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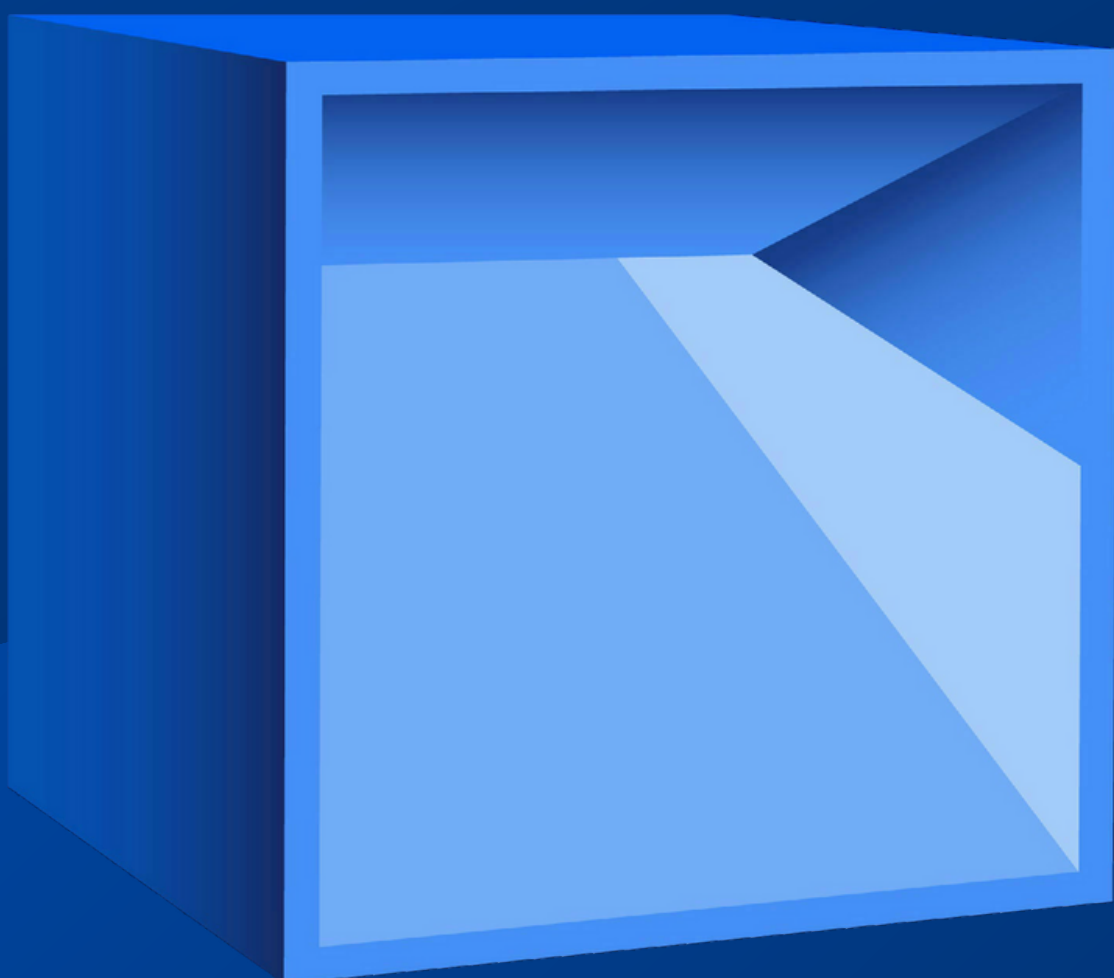

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SUMMARY

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Estimation of sharp geometric inequality in D_α-homothetically deformed Kenmotsu manifolds

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ABSTRACT

In this article, we investigate the Kenmotsu manifold when applied to a D_α-homothetic deformation. Then, given a submanifold in a D_α-homothetically deformed Kenmotsu manifold, we derive the generalized Wintgen inequality. Additionally, we find this inequality for submanifolds such as slant, invariant, and anti-invariant in the same ambient space.

RESUMEN

En este artículo estudiamos la variedad de Kenmotsu cuando se aplica a una deformación D_α-homotética. Luego, dada una subvariedad en una variedad de Kenmotsu D_α-homotéticamente deformada, derivamos la desigualdad de Wintgen generalizada. Adicionalmente, encontramos esta desigualdad para subvariedades tales como oblicuas, invariantes y anti-invariantes en el mismo espacio ambiente.

Keywords and Phrases: Normalized scalar curvature, scalar curvature, mean curvature, D_α-homothetic deformation.

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1 Introduction

The Wintgen inequality is a sharp geometric inequality for surfaces in a 4-dimensional Euclidean space $\mathbb{E}^4$ involving Gauss curvature K (intrinsic invariants), normal curvature and square mean curvature (extrinsic invariants). The intrinsic and extrinsic curvature of a surface can be combined in the second fundamental form. This is a *quadratic form* in the tangent plane to the surface at a point.

Quadratic forms occupy a central place in various branches of mathematics, including number theory, linear algebra, group theory (orthogonal groups), differential geometry (the Riemannian metric, the second fundamental form), differential topology (intersection forms of four-manifolds), Lie theory (the Killing form), and statistics (where the exponent of a zero-mean multivariate normal distribution has the quadratic form $\mathbf{x}^{\mathsf{T}}\mathbf{\Sigma}^{-1}\mathbf{x}$).

P. Wintgen [25], proved that the Gauss curvature K, the normal curvature $K^{\perp}$ and the squared mean curvature $||H||^2$ for any surface $\mathcal{M}^2$ in E^4 satisfy the following inequality [1]:

$$||\mathcal{H}||^2 \geq K + |K^{\perp}|$$

and the equality holds if and only if the ellipse of curvature of $\tilde{\mathcal{M}}^2$ in $\mathbb{E}^4$ is a circle. Later, it was extended by I. V. Guadalupe *et al.* [12] for arbitrary codimension m in real space forms $\tilde{\mathcal{M}}^{(m+2)}(c)$ as

$$||\mathcal{H}||^2 + c \geq K + |K^{\perp}|.$$

In 1999, De Smet *et al.* conjectured the generalized Wintgen inequality for submanifolds in real space form. The conjecture is known as DDVV conjecture. It has been proved by Zhiqin Lu in [16] and Jianquan Ge-Zizhou Tang in [11], independently and differently. Ion Mihai [17, 18] established such inequality for Lagrangian submanifold in complex space form and for Legendrian submanifolds in Sasakian space forms. Since then numerous authors studied such inequality for several kinds of submanifolds in different ambient space forms (for example, see [3, 12, 19–22]).

In 1971, Kenmotsu investigated a class of contact Riemannian manifolds, named Kenmostu manifolds, which satisfy some special conditions [15]. After that Kenmotsu manifolds have been discussed by Jun *et al.* [14] and many authors.

In 1968 Tanno [24] introduced the notion of D-homothetic deformation (for more details see [23]). In [8] Carriazo and Martin-Molina studied D-homothetic deformation of generalized (k, μ)-space forms. De and Ghosh studied D-homothetic deformation of almost contact metric manifolds [10].

In the present article, we obtain the generalized Wintgen inequalities for submanifolds of a D_α-

homothetically deformed Kenmotsu manifold. We also discuss such inequality for various slant submanifolds as an application of the inequality obtained.

2 Preliminaries

An odd dimensional $(2n+1)$ smooth manifold $(\tilde{\mathcal{M}}, g)$ is said to be an almost contact metric manifold [5], if it admits a $(1,1)$-tensor field φ, a structure vector field ζ, a 1-form η and a Riemannian metric g such that [26]

$$\varphi^2 E = -E + \eta(E)\zeta, \tag{2.1}$$

$$\eta(\zeta) = 1, \quad \varphi(\zeta) = 0, \quad \eta \circ \varphi = 0, \tag{2.2}$$

$$\eta(E) = g(E, \zeta), \tag{2.3}$$

$$g(\varphi E, \varphi F) = g(E, F) - \eta(E)\eta(F), \tag{2.4}$$

for any vector fields E, F on $\tilde{\mathcal{M}}$.

If a contact metric manifold satisfies

$$(\tilde{\nabla}_E \varphi)F = -g(E, \varphi F)\zeta - \eta(F)\varphi E, \tag{2.5}$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to g, then $\tilde{\mathcal{M}}$ is called a *Kenmotsu manifold* [15].

An almost contact metric manifold is Kenmotsu manifold if and only if

$$\tilde{\nabla}_E \zeta = E - \eta(E)\zeta. \tag{2.6}$$

Moreover, we suppose that the Riemannian curvature tensor $\tilde{R}$, the Ricci tensor $\tilde{S}$ of type $(0,2)$ in Kenmotsu manifold $\tilde{\mathcal{M}}$ with respect to $\tilde{\nabla}$ satisfy [15]

$$(\tilde{\nabla}_E \eta)F = g(\varphi E, \varphi F) = g(E, F) - \eta(E)\eta(F), \tag{2.7}$$

$$(\tilde{\nabla}_\zeta \eta)F = 0, \tag{2.8}$$

$$\tilde{R}(E, F)\zeta = \eta(E)F - \eta(F)E, \tag{2.9}$$

$$\tilde{R}(\zeta, E)F = \eta(F)E - g(E, F)\zeta, \tag{2.10}$$

$$\tilde{R}(\zeta, E)\zeta = -\tilde{R}(E, \zeta)\zeta = E - \eta(E)\zeta, \tag{2.11}$$

$$\eta(\tilde{R}(E,F)G) = g(E,G)\eta(F) - g(F,G)\eta(E), \tag{2.12}$$

$$\tilde{S}(E,\zeta) = -2n\eta(E), \tag{2.13}$$

$$\tilde{S}(\varphi E,\varphi F) = \tilde{S}(E,F) + 2n\eta(E)\eta(F). \tag{2.14}$$

An odd dimensional Kenmotsu manifold $\tilde{\mathcal{M}}(\varphi,\zeta,\eta,g)$ is said to be η-Einstein manifold if $\tilde{S}$ is of the form

$$\tilde{S} = ag + b\eta\otimes\eta,$$

where a and b are smooth functions on $\tilde{\mathcal{M}}$.

Definition 2.1 ([24])**.** *If an $(2n+1)$-dimensional contact metric manifold $\tilde{\mathcal{M}}$ with almost contact metric structure (φ,ζ,η,g) is transformed into $(\varphi^\sharp,\zeta^\sharp,\eta^\sharp,g^\sharp)$, where*

$$\varphi^\sharp = \varphi, \quad \zeta^\sharp = \frac{1}{\alpha}\zeta, \quad \eta^\sharp = \alpha\eta, \quad g^\sharp = \alpha g + \alpha(\alpha-1)\eta\otimes\eta \tag{2.15}$$

and α is a positive constant, then the transformation is called a D_α-homothetic deformation.

The relation between the Levi-Civita connection $\tilde{\nabla}$ of g and the Levi-Civita connection $\nabla^\sharp$ of $g^\sharp$ is given by [2]

$$\nabla^\sharp_E F = \tilde{\nabla}_E F + \frac{\alpha-1}{\alpha} g(\varphi E,\varphi F)\zeta \tag{2.16}$$

for all vector fields E, F on $\tilde{\mathcal{M}}$.

3 Curvature tensor on Kenmotsu manifold under a D_α-homothetic deformation

Let $\tilde{\mathcal{M}}$ be Kenmotsu manifold of dimension $(2n+1)$. The curvature tensor $\mathcal{R}^\sharp$ of $\tilde{\mathcal{M}}$ under a D_α-homothetic deformation $\nabla^\sharp$ is defined by [13]

$$\mathcal{R}^\sharp(E,F)G = \nabla^\sharp{}_E\nabla^\sharp{}_F G - \nabla^\sharp{}_F\nabla^\sharp{}_E G - \nabla^\sharp{}_{[E,F]}G. \tag{3.1}$$

In the work by Blaga [4], the curvature tensors of forms $(1,3)$ and $(0,4)$, along with the Ricci curvature tensor and scalar curvature are presented.

$$\mathcal{R}^\sharp(E,F,G,H) = \tilde{R}(E,F,G,H) + \frac{\alpha-1}{\alpha}[g(\varphi F,\varphi G)g(E,H) - g(\varphi E,\varphi G)g(F,H)], \tag{3.2}$$

$$\mathcal{R}^{\sharp}(E,F,G,H) = \alpha\tilde{R}(E,F,G,H) + (\alpha-1)\Big\{\eta(G)[\eta(E)g(F,H) - \eta(F)g(E,H)] - g(E,G)[g(F,H) - \eta(F)\eta(H)] + g(F,G)[g(E,H) - \eta(E)\eta(H)]\Big\}, \tag{3.3}$$

$$S^{\sharp}(E,F) = \tilde{S}(E,F) + 2n\left(\frac{\alpha-1}{\alpha}\right)g(\varphi E, \varphi F), \tag{3.4}$$

where $S^{\sharp}$ and $\tilde{S}$ indicate Ricci curvature tensors with respect to $\nabla^{\sharp}$ and $\tilde{\nabla}$. Also, the scalar curvatures $\tau^{\sharp}$ and $\tilde{\tau}$ with respect to $\nabla^{\sharp}$ and $\tilde{\nabla}$ are related by

$$\tau^{\sharp} = \frac{1}{\alpha}\tilde{\tau} + \frac{2n(2n+1)(\alpha-1)}{\alpha^2}. \tag{3.5}$$

Thus, we have the following result:

Proposition 3.1. *In an η-Einstein Kenmotsu manifold of dimension $(2n+1)$, the Ricci tensor is given by*

$$\tilde{S}(E,F) = \left[\frac{\tilde{\tau}+2n}{2n}\right]g(E,F) + \left[-(2n+1) - \frac{\tilde{\tau}}{2n}\right]\eta(E)\eta(F),$$

for any vector fields E, F on $\tilde{\mathcal{M}}$. Here $\tilde{Q}$ is the Ricci operator defined by $\tilde{S}(E,F) = g(\tilde{Q}E, F)$.

By equation (3.4) and Proposition 3.1, we have

Theorem 3.2. *Let $\tilde{\mathcal{M}}(\varphi,\zeta,\eta,g)$ be a $(2n+1)$-dimensional η-Einstein Kenmotsu manifold. Then the manifold $\tilde{\mathcal{M}}(\varphi^{\sharp},\zeta^{\sharp},\eta^{\sharp},g^{\sharp})$ is again an η-Einstein manifold under a D_α-homothetic deformation with*

$$S^{\sharp}(E,F) = \left[\frac{\tilde{\tau}+2n}{2n} + 2n\frac{\alpha-1}{\alpha}\right]g(E,F) + \left[2n\frac{1-\alpha}{\alpha} - (2n+1) - \frac{\tilde{\tau}}{2n}\right]\eta(E)\eta(F),$$

for all $E, F \in \Gamma(\tilde{\mathcal{M}})$.

4 Wintgen inequality for submanifolds in Kenmotsu manifold under D_α-homothetic deformation

The present section deals with the derivation of generalized Wintgen inequalities for submanifolds in D_α-homothetically deformed Kenmotsu manifold.

Let $\mathcal{M}$ be m-dimensional submanifold of $(2n+1)$-dimensional D_α-homothetically deformed Kenmotsu manifolds $\tilde{\mathcal{M}}$. Let ∇ and $\nabla^{\perp}$ represent the induced connections on the tangent bundle $T\mathcal{M}$ and $T^{\perp}\mathcal{M}$ of $\mathcal{M}$, respectively and denote by h the second fundamental form of $\mathcal{M}$ for all

$E, F \in \Gamma(T\mathcal{M})$ and $N \in \Gamma(T^{\perp}\mathcal{M})$, recall the Gauss and Weingarten formulas by

$$\tilde{\nabla}_E F = \nabla_E F + h(E, F),$$

and

$$\tilde{\nabla}_E N = -A_N E + \nabla^{\perp}_E N,$$

where A_N is used for notation of the shape operator of $\mathcal{M}$ with respect to N. The following equation is well known

$$g(A_N E, F) = g(h(E, F), N), \quad \text{for all} \;\; E, F \in \Gamma(T\mathcal{M}), \quad N \in \Gamma(T^{\perp}\mathcal{M}).$$

Let $\mathcal{R}$ is the Riemannian curvature tensor of $\mathcal{M}$. Then we recall the equation of Gauss given by

$$\tilde{R}(E, F, G, H) = \mathcal{R}(E, F, G, H) - g(h(E, H), h(F, G)) + g(h(E, G), h(F, H)), \tag{4.1}$$

for all $E, F, G, H \in \Gamma(T\mathcal{M})$.

On combining (3.2) and (4.1), we arrive at

$$\begin{aligned}\mathcal{R}^{\sharp}(E, F, G, H) = {} & \mathcal{R}(E, F, G, H) - g(h(E, H), h(F, G)) + g(h(E, G), h(F, H)) \\ & + \frac{\alpha - 1}{\alpha}[g(\varphi F, \varphi G) g(E, H) - g(\varphi E, \varphi G) g(F, H)],\end{aligned}$$

which gives

$$\begin{aligned}\mathcal{R}(E, F, G, H) = {} & \mathcal{R}^{\sharp}(E, F, G, H) + g(h(E, H), h(F, G)) - g(h(E, G), h(F, H)) \\ & - \left(\frac{\alpha - 1}{\alpha}\right)[g(\varphi F, \varphi G) g(E, H) - g(\varphi E, \varphi G) g(F, H)].\end{aligned} \tag{4.2}$$

Assume that $\{e_1, \ldots, e_m\}$ and $\{e_{m+1}, \ldots, e_{2n+1}\}$ represent local orthonormal tangent frame of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ and a local orthonormal normal frame of the normal bundle $T^{\perp}\mathcal{M}$ of $\mathcal{M}$ in $\tilde{\mathcal{M}}$. Define the mean curvature vector $\mathcal{H}$ of $\mathcal{M}$ by

$$\mathcal{H} = \sum_{i=1}^{m} \frac{1}{m} h(e_i, e_i) \tag{4.3}$$

and squared norm of second fundamental form by

$$||h||^2 = \sum_{i,j=1}^{m} g\big(h(e_i, e_j), h(e_i, e_j)\big)^2. \tag{4.4}$$

Here we note that a submanifold $\mathcal{M}$ in $\tilde{\mathcal{M}}$ is called *minimal* if $\mathcal{H} = 0$.

We write the scalar curvature τ of $\mathcal{M}$ at $p \in \mathcal{M}$ as

$$\tau = \sum_{1\leq i<j\leq m} \mathcal{R}(e_i, e_j, e_j, e_i) \tag{4.5}$$

and define the normalized scalar curvature ρ of $\mathcal{M}$ by

$$\rho = \frac{2\tau}{m(m-1)} = \frac{2}{m(m-1)} \sum_{1\leq i<j\leq m} \mathcal{K}(e_i \wedge e_j), \tag{4.6}$$

where $\mathcal{K}$ is the sectional curvature function on $\mathcal{M}$.

The scalar normal curvature $\mathcal{K}_{nor}$ in terms of the components of the second fundamental form by the following expression [17]:

$$\mathcal{K}_{nor} = \sum_{1\leq r<s\leq 2n-m+1} \sum_{1\leq i<j\leq m} \left(\sum_{k=1}^{m} h^r_{jk} h^s_{ik} - h^r_{ik} h^s_{jk} \right)^2. \tag{4.7}$$

We also have the following relation for the normalized scalar normal curvature [17]

$$\rho_{nor} = \frac{2}{m(m-1)} \sqrt{\mathcal{K}_{nor}}. \tag{4.8}$$

Now, we prove the generalized Wintgen inequality for submanifolds of D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$.

Theorem 4.1. *Let $\mathcal{M}$ be an m-dimensional submanifold of a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$. Then*

$$\rho - \rho^\sharp + \rho_{nor} \leq ||\mathcal{H}||^2 - \left(\frac{m-1}{m}\right)\left(\frac{\alpha-1}{\alpha}\right), \tag{4.9}$$

where $\rho^\sharp$ denotes the normalized scalar curvature with respect to $\nabla^\sharp$.

Moreover, the equality case holds uniformly in (4.9) if and only if the shape operators A_r, $r = \{1, \ldots, 2n-m+1\}$ take the following forms with the suitable orthonormal frames:

$$A = \begin{pmatrix} \mu_1 & \mu & 0 & \ldots & 0 \\ \mu & \mu_1 & 0 & \ldots & 0 \\ 0 & 0 & \mu_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mu_1 \end{pmatrix}, \tag{4.10}$$

$$A_2 = \begin{pmatrix} \mu_2+\mu & 0 & 0 & \dots & 0 \\ 0 & \mu_2-\mu & 0 & \dots & 0 \\ 0 & 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu_2 \end{pmatrix}, \tag{4.11}$$

$$A_3 = \begin{pmatrix} \mu_3 & 0 & 0 & \dots & 0 \\ 0 & \mu_3 & 0 & \dots & 0 \\ 0 & 0 & \mu_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu_3 \end{pmatrix}, \quad A_4 = \cdots = A_{2n-m+1} = 0, \tag{4.12}$$

for some suitable orthonormal basis $\{e_1, \dots, e_m\}$ *of* $T_p\mathcal{M}$ *and* $\{E_1, \dots, E_{2n-m+1}\}$ *of* $T_p^{\perp}\mathcal{M}$*. Here* μ_1, μ_2, μ_3*, and* μ *are real numbers.*

Proof. Assume that $\{e_1, \dots, e_m\}$ and $\{e_{m+1}, \dots, e_{2n+1} = \zeta\}$ denote the local orthonormal tangent frame and local orthonormal normal frame on $\mathcal{M}$ respectively. Then, in view of (4.2), we have

$$\begin{aligned} \tau &= \sum_{1\le i<j\le m} \mathcal{R}(e_i, e_j, e_j, e_i) \\ &= \sum_{1\le i<j\le m} \Big\{ R^{\sharp}(e_i, e_j, e_j, e_i) + g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_j, e_i)) \\ &- \left(\frac{\alpha-1}{\alpha}\right)\Big[g(\varphi e_j, \varphi e_j) g(e_i, e_i) - g(\varphi e_i, \varphi e_j) g(e_j, e_i)\Big]\Big\} \\ &= \tau^{\sharp} - (m-1)^2\left(\frac{\alpha-1}{2\alpha}\right) + \sum_{r=1}^{2n-m+1} \sum_{1\le i<j\le m} \Big[h^r_{ii} h^r_{jj} - (h^r_{ij})^2 \Big]. \end{aligned} \tag{4.13}$$

On the other hand, we note that

$$\begin{aligned} m^2 ||\mathcal{H}||^2 &= \sum_{r=1}^{2n-m+1} \Big(\sum_{i=1}^{m} h^r_{ii} \Big)^2 \\ &= \frac{1}{m-1} \sum_{r=1}^{2n-m+1} \sum_{1\le i<j\le m} \Big(h^r_{ii} - h^r_{jj} \Big)^2 + \frac{2m}{m-1} \sum_{r=1}^{2n-m+1} \sum_{1\le i<j\le m} h^r_{ii} h^r_{jj}. \end{aligned} \tag{4.14}$$

But from [16], it is known

$$\begin{aligned}&\sum_{r=1}^{2n-m+1}\sum_{1\le i<j\le m}(h_{ii}^r-h_{jj}^r)^2+2m\sum_{r=1}^{2n-m+1}\sum_{1\le i<j\le m}(h_{ij}^r)^2\\ \ge\ & 2m\left[\sum_{1\le r<s\le 2n-m}\sum_{1\le i<j\le m}\left(\sum_{k=1}^{m}\left(h_{jk}^r h_{ik}^s-h_{ik}^r h_{jk}^s\right)\right)^2\right]^{\frac{1}{2}}. \qquad (4.15)\end{aligned}$$

On combining (4.14), (4.15) and (4.7), we have

$$\begin{aligned}m^2||\mathcal{H}||^2-m^2\rho_{nor} &\ge \frac{2m}{m-1}\sum_{r=1}^{2n-m+1}\sum_{1\le i<j\le m}\Big(h_{ii}^r h_{jj}^r-(h_{ij}^r)^2\Big)\\ &= \frac{2m}{m-1}\left[\tau-\tau^\sharp+(m-1)^2\left(\frac{\alpha-1}{2\alpha}\right)\right]. \qquad (4.16)\end{aligned}$$

Hence, by substituting (4.8), (4.13) into (4.16), we arrive

$$||\mathcal{H}||^2-\rho_{nor}\ge\rho-\rho^\sharp+\left(\frac{m-1}{m}\right)\left(\frac{\alpha-1}{\alpha}\right),$$

whereby proving the inequality (4.9). □

An immediate consequence of the Theorem 4.1 yields the following:

Corollary 4.2. *Let $\mathcal{M}$ be a minimal m-dimensional submanifold in a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$. Then*

$$\rho-\rho^\sharp+\rho_{nor}+\left(\frac{m-1}{m}\right)\left(\frac{\alpha-1}{\alpha}\right)\le 0.$$

5 Wintgen inequality for θ-slant submanifolds in Kenmotsu manifold under D_α-homothetic deformation

Let $\mathcal{M}$ be a submanifold of a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$. For each nonzero vector U tangent to $\tilde{\mathcal{M}}$ at any point p if the slant angle between $T\mathcal{M}$ and φU is independent of the choice of $p\in\mathcal{M}$, then $\mathcal{M}$ is said to be *slant submanifold*. Observe that submanifold $\mathcal{M}$ becomes φ-*invariant* and φ-*anti-invariant* if the slant angle $\theta=0$ and $\theta=\frac{\pi}{2}$, respectively. A slant submanifold which is neither invariant nor anti-invariant is called proper slant (or θ-slant proper) submanifold.

Recall the results of [6, 7, 9] the following properties of slant submanifolds in an almost contact metric manifolds holds.

Theorem 5.1 ([7]). *Let $\mathcal{M}$ be a submanifold of an almost contact metric manifold $(M, \varphi, \eta, \zeta, g)$ such that $\zeta \in \Gamma(T\mathcal{M})$. Then*

(1) *$\mathcal{M}$ is slant if and only if there exists a constant $\delta \in [0,1]$ such that $P^2 = -\delta(I - \eta \otimes \zeta)$. Furthermore, if the θ is the slant angle of M, then $\delta = \cos^2\theta$*

(2) *$g(PU, PV) = \cos^2\theta[g(U,V) - \eta(U)\eta(V)]$, for any $U, V \in \Gamma(T\mathcal{M})$.*

Now, we prove the generalized Wintgen inequality for θ-slant submnaifolds of D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$.

Theorem 5.2. *Let $\mathcal{M}$ be an m-dimensional θ-slant submanifold of a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$. Then*

$$\rho_{nor} + \rho - \rho^\sharp \leq ||\mathcal{H}||^2 - \cos^2\theta\left(\frac{m-1}{m}\right)\left(\frac{\alpha-1}{\alpha}\right). \tag{5.1}$$

Proof. Suppose that the local orthonormal tangent frame field on $\mathcal{M}$ is as follows: $\{e_1, e_2 = \sec\theta P e_1, \ldots, e_{m-2}, e_{m-1} = \sec\theta P e_{m-2}, e_m = \zeta\}$ and the local orthonormal normal frame field on $\mathcal{M}$ is given by $\{e_{m+1}, \ldots, e_{2n+1}\}$. Then we have

$$\begin{aligned}
\tau &= \sum_{1\leq i<j\leq m} \mathcal{R}(e_i, e_j, e_j, e_i) \\
&= \sum_{1\leq i<j\leq m} \Big\{ R^\sharp(e_i, e_j, e_j, e_i) + g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_j, e_i)) \\
&- \left(\frac{\alpha-1}{\alpha}\right)\big[g(Pe_j, Pe_j)g(e_i, e_i) - g(Pe_i, Pe_j)g(e_j, e_i)\big]\Big\} \\
&= \tau^\sharp - (m-1)^2\cos^2\theta\left(\frac{\alpha-1}{2\alpha}\right) + \sum_{r=1}^{2n-m+1}\sum_{1\leq i<j\leq m}\Big[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\Big].
\end{aligned} \tag{5.2}$$

By using similar arguments as in the proof of Theorem 4.1 and (5.2), we get the desired inequality (5.1). □

Remark 5.3. *In case of an m-dimensional θ-slant submanifold $\mathcal{M}$ of a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$. The equality case holds uniformly if and only if the shape operators take the following forms with the suitable orthonormal frames as in Theorem 4.1.*

Now, we classify the geometrical bearing of invariant and anti-invariant submanifolds a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$ in terms of slant angle θ and in light of Theorem 5.2.

If $\mathcal{M}$ is an invariant submanifold, then $\theta = 0$. Then we turn up

Corollary 5.4. *Let $\mathcal{M}$ be an m-dimensional invariant submanifold of a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$. Then*

$$\rho_{nor} + \rho - \rho^\sharp \leq ||\mathcal{H}||^2 - \left(\frac{m-1}{m}\right)\left(\frac{\alpha-1}{\alpha}\right). \tag{5.3}$$

If $\mathcal{M}$ is an ant-invariant submanifold, then $\theta = \frac{\pi}{2}$. Then we have

Corollary 5.5. *Let $\mathcal{M}$ be an m-dimensional anti-invariant submanifold of a D_α-homothetically deformed Kenmotsu manifold $\tilde{\mathcal{M}}$ of dimension $(2n+1)$. Then*

$$\rho_{nor} + \rho - \rho^\sharp \leq ||\mathcal{H}||^2. \tag{5.4}$$

Remark 5.6. *The equality case holds uniformly if and only if the shape operators take the following forms with the suitable orthonormal frames as in Theorem 4.1.*

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On the solution of $\mathcal{T}-$controllable abstract fractional differential equations with impulsive effects

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ABSTRACT

In this research article, we delimitate the definition of mild solution for abstract fractional differential equations with state-dependent delay (AFDEw/SDD) of order $\alpha \in (1, 2)$ with impulsive effects and compare the solution to the second-order impulsive differential equations. Further, we obtain sufficient conditions of the existence of mild solution for instantaneous and non-instantaneous impulsive fractional functional differential inclusions with state-dependent delay (IFDIw/SDD) using the multi-valued fixed point theory and operator techniques. Furthermore, we study the trajectory controllability ($\mathcal{T}-$controllability) of the AFDEw/SDD. At last, we present some examples to illustrate the sufficient conditions involving partial and ordinary derivatives.

RESUMEN

En este artículo de investigación, delimitamos la definición de solución mild para ecuaciones diferenciales fraccionarias con retardo dependiente del estado (AFDEw/SDD) de orden $\alpha \in (1, 2)$ con efectos impulsivos y comparamos la solución con aquellas de ecuaciones diferenciales impulsivas de segundo orden. Además obtenemos condiciones suficientes para la existencia de soluciones mild de inclusiones funcionales diferenciales fraccionales instantánea y no-instantáneamente impulsivas con retardo dependiente del estado (IFDIw/SDD) usando la teoría de punto fijo multivaluados y técnicas de operadores. Más aún, estudiamos la controlabilidad por trayectoria ($\mathcal{T}-$controlabilidad) de los AFDEw/SDD. Finalmente, presentamos algunos ejemplos para ilustrar las condiciones suficientes que involucran derivadas parciales y ordinarias.

Keywords and Phrases: Fractional differential equation, functional-differential equations with fractional derivatives, initial value problems, fixed point theorems, controllability.

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1 Introduction

In the last few decades, many researchers paid attention on impulsive differential equations, because the models subject to abrupt changes are not described by classical models, so such type equations simulated in term of impulsive models. In the nature, there are lots of systems in which the time evolution of the state variable depends on the past history in some arbitrary way subject to abrupt changes are modeled in impulsive functional differential equations, see [12–14, 16, 19, 41, 43] for update. These equations arise in several fields of science and engineering which describe the evolution processes. The impulsive effects may be instantaneous or non-instantaneous (more details [2, 25, 37]) which is shown in many biological phenomena involving thresholds, optimal control models in economics, etc.

The reason of receiving great attention of fractional calculus is that it describes the memory and hereditary property. Due to this property fractional mathematical models give the more realistic and practical results than the ordinary models. For the fractional calculus and its applications see the monographs and papers [7, 30, 31, 34, 38–40] and references therein. Further, more specific type of functional differential equations are state dependent delay equations which arise in applied model when traditional simplifications are abandoned. For recent development theory of functional differential equations with state dependent delay reader can see the papers [1, 6, 8, 17, 18, 21] and references therein.

In additional, fractional differential inclusion is the generalization of fractional differential equation; therefore, all problems which contain the property of solution such as existence, uniqueness, stability, periodicity and controllability are presented in the theory of inclusion. A differential inclusion usually has many solutions which start from a given point and pass through others. It is recently seen that new issue appear in the differential inclusion for the investigation of topological properties of the set of solution, and selection of solutions. One can see the articles [9, 10, 15] for more info about this hot topic.

In this appraise, we describe the existence of solution for fractional order case. Feckan *et al.* [19] gave the suitable definition of solution for impulsive nonlinear fractional differential equation of order $\alpha \in (0,1)$, and Wang [43] extended the problem considered in [19] for the order $\alpha \in (1,2)$. Wang *et al.* [41] defined the mild solution using the probability density function for impulsive fractional evolution equations of order $\alpha \in (0,1)$, and motivated by [41] authors [16] extended the definition of mild solution for neutral impulsive fractional functional differential equation with order $\alpha \in (0,1)$ using analytic operator theory. Shu *et al.* [40] determined the definition of mild solution for fractional differential equations with nonlocal conditions to order $\alpha \in (1,2)$ without impulse. The existence results of mild solution for impulsive fractional differential inclusions with nonlocal conditions investigated by Wang *et al.* [42] when the linear part is a fractional sectorial operator for convex and nonconvex of nonlinear term. Liu and Ahmad [32] analyzed an impulsive multiterm fractional differential equations with single and multiple base points for Caputo's fractional derivative. Recently, Feckan *et al.* [20] proposed two type Caputo's fractional derivative named as generalized Caputo's derivative for single base point with the lower bound at zero and classical Caputo's derivative for multiple base points with lower bounded at non-zero.

Controllability is one of the contemplated properties of fractional dynamical systems (FDSs) that confirm the steering of a FDS from an arbitrary initial state to a desired arbitrary final state via a set of certain admissible control. In 1963, Kalman [28], first time gave the notion of controllability. Based on the available literature, we found that there are various concepts of controllability, some like

- approximate controllability (any state vector may be steered arbitrarily close to another state vector)
- exact controllability (any pair of state vectors may be connected by a trajectory)
- the null controllability (any state vector may be steered to 0)
- $\mathcal{T}-$controllability (we look for a control which steers the system along a prescribed trajectory rather than a control steering a given initial state to desired final state.)

It is obvious that $\mathcal{T}-$controllability is a stronger notion than other controllability notions. For example: To launch a rocket in space sometimes it may be desirable a precise path along with desired destination for cost effectiveness and so on, which is based on $\mathcal{T}-$controllability notation. For more details on $\mathcal{T}-$controllability one can see the papers [11, 23, 27, 35] and reference therein.

We found that there is no literature available on existence of mild solution for instantaneous and non-instantaneous impulsive fractional differential inclusion of order $\alpha \in (1,2)$. By inspiration of works [11, 16, 19, 23, 27, 29, 33, 35, 36, 40, 41, 43–45], we consider the following fractional functional differential inclusion with instantaneous and non-instantaneous impulsive effects.

First, we obtain the sufficient conditions of existence of mild solution for the following problem with instantaneous impulse

$$\begin{aligned} {}_0^CD_t^{\alpha}u(t) &\in Au(t) + f(t, u_{\rho(t,u_t)}), \quad 0 < t \leq T, \quad t \neq t_k, \quad k = 1, 2, \dots, m, && (1.1)\\ u(t) &= \phi(t), \quad t \in (-\infty, 0]; \quad u'(0) = u_0 \in X, && (1.2)\\ \Delta u(t_k) &= I_k(u(t_k^-)); \quad \Delta u'(t_k) = J_k(u(t_k^-)), && (1.3) \end{aligned}$$

where ${}_0^CD_t^{\alpha}$ denotes the generalized Caputo's fractional derivative of order $\alpha \in (1,2)$ for the state $u(t)$ belong to complex Banach space X and $A : D(A) \subset X \to X$ is the closed linear densely defined operator of sectorial type defined on X. The functions $f : [0,T] \times \mathfrak{B}_e \to \mathcal{F}(X)$; $\rho : [0,T] \times \mathfrak{B}_e \to (-\infty, T]$; $\phi(t) : (-\infty, 0] \to X$ satisfy some assumptions, and $\phi(t)$ in to a abstract phase space $\mathfrak{B}_e$.

The notation $(0,T]$ denotes operational interval such that $0 \leq t_0 < t_1 < \cdots < t_m < t_{m+1} \leq T < \infty$. The history function $u_t : (-\infty, 0] \to X$ defined by $u_t(\theta) = u(t+\theta),\ \theta \in (-\infty, 0]$ belongs to $\mathfrak{B}_e$ and $u'(t)$ denotes the ordinary derivative of $u(t)$. The jump functions $I_k, J_k \in C(X,X),\ k = 1,2,\ldots,m$, are bounded and $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$ where $u(t_k^+)$ and $u(t_k^-)$ represent the right-hand and left-hand limits of $u(t)$ at $t = t_k$ with $u(t_k^-) = u(t_k)$. Also, we have $\triangle u'(t_k) = u'(t_k^+) - u'(t_k^-)$ where $u'(t_k^+)$ and $u'(t_k^-)$ represent the right-hand and left-hand limits of $u'(t)$ at $t = t_k$, also we take $u'(t_k^-) = u'(t_k)$ respectively.

Second, we give the sufficient conditions for problem with non-instantaneous impulsive fractional functional differential equation

$$ {}_0^C D_t^\alpha u(t) = Au(t) + f(t, u_{\rho(t,u_t)}, Bu_{\rho(t,u_t)}), \quad t \in (s_i, t_{i+1}] \subseteq (0,T], \quad i = 0,1,\ldots,N, \tag{1.4} $$
$$ u(t) = g_i(t, u(t)), \quad u'(t) = q_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,N, \tag{1.5} $$
$$ u(t) + G(u) = \phi(t), \quad t \in (-\infty, 0] \quad u'(0) = u_1 \in X, \tag{1.6} $$

where ${}_0^C D_t^\alpha$ is classical Caputo's fractional derivative. $f : [0,T] \times \mathfrak{B}_e \times \mathfrak{B}_e \to X, G : X \to X$ are given functions and satisfy some assumptions and the term $Bu_{\rho(t,u_t)}$ is given by $Bu_{\rho(t,u_t)} = \int_0^t K(t,s)(u_{\rho(s,u_s)})\,ds$ where $K \in C(D, \mathbb{R}^+)$ is the set of all positive functions which are continuous on $D = \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t < T\}$ and $B^* = \sup_{t\in[0,t]} \int_0^t K(t,s)\,ds < \infty$. Here $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_N \leq s_N \leq t_{N+1} = T$ are pre-fixed numbers, and $g_i, q_i \in C((t_i, s_i] \times X; X)$ for all $i = 1,2,\ldots,N$. The nonlocal condition $G(u)$ defined as $G(u) = \sum_{k=1}^r c_k u(t_k)$, where $c_k, k = 1,\ldots,r$, are given constants and $0 < t_1 < t_2 < \cdots < t_r < T$ respectively.

Finally, we consider nonlinear fractional delay differential equation with non-local condition and provide some sufficient conditions for $\mathcal{T}-$controllability for the equation of the form:

$$ {}_0^C D_t^\alpha u(t) = Au(t) + \text{Ḃ}\varpi(t) + f(t, u_{\rho(t,u_t)}, Bu_{\rho(t,u_t)}),\ t \in (s_i, t_{i+1}] \subseteq (0,T],\ i = 0,1,\ldots,N, \tag{1.7} $$
$$ u(t) = g_i(t, u(t)), \quad u'(t) = q_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,N, \tag{1.8} $$
$$ u(t) + G(u) = \phi(t), \quad t \in (-\infty, 0] \quad u'(0) = u_1 \in X, \tag{1.9} $$

The linear operator Ḃ : $\mathcal{U}$(Banach space) $\to X$ is a bounded operator and $\varpi(t) \in \mathcal{L}^2(J, \mathcal{U})$ is a control function of the system.

Moreover, a strong motivation to study the model problem (1.1), (1.4) and, (1.7) with aftereffect and subject to impulsive conditions (1.3), (1.5) and (1.9) comes from physics because this model represents the inverse heat condition problem. In this paper, we have used the standard fixed point technique taking generalized and classical Caputo's fractional derivative in abstract phase space to established the results.

Further, motivation is that in dynamical models, generally we assume that the linear or non-linear terms are smooth or continuous functions. However, in many modern models, the underlying dynamical models are not necessarily even continuous. For examples, models of friction and Low dimensional climate models do not belong to above models so to remove the restriction or for non-smooth systems with the discontinuous terms are frequently remodeled as a differential inclusion. This is the advantage to study the qualitative analysis of this paper.

A strong motivation to prove the existence results that the knowledge of existence does not prove the uniqueness of solutions also. For example, we have some fractional differential equation model like ${}_{0}^{C}D_t^{1/2}x(t) = x^{1/2}(t)$ with initial condition $x(0) = 0$ for $t \in [0, T]$ has a trivial solution $x \equiv 0$ and non trivial solution $x(t) = \frac{\pi}{4}t$. This shows that the solution obviously exists and is not unique because it fails to satisfy the Lipschitz continuity condition. Hence, in a differential equation, solution can exist and can be not unique. In other words, the knowledge of existence does not ensure the uniqueness of the solution.

Further information about this work, it has five sections. Section 2 provides some basic definitions, theorems, notations and lemmas. Section 3 is equipped with existence results of the mild solution for the considered problems (1.1)-(1.6). Section 4 contributes to the Trajectory controllability results for the considered fractional delay differential equation. In Section 5 examples are provided to illustrate our results.

2 Preliminaries

Let X be a arbitrary complex Banach space with norm $\|\cdot\|_X$ and $L(X)$ denotes the Banach space of bounded linear operators from X into X with norm $\|\cdot\|_{L(X)}$ and both are equipped with its natural topology. Let $C([0, T], X)$ be the space of all real valued (or complex valued) continuous functions from $[0, T]$ into X with the sup norm

$$\|u\|_{C([0,T],X)} = \sup_{t\in[0,T]} \{\|u(t)\|_X : u \in C([0, T], X)\}.$$

is a Banach space.

For the general setting of abstract phase space $\mathfrak{B}_e, \mathfrak{B}'_e$ with impulse effects we refer the work [16,24] and for further notations like ${}_{a}^{C}D_t^{\alpha}$ (Caputo's derivative), ${}_{a}\mathcal{J}_t^{\alpha}$ (Riemann-Liouville integral) and $E_{\alpha,\beta}(\cdot)$ (Mittag-Leffler function) we refer [34, 38]. For $A : D(A) \subseteq X \to X$ (Sectorial operator) see [40], and for $S_\alpha(t), T_\alpha(t)$ (Operators) [40] particular case of $W_{\alpha,\beta}(t)$ (Operator functions) we refer [22] respectively.

Let $\mathcal{T}$ be the set of all functions $\vartheta(\cdot) \in \mathfrak{B}'_e$ defined on $J = [0, T]$ such that $\vartheta(0) = \phi(0),\ \vartheta'(0) = u_1$ and $\vartheta(T) = \phi_T,\ \vartheta'(T) = u_T$ for all $t \in J$ and the fractional derivative ${}^{C}D_t^{\alpha}\vartheta(t)$ exist almost everywhere. The set $\mathcal{T}$ is called the set of all feasible trajectories for the fractional dynamical system.

Lemma 2.1 ([24]). *Let* $u : (-\infty, T] \to X$ *be a function such that* $u_0 = \phi, u\,|_{(t_k,t_{k+1}]} \in C^2((t_k, t_{k+1}], X)$, *then for all* $t \in (t_k, t_{k+1}]$, *the following conditions hold:*

(C_1) $u_t \in \mathfrak{B}_e$.

(C_2) $\|u(t)\|_X \leq H\|u_t\|_{\mathfrak{B}_e}$.

(C_3) $\|u_t\|_{\mathfrak{B}_e} \leq K(t)\sup\{\|u(s)\| : 0 \leq s \leq t\} + M(t)\|\phi\|_{\mathfrak{B}_e}$, *where* $H > 0$ *is constant;* $K, M : [0,\infty) \to [0,\infty)$, $K(\cdot)$ *is continuous,* $M(\cdot)$ *is locally bounded and* K, M *are independent of* $u(t)$.

(C_4) *The function* $t \mapsto \phi_t$ *is well defined and continuous from the set*

$$\Re(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in [0,T] \times \mathfrak{B}_e\}$$

into $\mathfrak{B}_e$ *and there exists a continuous and bounded function* $J^\phi : \Re(\rho^-) \to (0,\infty)$ *such that* $\|\phi_t\|_{\mathfrak{B}_e} \leq J^\phi(t)\|\phi\|_{\mathfrak{B}_e}$ *for every* $t \in \Re(\rho^-)$.

Lemma 2.2 ([8]). *Let* $u : (-\infty, T] \to X$ *be function such that* $u_0 = \phi,\ u\,|_{(t_k,t_{k+1}]} \in C^2((t_k, t_{k+1}], X)$ *and if* (C_4) *hold, then*

$$\|u_s\|_{\mathfrak{B}_e} \leq (M_e + J^\phi)\|\phi\|_{\mathfrak{B}_e} + K_e \sup\{\|u(\theta)\|;\ \theta \in [0, \max\{0,s\}]\},\ s \in \Re(\rho^-) \cup (t_k, t_{k+1}],$$

where $J^\phi = \sup_{t\in\Re(\rho^-)} J^\phi(t)$, $M_e = \sup_{s\in[0,T]} M(s)$ *and* $K_e = \sup_{s\in[0,T]} K(s)$.

To use the multi-valued analysis that is discussed in reference [9], we have some properties which are required to prove our main result. Denote by $\mathcal{F}(X) = \{Y \subset X : Y \neq \emptyset\}, \mathcal{F}_{cl}(X) = \{Y \subset \mathcal{F}(X) : Y \text{ is closed}\}, \mathcal{F}_b(X) = \{Y \subset \mathcal{F}(X) : Y \text{ is bounded}\}, \mathcal{F}_{cv}(X) = \{Y \subset \mathcal{F}(X) : Y \text{ is convex}\}, \mathcal{F}_{cp}(X) = \{Y \subset \mathcal{F}(X) : Y \text{ is compact}\}$.

A multi-valued map $\mathcal{G} : X \to \mathcal{F}(X)$ is convex (closed) valued if $\mathcal{G}(x)$ is convex (closed) for all $x \in X$. $\mathcal{G}$ is bounded on bounded sets if $\mathcal{G}(B) = \cup_{x\in B}\mathcal{G}(x)$ is bounded in X for any bounded set B of $\mathcal{F}(X)$ (*i.e.* $\sup_{x\in B}\{\sup\{\|y\| : y \in \mathcal{G}(x)\}\} < \infty$).

A multi-valued map $\mathcal{G} : [0,T] \to P_{cl}(X)$ is said to be measurable if for each $y \in X$ the function $Y : [0,T] \to \mathbb{R}$ defined by

$$Y(t) = d(y, G(t)) = \inf\{|y - z| : z \in \mathcal{G}(t)\},$$

belongs to $L^1([0,T],\mathbb{R})$.

Definition 2.3 ([9]). *A multi-valued map $F : [0,T] \times X \to \mathcal{F}(X)$ is Carathe̋ódory if*

(i) $t \to F(t,u)$ is measurable for each $u \in X$, and

(ii) $u \to F(t,u)$ is upper semi continuous (u.s.c.) for almost all $t \in [0,T]$.

For each $y \in C([0,T],X)$, define the set of selections for F by

$$S_{Fy} = \{v \in L^1([0,T],X) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,T]\}.$$

Let (X,d) be a metric space induced by the norm space $(X, \|\cdot\|_X)$. Consider $H_d : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}_+ \cup \infty$ given by

$$H_d(A,B) = \max\left\{ \sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b) \right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{F}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{F}_{cl}(X), H_d)$ is a generalized metric space.

Definition 2.4 ([9]). *A multi-valued operator $\mathcal{N} : X \to \mathcal{F}_{cl}(X)$ is called:*

(i) $\gamma-$Lipschitz if there exists $\gamma > 0$ such that

$$H_d(\mathcal{N}(x), \mathcal{N}(y)) \le \gamma d(x,y) \quad \text{for all } x,y \in X;$$

(ii) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Lemma 2.5 ([9]). *Let (X,d) be a complete metric space. If $\mathcal{N} : X \to \mathcal{F}_{cl}(X)$ is a contraction, then Fix $\mathcal{N} \neq \emptyset$.*

Lemma 2.6 ([9]). *Let f satisfy the uniform Holder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator of the type (M,θ,α,μ). Consider differential equation of order $\alpha \in (1,2)$ with instantaneous impulse*

$$\begin{aligned}
{}_0^C D_t^\alpha u(t) &= Ay(t) + f(t), \quad t \in [0,T], \quad t \neq t_k, && (2.1)\\
u(0) &= u_0 \in X; \quad u'(0) = u_1 \in X, && (2.2)\\
\Delta u(t_k) &= I_k(u(t_k^-)); \quad \Delta u'(t_k) = J_k(u(t_k^-)), \quad t \neq t_k, \quad k = 1,2,\ldots,m. && (2.3)
\end{aligned}$$

and with non-instantaneous impulse

$$\begin{aligned}
{}_a^C D_t^\alpha u(t) &= Au(t) + f(t), \quad t \in (s_i, t_{i+1}] \subset J = (a,T], \quad a \ge 0, \quad i = 0,1,\ldots,N, && (2.4)\\
u(a) &= u_0 \in X; \quad u'(a) = u_1 \in X, && (2.5)\\
u(t) &= g_i(t,u(t)); \quad u'(t) = q_i(t,u(t)), \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,N. && (2.6)
\end{aligned}$$

Then a function $u(t) \in PC([0,T],X)$ is a solution of the system (2.1)-(2.3) if it satisfies following integral equation

$$u(t) = \begin{cases} S_\alpha(t)u_0 + u_1 \int_0^t S_\alpha(s)ds + \int_0^t T_\alpha(t-s)f(s)ds, & t \in (0,t_1] \\ S_\alpha(t)u_0 + u_1 \int_0^t S_\alpha(s)ds + \sum_{i=1}^k S_\alpha(t-t_i)I_i(u(t_i^-)) \\ + \sum_{i=1}^k J_i(u(t_i^-)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_0^t T_\alpha(t-s)f(s)ds, & t \in (t_k, t_{k+1}], \end{cases} \tag{2.7}$$

and a function $u(t) \in PC([a,T],X)$ is a solution of system (2.4)-(2.6) if it satisfies the following integral equation

$$u(t) = \begin{cases} S_\alpha(t-a)u_0 + u_1 \int_a^t S_\alpha(s-a)ds + \int_a^t T_\alpha(t-s)f(s)ds & t \in (a,t_1], \\ S_\alpha(t-s_i)g_i(s_i,u(s_i)) + q_i(s_i,u(s_i)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_{s_i}^t T_\alpha(t-s)f(s)ds & t \in (s_i, t_{i+1}] \end{cases} \tag{2.8}$$

Remark 2.7. *The α-resolvent family $T_\alpha(t)$ associated with solution operator $S_\alpha(t)$ can be defined as*

$$\int_0^t S_\alpha(\theta)x\,d\theta = {}_0\mathcal{J}_t^1 S_\alpha(\theta)x\,d\theta; \quad T_\alpha(t)x = {}_0\mathcal{J}_t^{\alpha-1} S_\alpha(\theta)x\,d\theta, \quad x \in X, \quad t \in [0,T].$$

For the special case when $\alpha \to 2$, we get following results

(1) *$T_\alpha(t)$ is the cosine function $C(t)$ and $S_\alpha(t)$ is the sine function $S(t)$ defined as*

$$S(t)x = \int_0^t C(\theta)x\,d\theta, \quad x \in X, \quad t \in [0,T]$$

(2) *Solution of system (2.1)-(2.3) for $t \in (0,T]$ can be reduced as*

$$u(t) = \begin{cases} C(t)u_0 + S(t)u_1 + \int_0^t S(t-s)f(s)ds & t \in (0,t_1] \\ C(t)u_0 + S(t)u_1 + \sum_{i=1}^k C(t-t_i)I_i(u(t_i^-)) \\ + \sum_{i=1}^k S(t-t_i)J_i(u(t_i^-)) + \int_0^t S(t-s)f(s)ds & t \in (t_k, t_{k+1}], \end{cases}$$

which is the same as Definition 2.1 in [26].

