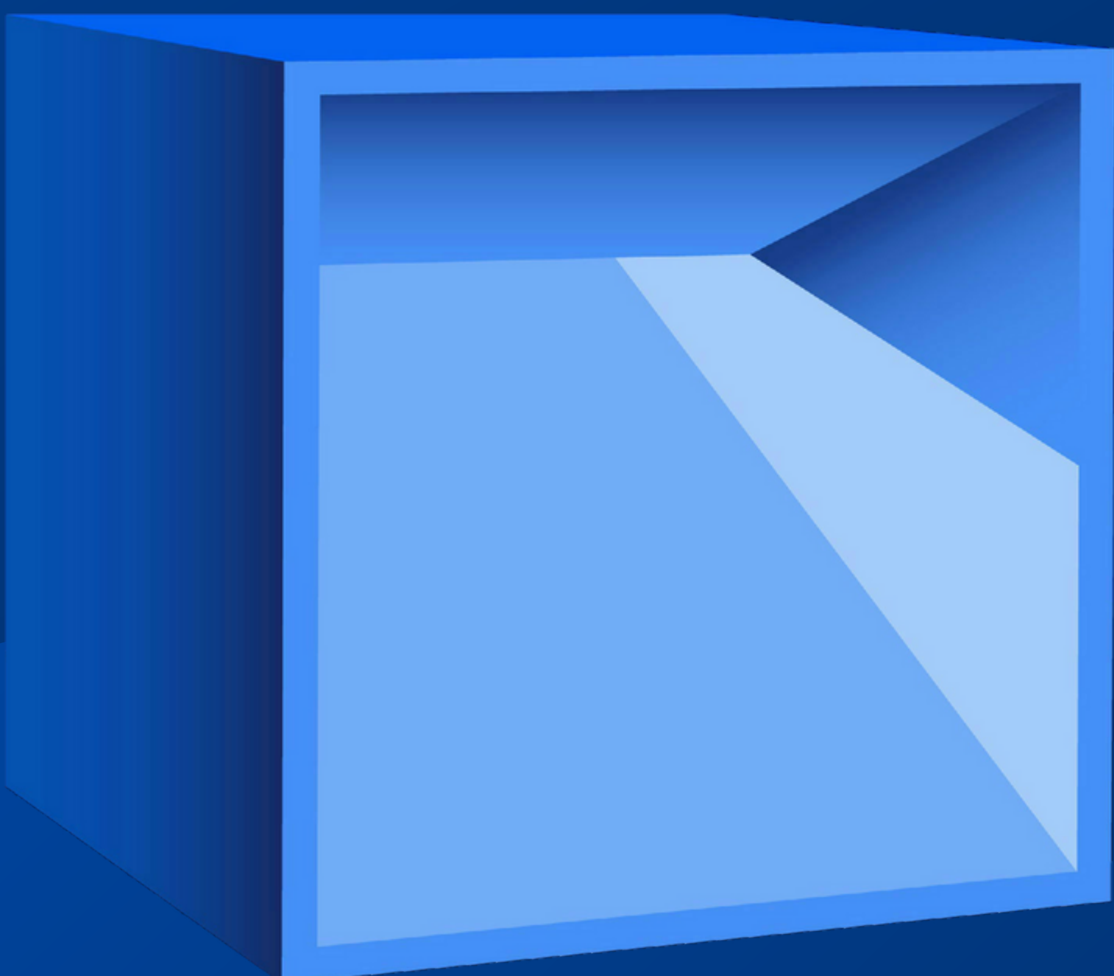
ISSN 0719-0646
ONLINE VERSION

UNIVERSIDAD
DE LA FRONTERA

VOLUME 26 · ISSUE 1
2024

Cubo

A Mathematical Journal



Departamento de Matemática y Estadística
Facultad de Ingeniería y Ciencias
Temuco - Chile

www.cubo.ufro.cl

CUBO
A Mathematical Journal

CUBO, A Mathematical Journal, is a scientific journal founded in 1985, and published by the Department of Mathematics and Statistics of the Universidad de La Frontera, Temuco, Chile. CUBO appears in three issues per year and is indexed in the Web of Science, Scopus, MathSciNet, zbMATH Open, DOAJ, SciELO-Chile, Dialnet, REDIB, Latindex and MIAR. The journal publishes original results of research papers, preferably not more than 20 pages, which contain substantial results in all areas of pure and applied mathematics.

