2.3) *it is not necessary to check all cases with* $e > 0$ *and* $e(n + \binom{n+2}{n}) + f(2n+1) + g(n+1) \in \{h^0(\mathcal{L}) + n - 1, h^0(\mathcal{L}) + n\}$ *(Remark 3.2).*

Remark 2.6. *Let* X *be an integral projective variety,* L *a line bundle on* X *and* $V \subseteq H^0(L)$*. Set* $n := \dim X$*. Fix a general* $p \in X_{\mathrm{reg}}$*. The function* $f : \mathbb{N} \longrightarrow \mathbb{N}$ *defined by the formula* $f(m) := \dim V(-mp)$ *is non-increasing. Since we take* p *general in* X_{reg}*, the semicontinuity theorem for cohomology shows that this function does not depend upon the choice of the general* p*. Consider its first difference* $g : \mathbb{N} \longrightarrow \mathbb{N}$*, i.e. set* $g(0) := f(0) = \dim V$ *and* $g(m) = f(m-1) - f(m)$ *for all* $m > 0$*.*

Observation 1: *If* $f(m) \neq 0$*, then* $g(m+1) > 0$*, i.e.* $f(m+1) < f(m)$*, unless* $f(m) = 0$ *([13, Proposition 2.3]).*

Now we fix an arbitrary $o \in X_{\mathrm{reg}}$*, set* $R := \mathcal{O}_{X,o}$ *and call* μ *the maximal ideal of the local ring* R*. Thus* $R/\mu \cong \mathbb{K}$ *and, since* X *is smooth at* o*, the graded ring* $GR_o := \bigoplus_{t \ge 0} \mu^t/\mu^{t+1}$ *(with the convention* $\mu^0 = R$*) is isomorphic to a polynomial ring in* n *variables over* $\mathbb{K}$*. Taking a regular system of parameters* $t_1, \dots, t_n$*, we may see each* μ^m/μ^{m+1} *as the* $\mathbb{K}$*-vector space of all degree* m *homogeneous polynomials in the variables* $t_1, \dots, t_n$*. Thus* $\dim_{\mathbb{K}} \mu^m/\mu^{m+1} = \binom{n+m-1}{n-1}$*. Set* $f_o(m) := \dim V(-mo)$ *and* $g_o(m) := f_o(m+1) - f_o(m)$*. There is an evaluation map* $e_{o,m} : V(-m)/V(-(m+1)o) \longrightarrow \mu^m/\mu^{m+1}$ *and* $g_o(m)$ *is the rank of the evaluation map* $e_{o,m}$*. For a general* o *we write* e_m *instead of* $e_{o,m}$*. For any integer* v *such that* $0 \le v \le \binom{n+m-1}{n-1}$ *let* $G(v, \mu^m/\mu^{m+1})$ *denote the Grassmannian of all* v*-dimensional linear subspaces of* μ^m/μ^{m+1}*. Call* $\pi : \mu^m \longrightarrow \mu^{m+1}$ *the quotient map. Fix* v *and* $W \in G(v, \mu^m/\mu^{m+1})$*. Set* $I_W := \pi^{-1}(o) \subset R$ *and* $Z_W := \mathrm{Spec}(R/I_W)$*. Note that* Z_W *is a connected degree* 0 *subscheme of* X *of degree* $\binom{n+m}{n} + \dim W$*. The integer* $\dim V(-Z_W)$ *is the number of conditions that* Z_W *imposes to* V*.*

We have $\mu^{k+1} \subseteq I_Z \subset \mu^{k+1}$. *The integer* $\dim V(-Z_W)$ *depends on the integers* $g_o(m)$ *and* $\dim W$ *and on the position of* W *with respect to the linear subspace* $\mathrm{Im}(e_{m,o})$. *Concerning the integer* $\dim V(-Z_W)$ *we only know the trivial inequalities coming from the Grassmann formula. Since the Grassmannian is an irreducible variety, it makes sense to speak about the general element of* $G(v, \mu^m/\mu^{m+1})$. *For such a general* W *we have* $\dim V(-Z_W) = \dim V(-mo) - \min\{v, g_o(m)\}$. *For a general* o *we have* $g_m(o) > 0$. *In this case any* W *of positive dimension imposes at least one condition to* $V(-mo)$. *The* m**-spread** $\mathrm{sp}_m(X,V)$ *of* (X,V) *at its general point is the minimal integer* x *such that* $0 \le x \le n$ *and there is an* x*-dimensional linear subspace* $E \subseteq \mathbb{K}[t_1,\dots,t_n]$ *such that* $\mathrm{Im}(e_m) \subseteq S^m(E)$. *Obviously* $\mathrm{sp}_m(X,V) \le \min\{n, g(m)\}$. *When* $g(m) < n$ *the pair* (X,V) *has a very particular behaviour ([5, Proposition 1]). We do not know (we lack an integrability condition) if something similar is true just assuming* $\mathrm{sp}_m(X,V) < n$.