(3) *Solution of system (2.4)-(2.6) for $t \in (a,T]$ can be reduced as*

$$u(t) = \begin{cases} C(t-a)u_0 + u_1 \int_a^t S(s-a)ds + \int_a^t S(t-s)f(s)ds & t \in (a,t_1], \\ C(t-s_i)g_i(s_i,u(s_i)) + q_i(s_i,u(s_i)) \int_{t_i}^t S(s-t_i)ds + \int_{s_i}^t S(t-s)f(s)ds & t \in (s_i, t_{i+1}] \end{cases}$$

which is the same as Definition 2.1 in [25].

Definition 2.8. *A function $u : (-\infty, T] \to X$ such that $u \in \mathfrak{B}'_e$, is called a mild solution of problem* (1.1)-(1.3) *if $u(0) = \phi(0)$ and it satisfies the following integral equation*

$$u(t) = \begin{cases} S_\alpha(t)\phi(0) + u_0 \int_0^t S_\alpha(s)ds + \int_0^t T_\alpha(t-s)f(s, u_{\rho(s,u_s)})ds, & t \in (0, t_1] \\ S_\alpha(t)\phi(0) + u_0 \int_0^t S_\alpha(s)ds + \sum_{i=1}^k S_\alpha(t-t_i)I_i(u(t_i^-)) & \\ + \sum_{i=1}^k J_i(u(t_i^-)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_0^t T_\alpha(t-s)f(s, u_{\rho(s,u_s)})ds, & t \in (t_k, t_{k+1}]. \end{cases}$$

Definition 2.9. *A function $u : (-\infty, T] \to X$ such that $u \in \mathfrak{B}'_e$ is called a mild solution of the problem (1.4)-(1.6) if $u(0) = \phi(0) - G(u)$ and satisfies the following integral equation*

$$u(t) = \begin{cases} (\phi(0) - G(u))S_\alpha(t) + u_1 \int_0^t S_\alpha(s)ds & \\ + \int_0^t T_\alpha(t)f(s, u_{\rho(s,u_s)}, Bu_{\rho(s,u_s)})ds, & t \in (0, t_1], \\ g_i(s_i, u(s_i))S_\alpha(t-s_i) + q_i(s_i, u(s_i)) \int_{t_i}^t S_\alpha(s-t_i)ds & \\ + \int_{s_i}^t T_\alpha(t-s)f(s, u_{\rho(s,u_s)}, Bu_{\rho(s,u_s)})ds, & t \in (s_i, t_{i+1}], \end{cases}$$

for $i = 1, 2, \ldots, N$.

Definition 2.10. *The system* (1.1) *is said to be $\mathcal{T}-$*controllable *if for any $u(\cdot) \in \mathcal{T}$ there exists a control function $\varpi(t) \in \mathcal{L}^2(J, \mathcal{U})$ such that the corresponding solution $u(\cdot)$ of Eq.* (1.1) *satisfies $u(t) = \vartheta(t)$ almost everywhere.*

Definition 2.11. *A function $u : (-\infty, T] \to X$ such that $u \in \mathfrak{B}'_e$ is called a mild solution of the problem* (1.7)-(1.9) *if $u(0) = \phi(0) - G(u)$ and satisfies the following integral equation*

$$u(t) = \begin{cases} (\phi(0) - G(u))S_\alpha(t) + u_1 \int_0^t S_\alpha(s)ds & \\ + \int_0^t T_\alpha(t-s)[\mathcal{B}\varpi(s) + f(s, u_{\rho(s,u_s)}, Bu_{\rho(s,u_s)})]ds, & t \in (0, t_1], \\ g_i(s_i, u(s_i))S_\alpha(t-s_i) + q_i(s_i, u(s_i)) \int_{t_i}^t S_\alpha(s-t_i)ds & \\ + \int_{s_i}^t T_\alpha(t-s)[\mathcal{B}\varpi(s) + f(s, u_{\rho(s,u_s)}, Bu_{\rho(s,u_s)})]ds, & t \in (s_i, t_{i+1}], \end{cases}$$

for $i = 1, 2, \ldots, N$.

3 Existence result of mild solution

In this section, we shall establish the existence result of solution for the problems (1.1)-(1.6) for the both case of impulsive effects and also prove the continuous dependent of solution on initial conditions. Further, if A is a sectorial operator then strongly continuous functions are bounded - *i.e.*,

$$\|S_\alpha(t)\|_{L(X)} \leq M; \quad \|T_\alpha(t)\|_{L(X)} \leq M.$$

3.1 Instantaneous case

In this case, we prove the existence of mild solution for problem (1.1)-(1.3) with a non-convex valued right-hand side. Due to this analysis we can make the following assumptions:

(H_1) $f : [0,T] \times \mathfrak{B}_e \to \mathcal{F}_{cp}(X)$ is Carathéodory and has the property that $f(\cdot, \psi) : [0,T] \to \mathcal{F}_{cp}(X)$ is measurable, for each $\psi \in \mathfrak{B}_e$.

(H_2) There exists $l \in L^1([0,T], \mathbb{R}^+)$ such that

$$H_d(f(t,\psi), f(t,\xi)) \leq l(t)\|\psi - \xi\|_{\mathfrak{B}_e} \quad \text{for every} \quad \psi, \xi \in \mathfrak{B}_e$$

and

$$d(0, f(t,0)) \leq l(t) \quad \text{a.e.} \quad t \in [0,T].$$

Our result is based on contraction multi-valued fixed point theorem given by Covitz and Nadler [15].

Theorem 3.1. *Let the assumptions* (H_1) *and* (H_2) *hold. Then problem* (1.1)-(1.3) *has at least one mild solution* $u(t)$ *on* $[0,T]$.

Proof. Consider the space $\mathfrak{B}''_e = \{u \in \mathfrak{B}'_e : u(0) = \phi(0)\}$ and $y(t) = \phi(t)$ for $t \in (-\infty, 0]$ endowed with the uniform convergence topology. We shall show that $\mathcal{P}$ has fixed points, where the multi-valued operator $\mathcal{P} : \mathfrak{B}''_e \to \mathcal{F}(\mathfrak{B}''_e)$ defined as $\mathcal{P}(u) = \{\bar{e} \in \mathfrak{B}''_e\}$ with

$$\bar{e}(t) = \begin{cases} S_\alpha(t)\phi(0) + u_0 \int_0^t S_\alpha(s)ds + \int_0^t T_\alpha(t-s)v(s)ds, & t \in (0, t_1], \\ S_\alpha(t)\phi(0) + u_0 \int_0^t S_\alpha(s)ds + \sum_{i=1}^k S_\alpha(t-t_i)I_i(u(t_i^-)) & \\ + \sum_{i=1}^k J_i(u(t_i^-)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_0^t T_\alpha(t-s)v(s)ds, & t \in (t_k, t_{k+1}], \end{cases}$$

where $v(s) \in S_{f,\bar{u}_{\rho(s,\bar{u}_s)}}$ for $t \in [0,T]$ and $\bar{u} : (-\infty, T] \to X$ is such that $\bar{u}(0) = \phi(0)$ and $\bar{u} = u$ on $[0,T]$. We shall show that $\mathcal{P}$ has fixed points. Let $\mathcal{P}(u) \in \mathcal{F}_{cl}(\mathfrak{B}''_e)$ for all $u \in \mathfrak{B}''_e$. Let $\{u_n\}_{n\geq 0} \in \mathcal{P}(u)$ be such that $u_n \to u \in \mathfrak{B}'''_e$. Then there exists $v_n \in S_{f,\bar{u}_{\rho(s,\bar{u}_s)}}$ such that, for each $t \in (t_k, t_{k+1}]$,

$$u_n(t) = \begin{cases} S_\alpha(t)\phi(0) + u_0 \int_0^t S_\alpha(s)ds + \sum_{i=1}^k S_\alpha(t-t_i)I_i(u_n(t_i^-)) \\ + \sum_{i=1}^k J_i(u_n(t_i^-)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_0^t T_\alpha(t-s)v_n(s)ds. \end{cases}$$

Using the fact that f has compact values, we may pass to a subsequence if necessary to obtain that v_n converges to v in $L^1([0,T], X)$ and hence $v \in S_{f,\bar{u}_{\rho(s,\bar{u}_s)}}$. Thus, for each $t \in (t_k, t_{k+1}]$

$$u_n(t) \to u(t) = \begin{cases} S_\alpha(t)\phi(0) + u_0 \int_0^t S_\alpha(s)ds + \sum_{i=1}^k S_\alpha(t-t_i)I_i(u(t_i^-)) \\ + \sum_{i=1}^k J_i(u(t_i^-)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_0^t T_\alpha(t-s)v(s)ds, \end{cases}$$

which implies that $u \in \mathcal{P}(u)$.

There exist $\gamma < 1$ such that

$$H_d(f(u_1), f(u_2)) \leq \gamma \|u_1 - u_2\|_{\mathfrak{B}_e'''} \quad \text{for all} \quad u_1, u_2 \in \mathfrak{B}_e''.$$

Let $u_1, u_2 \in \mathfrak{B}_e''$ and $\bar{e} \in \mathcal{P}(u)$. Then there exists $v(t) \in f(t, \bar{u}_{\rho(t,\bar{u}_t)})$ such that, for each $t \in (t_k, t_{k+1}]$,

$$\bar{e}(t) = \begin{cases} S_\alpha(t)\phi(0) + u_0 \int_0^t S_\alpha(s)ds + \sum_{i=1}^k S_\alpha(t-t_i)I_i(u(t_i^-)) \\ + \sum_{i=1}^k J_i(u(t_i^-)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_0^t T_\alpha(t-s)v(s)ds. \end{cases}$$

From (H_2) it follows that

$$H_d(f(t, \bar{u_1}_{\rho(t,\bar{u_1}_t)}), f(t, \bar{u_2}_{\rho(t,\bar{u_2}_t)})) \leq l(t)\|u_1 - u_2\|_{\mathfrak{B}_e''}.$$

Hence, there exists $w \in f(t, \bar{u}_{\rho(t,\bar{u}_t)})$ such that

$$\|v - w\|_{\mathfrak{B}_e''} \leq l(t)\|u_1 - u_2\|_{\mathfrak{B}_e''}.$$

Consider $U : [0,T] \to \mathcal{F}(X)$ given by

$$U(t) = \{w \in X : \|v - w\| \leq l(t)\|u_1 - u_2\|_{\mathfrak{B}_e''}\}.$$

Since the multi-valued operator $V(t) = U(t) \cap f(t, \bar{u_2}_{\rho(t,\bar{u_2}_t)})$ is measurable [10], there exists a function $v_2(t)$ which is a measurable selection for V. Thus, $\bar{v}(t) \in f(t, \bar{u_2}_{\rho(t,\bar{u_2}_t)})$ and for each $t \in (t_k, t_{k+1}]$,

$$v(t) - \bar{v}(t) \leq l(t)\|u_1 - u_2\|_{\mathfrak{B}_e''}.$$

For each $t \in (t_k, t_{k+1}]$ we define

$$\bar{e}(t) = \begin{cases} S_\alpha(t)\phi(0) + u_0 \int_0^t S_\alpha(s)ds + \sum_{i=1}^k S_\alpha(t-t_i)I_i(u(t_i^-)) \\ + \sum_{i=1}^k J_i(u(t_i^-)) \int_{t_i}^t S_\alpha(s-t_i)ds + \int_0^t T_\alpha(t-s)\bar{v}(s)ds. \end{cases}$$

Then, we have

$$\begin{aligned} \|e(t) - \bar{e}(t)\|_{\mathfrak{B}_e''} &\leq \int_0^t \|T_\alpha(t-s)\|_{L(X)}\|v(s) - \bar{v}(s)\|ds \leq M\int_0^t l(s)\|u_1 - u_2\|ds \\ &\leq \int_0^t \bar{l}(s)\|u_1 - u_2\|ds \leq \frac{1}{\tau}e^{\tau L(t)}\|u_1 - u_2\|_{\mathfrak{B}_e''}, \end{aligned}$$

where $\tau > 1,\ L(t) = \int_0^t Ml(s)ds$ and $\|\cdot\|_{\mathfrak{B}_e''}$ is the Bielecki-type norm on $\mathfrak{B}_e''$ defined by

$$\|u\|_{\mathfrak{B}_e''} = \sup\{e^{-\tau L(t)}\|u(t)\| : t \in [0,T]\}.$$

Therefore

$$\|e(t) - \bar{e}(t)\|_{\mathfrak{B}_e''} \leq \frac{1}{\tau}\|u_1 - u_2\|_{\mathfrak{B}_e''}.$$

Obtained by interchanging of u_1 and u_2, and by an analogous relation, it follows that

$$H_d(\mathcal{P}(u_1), \mathcal{P}(u_2)) \leq \frac{1}{\tau}\|u_1 - u_2\|_{\mathfrak{B}_e''},$$

which implies that $\mathcal{P}$ is a contraction, and thus, by Lemma 2.5 there exists a fixed point $u(t) \in \mathfrak{B}_e''$, which is a mild solution to the problem (1.1)-(1.3). This completes the proof. □

3.2 Non-instantaneous Case

In this case, we shall establish the existence result of solution for the problem (1.4)-(1.6). Now, we introduce the following assumption.

(H_3) The function f is jointly continuous and there exist positive constants L_{f1}, L_{f2} such that

$$\|f(t,\psi,\mu) - f(t,\xi,\nu)\|_X \leq L_{f1}\|\psi - \xi\|_{\mathfrak{B}_e} + L_{f2}\|\mu - \nu\|_{\mathfrak{B}_e}, \quad \forall\ \psi,\xi,\mu,\nu \in \mathfrak{B}_e.$$

(H_4) The functions g_i, q_i and G are continuous and there exist positive constants L_{g_i}, L_{q_i} and L_G such that

$$\|g_i(t,x) - g_i(t,y)\|_X \leq L_{g_i}\|x - y\|_X; \quad \|q_i(t,x) - q_i(t,y)\|_X \leq L_{q_i}\|x - y\|_X;$$

$$\|G(x) - G(y)\|_X \leq L_G\|x - y\|_X,$$

for all $x, y \in X,\ t \in (t_i, s_i]$ and each $i = 1, 2, \ldots, N$.

Theorem 3.2. *If the assumptions (H_3) and (H_4) hold and constant*

$$\Delta = (\delta + TMK_e(L_{f1} + B^* L_{f2})) < 1,$$

where $\delta = \max\{L_G M, L_{g_i}M + L_{q_i}MT\}$ for $i = 1, \ldots, N$. Then there exists a unique mild solution $u(t)$ of the problem (1.4)-(1.6) on $[0,T]$.

Proof. Consider the space $\mathfrak{B}''_e$ as given in Theorem 3.1 and we define an operator $\mathcal{P}:\mathfrak{B}''_e \to \mathfrak{B}''_e$ as

$$\mathcal{P}u(t) = \begin{cases} (\phi(0) - G(\bar{u}))S_\alpha(t) + u_1 \int_0^t S_\alpha(s)ds \\ + \int_0^t T_\alpha(t-s) f(s, \bar{u}_{\rho(s,\bar{u}_s)}, B\bar{u}_{\rho(s,\bar{u}_s)})ds, & t \in (0, t_1], \\ g_i(s_i, \bar{u}(s_i))S_\alpha(t - s_i) + q_i(s_i, \bar{u}(s_i)) \int_{t_i}^t S_\alpha(s - t_i)ds \\ + \int_{s_i}^t T_\alpha(t-s) f(s, \bar{u}_{\rho(s,\bar{u}_s)}, B\bar{u}_{\rho(s,\bar{u}_s)})ds, & t \in (s_i, t_{i+1}], \end{cases} \tag{3.1}$$

where $\bar{u} : (-\infty, T] \to X$ is such that $u(\bar{0}) = \phi(0) - G(\bar{u}), u'(\bar{0}) = u_1$ and $\bar{u} = u$ on $[0, T]$. We shall show that the operator $\mathcal{P}$ has a fixed point. So let $u(t), u^*(t) \in \mathfrak{B}'_e$ for $t \in (0, t_1]$, we get

$$\begin{aligned} \|\mathcal{P}u - \mathcal{P}u^*\|_{\mathfrak{B}'_e} &\leq \|G(\bar{u}) - G(\bar{u}^*)\| \|S_\alpha(t)\|_{L(X)} + \int_0^t \|T_\alpha(t-s)\|_{L(X)} \\ &\quad \times \|f(s, \bar{u}_{\rho(s,\bar{u}_s)}, B\bar{u}_{\rho(s,\bar{u}_s)}) - f(s, \bar{u}^*_{\rho(s,\bar{u}^*_s)}, B\bar{u}^*_{\rho(s,\bar{u}^*_s)})\|_X ds, \\ \|\mathcal{P}u - \mathcal{P}u^*\|_X &\leq \{L_G M + TMK_e(L_{f1} + B^* L_{f2})\} \|u - u^*\|_X. \end{aligned}$$

For $t \in (s_i, t_{i+1}]$, we have

$$\begin{aligned} \|\mathcal{P}u - \mathcal{P}u^*\|_{\mathfrak{B}'_e} &\leq \|g_i(s_i, \bar{u}(s_i)) - g_i(s_i, \bar{u}^*(s_i))\|_X \|S_\alpha(t-s)\|_{L(X)} \\ &\quad + \|q_i(s_i, \bar{u}(s_i)) - q_i(s_i, \bar{u}^*(s_i))\|_X \int_0^t \|S_\alpha(t-s)\|_{L(X)} ds \\ &\quad + \int_{s_i}^t \|T_\alpha(t-s)\|_{L(X)} \|f(s, \bar{u}_{\rho(s,\bar{u}_s)}, B\bar{u}_{\rho(s,\bar{u}_s)}) - f(s, \bar{u}^*_{\rho(s,\bar{u}^*_s)}, B\bar{u}^*_{\rho(s,\bar{u}^*_s)})\|_X ds, \\ \|\mathcal{P}u - \mathcal{P}u^*\|_X &\leq (L_{g_i} M + L_{q_i} MT + TMK_e(L_{f1} + B^* L_{f2})) \|u - u^*\|_X. \end{aligned}$$

Let $\delta = \max\{L_G M, L_{g_i} M + L_{q_i} MT\}$, then for all $t \in [0, T]$, we obtain

$$\|\mathcal{P}u - \mathcal{P}u^*\|_X \leq (\delta + TMK_e(L_{f1} + B^* L_{f2})) \|u - u^*\|_X.$$

We have

$$\|\mathcal{P}u - \mathcal{P}u^*\|_X \leq \Delta \|u - u^*\|_X.$$

Since $\Delta < 1$, which implies that $\mathcal{P}$ is a contraction map and there exists a unique fixed point $u(t)$ which is the mild solution of system (1.4)-(1.6) on $[0, T]$. □

3.3 Continuous Dependence of Mild Solutions

This section is concerned with continuous dependence of mild solutions consider the system (1.4)-(1.6).

Theorem 3.3. *Suppose that the assumptions (H_3) and (H_4) are satisfied and the following condition hold:*

$$\big[\max\{ML_G, ML_{g_i} + MTL_{q_i}\} + MT(L_{f_1} + L_{f_2}B^*)(M_e + J^\phi)\big] < 1.$$

Then for each ϕ, ϕ^*, let u, u^* be the corresponding mild solutions of the system (1.4)-(1.6), then the following inequalities hold:

$$\begin{aligned} \|u - u^*\|_X &\leq \frac{MT(M + L_{f_1} + L_{f_2}B^*)}{1 - [ML_G + MT(L_{f_1} + L_{f_2}B^*)(M_e + J^\phi)]}\|\phi - \phi^*\|, \quad t \in (0, t_1], \\ \|u - u^*\|_X &\leq \frac{MT(M + L_{f_1} + L_{f_2}B^*)}{1 - [ML_{g_i} + MTL_{q_i} + MT(L_{f_1} + L_{f_2}B^*)(M_e + J^\phi)]}\|\phi - \phi^*\|, \quad t \in (s_i, t_{i+1}], \end{aligned}$$

for $i = 1, 2, \ldots, N$.

Proof. The proof is similar as Theorem 3.2. □

4 Trajectory Controllability

This section deals with the $\mathcal{T}-$controllability results of the considered nonlinear fractional delay differential equation with non-local condition and non-instantaneous impulses.

Theorem 4.1. *Let the assumption (H_3) and (H_4) hold, then problem (1.7)-(1.9) is $\mathcal{T}-$controllable on $[0, T]$.*

Proof. Let $\vartheta(t)$ be any given trajectory in $\mathcal{T}$ and we choose the feedback control $\varpi(t)$ given as

$$\varpi(t) = Ş^{-1}[{}^C_0D^\alpha_t\vartheta(t) - A\vartheta(t) - f(t, \vartheta_{\rho(t,\vartheta_t)}, B\vartheta_{\rho(t,\vartheta_t)})], \quad t \in (s_i, t_{i+1}] \subseteq (0, T]. \tag{4.1}$$

Plugging the control $\varpi(t)$ from Eq. (4.1) in Eq. (1.7) and we get

$${}^C_0D^\alpha_t u(t) = Au(t) + f(t, u_{\rho(t,u_t)}, Bu_{\rho(t,u_t)}) + {}^C_0D^\alpha_t\vartheta(t) - A\vartheta(t) - f(t, \vartheta_{\rho(t,\vartheta_t)}, B\vartheta_{\rho(t,\vartheta_t)}),$$
$$t \in (s_i, t_{i+1}] \subseteq (0, T].$$

From the equation above, we have

$$ {}_0^C D_t^\alpha [u(t)-\vartheta(t)] = A[u(t)-\vartheta(t)] + f(t,u_{\rho(t,u_t)},Bu_{\rho(t,u_t)}) - f(t,\vartheta_{\rho(t,\vartheta_t)},B\vartheta_{\rho(t,\vartheta_t)}), \quad t\in(s_i,t_{i+1}]\subseteq(0,T]. $$

Again, if we choose $\chi(t)=u(t)-\vartheta(t)$, without loss of generality, then our original problem (1.7)-(1.9) is modified as follows:

$$ {}_0^C D_t^\alpha \chi(t) = A\chi(t) + f(t,u_{\rho(t,u_t)},Bu_{\rho(t,u_t)}) - f(t,\vartheta_{\rho(t,\vartheta_t)},B\vartheta_{\rho(t,\vartheta_t)}), \quad t\in(s_i,t_{i+1}]\subseteq(0,T], \quad i=0,1,\ldots,N, \tag{4.2} $$

$$ \chi(t) = g_i(t,u(t)) - g_i(t,\vartheta(t)), \quad \chi'(t) = q_i(t,u(t)) - q_i(t,\vartheta(t)), \tag{4.3} $$

$$ t\in(t_i,s_i], \quad i=1,2,\ldots,N, \tag{4.4} $$

$$ \chi(t) = -G(u)+G(\vartheta), \quad t\in(-\infty,0], \quad \chi'(0)=0. \tag{4.5} $$

The mild solution of the problem (4.2)-(4.5) is given by

$$ \chi(t) = \begin{cases} (-G(u)+G(\vartheta))S_\alpha(t) + \int_0^t T_\alpha(t-s)[f(s,u_{\rho(s,u_s)},Bu_{\rho(s,u_s)}) - f(s,\vartheta_{\rho(s,\vartheta_s)},B\vartheta_{\rho(s,\vartheta_s)})]ds, \\ \hfill t\in(0,t_1], \\ S_\alpha(t-s_i)[g_i(t,u(t))-g_i(t,\vartheta(t))] + \int_{t_i}^t S_\alpha(s-t_i)ds[q_i(t,u(t))-q_i(t,\vartheta(t))] \\ + \int_{s_i}^t T_\alpha(t-s)[f(s,u_{\rho(s,u_s)},Bu_{\rho(s,u_s)}) - f(s,\vartheta_{\rho(s,\vartheta_s)},B\vartheta_{\rho(s,\vartheta_s)})]ds, \quad t\in(s_i,t_{i+1}], \end{cases} $$

For the trajectory control, we will show that $\|\chi(t)\|=0$. Now, without loss of generality, we consider the subinterval $(s_i,t_{i+1}]$, to estimate

$$ (L_{g_i}M + L_{q_i}MT + TMK_e(L_{f1}+B^*L_{f2}))\|u-u^*\|_X. $$

$$ \begin{aligned} \|\chi(t)\| &\le \|S_\alpha(t-s_i)\|\|g_i(t,u(t))-g_i(t,\vartheta(t))\| + \int_{t_i}^t \|S_\alpha(s-t_i)\|ds\|q_i(t,u(t))-q_i(t,\vartheta(t))\| \\ &\quad + \int_{s_i}^t \|T_\alpha(t-s)\|\|f(s,u_{\rho(s,u_s)},Bu_{\rho(s,u_s)}) - f(s,\vartheta_{\rho(s,\vartheta_s)},B\vartheta_{\rho(s,\vartheta_s)})\|ds, \\ &\le L_{g_i}M\|\chi(t)\| + L_{q_i}M\int_{t_i}^t |\chi(t)\|ds + MK_e(L_{f1}+B^*L_{f2})\int_{t_i}^t \|\chi(t)\|ds \\ &= L_{g_i}M\|\chi(t)\| + [L_{q_i}M + MK_e(L_{f1}+B^*L_{f2})]\int_{t_i}^t \|\chi(t)\|ds \\ &= \Phi\|\chi(t)\| + \Psi\int_{t_i}^t \|\chi(s)\|ds, \end{aligned} $$

where $\Phi = L_{g_i}M$, $\Psi = [L_{q_i}M + MK_e(L_{f1}+B^*L_{f2})]$ are constants. Now, applying Gronwall's inequality, we get

$$\chi(t) = 0.$$

Hence $u(t) = \vartheta(t)$ almost everywhere. Thus, the control problem (1.7)-(1.9) is $\mathcal{T}-$controllable. □

5 Examples

This section contains examples to validate the derived results (existence and $\mathcal{T}-$controllability) of the considered systems.

5.1 Example

To prove the theoretical existence result, we shall consider the following impulsive fractional order partial differential inclusion of the form

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} \in \frac{\partial^2 u(t,x)}{\partial y^2} + \int_{-\infty}^{t} e^{2(s-t)} \cos\left(\frac{u(s-\rho_1(s)\rho_2(\|u\|),x)}{16}\right) ds, \quad t \neq \frac{1}{2}, \quad (5.1)$$

$$u(t,0) = u(t,\pi) = 0; \quad u'(t,0) = u'(t,\pi) = 0, \quad t \geq 0, \quad (5.2)$$

$$u(t,x) = \phi(t,x); \quad u'(0,x) = u_0, \quad t \in (-\infty,0], \quad x \in [0,\pi], \quad (5.3)$$

$$\Delta u|_{t=\frac{1}{2}} = \int_{-\infty}^{\frac{1}{2}} g\left(\frac{1}{2}-s\right) u(s,x)\,ds; \quad \Delta u'|_{t=\frac{1}{2}} = \int_{-\infty}^{\frac{1}{2}} q\left(\frac{1}{2}-s\right) u(s,x)\,ds, \quad (5.4)$$

are fixed numbers and $\phi(t) \in \mathfrak{B}_e$. Let $X = L^2[0,\pi]$ and define the operator $A : D(A) \subset X \to X$ by $Aw = w''$ with the domain $D(A) := \{w \in X : \ w, w' \text{ are absolutely continuous, } w'' \in X, w(0) = 0 = w(\pi)\}$. Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A),$$

where $w_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in X and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t} (\omega, \omega_n)\omega_n, \quad \text{for all } \omega \in X, \quad \text{and every } t > 0.$$

Let $h(s) = e^{2s},\ s < 0$ then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2} < \infty$, for $t \in (-\infty, 0]$ and define

$$\|\phi\|_{\mathfrak{B}_e} = \int_{-\infty}^{0} h(s) \sup_{\theta\in[s,0]} \|\phi(\theta)\|_{L^2} ds.$$

Hence for $(t,\phi) \in [0,1] \times \mathfrak{B}_e$, where $\phi(\theta)(x) = \phi(\theta,x),\ (\theta,x) \in (-\infty,0] \times [0,\pi]$. We assume that $\rho_i : [0,\infty) \to [0,\infty)$, $i = 1,2$, are continuous functions.

Set $u(t)(x) = u(t,x)$, and $\rho(t,\phi) = \rho_1(t)\rho_2(\|\phi(0)\|)$ we have

$$f(t,\phi)(x) = \int_{-\infty}^{0} e^{2(s)} \cos\left(\frac{\phi}{16}\right) ds.$$

Then with above setting the problem (5.1)-(5.4) can be written in the abstract form of equation (1.1)-(1.3). Further, we can estimate

$$\begin{aligned} \|f(t,\phi)(x) - f(t,\varphi)(x)\|_{L^2} &= \left[\int_0^{\pi} \left\{\int_{-\infty}^{0} e^{2(s)} \left\|\cos\left(\frac{\phi}{16}\right) - \cos\left(\frac{\varphi}{16}\right)\right\| ds\right\}^2 dx\right]^{\frac{1}{2}} \\ &\leq \frac{1}{16}\left[\int_0^{\pi} \left\{\int_{-\infty}^{0} e^{2(s)}(\|\phi-\varphi\|_{L^2}) ds\right\}^2 dx\right]^{\frac{1}{2}} \leq \frac{\sqrt{\pi}}{16}\|\phi-\varphi\|_{\mathfrak{B}_e}. \end{aligned}$$

This shows that the multivalued map f follows the assumption H_2. This implies that there exists at least one mild solution of problem (5.1)-(5.4).

5.2 Example

Consider the following fractional order functional differential equation

$$\begin{aligned} \frac{\partial^\alpha}{\partial t^\alpha} u(t,x) &= \frac{\partial^2}{\partial y^2} u(t,x) + \int_{-\infty}^{t} e^{2(\nu - t)} \frac{u(\nu - \sigma(\|u\|), x)}{24} \, d\nu \\ &+ \int_0^t \cos(t-s) \int_{-\infty}^{\xi} e^{2(\nu-\xi)} \frac{u(\nu - \sigma(\|u\|), x)}{25} \, d\nu \, ds, \\ &\quad (t,x) \in \cup_{i=1}^{N} [s_i, t_{i+1}] \times [0,\pi], \quad (5.5) \\ u(t,0) &= u(t,\pi) = 0, \quad t \geq 0, \quad (5.6) \\ u(t,x) + \sum_{k=1}^{r} c_k u(s_k, x) &= \phi(t,x), \quad t \in (-\infty, 0]; \quad u'(t,x) = 0, \quad x \in [0,\pi], \quad (5.7) \\ u(t,x) &= G_i(t,y); \quad u'(t,x) = H_i(t,y), \quad t \in (t_i, s_i], \quad (5.8) \end{aligned}$$

are fixed numbers and $\phi \in \mathfrak{B}_e$. Setting $u(t)(x) = u(t,x)$, and

$$\rho(t,\phi) = t - \sigma(\|\phi(0)\|), \quad (t,\phi) \in [0,T] \times \mathfrak{B}_e,$$

we have

$$f(t,\phi,B\phi) = \int_{-\infty}^{0} e^{2(\nu)} \frac{\phi}{24} d\nu + \int_0^t \cos(t-s) \int_{-\infty}^{0} e^{2(\nu)} \frac{\phi}{25} \, d\nu \, ds,$$

$$g_i(t,y) = G_i(t,y); \quad q_i(t,y) = H_i(t,y), \quad G(y) = \sum_{k=1}^{r} c_k u(s_k, x).$$

Then the above equations (5.5)-(5.8) can be written in the abstract form as (1.4)-(1.6). Furthermore, we can see that for $(t,\phi,B\phi),(t,\psi,B\psi)\in[0,T]\times\mathfrak{B}_e\times\mathfrak{B}_e$, may verify that

$$\begin{aligned}\|f(t,\phi,B\phi)-f(t,\psi,B\psi)\|_{L^2} &\le \Bigg[\int_0^\pi\left\{\int_{-\infty}^0 e^{2(s)}\left\|\frac{\phi}{24}-\frac{\psi}{24}\right\|ds\right\}^2 dy\\ &\quad+\int_0^\pi\left\{\left\|\int_0^t\cos(t-s)\int_{-\infty}^0 e^{2(\nu)}\frac{\phi}{25}-\frac{\psi}{25}d\nu\,ds\right\|\right\}dy\Bigg]^{1/2}\\ &\le\Bigg[\int_0^\pi\left\{\frac{1}{24}\int_{-\infty}^0 e^{2(s)}\sup\|\phi-\psi\|ds\right\}^2 dy\\ &\quad+\int_0^\pi\left\{\frac{1}{25}\int_{-\infty}^0 e^{2(s)}\sup\|\phi-\psi\|ds\right\}^2 dy\Bigg]^{1/2}\\ &\le\frac{\sqrt{\pi}}{24}\|\phi-\psi\|+\frac{\sqrt{\pi}}{25}\|\phi-\psi\|.\end{aligned}$$

Hence, function f satisfies (H_3). Similarly, we can show that the functions g_i, q_i, $h(y)$ satisfy (H_4). All the condition of Theorem 3.2 have fulfilled, so we deduced that the system (5.5)-(5.8) has a unique mild solution on $[0,T]$.

5.3 Example

Consider the following example for fractional functional ordinary differential equation

$$\begin{aligned}{}_0^C D_t^\alpha u(t) &= u(t)+\frac{e^t u(t-\sigma(u(t)))+2}{1+u^2(t-\sigma(u(t)))}+\int_0^t\sin(t-s)u(s-\sigma(u(s)))ds,\quad t\in(0,1], &(5.9)\\ u(t) &+ \sum_{k=1}^r c_k u(s_k)=\frac{1}{2},\quad t\in(-\infty,0],\quad u'(t)=0, &(5.10)\\ u(t) &= \frac{u(t)}{16(1+u(t))};\quad u'(t)=\frac{u(t)}{25(1+u(t))},\quad t\in(1,2], &(5.11)\end{aligned}$$

where ${}_0^C D_t^\alpha$ is classical Caputo's fractional derivative of order $\alpha\in(1,2), 0=t_0=s_0<t_1=1<s_1=2$ are prefixed numbers and $\frac{1}{2}\in\mathfrak{B}_e$. Setting

$$\begin{aligned}\rho(t,\varphi) &= t-\sigma(\varphi(0)),\\ f(t,\varphi,B\varphi) &= \frac{e^t\varphi+2}{1+\varphi^2}+\int_0^t\sin(t-s)\varphi\,ds,\\ g_i(t,y) &= \frac{u(t)}{16(1+u(t))};\quad q_i(t,y)=\frac{u(t)}{25(1+u(t))},\quad G(y)=\sum_{k=1}^r c_k u(s_k),\end{aligned}$$

then the problem (5.9)-(5.11) can be written in the abstract form as (1.4)-(1.6), which implies that the system (5.9)-(5.11) has a unique mild solution on $[0,2]$.

5.4 Example

Consider the following control system

$$\begin{aligned}\frac{\partial^\alpha}{\partial t^\alpha}u(t,x) &= \frac{\partial^2}{\partial y^2}u(t,x)+\int_{-\infty}^{t} e^{4(\nu-t)}\frac{u(\nu-\sigma(\|u\|),x)}{12}\,d\nu+14\varpi(t,x) \\ &+ \int_0^t \sin(t-s)\int_{-\infty}^{\xi} e^{4(\nu-\xi)}\frac{u(\nu-\sigma(\|u\|),x)}{28}\,d\nu\,ds, \quad (t,x)\in\cup_{i=1}^{N}[s_i,t_{i+1}]\times[0,\pi],\end{aligned} \tag{5.12}$$

with initial, history and impulsive conditions given as (5.6)-(5.8). With these settings as given in example 5.2, the problem (5.12) with conditions (5.6)-(5.8) can be written in the abstract form of equation (1.7)-(1.9). Therefore the problem (5.12) is $\mathcal{T}-$controllable on J.

Thus, examples provided in this paper demonstrate the authenticity of our results. In first example, we considered fractional order partial differential inclusion with instantaneous impulsive and showed that considered problem has least one mild solution. Non-instantaneous impulse with partial derivative and nonlocal condition is taken in second examples and proved that there exists a unique mild solution for it. In third example, we considered the functional ordinary differential equation with infinite delay and demonstrate the uniqueness of mild solution for the system.

6 Conclusion

In this investigation, we observed that the Definition 2.8 is more reasonable and suitable by using the generalized Caputo's derivative in compare to classical and it is generalized form. Furthermore, we have proved the existence, uniqueness and continuous dependence results of mild solutions for fractional differential inclusion and equations with state dependent delay subject to instantaneous and non-instantaneous impulse. We showed $\mathcal{T}-$controllability. Also, we have illustrated the existence and $\mathcal{T}-$controllability theory from some examples.

7 Conflict of Interests

The authors declare that there is no conflict of interest regarding the publication of this article.

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On a class of fractional $p(\cdot,\cdot)-$Laplacian problems with sub-supercritical nonlinearities

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ABSTRACT

This paper is devoted to study a class of nonlocal variable exponent problems involving fractional $p(\cdot,\cdot)$-Laplacian operator. Under appropriate conditions, some new results on the existence and nonexistence of solutions are established via variational approach and Pohozaev's fibering method.

RESUMEN

Este artículo está dedicado al estudio de una clase de problemas no locales con exponente variable que involucran al operador $p(\cdot,\cdot)$-Laplaciano fraccionario. Bajo condiciones apropiadas se establecen algunos resultados nuevos sobre la existencia y no existencia de soluciones a través de un enfoque variacional y el método de fibración de Pohozaev.

Keywords and Phrases: Fractional $p(\cdot,\cdot)$-Laplacian operator; sub-supercritical nonlinearities; variational methods; Pohozaev's fibering method.

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1 Introduction

In the present paper, we are interested in the existence of solutions for the following problem

$$\begin{cases} M\left(T_u\right)\left(-\Delta_{p(\cdot,\cdot)}\right)^s u + w(x)|u|^{p(x,x)-2}u = \lambda a(x)|u|^{q(x)-2}u - \varepsilon b(x)|u|^{r(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \qquad (P_{\lambda,\varepsilon}^M)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p : \overline{\Omega}\times\overline{\Omega} \to (1,+\infty)$, $q, r : \overline{\Omega} \to (1,+\infty)$ are continuous functions, $s \in (0,1)$ with $N > sp(x,y)$ for all $(x,y) \in \overline{\Omega}$, $\lambda, \varepsilon > 0$ are parameters, $a, b, w \in L^\infty(\Omega)$, M models a Kirchhoff coefficient,

$$T_u = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx\, dy$$

and $\left(-\Delta_{p(\cdot,\cdot)}\right)^s$ is the fractional $p(\cdot,\cdot)$-Laplacian defined as

$$\left(-\Delta_{p(x,\cdot)}\right)^s u(x) = p.v. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}} dy, \quad x \in \mathbb{R}^N,$$

where $p.v.$ is used as abbreviation in the principal value sense.

In the past few decades, nonlinear problems involving nonlocal and pseudo-differential operators have gained considerable popularity and importance. The interest in investigating such problems is stimulated by their applications in numerous fields of applied sciences, such as the description of some phenomena in physics and engineering, population dynamics, finance, chemical reaction design, optimization, minimal surfaces and game theory (see [12, 29, 32, 38]). Moreover, differential equations and variational problems with variable exponent have a strong physical motivation. As can be seen in [5, 22, 35], they emerge from the mathematical description of the dynamics fluids like the electrorhelogical and the thermorheological. They also appear in elastic mechanics, image restoration and biology (see [14, 16, 37, 43]). Some recent results on $p(\cdot,\cdot)$-Laplacian problems can be found in [1, 4, 6, 13, 15, 19, 25, 27, 30, 36, 42].

Recently, great attention has been focused in extending some results on $p(\cdot,\cdot)$-Laplacian problems to the fractional case. For example, we cite [11, 26]. In [26] Kaufmann *et al.* introduced the fractional Sobolev space with variable exponent, and established the existence and uniqueness of solutions for the fractional $p(\cdot,\cdot)$-Laplacian problem

$$\begin{cases} \left(-\Delta_{p(\cdot,\cdot)}\right)^s u + |u|^{q(x)-2}u = f(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Bahrouni *et al.* [11] established some results on the following fractional $p(\cdot,\cdot)$-Laplacian equation with the nonlocal Robin boundary condition

$$\begin{cases} \left(-\Delta_{p(\cdot,\cdot)}\right)^s u + |u|^{p(x,x)-2}u = f(x,u) & \text{in} \quad \Omega \\ \mathcal{N}_{s,p(\cdot,\cdot)}u + \beta(x)|u|^{p(x,x)-2}u = g(x) & \text{on} \quad \mathbb{R}^N \backslash \overline{\Omega}, \end{cases}$$

where $\mathcal{N}_{s,p(\cdot,\cdot)}$ is the nonlinear modification of the following Neumann boundary condition

$$\mathcal{N}_s u(x) := c_{N,s} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N \backslash \overline{\Omega},$$

which was first introduced by Dipierro *et al.* in [17]. The latter nonlocal normal derivative is used in [18] to describe the diffusion of a biological population living in an ecological niche and subject to both local and nonlocal dispersals.

We also refer the reader to [9, 10, 23, 24] for more information.

Problem $(P^M_{\lambda,\varepsilon})$ is a fractional version related to the following hyperbolic equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

which was initially introduced by Kirchhoff [28] as a generalization of the classical D'Alembert wave equation taking into consideration the change in length of the strings produced by transverse vibrations. For additional discussions and physical phenomena described by nonlinear vibration theory, we mention [31]. It was mainly after the work [21], where Fiscella and Valdinoci proposed a stationary fractional Kirchhoff model, that the existence and multiplicity of solutions for Kirchhoff-type fractional p-Laplacian and $p(\cdot,\cdot)$-Laplacian problems were well investigated by many authors, one can see [8, 34, 39, 41, 44]. In particular, Zhang *et al.* [41] studied the following problem

$$\begin{cases} M\left(T_u\right) \left(-\Delta_{p(\cdot,\cdot)}\right)^s u = f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.1}$$

By means of variational methods and mountain pass theorem, they proved the existence of at least one nontrivial solution for (1.1). In [2], Akkoyunlu and Ayazoglu considered the following fractional p-Kirchhoff problem with potential

$$M(||u||^p) \left((-\Delta)^s_p u + V(x)|u|^{p-2}u\right) = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where

$$||u||^p = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy + \int_{\mathbb{R}^N} V(x)|u|^p dx.$$

By using the variational approach, (S_+) mapping theory and Krasnoselskii's genus theory, the authors have established the existence of infinitely many nontrivial weak solutions. After that, the equation (1.2) was generalized by Ayazoglu *et al.* in [7] considering the following fractional Schrödinger-Kirchhoff equation

$$M(A_{s,q(\cdot),p(\cdot,\cdot)}(u))\left((-\Delta)^s_{p(\cdot,\cdot)}u+V(x)|u|^{q(x)-2}u\right)=f(x,u)\quad \text{in } \mathbb{R}^N,$$

where

$$A_{s,q(\cdot),p(\cdot,\cdot)}(u)=\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy+\int_{\mathbb{R}^N}\frac{V(x)}{q(x)}|u|^{q(x)}dx,$$

$N\geq 2$, $M:(0,+\infty)\to(1,\infty)$ is a continuous and monotone Kirchhoff function, $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and V is a potential function. They obtained the existence and multiplicity of solutions by applying the variational approach combined with Mountain Pass Theorem and Krasnoselskii's genus theory.

Inspired by the above cited papers, we will consider problem $(P^M_{\lambda,\varepsilon})$ with sub-supercritical nonlinearities, and prove the existence of solutions via the variational methods combined with the fibering method that was introduced by Pohozaev [33]. We also give the behavior of the solution for problem $(P_{\lambda,\varepsilon})$, and so of the energy functional associated, as $\varepsilon\to 0$. The Pohozaev's fibering method is centered on representing solutions in the form $u=tv$, where t is a real number ($t\neq 0$), and $v\in X\setminus\{0\}$, satisfying the condition:

$$\frac{\partial\Phi}{\partial t}(t,v)=0. \tag{1.3}$$

Here, Φ denotes a functional defined on $\mathbb{R}\times X$. Consequently, the fundamental concept of the Pohozaev's fibering method involves embedding the space X of the original problem within the larger space $\mathbb{R}\times X$ and subsequently exploring the new problem of conditional solvability within the $\mathbb{R}\times X$ space, subject to the condition (1.3).

2 Preliminaries

At first, we give some useful notations and basic results on variable exponent Lebesgue spaces that will be used in proving the main theorems (see [20]). We denote by $C_+(\overline{\Omega})$ the set of all continuous functions $q:\overline{\Omega}\to(1,\infty)$. For $q\in C_+(\overline{\Omega})$, we write

$$q^+:=\max_{x\in\overline{\Omega}}q(x)\quad\text{and}\quad q^-:=\min_{x\in\overline{\Omega}}q(x).$$

Define the variable exponent Lebesgue space as follows:

$$L^{q(\cdot)}(\Omega)=\left\{u:\Omega\to\mathbb{R}\text{ measurable }:\int_\Omega|u|^{q(x)}dx<\infty\right\}.$$

$L^{q(\cdot)}(\Omega)$ endowed with the norm

$$||u||_{q(\cdot)} = \inf\left\{\tau > 0 : \int_\Omega \left|\frac{u}{\tau}\right|^{q(x)} dx \leq 1\right\}.$$

is a separable and reflexive Banach space. Let $L^{q'(\cdot)}(\Omega)$ be the conjugate space of $L^{q(\cdot)}(\Omega)$ with $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$. Then the following Hölder-type inequality holds.

Lemma 2.1 ([20])**.** *Let $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{q'(\cdot)}(\Omega)$. Then*

$$\int_\Omega |uv|\, dx \leq \left(\frac{1}{q^-} + \frac{1}{(q')^-}\right) ||u||_{q(\cdot)}||v||_{q'(\cdot)}.$$

On the space $L^{q(\cdot)}(\Omega)$, we consider the modular function given by

$$\rho_{q(\cdot)}(u) = \int_\Omega |u|^{q(x)} dx.$$

Lemma 2.2 ([20])**.** *For any $u \in L^{q(\cdot)}(\Omega)$, we have*

$$\min\left(||u||_{q(\cdot)}^{q^-}, ||u||_{q(\cdot)}^{q^+}\right) \leq \rho_{q(\cdot)}(u) \leq \max\left(||u||_{q(\cdot)}^{q^-}, ||u||_{q(\cdot)}^{q^+}\right).$$

Lemma 2.3 ([20])**.** *Let $u \in L^{q(\cdot)}(\Omega)$ and $\{u_n\} \subset L^{q(\cdot)}(\Omega)$. Then the following properties are equivalent:*

(1) $\lim_{n\to\infty} ||u_n - u||_{q(\cdot)} = 0$;

(2) $\lim_{n\to\infty} \rho_{q(\cdot)}(u_n - u) = 0$.

Lemma 2.4 ([3])**.** *Let $q, r \in C_+(\overline{\Omega})$ with $q(x) \leq r(x)$ in Ω and $u \in L^{r(\cdot)}(\Omega)$. Then $|u|^{q(\cdot)} \in L^{\frac{r(\cdot)}{q(\cdot)}}(\Omega)$ and*

$$\||u|^{q(\cdot)}\|_{\frac{r(\cdot)}{q(\cdot)}} \leq \max\left(\|u\|_{r(\cdot)}^{q^+}, \|u\|_{r(\cdot)}^{q^-}\right).$$

Next, we define the convenient variable exponent fractional Sobolev space to supply a variational structure for handling our problems. Let $p : \overline{\Omega} \times \overline{\Omega} \to (1, \infty)$ be as mentioned above and put

$$\overline{p}(x) = p(x, x) \quad \text{for all} \quad x \in \overline{\Omega}.$$

Let $W^{s,p(\cdot,\cdot)}(\Omega)$ be the variable exponent fractional Sobolev space defined as follows:

$$\mathbb{W} := W^{s,p(\cdot,\cdot)}(\Omega) = \left\{u \in L^{\overline{p}(\cdot)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\xi^{p(x,y)}|x - y|^{N+sp(x,y)}}\, dx\, dy < \infty, \text{ for some } \xi > 0\right\}.$$

Equip $\mathbb{W}$ with the norm

$$||u||_{\mathbb{W}} = [u]_{\mathbb{W}} + ||u||_{\overline{p}(\cdot)},$$

where

$$[u]_{\mathbb{W}} = \inf\left\{\xi > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\xi^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy \le 1\right\}.$$

Then $(\mathbb{W}, ||u||_{\mathbb{W}})$ is a Banach space. For any $u \in \mathbb{W}$, we set

$$\rho_{p,\overline{p}}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_{\Omega} |u|^{\overline{p}(x)}dx$$

and

$$||u||_{p,\overline{p}} = \inf\left\{\xi > 0 : \rho_{p,\overline{p}}\left(\frac{u}{\xi}\right) \le 1\right\}.$$

The norm $||\cdot||_{p,\overline{p}}$ is equivalent to $||\cdot||_{\mathbb{W}}$. Furthermore, from [41, Lemma 2.2], $(\mathbb{W}, ||\cdot||_{\mathbb{W}})$ is uniformly convex and hence $\mathbb{W}$ is a reflexive Banach space. The following lemma states the compactness of the embedding $\mathbb{W}$ into the variable exponent Lebesgue spaces.

Lemma 2.5 ([40, 41])**.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $s \in (0,1)$. Assume that $p : \overline{\Omega}\times\overline{\Omega} \to (1,\infty)$ is continuous and symmetric (i.e. $p(x,y) = p(y,x)$) with $sp(x,y) < N$ for all $x, y \in \overline{\Omega}$. Let $q \in C_+(\overline{\Omega})$ such that*

$$q(x) < p_s^*(x) := \frac{N\overline{p}(x)}{N - s\overline{p}(x)} \quad \textit{for all} \quad x \in \overline{\Omega}.$$

Then, there exists $C = C(N, s, p, q, \Omega)$ such that

$$||u||_{q(\cdot)} \le C||u||_{\mathbb{W}} \quad \textit{for all} \quad u \in \mathbb{W}.$$

Therefore, the space $\mathbb{W}$ is continuously embedded into $L^{q(\cdot)}(\Omega)$. Moreover, this embedding is compact.

Due to the presence of the Dirichlet boundary condition $u = 0$ in $\mathbb{R}^N\backslash\Omega$, we need to encode this condition in the weak formulation of $(P^M_{\lambda,\varepsilon})$ and $(P_{\lambda,\varepsilon})$. For this, let us define the new space

$$\mathbb{X} := X^{s,p(\cdot,\cdot)}(\Omega) = \left\{u : \mathbb{R}^N \to \mathbb{R},\ u|_\Omega \in L^{\overline{p}(\cdot)}(\Omega),\ \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\xi^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < \infty,\ \text{for some } \xi > 0\right\},$$

where $\mathcal{Q} = \mathbb{R}^N \times \mathbb{R}^N\backslash(\Omega^c \times \Omega^c)$. Endow $\mathbb{X}$ with the norm

$$||u||_{\mathbb{X}} = [u]_{\mathbb{X}} + ||u||_{\overline{p}(\cdot)},$$

where

$$[u]_{\mathbb{X}} = \inf\left\{\xi > 0 : \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\xi^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy \le 1\right\}.$$

In the same way $(\mathbb{X}, ||\cdot||_{\mathbb{X}})$ is a separable reflexive Banach space.

Since the variable exponent p, $\overline{p}$ and q are continuous, we can extend p to $\mathbb{R}^N \times \mathbb{R}^N$ and $\overline{p}, q$ to

$\mathbb{R}^N$ continuously with conditions given in Lemma 2.5. Let $\mathbb{X}_0$ be the linear space:

$$\mathbb{X}_0 = \{u \in \mathbb{X} : u = 0 \quad \text{a.e. in } \mathbb{R}^N \backslash \Omega\}$$

equipped with the norm

$$||u||_{\mathbb{X}_0} = [u]_{\mathbb{X}} = \inf\left\{\xi > 0 : \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{\xi^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dx\, dy \leq 1\right\}.$$

Obviously, $(\mathbb{X}_0, ||\cdot||_{\mathbb{X}_0})$ is a reflexive Banach space. Set

$$\rho_0(u) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy \quad \text{for all } u \in \mathbb{X}_0.$$

Lemma 2.6 ([41])**.** *For all $u, u_n \in \mathbb{X}_0$, the following properties hold true:*

(1) $||u||_{\mathbb{X}_0} > 1 \implies ||u||_{\mathbb{X}_0}^{p^-} \leq \rho_0(u) \leq ||u||_{\mathbb{X}_0}^{p^+}$;

(2) $||u||_{\mathbb{X}_0} \leq 1 \implies ||u||_{\mathbb{X}_0}^{p^+} \leq \rho_0(u) \leq ||u||_{\mathbb{X}_0}^{p^-}$;

(3) $||u_n - u||_{\mathbb{X}_0} \to 0 \iff \rho_0(u_n - u) \to 0.$

Lemma 2.7 ([41])**.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $s \in (0,1)$. Assume that $p : \overline{\Omega} \times \overline{\Omega} \to (1,\infty)$ is continuous and symmetric with $sp(x,y) < N$ for all $x, y \in \overline{\Omega}$. Let $q \in C_+(\overline{\Omega})$ such that*

$$q(x) < p_s^*(x) := \frac{N\overline{p}(x)}{N - s\overline{p}(x)} \quad \textit{for all} \quad x \in \overline{\Omega}.$$

Then, there exists $C = C(N, s, p, q, \Omega) > 0$ such that

$$||u||_{q(\cdot)} \leq C||u||_{\mathbb{X}_0} \quad \textit{for all} \quad u \in \mathbb{X}_0.$$

Therefore, the space $\mathbb{X}_0$ is continuously embedded into $L^{q(\cdot)}(\Omega)$. Moreover, this embedding is compact.