EDITORIAL BOARD

CUBO
A MATHEMATICAL JOURNAL
Universidad de La Frontera
Volume 26/Nº1 – APRIL 2024

SUMMARY

CUBO, A Mathematical Journal
Vol. 26, no. 01, pp. 1–19, April 2024
DOI: 10.56754/0719-0646.2601.001

Frame's Types of Inequalities and Stratification

Branko Malešević[1,✉]
Dimitrije Jovanović[1]

[1] *School of Electrical Engineering, University of Belgrade, Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia.*
maleseviс@etf.rs✉
dickica@hotmail.com

ABSTRACT

In this paper we examine some inequalities of Frame's type on the interval $(0, \pi/2)$. By observing this domain we simply obtain the results using the appropriate families of stratified functions and MTP - Mixed Trigonometric Polynomials. Additionally, from those families we specify a minimax approximant as a function with some optimal properties.

RESUMEN

En este artículo examinamos algunas desigualdades de tipo Frame en el intervalo $(0, \pi/2)$. Observando este dominio simplemente obtenemos los resultados usando las familias apropiadas de funciones estratificadas y PTM - Polinomios Trigonométricos Mezclados. Adicionalmente, a partir de esas familias, especificamos un aproximante minimax como una función con algunas propiedades optimales.

Keywords and Phrases: Frame's type inequalities, stratified families of functions, mixed trigonometric polynomial functions.

2020 AMS Mathematics Subject Classification: 33B10, 26D05.

Published: 19 March, 2024
Accepted: 04 January, 2024
Received: 22 August, 2023

1 Introduction

This paper deals with some inequalities that are discussed in [10, 19], see also the monograph [11, part 3.4.20]. In [1, 13] is stated the Cusa-Huygens approximation:

$$x \approx \frac{3\sin x}{2+\cos x}, \quad \text{for} \quad x \in (0,\pi),$$

which in the paper [9] is specified using families of stratified functions on the domain $(0, \pi/2)$. L. Zhu in [19] gives the following two inequalities:

$$x - \frac{3\sin x}{2+\cos x} > \frac{1}{180}x^5, \quad \text{for} \quad x \in (0,\pi) \tag{1.1}$$

and

$$x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) > \frac{1}{2100}x^7, \quad \text{for} \quad x \in (0,\pi), \tag{1.2}$$

and names them Frame's inequalities. In the monograph *Analytic Inequalities* by D.S. Mitrinović [11, part 3.4.20.] inequalities (1.1) and (1.2) appear with the wrong relation, which L. Zhu corrects in [19].

Based on inequality (1.1) the following assertion is proved in the paper [10].

Theorem 1.1. *The following inequalities are true:*

$$\frac{1}{180}\,x^5 \le x - \frac{3\sin x}{2+\cos x} \le \frac{1}{m_1}\,x^5\,, \tag{1.3}$$

where $x \in [0,\pi]$ *and* $m_1 = 92.96406\ldots = 1/f(x_0)$. *The value* $f(x_0)$ *is determined for the function*

$$f(x) = \left(x - \frac{3\sin x}{2+\cos x}\right) \Big/ x^5 : (0,\pi) \longrightarrow \mathbb{R}$$

at the point $x_0 = 2.83982\ldots$ *at which the function reaches its maximum* $f(x_0)$ *on the interval* $(0,\pi)$. *The equality in* (1.3) *holds for both sides when* $x = 0$ *and holds for the right hand side when* $x = x_0$.

Inequality (1.3) is used to estimate the Cusa-Huygens function $\varphi(x) = x - \dfrac{3\sin x}{2+\cos x}$ over $(0,\pi)$ [10].

The motivation for this paper is to improve the previous results, by finding the minimax approximant for unconsidered values of parameters. We will observe the shorter interval $(0, \pi/2)$, for a more precise estimate in the origin's neighbourhood. The used approach combines the concept of stratification [9] with a method for proving MTP inequalities [8]. This way we can simply prove the known results, and also establish novel ones. Analogously, this procedure can be applied to consider other types of MTP inequalities. In addition, it is possible to apply this approach in solving concrete practical problems such as in [5] and [12].

This paper is organized as follows. The required theoretical background is presented in section 2. In subsection 2.1 are given definitions of stratification and the minimax approximant, as well as Nike theorem in two forms. In subsection 2.2 is explained the used method for proving MTP inequalities. In section 3 are analyzed two inequalities of Frame's type using stratification and MTP method. In subsection 3.1 are given improved results regarding the inequality (1.1). In subsection 3.2 are given improved results regarding the inequality (1.2), obtained analogously to subsection 3.1. Section 4 concludes the paper.