3 Examples

In this section we take $X := \mathbb{P}^1 \times \mathbb{P}^1$. As a warming up for the next section we give 2 cases (Propositions 3.4 and 3.5) in which the congruence classes for some of the integer a, b of the line bundle $\mathcal{O}_X(a,b)$ greatly help and then a case (Proposition 3.6) which shows how to use the lucky cases to prove more general ones. We also show how to handle some zero-dimensional schemes with a very particular shape (Lemmas 3.7 and 3.8 and Proposition 3.9).

Remark 3.1. *Fix integers* $a \ge b > 0$. *Let* $W \subset X$ *be a general union of* c 2*-points. Then either* $h^0(\mathcal{I}_W(a,b)) = 0$ *or* $h^1(\mathcal{I}_W(a,b)) = 0$, *except in the case* $b = 2$, a *even and* $c = a/2+1$ *([10, 16, 17]). In the exceptional case* $h^0(\mathcal{I}_W(a,b)) = h^1(\mathcal{I}_W(a,b)) = 1$ *and* $|\mathcal{I}_W(a,2)| = \{2C\}$ *where* $C \cong \mathbb{P}^1$ *and* $\{C\} = |\mathcal{I}_W(a/2,1)|$.

The following observation simplifies many proofs and it is essential to do by computer in a cheap way some small degrees cases to be used for inductive proofs for other joins.

Remark 3.2. *Fix positive integers* a, b *and* w *and a zero-dimensional scheme* $W \subset X$ *such that* $\deg(W) = w$ *and* $h^1(\mathcal{I}_W(a,b)) > 0$. *To prove that for all integers* $c \in \mathbb{N}$ *a general union* Z *of* W *and* c 2*-points satisfies either* $h^1(\mathcal{I}_Z(a,b)) = 0$ *or* $h^0(\mathcal{I}_W(a,b)) = 0$ *it is sufficient to check the integers* $c \in \{\lfloor((a+1)(b+1)-w)/3\rfloor, \lceil((a+1)(b+1)-w)/3\rceil\}$. *Hence it is sufficient to check all* $c \in \mathbb{N}$ *such that*

$$(a+1)(b+1) - 2 \le w + 3c \le (a+1)(b+1) + 2 \qquad (3.1)$$

We can do better. Indeed, any 2*-point at a general* $p \in X$ *contains a general connected degree* 2 *zero-dimensional scheme* v. *Thus for any* $V \subseteq H^0(\mathcal{O}_X(a,b))$ *we have* $\dim V(-2p) \le \min\{0, \dim V - 2\}$. *Thus it is sufficient to check all integers* c *such that*

$$(a+1)(b+1) - 1 \le w + 3c \le (a+1)(b+1) + 1 \qquad (3.2)$$

Now assume $w+3c=(a+1)(a+1)+1$ *and that we know that* $h^1(\mathcal{I}_{W\cup E}(a,b))=0$ *for all unions of* c' *general* 2*-points with* c' *satisfying* $w+3c'\le(a+1)(b+1)$*. Let* $E\subset X$ *be a general union of* $c-1$ 2*-points. Thus* $h^0(\mathcal{I}_{W\cup E}(a,b))=2$*. Thus a general union* E' *of* E *and a general* 2*-point satisfies* $h^0(\mathcal{I}_{W\cup E'}(a,b))=0$*. Thus it is sufficient to check all integers* c *such that*

$$(a+1)(b+1)-1\le w+3c\le(a+1)(b+1) \tag{3.3}$$

Thus it is sufficient to check the integer $c:=\lfloor((a+1)(b+1)-w)/3\rfloor$*.*

Lemma 3.3. *Fix* $(a,b)\in\mathbb{N}^2$*,* $L\in|\mathcal{O}_X(1,0)|$ *and a zero-dimensional scheme* $W\subset X$ *such that* $h^1(\mathcal{I}_W(a,b))=0$*. Set* $u:=\deg(W\cap L)$*. Let* $E\subset L$ *be a zero-dimensional scheme such that* $E\cap W=\emptyset$ *and set* $x:=\deg(E)$*. Assume* $h^1(\mathcal{I}_{\mathrm{Res}_L(W)}(a-1,b))\le b+1-u$*. Then* $h^1(\mathcal{I}_{W\cup E}(a,b))=\max\{0,h^0(\mathcal{I}_W(a,b))-x\}$*.*

Proof. First assume $x=b+1-u$. Thus $h^i(L,\mathcal{I}_{L\cap(W\cup E)}(a,b))=0$, $i=0,1$. Hence $h^i(\mathcal{I}_{W\cup E}(a,b))=h^i(\mathcal{I}_{\mathrm{Res}_L(W)}(a-1,b))$, $i=0,1$.