Remark 2.8. *Since $1 < \overline{p}(x) = p(x,x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, by Lemma 2.7, the norms $||\cdot||_{\mathbb{X}_0}$ and $||\cdot||_{\mathbb{X}}$ are equivalent in $\mathbb{X}_0$.*

We look for solutions of problems $(P_{\lambda,\varepsilon}^M)$ and $(P_{\lambda,\varepsilon})$ in the separable reflexive Banach space $X = \mathbb{X}_0 \cap L^{r(\cdot)}(\Omega)$ which is equipped with the norm

$$||u||_X = ||u||_{\mathbb{X}} + ||u||_{r(\cdot)}.$$

3 Hypotheses and main results

Before stating what we believe that are the main contributions, we first list some assumptions on the data of $(P^M_{\lambda,\varepsilon})$. Concerning the Kirchhoff function $M : \mathbb{R}^+ \to \mathbb{R}^+$, we use the following two assumptions:

(M_0) M is a C^1 nondecreasing function;

(M_1) M is a continuous function such that $M(t) \geq m_0 > 0$ for all $t > 0$.

For the functions a, b, w, p, q and r, we make the following hypotheses:

(H_1) $q, r : \overline{\Omega} \to (1, \infty)$ and $p : \overline{\Omega} \times \overline{\Omega} \to (1, \infty)$ are continuous such that $sp(x, y) < N$, $p(x, y) = p(y, x)$ and $q(x) < p^*_s(x) < r^- := \min_{x \in \overline{\Omega}} r(x)$ for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$, where

$$p^*_s(x) := \frac{Np(x, x)}{N - sp(x, x)};$$

(H_2) $a, b, w \in L^\infty(\Omega)$ with b and w are nonnegative and $|\Omega^+_a| > 0$, where $\Omega^+_a = \{x \in \Omega : a(x) > 0\}$;

(H_3) $ab^{-\frac{q(\cdot)}{r(\cdot)}} \in L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\Omega^+_a)$;

(H_4) $q^-(r^- - q^+) < p^+(r^- - p^-)$ and $r^+ \leq \min\left\{ \frac{q^- p^+ (q^+ - p^-)}{p^+(r^- - p^-) - q^-(r^- - q^+)}, \frac{q^-(r^- - p^-)}{q^+ - p^-} \right\}$;

The main results can be stated as follows.

Theorem 3.1. *Assume that $(M_0) - (M_1)$ and $(H_1) - (H_2)$ hold. If $q^+ < p^-$, then problem $(P^M_{\lambda,\varepsilon})$ admits at least one nontrivial solution.*

Theorem 3.2. *Assume that (M_1) and $(H_1) - (H_2)$ hold. If $p^+ < q^-$, $a(x) \geq 0$ for a.e. $x \in \Omega$ and $b(x) > b_0 > 0$ for a.e. $x \in \Omega$, then for all $\varepsilon > 0$ there exists $\lambda_\varepsilon > 0$ such that problem $(P^M_{\lambda,\varepsilon})$ has no nontrivial solution for all $\lambda \in (0, \lambda_\varepsilon)$.*

The following two theorems concern problem $(P^M_{\lambda,\varepsilon})$ with $M \equiv 1$, that is,

$$\begin{cases} \left(-\Delta_{p(\cdot,\cdot)}\right)^s u + w(x)|u|^{p(x,x)-2}u = \lambda a(x)|u|^{q(x)-2}u - \varepsilon b(x)|u|^{r(x)-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \qquad (P_{\lambda,\varepsilon})$$

Theorem 3.3. *Assume that $(H_1) - (H_4)$ hold. If $q(\cdot) = q$ is constant with $p^+ < q$ or $p(\cdot)$ and $r(\cdot)$ are constants, then for all $\varepsilon > 0$ there exists $\lambda^*_\varepsilon > 0$ such that problem $(P_{\lambda,\varepsilon})$ admits at least one nontrivial solution provided $\lambda > \lambda^*_\varepsilon$.*

Theorem 3.4. *Assume that $(H_1) - (H_4)$ hold and $q(\cdot) = q$ is constant with $p^+ < q$ or $p(\cdot)$ and $r(\cdot)$ are constants. Let $\varepsilon_0 > 0$ and $\lambda > \lambda^*_{\varepsilon_0}$. Then, there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that for all*

$\varepsilon \in (0, \varepsilon_1)$, problem $(P_{\lambda,\varepsilon})$ admits at least one nontrivial solution u_ε verifying $||u_\varepsilon||_X \to +\infty$ and $\mathcal{I}_\varepsilon(u_\varepsilon) \to -\infty$ as $\varepsilon \to 0$, where $\mathcal{I}_\varepsilon$ is the associated energy functional to $(P_{\lambda,\varepsilon})$.

Remark 3.5. *The conclusions of Theorems 3.1 and 3.2 also hold for problem $(P_{\lambda,\varepsilon})$.*

4 Proof of theorems

Proof of Theorem 3.1. It is well known that the weak solution of $(P^M_{\lambda,\varepsilon})$ corresponds to the critical point of the energy functional defined on X by

$$\begin{aligned}\mathcal{I}_\varepsilon(u) &= \widehat{M}\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right) + \int_\Omega \frac{w(x)}{\overline{p}(x)}|u|^{\overline{p}(x)}dx \\ &\quad -\lambda\int_\Omega \frac{a(x)}{q(x)}|u|^{q(x)}dx + \varepsilon\int_\Omega \frac{b(x)}{r(x)}|u|^{r(x)}dx, \end{aligned} \tag{4.1}$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$. By standard arguments, one can verify that $\mathcal{I}_\varepsilon \in C^1(X,\mathbb{R})$. For any $(t,v) \in (0,\infty)\times X$, we define

$$\begin{aligned}\Phi_\varepsilon(t,v) &:= \mathcal{I}_\varepsilon(tv) \\ &= \widehat{M}\left(\int_{\mathbb{R}^{2N}} t^{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right) + \int_\Omega \frac{w(x)}{\overline{p}(x)}t^{\overline{p}(x)}|v|^{\overline{p}(x)}dx \\ &\quad -\lambda\int_\Omega \frac{a(x)}{q(x)}t^{q(x)}|v|^{q(x)}dx + \varepsilon\int_\Omega \frac{b(x)}{r(x)}t^{r(x)}|v|^{r(x)}dx.\end{aligned}$$

Observe that if $u = tv$ is a nontrivial critical of $\mathcal{I}_\varepsilon$, then $\dfrac{\partial \Phi_\varepsilon}{\partial t}(t,v) = 0$. Moreover, if for each $v \in X\setminus\{0\}$, there is a unique $t = t(v)$ satisfying

$$\frac{\partial \Phi_\varepsilon}{\partial t}(t,v) = 0 \tag{4.2}$$

and $t : v \mapsto t(v)$ is continuously differentiable on $X\setminus\{0\}$, we can infer that

$$\widetilde{\mathcal{I}}_\varepsilon(v) := \mathcal{I}_\varepsilon(t(v)v)$$

is a well-defined C^1 functional. The following result plays a key role in the proof of our main theorem.

Lemma 4.1 ([33])**.** *Let $\Psi : X \to \mathbb{R}$ be a functional of class C^1 on $X\setminus\{0\}$ verifying*

$$\langle \Psi'(v), v\rangle \neq 0 \quad \textit{if} \quad \Psi(v) = 1.$$

If v is a conditional critical point of $\widetilde{I}_\varepsilon$ under the constraint $\Psi(v) = 1$, then $u := t(v)v$ is a critical point of $\mathcal{I}_\varepsilon$.

Consider the functional $\Psi_\varepsilon : X \to \mathbb{R}$ given by

$$\begin{aligned}\Psi_\varepsilon(v) &= M\left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad + \int_\Omega w(x)|v|^{\overline{p}(x)}dx + \varepsilon\int_\Omega b(x)|v|^{r(x)}dx. \qquad (4.3)\end{aligned}$$

It is obvious that Ψ_ε satisfies hypotheses of Lemma 4.1, therefore the problem of finding solutions of $(P^M_{\lambda,\varepsilon})$ will be reduced to that of locating the critical points of $\widetilde{I}_\varepsilon$ on the set

$$\mathcal{U}_\varepsilon = \{v \in X : \Psi_\varepsilon(v) = 1\}.$$

Note that (4.2) is equivalent to

$$\begin{aligned}\varphi_v(t) &:= M\left(\int_{\mathbb{R}^{2N}} \frac{t^{p(x,y)}|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t^{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad + \int_\Omega t^{\overline{p}(x)}w(x)|v|^{\overline{p}(x)}dx + \varepsilon\int_\Omega t^{r(x)}b(x)|v|^{r(x)}dx - \lambda\int_\Omega t^{q(x)}a(x)|v|^{q(x)}dx \\ &= 0. \qquad (4.4)\end{aligned}$$

Let

$$\Theta_a := \left\{v \in X : \int_\Omega a(x)|v|^{q(x)}dx > 0\right\}.$$

Claim 4.2. *For any $v \in \Theta_a$, equation (4.4) admits a unique positive solution $t(v)$. Moreover, $\varphi_v(t) < 0$ for all $t < t(v)$ and $\varphi_v(t) > 0$ for all $t > t(v)$.*

Indeed, by (M_0), for all $t \geq 1$,

$$\begin{aligned}\varphi_v(t) &\geq t^{p^-}M\left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad + t^{p^-}\int_\Omega w(x)|v|^{\overline{p}(x)}dx + \varepsilon t^{r^-}\int_\Omega b(x)|v|^{r(x)}dx - \lambda t^{q^+}\int_\Omega a(x)|v|^{q(x)}dx\end{aligned}$$

and for all $0 < t \leq 1$,

$$\begin{aligned}\varphi_v(t) &\leq t^{p^-}M\left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad + t^{p^-}\int_\Omega w(x)|v|^{\overline{p}(x)}dx + \varepsilon t^{r^-}\int_\Omega b(x)|v|^{r(x)}dx - \lambda t^{q^+}\int_\Omega a(x)|v|^{q(x)}dx\end{aligned}$$

Since $q^+ < p^-$, we can choose $t_\infty > 1$ such that $\varphi_v(t_\infty) > 0$ and by (H_2), we can find $0 < t_0 \leq 1$ satisfying $\varphi_v(t_0) \leq 0$. Therefore, by virtue of the continuity of φ_v, equation (4.4) has at least one solution $t(v) > 0$. The uniqueness of $t(v)$ follows from (H_2) and using the fact that $q^+ < p^-$ and

M is nondecreasing. Furthermore, for all $t < t(v)$,

$$\begin{aligned} &M\left(\int_{\mathbb{R}^{2N}} \frac{t^{p(x,y)}|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t^{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &+\int_\Omega t^{\overline{p}(x)}w(x)|v|^{\overline{p}(x)}dx+\varepsilon\int_\Omega t^{r(x)}b(x)|v|^{r(x)}dx \\ &<\lambda\int_\Omega t^{q(x)}a(x)|v|^{q(x)}dx \end{aligned} \tag{4.5}$$

and for all $t > t(v)$,

$$\begin{aligned} &M\left(\int_{\mathbb{R}^{2N}} \frac{t^{p(x,y)}|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t^{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &+\int_\Omega t^{\overline{p}(x)}w(x)|v|^{\overline{p}(x)}dx+\varepsilon\int_\Omega t^{r(x)}b(x)|v|^{r(x)}dx \\ &>\lambda\int_\Omega t^{q(x)}a(x)|v|^{q(x)}dx. \end{aligned} \tag{4.6}$$

Then, the function $t : v \mapsto t(v)$ is well defined, and by applying the implicit function theorem, we deduce that $t(\cdot) \in C^1(X\setminus\{0\},(0,+\infty))$. If $v \in \mathcal{U}_\varepsilon \cap \Theta_a$ and $t(v) \geq 1$, it holds from (H_1), the nondecreasing of M and (4.4) that

$$\begin{aligned} t(v)^{p^-} &= t(v)^{p^-}\Psi_\varepsilon(v) \\ &= M\left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t(v)^{p^-}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad +t(v)^{p^-}\int_\Omega w(x)|v|^{\overline{p}(x)}dx+\varepsilon t(v)^{p^-}\int_\Omega b(x)|v|^{r(x)}dx \\ &\leq M\left(\int_{\mathbb{R}^{2N}} t(v)^{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t(v)^{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad +\int_\Omega t(v)^{\overline{p}(x)}w(x)|v|^{\overline{p}(x)}dx+\varepsilon\int_\Omega t(v)^{r(x)}b(x)|v|^{r(x)}dx \\ &= \lambda\int_\Omega t(v)^{q(x)}a(x)|v|^{q(x)}dx \\ &\leq \lambda t(v)^{q^+}\int_\Omega a(x)|v|^{q(x)}dx, \end{aligned}$$

thus

$$t(v)^{p^- - q^+} \leq \lambda\int_\Omega a(x)|v|^{q(x)}dx.$$

This shows that $t(\cdot)$ is bounded in $\mathcal{U}_\varepsilon \cap \Theta_a$. Since M is nondecreasing, $\widehat{M}(\tau) \leq \tau M(\tau)$ for all $\tau \geq 0$.

Then, by (H_1) and (4.4) for any $v \in \mathcal{U}_\varepsilon \cap \Theta_a$, we have

$$\begin{aligned}
\widetilde{I}_\varepsilon(v) &\leq \frac{1}{p^-} M\left(\int_{\mathbb{R}^{2N}} t^{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\, dx\, dy\right) \int_{\mathbb{R}^{2N}} t^{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy \\
&\quad + \frac{1}{p^-} \int_\Omega t^{\overline{p}(x)} w(x)|v|^{\overline{p}(x)} dx + \frac{\varepsilon}{r^-} \int_\Omega t^{r(x)} b(x)|v|^{r(x)} dx - \frac{\lambda}{q^+} \int_\Omega t^{q(x)} a(x)|v|^{q(x)} dx \\
&= \left(\frac{1}{p^-} - \frac{1}{q^+}\right) M\left(\int_{\mathbb{R}^{2N}} t^{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\, dx\, dy\right) \int_{\mathbb{R}^{2N}} t^{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy \\
&\quad + \left(\frac{1}{p^-} - \frac{1}{q^+}\right) \int_\Omega t^{\overline{p}(x)} w(x)|v|^{\overline{p}(x)} dx + \varepsilon\left(\frac{1}{r^-} - \frac{1}{q^+}\right) \int_\Omega t^{r(x)} b(x)|v|^{r(x)} dx \\
&< 0.
\end{aligned}$$

Then

$$\alpha_0 := \inf_{v \in \mathcal{U}_\varepsilon \cap \Theta_a} \widetilde{I}_\varepsilon(v) < 0.$$

Let $\{v_n\} \subset \mathcal{U}_\varepsilon \cap \Theta_a$ be a sequence such $\widetilde{I}_\varepsilon(v_n) \to \alpha_0$. From (M_1), we have

$$1 = \Psi_\varepsilon(v_n) \geq m_0 \int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy,$$

thus from Lemma 2.6, we deduce that $\{v_n\}$ is bounded in $\mathbb{X}_0$. Therefore, up to a subsequence, we may assume that

$$\begin{cases} v_n \rightharpoonup v_0 \text{ in } \mathbb{X}_0, \\ v_n \to v_0 \text{ in } L^{\overline{p}(\cdot)}(\Omega) \text{ and } L^{q(\cdot)}(\Omega), \\ v_n \to v_0 \text{ a.e. in } \Omega. \end{cases} \tag{4.7}$$

We may also assume that $t(v_n) \to t_0$, since $\{t(v_n)\}$ is bounded. Then

$$\widehat{M}\left(\int_{\mathbb{R}^{2N}} t_0^{p(x,y)} \frac{|v_0(x)-v_0(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\, dx\, dy\right) \leq \liminf_{n\to+\infty} \widehat{M}\left(\int_{\mathbb{R}^{2N}} t(v_n)^{p(x,y)} \frac{|v_n(x)-v_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\, dx\, dy\right),$$

$$\lim_{n\to+\infty} \int_\Omega \frac{t(v_n)^{\overline{p}(x)} w(x)}{p(x)} |v_n|^{\overline{p}(x)} dx = \int_\Omega \frac{t_0^{\overline{p}(x)} w(x)}{\overline{p}(x)} |v_0|^{\overline{p}(x)} dx,$$

$$\lim_{n\to+\infty} \int_\Omega \frac{t(v_n)^{q(x)} a(x)}{q(x)} |v_n|^{q(x)} dx = \int_\Omega \frac{t_0^{q(x)} a(x)}{q(x)} |v_0|^{q(x)} dx$$

and

$$\int_\Omega \frac{t_0^{r(x)} b(x)}{r(x)} |v_0|^{r(x)} dx \leq \liminf_{n\to+\infty} \int_\Omega \frac{t(v_n)^{r(x)} b(x)}{r(x)} |v_n|^{r(x)} dx.$$

Therefore

$$\mathcal{I}_\varepsilon(t_0 v_0) \leq \liminf_{n\to+\infty} \mathcal{I}_\varepsilon(t(v_n)v_n) = \liminf_{n\to+\infty} \widetilde{I}_\varepsilon(v_n) = \alpha_0 < 0, \tag{4.8}$$

from which, we deduce that $v_0 \neq 0$ and $t_0 > 0$. Recall that the pair $(t(v_n), v_n)$ verifies (4.4), so by sending n to $+\infty$ and using (4.7), we arrive at

$$M\left(\int_{\mathbb{R}^{2N}} t_0^{p(x,y)} \frac{|v_0(x)-v_0(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t_0^{p(x,y)} \frac{|v_0(x)-v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \tag{4.9}$$

$$+\int_\Omega t_0^{\overline{p}(x)} w(x)|v_0|^{\overline{p}(x)}dx + \varepsilon\int_\Omega t_0^{r(x)} b(x)|v_0|^{r(x)}dx \tag{4.10}$$

$$\le \lambda\int_\Omega t_0^{q(x)} a(x)|v_0|^{q(x)}dx. \tag{4.11}$$

Thus $\int_\Omega a(x)|t_0v_0|^{q(x)}dx > 0$. Furthermore, $t_0v_0 \in L^{r(\cdot)}(\Omega)$, and hence $t_0v_0 \in X$. In view of Claim 4.2 and (4.9), we have $t_0 \le t(v_0)$. Suppose by contradiction that $t_0 < t(v_0)$. Let $\psi_{v_0} : t \mapsto \mathcal{I}_\varepsilon(tv_0)$. Then $t\psi'_{v_0}(t) = \varphi_{v_0}(t)$, therefore by Claim 4.2, $t\psi'_{v_0}(t) < 0$ for all $0 < t < t(v_0)$, which yields that the function ψ_{v_0} is decreasing on $[0, t(v_0)]$. It follows from (4.8) that

$$\widetilde{\mathcal{I}}_\varepsilon(v_0) = \mathcal{I}_\varepsilon(t(v_0)v_0) < \mathcal{I}_\varepsilon(t_0v_0) \le \alpha_0. \tag{4.12}$$

By definition of $t(\cdot)$, for any $\tau > 0$, we have

$$M\left(\int_{\mathbb{R}^{2N}} \frac{t(\tau v_0)^{p(x,y)}|\tau(v_0(x)-v_0(y))|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} t(\tau v_0)^{p(x,y)} \frac{|\tau(v_0(x)-v_0(y))|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy$$
$$+\int_\Omega t(\tau v_0)^{\overline{p}(x)} w(x)|\tau v_0|^{\overline{p}(x)}dx + \varepsilon\int_\Omega t(\tau v_0)^{r(x)} b(x)|\tau v_0|^{r(x)}dx$$
$$= \lambda\int_\Omega t(\tau v_0)^{q(x)} a(x)|\tau v_0|^{q(x)}dx,$$

so that

$$M\left(\int_{\mathbb{R}^{2N}} \frac{(\tau t(\tau v_0))^{p(x,y)}|v_0(x)-v_0(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} (\tau t(\tau v_0))^{p(x,y)} \frac{|v_0(x)-v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy$$
$$+\int_\Omega (\tau t(\tau v_0))^{\overline{p}(x)} w(x)|v_0|^{\overline{p}(x)}dx + \varepsilon\int_\Omega (\tau t(\tau v_0))^{r(x)} b(x)|v_0|^{r(x)}dx$$
$$= \lambda\int_\Omega (\tau t(\tau v_0))^{q(x)} a(x)|v_0|^{q(x)}dx.$$

Hence, by the uniqueness of the solution $t(v_0)$ of equation (4.4), we have

$$\tau t(\tau v_0) = t(v_0). \tag{4.13}$$

We next choose $\tau > 0$ such that $\tau v_0 \in \mathcal{U}_\varepsilon$. From (4.12) and (4.13), we obtain

$$\widetilde{\mathcal{I}}_\varepsilon(\tau v_0) = \mathcal{I}_\varepsilon(t(\tau v_0)\tau v_0) = \mathcal{I}_\varepsilon(t(v_0)v_0) = \widetilde{\mathcal{I}}_\varepsilon(v_0) < \alpha_0,$$

which contradicts the definition of α_0, and consequently $t_0 = t(v_0)$. By (4.8) and (4.13), we have

$$\alpha_0 \le \widetilde{\mathcal{I}}_\varepsilon(\tau v_0) = \mathcal{I}_\varepsilon(t(\tau v_0)\tau v_0) = \mathcal{I}_\varepsilon(t(v_0)v_0) = \widetilde{\mathcal{I}}_\varepsilon(v_0) \le \alpha_0,$$

thus $\widetilde{\mathcal{I}}_\varepsilon(v_0) = \alpha_0$. Hence v_0 is a conditional critical point of $\widetilde{\mathcal{I}}_\varepsilon$. Applying Lemma 4.1, we conclude that $u := t(v_0)v_0$ is a solution of $(P^M_{\lambda,\varepsilon})$. The proof of Theorem 3.1 is finished.

Proof of Theorem 3.2. Suppose that problem $(P^M_{\lambda,\varepsilon})$ has a nontrivial solution u. Then, taking u as a test function,

$$\begin{aligned} &M\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &+\int_\Omega w(x)|u|^{\overline{p}(x)}dx + \varepsilon\int_\Omega b(x)|u|^{r(x)}dx = \lambda\int_\Omega a(x)|u|^{q(x)}dx \end{aligned} \tag{4.14}$$

Since $b(x) > b_0 > 0$, for a.e. $x \in \Omega$, by Young's inequality, we can write

$$\begin{aligned} \lambda\int_\Omega a(x)|u|^{q(x)}dx &\le \varepsilon\int_\Omega \frac{q(x)}{r(x)}b(x)|u|^{r(x)}dx + \int_\Omega \frac{r(x)-q(x)}{r(x)}\varepsilon^{\frac{-q(x)}{r(x)-q(x)}}(\lambda a(x))^{\frac{r(x)}{r(x)-q(x)}}b(x)^{\frac{q(x)}{q(x)-r(x)}}dx \\ &\le \frac{\varepsilon q^+}{r^-}\int_\Omega b(x)|u|^{r(x)}dx + \frac{r^+-q^-}{r^-}\int_\Omega \varepsilon^{\frac{-q(x)}{r(x)-q(x)}}(\lambda a(x))^{\frac{r(x)}{r(x)-q(x)}}b(x)^{\frac{q(x)}{q(x)-r(x)}}dx \\ &\le \frac{\varepsilon q^+}{r^-}\int_\Omega b(x)|u|^{r(x)}dx + \frac{r^+-q^-}{r^-}\varepsilon^{-\kappa}\lambda^\varrho||a||_\infty^\gamma\int_\Omega b(x)^{\frac{r(x)}{r(x)-q(x)}}dx, \end{aligned}$$

where

$$\kappa := \begin{cases} \frac{q^+}{r^--q^+} & \text{if} \quad \varepsilon \le 1 \\ \frac{q^-}{r^+-q^-} & \text{if} \quad \varepsilon > 1, \end{cases} \qquad \varrho := \begin{cases} \frac{r^-}{r^+-q^-} & \text{if} \quad \lambda < 1 \\ \frac{r^+}{r^--q^+} & \text{if} \quad \lambda \ge 1 \end{cases}$$

and

$$\gamma := \begin{cases} \frac{r^-}{r^+-q^-} & \text{if} \quad ||a||_\infty < 1 \\ \frac{r^+}{r^--q^+} & \text{if} \quad ||a||_\infty \ge 1. \end{cases}$$

It holds then from (4.14) that

$$\begin{aligned} &M\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\le \frac{\varepsilon(q^+-r^-)}{r^-}\int_\Omega b(x)|u|^{r(x)}dx + \frac{r^+-q^-}{r^-}\varepsilon^{-\kappa}\lambda^\varrho||a||_\infty^\gamma\int_\Omega b(x)^{\frac{r(x)}{r(x)-q(x)}}dx \\ &\le \frac{r^+-q^-}{r^-}\varepsilon^{-\kappa}\lambda^\varrho||a||_\infty^\gamma\int_\Omega b(x)^{\frac{r(x)}{r(x)-q(x)}}dx, \end{aligned} \tag{4.15}$$

since $q^+ < r^-$. On the other hand, by Lemmas 2.2, 2.6 and 2.7, for some $C_0 > 0$, we have

$$\int_\Omega |u|^{q(x)}dx \le C_0\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\right)^\vartheta, \tag{4.16}$$

where

$$\vartheta := \begin{cases} \frac{q^-}{p^+} & \text{if} \quad ||u||_{q(x)} \leq 1 \text{ and } ||u||_{\mathbb{X}_0} \leq 1 \\ \frac{q^+}{p^+} & \text{if} \quad ||u||_{q(x)} > 1 \text{ and } ||u||_{\mathbb{X}_0} \leq 1 \\ \frac{q^-}{p^-} & \text{if} \quad ||u||_{q(x)} \leq 1 \text{ and } ||u||_{\mathbb{X}_0} > 1 \\ \frac{q^+}{p^-} & \text{if} \quad ||u||_{q(x)} > 1 \text{ and } ||u||_{\mathbb{X}_0} > 1. \end{cases}$$

Note that $\vartheta > 1$, since $p^+ < q^-$. From (M_1), (4.14) and (4.16), we get

$$\begin{aligned} m_0 \left(\frac{1}{C_0 ||a||_\infty} \int_\Omega a(x)|u|^{q(x)} dx \right)^{\frac{1}{\vartheta}} &\leq M \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} \, dx \, dy \right) \\ &\quad \times \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy \\ &\leq \lambda \int_\Omega a(x)|u|^{q(x)} dx, \end{aligned} \tag{4.17}$$

which implies

$$\left(\frac{m_0^\vartheta}{\lambda C_0 ||a||_\infty} \right)^{\frac{1}{\vartheta-1}} \leq m_0 \left(\frac{1}{C_0 ||a||_\infty} \int_\Omega a(x)|u|^{q(x)} dx \right)^{\frac{1}{\vartheta}}. \tag{4.18}$$

Combining (4.15), (4.17) and (4.18), we obtain

$$\left(\frac{m_0^\vartheta}{\lambda C_0 ||a||_\infty} \right)^{\frac{1}{\vartheta-1}} \leq \frac{r^+ - q^-}{r^-} \varepsilon^{-\kappa} \lambda^\varrho ||a||_\infty^\gamma \int_\Omega b(x)^{\frac{r(x)}{r(x)-q(x)}} dx,$$

hence

$$\lambda \geq \lambda_\varepsilon := \left(\frac{r^- \varepsilon^\kappa m_0^{\frac{\vartheta}{\vartheta-1}}}{C_0^{\frac{1}{\vartheta-1}} ||a||_\infty^{\frac{\gamma(\vartheta-1)+1}{\vartheta-1}} (r^+ - q^-) \int_\Omega b(x)^{\frac{r(x)}{r(x)-q(x)}} dx} \right)^{\frac{\vartheta-1}{\varrho(\vartheta-1)+1}},$$

and the proof of Theorem 3.2 is completed.

Proof of Theorem 3.3. Assume $q(\cdot) = q$ is constant. For $v \in \Theta_a$ and $t > 0$, we set

$$\Upsilon_{\varepsilon,v}(t) := \frac{\int_{\mathbb{R}^{2N}} \frac{t^{p(x,y)-q}|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy + \int_\Omega t^{\overline{p}(x)-q} w(x)|v|^{\overline{p}(x)} dx + \varepsilon \int_\Omega t^{r(x)-q} b(x)|v|^{r(x)} dx}{\int_\Omega a(x)|v|^q dx},$$

$$F(v) := \frac{\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy + \int_\Omega w(x)|v|^{\overline{p}(x)} dx}{\int_\Omega a(x)|v|^q dx}$$

and

$$H(v) := \frac{\varepsilon \int_\Omega b(x)|v|^{r(x)} dx}{\int_\Omega a(x)|v|^q dx}.$$

Then

$$\begin{cases} t^{p^- - q} F(v) + t^{r^- - q} H(v) \leq \Upsilon_{\varepsilon,v}(t) \leq t^{p^+ - q} F(v) + t^{r^+ - q} H(v) & \text{if} \quad t \geq 1 \\ t^{p^+ - q} F(v) + t^{r^+ - q} H(v) \leq \Upsilon_{\varepsilon,v}(t) \leq t^{p^- - q} F(v) + t^{r^- - q} H(v) & \text{if} \quad t < 1. \end{cases} \tag{4.19}$$

Having in mind that $p^+ < q < r^-$, it follows that

$$\lim_{t\to 0^+} \Upsilon_{\varepsilon,v}(t) = \lim_{t\to+\infty} \Upsilon_{\varepsilon,v}(t) = +\infty. \tag{4.20}$$

On the other hand, it is not difficult to see that the function $\Upsilon_{\varepsilon,v}$ admits a global minimum $t^*(v)$, which is a unique solution of the equation

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{(q-p(x,y))\, t^{p(x,y)}|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega (q-\overline{p}(x))\, t^{\overline{p}(x)} w(x)|v|^{\overline{p}(x)}dx\\ &= \varepsilon \int_\Omega (r(x)-q)\, t^{r(x)} b(x)|v|^{r(x)}dx.\end{aligned} \tag{4.21}$$

By (4.20), for $\lambda > 0$ large enough, there are exactly two positive reals $t_1(v) < t^*(v) < t_2(v)$ such that $\Upsilon_{\varepsilon,v}(t_1(v)) = \Upsilon_{\varepsilon,v}(t_2(v)) = \lambda$. Clearly $t_1(v)$ and $t_2(v)$ satisfy (4.4) with $M \equiv 1$, and $t(v) := t_2(v)$ increases as λ increases or ε decreases. Let

$$\Theta_a^\varepsilon(\lambda) := \{v \in \Theta_a : \lambda > \Upsilon_{\varepsilon,v}(t^*(v))\}.$$

Then, for λ sufficiently large, $\Theta_a^\varepsilon(\lambda) \neq \emptyset$. By (4.21), for $v \in \Theta_a^\varepsilon(\lambda)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{p(x,y)t^*(v)^{p(x,y)}|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega \overline{p}(x)t^*(v)^{\overline{p}(x)} w(x)|v|^{\overline{p}(x)}dx\\ &+\varepsilon \int_\Omega r(x)t^*(v)^{r(x)} b(x)|v|^{r(x)}dx < \lambda q t^*(v)^q \int_\Omega a(x)|v|^q dx,\end{aligned}$$

it holds then

$$t^*(v) < \begin{cases} \left(\frac{\lambda q \int_\Omega a(x)|v|^q dx}{\varepsilon r^- \int_\Omega b(x)|v|^{r(x)}dx}\right)^{\frac{1}{r^- - q}} & \text{if} \quad t^*(v) \geq 1 \\ \left(\frac{\lambda q \int_\Omega a(x)|v|^q dx}{\varepsilon r^- \int_\Omega b(x)|v|^{r(x)}dx}\right)^{\frac{1}{r^+ - q}} & \text{if} \quad t^*(v) < 1. \end{cases} \tag{4.22}$$

Claim 4.3. *If $v \in \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)$, then*

$$1 < \varepsilon \int_\Omega b(x)|v|^{r(x)}dx + \beta \left(\int_\Omega b(x)|v|^{r(x)}dx\right)^\theta.$$

for some $\beta > 0$ and

$$\theta := \begin{cases} \frac{q(r^- - p^-) - r^+(q-p^-)}{r^+(r^- - q)} & \textit{if} \quad ||v||_q < 1 \ \textit{and}\ t^*(v) \geq 1, \\ \frac{p^+}{r^+} & \textit{if} \quad ||v||_q < 1 \ \textit{and}\ t^*(v) < 1, \\ \frac{q(r^+ - p^+) - r^+(q-p^+)}{r^+(r^+ - q)} & \textit{if} \quad ||v||_q \geq 1 \ \textit{and}\ t^*(v) < 1, \\ \frac{q(r^- - p^-) - r^+(q-p^-)}{r^+(r^- - q)} & \textit{if} \quad ||v||_q \geq 1 \ \textit{and}\ t^*(v) \geq 1. \end{cases}$$

We just prove the case $||v||_q < 1$ and $t^*(v) \geq 1$, since others cases can be treated similarly. In fact, we have $\Upsilon_{\varepsilon,v}(t^*(v)) < \lambda$, thus

$$\int_{\mathbb{R}^{2N}} \frac{t^*(v)^{p(x,y)}|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega t^*(v)^{\overline{p}(x)} w(x)|v|^{\overline{p}(x)}dx < \lambda t^*(v)^q \int_\Omega a(x)|v|^q dx, \tag{4.23}$$

which yields

$$t^*(v)^{p^-}\left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega w(x)|v|^{\overline{p}(x)}dx\right) < \lambda t^*(v)^q \int_\Omega a(x)|v|^q dx. \tag{4.24}$$

Taking into account that $\Psi_\varepsilon(v) = 1$, from (4.3) with $M \equiv 1$ and (4.24), we get

$$1 - \varepsilon\int_\Omega b(x)|v|^{r(x)}dx < \lambda t^*(v)^{q-p^-}\int_\Omega a(x)|v|^q dx$$

and hence in view of (4.22),

$$\left(\varepsilon\int_\Omega b(x)|v|^{r(x)}dx\right)^{\frac{q-p^-}{r^--q}}\left(1-\varepsilon\int_\Omega b(x)|v|^{r(x)}dx\right) < \left(\frac{q}{r^-}\right)^{\frac{q-p^-}{r^--q}}\left(\lambda\int_\Omega a(x)|v|^q dx\right)^{\frac{r^--p^-}{r^--q}}. \tag{4.25}$$

By Lemmas 2.1, 2.4 and (H_3), we can find $C_1 > 0$ such that

$$\int_\Omega a(x)|v|^q dx \le C_1\left(\int_\Omega b(x)|v|^{r(x)}dx\right)^{\frac{q}{r^+}}. \tag{4.26}$$

Combining this inequality with (4.25), we deduce

$$1 < \varepsilon\int_\Omega b(x)|v|^{r(x)}dx + \beta\left(\int_\Omega b(x)|v|^{r(x)}dx\right)^{\frac{q(r^--p^-)-r^+(q-p^-)}{r^+(r^--q)}}$$

and the claim follows. Therefore, for some $C_2 > 0$,

$$\int_\Omega b(x)|v|^{r(x)}dx > C_2 \quad \text{for all} \quad v \in \Theta_a^\varepsilon(\lambda).$$

So, according to (4.22) and (4.26), the set $\{t(v) : v \in \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)\}$ is bounded above. Let v_1 be fixed in $\mathcal{U}_\varepsilon$. Then, $v_1 \in \Theta_a^\varepsilon(\lambda)$ for all $\lambda > \lambda_\varepsilon^1 := \Upsilon_{\varepsilon,v_1}(t^*(v_1))$. From (4.4) with $M \equiv 1$, we have

$$\begin{aligned}\widetilde{\mathcal{I}}_\varepsilon(v_1) \le &\left(\frac{1}{p^-}-\frac{1}{r^-}\right)\int_{\mathbb{R}^{2N}} t(v_1)^{p(x,y)}\frac{|v_1(x)-v_1(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &+\left(\frac{1}{p^-}-\frac{1}{r^-}\right)\int_\Omega t(v_1)^{\overline{p}(x)}w(x)|v_1|^{\overline{p}(x)}dx - \lambda\left(\frac{1}{q}-\frac{1}{r^-}\right)\int_\Omega t(v_1)^q a(x)|v_1|^q dx.\end{aligned} \tag{4.27}$$

Recalling that $\lambda \mapsto t_\lambda(v_1) := t(v_1)$ increases as λ increases and $p^- \le p^+ < q < r^-$, we choose $\lambda_\varepsilon^2 > 0$ large enough such that for all $\lambda > \lambda_\varepsilon^2$, $\widetilde{\mathcal{I}}_\varepsilon(v_1) < 0$. Hence, for all $\lambda > \lambda_\varepsilon^* := \max(\lambda_\varepsilon^1, \lambda_\varepsilon^2)$, $\alpha_1 := \inf\limits_{v\in\mathcal{U}_\varepsilon\cap\Theta_a^\varepsilon(\lambda)} \widetilde{\mathcal{I}}_\varepsilon(v) < 0$. Now, we show that the minimum of $\widetilde{\mathcal{I}}_\varepsilon$ is achieved in $\mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)$ with

$\lambda > \lambda_\varepsilon^*$. Indeed, let $\{v_n\} \subset \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)$ such that $\widetilde{\mathcal{I}}_\varepsilon(v_n) \to \alpha_1$. Since $\{v_n\}$ is bounded in $\mathbb{X}_0$, going to a subsequence if necessary, there exists $v_0 \in \mathbb{X}_0$ satisfying (4.7). As previously argued in the proof of Theorem 3.1, we deduce that $v_0 \neq 0$, $v_0 \in L^{r(\cdot)}(\Omega)$ and $\{t(v_n)\}$ converges to $t_0 = t(v_0) > 0$ with

$$\widetilde{\mathcal{I}}_\varepsilon(v_0) = \mathcal{I}_\varepsilon(t(v_0)v_0) = \mathcal{I}_\varepsilon(t_0 v_0) \leq \alpha_1. \tag{4.28}$$

Since $\{t^*(v_n)\}$ is also bounded, up to a subsequence, $t^*(v_n) \to t_0^*$. By (4.19) and direct computation, we obtain

$$\begin{aligned}
\Upsilon_{\varepsilon,v_0}(t^*(v_n)) &\geq \min_{t>0}\left(t^{p^- - q}F(v_0) + t^{r^- - q}H(v_0)\right)\\
&= \left[\left(\frac{q-p^-}{r^- - q}\right)^{\frac{p^- - q}{r^- - p^-}} + \left(\frac{q-p^-}{r^- - q}\right)^{\frac{r^- - q}{r^- - p^-}}\right] F(v_0)^{\frac{r^- - q}{r^- - p^-}} H(v_0)^{\frac{q-p^-}{r^- - p^-}} \text{ if } t^*(v_n) \geq 1\\
\Upsilon_{\varepsilon,v_0}(t^*(v_n)) &\geq \min_{0<t<1}\left(t^{p^+ - q}F(v_0) + t^{r^+ - q}H(v_0)\right)\\
&= \left[\left(\frac{q-p^+}{r^+ - q}\right)^{\frac{p^+ - q}{r^+ - p^+}} + \left(\frac{q-p^+}{r^+ - q}\right)^{\frac{r^+ - q}{r^+ - p^+}}\right] F(v_0)^{\frac{r^+ - q}{r^+ - p^+}} H(v_0)^{\frac{q-p^+}{r^+ - p^+}} \text{ if } t^*(v_n) < 1.
\end{aligned}$$

Therefore, passing to the limit as $n \to +\infty$, we get $\Upsilon_{\varepsilon,v_0}(t_0^*) > 0$, thus $t_0^* > 0$. On the other hand, by (4.7) and Fatou's lemma, we entail $\lambda \geq \Upsilon_{\varepsilon,v_0}(t_0^*) \geq \Upsilon_{\varepsilon,v_0}(t^*(v_0))$. Suppose by contradiction that $\lambda = \Upsilon_{\varepsilon,v_0}(t^*(v_0))$. We have $\lambda = \Upsilon_{\varepsilon,v_n}(t(v_n))$, thus

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{t(v_n)^{p(x,y)}|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega t(v_n)^{\overline{p}(x)}w(x)|v_n|^{\overline{p}(x)}dx\\
&+\varepsilon\int_\Omega t(v_n)^{r(x)}b(x)|v_n|^{r(x)}dx = \lambda t(v_n)^q\int_\Omega a(x)|v_n|^q dx,
\end{aligned}$$

and so, by (4.7),

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{t(v_0)^{p(x,y)}|v_0(x)-v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega t(v_0)^{\overline{p}(x)}w(x)|v_0|^{\overline{p}(x)}dx\\
&+\varepsilon\int_\Omega t(v_0)^{r(x)}b(x)|v_0|^{r(x)}dx \leq \lambda t(v_0)^q\int_\Omega a(x)|v_0|^q dx,
\end{aligned}$$

which means that $\Upsilon_{\varepsilon,v_0}(t^*(v_0)) = \lambda \geq \Upsilon_{\varepsilon,v_0}(t(v_0))$. Therefore,

$$t^*(v_0) = t(v_0) = t_0. \tag{4.29}$$

From (4.21), we have

$$\begin{aligned}
&\left(q-p^-\right)\left(\int_{\mathbb{R}^{2N}} \frac{t^*(v_0)^{p(x,y)}|v_0(x)-v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega t^*(v_0)^{\overline{p}(x)}w(x)|v_0|^{\overline{p}(x)}dx\right)\\
&\geq \left(r^- - q\right)\varepsilon\int_\Omega t^*(v_0)^{r(x)}b(x)|v_0|^{r(x)}dx.
\end{aligned}\tag{4.30}$$

By virtue of (4.4) with $M \equiv 1$, (4.29)-(4.30) and (H_4), we get

$$\begin{aligned}
\alpha_1 &= \lim_{n\to+\infty} \mathcal{I}_\varepsilon(t(v_n)v_n) \geq \mathcal{I}_\varepsilon(t(v_0)v_0) = \mathcal{I}_\varepsilon(t^*(v_0)v_0) \\
&\geq \frac{\varepsilon}{q}\left(\frac{(q-p^+)(r^- - q)}{p^+(q-p^-)} - \frac{r^+ - q}{r^+}\right)\int_\Omega t^*(v_0)^{r(x)} b(x)|v_0|^{r(x)}dx \\
&= \frac{\varepsilon\left(qp^+(q-p^-) - r^+\left[p^+(r^- - q) - q(r^- - q)\right]\right)}{r^+p^+q(q-p^-)}\int_\Omega t^*(v_0)^{r(x)} b(x)|v_0|^{r(x)}dx \\
&\geq 0,
\end{aligned}$$

which contradicts $\alpha_1 < 0$. Then $\lambda > \Upsilon_{\varepsilon,v_0}(t^*(v_0))$, and consequently $v_0 \in \Theta_a^\varepsilon(\lambda)$. We choose $\tau > 0$ such that $\tau v_0 \in \mathcal{U}_\varepsilon$. Using the uniqueness of the solution $t^*(v_0)$ of equation (4.21), we infer $\tau t^*(\tau v_0) = t^*(v_0)$. Therefore $\Upsilon_{\varepsilon,\tau v_0}(t^*(\tau v_0)) = \Upsilon_{\varepsilon,v_0}(t^*(v_0)) < \lambda$, thus $\tau v_0 \in \Theta_a^\varepsilon(\lambda)$. Hence $\tau v_0 \in \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)$. It holds from (4.13) and (4.28) that

$$\alpha_1 \leq \widetilde{\mathcal{I}}_\varepsilon(\tau v_0) = \mathcal{I}_\varepsilon(t(\tau v_0)\tau v_0) = \mathcal{I}_\varepsilon(t(v_0)v_0) = \widetilde{\mathcal{I}}_\varepsilon(v_0) \leq \alpha_1,$$

thus $\widetilde{\mathcal{I}}_\varepsilon(v_0) = \alpha_1$. Thanks again to Lemma 4.1, we see that $u := t(v_0)v_0$ is a solution of $(P_{\lambda,\varepsilon})$.

Suppose now that $p(x,y) = p$, $r(x) = r$ are constant and $q(x)$ varies. Let

$$\Gamma_v(t) := A(v) + \varepsilon t^{r-p}B(v) - \lambda\int_\Omega t^{q(x)}a(x)|v|^{q(x)-p}dx,$$

where

$$A(v) := \int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_\Omega w(x)|v|^p dx$$

and

$$B(v) := \int_\Omega b(x)|v|^r dx.$$

Then Γ_v is continuous, $\Gamma_v(0) = A(v) > 0$ and $\Gamma_v(t) \to +\infty$ as $t \to +\infty$, since $p < q(x) < r$ for all $x \in \Omega$. On the other hand, for λ large enough, we have $\inf\limits_{t>0}\Gamma_v(t) < 0$. Therefore, by (H_2), there are exactly two positive reals $t_1(v) < t_2(v)$ such that $\Gamma_v(t_1(v)) = \Gamma_v(t_2(v)) = 0$. So, by using the same arguments as above, we obtain a solution of $(P_{\lambda,\varepsilon})$. The proof of Theorem 3.3 is completed.

Proof of Theorem 3.4. Let $\varepsilon_0 > 0$. In view of Theorem 3.3, for $\lambda > \lambda^*_{\varepsilon_0}$, problem $(P_{\lambda,\varepsilon})$ with $\varepsilon = \varepsilon_0$ admits a solution $u_{\varepsilon_0} = t(v_{\varepsilon_0})v_{\varepsilon_0}$ with $v_{\varepsilon_0} \in \Theta_a^{\varepsilon_0}(\lambda)$. In the case $q(x) = q$, for all $\varepsilon \in (0,\varepsilon_0)$, problem $(P_{\lambda,\varepsilon})$ has a solution $u_\varepsilon = t(v_\varepsilon)v_\varepsilon$. In fact, from (4.19), we have

$$\begin{aligned}
\Upsilon_{\varepsilon,v}(t^*(v)) \quad &\leq \quad \left[\left(\frac{q-p^+}{r^+-q}\right)^{\frac{p^+-q}{r^+-p^+}} + \left(\frac{q-p^+}{r^+-q}\right)^{\frac{r^+-q}{r^+-p^+}}\right] F(v)^{\frac{r^+-q}{r^+-p^+}} \\
&\times\left(\frac{\int_\Omega b(x)|v|^{r(x)}dx}{\int_\Omega a(x)|v|^q dx}\right)^{\frac{q-p^+}{r^+-p^+}} \varepsilon^{\frac{q-p^+}{r^+-p^+}} \quad \text{if} \quad t^*(v) \geq 1
\end{aligned}$$

and

$$\Upsilon_{\varepsilon,v}(t^*(v)) \leq \left[\left(\frac{q-p^-}{r^--q}\right)^{\frac{p^--q}{r^--p^-}} + \left(\frac{q-p^-}{r^--q}\right)^{\frac{r^--q}{r^--p^-}}\right] F(v)^{\frac{r^--q}{r^--p^-}}$$
$$\times \left(\frac{\int_\Omega b(x)|v|^{r(x)}dx}{\int_\Omega a(x)|v|^q dx}\right)^{\frac{q-p^-}{r^--p^-}} \varepsilon^{\frac{q-p^-}{r^--p^-}} \quad \text{if} \quad t^*(v) < 1.$$

Since $p^- \leq p^+ < q < r^-$, $\Upsilon_{\varepsilon,v}(t^*(v)) \downarrow 0$ as $\varepsilon \downarrow 0$. Thus $\lambda > \Upsilon_{\varepsilon,v_{\varepsilon_0}}(t^*(v_{\varepsilon_0}))$ for any $\varepsilon \in (0,\varepsilon_0)$. Hence $v_{\varepsilon_0} \in \Theta_a^\varepsilon(\lambda)$. By (4.21), we have

$$\min\left((t_\varepsilon^*(v_{\varepsilon_0}))^{p^-}, (t_\varepsilon^*(v_{\varepsilon_0}))^{p^+}\right) \int_{\mathbb{R}^{2N}} \frac{(q-p(x,y))\,|v_{\varepsilon_0}(x)-v_{\varepsilon_0}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy$$
$$\leq \varepsilon \max\left((t_\varepsilon^*(v_{\varepsilon_0}))^{r^-}, (t_\varepsilon^*(v_{\varepsilon_0}))^{r^+}\right) \int_\Omega (r(x)-q)\,b(x)|v_{\varepsilon_0}|^{r(x)}dx,$$

which yields

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^{2N}} \frac{(q-p(x,y))\,|v_{\varepsilon_0}(x)-v_{\varepsilon_0}(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy$$
$$\leq \max\left((t_\varepsilon^*(v_{\varepsilon_0}))^{r^+-p^-}, t_\varepsilon^*(v_{\varepsilon_0})^{r^--p^+}\right) \int_\Omega (r(x)-q)\,b(x)|v_{\varepsilon_0}|^{r(x)}dx.$$

It holds that $t_\varepsilon^*(v_{\varepsilon_0}) \to +\infty$ as $\varepsilon \to 0$, since $p^+ < r^-$. Noting that $t_\varepsilon^*(v_{\varepsilon_0}) < t_\varepsilon(v_{\varepsilon_0})$, we deduce that $t_\varepsilon(v_{\varepsilon_0}) \to +\infty$ as $\varepsilon \to 0$. Therefore, in view of (4.27), for some $\varepsilon_1 \in (0,\varepsilon_0)$ small enough, $\widetilde{\mathcal{I}}_\varepsilon(v_{\varepsilon_0}) < 0$ for all $\varepsilon \in (0,\varepsilon_1)$. Let $\tau > 0$ such that $\tau v_{\varepsilon_0} \in \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)$. Since $\widetilde{\mathcal{I}}_\varepsilon(\tau v_{\varepsilon_0}) = \widetilde{\mathcal{I}}_\varepsilon(v_{\varepsilon_0}) < 0$,

$$\inf_{v \in \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)} \widetilde{\mathcal{I}}_\varepsilon(v) < 0 \quad \text{for all} \quad \varepsilon \in (0,\varepsilon_1).$$

Through a similar reasoning to that of Theorem 3.1, we can show that for any $\varepsilon \in (0,\varepsilon_1)$, problem $(P_{\lambda,\varepsilon})$ has a solution $u_\varepsilon = t_\varepsilon(v_\varepsilon)v_\varepsilon$, with $v_\varepsilon \in \mathcal{U}_\varepsilon \cap \Theta_a^\varepsilon(\lambda)$. Moreover, $\mathcal{I}_\varepsilon(u_\varepsilon) = \widetilde{\mathcal{I}}_\varepsilon(v_\varepsilon) \to -\infty$ as $\varepsilon \to 0$. By (4.1) with $M \equiv 1$ and (4.16), we conclude that $||u_\varepsilon||_X \to +\infty$ as $\varepsilon \to 0$. The proof of Theorem 3.4 is completed.

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Multiple general sigmoids based Banach space valued neural network multivariate approximation

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ABSTRACT

Here we present multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We treat also the case of approximation by iterated operators of the last four types. These approximations are derived by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives. Our multivariate operators are defined by using a multidimensional density function induced by several different among themselves general sigmoid functions. This is done on the purpose to activate as many as possible neurons. The approximations are pointwise and uniform. The related feed-forward neural network is with one hidden layer. We finish with related L_p approximations.

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RESUMEN

Presentamos aproximaciones multivariadas cuantitativas de funciones multivariadas continuas con valores en un espacio de Banach definidas en una caja o en $\mathbb{R}^N$, $N \in \mathbb{N}$, a través de operadores de redes neuronales multivariados normalizados, de cuasi-interpolación, de tipo Kantorovich y de tipo cuadratura. También tratamos el caso de aproximación usando operadores iterados de los últimos cuatro tipos. Estas aproximaciones se derivan estableciendo desigualdades multidimensionales de tipo Jackson que involucran el módulo de continuidad multivariado de la función comprometida o sus derivadas de Fréchet de alto orden. Nuestros operadores multivariados son definidos usando una función de densidad multidimensional inducida por varias funciones sigmoidales generales diferentes entre sí. Esto se hace con el propósito de activar la mayor cantidad de neuronas posible. Las aproximaciones son puntutales y uniformes. La red neuronal prealimentada relacionada tiene un nivel oculto. Concluimos con aproximaciones L_p relacionadas.

Keywords and Phrases: General sigmoid functions, multivariate neural network approximation, quasi-interpolation operator, Kantorovich type operator, quadrature type operator, multivariate modulus of continuity, abstract approximation, iterated approximation, L_p approximation.

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1 Introduction

The author in [2, 3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

For this article the author is motivated by the article [14] of Z. Chen and F. Cao, also by [4–12, 15, 16].

The author here performs multivariate multiple general sigmoid functions based neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$. Also he does iterated and L_p approximations. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order Fréchet derivative and given by very tight multidimensional Jackson type inequalities.

The author here comes up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or $\mathbb{R}^N$, as well as Kantorovich type and quadrature type related operators on $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density functions induced by multiple general sigmoid functions and defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is a general sigmoid function, but here we use a multiple number of them simultaneously for the first time, so we can activate a maximum number of neurons. About neural networks read [17–19].

2 Basics

Let $i = 1, \ldots, N \in \mathbb{N}$ and $h_i : \mathbb{R} \to [-1, 1]$ be a general sigmoid function, such that it is strictly increasing, $h_i(0) = 0$, $h_i(-x) = -h_i(x)$, $h_i(+\infty) = 1$, $h_i(-\infty) = -1$. Also h_i is strictly convex over $(-\infty, 0]$ and striclty concave over $[0, +\infty)$, with $h_i^{(2)} \in C(\mathbb{R}, [-1, 1])$.

We consider the activation function

$$\psi_i(x) := \frac{1}{4}\left(h_i(x+1) - h_i(x-1)\right), \quad x \in \mathbb{R}, \quad i = 1, \ldots, N. \tag{2.1}$$

As in [11, p. 285], we get that $\psi_i(-x) = \psi_i(x)$, thus ψ_i is an even function. Since $x + 1 > x - 1$, then $h_i(x+1) > h_i(x-1)$, and $\psi_i(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi_i(0) = \frac{h_i(1)}{2}, \quad i = 1, \ldots, N. \tag{2.2}$$

Let $x > 1$, we have that

$$\psi_i'(x) = \frac{1}{4}\left(h_i'(x+1) - h_i'(x-1)\right) < 0,$$

by h_i' being strictly decreasing over $[0, +\infty)$.

Let now $0 < x < 1$, then $1 - x > 0$ and $0 < 1 - x < 1 + x$. It holds $h_i'(x-1) = h_i'(1-x) > h_i'(x+1)$, so that again $\psi_i'(x) < 0$. Consequently ψ_i is strictly decreasing on $(0, +\infty)$.

Clearly, ψ_i is strictly increasing on $(-\infty, 0)$, and $\psi_i'(0) = 0$.

See that

$$\lim_{x \to +\infty} \psi_i(x) = \frac{1}{4}\left(h_i(+\infty) - h_i(+\infty)\right) = 0, \tag{2.3}$$

and

$$\lim_{x \to -\infty} \psi_i(x) = \frac{1}{4}\left(h_i(-\infty) - h_i(-\infty)\right) = 0. \tag{2.4}$$

That is the x-axis is the horizontal asymptote on ψ_i.

Conclusion, ψ is a bell symmetric function with maximum

$$\psi_i(0) = \frac{h_i(1)}{2}.$$

We need

Theorem 2.1. *We have that*

$$\sum_{i=-\infty}^{\infty} \psi_i(x - i) = 1, \quad \forall x \in \mathbb{R}, \quad i = 1, \ldots, N. \tag{2.5}$$

Proof. As exactly the same as in [11, p. 286], is omitted. ∎

Theorem 2.2. *It holds*

$$\int_{-\infty}^{\infty} \psi_i(x)\,dx = 1, \quad i = 1, \ldots, N. \tag{2.6}$$

Proof. Similar to [11, p. 287]. It is omitted. □

Thus $\psi_i(x)$ is a density function on $\mathbb{R}$, $i = 1, \ldots, N$.

We give

Theorem 2.3. *Let* $0 < \alpha < 1$, *and* $n \in \mathbb{N}$ *with* $n^{1-\alpha} > 2$. *It holds*

$$\sum_{\substack{k=-\infty \\ |nx-k| \geq n^{1-\alpha}}}^{\infty} \psi_i(nx-k) < \frac{\left(1 - h_i\left(n^{1-\alpha} - 2\right)\right)}{2}, \quad i = 1, \ldots, N. \tag{2.7}$$

Notice that

$$\lim_{n \to +\infty} \frac{\left(1 - h_i\left(n^{1-\alpha} - 2\right)\right)}{2} = 0, \quad i = 1, \ldots, N.$$

Proof. Let $x \geq 1$. That is $0 \leq x - 1 < x + 1$. Applying the mean value theorem we get

$$\psi_i(x) \overset{(2.1)}{=} \frac{1}{4} \cdot 2 \cdot h_i'(\xi) = \frac{h_i'(\xi)}{2}, \tag{2.8}$$

for some $x - 1 < \xi < x + 1$.

Since h_i' is strictly decreasing we obtain $h_i'(\xi) < h_i'(x-1)$ and

$$\psi_i(x) < \frac{h_i'(x-1)}{2}, \quad \forall\, x \geq 1. \tag{2.9}$$

Therefore we have

$$\begin{aligned}
\sum_{\substack{k=-\infty \\ |nx-k| \geq n^{1-\alpha}}}^{\infty} \psi_i(nx-k) &= \sum_{\substack{k=-\infty \\ |nx-k| \geq n^{1-\alpha}}}^{\infty} \psi_i(|nx-k|) < \frac{1}{2} \sum_{\substack{k=-\infty \\ |nx-k| \geq n^{1-\alpha}}}^{\infty} h_i'(|nx-k|-1) \\
&\leq \frac{1}{2} \int_{(n^{1-\alpha}-1)}^{+\infty} h_i'(x-1)\,d(x-1) = \frac{1}{2}\left(h_i(x-1)\big|_{(n^{1-\alpha}-1)}^{+\infty}\right) \\
&= \frac{1}{2}\left[h_i(+\infty) - h_i\left(n^{1-\alpha}-2\right)\right] = \frac{1}{2}\left(1 - h_i\left(n^{1-\alpha}-2\right)\right).
\end{aligned} \tag{2.10}$$

The claim is proved. □

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We further give

Theorem 2.4. *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_i (nx-k)} < \frac{1}{\psi_i (1)}, \quad \forall \ x \in [a,b], \quad i = 1, \ldots, N. \tag{2.11}$$

Proof. As similar to [11, p. 289] is omitted. ☐

Remark 2.5. *We have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_i (nx-k) \neq 1, \quad i = 1, \ldots, N, \tag{2.12}$$

for at least some $x \in [a,b]$.

See [11, p. 290], same reasoning.

Note 2.6. *For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds (by* (2.5)*)*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_i (nx-k) \leq 1, \quad i = 1, \ldots, N. \tag{2.13}$$

We make

Remark 2.7. *We define*

$$Z(x_1, \ldots, x_N) := Z(x) := \prod_{i=1}^{N} \psi_i (x_i), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N, \quad N \in \mathbb{N}. \tag{2.14}$$

It has the properties:

(i)

$$Z(x) > 0, \quad \forall x \in \mathbb{R}^N, \tag{2.15}$$

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} Z(x_1-k_1, \ldots, x_N-k_N) =$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \prod_{i=1}^{N} \psi_i (x_i-k_i) = \prod_{i=1}^{N} \left(\sum_{k_i=-\infty}^{\infty} \psi_i (x_i-k_i) \right) \overset{(2.5)}{=} 1.$$

Hence

$$\sum_{k=-\infty}^{\infty} Z(x-k) = 1. \tag{2.16}$$

That is

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) = 1, \quad \forall x \in \mathbb{R}^N, \quad n \in \mathbb{N}. \tag{2.17}$$

and

(iv)

$$\int_{\mathbb{R}^N} Z(x)\,dx = \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} \psi_i(x_i)\right) dx_1 \cdots dx_N = \prod_{i=1}^{N} \left(\int_{-\infty}^{\infty} \psi_i(x_i)\,dx_i\right) \overset{(2.6)}{=} 1, \tag{2.18}$$

thus

$$\int_{\mathbb{R}^N} Z(x)\,dx = 1, \tag{2.19}$$

that is Z is a multivariate density function.

Here denote $\|x\|_\infty := \max\{|x_1|, \ldots, |x_N|\}$, $x \in \mathbb{R}^N$, *also set* $\infty := (\infty, \ldots, \infty)$, $-\infty := (-\infty, \ldots, -\infty)$ *upon the multivariate context, and*

$$\lceil na \rceil := (\lceil na_1 \rceil, \ldots, \lceil na_N \rceil)$$

$$\lfloor nb \rfloor := (\lfloor nb_1 \rfloor, \ldots, \lfloor nb_N \rfloor),$$

where $a := (a_1, \ldots, a_N)$, $b := (b_1, \ldots, b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \psi_i(nx_i - k_i)\right) = \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^{N} \psi_i(nx_i - k_i)\right)$$
$$= \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi_i(nx_i - k_i)\right). \tag{2.20}$$

For $0 < \beta < 1$ *and* $n \in \mathbb{N}$, *a fixed* $x \in \mathbb{R}^N$, *we have that*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{\substack{k=\lceil na \rceil \\ \left\|\frac{k}{n}-x\right\|_\infty \le \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k) + \sum_{\substack{k=\lceil na \rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k). \tag{2.21}$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\|\frac{k}{n} - x\right\|_\infty > \frac{1}{n^\beta}$ *implies that there exists at least one* $\left|\frac{k_r}{n} - x_r\right| > \frac{1}{n^\beta}$, *where* $r \in \{1, \ldots, N\}$.

(v) *We notice that*

$$\begin{aligned}
\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor} Z(nx-k) &= \sum_{\substack{k_1=\lceil na_1\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb_1\rfloor} \cdots \sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} \left(\prod_{i=1}^{N} \psi_i(nx_i-k_i)\right) \\
&= \prod_{i=1}^{N} \left(\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb_i\rfloor} \psi_i(nx_i-k_i) \right) \\
&\leq \left(\prod_{\substack{i=1 \\ i\neq r}}^{N} \left(\sum_{k_i=-\infty}^{\infty} \psi_i(nx_i-k_i) \right) \right) \left(\sum_{\substack{k_r=\lceil na_r\rceil \\ \left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^{\beta}}}}^{\lfloor nb_r\rfloor} \psi_r(nx_r-k_r) \right) \\
&= \left(\sum_{\substack{k_r=\lceil na_r\rceil \\ \left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^{\beta}}}}^{\lfloor nb_r\rfloor} \psi_r(nx_r-k_r) \right) \qquad (2.22) \\
&\leq \sum_{\substack{k_r=-\infty \\ \left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^{\beta}}}}^{\infty} \psi_r(nx_r-k_r) = \sum_{\substack{k_r=-\infty \\ |nx_r-k_r|>n^{1-\beta}}}^{\infty} \psi_r(nx_r-k_r) \\
&\overset{(2.7)}{<} \frac{1-h_r\left(n^{1-\beta}-2\right)}{2} \leq \max_{i\in\{1,\ldots,N\}} \left(\frac{1-h_i\left(n^{1-\beta}-2\right)}{2} \right),
\end{aligned}$$

where $0<\beta<1$.

That is we get:

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor} Z(nx-k) < \max_{i\in\{1,\ldots,N\}} \left(\frac{1-h_i\left(n^{1-\beta}-2\right)}{2} \right), \qquad (2.23)$$

$0<\beta<1$, *with* $n\in\mathbb{N}: n^{1-\beta}>2$, $\forall\ x\in\prod_{i=1}^{N}[a_i,b_i]$.

(vi) *It is clear that*

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty} Z(nx-k) < \max_{i\in\{1,\ldots,N\}} \left(\frac{1-h_i\left(n^{1-\beta}-2\right)}{2} \right), \qquad (2.24)$$

$0<\beta<1$, $n\in\mathbb{N}: n^{1-\beta}>2$, $\forall\ x\in\prod_{i=1}^{N}[a_i,b_i]$.

(*vii*) *By Theorem 2.4 we get that*

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} = \frac{1}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi_i(nx_i - k_i) \right)} < \frac{1}{\prod_{i=1}^{N} \psi_i(1)},$$

thus

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} < \frac{1}{\prod_{i=1}^{N} \psi_i(1)}, \tag{2.25}$$

$\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right),\ \ n \in \mathbb{N}.$

Furthermore it holds

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \lim_{n \to \infty} \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi_i(nx_i - k_i) \right) = \prod_{i=1}^{N} \left(\lim_{n \to \infty} \sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi_i(nx_i - k_i) \right) \neq 1, \tag{2.26}$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

We state

Definition 2.8. *We denote by*

$$\delta_N(\beta, n) := \max_{i \in \{1, \ldots, N\}} \left(\frac{1 - h_i\left(n^{1-\beta} - 2\right)}{2} \right), \tag{2.27}$$

where $0 < \beta < 1$.

We make

Remark 2.9. *Here* $\left(X, \|\cdot\|_\gamma \right)$ *is a Banach space.*

Let $f \in C\left(\prod_{i=1}^{N} [a_i, b_i], X \right)$, $x = (x_1, \ldots, x_N) \in \prod_{i=1}^{N} [a_i, b_i]$, $n \in \mathbb{N}$ *such that* $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \ldots, N$.

We introduce and define the following multivariate linear normalized neural network operator ($x := (x_1, \ldots, x_N) \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$*):*

$$A_n(f, x_1, \ldots, x_N) := A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}$$

$$= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \psi_i(nx_i - k_i) \right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi_i(nx_i - k_i) \right)}. \tag{2.28}$$

For large enough $n \in \mathbb{N}$ *we always obtain* $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \dots, N$. *Also* $a_i \leq \frac{k_i}{n} \leq b_i$, *iff* $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, \dots, N$.

When $g \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$ we define the companion operator

$$\widetilde{A}_n (g, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}. \tag{2.29}$$

Clearly $\widetilde{A}_n$ is a positive linear operator. We have that

$$\widetilde{A}_n (1, x) = 1, \quad \forall x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right).$$

Notice that $A_n (f) \in C\left(\prod_{i=1}^{N} [a_i, b_i], X\right)$ and $\widetilde{A}_n (g) \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

Furthermore it holds

$$\|A_n (f, x)\|_\gamma \leq \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\|f\left(\frac{k}{n}\right)\right\|_\gamma Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} = \widetilde{A}_n \left(\|f\|_\gamma, x\right), \tag{2.30}$$

$\forall\ x \in \prod_{i=1}^{N} [a_i, b_i]$. Clearly $\|f\|_\gamma \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

So, we have that

$$\|A_n (f, x)\|_\gamma \leq \widetilde{A}_n \left(\|f\|_\gamma, x\right), \tag{2.31}$$

$\forall\ x \in \prod_{i=1}^{N} [a_i, b_i]$, $\forall\ n \in \mathbb{N}$, $\forall\ f \in C\left(\prod_{i=1}^{N} [a_i, b_i], X\right)$.

Let $c \in X$ and $g \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$, then $cg \in C\left(\prod_{i=1}^{N} [a_i, b_i], X\right)$.

Furthermore it holds

$$A_n (cg, x) = c\widetilde{A}_n (g, x), \quad \forall x \in \prod_{i=1}^{N} [a_i, b_i]. \tag{2.32}$$

Since $\widetilde{A}_n (1) = 1$, we get that

$$A_n (c) = c, \quad \forall c \in X. \tag{2.33}$$

We call $\widetilde{A}_n$ the companion operator of A_n.

For convenience we call

$$\begin{aligned} A_n^* (f, x) &:= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k) \\ &= \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \psi_i (nx_i - k_i)\right), \end{aligned} \tag{2.34}$$

$\forall\ x \in \left(\prod_{i=1}^{N}\left[a_i, b_i\right]\right).$

That is

$$A_n\left(f, x\right) := \frac{A_n^*\left(f, x\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)}, \tag{2.35}$$

$\forall\ x \in \left(\prod_{i=1}^{N}\left[a_i, b_i\right]\right)$, $n \in \mathbb{N}$.

Hence

$$A_n\left(f, x\right) - f\left(x\right) = \frac{A_n^*\left(f, x\right) - f\left(x\right)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)}. \tag{2.36}$$

Consequently we derive

$$\left\|A_n\left(f, x\right) - f\left(x\right)\right\|_{\gamma} \overset{(2.25)}{\leq} \left(\prod_{i=1}^{N}\psi_i\left(1\right)\right)^{-1}\left\|A_n^*\left(f, x\right) - f\left(x\right)\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)\right\|_{\gamma}, \tag{2.37}$$

$\forall\ x \in \left(\prod_{i=1}^{N}\left[a_i, b_i\right]\right).$

We will estimate the right hand side of (2.37).

For the last and others we need

Definition 2.10 ([11, p. 274]). *Let M be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$, and $\left(X, \|\cdot\|_{\gamma}\right)$ be a Banach space. Let $f \in C\left(M, X\right)$. We define the first modulus of continuity of f as*

$$\omega_1\left(f, \delta\right) := \sup_{\substack{x,y \in M \\ \|x-y\|_p \leq \delta}} \left\|f\left(x\right) - f\left(y\right)\right\|_{\gamma}, \quad 0 < \delta \leq diam\left(M\right). \tag{2.38}$$

If $\delta > diam\left(M\right)$, then

$$\omega_1\left(f, \delta\right) = \omega_1\left(f, diam\left(M\right)\right). \tag{2.39}$$

Notice $\omega_1\left(f, \delta\right)$ is increasing in $\delta > 0$. For $f \in C_B\left(M, X\right)$ (continuous and bounded functions) $\omega_1\left(f, \delta\right)$ is defined similarly.

Lemma 2.11 ([11, p. 274]). *We have $\omega_1\left(f, \delta\right) \to 0$ as $\delta \downarrow 0$, iff $f \in C\left(M, X\right)$, where M is a convex compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$.*

Clearly we have also: $f \in C_U\left(\mathbb{R}^N, X\right)$ (uniformly continuous functions), iff $\omega_1\left(f, \delta\right) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (2.38). The space $C_B\left(\mathbb{R}^N, X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

When $f \in C_B\left(\mathbb{R}^N, X\right)$ we define,

$$B_n(f,x) := B_n(f,x_1,\ldots,x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k)$$
$$:= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n},\frac{k_2}{n},\ldots,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\psi_i(nx_i-k_i)\right), \tag{2.40}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B\left(\mathbb{R}^N, X\right)$ we define the multivariate Kantorovich type neural network operator

$$C_n(f,x) := C_n(f,x_1,\ldots,x_N) := \sum_{k=-\infty}^{\infty}\left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt\right) Z(nx-k) = \tag{2.41}$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}\left(n^N\int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}}\cdots\int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1,\ldots,t_N)\,dt_1\ldots dt_N\right)\cdot\left(\prod_{i=1}^{N}\psi_i(nx_i-k_i)\right),$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f,x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1,\ldots,\theta_N) \in \mathbb{N}^N$, $r = (r_1,\ldots,r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1,r_2,\ldots,r_N} \geq 0$, such that $\sum\limits_{r=0}^{\theta} w_r = \sum\limits_{r_1=0}^{\theta_1}\sum\limits_{r_2=0}^{\theta_2}\cdots\sum\limits_{r_N=0}^{\theta_N} w_{r_1,r_2,\ldots,r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}(f) := \delta_{n,k_1,k_2,\ldots,k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n}+\frac{r}{n\theta}\right)$$
$$= \sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}\cdots\sum_{r_N=0}^{\theta_N} w_{r_1,r_2,\ldots,r_N} f\left(\frac{k_1}{n}+\frac{r_1}{n\theta_1},\frac{k_2}{n}+\frac{r_2}{n\theta_2},\ldots,\frac{k_N}{n}+\frac{r_N}{n\theta_N}\right), \tag{2.42}$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1},\frac{r_2}{\theta_2},\ldots,\frac{r_N}{\theta_N}\right)$.

We set

$$D_n(f,x) := D_n(f,x_1,\ldots,x_N) := \sum_{k=-\infty}^{\infty}\delta_{nk}(f)\,Z(nx-k) \tag{2.43}$$
$$= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}\delta_{n,k_1,k_2,\ldots,k_N}(f)\left(\prod_{i=1}^{N}\psi_i(nx_i-k_i)\right), \quad \forall x \in \mathbb{R}^N.$$

In this article we study the approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I.

3 Multivariate general sigmoid neural network approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

Theorem 3.1. *Let* $f \in C\left(\prod_{i=1}^{N}[a_i, b_i], X\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$. *Then*

1)
$$\|A_n(f, x) - f(x)\|_\gamma \leq \left(\prod_{i=1}^{N}\psi_i(1)\right)^{-1}\left[\omega_1\left(f, \frac{1}{n^\beta}\right) + 2\delta_N(\beta, n)\left\|\|f\|_\gamma\right\|_\infty\right] =: \lambda_1(n), \tag{3.1}$$

and

2)
$$\left\|\|A_n(f) - f\|_\gamma\right\|_\infty \leq \lambda_1(n). \tag{3.2}$$

We notice that $\lim_{n\to\infty} A_n(f) \overset{\|\cdot\|_\gamma}{=} f$, *pointwise and uniformly.*

Above ω_1 *is with respect to* $p = \infty$ *and the speed of convergence is* $\max\left(\frac{1}{n^\beta}, \delta_N(\beta, n)\right)$.

Proof. As similar to [12] is omitted. Use of (2.37). □

We make

Remark 3.2 ([11, pp. 263–266]). *Let* $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $N \in \mathbb{N}$; *where* $\|\cdot\|_p$ *is the* L_p*-norm,* $1 \leq p \leq \infty$. $\mathbb{R}^N$ *is a Banach space, and* $\left(\mathbb{R}^N\right)^j$ *denotes the* j*-fold product space* $\mathbb{R}^N \times \cdots \times \mathbb{R}^N$ *endowed with the max-norm* $\|x\|_{(\mathbb{R}^N)^j} := \max_{1\leq\lambda\leq j}\|x_\lambda\|_p$, *where* $x := (x_1, \ldots, x_j) \in \left(\mathbb{R}^N\right)^j$.

Let $\left(X, \|\cdot\|_\gamma\right)$ *be a general Banach space. Then the space* $L_j := L_j\left(\left(\mathbb{R}^N\right)^j; X\right)$ *of all* j*-multilinear continuous maps* $g : \left(\mathbb{R}^N\right)^j \to X$, $j = 1, \ldots, m$, *is a Banach space with norm*

$$\|g\| := \|g\|_{L_j} := \sup_{\left(\|x\|_{(\mathbb{R}^N)^j}=1\right)}\|g(x)\|_\gamma = \sup\frac{\|g(x)\|_\gamma}{\|x_1\|_p \cdots \|x_j\|_p}. \tag{3.3}$$

Let M *be a non-empty convex and compact subset of* $\mathbb{R}^k$ *and* $x_0 \in M$ *is fixed.*

Let O *be an open subset of* $\mathbb{R}^N : M \subset O$. *Let* $f : O \to X$ *be a continuous function, whose Fréchet derivatives (see [20])* $f^{(j)} : O \to L_j = L_j\left(\left(\mathbb{R}^N\right)^j; X\right)$ *exist and are continuous for* $1 \leq j \leq m$, $m \in \mathbb{N}$.

Call $(x - x_0)^j := (x - x_0, \ldots, x - x_0) \in (\mathbb{R}^N)^j$, $x \in M$.

We will work with $f|_M$.

Then, by Taylor's formula [13], [20, p. 124] , we get

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} + R_m(x, x_0), \quad \text{all } x \in M, \tag{3.4}$$

where the remainder is the Riemann integral

$$R_m(x, x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!} \left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0) \right) (x - x_0)^m du, \tag{3.5}$$

here we set $f^{(0)}(x_0)(x - x_0)^0 = f(x_0)$.

We consider

$$w := \omega_1\left(f^{(m)}, h\right) := \sup_{\substack{x,y \in M \\ \|x-y\|_p \le h}} \left\| f^{(m)}(x) - f^{(m)}(y) \right\|, \tag{3.6}$$

$h > 0$.

We obtain

$$\begin{aligned} &\left\| \left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0) \right) (x - x_0)^m \right\|_\gamma \le \\ &\left\| f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0) \right\| \cdot \|x - x_0\|_p^m \le w \|x - x_0\|_p^m \left\lceil \frac{u \|x - x_0\|_p}{h} \right\rceil, \end{aligned} \tag{3.7}$$

by [1, Lemma 7.1.1, p. 208], where $\lceil \cdot \rceil$ *is the ceiling.*

Therefore for all $x \in M$ *(see [1, pp. 121-122]):*

$$\|R_m(x, x_0)\|_\gamma \le w \|x - x_0\|_p^m \int_0^1 \left\lceil \frac{u \|x - x_0\|_p}{h} \right\rceil \frac{(1-u)^{m-1}}{(m-1)!} du = w \Phi_m\left(\|x - x_0\|_p\right) \tag{3.8}$$

by a change of variable, where

$$\Phi_m(t) := \int_0^{|t|} \left\lceil \frac{s}{h} \right\rceil \frac{(|t| - s)^{m-1}}{(m-1)!} ds = \frac{1}{m!} \left(\sum_{j=0}^{\infty} (|t| - jh)_+^m \right), \quad \forall t \in \mathbb{R}, \tag{3.9}$$

is a (polynomial) spline function, see [1, p. 210-211].

Also from there we get

$$\Phi_m(t) \le \left(\frac{|t|^{m+1}}{(m+1)!h} + \frac{|t|^m}{2m!} + \frac{h|t|^{m-1}}{8(m-1)!} \right), \quad \forall t \in \mathbb{R}, \tag{3.10}$$

with equality true only at $t = 0$.

Therefore it holds

$$\|R_m(x,x_0)\|_\gamma \leq w\left(\frac{\|x-x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|x-x_0\|_p^m}{2m!} + \frac{h\|x-x_0\|_p^{m-1}}{8(m-1)!}\right), \quad \forall x \in M. \tag{3.11}$$

We have found that

$$\left\|f(x) - \sum_{j=0}^m \frac{f^{(j)}(x_0)(x-x_0)^j}{j!}\right\|_\gamma \leq \omega_1\left(f^{(m)},h\right)\left(\frac{\|x-x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|x-x_0\|_p^m}{2m!} + \frac{h\|x-x_0\|_p^{m-1}}{8(m-1)!}\right) < \infty, \tag{3.12}$$

$\forall\ x, x_0 \in M$.

Here $0 < \omega_1\left(f^{(m)},h\right) < \infty$, *by* M *being compact and* $f^{(m)}$ *being continuous on* M.

One can rewrite (3.12) *as follows:*

$$\left\|f(\cdot) - \sum_{j=0}^m \frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!}\right\|_\gamma \leq \omega_1\left(f^{(m)},h\right)\left(\frac{\|\cdot-x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|\cdot-x_0\|_p^m}{2m!} + \frac{h\|\cdot-x_0\|_p^{m-1}}{8(m-1)!}\right), \tag{3.13}$$

$\forall x_0 \in M$, *a pointwise functional inequality on* M.

Here $(\cdot - x_0)^j$ *maps* M *into* $\left(\mathbb{R}^N\right)^j$ *and it is continuous, also* $f^{(j)}(x_0)$ *maps* $\left(\mathbb{R}^N\right)^j$ *into* X *and it is continuous. Hence their composition* $f^{(j)}(x_0)(\cdot - x_0)^j$ *is continuous from* M *into* X.

Clearly $f(\cdot) - \sum_{j=0}^m \frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!} \in C(M,X)$, *hence* $\left\|f(\cdot) - \sum_{j=0}^m \frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!}\right\|_\gamma \in C(M)$.

Let $\left\{\widetilde{L}_N\right\}_{N\in\mathbb{N}}$ *be a sequence of positive linear operators mapping* $C(M)$ *into* $C(M)$.

Therefore we obtain

$$\left(\widetilde{L}_N\left(\left\|f(\cdot) - \sum_{j=0}^m \frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!}\right\|_\gamma\right)\right)(x_0) \leq$$

$$\omega_1\left(f^{(m)},h\right)\left[\frac{\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)}{(m+1)!h} + \frac{\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^m\right)\right)(x_0)}{2m!} + \frac{h\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{m-1}\right)\right)(x_0)}{8(m-1)!}\right], \tag{3.14}$$

$\forall\ N \in \mathbb{N}$, $\forall\ x_0 \in M$.

Clearly (3.14) is valid when $M = \prod_{i=1}^N [a_i, b_i]$ and $\widetilde{L}_n = \widetilde{A}_n$, see (2.29).

All the above is preparation for the following theorem, where we assume Fréchet differentiability of functions.

This will be a direct application of Theorem 10.2 in [11, pp. 268-270]. The operators A_n, $\widetilde{A}_n$ fulfill its assumptions, see (2.28), (2.29), (2.31), (2.32) and (2.33).

We present the following high order approximation results.

Theorem 3.3. *Let O open subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$, such that $\prod_{i=1}^{N}[a_i,b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O,X)$, the space of m-times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^{N}[a_i,b_i]}$. Let $x_0 \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$ and $r>0$. Then*

1)

$$\left\|\left(A_n(f)\right)(x_0) - \sum_{j=0}^{m}\frac{1}{j!}\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}$$
$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right], \tag{3.15}$$

2) *additionally if $f^{(j)}(x_0) = 0$, $j = 1, \ldots, m$, we have*

$$\left\|\left(A_n(f)\right)(x_0) - f(x_0)\right\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)} \tag{3.16}$$
$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],$$

3)

$$\left\|\left(A_n(f)\right)(x_0) - f(x_0)\right\|_\gamma \leq \sum_{j=1}^{m}\frac{1}{j!}\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma +$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)} \tag{3.17}$$
$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],$$

4)

$$\left\| \|A_n(f) - f\|_\gamma \right\|_{\infty, \prod_{i=1}^{N}[a_i,b_i]} \leq \sum_{j=1}^{m} \frac{1}{j!} \left\| \left\| \left(A_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]}$$

$$+ \frac{\omega_1 \left(f^{(m)}, r \left\| \left(\widetilde{A}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]}^{\frac{1}{m+1}} \right)}{rm!}$$

$$\left\| \left(\widetilde{A}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]}^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right]. \quad (3.18)$$

We need

Lemma 3.4. *The function* $\left(\widetilde{A}_n \left(\|\cdot - x_0\|_p^m \right) \right)(x_0)$ *is continuous in* $x_0 \in \left(\prod_{i=1}^{N}[a_i,b_i] \right)$, $m \in \mathbb{N}$.

Proof. By Lemma 10.3, [11, p. 272]. □

We make

Remark 3.5. *By [11, Remark 10.4, p. 273], we get that*

$$\left\| \left(\widetilde{A}_n \left(\|\cdot - x_0\|_p^k \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]} \leq \left\| \left(\widetilde{A}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]}^{\left(\frac{k}{m+1}\right)}, \quad (3.19)$$

for all $k = 1, \ldots, m$.

We give

Corollary 3.6 (to Theorem 3.3, case of $m = 1$)**.** *Then*

1)

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq \left\| \left(A_n \left(f^{(1)}(x_0)(\cdot - x_0) \right) \right)(x_0) \right\|_\gamma$$

$$+ \frac{1}{2r} \omega_1 \left(f^{(1)}, r \left(\left(\widetilde{A}_n \left(\|\cdot - x_0\|_p^2 \right) \right)(x_0) \right)^{\frac{1}{2}} \right) \left(\left(\widetilde{A}_n \left(\|\cdot - x_0\|_p^2 \right) \right)(x_0) \right)^{\frac{1}{2}} \left[1 + r + \frac{r^2}{4} \right], \quad (3.20)$$

2)

$$\left\| \|(A_n(f)) - f\|_\gamma \right\|_{\infty, \prod_{i=1}^{N}[a_i,b_i]} \leq \left\| \left\| \left(A_n \left(f^{(1)}(x_0)(\cdot - x_0) \right) \right)(x_0) \right\|_\gamma \right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]} +$$

$$\frac{1}{2r} \omega_1 \left(f^{(1)}, r \left\| \left(\widetilde{A}_n \left(\|\cdot - x_0\|_p^2 \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]}^{\frac{1}{2}} \right)$$

$$\left\| \left(\widetilde{A}_n \left(\|\cdot - x_0\|_p^2 \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]}^{\frac{1}{2}} \left[1 + r + \frac{r^2}{4} \right], \quad r > 0. \quad (3.21)$$

We make

Remark 3.7. *We estimate* ($0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$),

$$\widetilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)}$$

$$\overset{(2.25)}{<} \left(\prod_{i=1}^{N} \psi_i(1) \right)^{-1} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) \tag{3.22}$$

$$= \left(\prod_{i=1}^{N} \psi_i(1) \right)^{-1} \left\{ \sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x_0 \right\|_\infty \le \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) \right.$$

$$\left. + \sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x_0 \right\|_\infty > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) \right\}$$

$$\overset{(2.23)}{\le} \left(\prod_{i=1}^{N} \psi_i(1) \right)^{-1} \left\{ \frac{1}{n^{\alpha(m+1)}} + \delta_N(\alpha, n) \|b - a\|_\infty^{m+1} \right\}, \tag{3.23}$$

(where $b - a = (b_1 - a_1, \ldots, b_N - a_N)$*).*

We have proved that ($\forall \ x_0 \in \prod_{i=1}^{N} [a_i, b_i]$)

$$\widetilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) < \left(\prod_{i=1}^{N} \psi_i(1) \right)^{-1} \left\{ \frac{1}{n^{\alpha(m+1)}} + \delta_N(\alpha, n) \|b - a\|_\infty^{m+1} \right\} =: \varphi_1(n) \tag{3.24}$$

($0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$).

And, consequently it holds

$$\left\| \widetilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} < \left(\prod_{i=1}^{N} \psi_i(1) \right)^{-1} \left\{ \frac{1}{n^{\alpha(m+1)}} + \delta_N(\alpha, n) \|b - a\|_\infty^{m+1} \right\}$$

$$= \varphi_1(n) \to 0, \quad \text{as } n \to +\infty. \tag{3.25}$$

So, we have that $\varphi_1(n) \to 0$, *as* $n \to +\infty$. *Thus, when* $p \in [1, \infty]$, *from Theorem 3.3 we have the convergence to zero in the right hand sides of parts (1), (2).*

Next we estimate $\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma$.

We have that

$$\left(\widetilde{A}_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0) = \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f^{(j)}(x_0)\left(\frac{k}{n}-x_0\right)^j Z(nx_0-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx_0-k)}. \tag{3.26}$$

When $p=\infty$, $j=1,\ldots,m$, we obtain

$$\left\|f^{(j)}(x_0)\left(\frac{k}{n}-x_0\right)^j\right\|_\gamma \leq \left\|f^{(j)}(x_0)\right\|\left\|\frac{k}{n}-x_0\right\|_\infty^j. \tag{3.27}$$

We further have that

$$\left\|\left(\widetilde{A}_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma \overset{(2.25)}{<}$$

$$\left(\prod_{i=1}^N \psi_i(1)\right)^{-1}\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f^{(j)}(x_0)\left(\frac{k}{n}-x_0\right)^j\right\|_\gamma Z(nx_0-k)\right) \leq$$

$$\left(\prod_{i=1}^N \psi_i(1)\right)^{-1}\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f^{(j)}(x_0)\right\|\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right) = \tag{3.28}$$

$$\left(\prod_{i=1}^N \psi_i(1)\right)^{-1}\left\|f^{(j)}(x_0)\right\|\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right) =$$

$$\left(\prod_{i=1}^N \psi_i(1)\right)^{-1}\left\|f^{(j)}(x_0)\right\|\left\{\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x_0\right\|_\infty \leq \frac{1}{n^\alpha}}}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right.$$

$$\left. + \sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x_0\right\|_\infty > \frac{1}{n^\alpha}}}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right\} \overset{(2.23)}{\leq} \tag{3.29}$$

$$\left(\prod_{i=1}^N \psi_i(1)\right)^{-1}\left\|f^{(j)}(x_0)\right\|\left\{\frac{1}{n^{\alpha j}} + \delta_N(\alpha,n)\|b-a\|_\infty^j\right\} \to 0, \quad \text{as } n\to\infty.$$

That is

$$\left\|\left(\widetilde{A}_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma \to 0, \quad \text{as } n\to\infty.$$

Therefore when $p = \infty$, for $j = 1, \ldots, m$, we have proved:

$$\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma < \left(\prod_{i=1}^{N} \psi_i(1) \right)^{-1} \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \delta_N(\alpha, n) \|b - a\|_\infty^j \right\}$$
$$\leq \left(\prod_{i=1}^{N} \psi_i(1) \right)^{-1} \left\| f^{(j)} \right\|_\infty \left\{ \frac{1}{n^{\alpha j}} + \delta_N(\alpha, n) \|b - a\|_\infty^j \right\}$$
$$=: \varphi_{2j}(n) < \infty, \tag{3.30}$$

and converges to zero, as $n \to \infty$.

We conclude:

In Theorem 3.3, the right hand sides of (3.26) and (3.18) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Also in Corollary 3.6, the right hand sides of (3.20) and (3.21) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Conclusion 3.8. *We have proved that the left hand sides of* (3.15), (3.16), (3.17), (3.18) *and* (3.20), (3.21) *converge to zero as $n \to \infty$, for $p \in [1, \infty]$. Consequently $A_n \to I$ (unit operator) pointwise and uniformly, as $n \to \infty$, where $p \in [1, \infty]$. In the presence of initial conditions we achieve a higher speed of convergence, see* (3.16)*. Higher speed of convergence happens also to the left hand side of* (3.15).

We give

Corollary 3.9 (to Theorem 3.3)**.** *Let O open subset of $\left(\mathbb{R}^N, \|\cdot\|_\infty \right)$, such that $\prod_{i=1}^{N} [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma \right)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O, X)$, the space of m-times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^{N} [a_i, b_i]}$. Let $x_0 \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$ and $r > 0$. Here $\varphi_1(n)$ as in* (3.24) *and $\varphi_{2j}(n)$ as in* (3.30)*, where $n \in \mathbb{N} : n^{1-\alpha} > 2$, $0 < \alpha < 1$, $j = 1, \ldots, m$. Then*

1)

$$\left\| (A_n(f))(x_0) - \sum_{j=0}^{m} \frac{1}{j!} \left(A_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \leq$$
$$\frac{\omega_1 \left(f^{(m)}, r (\varphi_1(n))^{\frac{1}{m+1}} \right)}{r m!} (\varphi_1(n))^{\left(\frac{m}{m+1} \right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{m r^2}{8} \right], \tag{3.31}$$

2) *additionally, if* $f^{(j)}(x_0) = 0$, $j = 1, \ldots, m$, *we have*

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq \frac{\omega_1\left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}}\right)}{rm!}(\varphi_1(n))^{\left(\frac{m}{m+1}\right)}\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right], \tag{3.32}$$

3)

$$\left\|\|A_n(f) - f\|_\gamma\right\|_{\infty, \prod_{i=1}^N [a_i, b_i]} \leq \sum_{j=1}^m \frac{\varphi_{2j}(n)}{j!} + \frac{\omega_1\left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}}\right)}{rm!}(\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \cdot \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right] =: \varphi_3(n) \to 0, \quad \text{as } n \to \infty. \tag{3.33}$$

We continue with

Theorem 3.10. *Let* $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, ω_1 *is for* $p = \infty$. *Then*

1)
$$\|B_n(f, x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + 2\delta_N(\beta, n)\left\|\|f\|_\gamma\right\|_\infty =: \lambda_2(n), \tag{3.34}$$

2)
$$\left\|\|B_n(f) - f\|_\gamma\right\|_\infty \leq \lambda_2(n). \tag{3.35}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, *we obtain* $\lim_{n\to\infty} B_n(f) = f$, *uniformly. The speed of convergence above is* $\max\left(\frac{1}{n^\beta}, \delta_N(\beta, n)\right)$.

Proof. As similar to [12] is omitted. □

We give

Theorem 3.11. *Let* $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, ω_1 *is for* $p = \infty$. *Then*

1)
$$\|C_n(f, x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + 2\delta_N(\beta, n)\left\|\|f\|_\gamma\right\|_\infty =: \lambda_3(n), \tag{3.36}$$

2)
$$\left\|\|C_n(f) - f\|_\gamma\right\|_\infty \leq \lambda_3(n). \tag{3.37}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, *we obtain* $\lim_{n\to\infty} C_n(f) = f$, *uniformly.*

Proof. As similar to [12] is omitted. □

We also present

Theorem 3.12. *Let* $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, ω_1 *is for* $p = \infty$. *Then*

1)

$$\left\| D_n(f, x) - f(x) \right\|_\gamma \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + 2\delta_N(\beta, n) \left\| \|f\|_\gamma \right\|_\infty = \lambda_4(n), \tag{3.38}$$

2)

$$\left\| \left\| D_n(f) - f \right\|_\gamma \right\|_\infty \leq \lambda_4(n). \tag{3.39}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, *we obtain* $\lim_{n\to\infty} D_n(f) = f$, *uniformly.*

Proof. As similar to [12] is omitted. □

We make

Definition 3.13. *Let* $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, *where* $\left(X, \|\cdot\|_\gamma\right)$ *is a Banach space. We define the general neural network operator*

$$F_n(f, x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z(nx - k) = \begin{cases} B_n(f, x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f, x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt, \\ D_n(f, x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases} \tag{3.40}$$

Clearly $l_{nk}(f)$ is an X-valued bounded linear functional such that $\|l_{nk}(f)\|_\gamma \leq \left\| \|f\|_\gamma \right\|_\infty$.

Hence $F_n(f)$ is a bounded linear operator with $\left\| \|F_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty$.

We need

Theorem 3.14. *Let* $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \geq 1$. *Then* $F_n(f) \in C_B\left(\mathbb{R}^N, X\right)$.

Proof. Very lengthy and as similar to [12] is omitted. □

Remark 3.15. *By* (2.28) *it is obvious that* $\left\| \|A_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty < \infty$, *and* $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, *given that* $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Call L_n *any of the operators* A_n, B_n, C_n, D_n.

Clearly then

$$\left\| \left\| L_n^2(f) \right\|_\gamma \right\|_\infty = \left\| \left\| L_n(L_n(f)) \right\|_\gamma \right\|_\infty \leq \left\| \left\| L_n(f) \right\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty, \tag{3.41}$$

etc.

Therefore we get

$$\left\| \left\| L_n^k (f) \right\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty, \quad \forall\ k \in \mathbb{N}, \tag{3.42}$$

the contraction property.

Also we see that

$$\left\| \left\| L_n^k (f) \right\|_\gamma \right\|_\infty \leq \left\| \left\| L_n^{k-1} (f) \right\|_\gamma \right\|_\infty \leq \cdots \leq \left\| \left\| L_n (f) \right\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty. \tag{3.43}$$

Here L_n^k are bounded linear operators.

Notation 3.16. *Here $N \in \mathbb{N}$, $0 < \beta < 1$. Denote by*

$$c_N := \begin{cases} \left(\prod_{i=1}^{N} \psi_i (1) \right)^{-1}, & \text{if} \quad L_n = A_n, \\ 1, \quad \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{3.44}$$

$$\varphi (n) := \begin{cases} \frac{1}{n^\beta}, & \text{if } L_n = A_n, \quad B_n, \\ \frac{1}{n} + \frac{1}{n^\beta}, & \text{if } L_n = C_n, D_n, \end{cases} \tag{3.45}$$

$$\Omega := \begin{cases} C\left(\prod_{i=1}^{N} [a_i, b_i], X \right), & \text{if } L_n = A_n, \\ C_B\left(\mathbb{R}^N, X\right), & \text{if} \quad L_n = B_n, C_n, D_n, \end{cases} \tag{3.46}$$

and

$$Y := \begin{cases} \prod_{i=1}^{N} [a_i, b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases} \tag{3.47}$$

We give the condensed

Theorem 3.17. *Let $f \in \Omega$, $0 < \beta < 1$, $x \in Y$; n, $N \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*

(i)

$$\|L_n (f, x) - f (x)\|_\gamma \leq c_N \left[\omega_1 (f, \varphi (n)) + 2\delta_N (\beta, n) \left\| \|f\|_\gamma \right\|_\infty \right] =: \tau (n), \tag{3.48}$$

where ω_1 is for $p = \infty$,

(ii)

$$\left\| \|L_n (f) - f\|_\gamma \right\|_\infty \leq \tau (n) \to 0, \text{ as } n \to \infty. \tag{3.49}$$

For f uniformly continuous and in Ω we obtain

$$\lim_{n \to \infty} L_n (f) = f,$$

pointwise and uniformly.

Proof. By Theorems 3.1, 3.10, 3.11, 3.12. □

Next we do iterated neural network approximation (see also [9]).

We make

Remark 3.18. *Let $r \in \mathbb{N}$ and L_n as above. We observe that*

$$L_n^r f - f = \left(L_n^r f - L_n^{r-1} f\right) + \left(L_n^{r-1} f - L_n^{r-2} f\right) +$$

$$\left(L_n^{r-2} f - L_n^{r-3} f\right) + \cdots + \left(L_n^2 f - L_n f\right) + \left(L_n f - f\right).$$

Then

$$\left\| \|L_n^r f - f\|_\gamma \right\|_\infty \le \left\| \|L_n^r f - L_n^{r-1} f\|_\gamma \right\|_\infty + \left\| \|L_n^{r-1} f - L_n^{r-2} f\|_\gamma \right\|_\infty +$$

$$\left\| \|L_n^{r-2} f - L_n^{r-3} f\|_\gamma \right\|_\infty + \cdots + \left\| \|L_n^2 f - L_n f\|_\gamma \right\|_\infty + \left\| \|L_n f - f\|_\gamma \right\|_\infty =$$

$$\left\| \|L_n^{r-1} (L_n f - f)\|_\gamma \right\|_\infty + \left\| \|L_n^{r-2} (L_n f - f)\|_\gamma \right\|_\infty + \left\| \|L_n^{r-3} (L_n f - f)\|_\gamma \right\|_\infty$$

$$+ \ldots + \left\| \|L_n (L_n f - f)\|_\gamma \right\|_\infty + \left\| \|L_n f - f\|_\gamma \right\|_\infty \le r \left\| \|L_n f - f\|_\gamma \right\|_\infty. \tag{3.50}$$

That is

$$\left\| \|L_n^r f - f\|_\gamma \right\|_\infty \le r \left\| \|L_n f - f\|_\gamma \right\|_\infty. \tag{3.51}$$

We give

Theorem 3.19. *All here as in Theorem 3.17 and $r \in \mathbb{N}$, $\tau(n)$ as in* (3.48). *Then*

$$\left\| \|L_n^r f - f\|_\gamma \right\|_\infty \le r\tau(n). \tag{3.52}$$

So that the speed of convergence to the unit operator of L_n^r is not worse than of L_n.

Proof. By (3.51) and (3.49). □

We make

Remark 3.20. *Let $m_1, \ldots, m_r \in \mathbb{N} : m_1 \le m_2 \le \cdots \le m_r$, $0 < \beta < 1$, $f \in \Omega$. Then $\varphi(m_1) \ge \varphi(m_2) \ge \cdots \ge \varphi(m_r)$, φ as in* (3.45).

Therefore

$$\omega_1(f, \varphi(m_1)) \ge \omega_1(f, \varphi(m_2)) \ge \cdots \ge \omega_1(f, \varphi(m_r)). \tag{3.53}$$

Assume further that $m_i^{1-\beta} > 2$, $i = 1, \ldots, r$. Then

$$\delta_N(\beta, m_1) \ge \delta_N(\beta, m_2) \ge \cdots \ge \delta_N(\beta, m_r). \tag{3.54}$$

Let L_{m_i} as above, $i=1,\dots,r$, all of the same kind.

We write

$$\begin{gathered}L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_2}\left(L_{m_1}f\right)\right)\right)-f=\\ L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_2}\left(L_{m_1}f\right)\right)\right)-L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_2}f\right)\right)+\\ L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_2}f\right)\right)-L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_3}f\right)\right)+\\ L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_3}f\right)\right)-L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_4}f\right)\right)+\dots+\\ L_{m_r}\left(L_{m_{r-1}}f\right)-L_{m_r}f+L_{m_r}f-f=\\ L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_2}\right)\right)\left(L_{m_1}f-f\right)+L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_3}\right)\right)\left(L_{m_2}f-f\right)+\\ L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_4}\right)\right)\left(L_{m_3}f-f\right)+\dots+L_{m_r}\left(L_{m_{r-1}}f-f\right)+L_{m_r}f-f.\end{gathered}\tag{3.55}$$

Hence by the triangle inequality property of $\left\|\|\cdot\|_\gamma\right\|_\infty$ we get

$$\begin{aligned}\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_2}\left(L_{m_1}f\right)\right)\right)-f\right\|_\gamma\right\|_\infty\le&\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_2}\right)\right)\left(L_{m_1}f-f\right)\right\|_\gamma\right\|_\infty\\ &+\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_3}\right)\right)\left(L_{m_2}f-f\right)\right\|_\gamma\right\|_\infty\\ &+\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_4}\right)\right)\left(L_{m_3}f-f\right)\right\|_\gamma\right\|_\infty\\ &+\dots+\left\|\left\|L_{m_r}\left(L_{m_{r-1}}f-f\right)\right\|_\gamma\right\|_\infty+\left\|\left\|L_{m_r}f-f\right\|_\gamma\right\|_\infty\end{aligned}$$

(repeatedly applying (3.41)*)*

$$\begin{aligned}&\le\left\|\left\|L_{m_1}f-f\right\|_\gamma\right\|_\infty+\left\|\left\|L_{m_2}f-f\right\|_\gamma\right\|_\infty+\left\|\left\|L_{m_3}f-f\right\|_\gamma\right\|_\infty\\ &+\dots+\left\|\left\|L_{m_{r-1}}f-f\right\|_\gamma\right\|_\infty+\left\|\left\|L_{m_r}f-f\right\|_\gamma\right\|_\infty=\sum_{i=1}^{r}\left\|\left\|L_{m_i}f-f\right\|_\gamma\right\|_\infty.\end{aligned}\tag{3.56}$$

That is, we proved

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_2}\left(L_{m_1}f\right)\right)\right)-f\right\|_\gamma\right\|_\infty\le\sum_{i=1}^{r}\left\|\left\|L_{m_i}f-f\right\|_\gamma\right\|_\infty.\tag{3.57}$$

We give

Theorem 3.21. *Let $f\in\Omega$; N, $m_1,m_2,\dots,m_r\in\mathbb{N}:m_1\le m_2\le\dots\le m_r$, $0<\beta<1$; $m_i^{1-\beta}>2$, $i=1,\dots,r$, $x\in Y$, and let $(L_{m_1},\dots,L_{m_r})$ as $(A_{m_1},\dots,A_{m_r})$ or $(B_{m_1},\dots,B_{m_r})$ or $(C_{m_1},\dots,C_{m_r})$ or $(D_{m_1},\dots,D_{m_r})$, $p=\infty$. Then*

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_2}\left(L_{m_1}f\right)\right)\right)(x)-f(x)\right\|_\gamma\le\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(\dots L_{m_2}\left(L_{m_1}f\right)\right)\right)-f\right\|_\gamma\right\|_\infty$$

$$\leq \sum_{i=1}^{r} \left\| \left\| L_{m_i} f - f \right\|_{\gamma} \right\|_{\infty} \leq c_N \sum_{i=1}^{r} \left[\omega_1 \left(f, \varphi \left(m_i \right) \right) + 2\delta_N \left(\beta, m_i \right) \left\| \left\| f \right\|_{\gamma} \right\|_{\infty} \right]$$

$$\leq r c_N \left[\omega_1 \left(f, \varphi \left(m_1 \right) \right) + 2\delta_N \left(\beta, m_1 \right) \left\| \left\| f \right\|_{\gamma} \right\|_{\infty} \right]. \quad (3.58)$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of L_{m_1}.

Proof. Using (3.57), (3.53), (3.54) and (3.48), (3.49). □

We continue with

Theorem 3.22. *Let all as in Corollary 3.9, and* $r \in \mathbb{N}$. *Here* $\varphi_3(n)$ *is as in* (3.33). *Then*

$$\left\| \left\| A_n^r f - f \right\|_{\gamma} \right\|_{\infty} \leq r \left\| \left\| A_n f - f \right\|_{\gamma} \right\|_{\infty} \leq r \varphi_3 (n). \quad (3.59)$$

Proof. By (3.51) and (3.33). □

Next we present some L_{p_1}, $p_1 \geq 1$, approximation related results.

Theorem 3.23. *Let* $p_1 \geq 1$, $f \in C\left(\prod_{i=1}^{N} [a_i, b_i], X \right)$, $0 < \beta < 1$; $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, *and* $\lambda_1(n)$ *as in (3.1),* ω_1 *is for* $p = \infty$. *Then*

$$\left\| \left\| A_n (f) - f \right\|_{\gamma} \right\|_{p_1, \prod_{i=1}^{N} [a_i, b_i]} \leq \lambda_1 (n) \left(\prod_{i=1}^{N} (b_i - a_i) \right)^{\frac{1}{p_1}}. \quad (3.60)$$

We notice that $\lim_{n \to \infty} \left\| \left\| A_n (f) - f \right\|_{\gamma} \right\|_{p_1, \prod_{i=1}^{N} [a_i, b_i]} = 0$.