If $x<b+1-u$, then we reduce the proof to the case just proved taking instead of E the union of E and $b+1-u-x$ points.

Now assume $x>b+1$. Instead of E we use any subscheme $E'\subset E$ such that $\deg(E')=b+1-u$. □

Proposition 3.4. *Fix positive integers* a *and* b *such that* a *is odd and* $b\equiv 4 \pmod 5$*. Then for all* $c>0$ *the join of* c *copies of* $J(1)$ *has the expected dimension* $\min\{(a+1)(b+1)-1,5c-1\}$*.*

Proof. Let $Z\subset X$ be a general union of c elements of $\mathcal{Z}(1)$. It is sufficient to do the case $c=(a+1)(b+1)/5$ and prove that $h^i(\mathcal{I}_Z(a,b))=0$, $i=0,1$. We fix $L\in|\mathcal{O}_X(1,0)|$. Let $Z'\subset X$ be a general union of $(a-1)(b+1)/5$ elements of $\mathcal{Z}(1)$ with the convention $Z'=\emptyset$ if $a=1$. Take a general $A\cup B\subset L$ such that $\#A=\#B=(b+1)/5$ and $A\cap B=\emptyset$. Let W be the union of Z' and the scheme Z'' obtained in the following way. We degenerate $(b+1)/5$ connected components of Z to elements of $Z(p,1)$, $p\in A$, with respect to a tangent vector not tangent to L and $(b+1)/5$ connected components of Z to elements of $Z(p,1)$, $p\in B$, with respect to the tangent vector of L. Remark 2.2 gives $\deg(Z''\cap L)=b+1$ and $\deg(\mathrm{Res}_L(Z)\cap L)=b+1$. Thus using twice the residual exact sequence of L we get that it is sufficient to prove that $h^i(\mathcal{I}_{Z'}(a-1,b))=0$, $i=0,1$. This is true if $a=1$ (and hence $Z'=\emptyset$), while if $a\ge 3$ we use induction on a. □

Proposition 3.5. *Fix positive integers* a *and* b *such that* $b\equiv 4 \pmod 5$ *and* $a\ge 3$*. The join of* $(b+1)/5$ *copies of* $J(2)$ *and an arbitrary number,* c*, of copies of* X *has the expected dimension.*

Proof. Let Z_1 be a general union of c 2-points. Since each element of $J(2)$ has degree 8, it is sufficient to check the positive integers c such that $8(b+1)/5+3c\le(a+1)(b+1)$ (Remark 3.2). Fix $L\in|\mathcal{O}_X(1,0)|$. For every $p\in L$, let $E(p)$ be an element of $Z(p,2)$ with as tangent vector the one associated to L. We have $\deg(E(p) \cap L) = 4$, $\deg(\mathrm{Res}_L(E(p)) \cap L) = 3$ and $\mathrm{Res}_L(\mathrm{Res}_L(E(p))) = \{p\}$. Set $E := \bigcup_{p \in A} E(p)$. By semicontinuity to prove the proposition for the integer c it is sufficient to prove that $h^j(\mathcal{I}_{E \cup Z_1}(a,b)) = 0$.

Take a general $A \cup B \subset L$ such that $\#A = \#B = (b+1)/5$ and $A \cap B = \emptyset$. Let E be the union of all $E(p)$, $p \in A$. By semicontinuity to prove the proposition for the integer c it is sufficient to prove $h^1(\mathcal{I}_{E \cup Z_1}(a,b)) = 0$. Let $(2B, L)$ denote the union of all 2-points of L with a point of B as their reduction. We apply the Differential Horace Lemma for 2-points at each $p \in B$. Since $h^i(L, \mathcal{I}_{B \cup (E \cap L)}(a,b)) = 0$, $i = 0,1$, the Differential Horace Lemma gives that to prove that $h^1(\mathcal{I}_{E \cup E'}(a,b)) = 0$ for a general union E' of $2(b+1)/5$ 2-points, it is sufficient to prove $h^1(\mathcal{I}_{\mathrm{Res}_L(E) \cup (2B,L)}(a-1,b)) = 0$. Since $\deg((\mathrm{Res}_L(E) \cup (2B,L)) = b+1$, it is sufficient to prove $h^1(\mathcal{I}_A(a-2,b)) = 0$. Since $a \geq 2$, we proved the case $c \leq 2(b+1)/5$.

Now assume $c > 2(b+1)/5$ and set $x := c - 2(b+1)/5$. Let $W \subset X$ be a general union of x 2-points. Either $h^0(\mathcal{I}_W(a-2,b)) = 0$ or $h^1(\mathcal{I}_W(a-2,b)) = 0$ or a and b are even, $x = b+1$ and $h^1(\mathcal{I}_W(a-2,b)) = h^0(\mathcal{I}_W(a-2,b)) = 1$ (Remark 3.1).

If $h^0(\mathcal{I}_W(a-2,b)) \leq 1$, then $h^0(\mathcal{I}_{E' \cup W}(a,b)) = 0$, concluding the proof in this case.

Now assume $h^1(\mathcal{I}_W(a-2,b)) = 0$ and hence $h^0(\mathcal{I}_W(a-2,b)) = (a-1)(b+1) - 3x$. To prove this case we need to prove that either $h^0(\mathcal{I}_{W \cup A}(a-2,b)) = 0$ or $h^1(\mathcal{I}_{W \cup A}(a-2,b) = 0$. Since W is general, $W \cap L = \emptyset$. Since A is general in L and $\#A = (b+1)/5$, it is sufficient to prove that $h^0(\mathcal{I}_W(a-3,b)) \leq \min\{0, (a-1)(b+1) - 3x - (b+1)/5\}$.

If $a \geq 4$, the inequality $h^0(\mathcal{I}_W(a-3,b)) \leq \min\{0, (a-1)(b+1) - 3x - (b+1)/5\}$ is true, because either $h^0(\mathcal{I}_W(a-3,b)) = 0$ or $h^1(\mathcal{I}_W(a-3,b)) = 0$ or $h^0(\mathcal{I}_W(a-3,b)) = h^1(\mathcal{I}_W(a-3,b)) = 1$ (Remark 3.1).

Now assume $a = 3$. We have $h^0(\mathcal{I}_W(1,b)) = 2b+2-3x$ and $h^0(\mathcal{I}_W(0,b)) = b+1 < h^0(\mathcal{I}_W(1,b)) - (b+1)/5$. □

Fix the bidegree a, b of the line bundle $\mathcal{O}_X(a,b)$. Instead of a prescribed number of copies of $J(2)$ we may use an arbitrary, but small with respect to b, number of copies of $J(2)$ as in the following statement (taking all possible $e \leq b-2$ with $e \equiv 4 \pmod 5$).

Proposition 3.6. *Fix positive integers a, e and b such that $e \equiv 4 \pmod 5$, $a \geq 3$ and $b \geq e+4$. The join of $(e+1)/5$ copies of $J(2)$ and an arbitrary number, c, of copies of X has the expected dimension.*

Proof. Let Z_1 be a general union of s 2-points. Since each element of $J(2)$ has degree 8, it is sufficient to check the positive integer c such that $8(e+1)/5 + 3c \leq (a+1)(b+1)$ (Remark 3.2).

Fix $L \in |\mathcal{O}_X(1,0)|$. For every $p \in L$ let $E(p)$ be an element of $Z(p,2)$ with as tangent vector the one associated to L. We have $\deg(E(p) \cap L) = 4$, $\deg(\mathrm{Res}_L(E(p)) \cap L) = 3$ and $\mathrm{Res}_L(\mathrm{Res}_L(E(p))) = \{p\}$.

Set $E_1 := \bigcup_{p\in A} E(p)$. Set $e_1 := \lfloor (b-e)/2\rfloor$, $f_1 := b-e-2e_1$, $e_2 := \lfloor (b-e-e_1-2f_1)/2\rfloor$ and $f_2 := b-e-e_1-f_1-2f_1$. Note that $0\le f_1\le 1$ and $0\le f_2\le 1$. Since $b\ge e+2$, $(e_1,f_1)\ne (0,1)$ and hence $2e_1+f_1\ge e_1+2f_1$. Since $b\ge e+4$, we have $e_1+2f_1\ge 2$. Take a general $A\cup B\cup E\cup F\subset L$ such that $\#A=\#B=(e+1)/10$, $\#E=e_1$, $\#F=f_1$ and the sets A, B, E and F are pairwise disjoint. Let U be the union of all $E(p)$, $p\in A$. By semicontinuity to prove the proposition for the integer c it is sufficient to prove $h^j(\mathcal{I}_{U\cup Z_1}(a,b))=0$.

Let $(2B,L)$ (resp. $(2F,L)$, resp. $(2E,L)$) denote the union of all 2-points of L with a point of B (resp. F) as their reduction. We apply the Differential Horace Lemma for 2-points at each $p\in B\cup F$, while add all 2-points $2p$ of X with $p\in E$. Since $h^i(L,\mathcal{I}_{B\cup(U\cap L)\cup(2E,L)\cup F}(a,b))=0$, $i=0,1$, the Differential Horace Lemma gives that to prove that $h^1(\mathcal{I}_{U\cup U'}(a,b))=0$ for a general union U' of $(e+1)/10+e_1+f_1$ 2-points, it is sufficient to prove $h^1(\mathcal{I}_{\mathrm{Res}_L(E_1)\cup(2B,L)\cup(E,L)\cup(2F,L)}(a-1,b))=0$. We have $\deg(\mathrm{Res}_L(E)\cap L)+\deg((2B,L))+\deg((2F,L))\le b+1$. Thus the intersection τ of $\mathrm{Res}_L(E)\cup(2B,L)\cup(E,L)\cup(2F,L)$ with L satisfies $h^1(L,\mathcal{I}_\tau(a-1,b))=0$, while its residue is A.