Proof. Obvious, by integrating (3.1), etc. □

It follows

Theorem 3.24. *Let* $p_1 \geq 1$, $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$; $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, *and* ω_1 *is for* $p = \infty$; $\lambda_2(n)$ *as in* (3.34) *and* K *a compact subset of* $\mathbb{R}^N$. *Then*

$$\left\| \left\| B_n (f) - f \right\|_{\gamma} \right\|_{p_1, K} \leq \lambda_2 (n) |K|^{\frac{1}{p_1}}, \quad (3.61)$$

where $|K| < \infty$, *is the Lebesgue measure of* K.

We notice that $\lim_{n \to \infty} \left\| \left\| B_n (f) - f \right\|_{\gamma} \right\|_{p_1, K} = 0$, *for* $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right) \right)$.

Proof. By integrating (3.34), etc. □

Next come

Theorem 3.25. *All as in Theorem 3.24, but now we use* $\lambda_3(n)$ *of* (3.36). *Then*

$$\left\| \|C_n(f) - f\|_\gamma \right\|_{p_1,K} \leq \lambda_3(n) |K|^{\frac{1}{p_1}}. \tag{3.62}$$

We have that $\lim_{n\to\infty} \left\| \|C_n(f) - f\|_\gamma \right\|_{p_1,K} = 0$, *for* $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$.

Proof. By (3.36). □

Theorem 3.26. *All as in Theorem 3.24, but now we use* $\lambda_4(n)$ *of* (3.38). *Then*

$$\left\| \|D_n(f) - f\|_\gamma \right\|_{p_1,K} \leq \lambda_4(n) |K|^{\frac{1}{p_1}}. \tag{3.63}$$

We have that $\lim_{n\to\infty} \left\| \|D_n(f) - f\|_\gamma \right\|_{p_1,K} = 0$, *for* $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$.

Proof. By (3.38). □

Application 3.27. *A typical application of all of our results is when* $\left(X, \|\cdot\|_\gamma\right) = (\mathbb{C}, |\cdot|)$, *where* $\mathbb{C}$ *are the complex numbers.*

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A nice asymptotic reproducing kernel

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ABSTRACT

We extend the assertion of Problem 12340 in Amer. Math. Monthly 129 (2022), 686, by deriving some additional asymptotic behaviour of that special kernel.

RESUMEN

Extendemos la formulación del Problema 12340 en Amer. Math. Monthly 129 (2022), 686, derivando un comportamiento asintótico adicional de dicho núcleo especial.

Keywords and Phrases: Integral kernel, reproducing kernel, good kernel, summability kernel, real analysis.

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1 An integral operator

The solution to Problem 12340 in [1, p. 686] tells us that for $g : [0,1] \to \mathbb{R}$ continuous,

$$\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n}\, dx = \frac{\pi}{4}\, g\left(\frac{1}{2}\right). \tag{1.1}$$

We shall prove the following additional properties:

Proposition 1.1. *Let $f \in L^1_{loc}(\mathbb{R})$ and for $t > 0$ and $x \in [0,1]$, let*

$$k_t(x) := \frac{1}{\pi}\,\frac{1}{2^t}\,\frac{4t}{x^t + (1-x)^t}.$$

Then

(i) $\lim\limits_{t\to\infty} k_t(x) = \begin{cases} 0 & \text{if } x \neq 1/2 \\ \infty & \text{if } x = 1/2. \end{cases}$

(ii) $\lim\limits_{t\to\infty} \int_0^1 k_t(x)\, dx = 1.$

(iii) $\lim\limits_{t\to\infty} \int_0^1 k_t(x) f\left(s - \frac{1}{2} + x\right)\, dx = f(s)$ *for each continuity point s of f.*

(iv) $\lim\limits_{t\to\infty} \int_{-\infty}^{\infty} K_t\left(y - s + \frac{1}{2}\right) f(y)\, dy = f(s)$ *for each continuity point s of f, where K_t coincides with k_t extended outside $[0,1]$ by 0.*

Thus we may call $k(t,x) := k_t(x)$ a *shifted asymptotic reproducing kernel* for $L^1[0,1]$ or $C[0,1]$ for example (see also at the end of this note).

Proof. (i) is evident

(ii) We show, more generally, that for any continuous function g on $[0,1]$ we have

$$\lim_{t\to\infty} \int_0^1 k_t(x) g(x)\, dx = g(1/2). \tag{1.2}$$

Note that (1.1) is just the discrete version of (1.2) by taking $t = n$ [1]. So, to prove (1.2), we split the integral into two parts and use two different changes of variables. Let $t \geq 1$. Then

[1] This was submitted by myself and Rudolf Rupp as solution to the Monthly problem above.

$$\begin{aligned}
I_t &:= \frac{t}{2^t}\int_0^{1/2} \underbrace{\frac{g(x)}{x^t+(1-x)^t}\,dx}_{x=:\frac{1}{2}-\frac{s}{2t}} + \frac{t}{2^t}\int_{1/2}^{1} \underbrace{\frac{g(x)}{x^t+(1-x)^t}\,dx}_{x=:\frac{1}{2}+\frac{s}{2t}} \\
&= \frac{t}{2^t}\int_0^t \frac{g(\frac12-\frac{s}{2t})}{(\frac12-\frac{s}{2t})^t+(\frac12+\frac{s}{2t})^t}\,\frac{1}{2t}ds + \frac{t}{2^t}\int_0^t \frac{g(\frac12+\frac{s}{2t})}{(\frac12+\frac{s}{2t})^t+(\frac12-\frac{s}{2t})^t}\,\frac{1}{2t}ds \\
&= \frac12\int_0^t \frac{g(\frac12-\frac{s}{2t})+g(\frac12+\frac{s}{2t})}{(1-\frac{s}{t})^t+(1+\frac{s}{t})^t}\,ds.
\end{aligned}$$

Note that $t\mapsto(1+\frac{s}{t})^t$ is increasing; so the integrand is dominated for $s\geq 1$ by

$$\frac{||g||_\infty}{(1+\frac{s}{2})^2}\leq ||g||_\infty 4s^{-2}.$$

Hence, as $t\to\infty$,

$$\begin{aligned}
\lim_{t\to\infty} I_t &= \frac12 2g(1/2)\int_0^\infty \frac{1}{e^{-s}+e^s}\,ds \\
&= g(1/2)\int_0^\infty \frac{e^s}{1+(e^s)^2}\,ds \\
&= g(1/2)\big[\arctan e^s\big]_0^\infty \\
&= g(1/2)\left(\frac{\pi}{2}-\frac{\pi}{4}\right) \\
&= \frac{\pi}{4}g(1/2).
\end{aligned}$$

If we take $g\equiv 1$, we finally obtain (ii):

$$\int_0^1 k_t(x)dx = \frac{4}{\pi}I_t \to 1.$$

(iii) Let $f\in L^1[0,1]$ and suppose that $1/2$ is a continuity point of f. Given $\epsilon>0$, choose $\delta>0$ so that $|f(x)-f(1/2)|<\epsilon$ for $|x-1/2|<\delta$. For $0\leq x\leq 1$ and $t\geq 1$, let $h(x):=x^t+(1-x)^t$. Then h is a convex function with minimum at $x=1/2$. Hence, whenever $0<\delta<1/2$, the condition $|x-1/2|\geq\delta$ with $0\leq x\leq 1$ implies that

$$x^t+(1-x)^t\geq(1/2+\delta)^t+(1/2-\delta)^t.$$

Thus, as $\delta\neq 0$,

$$\frac{t}{2^t}\,\frac{1}{x^t+(1-x)^t}\leq\frac{t}{(1+2\delta)^t+(1-2\delta)^t}=: m_t\to 0 \quad\text{as}\quad t\to\infty.$$

Consequently $\left|\int_0^1 k_t(x)f(x)\,dx - \int_0^1 k_t(x)f(1/2)\,dx\right|$

$$\begin{aligned}
&\leq \left|\left(\int_{\substack{|x-1/2|\geq\delta\\0\leq x\leq 1}} + \int_{|x-1/2|\leq\delta}\right) k_t(x)|f(x)-f(1/2)|\,dx\right|\\
&\leq \frac{4\,m_t}{\pi}\int_{\substack{|x-1/2|\geq\delta\\0\leq x\leq 1}} \big(|f(x)|+|f(1/2)|\big)dx + \epsilon\int_{|x-1/2|\leq\delta} k_t(x)dx\\
&\leq \frac{4\,m_t}{\pi}\big(||f||_1+|f(1/2)|\big)+\epsilon\int_0^1 k_t(x)dx\\
&\leq 2\epsilon \quad \text{for } t\geq t_0.
\end{aligned}$$

As $\lim\limits_{t\to\infty}\int_0^1 k_t(x)f\left(\frac{1}{2}\right)\,dx = f\left(\frac{1}{2}\right)$ by (ii), we deduce that

$$\lim_{t\to\infty}\int_0^1 k_t(x)f(x)\,dx = f\left(\frac{1}{2}\right).$$

If $f\in L^1_{loc}(\mathbb{R})$ satisfies the assumptions above, we put $F(x) := f(s-\frac{1}{2}+x)$. Then $F\in L^1[0,1]$ and $1/2$ is a continuity point of F. Hence

$$\lim_{t\to\infty}\int_0^1 k_t(x)F(x)\,dx = F(1/2) = f(s).$$

(iv) is obtained from (iii) by a linear change of the variable. □

We may ask what happens if s is a jump point. Do we have a similar behaviour as in the Dirichlet-Jordan Theorem for Fourier series?

It is interesting to discuss the relations that exist between our shifted asymptotic reproducing kernel $k_t(x) = k(t,x)$ and the so-called "summability kernels" in [2, p. 9], respectively "good kernels" in [3, p. 48], the most prominent examples being the Fejér kernel and the Poisson kernel for $L^1(\mathbb{T})$ concerning 2π-periodic functions. In fact, using suitable transformations, in particular the new variable $y = 2\pi(x-\frac{1}{2})$, equivalently $x = \frac{1}{2}+\frac{y}{2\pi}$, we get the following relations (we restrict w.l.o.g. to the discrete case): let $I_n := \int_0^1 k_n(x)dx$ and

$$K_n^*(y) := I_n^{-1}\cdot k_n\left(\frac{1}{2}+\frac{y}{2\pi}\right), \quad -\pi\leq y<\pi,$$

and extend this function 2π-periodically. Then K_n^* is continuous on $\mathbb{R}$ as

$$K_n^*(-\pi) = \lim_{y\to-\pi}K_n^*(y) = k_n(0) = k_n(1) = \lim_{y\to\pi}K_n^*(y) = K_n^*(\pi).$$

Observe that for $|y| < \pi$,

$$K_n^*(y) = I_n^{-1} \cdot \frac{4n\pi^{n-1}}{(\pi+y)^n + (\pi-y)^n}.$$

Moreover, $K_n^* \geq 0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n^*(y)dy = 1,$$

and, by the proof of (iii) and (ii),

$$\int_{\delta\leq|y|\leq\pi} K_n^*(y)dy \to 0 \quad \text{as } n \to \infty$$

for every $\delta > 0$, δ small. Hence, according to [3, p. 48], (K_n^*) is a family of good kernels. Consequently, by [3, Theorem 4.1, p. 49],

$$(f * K_n^*)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)K_n^*(y)dy \to f(x)$$

for every continuity point x of $f \in L_{loc}(\mathbb{R})$, f 2π-periodic.

Readers having a good command of the Chinese language (unfortunately I don't), may also consult the classroom survey [4] for studies on summability/good kernels.

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Note on the F_0-spaces

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ABSTRACT

A rationally elliptic space X is called an F_0-space if its rational cohomology is concentrated in even degrees. The aim of this paper is to characterize such a space in terms of the homotopy groups of its skeletons as well as the rational cohomology of its Postnikov sections.

RESUMEN

Un espacio racionalmente elíptico X se llama un espacio F_0 si su cohomología racional está concentrada en grados pares. El propósito de este artículo es caracterizar dichos espacios en términos de los grupos de homotopía tanto de sus esqueletos como de la cohomología racional de sus secciones de Postnikov.

Keywords and Phrases: Rationaly elliptic space, Sullivan model, Quillen model, Whitehead exact sequence, F_0-space.

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1 Introduction

Along this paper space means a simply connected CW-complex X of finite type, *i.e.*, dim $H^n(X;\mathbb{Q}) < \infty$ for all n. A space X is called rationally elliptic if both the graded vector spaces $H^*(X;\mathbb{Q})$ and $\pi_*(X)\otimes\mathbb{Q}$ are finite dimensional. Furthermore, if $H^{\mathrm{odd}}(X;\mathbb{Q})=0$, then X is called an F_0-space. For instance, products of even spheres, complex Grassmannian manifolds and homogeneous spaces G/H such that rank $G=$ rank H are F_0-spaces.

Given a rationally elliptic space X. For any positive integer n, let $X^{[n]}$ denote the n-Postnikov section of X and X^n its n-skeleton. The aim of this paper is to characterize an F_0-space in terms of the homotopy groups of its skeletons and the rational cohomology of its Postnikov sections. More precisely, let:

$$\Gamma_n(X)=\ker(\pi_n(X^n)\otimes\mathbb{Q}\longrightarrow\pi_n(X^n;X^{n-1})\otimes\mathbb{Q}),\quad n\geq 2.$$

By exploiting the properties of the Whitehead exact sequences associated respectively with the Sullivan model and the Quillen model of X, we prove the following result

Theorem 1.1. *Let X be a rationally elliptic space. If $\pi_{\mathrm{even}}(X)\otimes\mathbb{Q}\neq 0$, then the following statements are equivalent.*

(1) *X is an F_0-space.*

(2) $\Gamma_{2n}(X)=0$ *for all* $n\geq 1$.

(3) $H^{2n+1}(X^{[2n-1]};\mathbb{Q})=0$ *for all* $n\geq 1$.

Note that if X is a (non-trivial) rationally elliptic space such that $\pi_{\mathrm{even}}(X)\otimes\mathbb{Q}=0$, then X cannot be an F_0-space as it is mentioned in Remark 3.3.

We show our results using standard tools of rational homotopy theory by working algebraically on the models of Quillen and Sullivan of X. We refer to [8] for a general introduction to these techniques. We recall some of the notation here. By a Sullivan algebra we mean a free graded commutative algebra ΛV, for some finite-type graded vector space $V=(V^{\geq 2})$, *i.e.*, dim $V^n<\infty$ for all $n\geq 2$, together with a differential ∂ of degree $+1$ that is decomposable, *i.e.*, satisfies $\partial : V\to\Lambda^{\geq 2}V$. Here $\Lambda^{\geq 2}V$ denotes the graded vector space spanned by all the monomials $v_1\cdots v_r$ such that $v_1,\ldots,v_r\in V$ and $r\geq 2$.

Every space X has a corresponding Sullivan algebra called the Sullivan model of X, unique up to isomorphism, that encodes the rational homotopy of X. In particular, we have the following identifications valid for every $n\geq 2$,

$$H^n(X;\mathbb{Q})\cong H^n(\Lambda V),\quad V^n\cong \mathrm{Hom}(\pi_n(X)\otimes\mathbb{Q},\mathbb{Q}).\tag{1.1}$$

Dually, by a free differential graded Lie algebra $(\mathbb{L}(W), \delta)$ (DGL for short), we mean a free graded Lie algebra $\mathbb{L}(W)$, for some finite-type vector space $W = (W_{\geq 1})$, together with a decomposable differential δ of degree -1, *i.e.*, $\delta(W) \to \mathbb{L}^{\geq 2}(W)$. Here $\mathbb{L}^{\geq 2}(W)$ denotes the graded vector space spanned by all the brackets of lengths ≥ 2.

Every space X has a corresponding DGL, called the Quillen model of X, unique up to isomorphism, and which determines completely the rational homotopy type of X. In particular, we have the following identifications valid for every $n \geq 2$,

$$\pi_n(X) \otimes \mathbb{Q} \cong H_{n-1}(\mathbb{L}(W)), \quad H_n(X;\mathbb{Q}) \cong W_{n-1}. \tag{1.2}$$

2 Whitehead exact sequences in rational homotopy theory

2.1 Whitehead exact sequence of a DGL

Let $(\mathbb{L}(W), \delta)$ be a DGL. For any positive integer n, we define the linear maps

$$j_n : H_n(\mathbb{L}(W_{\leq n})) \to W_n, \quad b_n : W_n \to H_{n-1}(\mathbb{L}(W_{\leq n-1})),$$

by setting

$$j_n([w + y]) = w, \quad b_n(w) = [\delta(w)], \tag{2.1}$$

were $[\delta(w)]$ denotes the homology class of $\delta(w)$ in the sub-Lie algebra $\mathbb{L}_{n-1}(W_{\leq n-1})$. Recall that if $x \in H_n(\mathbb{L}(W_{\leq n}))$, then $x = [w + y]$, where $w \in W_n$, $y \in \mathbb{L}_n(W_{\leq n-1})$ and $\delta(w + y) = 0$.

To every DGL $(\mathbb{L}(W), \delta)$, we can assign (see [2, 6, 7] for more details) the following long exact sequence

$$\cdots \to W_{n+1} \xrightarrow{b_{n+1}} \Gamma_n \to H_n(\mathbb{L}(W)) \xrightarrow{h_n} W_n \xrightarrow{b_n} \cdots \tag{2.2}$$

called the Whitehead exact sequence of $(\mathbb{L}(W), \delta)$, where

$$\Gamma_n = \ker(j_n : H_n(\mathbb{L}(W_{\leq n})) \to W_n), \quad \forall n. \tag{2.3}$$

Remark 2.1. *If $(\mathbb{L}(W), \delta)$ is the Quillen model of a space X, then by the properties of this model, the DGL $(\mathbb{L}(W_{\leq n}), \delta)$ can be chosen as the Quillen model of the $(n+1)$-skeleton X^{n+1}. Thus, we derive the following identification*

$$\Gamma_{n+1}(X) \cong \Gamma_n, \quad \forall n \geq 1 \tag{2.4}$$

where

$$\Gamma_{n+1}(X) = \ker(\pi_{n+1}(X^{n+1}) \otimes \mathbb{Q} \longrightarrow \pi_{n+1}(X^{n+1}; X^n) \otimes \mathbb{Q}). \tag{2.5}$$

2.2 Whitehead exact sequence of a Sullivan algebra

Likewise, let $(\Lambda V, \partial)$ be a Sullivan algebra. In [1, 4, 5], it is shown that with $(\Lambda V, \partial)$, we can associate the following long exact sequence

$$\cdots \to V^n \xrightarrow{b^n} H^{n+1}(\Lambda V^{\leq n-1}) \longrightarrow H^{n+1}(\Lambda V) \longrightarrow V^{n+1} \xrightarrow{b^{n+1}} \cdots \tag{2.6}$$

called the Whitehead exact sequence of $(\Lambda V, \partial)$. Recall that the linear map b^n is defined by setting $b^n(v) = [\partial(v)]$. Here $[\partial(v)]$ denotes the cohomology class of $\partial(v) \in \Lambda V^{\leq n-1}$.

Remark 2.2. *If $(\Lambda V, \partial)$ is the Sullivan minimal of a given space X, then by virtue of the properties of this model, $(\Lambda V^{\leq n-1}, \partial)$ can be chosen as the Sullivan model of the $(n-1)$-Postnikov section $X^{[n-1]}$. Thus, we derive the following identification*

$$H^{n+1}(X^{[n-1]}; \mathbb{Q}) \cong H^{n+1}(\Lambda V^{\leq n-1}), \quad \forall n \geq 2. \tag{2.7}$$

Proposition 2.3. *If $(\Lambda V, \partial)$ is the Sullivan model of a space X and $(\mathbb{L}(W), \delta)$ its Quillen model, then we have*

$$\Gamma_n = H^{n+2}(\Lambda V^{\leq n}), \quad \forall n \geq 2. \tag{2.8}$$

where Γ_n is defined in (2.3).

Proof. Applying the exact functor $\mathrm{Hom}(\cdot, \mathbb{Q})$ to the exact sequence (2.2) we obtain

$$\cdots \leftarrow \mathrm{Hom}(W_{n+1}, \mathbb{Q}) \leftarrow \mathrm{Hom}(\Gamma_n, \mathbb{Q}) \leftarrow \mathrm{Hom}(H_n(\mathbb{L}(W)), \mathbb{Q}) \leftarrow \mathrm{Hom}(W_n, \mathbb{Q}) \xleftarrow{b_n} \cdots \tag{2.9}$$

Taking into account that by virtues of the Quillen and Sullivan models we have

- Any vector space involved in this paper is of finite dimension which implies that it has the same dimension as its dual.
- $\mathrm{Hom}(W_n, \mathbb{Q}) \cong H^{n+1}(\Lambda V) \cong H^{n+1}(X; \mathbb{Q})$ for all $n \geq 1$.
- $\mathrm{Hom}(H_n(\mathbb{L}(W)); \mathbb{Q}) \cong V^{n+1} \cong \pi_{n+1}(X) \otimes \mathbb{Q}$ for all $n \geq 1$.
- The two maps $H^{n+1}(\Lambda V) \to V^{n+1}$ and $\mathrm{Hom}(W_n, \mathbb{Q}) \to \mathrm{Hom}(H_n(\mathbb{L}(W)), \mathbb{Q})$ appearing in (2.6) and (2.9) are the same linear map because they can be identified with the following linear map

  $$\mathrm{Hom}(H_{n+1}(X; \mathbb{Q}); \mathbb{Q}) \cong H^{n+1}(X; \mathbb{Q}) \to \mathrm{Hom}(\pi_{n+1}(X) \otimes \mathbb{Q}; \mathbb{Q}),$$

  which is the dual of the Hurewicz homomorphism $\pi_{n+1}(X) \otimes \mathbb{Q} \to H_{n+1}(X; \mathbb{Q})$. Here we use the well-known universal coefficient theorem.

Finally, by comparing the sequences (2.6), (2.9) we get (2.8). □

Corollary 2.4. *If X is a given space, then*

$$\Gamma_{n+1}(X) \cong H^{n+2}(X^{[n]};\mathbb{Q}), \quad \textit{as vector spaces}, \quad \forall n \geq 1. \tag{2.10}$$

Proof. It suffices to apply the identifications (2.4), (2.7) and Proposition 2.3 to the Sullivan model and the Quillen model of the space X. □

3 The main result

As it is stated in the introduction, a space X is called rationally elliptic if both the graded vector spaces $H^*(X;\mathbb{Q})$ and $\pi_*(X) \otimes \mathbb{Q}$ are finite dimensional.

Proposition 3.1 ([8, Proposition 32.10])**.** *If X is a rationally elliptic space and $(\Lambda V, \partial)$ its Sullivan model, then* $\dim H^{\text{even}}(\Lambda V) \geq \dim H^{\text{odd}}(\Lambda V)$. *Furthermore, the following statements are equivalent*

(1) *X is an F_0-space.*

(2) $\dim V^{\text{even}} = \dim V^{\text{odd}}$ *and $(\Lambda V, \partial)$ is pure, i.e., $\partial(V^{\text{even}}) = 0$ and $\partial(V^{\text{odd}}) \subseteq \Lambda V^{\text{even}}$.*

Using the identification (1.1) and (1.2), we can translate the above Proposition in terms of the Model of the Quillen. Thus, we have the following result.

Proposition 3.2. *If $(\mathbb{L}(W), \delta)$ is the Quillen model of a rationally elliptic space X, then* $\dim W_{\text{odd}} \geq \dim W_{\text{even}}$. *Moreover, the following statements are equivalent*

(1) *X is an F_0-space.*

(2) $W_{\text{even}} = 0$.

(3) $H_{\text{even}}(\mathbb{L}(W)) = H_{\text{even}}(\mathbb{L}(W))$.

Subsequently, we need the following obvious remark.

Remark 3.3. *Let $(\mathbb{L}(W), \delta)$ be the Quillen model of a rationally elliptic space X.*

(1) *If $W_{\text{odd}} = 0$, then X is rationally trivial. Indeed, Since X is a rationally elliptic space, using Proposition 3.1, it follows that* $\dim W_{\text{odd}} \geq \dim W_{\text{even}}$. *Hence, if $W_{\text{odd}} = 0$, then $W = W_{\text{odd}} \oplus W_{\text{even}} = 0$ implying that X is rationally trivial.*

(2) *If X is a (non-trivial) rationally elliptic space such that $\pi_{\text{even}}(X) \otimes \mathbb{Q} = 0$, then X cannot be an F_0-space. Indeed, if so, then we must have*

$$\dim\, \pi_{\text{even}}(X) \otimes \mathbb{Q} = \dim\, \pi_{\text{odd}}(X) \otimes \mathbb{Q}.$$

Therefore, $\dim\, \pi_*(X) \otimes \mathbb{Q} = \dim\, \pi_{\text{even}}(X) \otimes \mathbb{Q} + \dim\, \pi_{\text{odd}}(X) \otimes \mathbb{Q} = 0$. *As a result, X is rationally trivial.*

Proposition 3.4. *Let $(\mathbb{L}(W), \delta)$ be the Quillen model of a rationally elliptic space such that $H_{\text{odd}}(\mathbb{L}(W)) \neq 0$. If $\Gamma_{\text{odd}} = 0$, then $W_{\text{even}} = 0$.*

Proof. Assume by contradiction that $W_{\text{even}} \neq 0$ and let $w_0 \in W_{\text{even}}$ such that

$$|w_0| = \max\{|w|, \quad w \in W_{\text{even}}\}. \tag{3.1}$$

Let us consider the Whitehead exact sequence (2.2) of $(\mathbb{L}(W), \delta)$. Since $\Gamma_{\text{odd}} = 0$, it follows that $b_{|w_0|}(w_0) = 0$ and from the relation (2.1) there exists a decomposable element in $q_0 \in \mathbb{L}(W)$ such that $\delta(w_0 + q_0) = 0$.

Next, as $H_{\text{odd}}(\mathbb{L}(W)) \neq 0$, there exists a non-trivial homology class $\{w + y\} \in H_{2m+1}(\mathbb{L}(W))$, where $w \in W_{2m+1}$ and y is a decomposable element in $\mathbb{L}_{2m+1}(W)$, for a certain $m \in \mathbb{N}$.

Therefore, the bracket $[w_0 + q_0, w + y]$ is a decomposable cycle of degree $|w_0| + 2m - 1$ providing a homology class in the vector space

$$\Gamma_{|w_0|+2m+1} \subset H_{|w_0|+2m+1}(\mathbb{L}(W_{\leq |w_0|+2m+1})).$$

It is worth noting that as $|w_0|$ is even, then $|w_0| + 2m + 1$ is odd and by taking into account the relation (3.1), the cycle $[w_0 + q_0, w + y]$ cannot be a boundary in $\Gamma_{|w_0|+2m+1}$ implying that $\Gamma_{\text{odd}} \neq 0$. Contradiction. □

Corollary 3.5. *Let X be a rationally elliptic space such that $\pi_{\text{even}}(X) \otimes \mathbb{Q} \neq 0$ and let $\Gamma_*(X)$ as in* (2.5). *If $\Gamma_{\text{even}}(X) = 0$, then X is an F_0-space.*

Proof. Working algebraically, let $(\mathbb{L}(W), \delta)$ be the Quillen model of X. Since $\Gamma_{\text{even}}(X) = 0$, the identifications (2.4) implies that $\Gamma_{\text{odd}} = 0$. Next, by applying Proposition 3.4, it follows that $W_{\text{even}} = 0$ and by the identifications (1.2), we deduce that $H^{\text{even}}(X; \mathbb{Q}) = 0$. Hence, X is an F_0-space. □

Corollary 3.5 implies the following result which gives a characterization of an F_0-space X in terms of the homotopy groups of its skeletons.

Corollary 3.6. *Let X be a rationally elliptic space such that $\pi_{\text{even}}(X)\otimes\mathbb{Q}\neq 0$. If*

$$\pi_{2n}(X^{2n})\otimes\mathbb{Q}=0,\quad \forall n\geq 1, \tag{3.2}$$

then X is an F_0-space.

Proof. First, according to (2.5), we know that $\Gamma_{2n}(X)\subset\pi_{2n}(X^{2n})\otimes\mathbb{Q}$ for all $n\geq 1$. Therefore, the relation (3.2) implies that $\Gamma_{\text{even}}(X)=0$. Then, it suffices to apply Corollary 3.5. □

The next result gives characterization of an F_0-space X in terms of the rational cohomology of its Postnikov sections.

Corollary 3.7. *Let X be a rationally elliptic space such that $\pi_{\text{even}}(X)\otimes\mathbb{Q}\neq 0$. If*

$$H^{2n+1}(X^{[2n-1]};\mathbb{Q})=0,\quad \forall n\geq 1,$$

then X is an F_0-space.

Proof. First, by Corollary 2.4, if $H^{2n+1}(X^{[2n-1]};\mathbb{Q})=0$ for all n, then $\Gamma_{\text{even}}(X)=0$. Next, it suffices to apply Corollary 3.6. □

Proposition 3.8. *If X is an F_0-space, then $\Gamma_{2n}(X)=0,\quad \forall n\geq 1$.*

Proof. Let $(\Lambda V,\partial)$ be the Sullivan model of X. By (2.6), the Whitehead exact sequence of $(\Lambda V,\partial)$ can be written as

$$\cdots\to V^{2n}\xrightarrow{b^{2n}} H^{2n+1}(\Lambda V^{\leq 2n-1})\longrightarrow H^{2n+1}(\Lambda V)\longrightarrow V^{2n+1}\xrightarrow{b^{2n+1}}\cdots$$

As X is an F_0-space, then by Proposition 3.1, the Sullivan model $(\Lambda V,\partial)$ of X satisfies $H^{\text{odd}}(\Lambda V)=0$ and $\partial(V^{\text{even}})=0$, it follows that the maps $b^{\text{even}}=0$. Consequently, $H^{2n+1}(\Lambda V^{\leq 2n-1})=0$ for every $n\geq 1$. Hence, the result follows from the formula (2.10). □

Proof of Theorem 1.1. It follows from Corollaries 3.5, 3.7 and Proposition 3.8 after taking Remark 3.3 into account. □

Conflict of interest

The author has not disclosed any competing interests.

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Quotient rings satisfying some identities

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ABSTRACT

This paper investigates the commutativity of the quotient ring $\mathcal{R}/P$, where $\mathcal{R}$ is an associative ring with a prime ideal P, and the possibility of forms of derivations satisfying certain algebraic identities on $\mathcal{R}$. We provide some results for strong commutativity-preserving derivations of prime rings.

RESUMEN

Este artículo investiga la conmutatividad del anillo cociente $\mathcal{R}/P$, donde $\mathcal{R}$ es un anillo asociativo con un ideal primo P, y la posibilidad de formas de derivaciones que satisfacen ciertas identidades algebraicas en $\mathcal{R}$. Entregamos algunos resultados para derivaciones de anillos primos que preservan la conmutatividad fuerte.

Keywords and Phrases: Derivations, prime ideals, prime rings.

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1 Introduction

In all that follows, $\mathcal{R}$ always denotes an associative ring with center $Z(\mathcal{R})$ and $\mathcal{C}$ is the extended centroid of $\mathcal{R}$ (we refer the reader to [3] for more information about these objects). As usual, the symbols $[s,t]$ and $s\circ t$ denote the commutator $st-ts$ and the anticommutator $st+ts$, respectively. Recall that a ring $\mathcal{R}$ is prime if $x\mathcal{R}y=\{0\}$ implies $x=0$ or $y=0$, and $\mathcal{R}$ is semiprime if $x\mathcal{R}x=\{0\}$ implies $x=0$.

A map $\mathcal{D}:\mathcal{R}\to\mathcal{R}$ is called a multiplicative derivation if $\mathcal{D}(xy)=\mathcal{D}(x)y+x\mathcal{D}(y)$ for all $x,y\in\mathcal{R}$, if $\mathcal{D}$ is also additive, we say that $\mathcal{D}$ is a derivation of $\mathcal{R}$.

The study of commutativity preserving mappings has been an active research area in matrix theory, operator theory, and ring theory (see [8, 19] for references). According to [5], let $\mathcal{S}$ be a subset of $\mathcal{R}$. A map $\varphi:\mathcal{R}\to\mathcal{R}$ is said to be strongly commutativity preserving (SCP) on $\mathcal{S}$ if $[\varphi(x),\varphi(y)]=[x,y]$ for all $x,y\in\mathcal{S}$. In [4], Bell and Daif investigated commutativity in rings admitting a derivation which is SCP on a nonzero right ideal. In particular, they proved that if a semiprime ring $\mathcal{R}$ admits a derivation $\mathcal{D}$ satisfying $[\mathcal{D}(x),\mathcal{D}(y)]=[x,y]$ for all x,y in a right ideal I of $\mathcal{R}$, then $I\subseteq Z(\mathcal{R})$ (see [9] for more information). In particular, $\mathcal{R}$ is commutative if $I=\mathcal{R}$. Later, Deng and Ashraf [10] proved that if there exists a derivation $\mathcal{D}$ of a semiprime ring $\mathcal{R}$ and a map $\varphi:I\to\mathcal{R}$ defined on a non-zero ideal I of $\mathcal{R}$ such that $[\varphi(x),\mathcal{D}(y)]=[x,y]$ for all $x,y\in I$, then $\mathcal{R}$ contains a non-zero central ideal. In particular, they showed that $\mathcal{R}$ is commutative if $I=\mathcal{R}$. Recently, this result was extended to Lie ideals and symmetric elements of prime rings by Lin and Liu in [12] and [13]. There is also a growing literature on strong commutativity preserving (SCP) maps and derivations (for references see [4, 8, 16], etc.) In [1], Ali *et al.* showed that if $\mathcal{R}$ is a semiprime ring and f is an endomorphism that is a strong commutativity preserving (simple, SCP) map on a non-zero ideal U of $\mathcal{R}$, then f commutes on U. In [18], Samman proved that an epimorphism of a semiprime ring is strongly commutativity preserving if and only if it is centralizing. Derivations and SCP mappings have been extensively studied in the context of operator algebras, prime rings, and semiprime rings. Many related generalizations of these results can be found in the literature (see for example [8, 11, 14, 15, 17]).

In this paper, we discuss the notion of a derivation that satisfies one of the following conditions:

(*i*) $[\mathcal{D}(x),\mathcal{D}(y)]+H([x,y])\in P$, for all $x,y\in\mathcal{R}$,

(*ii*) $\mathcal{D}(x)\circ\mathcal{D}(y)+H(x\circ y)\in P$, for all $x,y\in\mathcal{R}$,

(*iii*) $[\mathcal{D}(x),F(y)]+H(x\circ y)\in P$, for all $x,y\in\mathcal{R}$,

(*iv*) $\mathcal{D}(x)\circ\mathcal{D}(y)+H([x,y])\in P$, for all $x,y\in\mathcal{R}$,

where P is a prime ideal of $\mathcal{R}$, $\mathcal{D}$ is a derivation, and H is a multiplier of $\mathcal{R}$.

2 Results

In this section, we discuss some well-known results in the rings theory, which will be used in the following sections.

(*i*) $[x, yz] = y[x, z] + [x, y]z$.

(*ii*) $[xy, z] = [x, z]y + x[y, z]$.

(*iii*) $xy \circ z = (x \circ z)y + x[y, z] = x(y \circ z) - [x, z]y$.

(*iv*) $x \circ yz = y(x \circ z) + [x, y]z = (x \circ y)z + y[z, x]$.

Lemma 2.1 ([2], Lemma 2.1)**.** *Let $\mathcal{R}$ be a ring, P be a prime ideal of $\mathcal{R}$, and $\mathcal{D}$ a derivation of $\mathcal{R}$. If $[\mathcal{D}(x), x] \in P$ for all $x \in \mathcal{R}$, then $\mathcal{D}(\mathcal{R}) \subseteq P$ or $\mathcal{R}/P$ is commutative.*

Lemma 2.2 ([6])**.** *Let $\mathcal{R}$ be a prime ring. If functions $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ and $\mathcal{G} : \mathcal{R} \to \mathcal{R}$ are such that $\mathcal{F}(x)y\mathcal{G}(z) = \mathcal{G}(x)y\mathcal{F}(z)$ for all $x, y, z \in \mathcal{R}$, and $\mathcal{F} \neq 0$, then there exists λ in the extended centroid of $\mathcal{R}$ such that $\mathcal{G}(x) = \lambda\mathcal{F}(x)$ for all $x \in \mathcal{R}$.*

The following two Lemmas are also used to prove our theorems. The primary goal is to establish a connection between the commutativity of rings $\mathcal{R}/\mathcal{P}$ and the behavior of their derivations.

Lemma 2.3. *Let $\mathcal{R}$ be a ring and P be a prime ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a derivation $\mathcal{D}$ such that $\mathcal{R}$ satisfies one of the following assertions:*

(*i*) *$[x, \mathcal{D}(y)] \in P$ for all $x, y \in \mathcal{R}$,*

(*ii*) *$x \circ \mathcal{D}(y) \in P$ for all $x, y \in \mathcal{R}$,*

then $\mathcal{D}(\mathcal{R}) \subseteq P$ or $\mathcal{R}/P$ is commutative.

Proof. (*i*) Suppose that

$$[x, \mathcal{D}(y)] \in P \quad \text{for all} \;\; x, y \in \mathcal{R}. \tag{2.1}$$

Replacing y by yt in (2.1), we obtain

$$\mathcal{D}(y)[x, t] + [x, \mathcal{D}(y)]t + y[x, \mathcal{D}(t)] + [x, y]\mathcal{D}(t) \in P \quad \text{for all} \quad x, y, t \in \mathcal{R}.$$

Using (2.1), we get

$$\mathcal{D}(y)[x, t] + [x, y]\mathcal{D}(t) + y[x, \mathcal{D}(t)] \in P \quad \text{for all} \quad x, y, t \in \mathcal{R}. \tag{2.2}$$

For $x = t$ in (2.2), it follows that

$$[t, y]\mathcal{D}(t) + y[t, \mathcal{D}(t)] \in P \quad \text{for all} \quad y, t \in \mathcal{R}. \tag{2.3}$$

Taking ry instead of y in (2.3) and using (2.3), we conclude that

$$[t, r]y\mathcal{D}(t) \in P \quad \text{for all} \quad r, y, t \in \mathcal{R}.$$

Equivalently,

$$[t, r]\mathcal{R}\mathcal{D}(t) \subseteq P \quad \text{for all} \quad r, t \in \mathcal{R}.$$

By primeness of P, we arrive at

$$[t, r] \in P \quad \text{or} \quad \mathcal{D}(t) \in P \quad \text{for all} \quad r, t \in \mathcal{R}. \tag{2.4}$$

If there exists $t_0 \in \mathcal{R}$ such that $\mathcal{D}(t_0) \in P$, then (2.3) implies that $y[t_0, \mathcal{D}(t_0)] \in P$ for all $y \in \mathcal{R}$ which implies that $[t_0, \mathcal{D}(t_0)]y[t_0, \mathcal{D}(t_0)] \in P$ for all $y \in \mathcal{R}$. Since P is prime, then $[t_0, \mathcal{D}(t_0)] \in P$. So, (2.4) becomes $[t, \mathcal{D}(t)] \in P$ for all $t \in P$, in this case Lemma 2.1 forces that $\mathcal{D}(\mathcal{R}) \subseteq P$ or $\mathcal{R}/P$ is commutative.

(*ii*) Using the same techniques as those used in the proof of (*i*) with minor modifications, we can easily arrive at our result. □

Lemma 2.4. *Let $\mathcal{R}$ be a ring and P be a prime ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a derivation D such that $\mathcal{R}$ satisfies any of the following assertions:*

(*i*) $[x, \mathcal{D}(x)] \in P$ *for all* $x \in \mathcal{R}$,

(*ii*) $x \circ \mathcal{D}(x) \in P$ *for all* $x \in \mathcal{R}$,

then $\mathcal{D}(\mathcal{R}) \subseteq P$ *or* $\mathcal{R}/P$ *is commutative.*

Proof. (*i*) Assuming that

$$[x, \mathcal{D}(x)] \in P \quad \text{for all} \quad x \in \mathcal{R}. \tag{2.5}$$

Linearizing Eq. (2.5), we obtain

$$[x, \mathcal{D}(y)] + [y, \mathcal{D}(x)] \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.6}$$

Replacing y by yx in (2.6), and using it with (2.5) we obtain

$$y[x, \mathcal{D}(x)] + [x, y]\mathcal{D}(x) \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.7}$$

Putting yz instead of y in (2.7), where $z \in \mathcal{R}$ and using it again, we get

$$[x,y]z\mathcal{D}(x) \in P \quad \text{for all} \quad x,y,z \in \mathcal{R}.$$

Since P is prime ideal of $\mathcal{R}$, we arrive at

$$[x,y] \in P \quad \text{or} \quad \mathcal{D}(x) \in P \quad \text{for all} \quad x,y \in \mathcal{R}. \tag{2.8}$$

Suppose that $\mathcal{D}(\mathcal{R}) \nsubseteq P$. There exists $x \in \mathcal{R}$ such that $\mathcal{D}(x) \notin P$. By (2.8), we get $[x,y] \in P$ for all $y \in \mathcal{R}$ which implies that $\bar{x} \in Z(\mathcal{R}/P)$. Let $z \in \mathcal{R}$ such that $\bar{z} \notin Z(\mathcal{R}/P)$. Then, there exists $y_0 \in \mathcal{R}$ such that $[z,y_0] \notin P$. Therefore, from (2.8), we find that $\mathcal{D}(z) \in P$. On the other hand, since $\mathcal{D}(x) \notin P$, we can derive $\mathcal{D}(x+z) \notin P$. Using (2.8) again, the last expression gives $[x+z,y] \in P$ for all $y \in \mathcal{R}$, which forces that $[z,y] \in P$ for all $y \in \mathcal{R}$, a contradiction.

(ii) Suppose that

$$x \circ \mathcal{D}(x) \in P \quad \text{for all} \quad x \in \mathcal{R}. \tag{2.9}$$

Linearizing (2.9), we get

$$x \circ \mathcal{D}(y) + y \circ \mathcal{D}(x) \in P \quad \text{for all} \quad x,y \in \mathcal{R}. \tag{2.10}$$

Substituting yx for y in (2.10), and using it again, we find that

$$y(x \circ \mathcal{D}(x)) + [x,y]\mathcal{D}(x) + y[x,\mathcal{D}(x)] \in P \quad \text{for all} \quad x,y \in \mathcal{R}. \tag{2.11}$$

Replacing y by yz in (2.11), where $z \in \mathcal{R}$ and using it again, we obtain

$$[x,y]z\mathcal{D}(x) \in P \quad \text{for all} \quad x,y,z \in \mathcal{R}. \tag{2.12}$$

Continuing with the same techniques as used in (i), and we get the required result. □

Corollary 2.5. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a nonzero derivation $\mathcal{D}$, then the following assertions are equivalent:*

- (i) $[x,\mathcal{D}(x)] = 0$ *for all* $x \in \mathcal{R}$.
- (ii) $\mathcal{R}$ *is commutative.*

Corollary 2.6. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a nonzero derivation $\mathcal{D}$ then the following assertions are equivalent:*

- (i) $x \circ \mathcal{D}(x) = 0$ *for all* $x \in \mathcal{R}$.

(*ii*) $\mathcal{R}$ *is commutative of characteristic equal* 2.

Proof. By Lemma 2.4 we get that $\mathcal{R}$ is commutative. In this case, our identity becomes $2xD(x) = 0$ for all $x, y \in \mathcal{R}$. Linearizing the last equation, we find $2xD(y) + 2yD(x) = 0$ for all $x, y \in \mathcal{R}$. Replacing y by yx, we obtain $2xD(yx) = 0$ for all $x, y \in \mathcal{R}$. This implies that $2xD(y)x = 0$ for all $x, y \in \mathcal{R}$. Replacing yt with y and using the last expression, we get $2x\mathcal{R}y\mathcal{R}D(t)\mathcal{R}x = \{0\}$ for all $x, y \in \mathcal{R}$. Since $D \neq 0$, we conclude that $2x = 0$ for all $x \in \mathcal{R}$. □

Theorem 2.7. *Let $\mathcal{R}$ be a ring and P a prime ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admits a multiplier H and a derivation $\mathcal{D}$ of $\mathcal{R}$ such that $\mathcal{D}(P) \subseteq P$. If $[\mathcal{D}(x), \mathcal{D}(y)] + H([x, y]) \in P$ for all $x, y \in \mathcal{R}$, then one of the following assertions holds:*

(*i*) $H(\mathcal{R}) \subseteq P$.

(*ii*) *There exists $\lambda \in C$ such that $\mathcal{D} - \lambda$ maps $\mathcal{R}$ into P with $(\lambda^2 + H)([x, y]) \in P$ for all $x, y \in \mathcal{R}$.*

(*iii*) $\mathcal{R}/P$ *is a commutative ring.*

Proof. Suppose that $\mathcal{R}/P$ is not a commutative ring and

$$[\mathcal{D}(x), \mathcal{D}(y)] + H([x, y]) \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.13}$$

Replacing x by xt in (2.13) and using it, we conclude that

$$\mathcal{D}(x)[t, \mathcal{D}(y)] + x[\mathcal{D}(t), \mathcal{D}(y)] + [x, \mathcal{D}(y)]\mathcal{D}(t) + H(x)[t, y] \in P \quad \text{for all} \quad x, y, t \in R. \tag{2.14}$$

Substituting ux for x in (2.14), we find that

$$\mathcal{D}(u)x[t, \mathcal{D}(y)] + u\mathcal{D}(x)[t, \mathcal{D}(y)] + ux[\mathcal{D}(t), \mathcal{D}(y)] + u[x, \mathcal{D}(y)]\mathcal{D}(t) + [u, \mathcal{D}(y)]x\mathcal{D}(t) + uH(x)[t, y] \in P. \tag{2.15}$$

Left-multiplying (2.14) by u and comparing it with (2.15), we get

$$(\mathcal{D}(u)x - u\mathcal{D}(x))[t, \mathcal{D}(y)] + u\mathcal{D}(x)[t, \mathcal{D}(y)] + [u, \mathcal{D}(y)]x\mathcal{D}(t) \in P \quad \text{for all} \quad x, y, u, t \in \mathcal{R}. \tag{2.16}$$

Taking $t = \mathcal{D}(y)$ in (2.16), we obtain $[u, \mathcal{D}(y)]\mathcal{R}\mathcal{D}(\mathcal{D}(y)) \subseteq P$ for all $u, y \in \mathcal{R}$.

By primeness of P, it follows that for each y in $\mathcal{R}$ either $[u, \mathcal{D}(y)] \in P$ for all $u \in \mathcal{R}$ or $\mathcal{D}(\mathcal{D}(y)) \in P$. Let $A = \{y \in \mathcal{R} \mid [u, \mathcal{D}(y)] \in P \text{ for all } u \in \mathcal{R}\}$ and $B = \{y \in \mathcal{R} \mid \mathcal{D}(\mathcal{D}(y)) \in P\}$. Clearly, A and B are additive subgroups of $\mathcal{R}$ such that $A \cup B = \mathcal{R}$. The fact that a group cannot be a union of two of its proper subgroups, forces us to conclude that either $\mathcal{R} = A$ or $\mathcal{R} = B$.

Assume that $\mathcal{R} = A$. Then by Lemma 2.3 (i) and our hypothesis, we get $\mathcal{D}(\mathcal{R}) \subseteq P$. In the latter case, from our assumption we get

$$[u, \mathcal{D}(yw)] = \mathcal{D}(y)[u, w] + [u, \mathcal{D}(y)]w + [u, y\mathcal{D}(w)] \in P \quad \text{for all} \quad y, u, w \in \mathcal{R}.$$

Since $[u, \mathcal{D}(yw)] \in P$ and $[u, \mathcal{D}(y)] \in P$ for all $w, u, y \in P$, it is easy to notice that $\mathcal{D}(y)[u, w] \in P$ for all $y, u, w \in \mathcal{R}$. From this, we can easily arrive at $\mathcal{D}(y)\mathcal{R}[u, w] \subseteq P$ for all $u, w, y \in \mathcal{R}$. Hence, it follows that $\mathcal{D}(\mathcal{R}) \subseteq P$. From our initial hypothesis (2.13), we get

$$H([x, y]) \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.17}$$

In (2.17), replacing x by xt and using it again, we find that $[x, y]H(t) \in P$ for all $x, y, t \in \mathcal{R}$. Replacing y by yr, where $r \in \mathcal{R}$, we get $[x, y]rH(t) \in P$ for all $x, y, r, t \in \mathcal{R}$, which implies that by the primeness of P that $H(\mathcal{R}) \subseteq P$.

Next, we consider the case $\mathcal{R} = B$, it follows that $\mathcal{D}(\mathcal{D}(y)) \in P$ for all $y \in \mathcal{R}$. It implies that for each $x, y \in \mathcal{R}$, we have $\mathcal{D}([\mathcal{D}(x), \mathcal{D}(y)]) \in P$. Applying d to equation (2.13) and using the condition $d(P) \subseteq P$, we get

$$\mathcal{D}(H([x, y])) \in P \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{2.18}$$

Replacing x by xy in (2.18) and using it, we find that

$$H([x, y])\mathcal{D}(y) \in P \quad \text{for all} \quad x, y \in \mathcal{R}.$$

Replacing x by xt and using it, we find $H([x, y])t\mathcal{D}(y) \in P$ for all $x, y, t \in \mathcal{R}$. Therefore, either $H([\mathcal{R}, \mathcal{R}]) \subseteq P$ or $\mathcal{D}(\mathcal{R}) \subseteq P$. If $H([\mathcal{R}, \mathcal{R}]) \subseteq P$, then as in (2.17) we have $H(\mathcal{R}) \subseteq P$. Let us suppose that $\mathcal{D}(\mathcal{R}) \subseteq P$, from (2.16) we have $(\mathcal{D}(u)x - u\mathcal{D}(x))[t, \mathcal{D}(y)] \in P$ for all $x, y, t, u \in \mathcal{R}$, which means that $(\mathcal{D}(u)x - u\mathcal{D}(x)) \in P$ for all $x, u \in \mathcal{R}$ or $[t, \mathcal{D}(y)] \in P$ for all $y, t \in \mathcal{R}$ (the second case is already discussed above). So, we assume that $\mathcal{D}(u)x - u\mathcal{D}(x) \in P$ for all $u, x \in \mathcal{R}$. Replacing u by uy, we get

$$\overline{\mathcal{D}(u)yI_{\mathcal{R}}(x)} = \overline{I_{\mathcal{R}}(u)y\mathcal{D}(x)} \quad \text{for all} \quad x, y, u \in \mathcal{R},$$

where $I_{\mathcal{R}}$ is the identity map of $\mathcal{R}$.

Using Lemma 2.2, there exists $\overline{\lambda} \in \overline{C}$ such that $\overline{\mathcal{D}(x)} = \overline{\lambda x}$ for all $x \in \mathcal{R}$. It implies that $\mathcal{D}(x) - \lambda x \in P$ for all $x \in \mathcal{R}$. Hence,

$$[\{\mathcal{D} - \lambda\}(x), \{\mathcal{D} + \lambda\}(y)] \in P.$$

In view of our hypothesis, we get $(\lambda^2 + H)([x, y]) \in P$ for all $x, y \in \mathcal{R}$. □

In the same way, we can get the following result:

Theorem 2.8. *Let $\mathcal{R}$ be a ring and P a prime ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a multiplier H and a derivation $\mathcal{D}$ with $\mathcal{D}(P) \subseteq P$, such that any one of the following assertions holds:*

(a) $\mathcal{D}(x) \circ \mathcal{D}(y) + H([x, y]) \in P$ *for all* $x, y \in \mathcal{R}$,

(b) $\mathcal{D}(x) \circ \mathcal{D}(y) + H(x \circ y) \in P$ *for all* $x, y \in \mathcal{R}$,

(c) $[\mathcal{D}(x), \mathcal{D}(y)] + H(x \circ y) \in P$ *for all* $x, y \in \mathcal{R}$,

then one of the following holds:

(i) $H(\mathcal{R}) \subseteq P$.

(ii) *There exists* $\lambda \in C$ *such that* $\mathcal{D} - \lambda$ *maps* $\mathcal{R}$ *into* P *with* $(\lambda^2 + H)([x, y]) \in P$ *for all* $x, y \in \mathcal{R}$.