If $c\le (e+1)/5+e_1+f_1$, then we get that the join has the expected dimension $e+1+3c-1$.

Assume for the moment $c\ge (e+1)/5+e_1+f_1+e_2+f_2$. Fix a general $G\cup H\subset L$ such that $\#G=e_2$ and $\#H=f_2\le 1$. We apply the Differential Horace Lemma to H (if $f_2=1$) and specialize e_2 2-points to the 2-points $2p$, $p\in G$. Let Z_2 be a general union of $c-(e_1+1)/5-e_1-f_1-e_2-f_2$. To prove that $h^1(\mathcal{I}_{U\cup Z_1}(a,b))=0$ it is sufficient to prove that $h^j(\mathcal{I}_{Z_2\cup A\cup F\cup(2H,L)}(a-2,b))=0$. This is done as in the proof of Proposition 3.5, even if $H\ne\emptyset$, by Remark 2.3 applied to L or by Lemma 3.3.

If $(e+1)/5+e_1+f_1<c<(e+1)/5+e_1+f_1+e_2+f_2$ (and hence $c\le (e+1)/5+e_1+f_1+e_2$) instead of G and H we take G' with $\#G'=c-(e+1)/5-e_1-f_1-e_2$ and $H'=\emptyset$. □

We explain why a general union of 2 m-points (plus other objects) are easy to handle.

Lemma 3.7. *Let $Z\subset X$ be a general union of 2 m-points, $m\ge 2$. Then $h^i(\mathcal{I}_Z(m,m-1))=0$, $i=0,1$.*

Proof. Fix $L\in|\mathcal{O}_X(0,1)|$ and $o,o'\in L$, $o\ne o'$. We take mo and apply the Differential Horace Lemma with respect to L and o'. Thus on L we add $\{o\}$, at the first residual with respect to L intersected with L we add $(2o,L)$ and so on. Thus the intersection with L of the union W of mo with this virtual scheme has degree $m+1$ and the same holds for the intersection of L with the first m residual with respect to L. Thus we get the lemma taking several residual exact sequences of L. □

In the same way we get the following result.

Lemma 3.8. *Let $Z\subset X$ be a general union of one m-point, $m>0$, and one $(m+1)$-point. Then $h^i(\mathcal{I}_Z(m,m))=0$, $i=0,1$.*

Proposition 3.9. *Fix integer $m \geq 3$, $c \geq 0$, and $a \geq b \geq m+3$. Let $Z \subset X$ be a general union of 2 m-points and c 2-points. Then either $h^0(\mathcal{I}_Z(a,b)) = 0$ or $h^1(\mathcal{I}_Z(a,b)) = 0$.*

Proof. It is sufficient to check the positive integers c such that $(m+1)m + 3c \leq (a+1)(b+1)$ (Remark 3.2). Fix $L \in |\mathcal{O}_X(0,1)|$ and $o, o' \in L$, $o \neq o'$. We take mo and apply the Differential Horace Lemma with respect to L and o'. Thus on L we add $\{o\}$, at the first residual with respect to L intersected with L we add $(2o, L)$ and so on. Thus the intersection with L of the union W of mo with this virtual scheme has degree $m+1$ and the same holds for the intersection of L with the first m residual with respect to L. We call W this virtual degeneration of a general union of 2 m-points. Recall (Lemma 3.7) that $h^i(\mathcal{I}_W(m, m-1)) = 0$, $i = 0, 1$. We set $W_0 := W$ and for each $i \geq 1$ define recursively the virtual scheme W_i by the formula $W_i := \mathrm{Res}_L(W_{i-1})$. Thus $W_j = \emptyset$ for all $j \geq m$ and $\deg(W_i \cap L) = m+1$ for all $i < m$. The proof of Lemma 3.7 gives $h^i(\mathcal{I}_{W_j}(m, m-1-j)) = 0$, $0 \leq j \leq m$.

Set $e := \lfloor (a-m)/2 \rfloor$ and $f := a - m - 2e$. Fix a general $A \cup B \subset L$ such that $\#A = e$, $\#B = f$ and $A \cap B = \emptyset$.

We call H_i, $0 \leq i \leq m$, the assertion that a general union of W_i and an arbitrary number of 2-points has the expected postulation with respect to $\mathcal{O}_X(a, b-i)$. The case $i = m$ is true by Remark 3.1. Since H_0 proves the proposition for c, we prove all H_i by descending induction on i, Thus (changing b and m) we may assume H_1.

Assume for the moment $c \geq e + f$. Let $E \subset X$ be a general union of $c - e - f$ 2-points. We take as e of the 2-points the 2-points $2p$, $p \in A$. If $f \neq 0$ we apply the Differential Horace Lemma to F. Since $2e + f + \deg(W \cap L) = a + 1$, the Differential Horace Lemma shows that to show that to prove the proposition it is sufficient to prove that $h^j(\mathcal{I}_{W_1 \cup E \cup A \cup (2F, L)}(a, b-1)) = 0$.

Claim 1: $h^1(\mathcal{I}_{W_1 \cup E}(a, b-1)) = 0$.

Proof of Claim 1: By the inductive assumption either $h^1(\mathcal{I}_{W_1 \cup E}(a, b-1)) = 0$ or $h^0(\mathcal{I}_{W_1 \cup E}(a, b-1)) = 0$. Since $\deg(W_1 \cup E) - e - 2f = \deg(W) + 3c \leq (a+1)(b+1)$, $h^1(\mathcal{I}_{W_1 \cup E}(a, b-1)) = 0$.

Claim 1 gives $h^0(\mathcal{I}_{W_1 \cup E}(a, b-1)) = (a+1)b - m(m+1) - 3(c - e - f)$. Claim 1 and Remark 2.3 applied to L or Lemma 3.3 show that to prove H_0 it is sufficient to prove that $h^0(\mathcal{I}_{W_1 \cup E}(a, b-2)) \leq \max\{0, (a+1)b - m(m+1) - 3(c-e-f) - e - 2f\}$. Since $2e + f = a - m$ and $f \leq 1$, $3e + 3f + 2 \leq 2(a+1)$. Thus $h^0(\mathcal{I}_{W_2 \cup E}(a, b-2)) = 0$.

In the case $c < e + f$ (and hence $c \leq e$) instead of A and B we take $B = \emptyset$ and A with $\#A = c$. □

4 The proofs

In this section we take $X = \mathbb{P}^1 \times \mathbb{P}^1$. Since $\dim X = 2$, for each $m \geq 0$, any $p \in X$ and any $Z \in \mathcal{Z}(m)$ we have $\deg((m+1)p) = \binom{m+2}{2}$ and $\deg(Z) = 2 + \binom{m+2}{2}$.

Remark 4.1. *Fix integers $a \geq b \geq 0$ and $z \geq 2$. Fix homogeneous coordinates x_0, x_1 and y_0, y_1 of $\mathbb{P}^1$. The vector space $H^0(\mathcal{O}_X(a,b))$ is formed by all $f \in \mathbb{K}[x_0, x_1, y_0, y_1]$ which are bihomogenous of bidegree (a,b), i.e. homogenous of degree a with respect to x_0, x_1 and homogeneous of degree b with respect to y_0, y_1. Thus $H^0(\mathcal{O}_X(a,b))$ has as a basis all monomials $x_0^{\alpha_0} x_1^{\alpha_1} y_0^{\beta_0} y_1^{\beta_1}$ such that $(\alpha_0, \alpha_1, \beta_0, \beta_1) \in \mathbb{N}^4$, $\alpha_0 + \alpha_1 = a$ and $\beta_0 + \beta_1 = b$. Fix $p \in X$ and choose bihomogeneous coordinates x_0, x_1, y_0, y_1 such that $p = ((1:0),(1:0))$. Set $x := x_1/x_0$ and $y := y_1/y_0$. The vector space $H^0(\mathcal{I}_{zp}(a,b))$ is isomorphic to the subspace of the polynomial ring $\mathbb{K}[x,y]$ with as a bases all monomials $x^u y^v$ with $u + v \geq z$, $0 \leq u \leq a$ and $0 \leq v \leq b$. Since $\deg(\mathcal{O}_{zp}) = \binom{z+1}{2}$ and $a \geq b$, $h^1(\mathcal{I}_{zp}(a,b)) = 0$ if and only if $b \geq z-1$. If $a > b = z-2$, then $h^1(\mathcal{I}_{zp}(a, z-2)) = 1$.*

Proposition 4.2. *Fix positive integers $a \geq b$ and m.*

(1) *If $b < m$, then $h^1(\mathcal{I}_Z(a,b)) > 0$ for all $Z \in \mathcal{Z}(m)$.*

(2) *If $b > m$, then $h^1(\mathcal{I}_Z(a,b)) = 0$ for all $Z \in \mathcal{Z}(m)$.*

(3) *If $a > m$, then $h^1(\mathcal{I}_Z(a,m)) = 0$ for a general $Z \in \mathcal{Z}(m)$.*

(4) *There is $Z \in \mathcal{Z}(m)$ such that $h^1(\mathcal{I}_Z(a,m)) > 0$.*

Proof. Fix $p \in X$. We consider $Z \in \mathcal{Z}(p,m)$. Thus $(m+1)p \subset Z \subset (m+2)p$. Parts (1) and (2) follow from Remark 4.1.