(iii) $\mathcal{R}/P$ *is a commutative ring.*

Proof. (a) By our assumption

$$\mathcal{D}(x) \circ \mathcal{D}(y) + H([x, y]) \in P \quad \text{for all} \quad x, y \in \mathcal{R}.$$

Replacing x by xt in the above expression and using it, we conclude that

$$\mathcal{D}(x)[t, \mathcal{D}(y)] + (x \circ \mathcal{D}(y))\mathcal{D}(t) + x[\mathcal{D}(t), \mathcal{D}(y)] + H(x)[t, y] \in P. \tag{2.19}$$

Substituting ux for x in (2.19), we find that

$$\begin{aligned} \mathcal{D}(u)x[t, \mathcal{D}(y)] + u\mathcal{D}(x)[t, \mathcal{D}(y)] + u(x \circ \mathcal{D}(y))\mathcal{D}(t) - [u, \mathcal{D}(y)]x\mathcal{D}(t) \\ +ux[\mathcal{D}(t), \mathcal{D}(y)] + uH(x)[t, y] \in P \quad \text{for all} \quad x, y, t, u \in \mathcal{R}. \end{aligned} \tag{2.20}$$

From the Left multiplying (2.19) by u and comparing with (2.20), we get

$$(\mathcal{D}(u)x - u\mathcal{D}(x))[t, \mathcal{D}(y)] + u\mathcal{D}(x)[t, \mathcal{D}(y)] - [u, \mathcal{D}(y)]x\mathcal{D}(t) \in P \quad \text{for all} \quad x, y, t, u \in \mathcal{R}. \tag{2.21}$$

We process using the same approach as in Theorem 2.7, and finally, we arrive at our result. We can reach the conclusions of (b) and (c) by using similar techniques as before, with the necessary variations of (c). □

It is easy to prove that the maps $I_{\mathcal{R}}$ and $-I_{\mathcal{R}}$ are multipliers of $\mathcal{R}$. We get the following results by replacing H with $\mp I_{\mathcal{R}}$:

Corollary 2.9. *Let $\mathcal{R}$ be a ring and P be a proper prime ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a derivation $\mathcal{D}$ with $\mathcal{D}(P) \subseteq P$, such that $\mathcal{R}$ satisfies one of the following assertions:*

(a) $[\mathcal{D}(x), \mathcal{D}(y)] \pm [x, y] \in P$ *for all* $x, y \in \mathcal{R}$,

(b) $\mathcal{D}(x) \circ \mathcal{D}(y) \pm [x, y] \in P$ *for all* $x, y \in \mathcal{R}$,

(c) $\mathcal{D}(x) \circ \mathcal{D}(y) \pm (x \circ y) \in P$ *for all* $x, y \in \mathcal{R}$,

(d) $[\mathcal{D}(x), \mathcal{D}(y)] \pm (x \circ y) \in P$ *for all* $x, y \in \mathcal{R}$,

then one of the following holds:

(i) *there exists* $\lambda \in C$ *such that* $\mathcal{D} - \lambda$ *maps* $\mathcal{R}$ *into* P *with* $(\lambda^2 \mp I)([x, y]) \in P$ *for all* $x, y \in \mathcal{R}$;

(ii) $\mathcal{R}/P$ *is a commutative ring.*

Replacing H by $-I_{\mathcal{R}}$ in the Theorem 2.7 and P by $\{0\}$, we get the following corollary:

Corollary 2.10. *If $\mathcal{R}$ is a prime ring admitting a strong commutativity preserving (SCP) derivation $\mathcal{D}$, then one of the following assertions holds:*

(1) *There exists* $\lambda \in C$ *such that* $\mathcal{D}(x) = \lambda x$ *for all* $x \in \mathcal{R}$ *with* $\lambda^2 = 1$;

(2) $\mathcal{R}$ *is a commutative ring.*

Replacing H by $-I_{\mathcal{R}}$ in the Theorem 2.8 and P by $\{0\}$, we obtain the following corollary:

Corollary 2.11. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a derivation $\mathcal{D}$, such that any one of the following assertions hold:*

(a) $\mathcal{D}(x) \circ \mathcal{D}(y) = [x, y]$ *for all* $x, y \in \mathcal{R}$,

(b) $\mathcal{D}(x) \circ \mathcal{D}(y) = x \circ y$ *for all* $x, y \in \mathcal{R}$,

(c) $[\mathcal{D}(x), \mathcal{D}(y)] = x \circ y$ *for all* $x, y \in \mathcal{R}$,

then one of the following holds:

(i) *There exists* $\lambda \in C$ *such that* $\mathcal{D}(x) = \lambda x$ *for all* $x \in \mathcal{R}$ *with* $\lambda^2 = 1$;

(ii) $\mathcal{R}$ *is a commutative ring.*

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Approximate controllability of non-instantaneous impulsive stochastic integrodifferential equations driven by Rosenblatt process via resolvent operators

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ABSTRACT

In this work, we investigate the existence of a mild solution and the approximate controllability of non-instantaneous impulsive stochastic integrodifferential equations driven by the Rosenblatt process in Hilbert space with the Hurst parameter $\mathsf{H} \in (1/2, 1)$. We achieve the result using the semigroup theory of bounded linear operators, Grimmer's resolvent operator theory, and stochastic analysis. Using Krasnoselskii's and Schauder's fixed point theorems, we demonstrate the existence of mild solutions and the approximate controllability of the system. Finally, an example shows the potential for significant results.

RESUMEN

En este trabajo investigamos la existencia de una solución mild y la controlabilidad aproximada de ecuaciones integro-diferenciales estocásticas no-instantáneas impulsivas dirigidas por el proceso de Rosenblatt en espacios de Hilbert con el parámetro de Hurst $\mathsf{H} \in (1/2, 1)$. Logramos este resultado usando la teoría de semigrupos de operadores lineales acotados, la teoría del operador resolvente de Grimmer y análisis estocástico. Usando los teoremas de punto fijo de Krasnoselskii y Schauder, demostramos la existencia de soluciones mild y la controlabilidad aproximada del sistema. Finalmente, un ejemplo muestra el potencial para resultados significativos.

Keywords and Phrases: Approximate controllability, fixed point theorem, Rosenblatt process, stochastic integrodifferential equations, resolvent operator, non-instantaneous impulses.

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1 Introduction

Stochastic differential equations have become an active field of study because of their various applications in fields such as electrical engineering, mechanics, medical biology, economic systems, etc. For more information, see [2, 11, 18, 29]. The mathematical description of the phenomenon under investigation must account for randomness since many real-world events, such as stock prices, heat conduction in memory materials, and rising population, are unpredictable or noisy. It has been demonstrated that stochastic differential systems are especially powerful methods for describing and understanding this kind of event. Stochastic differential systems theory has been applied to model various phenomena in this life. Numerous authors have also investigated the existence, uniqueness, stability, controllability, approximate controllability, and other qualitative and quantitative properties of SDEs and stochastic integrodifferential equations (SIEs) using stochastic analysis, the fixed point approach, and the concept of resolvent operators in the case of SIEs. See for example, [6, 9, 15, 17]. In the last decades the theory of impulsive partial equations or inclusions seems to be a natural description of many real processes that are exposed to some disturbances, the duration of which is insignificant in comparison to the duration of the process. In addition to impulsive effects, stochastic effects also exist in real systems. Thus, impulsive stochastic differential equations describing these dynamical systems subject to both impulsive and stochastic changes have attracted significant attention. In particular, the papers [3, 26, 40] have studied the existence of smooth solutions for certain impulsive neutral stochastic functional integrodifferential equations with infinite delay in Hilbert spaces.

Let us consider $(\zeta_n)_{n\in\mathbb{Z}}$ a stationary Gaussian sequence with correlation function holds $\mathfrak{R}(n) = \mathbb{E}(\zeta_0\zeta_n) = n^{\frac{2\mathsf{H}-2}{k}}\mathscr{L}(n)$, with $\mathsf{H}\in\left(\frac{1}{2},1\right)$ and $\mathscr{L}\to\infty$. Let $\mathscr{G}$ denote the Hermite function of rank H. Also, if $\mathscr{G}$ admits the following,

$$\mathscr{G}(\rho) = \sum_{j\geq 0} c_j \mathsf{H}_j(\rho), \quad c_j = \frac{1}{j!}\mathbb{E}(\mathscr{G}(\zeta_0\mathsf{H}(\zeta_0))),$$

then $\mathsf{H} = \min\{j | c_j \neq 0\} \geq 1$. $\mathsf{H}_j(\rho) = (-1)^j e^{\frac{\rho^2}{2}} \frac{\partial_j}{\partial\rho_j} e^{-\frac{\rho^2}{2}}$, where $\mathsf{H}_j(\rho)$ is the Hermite polynomial of degree j. Then by the Non-central Limit Theorem, $\frac{1}{n^{\mathsf{H}}}\sum_{[nt]}^{j=1}\mathscr{G}(\zeta_j)$ converges as $n\to\infty$ in the sense of finite-dimensional distributions to the process

$$\mathsf{R}_{\mathsf{K}}^{\mathsf{H}}(\rho) = c(\mathsf{H},\mathsf{K})\int_{\mathbb{R}}\int_0^1 \left(\prod_{j=1}^{\mathsf{K}}(\xi-\vartheta)_+^{-\left(\frac{1}{2}+\frac{1-\mathsf{H}}{\mathsf{K}}\right)}\right) dW(\vartheta_1)\cdots dW(\vartheta_{\mathsf{K}}), \tag{1.1}$$

The (1.1) is a Wiener integral of order K with respect to the standard Brownian motion $(W(\vartheta))_{\vartheta\in\mathbb{R}}$ and $c(\mathsf{H},\mathsf{K})$ is normalizing constant depends on H and K. The process $(\mathsf{R}_{\mathsf{K}}^{\mathsf{H}}(\rho))_{\rho\geq 0}$ is known as the Hermite process.

- If $\mathsf{K} = 1$, the process (1.1) is the fractional Brownian motion with Hurst index $\mathsf{H} \in \left(\frac{1}{2}, 1\right)$.

- If $\mathsf{K} = 2$, the process given by (1.1) is called the Rosenblatt process, and it is not a Gaussian process, see [36, 37].

Fractional Brownian motion is a Gaussian stochastic process, which depends on a parameter $\mathsf{H} \in (0, 1)$ called the Hurst index established by Kolmogorov [24]. For further reference on fractional Brownian motion, we refer the reader to [28]. There is another process like Rosenblatt's process with a non-Gaussian character, which contributes to the other properties for $\mathsf{H} > 1/2$, the long memory property. Self-similar processes with long-range dependence are seen in a variety of fields, including econometrics, internet traffic, hydrology, turbulence, and finance. The Rosenblatt process is a self-similar process with stationary increments that occurs as the limit of long-range-dependent stationary series. Still, it is not a Gaussian process, however, in real situations when the Gaussianity is not plausible for the model, one can use the Rosenblatt process. Comparatively, Rosenblatt process gains its interest due to its convolution of the dependence structures and the property of non-Gaussianity. Therefore, it seems stimulating to establish the SDEs with Rosenblatt process. Observations of stock price processes suggest that they are not self-similar. In particular, in [5, 22], the authors established the existence and uniqueness of mild solutions for stochastic differential equations driven by the Rosenblatt process with finite delay. Recently, in [7, 8, 34, 35, 38], the authors analyzed the stability and controllability of the stochastic functional differential equation driven by the Rosenblatt process. Also, many real-life phenomena and processes are characterized by abrupt changes in their state variable. These changes can be classified into two types: (i) In the first type, the changes take place over a relatively short period compared to the overall duration of the whole process, known as instantaneous impulses. (ii) In the second type, these changes start impulsively at certain times and remain active for certain intervals, known as non-instantaneous impulses. A well-known application of non-instantaneous impulses is the introduction of insulin into the bloodstream, which is an abrupt change. The resulting absorption is gradual because it remains active for a finite time interval. Models of this situation are created using differential and integrodifferential equations of non-instantaneous pulses detailed in [21, 23].

Approximate controllability refers to moving a system from an arbitrary initial state to a state arbitrarily close to a final state using only certain admissible controls. Recently, many authors have established results on the approximate controllability of first, second, and fractional-order differential equations with impulses; [1, 14, 32], and the references cited there. In references [12, 16, 39], the authors studied the approximate controllability of fractional stochastic Hilfer integrodifferential equations.

Motivated by this consideration, in this paper, we investigate the existence of mild solutions and approximate controllability of non-instantaneous impulsive stochastic integrodifferential equations driven by the Rosenblatt process having the following form:

$$\begin{cases} d\vartheta(\rho) = [\mathrm{A}\vartheta(\rho) + \int_0^\rho \mathbf{\Gamma}(\rho - s)\vartheta(s)ds + \mathsf{Bu}(\rho) + \mathsf{F}(\rho, \vartheta(\rho))]d\rho + \mathsf{G}(\rho, \vartheta(\rho))d\mathsf{R}^{\mathsf{H}}(\rho), \\ \qquad\qquad\qquad\qquad\qquad\qquad \rho \in \cup_{i=0}^{m}(s_i, \rho_{i+1}), \\ \vartheta(\rho) = \mathsf{p}_i(\rho, \vartheta(\rho_i^-)), \quad \rho \in \cup_{i=1}^{m}(\rho_i, s_i], \\ \vartheta(0) = \vartheta_0, \end{cases} \tag{1.2}$$

where $0 = \rho_0 = s_0 < \rho_1 < \cdots < s_m < \rho_{m+1} = \mathsf{b}$, $\mathcal{J} = [0, \mathsf{b}]$, $\vartheta(\cdot)$ takes values in the separable Hilbert space $\mathbb{H}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. $\mathrm{A} : \mathcal{D}(\mathrm{A}) \subset \mathbb{H} \to \mathbb{H}$ and $\mathbf{\Gamma}(\rho) : \mathcal{D}(\mathbf{\Gamma}(\rho)) \subset \mathbb{H} \to \mathbb{H}$ are closed linear unbounded operators with $\mathcal{D}(\mathbf{\Gamma}(\rho)) \supset \mathcal{D}(\mathrm{A})$. $\{\mathsf{R_H}(\rho)\}_{\rho \geq 0}$ is Q-Rosenblatt process with Hurst index $\mathsf{H} \in (\frac{1}{2}, 1)$ defined in a complete probability space $(\Omega, , \mathcal{F}, \{\mathcal{F}_\rho\}_{\rho \geq 0}; \mathbb{P})$ with values in a Hilbert space $\mathbb{K}$. The functions $\mathsf{p}_i(\rho, \vartheta(\rho_i^-))$ represent non-instantaneous impulses in the intervals $(\rho_i, s_i], i = 1, 2, \ldots, m$, and the functions $\mathsf{F} : [0, \mathsf{b}] \times \mathbb{H} \to \mathbb{H}$, $\mathsf{G} : [0, \mathsf{b}] \times \mathbb{H} \to L_0^2(\mathbb{K}, \mathbb{H})$ are appropriate functions wich will be specified later. The control function $\mathsf{u}(\cdot)$ is given in $L^2_{\mathcal{F}_\rho}([0, \mathsf{b}], \mathfrak{U})$ of admissible control functions, where $L^2_{\mathcal{F}_\rho}([0, \mathsf{b}], \mathfrak{U})$ is the Hilbert space of all $\mathcal{F}_\rho$-adopted, square integrable processes; $\mathfrak{U}$ is a Hilbert space; B is a bounded linear operator from $\mathfrak{U}$ into $\mathbb{H}$.

More specifically, our work focuses on developing a set of new, good criteria for the existence of mild solutions and approximate controllability of non-instantaneous impulsive stochastic integro-differential equations driven by the Rosenblatt process having the following abstract form (1.2).

The main contributions of our work, in particular, are summarized in the three aspects listed below:

- A new class of non-instantaneous impulsive partial stochastic integrodifferential equations driven by the Rosenblatt process in Hilbert spaces is formulated.
- Initially, we establish the existence and uniqueness of mild solutions of the system above using stochastic analysis theory and the fixed point technique combined with resolvent operator theory.
- In comparison to [6, 17, 23], we enhance the approach and ease the conditions.
- Non-instantaneous impulsive partial stochastic integrodifferential equations driven by the Rosenblatt process in Hilbert spaces have received little attention in the literature. In order to bridge this gap, we have looked into the approximate controllability of (1.2).

This paper is organized as follows. In Section 2, we give some preliminaries, basic definitions, and results, which will be used in the sequel. In Section 3, the existence and approximate controllability outcomes of the considered system (1.2) are discussed. Section 4 illustrates the derived theoretical results through an example. Section 5 presents the conclusion and future direction of works in the last part of this work.

2 Preliminaries

Throughout this paper, $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{H}$ represent the real separable Hilbert spaces and $(\Omega, \mathcal{F}, \{\mathcal{F}_\rho\}_{\rho\geq 0}; \mathbb{P})$ be a complete probability space with natural filtration $(\mathcal{F}_\rho)_{\rho\geq 0}$, where $\mathcal{F}_\rho$, the Random variables generate σ-algebra $\Big\{\beta^{\mathsf{H}}(s), W(s), s \in [0,\rho]\Big\}$ and $\mathbb{P}$-null sets. We denote by $L^2_{\mathcal{F}_\rho}([0,\mathsf{b}],\mathbb{H})$ the space of all square integrable and $\mathcal{F}_\rho$-adapted process from $[0,\mathsf{b}]$ to $\mathbb{H}$ and $\mathcal{L}(\mathbb{X},\mathbb{H})$, $\mathcal{L}(\mathbb{Y},\mathbb{H})$ are respectively, the space of bounded linear operators from $\mathbb{X}$ to $\mathbb{H}$ and $\mathbb{Y}$ to $\mathbb{H}$. For convenience, the same notation $\|\cdot\|$ is used to denote the norms in $\mathbb{X}$, $\mathbb{H}$, $\mathbb{Y}$, $\mathcal{L}(\mathbb{X},\mathbb{H})$ and $\mathcal{L}(\mathbb{Y},\mathbb{H})$ and the inner product of $\mathbb{X}$, $\mathbb{H}$, $\mathbb{Y}$ is denoted by $\langle\cdot,\cdot\rangle$.

Let $\mathbb{C}([0,\mathsf{b}], L^2(\Omega,\mathbb{H}))$ be the space of all continuous $\mathcal{F}_\rho$-adapted measurable processes from $[0,\mathsf{b}]$ to $L^2(\Omega,\mathbb{H})$ that satisfy $\sup\limits_{\rho\in[0,\mathsf{b}]} \mathbb{E}\|\vartheta(\rho)\|^2 < \infty$. Then, it is easy to see that $\mathbb{C}\Big([0,\mathsf{b}], L^2(\Omega,\mathbb{H})\Big)$ is a Banach space equipped with the following norm:

$$\|\vartheta\|_{\mathbb{C}} = \left(\sup_{\rho\in[0,\mathsf{b}]} \mathbb{E}\|\vartheta(\rho)\|^2\right)^{\frac{1}{2}}. \tag{2.1}$$

Let

$$\mathbf{V_q} = \Big\{\vartheta \in \mathbb{C}([0,\mathsf{b}], L^2(\Omega,\mathbb{H})) \,:\, \|\vartheta\|^2_{\mathbb{C}} \leq \mathsf{q}\Big\}. \tag{2.2}$$

2.1 Rosenblatt process

Consider a time interval $[0,\mathsf{b}]$ with arbitrary fixed horizon b and $\{\mathsf{R}^{\mathsf{H}}(\rho), \rho\in[0,\mathsf{b}]\}$ the one dimensional Rosenblatt process with parameter $\mathsf{H}\in(\frac{1}{2},1)$, R^{H} has the following integral representation [37]

$$\mathsf{R}_{\mathsf{H}}(\rho) = q(\mathsf{H})\int_0^\rho\int_0^\rho \left[\int_{\vartheta_1\vee\vartheta_2}^\rho \frac{\partial K^{\mathsf{H}'}}{\partial u}(u,\vartheta_1)\frac{\partial K^{\mathsf{H}'}}{\partial u}(u,\vartheta_2)du\right] dW_1(\vartheta_1)dW_1(\vartheta_2), \tag{2.3}$$

where $K^{\mathsf{H}}(\rho,s)$ is given by

$$K^{\mathsf{H}}(\rho,s) = c_{\mathsf{H}} s^{\frac{1}{2}-\mathsf{H}}\int_s^\rho (u-s)^{\mathsf{H}-3/2}u^{\mathsf{H}-1/2}du \quad \text{for} \quad \rho > s,$$

with

$$c_{\mathsf{H}} = \sqrt{\frac{\mathsf{H}(2\mathsf{H}-1)}{\beta(2-2\mathsf{H},\mathsf{H}-\frac{1}{2})}},$$

$\beta(\cdot,\cdot)$ denotes the Beta function, $K^{\mathsf{H}}(\rho,s)=0$ when $\rho\le s$, $\{W_1(\rho),\rho\in[0,\mathsf{b}]\}$ is a Brownian motion, $\mathsf{H}'=\frac{\mathsf{H}+1}{2}$ and $q(\mathsf{H})=\frac{1}{\mathsf{H}+1}\sqrt{\frac{\mathsf{H}}{2(2\mathsf{H}-1)}}$ is a normalizing constant. The covariance of the Rosenblatt process $\{\mathsf{R}_{\mathsf{H}}(\rho),\rho\in[0,\mathsf{b}]\}$ satisfies

$$\mathbb{E}(\mathsf{R}_{\mathsf{H}}(\rho)\mathsf{R}_{\mathsf{H}}(s))=\frac{1}{2}\left(s^{2\mathsf{H}}+\rho^{2\mathsf{H}}-|s-\rho|^{2\mathsf{H}}\right)$$

and this structure of $\{\mathsf{R}_{\mathsf{H}}(\rho)\}_{\rho\in[0,\mathsf{b}]}$ allows us to represent it as a Wiener integral.

Let $\mathsf{R}_Q^{\mathsf{H}}(\rho)$ be a $\mathbb{K}$-valued Rosenblatt process with covariance Q as

$$\mathsf{R}_Q^{\mathsf{H}}(\rho)=\mathsf{R}_Q(\rho)=\sum_{n=1}^{\infty}\sqrt{\delta_n}\xi_n(\rho)e_n,\quad \rho\ge 0.$$

Next, we introduce the space $L_0^2(\mathbb{K},\mathbb{H})$ of all Q-Hilbert-Schmidt operators $\Psi:\mathbb{K}\to\mathbb{H}$. Recall that $\Psi\in L(\mathbb{K},\mathbb{H})$ is called a Q-Hilbert-Schmidt operator if

$$\|\Psi\|=\sum_{n=1}^{\infty}\|\sqrt{\delta_n}\Psi e_n\|^2<\infty,$$

and that the space L_0^2 equipped with the inner product $<\phi,\psi>_{L_0^2}=\sum_{n=1}^{\infty}<\phi e_n,\psi e_n>$, is a Hilbert space.

Let $\rho:[0,\mathsf{b}]\to L^2(Q^{1/2}\mathbb{K},\mathbb{H})$ such that

$$\sum_{n=1}^{\infty}\|K_{\mathsf{H}}^*(\rho Q^{1/2}e_n)\|_{L^2([0,\mathsf{b}];\mathbb{H})}<\infty. \tag{2.4}$$

Definition 2.1 (Tudor [37]). *Let $\kappa(l):[0,\mathsf{b}]\to L^2(Q^{1/2}\mathbb{K},\mathbb{H})$ satisfy (2.4). In that case, the stochastic integral of κ with respect to the Rosenblatt process $\mathsf{R}_Q^{\mathsf{H}}(\rho)$ is defined for $\rho\ge 0$ as follows*

$$\int_0^\rho \kappa(l)d\mathsf{R}_Q^{\mathsf{H}}(l):=\sum_{n=1}^{\infty}\int_0^\rho \kappa(s)Q^{1/2}e_nd\mathsf{R}_n(l)=\sum_{n=1}^{\infty}\int_0^\rho\int_0^\tau (K_{\mathsf{H}}^*(\kappa Q^{1/2}e_n))(\vartheta_1,\vartheta_2)dW_1(\vartheta_1)dW_1(\vartheta_2).$$

Lemma 2.2 ([34]). *For any $\kappa:[0,\mathsf{b}]\to L^2(Q^{1/2}\mathbb{K},\mathbb{H})$ such that $\sum_{n=1}^{\infty}\|\kappa Q^{1/2}e_n\|_{L^{1/\mathsf{H}}([0,\mathsf{b}];\mathbb{V})}<\infty$ holds, and for any $\alpha,\beta\in[0,\mathsf{b}]$ with $\beta>\alpha$, we have*

$$\mathbb{E}\left\|\int_\alpha^\beta \kappa(\rho)d\mathsf{R}_Q(\rho)\right\|^2\le c_{\mathsf{H}}(\beta-\alpha)^{2\mathsf{H}-1}\sum_{n=1}^{\infty}\int_\alpha^\beta\|\kappa(\rho)Q^{1/2}e_n\|^2d\rho.$$

If, in addition,

$$\sum_{n=1}^{\infty}\|\kappa(\rho)Q^{1/2}e_n\| \text{ is uniformly convergent for } \rho\in[0,\mathsf{b}],$$

then, it holds that

$$\mathbb{E}\left\|\int_{\alpha}^{\beta}\kappa(\rho)d\mathsf{R}_Q(\rho)\right\|^2 \leq c_{\mathsf{H}}(\beta-\alpha)^{2\mathsf{H}-1}\int_{\alpha}^{\beta}\|\kappa(\rho)\|^2_{L^2(Q^{1/2}\mathbb{K},\mathbb{V})}d\rho.$$

For further references, we refer to [19, 37].

2.2 Integrodifferential equations in Banach spaces

We recall some knowledge of partial integrodifferential equations and the related resolvent operators. Let $\mathcal{D}$ be the Banach space $\mathcal{D}(\mathrm{A})$ equipped with the graph norm defined by

$$\|\vartheta\|_{\mathcal{D}} := \|\mathrm{A}\vartheta\| + \|\vartheta\| \quad \text{for } \vartheta \in \mathcal{D}.$$

We denote by $\mathcal{C}(\mathbb{R}^+, \mathcal{D})$, the space of all functions from $\mathbb{R}^+$ into $\mathcal{D}$ which are continuous. Let us consider the following system for further purposes:

$$\begin{cases} \vartheta'(\rho) &= \mathrm{A}\vartheta(\rho) + \displaystyle\int_0^{\rho}\mathbf{\Gamma}(\rho-s)\vartheta(s)ds \quad \text{for } \rho \geq 0 \\ \vartheta(0) &= \vartheta_0 \in \mathcal{D}. \end{cases} \tag{2.5}$$

Definition 2.3 ([20]). *A resolvent operator for equation* (2.5) *is a bounded linear operator valued function* $\Psi(\rho) \in \mathcal{L}(\mathbb{H})$ *for* $\rho \geq 0$*, having the following properties :*

(*i*) $\Psi(0) = I$ *(the identity map of* $\mathbb{H}$*) and* $\|\Psi(\rho)\| \leq Ne^{\beta\rho}$ *for some constants* $N > 0$ *and* $\beta \in \mathbb{R}$.

(*ii*) *For each* $\vartheta \in \mathbb{H}$, $\Psi(\rho)\vartheta$ *is strongly continuous for* $\rho \geq 0$.

(*iii*) *For* $\vartheta \in \mathbb{H}, \Psi(\cdot)\vartheta \in \mathbb{C}^1(\mathbb{R}^+;\mathbb{H}) \cap \mathbb{C}(\mathbb{R}^+;\mathcal{D})$ *and*

$$\Psi'(\rho)\vartheta = \mathrm{A}\Psi(\rho)\vartheta + \int_0^{\rho}\mathbf{\Gamma}(\rho-s)\Psi(s)\vartheta ds = \Psi(\rho)\mathrm{A}\vartheta + \int_0^{\rho}\Psi(\rho-s)\mathbf{\Gamma}(s)\vartheta ds, \quad \textit{for } \rho \in [0, \mathsf{b}].$$

Next, we assume A and $(\mathbf{\Gamma}(\rho))_{\rho\geq 0}$ satisfy the following conditions:

($\mathbf{R_1}$) The operator A is the infinitesimal generator of a strongly continuous semigroup $(\mathsf{T}(\rho))_{\rho\geq 0}$ on $\mathbb{H}$.

($\mathbf{R_2}$) For all $\rho \geq 0$, the operator $\mathbf{\Gamma}(\rho)$ is closed and linear from $\mathcal{D}(\mathrm{A})$ to $\mathbb{Y}$ and $\mathbf{\Gamma}(\rho) \in \mathcal{L}(\mathcal{B}, \mathbb{H})$. For any $\vartheta \in \mathbb{H}$, the map $\rho \mapsto \mathbf{\Gamma}(\rho)\vartheta$ is bounded, differentiable and the derivative $\rho \mapsto \mathbf{\Gamma}'(\rho)\vartheta$ is bounded and uniformly continuous for $\rho \geq 0$.

Theorem 2.4 ([20]). *Assume that* ($\mathbf{R_1}$)-($\mathbf{R_2}$) *hold. Then, there exists a unique resolvent operator of the Cauchy problem* (2.5).

We have the following useful results.

Theorem 2.5 ([13]). *Let the assumptions* $(\mathbf{R_1})$ *and* $(\mathbf{R_2})$ *be satisfied. Let the* C_0*-semigroup* $(\mathsf{T}(\rho))_{\rho\geq 0}$ *generated by* A *be compact for* $\rho > 0$. *Then the corresponding resolvent operator* $(\Psi(\rho))_{\rho\geq 0}$ *of equation* (1.2) *is also compact for* $\rho > 0$.

Lemma 2.6 ([13]). *Let the assumptions* $(\mathbf{R_1})$ *and* $(\mathbf{R_2})$ *be satisfied. Then, there exists a constant* $L = L(\mathsf{b})$ *such that*

$$\|\Psi(\rho+\varepsilon) - \Psi(\varepsilon)\Psi(\rho)\|_{\mathcal{L}(\mathbb{H})} \leq L(\varepsilon), \quad \text{for } 0 < \varepsilon \leq \rho \leq \mathsf{b}.$$

Based on these, we have the following Theorem establishing the equivalence between operator-norm continuity of the semigroup generated by A and the resolvent operator $(\Psi(\rho))_{\rho\geq 0}$ corresponding to the linear equation (2.5).

Theorem 2.7 ([25]). *Let* A *be the infinitesimal generator of a* C_0*-semigroup* $(\mathsf{T}(\rho))_{\rho\geq 0}$ *and let* $(\mathbf{\Gamma}(\rho))_{\rho\geq 0}$ *satisfy* $(\mathbf{R_2})$. *Then the resolvent operator* $(\Psi(\rho))_{\rho\geq 0}$ *for Eq.* (2.5) *is operator-norm continuous (or continuous in the uniform operator topology) for* $\rho > 0$ *if and only if* $(\mathsf{T}(\rho))_{\rho\geq 0}$ *is operator-norm continuous for* $\rho > 0$.

Now, we introduce the space $C_{\mathsf{b}} = \mathcal{PC}([0,\mathsf{b}], L^2(\Omega,\mathbb{H}))$ formed by all $\mathbb{H}-$valued stochastic processes

$$\{\vartheta(\rho), \rho\in[0,\mathsf{b}] \text{ such that } \vartheta|_{I_i}\in C(I_i,\mathbb{H}) \text{ for all } w\in\Omega, i=0,1,\dots,m, \text{ and there exist}$$
$$\vartheta(\rho_i^-) \text{ and } \vartheta(\rho_i^+), i=1,2,\dots,m \text{ with } \vartheta(\rho_i^-)=\vartheta(\rho_i) \text{ and } \sup_{\rho\in[0,\mathsf{b}]}\mathbb{E}\|\vartheta(\rho)\|^2<\infty\}$$

endowed with the norm

$$\|\vartheta\|_{\mathcal{PC}} = \left(\sup_{\rho\in[0,\mathsf{b}]}\mathbb{E}\|\vartheta(\rho)\|^2\right)^{1/2}, \tag{2.6}$$

where $I_i = (\rho_i, \rho_{i+1}]$, $i = 0,1,\dots,m$.

Now, we define the mild solution of Eq. (1.2) expressed by the resolvent operator $\Psi(\rho)$ as follows.

Definition 2.8. *A* $\mathbb{H}$*-valued stochastic process* $\vartheta\in C([0,\mathsf{b}], L^2(\Omega,\mathbb{H}))$ *is called a mild solution of the stochastic problem* (1.2)*, if*

(1) $\vartheta(\rho)$ *is* $\mathcal{F}_\rho-$*adapted and measurable for each* $\rho\geq 0$.

(2) $\vartheta(\rho)$ *has càdlàg paths on* $\rho\in[0,\mathsf{b}]$ *a.s. and for each* $\rho\in[0,\mathsf{b}], \vartheta(\rho)$ *satisfies* $\vartheta(\rho) = \mathsf{p}_i(\rho,\vartheta(\rho_i^-))$ *for all* $\rho\in(\rho_i, s_i]$, $i = 1,2,\dots,m$ *and* $\vartheta(\rho)$ *is the solution of the following integral equations*

$$\begin{aligned}\vartheta(\rho) &= \Psi(\rho)\vartheta_0 + \int_0^\rho \Psi(\rho - s)\ \mathsf{F}(s, \vartheta(s))\ \mathrm{d}s + \int_0^\rho \Psi(\rho - s)\ \mathsf{Bu}(s)\ \mathrm{d}s \\ &\quad + \int_0^\rho \Psi(\rho - s)G(s, \vartheta(s))d\mathsf{R}^{\mathsf{H}}(s)\,\mathrm{d}s, \quad \textit{for } \rho \in [0, \rho_1], \\ \vartheta(\rho) &= \Psi(\rho - s_i)\mathsf{p}_i(s_i, \vartheta(\rho_i^-)) + \int_{s_i}^\rho \Psi(\rho - s)\ \mathsf{F}(s, \vartheta(s))\ \mathrm{d}s + \int_{s_i}^\rho \Psi(\rho - s)\ \mathsf{Bu}(s)\ \mathrm{d}s \\ &\quad + \int_{s_i}^\rho \Psi(\rho - s)G(s, \vartheta(s))d\mathsf{R}^{\mathsf{H}}(s)\,\mathrm{d}s, \quad \textit{for } \rho \in [s_i, \rho_{i+1}], \quad i = 1, 2, \ldots, m.\end{aligned} \tag{2.7}$$

Let us denote the state value of the system (1.2) at the time ρ by $\vartheta_\rho = \vartheta(\rho; \vartheta_0, \mathsf{u})$ with respect to initial value ϑ_0 and the control function u. The set of all final states is known as reachable set of the system (1.2) and defined as $\mathfrak{M}(\mathsf{b}, \vartheta_0, \mathsf{u}) = \Big\{\vartheta_{\mathsf{b}} = \vartheta(\mathsf{b}; \vartheta_0, \mathsf{u}) : \mathsf{u} \in L^2([0, \mathsf{b}], \mathfrak{U})\Big\}$.

Definition 2.9. *Eq.* (1.2) *is said to be approximately controllable on the interval* $[0, \mathsf{b}]$*, if*

$$\overline{\mathfrak{M}(\mathsf{b}, \vartheta_0, \mathsf{u})} = L^2(\Omega, \mathbb{H}),$$

that is, for arbitrary $\varepsilon > 0$*, it is possible to steer the state from the point* ϑ_0 *to within a distance* ε *from all points in the state space* $L^2(\Omega, \mathbb{H})$ *at time* b.

To discuss the approximate controllability of system (1.2) we introduce the following operators.

(1) The controllability Grammian $\mathbf{\Pi}_0^{\mathsf{b}}$ is defined by:

$$\mathbf{\Pi}_{s_i}^{\rho_{i+1}} = \int_{s_i}^{\rho_{i+1}} \Psi(\rho_{i+1} - s)\mathsf{B}\,\mathsf{B}^*\Psi^*(\rho_{i+1} - s)ds,$$

where B^* and $\Psi^*(\rho)$ denote the adjoint of the operators B and $\Psi(\rho)$.

(2) $W(\gamma, \mathbf{\Pi}_{s_i}^{\rho_{i+1}}) = (\gamma\,\mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}$.

In the sequel we assume that the operator $W(\gamma, \mathbf{\Pi}_{s_i}^{\rho_{i+1}})$ satisfies

($\mathbf{H}_0$) $\gamma W(\gamma, \mathbf{\Pi}_{s_i}^{\rho_{i+1}}) \to 0$ as $\gamma \to 0^+$ in the strong operator topology.

The above condition ($\mathbf{H}_0$) is equivalent to the approximate controllability of the linear system.

$$\begin{cases} \dfrac{d\vartheta(\rho)}{d\rho} = \mathrm{A}\vartheta(\rho) + \displaystyle\int_0^\rho \mathbf{\Gamma}(\rho - s)\vartheta(s)ds + \mathsf{Bu}(\rho), & \rho \in [0, \mathsf{b}], \\ \vartheta(0) = \vartheta_0. \end{cases} \tag{2.8}$$

In fact, we have that

Theorem 2.10 ([4, 10]). *The following statements are equivalent:*

(*i*) *The control system* (2.8) *is approximately controllable on* $[0, \mathsf{b}]$.

(*ii*) $\mathsf{B}^*\Psi^*(\rho)\vartheta = 0$ *for all* $\rho \in [0, \mathsf{b}]$ *imply* $\vartheta = 0$.

(*iii*) *The condition* $(\mathbf{H}_0)$ *holds.*

Lemma 2.11 ([27]). *For any* $\vartheta_{\rho_{i+1}} \in L^2(\Omega, \mathcal{F}_{\rho_{i+1}}, \mathbb{H})$, *there exist* $\Phi_i \in L^2(\Omega;\ L^2([s_i, \rho_{i+1}]; L_2^0(\mathbb{Y}, \mathbb{H})))$, *such that* $\vartheta_{\rho_{i+1}} = \mathbb{E}\vartheta_{\rho_{i+1}} + \int_{s_i}^{\rho_{i+1}} \Phi_i(s) d\mathsf{R}^{\mathsf{H}}(s)$.

Our results are based on the following Krasnoselskii's and Schauder's fixed point theorem.

Theorem 2.12 (Krasnoselskii's theorem [32]). *Let $\mathcal{B}$ be a closed, bounded and convex subset of a Banach space $\mathbb{H}$, and let Φ_1, Φ_2 be maps of $\mathcal{B}$ into $\mathbb{H}$ such that $\Phi_1\vartheta_1 + \Phi_2\vartheta_2 \in \mathcal{B}$, for all $\vartheta_1, \vartheta_2 \in \mathcal{B}$. If Φ_1 is a contraction and Φ_2 is continuous and compact, then the equation $\vartheta = \Phi_1\vartheta + \Phi_2\vartheta$ has a solution on $\mathcal{B}$.*

Theorem 2.13 (Schauder's theorem [33]). *If $\mathcal{B}$ is a closed, bounded and convex subset of a Banach space $\mathbb{H}$ and $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is completely continuous, then $\mathcal{F}$ has a fixed point in $\mathcal{B}$.*

3 Approximate controllability results

This section proves the approximate controllability of the stochastic control system (1.2). Let $M = \sup_{\rho\in[0,\mathsf{b}]} \|\Psi(\rho)\|$. In order to establish the results, we impose the following hypotheses.

(C$_1$) $\mathsf{T}(\rho)$ is compact for $\rho > 0$.

(C$_2$) The maps $\mathsf{p}_i : \mathsf{b}_i \times \mathbb{H} \to \mathbb{H}$, $\mathsf{b}_i = [\rho_i, s_i]$, $i = 1, 2, \ldots, m$ are continuous functions and satisfy

(a) There exist constants $D_{\mathsf{p}_i} > 0$, $i = 1, 2, \ldots, m$, such that

$$\mathbb{E}\|\mathsf{p}_i(\rho, \vartheta)\|^2 \leq D_{\mathsf{p}_i}(1 + \mathbb{E}\|\vartheta\|^2), \quad \forall \rho \in \mathsf{b}_i \text{ and } \vartheta \in \mathbb{H}.$$

(b) There exist constants $R_{\mathsf{p}_i} > 0$, $i = 1, 2, \ldots, m$, such that

$$\mathbb{E}\|\mathsf{p}_i(\rho, \vartheta_1) - \mathsf{p}_i(\rho, \vartheta_2)\|^2 \leq R_{\mathsf{p}_i}\mathbb{E}\|\vartheta_1 - \vartheta_2\|^2, \quad \forall \rho \in \mathsf{b}_i \text{ and } \vartheta_1, \vartheta_2 \in \mathbb{H}.$$

(C$_3$) The map $\mathsf{F} : \mathsf{b}_0 \times \mathbb{H} \to \mathbb{H}$, $\mathsf{b}_0 = \bigcup_{i=0}^{m} [s_i, \rho_{i+1}]$ is a continuous function and satisfies

(a) There exists a constant $\mathfrak{M}_{\mathsf{F}} > 0$ such that

$$\mathbb{E}\|\mathsf{F}(\rho,\vartheta)\|^2 \leq \mathfrak{M}_{\mathsf{F}}(1+\mathbb{E}\|\vartheta\|^2), \quad \forall \rho \in \mathsf{b}_0 \text{ and } \vartheta \in \mathbb{H}.$$

(b) There exists a constant $R_{\mathsf{F}} > 0$ such that

$$\mathbb{E}\|\mathsf{F}(\rho,\vartheta_1) - \mathsf{F}(\rho,\vartheta_2)\|^2 \leq R_{\mathsf{F}}\mathbb{E}\|\vartheta_1 - \vartheta_2\|^2, \quad \forall \rho \in \mathsf{b}_0 \text{ and } \vartheta_1, \vartheta_2 \in \mathbb{H}.$$

($\mathbf{C}_4$) The map $\mathsf{G} : \mathsf{b}_0 \times \mathbb{H} \to L_2^0$, is a continuous function and satisfies

(a) There exists a constant $\mathfrak{M}_{\mathsf{G}} > 0$ such that

$$\mathbb{E}\|\mathsf{G}(\rho,\vartheta)\|^2 \leq \mathfrak{M}_{\mathsf{G}}(1+\mathbb{E}\|\vartheta\|^2), \quad \forall \rho \in \mathsf{b}_0 \text{ and } \vartheta \in \mathbb{H}.$$

(b) There exists a constant $R_{\mathsf{G}} > 0$ such that

$$\mathbb{E}\|\mathsf{G}(\rho,\vartheta_1) - \mathsf{G}(\rho,\vartheta_2)\|^2 \leq R_{\mathsf{G}}\mathbb{E}\|\vartheta_1 - \vartheta_2\|^2, \quad \forall \rho \in \mathsf{b}_0 \text{ and } \vartheta_1, \vartheta_2 \in \mathbb{H}.$$

($\mathbf{C}_5$) The following inequalities hold

(a) $\max\limits_{0\leq i\leq m} N_i < 1,$

(b) $\max\limits_{1\leq i\leq m} D_{\mathsf{p}_i} < 1,$

(c) $\max\limits_{1\leq i\leq m} \{M^2\|\mathsf{B}\|^2 R_{\mathsf{u}_0}\rho_1^2, R_{\mathsf{p}_i}, 2\left(M^2 R_{\mathsf{p}_i} + M^2\|\mathsf{B}\|^2 R_{\mathsf{u}_i}\rho_{i+1}^2\right)\} < 1.$

($\mathbf{C}_6$) The linear control system (2.8) is approximately controllable on $[0,\mathsf{b}]$.

For any $\gamma > 0$, we define the operator $\mathcal{S}^{(\gamma)} : C([0,\mathsf{b}], L^2(\Omega,\mathbb{H})) \to C([0,\mathsf{b}], L^2(\Omega,\mathbb{H}))$ by

$$(\mathcal{S}^{(\gamma)}\vartheta)(\rho) = \Psi(\rho)\vartheta_0 + \int_0^\rho \Psi(\rho-s)\,\mathsf{F}(s,\vartheta(s))\,\mathrm{ds} + \int_0^\rho \Psi(\rho-s)\,\mathsf{B}\mathsf{u}^{(\gamma)}(s,\vartheta)\,\mathrm{ds} \\ + \int_0^\rho \Psi(\rho-s)\,\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s), \quad \forall \rho \in [0,\rho_1]$$

and

$$(\mathcal{S}^{(\gamma)}\vartheta)(\rho) = \Psi(\rho-s_i)\mathsf{p}_i(s_i,\vartheta(\rho_i^-)) + \int_{s_i}^\rho \Psi(\rho-s)\,\mathsf{F}(s,\vartheta(s))\,\mathrm{ds} + \int_{s_i}^\rho \Psi(\rho-s)\,\mathsf{B}\mathsf{u}^{(\gamma)}(s,\vartheta)\,\mathrm{ds} \\ + \int_{s_i}^\rho \Psi(\rho-s)\,\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s), \quad \forall \rho \in [s_i,\rho_{i+1}],\ i = 1,2,\ldots,m,$$

where,

$$\begin{aligned}\mathsf{u}^{(\gamma)}(s,\vartheta) &= \mathsf{B}^*\Psi^*(\rho_{i+1}-s)(\gamma\mathrm{Id}+\mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\left\{\mathbb{E}\,\vartheta_{\rho_{i+1}}+\int_{s_i}^{\rho_{i+1}}\Phi_i(s)d\mathsf{R}^{\mathsf{H}}(s)-\Psi(\mathsf{b}-s_i)\mathsf{p}_i(s_i,\vartheta(\rho_i^-))\right\}\\ &\quad-\mathsf{B}^*\Psi^*(\rho_{i+1}-s)\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id}+\mathbf{\Pi}_{s_i}^{\rho_i+1})^{-1}\Psi(\rho_{i+1}-s)\,\mathsf{F}(s,\vartheta(s))\,\mathrm{ds}\\ &\quad-\mathsf{B}^*\Psi^*(\rho_{i+1}-s)\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id}+\mathbf{\Pi}_{s}^{\rho_{i+1}})^{-1}\Psi(\rho_{i+1}-s)\,\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s)\end{aligned}$$

and $J_0(0,\cdot)=\vartheta_0$, $\vartheta(\rho_{m+1})=\vartheta_{\rho_{m+1}}=\vartheta_{\mathsf{b}}$.

Lemma 3.1. *There exist positive constants R_{u_i} and R_{v_i} $i=0,1,\ldots,m$, such that for all $\vartheta_1,\vartheta_2\in C_{\mathsf{b}}$, we have*

$$\mathbb{E}\|\mathsf{u}^{(\gamma)}(\rho,\vartheta_1)-\mathsf{u}^{(\gamma)}(\rho,\vartheta_2)\|^2\le R_{\mathsf{u}_i}\|\vartheta_1-\vartheta_2\|_{\mathcal{PC}}, \tag{3.1}$$

$$\mathbb{E}\|\mathsf{u}^{(\gamma)}(\rho,\vartheta)\|^2\le R_{v_i}, \tag{3.2}$$

where

$$R_{\mathsf{u}_i}=3\,\frac{\|\mathsf{B}\|^2M^4}{\gamma^2}\left\{R_{\mathsf{p}_i}+(\rho_{i+1}-s_i)^2R_{\mathsf{F}}+2R_{\mathsf{G}}c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}}\right\}, \tag{3.3}$$

$$\begin{aligned}R_{v_i}&=\frac{4\|\mathsf{B}\|^2M^4}{\gamma^2}\left[\mathbb{E}\|\vartheta_{\rho_{i+1}}\|^2+D_{\mathsf{p}_i}(1+\mathfrak{M})+(\rho_{i+1}-s_i)^2\,D_{\mathsf{F}}(1+\mathfrak{M})\right.\\ &\quad\left.+\,c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}}D_{\mathsf{G}}(1+\mathfrak{M})\right],\quad \|\vartheta\|^2_{\mathcal{PC}}\le\mathfrak{M}.\end{aligned} \tag{3.4}$$

Proof. Let $\vartheta_1,\vartheta_2\in C_{\mathsf{b}}$

$$\begin{aligned}&\mathbb{E}\|\mathsf{u}^{(\gamma)}(s,\vartheta_2)-\mathsf{u}^{(\gamma)}(s,\vartheta_1)\|^2\\ &\quad\le\mathbb{E}\Big\|\mathsf{B}^*\Psi^*(\rho_{i+1}-s)(\gamma\mathrm{Id}+\mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\big\{\Psi(\mathsf{b}-s_i)\big[\mathsf{p}_i(s_i,\vartheta_1(\rho_i^-))-\mathsf{p}_i(s_i,\vartheta_2(\rho_i^-))\big]\big\}\\ &\qquad-\mathsf{B}^*\Psi^*(\rho_{i+1}-s)\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id}+\mathbf{\Pi}_{s_i}^{\rho_i+1})^{-1}\Psi(\rho_{i+1}-s)\,\big[\mathsf{F}(s,\vartheta_1(s))-\mathsf{F}(s,\vartheta_2(s))\big]\,\mathrm{ds}\\ &\qquad-\mathsf{B}^*\Psi^*(\rho_{i+1}-s)\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id}+\mathbf{\Pi}_{s}^{\rho_{i+1}})^{-1}\Psi(\rho_{i+1}-s)\,\big[\mathsf{G}(s,\vartheta_1(s))-\mathsf{G}(s,\vartheta_2(s))\big]d\mathsf{R}^{\mathsf{H}}(s)\Big\|^2\\ &\quad\le\frac{3\|\mathsf{B}\|^2M^2}{\gamma^2}\Big[M^2\mathbb{E}\big\|\mathsf{p}_i(s_i,\vartheta_1(\rho_i^-))-\mathsf{p}_i(s_i,\vartheta_2(\rho_i^-))\big\|^2\\ &\qquad+M^2(\rho_{i+1}-s_i)\int_{s_i}^{\rho_{i+1}}\mathbb{E}\|\mathsf{F}(s,\vartheta_1(s))-\mathsf{F}(s,\vartheta_2(s))\|^2\,\mathrm{ds}\\ &\qquad+M^2c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}-1}\int_{s_i}^{\rho_{i+1}}\mathbb{E}\|\mathsf{G}(s,\vartheta_1(s))-\mathsf{G}(s,\vartheta_2(s))\|^2\,\mathrm{ds}\Big]\\ &\quad\le\;3\,\frac{\|\mathsf{B}\|^2M^4}{\gamma^2}R_{\mathsf{p}_i}\mathbb{E}\|\vartheta_2-\vartheta_1\|^2+\;3(\rho_{i+1}-s_i)\,R_{\mathsf{F}}\frac{\|\mathsf{B}\|^2M^4}{\gamma^2}\int_{s_i}^{\rho_{i+1}}\mathbb{E}\|\vartheta_2(s)-\vartheta_1(s)\|^2\,\mathrm{ds}\\ &\qquad+6\;R_{\mathsf{G}}c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}-1}\frac{\|\mathsf{B}\|^2M^4}{\gamma^2}\int_{s_i}^{\rho_{i+1}}\mathbb{E}\|\vartheta_2(s)-\vartheta_1(s)\|^2\,\mathrm{ds}\end{aligned}$$

$$\leq 3\,\frac{\|\mathsf{B}\|^2\,M^4}{\gamma^2}\left\{R_{\mathsf{p}_i}+\ (\rho_{i+1}-s_i)^2\,R_{\mathsf{F}}+2\ R_{\mathsf{G}}c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}}\right\}\|\vartheta_2-\vartheta_1\|^2_{\mathcal{PC}}. \quad (3.5)$$

Hence,

$$\mathbb{E}\|\mathsf{u}^{(\gamma)}(s,\vartheta_2)-\mathsf{u}^{(\gamma)}(s,\vartheta_1)\|^2\leq R_{\mathsf{u}_i}\|\vartheta_2-\vartheta_1\|^2_{\mathcal{PC}}.$$

The proof of inequality (3.2) is

$$\begin{aligned}
\mathbb{E}\|\mathsf{u}^{(\gamma)}(s,\vartheta_2)\|^2 \leq \mathbb{E}\Big\| & \mathsf{B}^*\Psi^*(\rho_{i+1}-s)(\gamma\mathrm{Id}+\mathbf{\Pi}^{\rho_{i+1}}_{s_i})^{-1}\big\{\vartheta_{\rho_{i+1}}-\Psi(\mathsf{b}-s_i)\mathsf{p}_i(s_i,\vartheta(\rho_i^-))\big\}\\
& -\mathsf{B}^*\Psi^*(\rho_{i+1}-s)\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id}+\mathbf{\Pi}^{\rho_i+1}_{s_i})^{-1}\Psi(\rho_{i+1}-s)\ \mathsf{F}(s,\vartheta(s))\ \mathrm{ds}\\
& -\mathsf{B}^*\Psi^*(\rho_{i+1}-s)\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id}+\mathbf{\Pi}^{\rho_{i+1}}_{s})^{-1}\Psi(\rho_{i+1}-s)\,\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s)\Big\|^2\\
\leq \frac{4\|\mathsf{B}\|^2M^2}{\gamma^2}\Big[& M^2D_{\mathsf{p}_i}(1+\mathbb{E}\|\vartheta\|^2)+M^2(\rho_{i+1}-s_i)\int_{s_i}^{\rho_{i+1}}\ D_{\mathsf{F}}(1+\mathbb{E}\|\vartheta\|^2)\ \mathrm{ds}\\
& +M^2c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}-1}\int_{s_i}^{\rho_{i+1}}\ D_{\mathsf{G}}(1+\mathbb{E}\|\vartheta\|^2)\ \mathrm{ds}\Big]\\
\leq \frac{4\|\mathsf{B}\|^2M^4}{\gamma^2}\ \big[& D_{\mathsf{p}_i}(1+\mathbb{E}\|\vartheta\|^2)+(\rho_{i+1}-s_i)^2\ D_{\mathsf{F}}(1+\mathbb{E}\|\vartheta\|^2)\\
& c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}}\ D_{\mathsf{G}}(1+\mathbb{E}\|\vartheta\|^2)\ \big].\\
\leq \frac{4\|\mathsf{B}\|^2M^4}{\gamma^2}\ \big[& \mathbb{E}\|\vartheta_{\rho_{i+1}}\|^2+D_{\mathsf{p}_i}(1+\mathfrak{M})+(\rho_{i+1}-s_i)^2\ D_{\mathsf{F}}(1+\mathfrak{M})\\
& c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}}\ D_{\mathsf{G}}(1+\mathfrak{M})\ \big].
\end{aligned}$$

Hence,

$$\mathbb{E}\|\mathsf{u}^{(\gamma)}(s,\vartheta)\|^2\leq R_{v_i}.$$ □

Let the constant $\mathfrak{M}$ satisfy the inequality

$$\mathfrak{M}\geq\max_{1\leq i\leq m}\left[\frac{Q_0}{1-N_0},\ \ \frac{D_{\mathsf{p}_i}}{1-D_{\mathsf{p}_i}},\ \ \frac{Q_i}{1-N_i}\right], \quad (3.6)$$

where

$$\begin{aligned}
Q_i &= \frac{16\rho_{i+1}^4\|\mathsf{B}\|^4M^4}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_{i+1}}\|^2+\left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_{i+1}^2}{\gamma^2}\right)\big[4M^2D_{\mathsf{p}_i}+4M^2\rho_{i+1}^2\mathfrak{M}_{\mathsf{F}}+4c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}\big],\\
D_{J_0} &= 0,\\
N_i &= \left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_{i+1}^2}{\gamma^2}\right)\big\{4M^2D_{\mathsf{p}_i}+4M^2\rho_{i+1}^2\mathfrak{M}_{\mathsf{F}}+4c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}\big\}\\
N_0 &= \left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_1^2}{\gamma^2}\right)\big\{4M^2\rho_1^2\mathfrak{M}_{\mathsf{F}}+4c_{\mathsf{H}}\rho_1^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}\big\}\\
Q_0 &= \frac{16\rho_1^4\|\mathsf{B}\|^4M^4}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_1}\|^2+\left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_1^2}{\gamma^2}\right)\big[4M^2\mathbb{E}\|\vartheta_0\|^2+4M^2\rho_1^2\mathfrak{M}_{\mathsf{F}}+4c_{\mathsf{H}}\rho_1^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}\big].
\end{aligned}$$

Theorem 3.2. *Assume that hypotheses* $(\mathbf{C}_1)$–$(\mathbf{C}_5)$ *hold. Then the system* (1.2) *has at least one mild solution on* $[0, \mathsf{b}]$.