Let L denote the only element of $|\mathcal{O}_X(1,0)|$ passing through p.

(a) Now we prove part (4). Use L as L_Z to define the scheme Z. Note that $L \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_L(a,m)) = m$. Thus $h^0(L, \mathcal{O}_L(a,m)) = m+1$. By part (b) of Remark 2.2 we have $\deg(Z \cap L) = m+2$ and hence $h^1(L, \mathcal{I}_{Z\cap L, L}(a,m)) = 1$. Thus $h^1(\mathcal{I}_Z(a,m)) > 0$.

(b) Now we prove part (3). Thus $a > m$. By the semicontinuity theorem for cohomology it is sufficient to find one $Z \in \mathcal{Z}(p,m)$ such that $h^1(\mathcal{I}_Z(a,m)) = 0$. Take $Z \in \mathcal{Z}(p,m)$ whose tangent vector is not contained in L. We have $\deg(Z \cap L) = m+1$ and $\mathrm{Res}_L(Z) \in \mathcal{Z}(m-1)$ (even if $m = 1$) by Remark 2.2. Use the residual exact sequence of L and that $h^1(\mathcal{I}_{\mathrm{Res}_L(Z)}(a-1, m)) = 0$, because $a - 1 \geq m$. □

Proof of Proposition 1.2: The case $m = 2$, *i.e.* the case of $c+1$ general 2-points is described in Remark 3.1. Assume $m > 2$ and that the proposition is true for smaller multiplicities. In step (b) we check the case $m = 3$ and see that the exceptional case $b = 2$ and a even of the case $m = 2$ gives no problem for the inductive proof.

(a) Fix $L \in |\mathcal{O}_X(0,1)|$, $p \in L$, and take mp. For any $x \in \mathbb{Z}$ we have $h^0(\mathcal{O}_L(a,x)) = a+1$. Note that $\deg(mp \cap L) = m$ and $\mathrm{Res}_L(mp) = (m-1)p$. Since $\deg(mp) = (m+1)m/2$, it is sufficient we may assume $3c \le (a+1)(b+1) - (m+1)m/2$. Thus we need to prove that $h^1(\mathcal{I}_{mp\cup G}(a,b)) = 0$ for a general union of c 2-points. Set $e := \lfloor (a+1-m)/2 \rfloor$ and $f := a+1-m-2e$. Thus $0 \le f \le 1$. Assume for the moment $c \ge e+f$. Let $E \subset X$ be a general union of $c-e-f$ 2-points. Take a general $A \cup B \subset L$ such that $\#A = e$, $\#B = f$ and $A \cap B = \emptyset$. We degenerate e 2-points to the 2-points $2q$, $q \in A$ and, if $f = 1$, apply the Differential Horace Lemma to F. Since $m + 2e + f = a+1$, $\mathrm{Res}_L(mp) = (m-1)p$ and $\mathrm{Res}_L(q) = \{q\}$ for all $q \in A$, the Differential Horace Lemma shows that to prove $h^1(\mathcal{I}_{mp\cup G}(a,b)) = 0$ it is sufficient to prove $h^1(\mathcal{I}_{(m-1)p\cup E\cup A\cup(2B,L)}(a,b-1)) = 0$.

Claim 1: $h^1(\mathcal{I}_{(m-1)p\cup E}(a,b-1)) = 0$.

Proof of Claim 1: Since $b-1-(m-1) = b-m$, we may use the inductive assumption. We have $\deg(mp) + \deg(G) - \deg((m-1)p) - 2e + f = h^0(\mathcal{O}_X(a,b)) - h^0(\mathcal{O}_X(a,b-1))$. Thus to prove Claim 1 it is sufficient to observe that $e + 2f \le 2e + f$, which is true because $a+1-m \ge 2$ and $0 \le f \le 1$.

Claim 1 implies $h^0(\mathcal{I}_{(m-1)p\cup E}(a,b-1)) = (a+1)b - m(m-1)/2 - 3(c-e-f)$. Note that $\mathrm{Res}_L((m-1)p) = (m-2)p$ and that $\deg(A \cup (2B,L)) = e+2f$. By Lemma 2.3 to prove that $h^1(\mathcal{I}_{(m-1)p\cup E\cup A\cup(2B,L)}(a,b-1)) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_{(m-2)p\cup E}(a,b-2)) \le \max\{0, (a+1)b - m(m-1)/2 - 3(c-e-f) - e - 2f\}$. Recall that $(m+1)m/2 + 3c \le (a+1)(b+1)$. We have $\deg((m-1)p) + \deg(E) - \deg((m-2)p) - \deg(E) = m$. We have $a+1-(m-1)-e-2f \ge 0$, because $e > 0$, $f \le 1$ and hence $2e+f \ge e+2f$. Since $m-2-b+2 = m-2$, we may use the inductive assumption (or Remark 3.1 for $m = 4$ or that $h^0(\mathcal{I}_{p\cup E}(a,b-2)) = \max\{0, h^0(\mathcal{I}_E(a,b-2)) - 1\}$ for a general $p \in X$ if $m = 3$).

If $c < e+f$ (and hence $c \le e$) the proof works taking $B = \emptyset$ and $\#A = c$.

(b) Assume $m = 3$. Take L, p, A, B and E as in step (a). We first check Claim 1. Taking a general L and a general $p \in L$ and then taking a general E we see that $2p \cup E$ is a general union of $c-e-f+1$ 2-points of X. We have $a > b > 0$ and $\deg(2p \cup E) = 3c+3-3e-3f$ with $6 + 3c \le (a+1)(b+1)$ and $2e+f = a-2$. To conclude the proof of Claim 1 it is sufficient to use Remark 3.1 and that $e+2f \ge 2$. By Claim 1 to conclude it is sufficient to check that $h^0(\mathcal{I}_{p\cup E}(a,b-2)) \le \max\{0, (a+1)b - 3 - 3(c-e-f) - e - 2f\}$. The generality of p gives $h^0(\mathcal{I}_{p\cup E}(a,b-2)) = \max\{0, h^0(\mathcal{I}_E(a,b-2)) - 1\}$. Since $m = 3$, $b-2 > 0$. Remark 3.1 gives that either $h^0(\mathcal{I}_E(a,b-2)) \le 1$ or $h^0(\mathcal{I}_E(a,b-2)) = (a+1)(b-1) - 3(c-e-f)$. □

Proof of Theorem 1.1: Since the case $m = 0$ is true by Remark 3.1, we may use induction on m even if $m = 1$. Fix $L \in |\mathcal{O}_X(0,1)|$ and $o \in L$. Take $U \in \mathcal{Z}(o,m)$ with L not as its tangent vector. By (3.3) it is sufficient to take a positive integer c such that $\binom{m+1}{2} + 2 + 3c \le (a+1)(b+1)$ and prove that $h^1(\mathcal{I}_{U\cup W}(a,b)) = 0$ for a general union W of c 2-points. By part (a) of Remark 2.2 $W := \mathrm{Res}_L(U) \in \mathcal{Z}(o, m-1)$, L is not the tangent vector of W and $\deg(W \cap L) = m+1$. Set $e := \lfloor (a-m)/2 \rfloor$ and $f = a - m - 2e$. We have $0 \le f \le 1$. Since $a \ge m+2$, $e > 0$ and hence $2e + f \ge e + 2f$.

Claim 1: $h^1(\mathcal{I}_{G\cup E}(a, b-1)) = 0$.

Proof of Claim 1: We have $b - m = (b-1) - (m-1)$ and hence it is sufficient to use the inductive assumption. We have $\deg(U \cup W) - \deg(G \cup E) = m+1-3(e+f)$ and $h^0(\mathcal{O}_X(a,b)) - h^0(\mathcal{O}_X(a,b-1)) = a+1$. Since $a+1 = m+1+2e+f$, Claim 1 follows from the inductive assumption.

Claim 1 implies $h^0(\mathcal{I}_{G\cup E}(a,b-1)) = (a+1)b - \binom{m}{2} - 2 - 3(c-e-f)$. We have $G' = \mathrm{Res}_L(G) \in \mathcal{Z}(o, m-2)$ if $m \ge 2$ and $G' = \{o\}$ if $m = 1$. We use Lemma 2.4 applied to the image of the restriction map $H^0(\mathcal{I}_{G\cup E\cup A\cup(2B,L)}(a,b-1)) \longrightarrow H^0(\mathcal{O}_L(a,b-1))$. To conclude the proof for m, c, a and b using it is sufficient to prove that $h^0(\mathcal{I}_{G'\cup E}(a,b-2)) \le \max\{0, (a+1)b - \binom{m}{2} - 2 - 3(c-e-f) - e - 2f\}$ and that $\deg(G' \cup E) \le (a+1)(b-1)$. The first inequality follows from the inductive assumption, while the second one follows from the following facts: $\deg(U) + 3c \le (a+1)(b+1)$, $\deg(U) - \deg(G') = 2m+1$, $3c - \deg(E) = 3e + 3f$, $m + 1 + 2e + f = a + 1$ and $f \le 1$.

If $c < e + f$ (and hence $c \le e$) the proof works taking $B = \emptyset$ and $\#A = c$. □

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The author has no competing interest.

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7 Data availability

All proofs are contained in the body of the paper.

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