Proof. First, we define two operators Φ_1 and Φ_2 on

$$S_{\mathfrak{M}} = \{\vartheta \in C_{\mathsf{b}} : \|\vartheta\|^2_{\mathcal{PC}} \leq \mathfrak{M}\} \subseteq C_{\mathsf{b}}.$$

as follows

$$(\Phi_1\vartheta)(\rho) = \begin{cases} \Psi(\rho)\vartheta_0 + \int_0^\rho \Psi(\rho - s)\mathsf{B}\mathsf{u}^\gamma(s, \vartheta)ds, & \rho \in [0, \rho_1], \\ \mathsf{p}_i(\rho, \vartheta(\rho_i^-)), & \rho \in (\rho_i, s_i], \\ \Psi(\rho - s_i)\mathsf{p}_i(s_i, \vartheta(\rho_i^-)) + \int_{s_i}^\rho \Psi(\rho - s)\mathsf{B}\mathsf{u}^\gamma(s, \vartheta)ds & \rho \in (s_i, \rho_{i+1}], \end{cases}$$

and

$$(\Phi_2\vartheta)(\rho) = \begin{cases} \int_0^\rho \Psi(\rho - s)\mathsf{F}(s, \vartheta(s))ds + \int_0^\rho \Psi(\rho - s)\mathsf{G}(s, \vartheta(s))d\mathsf{R}^\mathsf{H}(s) & \rho \in [0, \rho_1], \\ 0 & \rho \in (\rho_i, s_i], \\ \int_{s_i}^\rho \Psi(\rho - s)\mathsf{F}(s, \vartheta(s))ds + \int_{s_i}^\rho \Psi(\rho - s)\mathsf{G}(s, \vartheta(s))d\mathsf{R}^\mathsf{H}(s) & \rho \in (s_i, \rho_{i+1}]. \end{cases}$$

The set $S_{\mathfrak{M}}$ is a bounded closed and convex set in C_{b}. Next, we prove that the operators Φ_1 and Φ_2 satisfy all the conditions of Krasnoselskii's theorem. For the sake of convenience, we split the proof into several steps.

Step 1. We prove that $\Phi_1\vartheta_1 + \Phi_2\vartheta_2 \in S_{\mathfrak{M}}$ for any $\vartheta_1, \vartheta_2 \in S_{\mathfrak{M}}$.

For any $\vartheta_1, \vartheta_2 \in S_{\mathfrak{M}}$ and $\rho \in [0, \rho_1]$, we have

$$\begin{aligned} \mathbb{E}\|(\Phi_1\vartheta_1)(\rho) + (\Phi_2\vartheta_2)(\rho)\|^2 &\leq 4\mathbb{E}\|\Psi(\rho)\vartheta_0\|^2 + 4\mathbb{E}\left\|\int_0^\rho \Psi(\rho - s)\mathsf{B}\mathsf{u}^\gamma(s, \vartheta)ds\right\|^2 \\ &+ 4\mathbb{E}\left\|\int_0^\rho \Psi(\rho - s)\mathsf{F}(s, \vartheta(s))ds\right\|^2 + 4\mathbb{E}\left\|\int_0^\rho \Psi(\rho - s)\mathsf{G}(s, \vartheta(s))d\mathsf{R}^\mathsf{H}(s)\right\|^2 \\ &\leq 4M^2\mathbb{E}\|\vartheta_0\|^2 + 4M^2\|\mathsf{B}\|^2\rho\int_0^\rho \mathbb{E}\left\|\mathsf{u}^\gamma(s, \vartheta)\right\|^2 ds \\ &+ 4M^2\int_0^\rho \mathbb{E}\left\|\mathsf{F}(s, \vartheta(s))\right\|^2 ds + 4M^2 c_\mathsf{H}\rho^{2\mathsf{H}-1}\int_0^\rho \mathbb{E}\left\|\mathsf{G}(s, \vartheta(s))\right\|^2 ds \\ &\leq 4M^2\mathbb{E}\|\vartheta_0\|^2 + \frac{4\|\mathsf{B}\|^4 M^4\rho_1^2}{\gamma^2}\Big[4\mathbb{E}\|\vartheta_{\rho_1}\|^2 + 4M^2\mathbb{E}\|\vartheta_0\|^2 + 4M^2\rho_1^2\, D_\mathsf{F}(1 + \mathfrak{M}) \\ &+ 4M^2 c_\mathsf{H}\rho_1^{2\mathsf{H}}\, D_\mathsf{G}(1 + \mathfrak{M})\Big] + 4M^2\rho_1^2\, D_\mathsf{F}(1 + \mathfrak{M}) + 4M^2 c_\mathsf{H}\rho_1^{2\mathsf{H}}\, D_\mathsf{G}(1 + \mathfrak{M}) \end{aligned}$$

$$\leq \frac{16\rho_1^4\|\mathsf{B}\|^4M^4}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_1}\|^2 + \left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_1^2}{\gamma^2}\right)$$
$$\times\left\{4M^2\mathbb{E}\|\vartheta_0\|^2 + 4M^2\rho_1^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M}) + 4c_{\mathsf{H}}\rho_1^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}(1+\mathfrak{M})\right\}.$$

Hence,

$$\mathbb{E}\|(\Phi_1\vartheta_1)(\rho)+(\Phi_2\vartheta_2)(\rho)\|^2 \leq Q_0 + N_0\mathfrak{M} \leq \mathfrak{M}. \tag{3.7}$$

For any $\vartheta_1,\vartheta_2\in S_{\mathfrak{M}}$, and $\rho\in(\rho_i,s_i]$, $i=1,2,\ldots,m$, we have

$$\mathbb{E}\|(\Phi_1\vartheta_1)(\rho)+(\Phi_2\vartheta_2)(\rho)\|^2 = \mathbb{E}\|\mathsf{p}_i(\rho,\vartheta_1(\rho_i^-))\|^2 \leq D_{\mathsf{p}_i}(1+\mathbb{E}\|\vartheta_1\|^2).$$

Hence,

$$\mathbb{E}\|(\Phi_1\vartheta_1)(\rho)+(\Phi_2\vartheta_2)(\rho)\|^2 \leq D_{\mathsf{p}_i}(1+\mathfrak{M}) \leq \mathfrak{M}. \tag{3.8}$$

For any $\vartheta_1,\vartheta_2\in S_{\mathfrak{M}}$, and $\rho\in(s_i,\rho_{i+1}]$, $i=1,2,\ldots,m$, we have

$$\begin{aligned}\mathbb{E}\|(\Phi_1\vartheta_1)(\rho)+(\Phi_2\vartheta_2)(\rho)\|^2 &\leq 4M^2\mathbb{E}\|\mathsf{p}_i(s_i,\vartheta(\rho_i^-))\|^2 + 4\mathbb{E}\left\|\int_{s_i}^{\rho}\Psi(\rho-s)\mathsf{B}\mathsf{u}^{\gamma}(s,\vartheta)ds\right\|^2\\
&+4\mathbb{E}\left\|\int_{s_i}^{\rho}\Psi(\rho-s)\|\mathsf{F}(s,\vartheta(s))ds\right\|^2 + 4\mathbb{E}\left\|\int_{s_i}^{\rho}\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s)\right\|^2\\
&\leq \frac{16\rho_{i+1}^4\|\mathsf{B}\|^4M^4}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_{i+1}}\|^2 + \left(1+\frac{3\|\mathsf{B}\|^4M^4\rho_{i+1}^2}{\gamma^2}\right)\\
&\times\left\{4M^2D_{\mathsf{p}_i}(1+\mathfrak{M}) + 4M^2\rho_{i+1}^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M}) + 4c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}(1+\mathfrak{M})\right\}\\
&\leq \frac{16\rho_{i+1}^4\|\mathsf{B}\|^4M^4}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_{i+1}}\|^2\\
&+\left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_{i+1}^2}{\gamma^2}\right)\left[4M^2D_{\mathsf{p}_i} + 4M^2\rho_{i+1}^2\mathfrak{M}_{\mathsf{F}} + 4c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}\right]\\
&+\left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_{i+1}^2}{\gamma^2}\right)\left\{4M^2D_{\mathsf{p}_i} + 4M^2\rho_{i+1}^2\mathfrak{M}_{\mathsf{F}} + 4c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}\right\}\mathfrak{M}.\end{aligned}$$

Hence,

$$\mathbb{E}\|(\Phi_1\vartheta_1)(\rho)+(\Phi_2\vartheta_2)(\rho)\|^2 \leq Q_i + N_i\mathfrak{M} \leq \mathfrak{M}. \tag{3.9}$$

Equations (3.7)–(3.9) implies that

$$\|\Phi_1\vartheta_1+\Phi_2\vartheta_2\|_{\mathcal{PC}}^2 \leq \mathfrak{M}.$$

Hence, $\Phi_1\vartheta_1+\Phi_2\vartheta_2\in S_{\mathfrak{M}}$.

Step 2. Φ_2 is continuous on $S_{\mathfrak{M}}$. Let $\{\vartheta_n\}_{n=1}^{\infty}$ be a sequence such that $\vartheta_n \to \vartheta$ in $S_{\mathfrak{M}}$.

For any $\rho \in (s_i,\, \rho_{i+1}]$, $i = 0, 1, \ldots, m$, we have

$$\begin{aligned}\mathbb{E}\|(\Phi_2\vartheta_n)(\rho) - (\Phi_2\vartheta)(\rho)\|^2 &\leq 2M^2\rho_{i+1}\int_{s_i}^{\rho}\mathbb{E}\|\mathsf{F}(s,\vartheta_n(s)) - \mathsf{F}(s,\vartheta(s))\|^2 ds\\ &\quad + 4c_{\mathsf{H}}M^2\rho_{i+1}^{2\mathsf{H}-1}\int_{s_i}^{\rho}\mathbb{E}\|\mathsf{G}(s,\vartheta_n(s) - \mathsf{G}(s,\vartheta(s))\|^2_{L_2^0} ds\\ &\leq \left[2M^2R_F\rho_{i+1}^2 + 4c_{\mathsf{H}}M^2\rho_{i+1}^{2\mathsf{H}}\right]\|\vartheta_n - \vartheta\|^2_{\mathcal{PC}}.\end{aligned}$$

Hence, $\mathbb{E}\|(\Phi_2\vartheta_n)(\rho) - (\Phi_2\vartheta)(\rho)\|^2 \to 0$ as $n \to \infty$, thus, Φ_2 is continuous on $S_{\mathfrak{M}}$.

Step 3. We show $\{(\Phi_2\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$ is equicontinuous.

For any $\tau_1, \tau_2 \in (s_i, \rho_{i+1}]$, $i = 0, 1, \ldots, m$, $\tau_1 < \tau_2$ and $\vartheta \in S_{\mathfrak{M}}$, we obtain

$$\begin{aligned}\mathbb{E}\|(\Phi_2\vartheta)(\tau_2) + (\Phi_2\vartheta)(\tau_1)\|^2 &\leq 4\mathfrak{M}_{\mathsf{F}}\tau_1\int_{s_i}^{\tau_1}\|\Psi(\tau_2 - s) - \Psi(\tau_1 - s)\|^2(1+\mathfrak{M})ds\\ &\quad + 4M^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M})(\tau_2-\tau_1)^2\\ &\quad + 8c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}-1}\mathfrak{M}_{\mathsf{G}}\int_{s_i}^{\tau_1}\|\Psi(\tau_2 - s) - \Psi(\tau_1 - s)\|^2(1+\mathfrak{M})ds\\ &\quad + 8M^2c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}-1}\mathfrak{M}_{\mathsf{G}}(\tau_2-\tau_1).\end{aligned} \tag{3.10}$$

We conclude that $\mathbb{E}\|(\Phi_2\vartheta)(\tau_2) - (\Phi_2\vartheta)(\tau_1)\|^2 \to 0$ as $\tau_2 \to \tau_1$, since the operator $\Psi(\rho)$ is compact, which implies the continuity of the operator $\Psi(\rho)$. Hence $\{(\Phi_2\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$ is equicontinuous. Also, clearly $\{(\Phi_2\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$ is bounded.

Step 4. We show that $\mathcal{Z}(\rho) = \{(\Phi_2\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$ is relatively compact in $\mathbb{H}$.

Clearly, $\mathcal{Z}(0) = \{0\}$ is compact. Let ε be a real number and $\rho \in (s_i, \rho_{i+1}]$, $i = 0, 1, \ldots, m$ be fixed with $0 < \varepsilon < \rho$. For any $\vartheta \in S_{\mathfrak{M}}$, we define

$$(\Phi_2^{\varepsilon}\vartheta)(\rho) = \begin{cases}\displaystyle\int_0^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\|F(s,\vartheta(s))ds + \int_0^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s) & \rho \in [0,\rho_1],\\ 0 & \rho \in (\rho_i, s_i],\\ \displaystyle\int_{s_i}^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{F}(s,\vartheta(s))ds + \int_{s_i}^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s) & \rho \in (s_i, \rho_{i+1}],\end{cases}$$

and

$$(\Phi_2^{*\varepsilon}\vartheta)(\rho)=\begin{cases}\Psi(\varepsilon)\displaystyle\int_0^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{F}(s,\vartheta(s))ds & \\ \qquad +\Psi(\varepsilon)\displaystyle\int_0^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s) & \rho\in[0,\rho_1],\\ 0 & \rho\in(\rho_i,s_i],\\ \Psi(\varepsilon)\displaystyle\int_{s_i}^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{F}(s,\vartheta(s))ds & \\ \qquad +\Psi(\varepsilon)\displaystyle\int_{s_i}^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s) & \rho\in(s_i,\rho_{i+1}].\end{cases}$$

By Lemma 2.6 and using the compactness of $\big(\Psi(\varepsilon)\big)_{\varepsilon>0}$, we deduce that the set $\mathcal{Z}^{\varepsilon}(\rho)=\{(\Phi_2^{\varepsilon}\vartheta)(\rho):\vartheta\in S_{\mathfrak{M}}\}$ is precompact in $\mathbb{H}$ for every ε, $0<\varepsilon<\rho$. Moreover, by Lemma 2.6 and Hölder's inequality, for every $\rho\in(0,\rho_1]$, we obtain:

$$\begin{aligned}
&\mathbb{E}\|(\Phi_2^{\varepsilon}\vartheta)(\rho)-\big(\Phi_2^{*\varepsilon}\vartheta\big)(\rho)\|^2\\
&\le 2\mathbb{E}\left\|\Psi(\varepsilon)\int_0^{\rho-\varepsilon}\Psi(\rho-s-\varepsilon)\mathsf{F}(s,\vartheta(s))ds-\int_0^{\rho-\varepsilon}\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds\right\|^2\\
&\quad+2\mathbb{E}\left\|\Psi(\varepsilon)\int_0^{\rho-\varepsilon}\Psi(\rho-s-\varepsilon)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)-\int_0^{\rho-\varepsilon}\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)\right\|^2\\
&=2\mathbb{E}\left\|\int_0^{\rho-\varepsilon}\big[\Psi(\varepsilon)\Psi(\rho-s-\varepsilon)-\Psi(\rho-s)\big]\mathsf{F}(s,\vartheta(s))ds\right\|^2\\
&\quad+2\mathbb{E}\left\|\int_0^{\rho-\varepsilon}\big[\Psi(\varepsilon)\Psi(\rho-s-\varepsilon)-\Psi(\rho-s)\big]\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)\right\|^2\\
&\le 2\mathbb{E}\int_0^{\rho-\varepsilon}\big\|\Psi(\varepsilon)\Psi(\rho-s-\varepsilon)-\Psi(\rho-s)\big\|^2\big\|\mathsf{F}(s,\vartheta(s))\big\|^2ds\\
&\quad+2\mathbb{E}\int_0^{\rho-\varepsilon}\big\|\Psi(\varepsilon)\Psi(\rho-s-\varepsilon)-\Psi(\rho-s)\big\|^2\big\|\mathsf{G}(s,\vartheta(s))\big\|^2d\mathsf{R}_Q^{\mathsf{H}}(s)\\
&\le 2L(\varepsilon)^2\rho\int_0^{\rho-\varepsilon}\mathbb{E}\big\|\mathsf{F}(s,z(s))\big\|^2ds+2L(\varepsilon)^2\int_0^{\rho-\varepsilon}\mathbb{E}\big\|\mathsf{G}(s,\vartheta(s))\big\|^2d\mathsf{R}_Q^{\mathsf{H}}(s)\\
&\le 3L(\varepsilon)^2\int_0^{\rho-\varepsilon}\mathfrak{M}_{\mathsf{F}}(1+\|\vartheta(s)\|^2)ds+3L(\varepsilon)^2c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}-1}\int_0^{\rho-\varepsilon}\mathfrak{M}_{\mathsf{G}}(1+\|\vartheta(s)\|^2)ds\\
&\le 3L(\varepsilon)^2\mathsf{b}^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M})+3L(\varepsilon)^2c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}}\mathfrak{M}_{\mathsf{G}}(1+\mathfrak{M})\underset{\varepsilon\to0}{\longrightarrow}0.
\end{aligned}$$

So the set $\mathcal{Z}^{\varepsilon}(\rho)=\{(\Phi_2^{\varepsilon}\vartheta)(\rho):\vartheta\in S_{\mathfrak{M}}\}$ is precompact in $\mathbb{H}$ by using the total boundedness.

Using this idea again, we obtain

$$\begin{aligned}
\mathbb{E}\|(\Phi_2\vartheta)(\rho)-(\Phi_2^{\varepsilon}\vartheta)(\rho)\|^2\le 3\mathbb{E}&\left\|\int_0^{\rho}\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds-\int_0^{\rho-\varepsilon}\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds\right\|^2\\
+3\mathbb{E}&\left\|\int_0^{\rho}\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)-\int_0^{\rho-\varepsilon}\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)\right\|^2
\end{aligned}$$

$$\leq 3\mathbb{E}\left\| \int_{\rho-\varepsilon}^{\rho} \Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds \right\|^2 + 3\mathbb{E}\left\| \int_{\rho-\varepsilon}^{\rho} \Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s) \right\|^2$$
$$\leq 3M^2\varepsilon\left[\int_{\rho-\varepsilon}^{\rho} \mathbb{E}\|\mathsf{F}(s,\vartheta(s))\|^2 ds + 2c_{\mathsf{H}}\varepsilon^{2\mathsf{H}-1}\int_{\rho-\varepsilon}^{\rho} \mathbb{E}\|\mathsf{G}(s,\vartheta(s))\|^2 ds \right]$$
$$\leq 2M^2\varepsilon^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M}) + 2M^2 c_{\mathsf{H}}\varepsilon^{2\mathsf{H}}\mathfrak{M}_{\mathsf{G}}(1+\mathfrak{M}) \to 0 \text{ as } \varepsilon \to 0.$$

Similarly, for any $\rho \in (e_i, \rho_{i+1}]$ with $i = 1, \ldots, m$. Let $e_i < \rho \leq s \leq \rho_{i+1}$ be fixed and let ε be a real number satisfying $0 < \varepsilon < \rho$. If we use Lemma 2.6 and compactness of $\big(\Psi(\varepsilon)\big)_{\varepsilon>0}$, we deduce that the set $\mathcal{Z}^{\varepsilon}(\rho)$ is precompact in $\mathbb{H}$ for every ε, $0 < \varepsilon < \rho$. Moreover, by Lemma 2.6 and Hölder's inequality, for every $\vartheta \in S_{\mathfrak{M}}$ we have:

$$\begin{aligned}
&\mathbb{E}\|(\Phi_2^{\varepsilon}\vartheta)(\rho) - \big(\Phi_2^{*\varepsilon}\vartheta\big)(\rho)\|^2 \\
&\leq 2\mathbb{E}\left\| \Psi(\varepsilon)\int_{s_i}^{\rho-\varepsilon} \Psi(\rho-s-\varepsilon)\mathsf{F}(s,\vartheta(s))ds - \int_{s_i}^{\rho-\varepsilon} \Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds \right\|^2 \\
&\quad + 2\mathbb{E}\left\| \Psi(\varepsilon)\int_{s_i}^{\rho-\varepsilon} \Psi(\rho-s-\varepsilon)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s) - \int_{s_i}^{\rho-\varepsilon} \Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s) \right\|^2 \\
&\leq 3\big(L(\varepsilon)\big)^2\left[\mathsf{b}\int_{s_i}^{\rho-\varepsilon} \mathbb{E}\|\|\mathsf{F}(s,\vartheta(s))\|^2 ds + c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}-1}\int_{s_i}^{\rho-\varepsilon} \mathbb{E}\|\mathsf{G}(s,\vartheta(s))\|^2 ds \right] \\
&\leq 3\big(L(\varepsilon)\big)^2\left[\mathsf{b}^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M}) + c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}}\mathfrak{M}_{\mathsf{G}}(1+\mathfrak{M}) \right] \underset{\varepsilon\to 0}{\longrightarrow} 0.
\end{aligned}$$

So the set $\mathcal{Z}^{\varepsilon}(\rho) = \{(\Phi_2^{\varepsilon}\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$ is precompact in $\mathbb{H}$ by using the total boundedness.

Using this idea again, we obtain

$$\begin{aligned}
\mathbb{E}\|(\Phi_2\vartheta)(\rho) - (\Phi_2^{\varepsilon}\vartheta)(\rho)\|^2 &\leq 2\mathbb{E}\left\| \int_{\rho-\varepsilon}^{\rho} \Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds \right\|^2 + 2\mathbb{E}\left\| \int_{\rho-\varepsilon}^{\rho} \Psi(\rho-s)\mathsf{G}(s)d\mathsf{R}_Q^{\mathsf{H}}(s) \right\|^2 \\
&\leq 2M^2\varepsilon^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M}) + 2M^2 c_{\mathsf{H}}\varepsilon^{2\mathsf{H}}\mathfrak{M}_{\mathsf{G}}(1+\mathfrak{M}) \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}$$

Therefore, as $\varepsilon \to 0$, there are precompact sets arbitrarily close to the set $\mathcal{Z}(\rho) = \{(\Phi_2\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$. Thus, the set $\mathcal{Z}(\rho) = \{(\Phi_2\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$ is precompact in $\mathbb{H}$. Finally, by the Arzelà-Ascoli theorem, we can conclude that the operator Φ_2 is continuous and compact.

Step 5. Φ_1 is a contraction.

For any $\vartheta_1, \vartheta_2 \in S_{\mathfrak{M}}$ and $\rho \in [0, \rho_1]$, we have

$$\mathbb{E}\|(\Phi_1\vartheta_1)(\rho) - (\Phi_1\vartheta_2)(\rho)\|^2 \leq M^2 R_{\mathsf{u}_0}\rho_1^2\|\vartheta_2 - \vartheta_1\|_{\mathcal{PC}}^2. \tag{3.11}$$

For any $\vartheta_1, \vartheta_2 \in S_{\mathfrak{M}}$ and $\rho \in (\rho_i, s_i]$, $i = 1, 2, \ldots, m$, we have

$$\mathbb{E}\|(\Phi_1\vartheta_1)(\rho) - (\Phi_1\vartheta_2)(\rho)\|^2 = \mathbb{E}\|\mathsf{p}_i(\rho, \vartheta_1(\rho_i^-)) - \mathsf{p}_i(\rho, \vartheta_2(\rho_i^-))\|^2 \leq R_{\mathsf{p}_i}\|\vartheta_2 - \vartheta_1\|_{\mathcal{PC}}^2. \tag{3.12}$$

For any $\vartheta_1, \vartheta_2 \in S_{\mathfrak{M}}$ and $\rho \in (s_i, \rho_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\mathbb{E}\|(\Phi_1\vartheta_1)(\rho) - (\Phi_1\vartheta_2)(\rho)\|^2 &\le 2M^2 R_{\mathsf{p}_i}\|\vartheta_2 - \vartheta_1\|^2_{\mathcal{PC}} + 2M^2\|\mathsf{B}\|^2 R_{\mathsf{u}_i}\rho^2_{i+1}\|\vartheta_2 - \vartheta_1\|^2_{\mathcal{PC}}\\ &= 2M^2\left(R_{\mathsf{p}_i} + \|\mathsf{B}\|^2 R_{\mathsf{u}_i}\rho^2_{i+1}\right)\|\vartheta_2 - \vartheta_1\|^2_{\mathcal{PC}}. \end{aligned} \tag{3.13}$$

Equations (3.11)–(3.13) and hypothesis $(\mathbf{C}_5)$ imply that Φ_1 is a contraction. The operators Φ_1, Φ_2 satisfy all the conditions of Theorem 2.12, then there exists a fixed point ϑ on $S_{\mathfrak{M}}$. Therefore, the system (1.2) has at least on mild solution on $[0, \mathsf{b}]$. □

Theorem 3.3. *Assume that hypotheses* $(\mathbf{C}_1)$–$(\mathbf{C}_6)$ *hold and the functions* F, G *are uniformly bounded on their respective domains. Then the system* (1.2) *is approximately controllable on* $[0, \mathsf{b}]$.

Proof. Let ϑ^γ be a fixed point of $\Phi_1 + \Phi_2$. By using the stochastic Fubini theorem, we obtain

$$\begin{aligned}\vartheta^{(\gamma)}(\rho_{i+1}) &= \vartheta_{\rho_{i+1}} - \gamma(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_{i+1}}_{s_i})^{-1}\left\{\mathbb{E}\,\vartheta_{\rho_{i+1}} - \Psi(\rho_{i+1} - s_i)\mathsf{p}_i(s_i, \vartheta(\rho_i^-))\right\}\\ &\quad - \gamma\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_{i+1}}_{s_i})^{-1}\Phi_i(s)d\mathsf{R}^{\mathsf{H}}(s) + \gamma\int_0^{\mathsf{b}}(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_i+1}_{s_i})^{-1}\Psi(\rho_{i+1} - s)\,\mathsf{F}(s, \vartheta(s))\,\mathrm{ds}\\ &\quad + \gamma\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_{i+1}}_{s_i})^{-1}\Psi(\rho_{i+1} - s)\,G(s, \vartheta(s))d\mathsf{R}^{\mathsf{H}}(s), \quad i = 0, 1, 2, \ldots, m. \end{aligned} \tag{3.14}$$

Moreover, hypotheses F and G are uniformly bounded. Then there are subsequences, still denoted by $\mathsf{F}(s, \vartheta^\gamma)$ and $\mathsf{G}(s, \vartheta^\gamma)$, which converge weakly to say $\mathsf{F}(s)$ and $\mathsf{G}(s)$ respectively in $\mathbb{H}$ and L^0_2. From the above equation, we obtain

$$\begin{aligned}\mathbb{E}\|\vartheta^{(\gamma)}(\rho_{i+1}) - \vartheta_{\rho_{i+1}}\|^2 &\le 7\mathbb{E}\left\|\gamma(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_{i+1}}_{s_i})^{-1}\mathbb{E}\vartheta_{\rho_{i+1}}\right\|^2\\ &\quad + 7\mathbb{E}\left\|\gamma\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_{i+1}}_{s_i})^{-1}\Phi_i(s)d\mathsf{R}^{\mathsf{H}}(s)\right\|^2\\ &\quad + 7\mathbb{E}\left\|\gamma(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_{i+1}}_{s_i})^{-1}\Psi(\rho_{i+1} - s_i)\mathsf{p}_i(s_i, \vartheta(\rho_i^-))\right\|^2\\ &\quad + 7\mathbb{E}\left\|\gamma\int_0^b(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_i+1}_{s_i})^{-1}\Psi(\rho_{i+1} - s)\,\mathsf{F}(s)\,\mathrm{ds}\right\|^2\\ &\quad + 7\mathbb{E}\left\|\gamma\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_{i+1}}_{s_i})^{-1}\Psi(\rho_{i+1} - s)\,\mathsf{G}(s)d\mathsf{R}^{\mathsf{H}}(s)\right\|^2\\ &\quad + 7\mathbb{E}\left\|\gamma\int_0^b(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_i+1}_{s_i})^{-1}\Psi(\rho_{i+1} - s)\,\mathsf{F}(s, \vartheta^\gamma(s))\,\mathrm{ds}\right\|^2\\ &\quad + 7\mathbb{E}\left\|\gamma\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_{i+1}}_{s_i})^{-1}\Psi(\rho_{i+1} - s)\,\mathsf{G}(s, \vartheta^\gamma(s))d\mathsf{R}^{\mathsf{H}}(s)\right\|^2. \end{aligned}$$

It follows from $(\mathbf{H}_0)$, for all $0 \le s \le \mathsf{b}$ the operator $\gamma(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_{i+1}}_s)^{-1} \to 0$ as $\gamma \to 0^+$, and $\|(\gamma\mathrm{Id} + \mathbf{\Pi}^{\rho_{i+1}}_s)^{-1}\|^2 \le 1$ and by using the Arzelà-Ascoli theorem, one can prove that the operator

$\bar{l}(\cdot) \to \int_{s_i}^{\rho_{i+1}} \Psi(\cdot - s)\bar{l}(s)ds$ is compact, we obtain

$$\mathbb{E}\|\vartheta^{(\gamma)}(\rho_{i+1}) - \vartheta_{\rho_{i+1}}\|^2 \to 0 \text{ as } \gamma \to 0^+.$$

This gives the approximate controllability of system (1.2) on $[0, \mathsf{b}]$. □

Now, we are going to prove the approximate controllability of the stochastic system (1.2) by using another method, namely Schauder's fixed point theorem with some other hypotheses, which are different from hypotheses of the Theorems 3.2 and 3.3. In order to establish the approximate controllability results, we impose the following hypotheses.

($\mathbf{C}_7$) $\mathsf{T}(\rho)$ is compact for $\rho > 0$.

($\mathbf{C}_8$) The function $\mathsf{F} : \mathcal{J} \times \mathbb{H} \to \mathbb{H}$ satisfy the following conditions

(a) for each $\rho \in \mathcal{J}$ the function $\mathsf{F}(\rho, \cdot) : \mathbb{H} \to \mathbb{H}$ is continuous for each $\vartheta \in \mathbb{H}$ the function $\mathsf{F}(\cdot, \vartheta) : \mathcal{J} \to \mathbb{H}$ is strongly measurable,

(b) for each positive number $\mathfrak{M}$, there exists $\mu_{\mathfrak{M}} \in L^1(\mathcal{J}, \mathbb{R}^+)$ such that

$$\sup_{\mathbb{E}\|\vartheta\|^2 \leq \mathfrak{M}} \mathbb{E}\|\mathsf{F}(\rho, \vartheta)\|^2 \leq \mu_{\mathfrak{M}}(\rho)$$

and there exists a $\Lambda_1 > 0$ such that

$$\lim_{\mathfrak{M} \longrightarrow \infty} \frac{\int_0^{\rho} \mu_{\mathfrak{M}}(s)d\rho}{\mathfrak{M}} = \Lambda_1 < \infty.$$

($\mathbf{C}_9$) The function $\mathsf{G} : \mathcal{J} \times L_2^0$ satisfies the following conditions

(a) for each $\rho \in \mathcal{J}$ the function $\mathsf{G}(\rho, \cdot) : \mathbb{H} \to L_2^0$ is continuous for each $\vartheta \in \mathbb{H}$ the function $\mathsf{G}(\cdot, \vartheta) : \mathcal{J} \to L_2^0$ is strongly measurable,

(b) for each positive number $\mathfrak{M}$, there exists $v_{\mathfrak{M}} \in L^1(\mathcal{J}, \mathbb{R}^+)$ such that

$$\sup_{\mathbb{E}\|\vartheta\|^2 \leq \mathfrak{M}} \mathbb{E}\|\mathsf{G}(\rho, \vartheta)\|^2_{L_2^0} \leq v_{\mathfrak{M}}(\rho)$$

and there exists a $\Lambda_2 > 0$ such that

$$\lim_{\mathfrak{M} \longrightarrow \infty} \frac{\int_0^{\rho} v_{\mathfrak{M}}(s)d\rho}{\mathfrak{M}} = \Lambda_2 < \infty.$$

Theorem 3.4. *Assume that hypotheses* ($\mathbf{C}_2$) *and* ($\mathbf{C}_7$)–($\mathbf{C}_9$) *hold. Then, the system* (1.2) *has at least one mild solution on* $[0, \mathsf{b}]$, *provided that*

$$\max_{1\le i\le m}\left[D_{\mathsf{p}_i}\left(\frac{4\|\mathsf{B}\|^2M^4\mathsf{b}^2}{\gamma^2}\right)\left(4M^2D_{\mathsf{p}_i}+4M^2\mathsf{b}\Lambda_1+8M^2c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}-1}\Lambda_2\right)\right]<1. \tag{3.15}$$

Proof. Consider a set

$$S'_{\mathfrak{M}}=\{\vartheta\in C_{\mathsf{b}}:\|\vartheta\|^2_{\mathcal{PC}}\le\mathfrak{M}\}\subseteq C_{\mathsf{b}},$$

where $\mathfrak{M}$ is constant. The set $S'_{\mathfrak{M}}$ is a bounded closed and convex set in C_{b}.

Now, we define an operator $\mathcal{F}$ on C_{b} by

$$(\mathcal{F}\vartheta)(\rho)=\begin{cases}\Psi(\rho)\vartheta_0+\int_0^\rho\Psi(\rho-s)\mathsf{B}\mathsf{u}^\gamma(s,\vartheta)ds\\ +\int_0^\rho\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds+\int_0^\rho\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s), & \rho\in[0,\rho_1],\\ \mathsf{p}_i(\rho,\vartheta(\rho_i^-)), & \rho\in(\rho_i,s_i],\\ \Psi(\rho-s_i)\mathsf{p}_i(s_i,\vartheta(\rho_i^-))+\int_{s_i}^\rho\Psi(\rho-s)\mathsf{B}\mathsf{u}^\gamma(s,\vartheta)ds\\ +\int_{s_i}^\rho\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds+\int_{s_i}^\rho\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s), & \rho\in(s_i,\rho_{i+1}].\end{cases}$$

Next, we prove that the operator $\mathcal{F}$ satisfies all Schauder's fixed point theorem conditions.

Now, we prove that there exists $\mathfrak{M}>0$ such that $\mathcal{F}(S'_{\mathfrak{M}})\subseteq S'_{\mathfrak{M}}$. If we assume that this assertion is false, then for any $\mathfrak{M}>0$, we can choose $\vartheta^{\mathfrak{M}}\in S'_{\mathfrak{M}}$ and $\rho\in[0,\mathsf{b}]$ such that $\mathbb{E}\|\mathcal{F}(\vartheta^{\mathfrak{M}})(\rho)\|^2>\mathfrak{M}$.

For any $\rho\in[0,\rho_1]$, we have

$$\begin{aligned}\mathfrak{M}<\mathbb{E}\|\mathcal{F}(\vartheta^{\mathfrak{M}})(\rho)\|^2\le\frac{16\|\mathsf{B}\|^2M^4\rho_1^2}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_1}\|^2+\left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_1^4}{\gamma^2}\right)\Big\{4M^2\mathbb{E}\|\vartheta_0\|^2\\ +4M^2\rho_1\int_0^\rho\mu_{\mathfrak{M}}(s)ds+8M^2c_{\mathsf{H}}\rho_1^{2\mathsf{H}-1}\int_0^\rho\upsilon_{\mathfrak{M}}(s)ds\Big\}.\end{aligned}$$

For any $\rho\in(\rho_i,s_i]$, $i=1,2,\dots,m$, we have

$$\mathfrak{M}<\mathbb{E}\|\mathcal{F}(\vartheta^{\mathfrak{M}})(\rho)\|^2=\mathbb{E}\|\mathsf{p}_i(\rho,\vartheta^{\mathfrak{M}}(\rho_i^-))\|^2\le D_{\mathsf{p}_i}(1+\mathfrak{M}).$$

Similarly, for $\rho\in(s_i,\rho_{i+1}]$, $i=1,2,\dots,m$, we have

$$\begin{aligned}\mathfrak{M}<\mathbb{E}\|\mathcal{F}(\vartheta^{\mathfrak{M}})(\rho)\|^2\le\frac{16\|\mathsf{B}\|^2M^4\rho_1^2}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_1}\|^2+\left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_{i+1}^2}{\gamma^2}\right)\Big\{4M^2D_{\mathsf{p}_i}(1+\mathfrak{M})\\ +4M^2\rho_{i+1}\int_0^\rho\mu_{\mathfrak{M}}(s)ds+8M^2\rho_{i+1}^{2\mathsf{H}-1}\int_0^\rho\upsilon_{\mathfrak{M}}(s)ds\Big\}\end{aligned}$$

From the above equations we have for $\rho \in [0, \mathsf{b}]$

$$\mathfrak{M} < \mathbb{E}\|(\mathcal{F}\vartheta^{\mathfrak{M}})(\rho)\|^2 \leq Q + D_{\mathsf{p}_i}\mathfrak{M} + \left(1 + \frac{4\|\mathsf{B}\|^4 M^4 \mathsf{b}^2}{\gamma^2}\right)\bigg\{4M^2 D_{\mathsf{p}_i}\mathfrak{M} + 4M^2\mathsf{b}\int_0^\rho \mu_{\mathfrak{M}}(s)ds + 8M^2 c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}-1}\int_0^\rho \upsilon_{\mathfrak{M}}(s)ds\bigg\},$$

where

$$Q = \max_{1\leq i\leq m}\left[\frac{16\|\mathsf{B}\|^2 M^2}{\gamma^2}[\rho_1^2\mathbb{E}\|\vartheta_{\rho_1}\|^2 + \rho_{i+1}^2\mathbb{E}\|\vartheta_{\rho_{i+1}}\|^2] + 4c_0M^2\mathbb{E}\|\vartheta_0\|^2 + D_{\mathsf{p}_i} + 4c_iM^2D_{\mathsf{p}_i}\right].$$

Dividing both sides of above by $\mathfrak{M}$ and taking $\mathfrak{M} \to \infty$, we obtain

$$1 < D_{\mathsf{p}_i} + \left(1 + \frac{4\|\mathsf{B}\|^4 M^4 \mathsf{b}^2}{\gamma^2}\right)\left(4M^2D_{\mathsf{p}_i} + 4M^2\mathsf{b}\Lambda_1 + 8M^2c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}-1}\Lambda_2\right).$$

This contradicts (3.15). Hence, there exists $\mathfrak{M} > 0$ such that $\mathcal{F}(S'_{\mathfrak{M}}) \subseteq S'_{\mathfrak{M}}$.

Adopting the method used in the Theorem 3.1 of the paper [31], one can easily show that $\mathcal{F}$ is a continuous operator. Hence, operator $\mathcal{F}$ satisfies all the conditions of the Theorem 2.13, then there exists a fixed point ϑ on $S'_{\mathfrak{M}}$. Therefore, the system (1.2) has at least one mild solution on $[0, \mathsf{b}]$. □

Theorem 3.5. *Assume that hypotheses* $(\mathbf{C}_2)$, $(\mathbf{C}_7)$–$(\mathbf{C}_9)$ *hold and the functions* F, G *are uniformly bounded on their respective domains. Then the stochastic system* (1.2) *is approximately controllable on* $[0, \mathsf{b}]$.

Proof. Using the same arguments as in the Theorem 3.3, one can prove the approximate controllability of stochastic system (1.2). □

Remark 3.6. *We can see that the hypotheses of the Theorem* (1.2) *and Theorem* (3.5) *are sufficient conditions but not necessary to prove the approximate controllability of the stochastic system* (1.2).

4 Application

For an illustration of the obtained theory, we consider the following stochastic integrodifferential system

$$\begin{cases} d\,z(\rho,\xi) &= \Big[\dfrac{\partial^2 z(\rho,\xi)}{\partial\xi^2} + \displaystyle\int_0^{\rho} \gamma(\rho-s)\dfrac{\partial^2 z(s,\xi)}{\partial\xi^2}ds + f_1(\rho, z(\rho,\xi)) + \mathsf{u}(\rho,\xi) \\ & \quad + g_1(\rho, z(\rho,\xi))\dfrac{d\mathsf{R}^{\mathsf{H}}(\rho)}{d\rho}\Big]\,d\rho, \quad 0\le\xi\le\pi, \quad \rho\in(2i,2i+1], \quad i=0,1,\dots,m, \\ z(\rho,0) &= z(\rho,\pi)=0, \quad \rho\ge 0, \\ z(\rho,\xi) &= \sin it.z((2i-1)^-,\xi), \quad \rho\in(2i-1,2i], \quad i=1,2,\dots,m, \\ z(0,\xi) &= z_0(\xi), \quad \xi\in[0,\pi]. \end{cases} \tag{4.1}$$

where $0=s_0=\rho_0<\rho_1<s_1<\cdots<s_m<\rho_{m+1}=\mathsf{b}<\infty$ with $\rho_1=1$, $s_i=2i$, $\rho_i=2i-1$, R^{H} is a Rosenblatt process. The functions f_1, g_1 and γ will be described below.

Let $\mathbb{H}=\mathbb{Y}=\mathbb{U}=L^2([0,\pi])$ with the norm $\|\cdot\|$. Define $\mathrm{A}:\mathcal{D}(\mathrm{A})\subset\mathbb{H}\to\mathbb{H}$ by $\mathrm{A}\vartheta=\vartheta''$ with domain

$$\mathcal{D}(\mathrm{A}) = H^2(0,\pi)\cap H_0^1(0,\pi).$$

The spectrum of A consists of the eigenvalues $-n^2$ for $n\in\mathbb{N}^\star$, with associated eigenvectors $e_n := \sqrt{\frac{2}{\pi}}\sin(n\vartheta)$, $(n=1,2,3,\dots)$. Furthermore, the set $\{e_n : n\in\mathbb{N}^\star\}$ is an orthogonal basis in $\mathbb{H}$. Then

$$\mathrm{A}\vartheta = \sum_{n=1}^{\infty} -n^2\langle\vartheta, e_n\rangle e_n, \quad \vartheta\in\mathbb{H}.$$

It is well known that A is the infinitesimal generator of a strongly continuous semigroup $\{\mathsf{T}(\rho)\}_{\rho\ge 0}$ on $\mathbb{H}$, which is compact and is given by

$$\mathsf{T}(\rho)\vartheta = \sum_{n=1}^{\infty} e^{-n^2\rho}\langle\vartheta, e_n\rangle e_n, \quad \vartheta\in\mathbb{H}.$$

In order to define the operator $Q:\mathbb{Y}\to\mathbb{Y}$, we choose a sequence $\{v_n\}_{n\ge1}\subset\mathbb{R}^+$, set $Qe_n=v_ne_n$ and assume that $Tr(Q)=\sum_{n=1}^{\infty}\sqrt{v_n}<\infty$. Define the process $\mathsf{R}_Q^{\mathsf{H}}(s)$ by

$$\mathsf{R}_Q^{\mathsf{H}}(s) = \sum_{n=1}^{\infty}\sqrt{v_n}\beta_n^{\mathsf{H}}(\rho)e_n$$

where $\mathsf{H}\in(\frac{1}{2},1)$ and $\{\beta_n^{\mathsf{H}}\}_{n\in\mathbb{N}}$ is a sequence of mutually independent two-sided one-dimensional fBm and an infinite dimensional space.

Let $\mathbf{\Gamma} : \mathcal{D}(\mathrm{A}) \subset \mathbb{H} \to \mathbb{H}$ be the operator defined by

$$\mathbf{\Gamma}(\rho)(\tilde{z}) = \gamma(\rho)\mathrm{A}\tilde{z} \quad \text{for} \quad \rho \geq 0 \quad \text{and} \quad \tilde{z} \in D(\mathrm{A}).$$

In order to rewrite system (4.1) in an abstract form in $\mathbb{H}$, we introduce the following notations

$$\begin{cases} \vartheta(\rho) &= z(\rho,\xi) \quad \text{for} \quad \rho \geq 0 \quad \text{and} \quad \xi \in [0,\pi], \\ \vartheta(0) &= z(0,\xi) \quad \text{for} \quad \xi \in [0,\pi], \end{cases}$$

and the bounded linear operator $\mathsf{B} : L^2([0,\pi]) \to L^2([0,\pi])$ as

$$\mathsf{Bu}(\rho)(\xi) = \mathsf{u}(\rho,\xi), \quad \rho \in [0,\mathsf{b}], \quad \xi \in [0,\pi].$$

Next, we define the functions $\mathsf{F} : \mathsf{b}_0 \times \mathbb{H} \to \mathbb{H}$ and $\mathsf{G} : \mathsf{b}_0 \times \mathbb{H} \to L_2(\mathbb{X},\mathbb{H})$ as

$$\mathsf{F}(\rho,\vartheta(\rho))(\xi) = f_1(\rho,\vartheta(\rho))(\xi), \quad \vartheta \in \mathbb{H}, \quad \xi \in [0,\pi], \tag{4.2}$$

$$\mathsf{G}(\rho,\vartheta)(\xi) = g_1(\rho,\vartheta(\rho))(\xi), \quad \vartheta \in \mathbb{H}, \quad \xi \in [0,\pi]. \tag{4.3}$$

The functions $\mathsf{p}_i : \mathsf{b}_i \times \mathbb{H} \to \mathbb{H}$ are given by $\mathsf{p}_i(\rho,\vartheta(t_i^-))(\xi) = \sin it.z((2i-1)^-,\xi)$. From the above choices of the functions and operator $\mathbf{\Gamma}(\rho)$ with $\mathsf{B} = Id$, the system (4.1) takes the following abstract form

$$\begin{cases} d\vartheta(\rho) = \mathrm{A}\vartheta(\rho) + \int_0^\rho \mathbf{\Gamma}(\rho-s)\vartheta(s)ds + \mathsf{F}(\rho,\vartheta(\rho)) + \mathsf{Bu}(\rho) + \mathsf{G}(s,\vartheta(s))\dfrac{d\mathsf{R}^{\mathsf{H}}(\rho)}{d\rho}, \quad \rho \in \cup_{i=0}^m (s_i,\rho_{i+1}), \\ \vartheta(\rho) = \mathsf{p}_i(\rho,\vartheta(\rho_i^-)), \quad \rho \in \cup_{i=1}^m (\rho_i,s_i], \\ \vartheta(0) = \vartheta_0. \end{cases} \tag{4.4}$$

Moreover, $\mathbf{\Gamma}(\rho)$ satisfies $(\mathbf{R_2})$ and hence, by Theorem 2.4, Eq. (2.5) has a resolvent operator $(\Psi(\rho))_{\rho\geq 0}$ on $\mathbb{H}$. In particular, if we take $\mathsf{F}(\rho,\vartheta(\rho))(\xi) = \dfrac{\sin\rho}{1+\sin\rho}\vartheta(\rho)(\xi)$, and $\mathsf{G}(\rho,\vartheta)(\xi) = \dfrac{\vartheta(\rho)(\xi)}{e(1+e^\rho)}$, we see that, F and G satisfy assumptions $(\mathbf{C_1})$ and $(\mathbf{C_2})$. Therefore all conditions of Theorem 3.2 are satisfied. Since the semigroup $\mathsf{T}(\rho)$ is compact for $\rho > 0$, it is clear from Theorem 2.5 that the resolvent operator $\Psi(\rho)$ is compact for all $\rho > 0$. Therefore, the associated linear system of (4.1) can not be exactly controllable but may be approximately controllable.

It remains now to verify that $(\mathbf{H}_0)$ is fulfilled. To this end, we have the following result:

Lemma 4.1 ([30])**.** *Let $\gamma(\rho) \in L^1(\mathbb{R}^+) \cap C^1(\mathbb{R}^+)$ with primitive $\mathbb{O}(\rho) \in L^1_{loc}(\mathbb{R}^+)$ such that $\mathbb{O}(\rho)$ is non-positive, non-decreasing and $\mathbb{O}(0) = -1$. If operator* A *is self-adjoint and positive semi-definite, then the resolvent operator $\Psi(\rho)$ associated to* (2.5) *is self-adjoint as well.*

By Lemma 4.1 above, the resolvent operator $\Psi(\rho)$ of (2.5) is self-adjoint. Thus

$$\Psi^*(\rho)y = \Psi(\rho)y, \quad y \in \mathbb{H}.$$

If $\Psi^*(\rho)y = 0$, for all $\rho \in \mathcal{J}$, thus

$$\Psi^*(\rho)y = \Psi(\rho)y = 0, \quad \rho \in \mathcal{J}.$$

It follows from the fact $\Psi(0) = \mathrm{Id}$ that $y = 0$, so by virtue of Theorem 2.10, $(\mathbf{H}_0)$ holds. Therefore, in view of Theorem 3.2 and Theorem 3.3, the stochastic integrodifferential system (4.4) is approximately controllable on $\mathcal{J}$.

Remark 4.2. *In this above example, if we choose* $\mathsf{F}(\rho, \vartheta) = \dfrac{1}{\rho^{1/3}} \sin \vartheta$*, we observe that* $F(\rho, \vartheta)$ *does not satisfy the Lipschitz condition* $(\mathbf{C}_3) - b$ *near* 0*, but it satisfies the hypotheses* $(\mathbf{C}_8)$ *(see [36]). With this setting, Theorem 3.2 can not be applied to the system* (4.4)*, but we can apply the Theorem 3.3 to the* (4.4)*.*

5 Conclusion

In this research, we investigated the approximate controllability for a class of non-instantaneous impulsive integrodifferential equations driven by the Rosenblatt process. The proposed results have been carried out using Grimmer resolvent operator, stochastic analysis theory, and fixed point techniques (Krasnoselskii's and Schauder's fixed point theorem). Finally, an example is provided to illustrate the applicability of our results. We believe our study can open new research routes in stochastic integrodifferential systems with state-dependent delay and fractional cases. This article will initiate future work in the above categories.

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On uniqueness of L-functions in terms of zeros of strong uniqueness polynomial

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ABSTRACT

In this article, we have mainly focused on the uniqueness problem of an L-function $\mathcal{L}$ with an L-function or a meromorphic function f under the condition of sharing the sets, generated from the zero set of some strong uniqueness polynomials. We have introduced two new definitions, which extend two existing important definitions of URSM and UPM in the literature and the same have been used to prove one of our main results. As an application of the result, we have exhibited a much improved and extended version of a recent result of Khoai-An-Phuong [13]. Our remaining results are about the uniqueness of L-function under weighted sharing of sets generated from the zeros of a suitable strong uniqueness polynomial, which improve and extend some results in [12].

RESUMEN

En este arículo nos hemos enfocado principalmente en el problema de unicidad de una L-función $\mathcal{L}$ con una L-función o una función meromorfa f bajo la condición de compartir los conjuntos, generados a partir del conjunto de ceros de algunos polinomios de unicidad fuerte. Hemos introducido dos definiciones nuevas, que extienden dos importantes definiciones existentes en la literatura de URSM y UPM, y las mismas han sido usadas para probar uno de nuestros resultados principales. Como una aplicación del resultado, exhibimos una versión mejorada y extendida de un resultado reciente de Khoai-An-Phuong [13]. Nuestros resultados restantes son sobre la unicidad de una L-función bajo la condición de compartir conjuntos pesados generados a partir de los ceros de un polinomio de unicidad fuerte apropiado, que mejora y extiende algunos resultados en [12].

Keywords and Phrases: Meromorphic function, strong uniqueness polynomial, uniqueness, shared sets, $\mathcal{L}$ function.

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1 Introduction

Riemann hypothesis can be generalized by replacing Riemann's zeta function by the formally similar, but much more general L-functions. Considering $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ as a prototype in 1989, Selberg defined a rather general class $\mathcal{S}$ of convergent Dirichlet series $L(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$ which are absolutely convergent for $Re(s)>1$. In the meantime, this so-called Selberg class L-function became important object of research as it plays a pivotal role in analytic number theory. An L-function in $\mathcal{S}$ need to satisfy the following axioms (see [18]):

(i) Ramanujan hypothesis: $a(n)\ll n^{\epsilon}$ for every $\epsilon>0$.

(ii) Analytic continuation: There is a non-negative integer k such that $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order.

(iii) Functional equation: $\mathcal{L}$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s)=\omega\overline{\Lambda_{\mathcal{L}}(1-\overline{s})},$$

where

$$\Lambda_{\mathcal{L}}(s)=\mathcal{L}(s)Q^s\prod_{j=1}^{K}\Gamma(\lambda_j s+\nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j,ω with $Re\ \nu_j\geq 0$ and $|\omega|=1$.

(iv) Euler product hypothesis: $\mathcal{L}$ can be written over prime as

$$\mathcal{L}(s)=\prod_{p}\exp\left(\sum_{k=1}^{\infty}b(p^k)/p^{ks}\right)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k)\ll p^{k\theta}$ for some $\theta<1/2$, where the product is taken over all prime numbers p. The degree $d_{\mathcal{L}}$ of an L-function $\mathcal{L}$ is defined to be

$$d_{\mathcal{L}}=2\sum_{j=1}^{K}\lambda_j,$$

where λ_j and K are respectively the positive real number and the positive integer as in axiom (iii) above.

In this paper we are going to discuss some results under the periphery of value distribution of L-functions in $\mathcal{S}$. Throughout this paper by an L-function we will mean an L-function of non-zero degree with the normalized condition $a(1)=1$. On the other hand, by meromorphic function f we mean meromorphic function in the whole complex plane $\mathbb{C}$. Let $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$, $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ and $\overline{\mathbb{N}}=\mathbb{N}\cup\{0\}$, where $\mathbb{C}$ and $\mathbb{N}$ denote the set of all complex numbers and natural numbers respectively and by $\mathbb{Z}$ we denote the set of all integers. Before entering into the detail literature, let us assume $\mathcal{M}(\mathbb{C})$ as the field of meromorphic functions over $\mathbb{C}$ and assume f, g be two non-constant meromorphic functions in $\mathcal{M}(\mathbb{C})$. The proofs of the theorems of the paper are heavily depending on Nevanlinna theory and we assume that the readers are familiar with the standard notations like the characteristic function $T(r,f)$, the proximity function $m(r,f)$, counting (reduced counting) function $N(r,f)$ ($\overline{N}(r,f)$) that are also explained in [9, 20]. By $S(r,f)$ we mean any quantity that satisfies $S(r,f) = O(\log(rT(r,f)))$ when $r \to \infty$, except possibly on a set of finite Lebesgue measure. When f has finite order, then $S(r,f) = O(\log r)$ for all r. For any $f \in \mathcal{M}(\mathbb{C})$, the order of f is defined as

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

2 Definitions

Before proceeding further, we require the following definitions.

Definition 2.1. *For some $a \in \mathbb{C} \cup \{\infty\}$, we define $E_f(S) = \cup_{a\in S}\{z : f(z) - a = 0\}$, where each point is counted according to its multiplicity. If we do not count the multiplicity then the set $\cup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$, then we say f and g share the set S Counting Multiplicity (CM). On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$ then we say f and g share the set S Ignoring Multiplicity (IM).*

The following definition is more generalized than Definition 2.1 and somehow been inspired from the idea in [11].

Definition 2.2. *Let S_1, $S_2 \subset \mathbb{C}$ and if $E_f(S_1) = E_g(S_2)$ ($\overline{E}_f(S_1) = \overline{E}_g(S_2)$) holds then we say that f, g have the same inverse image with respect to the sets S_1 and S_2 respectively, counting multiplicity (ignoring multiplicity) and abbreviated it as IICM $\{(S_1)(f),(S_2)(g)\}$ (IIIM$\{(S_1)(f),(S_2)(g)\}$).*

Definition 2.3 ([14])**.** *Let k be a positive integer, $b \in \mathbb{C}$ and $E_k(0; f - b)$ be the set of all zeros of $f - b$, where a zero of multiplicity p is counted p times if $p \le k$, and $k + 1$ times if $p > k$. If $E_k(0; f - b) = E_k(0; g - b)$, we say that $f - b$, $g - b$ share the 0 with weight k and we write it as $f - b$ and $g - b$ share $(0, k)$ or f and g share (b, k). For $S \subset \mathbb{C} \cup \{\infty\}$, we define $E_f(S, k) = \cup_{a\in S} E_k(a; f)$, where k is a non-negative integer or infinity. Clearly $E_f(S) = E_f(S, \infty)$. In particular, $E_f(S, k) = E_g(S, k)$ and $E_f(\{a\}, k) = E_g(\{a\}, k)$ implies f and g share the set S and the value a with weight k.*

Definition 2.4 ([14])**.** *Let $b \in \mathbb{C}$, by $N(r, b; f \mid\ge s)$ ($N(r, b; f \mid\le s)$) we denote the counting function of those zeros of $f - b$ of multiplicity $\ge s$ ($\le s$). Also $\overline{N}(r, b; f \mid\ge s)$ ($\overline{N}(r, b; f \mid\le s)$) are defined analogously.*

Definition 2.5 ([21]). *If for some set $S \subset \mathbb{C}$, $E_f(S) = E_g(S)$ implies $f = g$, then we will say S unique range set of meromorphic function and denote it as URSM.*

Definition 2.6. *If for two sets $S_1, S_2 \subset \mathbb{C}$, $E_f(S_1) = E_g(S_2)$ implies $f = g$, then we will say $\{S_1, S_2\}$ belong to the extended class unique range set of meromorphic function and we denote it by ECURSM. Similarly we can define extended class unique range set of L-function and denote it as ECURSL.*

Definition 2.7 ([4]). *A set $S \subset \mathbb{C}$ is called a unique range set for meromorphic (entire) functions with weight k if for any two non-constant meromorphic (entire) functions f and g, $E_f(S,k) = E_g(S,k)$ implies $f = g$. We write S is URSMk ($URSEk$) in short. In case of L-function it is reasonable to write it as $URSLk$.*

Definition 2.8 ([1]). *For a non-zero constant c, if $P(f) = cP(g)$ implies $f = g$ then P is called a strong uniqueness polynomial for meromorphic function and denote it by SUPM.*

Definition 2.9 ([15]). *A polynomial P is called a uniqueness polynomial for meromorphic functions if $P(f) = P(g)$ implies $f = g$ and we denote it as UPM.*

Definition 2.10. *Let P, Q be two polynomials of same degree. Now if $f = g$ for all f, g satisfying $P(f) = Q(g)$ then, then we call $\{P, Q\}$ belong to the the extended class of uniqueness polynomial of meromorphic function and denote it as ECUPM. Similarly we can define extended class of uniqueness polynomial of L-function and denote it as ECUPL.*

Definition 2.11 ([4]). *Let $P(z)$ be a polynomial of derivative index k, i.e., $P'(z)$ has mutually k distinct zeros given by $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$ respectively. Then $P(z)$ is said to satisfy the critical injection property if $P(d_i) \neq P(d_j)$ for $i \neq j$, where $i, j \in \{1, 2, \ldots, k\}$.*

3 Background and main results

Recently the value distributions of L-functions have been studied exhaustively by many researchers ([10, 16, 19], etc.). The value distribution of L-function is all about the roots of $\mathcal{L}(s) = c$. In 2007, regarding uniqueness problem of two $\mathcal{L}$ functions, Steuding [19] proved that the number of shared values can be reduced significantly than that happens in case of ordinary meromorphic function. Below we invoke the result.

Theorem 3.1 ([19]). *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two non-constant L-functions and $c \in \mathbb{C}$. If $E_{\mathcal{L}_1}(c) = E_{\mathcal{L}_2}(c)$ holds, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Since L-functions possess meromorphic continuations, it will be interesting to investigate under which conditions an L-function can share a set with an arbitrary meromorphic function. Inspired by the question of Gross [8] for meromorphic functions, Yuan-Li-Yi [22] proposed the analogous question for a meromorphic function f and an L-function $\mathcal{L}$ sharing one or two finite sets. Yuan-Li-Yi [22] answered the question by themselves by proving the following uniqueness result.

Theorem 3.2 ([22]). *Let f be a meromorphic function having finitely many poles in $\mathbb{C}$ and let $\mathcal{L} \in \mathcal{S}$ be a non-constant L-function. Let us consider the set $S = \{w : w^n + aw^m + b = 0\}$, where $(n, m) = 1$, $n > 2m + 4$. If $E_f(S) = E_{\mathcal{L}}(S)$, then we will have $f = \mathcal{L}$.*

Motivated by the results of [22], Khoai-An-Phuong [13] considered a different polynomial, whose zero set is not same with the set as in Theorem 3.2. Under the CM sharing of this set, they [13] obtained a uniqueness relation between an L-function and an arbitrary meromorphic function. In their paper, Khoai-An-Phuong ([13]) consider the polynomial.

$$P(z) = (m+n+1)\left(\sum_{i=0}^{n}\binom{m}{i}\frac{(-1)^i}{m+n+1-i}z^{m+n+1-i}a^i\right)+1, \tag{3.1}$$

and $(m+n+1)\left(\sum_{i=0}^{n}\binom{m}{i}\frac{(-1)^i}{m+n+1-i}\right)a^{n+m+1} \neq -1, -2$. Then $P'(z) = (n+m+1)z^n(z-a)^m$. In their recent paper, Khoai-An-Phuong ([13]) obtained the following result.

Theorem 3.3 ([13]). *Let f be a non-constant meromorphic function, $\mathcal{L}$ be an L-function, $P(z)$ be defined as in* (3.1) *and $S = \{z : P(z) = 0\}$. If $n \geq 2$, $m \geq 2$, $n + m \geq 8$, then $E_f(S) = E_{\mathcal{L}}(S)$ implies $f = \mathcal{L}$.*

Now from Theorem 3.3, the following questions are inevitable:

(1) The considered set in Theorem 3.3 is a particular one and it is clear by Example 6.1 given in the application part afterwards, that the set is actually a zero set of a particular SUPM. So it is obvious to explore, whether the set can be generalized by the set of zeros of an arbitrary SUPM.

(2) In Theorem 3.3, to obtain the uniqueness result between f and $\mathcal{L}$, the authors considered CM sharing of the set. So is it possible to relax the CM sharing of the set?

(3) The minimum cardinality of the set in Theorem 3.3 is nine. Is it possible to decrease the cardinality of the set?

In this article, inspired by Theorem 3.3, we have tried to explore and provide the best possible answer of the above questions. Before going to the result, let us consider the following polynomial,

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_t z^t, \tag{3.2}$$

of simple zeros with $P'(z) = (z - d_1)^{q_1}(z - d_2)^{q_2} \cdots (z - d_l)^{q_k}$, satisfying the following properties:

(*i*) $P(z)$ is a critically injective polynomial of degree $t \geq 5$ with simple zeros and the derivative index of it is $k \geq 2$ and for $k = 2$, $\min\{q_1, q_2\} \geq 2$.

(*ii*) $P(z)$ be a SUPM.

Theorem 3.4. *Let f be a non-constant meromorphic function, $\mathcal{L}$ be a non-constant L-function, and $P(z)$ be defined as in (3.2) satisfying properties (i) and (ii) such that $S = \{z : P(z) = 0\}$. Now if $E_f(S, 2) = E_{\mathcal{L}}(S, 2)$ and $t \geq 2k + 4$, then we have $f = \mathcal{L}$.*

In the application part of this article in Example 6.1 , we have considered a more general version of polynomial (3.1) and by means of Example 6.1 , we have shown that our result Theorem 3.4 improves Theorem 3.3. Also in [12], the authors explored the things in a new direction. They found some sufficient conditions for a general polynomial to be a uniqueness polynomial for L-function and found a general unique range sets for L-functions as well. The following result extends the perimeter of unique range sets for L-functions.

Theorem 3.5 ([12]). *Let $P(z)$ be a uniqueness polynomial for L-functions. Suppose that $P(z)$ has no multiple zeros, and $P(1) \neq 0$. Then the set $S = \{z : P(z) = 0\}$, is a unique range set for L-functions, counting multiplicities.*

From the statement of Theorem 3.5, it will be interesting to ponder over the answer of the following question:

Question 3.1. *What happens in Theorem 3.5, if $P(1) = 0$?*

In the following theorem, we will deal with the answer of the above question. In fact, in view of Definition 2.6 and Definition 2.10, we will re-investigate Theorem 3.5 under a broader perspective, so that the same theorem will automatically be included in our result and at the same time the question will be answered. Now for the next theorem let us consider $Z^{-}(\mathcal{L})$ to denote the set of trivial zeros of L in the negative half plane, where each zero is counted according to its multiplicity.

Theorem 3.6. *Let $S_1 = \{z : P_o(z) = 0\}$ and $S_2 = \{z : Q_o(z) = 0\}$ where P_o be a uniqueness polynomial of L-function and $Q_o = k_1 P_o + k_2$ and having no multiple zeros. If*

(*i*) *$k_2 = 0$ and either $P_o(1) \neq 0$ or $P_o(0) \neq 0$ together with $Z^{-}(\mathcal{L}_1) = Z^{-}(\mathcal{L}_2)$,*

(*ii*) *$k_2 \neq 0$ and $Z^{-}(\mathcal{L}_1) = Z^{-}(\mathcal{L}_2)$, $P_o(1) \neq P_o(0)$. Also either $P_o(1)Q_o(1) \neq 0$ or $P_o(0)Q_o(0) \neq 0$; then $\{P_o, Q_o\}$ belong to ECUPL and $\{S_1, S_2\}$ belong to ECURSL.*

Clearly in the above theorem, when $k_2 = 0$ and $P_o(1) \neq 0$, then Theorem 3.6 is actually Theorem 3.5. Hence this result is an extension of Theorem 3.5.

Considering the IM sharing of set, in [12] the following result was obtained.

Theorem 3.7 ([12]). *Let $P(z)$ be a strong uniqueness polynomial for L-functions, and assume that $P(z)$ has no multiple zeros, and the degree q, the derivative index k of P satisfy inequality $q \geq 2k+6$. Then the zero set of $P(z)$ is a unique range set, ignoring multiplicities, for L-functions.*

As usual it will be interesting to further reduce the cardinality of the set. In the next theorem, we will show that with the help of weighted sharing of weight two the cardinality of the range set can significantly be reduced.

Theorem 3.8. *Let $P(z)$ be a strong uniqueness polynomial for L-functions with simple zeros, of degree t and of derivative index k such that $t \geq 2k+3$. Then the set $S = \{z : P(z) = 0\}$ is URSL2.*

4 Lemma

Next, we present some lemmas that will be needed in the sequel. Henceforth, we denote by H, the following function :

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G}\right),$$

Lemma 4.1 ([5]). *Let $F = P(f)$ and $G = P(g)$ be non-constant meromorphic functions where $P(z)$ is defined same as in (3.2). Also let F, G share $(0, m)$ Then,*

$$N_E^{1)}(r, 0; F) \leq N(r, \infty; H) + S(r, F) + S(r, G).$$

Lemma 4.2. *Let F and G be defined same as in Lemma 4.1 and share $(0, m)$. Then,*

$$N(r, \infty; H) \leq \overline{N}_*(r, 0; F, G) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \sum_{i=1}^{k} \overline{N}(r, \alpha_i; f) + \sum_{i=1}^{k} \overline{N}(r, \alpha_i; g)$$
$$+ \overline{N}_0(r, 0; f') + \overline{N}_0(r, 0; g') + S(r, f) + S(r, g),$$

where $\overline{N}_0(r, 0; f')$ is the reduced counting function of those zeros of F' where $F \prod_{i=1}^{k}(f - \alpha_i) \neq 0$ and $\overline{N}_0(r, 0; g')$ is similarly defined and α_i , $i = 1, 2, \ldots, k$ are distinct zeros of $P'(z)$.

Proof. Here we are not giving the proof as the similar proof can be found in [14]. □

Lemma 4.3 ([3]). *Let F and G be non-constant meromorphic functions and let F, G share $(0, m)$. Then,*

$$\overline{N}(r, 0; F) + \overline{N}(r, 0; G) - N_E^{1)}(r, 0; F) + \left(m - \frac{1}{2}\right) \overline{N}_*(r, 0; F, G) \leq \frac{1}{2}[N(r, 0; F) + N(r, 0; G)].$$

Proof. Here we are not giving the proof as the similar proof can be found in [3]. □

Lemma 4.4 ([6])**.** *Let $P(z)$ be a polynomial defined as in (3.2) with property (i). Also assume f and g be two non-constant meromorphic functions such that,*

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1,$$

$c_0 \neq 0$. Then we will have $c_1 = 0$.

Lemma 4.5 ([7])**.** *Let $P(z)$ be a polynomial defined as in (3.2) with property (i). Then $P(z)$ will be a UPM if and only if*

$$\sum_{1 \leq l < m \leq k} q_l q_m > \sum_{l=1}^{k} q_l.$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $\max\{q_1, q_2, q_3\} \geq 2$ or when $k = 2$, $\min\{q_1, q_2\} \geq 2$ and $q_1 + q_2 \geq 5$, then also the above inequality holds.

Lemma 4.6 ([17])**.** *Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j},$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 4.7 ([20])**.** *Let $f, g \in M(\mathbb{C})$ and let $\rho(f)$, $\rho(g)$ be the order of f and g, respectively. Then*

$$\rho(f \cdot g) \leq \max\{\rho(f), \rho(g)\}.$$

Lemma 4.8. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two non-constant L-functions and for some $A > 0$, in $\sigma < -A$, $Z^-(\mathcal{L}_1) = Z^-(\mathcal{L}_2)$. Then we can find a infinite sequence of zeros in the same half plane of both $\mathcal{L}_i$, $i = 1, 2$.*

Proof. It is given that in $\sigma < -A$, $Z^-(\mathcal{L}_1) = Z^-(\mathcal{L}_2)$. From axiom (iii), let us assume

$$\mathcal{L}_i(s) = \chi_i(s)\overline{\mathcal{L}_i(1-\overline{s})}, \quad \text{where}$$
$$\chi_i(s) = \omega_i Q_i^{1-2s} \frac{\prod_{j=1}^{k_i} \Gamma(\lambda_{ij}(1-s) + \overline{\nu_{ij}})}{\prod_{j=1}^{k_i} \Gamma(\lambda_{ij} s + \nu_{ij})}, \quad \text{for } i = 1, 2.$$

In particular, in $\sigma < -A$, the poles of $\prod_{j=1}^{k_1}\Gamma(\lambda_{1j}s+\nu_{1j})$ and $\prod_{j=1}^{k_2}\Gamma(\lambda_{2j}s+\nu_{2j})$ must match, also the poles of $\prod_{j=1}^{k_1}\Gamma(\lambda_{1j}(1-s)+\overline{\nu_{1j}})$ and $\prod_{j=1}^{k_2}\Gamma(\lambda_{2j}(1-s)+\overline{\nu_{2j}})$ must match in $\sigma > A$. Also in $-A < \sigma < 0$, $\prod_{j=1}^{k_1}\Gamma(\lambda_{1j}s+\nu_{1j})$ and $\prod_{j=1}^{k_2}\Gamma(\lambda_{2j}s+\nu_{2j})$ can have finitely many poles. It follows that $\frac{\chi_1}{\chi_2}$ is a meromorphic function with finitely many poles and zeros. So here we can write it as $\frac{\chi_1(s)}{\chi_2(s)} = R(s)e^{as}$, where R is a rational function and a is a complex constant. Therefore here we have,

$$\mathcal{L}_1(s) = \chi_1(s)\overline{\mathcal{L}_1(1-\overline{s})},$$
$$\mathcal{L}_2(s) = \chi_1(s)R(s)e^{as}\overline{\mathcal{L}_2(1-\overline{s})}.$$

Then in some $\sigma < -B$, where $B \geq A$, it is possible to find a sequence $\{s_n(= -\frac{n+\nu_{1j}}{\lambda_{1j}})\}$ for some fixed j, of zeros of $\chi_1(s)$, which are also zeros of $\mathcal{L}_i(s)$ and $\mathcal{L}_i(1-\overline{s})$ never vanish in $\sigma > B$ for $i = 1, 2$. Also it can be seen that $Re(s_n) \to -\infty$ as $n \to \infty$. □

5 Proofs of the theorems

Proof of Theorem 3.4. Let us consider the following cases.

Case 1: First assume $H = 0$. Then integrating we have,

$$\frac{1}{P(\mathcal{L})} = \frac{c}{P(f)} + d, \tag{5.1}$$

where $c\ (\neq 0), d$ are constants. Clearly from Lemma 4.4 we have, $d = 0$. As from the hypothesis of the theorem we know $P(z)$ is a SUPM, from $P(f) = cP(\mathcal{L})$, we have $f = \mathcal{L}$.

Case 2: Next assume $H \neq 0$. Using the Second Fundamental Theorem we have,

$$\begin{aligned}(t-1)T(r,\mathcal{L}) &\leq \overline{N}(r,0;P(\mathcal{L})) + \overline{N}(r,\infty;\mathcal{L}) + S(r,\mathcal{L}) \\ &\leq \overline{N}(r,0;P(f)) + O(\log r) + S(r,\mathcal{L}) \\ &\leq nT(r,f) + S(r,\mathcal{L}).\end{aligned} \tag{5.2}$$

Similarly, we can have,

$$(t-2)T(r,f) \leq nT(r,\mathcal{L}) + S(r,f). \tag{5.3}$$

Clearly (5.2) and (5.3) we have, $\rho(f) = \rho(\mathcal{L}) = 1$ and hence $S(r,f) = S(r,\mathcal{L}) = S(r)$ (say).

Using the Second Fundamental theorem we have,

$$(t+k-1)(T(r,f)+T(r,\mathcal{L})) \leq \overline{N}(r,0;P(f))+\overline{N}(r,0;P(\mathcal{L}))+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L}) + \sum_{i=1}^{k}(\overline{N}(r,\alpha_i;f))+\overline{N}(r,\alpha_i;\mathcal{L}))-N_0(r,0;f')-N_0(r,0;\mathcal{L}')+S(r).$$

i.e.,

$$(t-1)T(r,\mathcal{L})+(t-2)T(r,f) \leq \overline{N}(r,0;P(f))\overline{N}(r,0;P(\mathcal{L}))-N_0(r,0;f') - N_0(r,0;\mathcal{L}')+S(r). \tag{5.4}$$

Using Lemmas 4.3, 4.1, 4.2 and 4.6, from (5.4) we have,

$$(t-1)T(r,\mathcal{L})+(t-2)T(r,f) \leq \frac{n}{2}\{T(r,f)+T(r,\mathcal{L})\}+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L}) + \sum_{i=1}^{k}(\overline{N}(r,\alpha_i,f)+\overline{N}(r,\alpha_i;\mathcal{L}))+S(r),$$

i.e.,

$$\left(\frac{t}{2}-2\right)T(r,f)+\left(\frac{t}{2}-1\right)T(r,\mathcal{L})+S(r) \leq kT(r,\mathcal{L})+(k+1)T(r,f)+S(r),$$

$$(t-2k-6)T(r,f)+(t-2-2k)T(r,\mathcal{L}) \leq S(r). \tag{5.5}$$

Using (5.2) we have

$$(t-2k-6)\frac{t-1}{t}T(r,\mathcal{L})+(t-2-2k)T(r,\mathcal{L}) \leq S(r). \tag{5.6}$$

From (5.6) for $t \geq 2k+4$ we arrive at a contradiction. □

Proof of Theorem 3.8. Let us consider two non-constant L-functions $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $E_{\mathcal{L}_1}(S,2) = E_{\mathcal{L}_2}(S,2)$ where S is the zero set of strong uniqueness polynomial for L-function. Also assume,

$$F = P(\mathcal{L}_1) \quad \text{and} \quad G = P(\mathcal{L}_2).$$

If $H = 0$, then from Case 1 of Theorem 3.4 we will have, $\mathcal{L}_1 = \mathcal{L}_2$. If $H \neq 0$, then proceeding similarly as done in (5.4), (5.5) we will have a contradiction for $t \geq 2k+3$. Hence finally we will have $\mathcal{L}_1 = \mathcal{L}_2$. □

Proof of Theorem 3.6. Let us assume for two non-constant L-functions, $\mathcal{L}_1$, $\mathcal{L}_2$; $E_{\mathcal{L}_1}(S_1) = E_{\mathcal{L}_2}(S_2)$. Clearly then we can set the auxiliary function

$$G = \frac{P_o(\mathcal{L}_1)}{Q_o(\mathcal{L}_2)} = (s-1)^l e^{p_1(s)}, \tag{5.7}$$

for some integer l and from Lemma 4.7 we will have $p_1(s) = as + b$, for some complex constants a, b. Now let us consider the following cases.

Case 1: First let $k_2 = 0$, *i.e.*, $Q_o = k_1 P_o$.

Subcase 1.1: $P_o(1) \neq 0$. Then,

$$G = \frac{(\mathcal{L}_1 - \alpha_1)(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - \alpha_1)(\mathcal{L}_2 - \alpha_2)\cdots(\mathcal{L}_2 - \alpha_t)} = k_1(s-1)^l e^{as+b}, \tag{5.8}$$

from (5.8) taking limit $\sigma \to +\infty$ we have,

$$\lim_{\sigma\to+\infty} k_1(s-1)^l e^{as+b} = 1,$$

which implies $a = l = 0$ and then simply $k_1 e^b = 1$. Finally we have, $\frac{P_o(\mathcal{L}_1)}{P_o(\mathcal{L}_2)} = 1$ and hence $\mathcal{L}_1 = \mathcal{L}_2$.

Subcase 1.2: Let us assume $P_o(1) = 0$ but $P_o(0) \neq 0$. Without loss of generality assume $\alpha_1 = 1$. Again $\mathcal{L}_i$ can be represented by a Dirichlet series, *i.e.*, $\mathcal{L}_i(s) = \sum_{n=1}^{\infty} \frac{a_i(n)}{n^s}$, $i = 1, 2$, absolutely convergent for $\sigma > 1$, where $a_i(1) = 1$. Also let n_1, n_2 be two integers such that $n_i = \min\{n\ (\geq 2) : a_i(n) \neq 0,\ i = 1, 2\}$. So,

$$\frac{\mathcal{L}_1 - 1}{\mathcal{L}_2 - 1} = \frac{\frac{1}{n_1^s}\left(a_1(n_1) + \sum_{n>n_1}^{\infty} a_1(n)(\frac{n_1}{n})^s\right)}{\frac{1}{n_2^s}\left(a_2(n_2) + \sum_{n>n_2}^{\infty} a_2(n)(\frac{n_2}{n})^s\right)} = \left(\frac{n_2}{n_1}\right)^s G_0(s), \tag{5.9}$$

where,

$$G_0(s) = \frac{a_1(n_1) + \sum_{n>n_1}^{\infty} a_1(n)(\frac{n_1}{n})^s}{a_2(n_2) + \sum_{n>n_2}^{\infty} a_2(n)(\frac{n_2}{n})^s}.$$

By the construction of $G_0(s)$ we note that $\lim_{\sigma\to+\infty} G_0(s) = \frac{a_1(n_1)}{a_2(n_2)}$. In view of (5.7), let us consider the following function

$$\begin{aligned} G_1 = G_0(s) \cdot \frac{(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - \alpha_2)\cdots(\mathcal{L}_2 - \alpha_t)} &= \frac{\mathcal{L}_1 - 1}{\mathcal{L}_2 - 1} \cdot q^s \cdot \frac{(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - \alpha_2)\cdots(\mathcal{L}_2 - \alpha_t)} \\ = q^s \frac{(\mathcal{L}_1 - 1)(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - 1)(\mathcal{L}_2 - \alpha_2)\cdots(\mathcal{L}_2 - \alpha_t)} &= q^s G = k_1 q^s (s-1)^l e^{as+b}, \end{aligned} \tag{5.10}$$

for some $q = \frac{n_1}{n_2} (\in \mathbb{Q}^+)$. We can write $q = e^{\log q} = e^{q'}$, then we can write it as,

$G_1 = k_1 q^s (s-1)^l e^{as+b} = k_1 (s-1)^l e^{(q'+a)s+b} = k_1 (s-1)^l e^{a's+b}$ where $a' = q' + a$. Let us consider $a' = a_1 + ia_2$ and $b = b_1 + ib_2$. With respect to the first equality of (5.10), taking limit $\sigma \to +\infty$, we have $\lim_{\sigma\to+\infty} G_1 = C_1$, for some constant $C_1 \in \mathbb{C}^*$. Next from the second and last equality of (5.10), taking limit $\sigma \to +\infty$, we have

$$\lim_{\sigma\to+\infty} \left| q^s \frac{(\mathcal{L}_1 - 1)}{(\mathcal{L}_2 - 1)} \cdot \frac{(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - \alpha_2)\cdots(\mathcal{L}_2 - \alpha_t)} \right| = |C_1| = \lim_{\sigma\to+\infty} |(s-1)^l e^{a's+\hat{b}}|$$
$$= \text{Constant} = \lim_{\sigma\to+\infty} |\sigma - 1 + it|^l e^{a_1\sigma - a_2 t + b_1}.$$

Therefore we must have $a_1 = 0 = l$, otherwise $\lim_{\sigma\to+\infty} |\sigma - 1 + it|^l e^{a_1\sigma - a_2 t + b_1} = \infty$ or 0 according as $a_1 >$ or < 0 and with the same argument it can be shown that $l = 0$. Also,

$$\lim_{\sigma\to+\infty} e^{-a_2 t + b_1} = |C_1|, \quad \forall t \in \mathbb{R},$$

implies $a_2 = 0$. Hence we have $a = a_1 + ia_2 = 0$ and $l = 0$. Therefore, $G_1 = k_1 e^b$ and from the last equality of (5.10), we get $G = q^{-s} k_1 e^b$, *i.e.*, from (5.8) we have

$$\frac{(\mathcal{L}_1 - 1)}{(\mathcal{L}_2 - 1)} \cdot \frac{(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - \alpha_2)\cdots(\mathcal{L}_2 - \alpha_t)} = q^{-s} k_1 e^b. \tag{5.11}$$

Now from Lemma 4.8, it is possible to find a sequence s_n of trivial zeros in $\sigma < -A$, whose real part diverges, *i.e.*, $Re(s_n) \to -\infty$, as $n \to \infty$. From (5.11) putting $s = s_n$ we have $q^{Re(-s_n)} |k_1| e^{Re(b)} = 1$, taking limit as $n \to \infty$ we will have $q^{Re(-s_n)} \to \infty$ or 0, according as $q > 1$ or < 1. So we must have $q = 1$ and hence $k_1 e^b = 1$. Therefore $P_o(\mathcal{L}_1) = P_o(\mathcal{L}_2) \implies \mathcal{L}_1 = \mathcal{L}_2$.

Case 2: Next let $k_2 \neq 0$. Then we can write G as,

$$G = \frac{P_o(\mathcal{L}_1)}{k_1 P_o(\mathcal{L}_2) + k_2} = \frac{(\mathcal{L}_1 - \alpha_1)(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_t)}{(\mathcal{L}_2 - \beta_1)(\mathcal{L}_2 - \beta_2)\cdots(\mathcal{L}_2 - \beta_t)} = (s-1)^l e^{as+b}. \tag{5.12}$$

Subcase 2.1: Let us assume $P_o(1).Q_o(1) \neq 0$. From (5.11) taking $\sigma \to +\infty$, we will have, $G = C =$ non-zero constant. Hence we have, $P_o(\mathcal{L}_1) = k_1' P_o(\mathcal{L}_2) + k_2'$. In view of Lemma 4.8, Putting $s = s_n$ we have, $k_2' = P_o(0)(1 - k_1')$.

Subcase 2.1.1: First let $P_o(0) = 0$, then $k_2' = 0$. Using the fact $P_o(1) \neq 0$, with the same argument as in Subcase 1.1 we will have, $P_o(\mathcal{L}_1) = P_o(\mathcal{L}_2)$ and hence $\mathcal{L}_1 = \mathcal{L}_2$.

Subcase 2.1.2: Next let $P_o(0) \neq 0$. Then we have $P_o(\mathcal{L}_1) - P_o(0) = k_1'(P_o(\mathcal{L}_2) - P_o(0))$. Taking $\sigma \to +\infty$ and noting that $P_o(0) \neq P_o(1)$, we have, $k_1' = 1$ and hence $k_2' = 0$. And the from Subcase 1.1 we will have the result.

Subcase 2.2: Assume $P_o(1)Q_o(1) = 0$ but $P_o(0)Q_o(0) \neq 0$.

Subcase 2.2.1: Let us assume $P_o(1) = 0 = Q_o(1)$. Without loss of generality assume $\alpha_1 = \beta_1 = 1$. Then proceeding similarly as done in Subcase 1.2 we will have $\frac{P_o(\mathcal{L}1)}{Q_o(\mathcal{L}_2)} =$ constant. Noting that $P_o(0) \neq 0$, like Subcase 2.1 we can show that the constant is 1 and so we have $\mathcal{L}_1 = \mathcal{L}_2$.

Subcase 2.2.2: Next let $P_o(1) = 0$ but $Q_o(1) \neq 0$. Then let $\alpha_1 = 1$ and we can have,

$$\mathcal{L}_1 - 1 = \frac{1}{n_1^s}\left(a_1(n_1) + \sum_{n>n_1}^{\infty} a_1(n)\left(\frac{n_1}{n}\right)^s\right) = \frac{1}{n_1^s}G_1(s),$$

where $G_1(s) = n_1^s(\mathcal{L}_1 - 1) = a_1(n_1) + \sum_{n>n_1}^{\infty} a_1(n)(\frac{n_1}{n})^s$ and $\lim_{\sigma\to+\infty} G_1 = a_1(n_1)$. Now, $G = \dfrac{(\mathcal{L}_1 - 1)(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_n)}{(\mathcal{L}_2 - \beta_1)(\mathcal{L}_2 - \beta_2)\cdots(\mathcal{L}_2 - \beta_n)} = (s-1)^l e^{as+b}$.
Let us set a function

$$\begin{aligned} G_2 &= G_1 \frac{(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_n)}{(\mathcal{L}_2 - \beta_1)(\mathcal{L}_2 - \beta_2)\cdots(\mathcal{L}_2 - \beta_n)} \\ &= n_1^s \frac{(\mathcal{L}_1 - 1)(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_n)}{(\mathcal{L}_2 - \beta_1)(\mathcal{L}_2 - \beta_2)\cdots(\mathcal{L}_2 - \beta_n)} = n_1^s G = (s-1)^l n_1^s e^{as+b}. \end{aligned} \tag{5.13}$$

Therefore we can write, $G_2 = (s-1)^l e^{a''s} e^b$, where $a'' = a + \log n_1$. Next the first equality of (5.13) implies, $\lim_{\sigma\to+\infty} G_2 =$ Constant. But $\lim_{\sigma\to+\infty} |(s-1)^l e^{a''s+b}| = 0$ or ∞, according as $Re(a'') <$ or > 0, it follows that $Re(a'') = 0$. Since the limit is independent of t, we will have $Im(a'') = 0$. With similar arguments we will have $l = 0$. Therefore $a'' = 0 = l$ and we will have from the last equality of (5.13),

$$G_2 = e^b \implies G = n_1^{-s} e^b$$

i.e.,

$$\frac{(\mathcal{L}_1 - 1)(\mathcal{L}_1 - \alpha_2)\cdots(\mathcal{L}_1 - \alpha_n)}{(\mathcal{L}_2 - \beta_1)(\mathcal{L}_2 - \beta_2)\cdots(\mathcal{L}_2 - \beta_n)} = n_1^{-s} e^b.$$

Proceeding similarly as in (5.11) we will have, $n_1 = 1$ and then we have $G = e^b =$ Constant. With the help of Subcase 2.1 we will have $\mathcal{L}_1 = \mathcal{L}_2$.

Subcase 2.2.3: Next let $P_o(1) \neq 0$ but $Q_o(1) = 0$, proceeding in a same way as done in Subcase 2.2.2 and then using Subcase 2.1 we will have $\mathcal{L}_1 = \mathcal{L}_2$.

Hence $\{P_o, Q_o\}$ belong to ECUPL and $\{S_1, S_2\}$ belong to ECURSL. □

6 Application

In this section, we show the application of Theorem 3.4. Not only that, next we are going to show that the much improved version of Theorem 3.3 falls under a special case of our Theorem 3.4. Below we explain this fact via the following example.

Example 6.1. We are going to consider a new polynomial of degree $m+n+1$ recently introduced in [2] as follows:

$$\begin{aligned} P(z) &= \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{m+n+1-j} z^{m+n+1-j} a^j \\ &+ \sum_{i=1}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \frac{(-1)^{i+j}}{m+n+1-i-j} z^{m+n+1-i-j} a^j b^i - c = Q(z) - c, \end{aligned} \tag{6.1}$$

where a and b be distinct such that $a, b \in \mathbb{C}$, $c \neq 0$, $Q(a), Q(b)$ and $m \geq n+2$ and $n \geq 2$. It is easy to verify that,

$$P'(z) = (z-a)^n (z-b)^m.$$

Clearly from the choice of c, $P(z)$ has only simple zeros. First we will show that (6.1) is critically injective, strong uniqueness polynomial with derivative index 2 with $m \geq n+2$ and $n \geq 2$. From Remark 1 [2, p. 506] it can be shown that $P(z)$ is critically injective polynomial. Next, let us assume for some constant $A \neq 0$ and for two non-constant meromorphic functions f, g with finitely many poles we have

$$P(f) = AP(g). \tag{6.2}$$

By Lemma 4.4, we get,

$$T(r,f) = T(r,g) + O(1). \tag{6.3}$$

Also here, $\overline{N}(r,\infty;f) = S(r,f) = \overline{N}(r,\infty;g) = S(r,g)$.

Now, consider the cases,

Case 1: Suppose $A \neq 1$. Then

$$P(f) + c = A(P(g) + c) - c(A-1), \tag{6.4}$$

i.e.,

$$Q(f) = AQ(g) - c(A-1) \implies Q(f) - Q(b) = AQ(g) - (Q(b) + c(A-1)).$$

Recall that the only zeros of $Q'(z)$ are a and b. So only possible multiple zeros of $\phi(z) =$

$AQ(z)-(Q(b)+c(A-1))$ could be a and b. First assume b is one multiple zero of $\phi(z)$. Thus $\phi(b)=0$, *i.e.*,

$$AQ(b)-(Q(b)+c(A-1))=0 \implies (Q(b)-c)(A-1)=0 \implies Q(b)=c,$$

a contradiction as $Q(b)\neq c$.

Next assume a is the multiple zero of $\phi(z)$. It is easy to see that $\phi(z)=(z-a)^{n+1}\phi_1(z)$, where $\phi_1(a)\neq 0$ and all zeros of $\phi_1(z)$ are simple, namely $\mu_j,\ j=1,2,\ldots,m$. Notice that $Q(z)-Q(b)=(z-b)^{m+1}\phi_2(z)$, where $\phi_2(b)\neq 0$ and all zeros of $\phi_2(z)$ are simple, namely $\nu_j,\ j=1,2,\ldots,n$. From (6.4) we have,

$$\overline{N}(r,b;f)+\sum_{i=1}^{n}\overline{N}(r,\nu_j;f)=\overline{N}(r,a;g)+\sum_{i=1}^{m}\overline{N}(r,\mu_j;g). \tag{6.5}$$

Now using the Second Fundamental Theorem we have,

$$\begin{aligned} mT(r,g) &\leq \overline{N}(r,a;g)+\sum_{i=1}^{m}\overline{N}(r,\mu_j;g)+\overline{N}(r,\infty;g)+S(r,g)\\ &\leq \overline{N}(r,b;f)+\sum_{i=1}^{n}\overline{N}(r,\nu_j;f)+S(r,g)\\ &\leq (n+1)T(r,f)=(n+1)T(r,g)+S(r,g), \end{aligned}$$

this contradicts the fact $m\geq n+2$.

Hence we see neither a nor b be the multiple zeros of $\phi(z)$ and hence all the zeros of $\phi(z)$ are simple, say $\delta_j,\ j=1,2,\ldots,m+n+1$. From (6.4) we have,

$$\begin{aligned} (m+n)T(r,g) &\leq \sum_{j=1}^{m+n+1}\overline{N}(r,\delta_j;g)+\overline{N}(r,\infty;g)+S(r,g)\\ &\leq \overline{N}(r,b;f)+\sum_{i=1}^{n}\overline{N}(r,\nu_j;f)+S(r,g)\\ &\leq (n+1)T(r,f)=(n+1)T(r,g)+S(r,g), \end{aligned}$$

a contradiction as $m\geq n+2$ and $n\geq 2$.

Case 2: Assume $A=1$.

$$P(f)=P(g).$$

Now the zeros of $P'(z)$ has multiplicities $m\geq 4,\ n\geq 2$ and $m+n\geq 6$. Hence from Lemma 4.5 we have from above, $f=g$. Now if we take $m=5,\ n=2$, then $P(z)$ becomes a polynomial of degree 8. So clearly from the above discussion if f be a meromorphic function and $\mathcal{L}$ be an L-function satisfying $E_f(S,2) = E_{\mathcal{L}}(S,2)$ such that the degree of $P(z)$ becomes $m+n+1 \geq 8$, then by Theorem 3.4, we have $f = \mathcal{L}$. As putting $b = 0$ in (6.1), we obtain the polynomial (3.1), for $m + n \geq 7$, the result in Theorem 3.3 holds as well. Clearly Theorem 3.4 is a three step improvements of Theorem 3.3:

(1) In *Theorem 3.4*, we have considered the zero set of an arbitrary SUPM satisfying properties (i) and (ii). By means of Example 6.1 we know that the polynomial (3.1) is itself a critically injective SUPM, so in terms of choice of SUPM, Theorem 3.4 is quite a generalization of Theorem 3.3.

(2) In the light of relaxation of sharing of the zero set Theorem 3.4 improves Theorem 3.3.

(3) Most importantly, it can be verified that the minimum cardinality of the considered set in Theorem 3.3 is nine, where as we have been able to show that in Theorem 3.3 still holds for the cardinality of the range set as $n + m + 1 \geq 8$. That is the cardinality of the range set in Theorem 3.3 can further be diminished.

Again since $\mathcal{L}$ can be analytically continued as a meromorphic function with only one pole, then from the above discussion it can be observed that if $\mathcal{L}_1$ and $\mathcal{L}_2$ share the zero set S of the polynomial (6.1) with weight two, *i.e.*, $E_{\mathcal{L}_1}(S,2) = E_{\mathcal{L}_2}(S,2)$ where $n + m + 1 \geq 7$, then according to Theorem 3.8 we will have $\mathcal{L}_1 = \mathcal{L}_2$.

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8 Conflict of interest

The authors declare that they have no conflicts of interest.

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A note on the structure of the zeros of a polynomial and Sendov's conjecture

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ABSTRACT

In this note we prove a result that highlights an interesting connection between the structure of the zeros of a polynomial $p(z)$ and Sendov's conjecture.

RESUMEN

En esta nota demostramos un resultado que da luces sobre una conexión interesante entre la estructura de los ceros de un polinomio $p(z)$ y la conjetura de Sendov.

Keywords and Phrases: Polynomials, zeros, critical points.

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1 Introduction

Let $p(z) := \sum_{j=0}^{n} a_j z^j$, where $a_j \in \mathbb{C}$ be a polynomial with complex coefficients. If we plot the zeros of a polynomial $p(z)$ and the zeros of its derivative $p'(z)$ (the critical points of $p(z)$) in the complex plane, there are interesting geometric relations between the two sets of points. To start with they have the same centroid. We also have the Gauss-Lucas Theorem which states that the critical points of a polynomial p lie in the convex hull of its zeros. Regarding the distribution of critical points of p within the convex hull of its zeros the well known Sendov's Conjecture asserts:

"If all the zeros of a polynomial p lie in $|z| \leq 1$ and if z_0 is any zero of $p(z)$, then there is a critical point of p in the disk $|z - z_0| \leq 1$."

The conjecture was posed by Bulgarian mathematician Blagovest Sendov in 1958, but is often attributed to Ilieff because of a reference in Hayman's *Research Problems in Function Theory* [6] in 1967. A large number of papers have been published on this conjecture (for details see [9]) but the general conjecture remains open. Rubinstein [10] in 1968 proved the conjecture for all polynomials of degree 3 and 4. In 1969 Schmeisser [11] showed that, if the convex hull containing all zeros of p has its vertices on $|z| = 1$, then p satisfies the conjecture (for the proof see [9, Theorem 7.3.4]). Later Schmeisser [12] also proved the conjecture for the Cauchy class of polynomials. In 1996 Borcea [2] showed that the conjecture holds true for polynomials with atmost six distinct zeros and in 1999 Brown and Xiang [3] proved the conjecture for polynomials of degree up to eight. Dégot [5] proved that for every zero (say) z_0 of a polynomial p there exists lower bound N_0 depending upon the modulus of z_0 such that $|z - z_0| \leq 1$ contains a critical point of p if $deg(p) > N_0$. Chalebgwa [4] gave an explicit formula for such a N_0. More recent work in this area includes that of Kumar [7], Sofi, Ahanger and Gardner [14], and Sofi and Shah [13]. As for the latest, Terence Tao [15] following on the work of Dégot [5], proved that the Sendov's conjecture holds for polynomials with sufficiently high degree.

In this paper we prove an interesting connection between the geometric structure of the zeros of a polynomial and Sendov's conjecture.

2 Statement and proof of the theorem

Theorem 2.1. *Let $p(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \geq 2$ with all its zeros $z_1, z_2, \ldots, z_n$ lying inside the closed unit disk. Suppose that for all $j = 1, 2, \ldots, n$*

$$\sum_{i=1, i\neq j}^{n} \left|1 - \frac{1}{z_j - z_i}\right|^2 \leq \sum_{i=1, i\neq j}^{n} \left|\frac{1}{z_j - z_i}\right|^2 \tag{2.1}$$

then $|z - z_j| \leq 1$ contains some critical point of p, that is, Sendov's conjecture holds for p.

[One prime (but not the only) example of polynomials satisfying the hypotheses of Theorem 2.1 are the polynomials whose zeros lie on a circle within the closed unit disk. In this case we may assume without loss of generality that $|z_i| = |z_j|$ for all $1 \leq i, j \leq n$ and that for a fixed but arbitrary $1 \leq j \leq n$, $0 < z_j \leq 1$. Hence for all $1 \leq i \leq n$

$$|z_i - (z_j - 1)| \leq |z_i| + |z_j - 1| = |z_i| + 1 - z_j = 1$$

and the required condition

$$\sum_{i=1, i\neq j}^{n} \left|1 - \frac{1}{z_j - z_i}\right|^2 \leq \sum_{i=1, i\neq j}^{n} \left|\frac{1}{z_j - z_i}\right|^2$$

is satisfied.]

Proof. Let $\zeta_1, \zeta_2 \ldots, \zeta_{n-1}$ be the critical points of p and assume to the contrary. Then there exists a zero of p say z_1 such that $|z_1 - \zeta_i| > 1$ for $1 \leq i \leq n-1$. We note that z_1 cannot be a repeated zero of p and hence $z_1 - z_i \neq 0$ for all $i = 2, 3, \ldots, n$ and

$$\frac{1}{|z_1 - \zeta_i|} < 1 \quad \text{for all } 1 \leq i \leq n-1.$$

Also we can write

$$p'(z) = na_n \prod_{i=1}^{n-1} (z - \zeta_i)$$

so that

$$\frac{p''(z)}{p'(z)} = \sum_{i=1}^{n-1} \frac{1}{z - \zeta_i}.$$

This gives

$$\frac{p''(z_1)}{p'(z_1)} = \sum_{i=1}^{n-1} \frac{1}{z_1 - \zeta_i}.$$

Hence

$$\left|\frac{p''(z_1)}{p'(z_1)}\right| = \left|\sum_{i=1}^{n-1} \frac{1}{z_1 - \zeta_i}\right| \leq \sum_{i=1}^{n-1} \frac{1}{|z_1 - \zeta_i|} < n-1.$$

That is

$$\left|\frac{p''(z_1)}{p'(z_1)}\right| < n-1. \tag{2.2}$$

Now suppose

$$p(z) = a_n(z - z_1)q(z), \quad \text{where } q(z) = \prod_{i=2}^{n} (z - z_i).$$

This gives

$$\frac{q'(z)}{q(z)} = \sum_{i=2}^{n} \frac{1}{z - z_i}$$

so that

$$\frac{q'(z_1)}{q(z_1)} = \sum_{i=2}^{n} \frac{1}{z_1 - z_i}.$$

Also

$$p'(z_1) = q(z_1) \quad \text{and} \quad p''(z_1) = 2q'(z_1).$$

Therefore from (2.2), we obtain

$$\left|\frac{2q'(z_1)}{q(z_1)}\right| = \left|\frac{p''(z_1)}{p'(z_1)}\right| < n-1$$

and hence

$$\left|\frac{q'(z_1)}{q(z_1)}\right| < \frac{n-1}{2}.$$

Thus

$$\left|\sum_{i=2}^{n} \frac{1}{z_1 - z_i}\right| < \frac{n-1}{2}. \tag{2.3}$$

Now

$$\mathfrak{Re}\left(\frac{1}{z_1 - z_i}\right) = \frac{1}{2} + \frac{1 - |z_1 - z_i - 1|^2}{2|z_1 - z_i|^2}$$

for all $i = 2, 3, \ldots, n$. This gives

$$\begin{aligned}
\sum_{i=2}^{n} \mathfrak{Re}\left(\frac{1}{z_1 - z_i}\right) &= \frac{n-1}{2} + \sum_{i=2}^{n} \frac{1 - |z_1 - z_i - 1|^2}{2|z_1 - z_i|^2} \\
&= \frac{n-1}{2} + \frac{1}{2}\left(\sum_{1=2}^{n}\left|\frac{1}{z_1 - z_i}\right|^2 - \sum_{i=2}^{n}\left|\frac{z_1 - z_i - 1}{z_1 - z_i}\right|^2\right) \\
&= \frac{n-1}{2} + \frac{1}{2}\left(\sum_{i=2}^{n}\left|\frac{1}{z_1 - z_i}\right|^2 - \sum_{1=2}^{n}\left|1 - \frac{1}{z_1 - z_i}\right|^2\right)
\end{aligned}$$

Now from (2.1)

$$\left(\sum_{i=2}^{n}\left|\frac{1}{z_1 - z_i}\right|^2 - \sum_{i=2}^{n}\left|1 - \frac{1}{z_1 - z_i}\right|^2\right) \geq 0$$

Therefore

$$\mathfrak{Re}\left(\sum_{i=2}^{n} \frac{1}{z_1 - z_i}\right) = \sum_{i=2}^{n} \mathfrak{Re}\left(\frac{1}{z_1 - z_i}\right) \geq \frac{n-1}{2}$$

and hence

$$\left|\sum_{i=2}^{n} \frac{1}{z_1 - z_i}\right| \geq \frac{n-1}{2}$$

which contradicts (2.3) and the contradiction proves the result. □

3 Declarations

Ethical Approval:

Not Applicable.

Conflict of Interest:

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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