2 Preliminaries

2.1 Stratification and Nike theorem

In this subsection we state relevant concepts and assertions from the paper [9].

The functions $\varphi_p(x)$, where $x \in (a,b) \subseteq \mathbb{R}$ and $p \in \mathbb{D} \subseteq \mathbb{R}^+$, are *increasingly stratified* if $p_1 > p_2 \iff \varphi_{p_1}(x) > \varphi_{p_2}(x)$ holds for each $x \in (a,b)$, and conversely, *decreasingly stratified* if $p_1 > p_2 \iff \varphi_{p_1}(x) < \varphi_{p_2}(x)$ holds for each $x \in (a,b)$ $(p_1, p_2 \in \mathbb{D})$.

Our aim is to determine the maximal subset $I \subseteq \mathbb{D}$ such that, for $p \in I$, we have $\varphi_p(x) > 0$ for each $x \in (a,b)$. Likewise, we want to determine the maximal subset $J \subseteq \mathbb{D}$ such that, for $p \in J$, we have $\varphi_p(x) < 0$ for each $x \in (a,b)$. We will assume that $\mathbb{D} = \mathbb{R}^+$, $I \cup J \subsetneqq \mathbb{D}$, $I \neq \emptyset$ and $J \neq \emptyset$. In that case, it is important to examine the sign of the function $\varphi_p(x)$ in terms of the parameter $p \in \mathbb{D} \setminus (I \cup J)$, for $x \in (a,b)$.

The value $\sup\limits_{x \in (a,b)} |\varphi_p(x)|$ is called *the approximation error* on the interval (a,b) and denoted by

$$d^{(p)} = \sup_{x \in (a,b)} |\varphi_p(x)|, \tag{2.1}$$

for $p \in \mathbb{D}$. Our aim is to determine the unique value of the parameter $p = p_0 \in \mathbb{D}$ for which the infimum of the error $d^{(p)}$ is attained:

$$d_0 = \inf_{p \in \mathbb{R}^+} \sup_{x \in (a,b)} |\varphi_p(x)|. \tag{2.2}$$

For such a value p_0 of the parameter p, the function $\varphi_{p_0}(x)$ is called *the minimax approximant* on (a,b).

If the family $\varphi_p(x)$ allows us to consider $x \in [a,b]$ and $p \in \mathbb{D} = [c,d] \subset \mathbb{R}^+$, then we have

$$d_0 = \min_{p\in[c,d]} \max_{x\in[a,b]} |\varphi_p(x)|\,.$$

The following assertions are proved in [9].

Theorem 2.1 ([9])**.** *Let $\varphi_p(x)$ be a family of functions that are continuous with respect to $x \in (a,b)$ for each $p \in \mathbb{R}^+$ and increasingly (decreasingly) stratified for $p \in \mathbb{R}^+$, and let $c, d \in \mathbb{R}^+$, where $c < d$. If:*

(a) $\varphi_c(x) < 0$ $(\varphi_c(x) > 0)$ and $\varphi_d(x) > 0$ $(\varphi_d(x) < 0)$ for each $x \in (a,b)$, and at the endpoints $\varphi_c(a+) = \varphi_d(a+) = 0$, $\varphi_c(b-) = 0$ $(\varphi_d(b-) = 0)$ and $\varphi_d(b-) \in \mathbb{R}^+$ $(\varphi_c(b-) \in \mathbb{R}^+)$ hold;

(b) the functions $\varphi_p(x)$ are continuous with respect to $p \in (c,d)$ for each $x \in (a,b)$ and $\varphi_p(b-)$ are also continuous with respect to $p \in (c,d)$;

(c) for each $p \in (c,d)$, there is a right neighbourhood of the point a in which $\varphi_p(x) < 0$;

(d) for each $p \in (c,d)$ the function $\varphi_p(x)$ has exactly one extremum at $t^{(p)}$ on (a,b), which is minimum;

then there is exactly one solution p_0, for $p \in \mathbb{R}^+$, to the following equation:

$$|\varphi_p(t^{(p)})| = \varphi_p(b-),$$

and for $d_0 = |\varphi_{p_0}(t^{(p_0)})| = \varphi_{p_0}(b-)$ we have

$$d_0 = \inf_{p\,\in\,\mathbb{R}^+} \sup_{x\,\in\,(a,b)} |\varphi_p(x)|\,.$$

Theorem 2.2 (Nike theorem, [7, 9])**.** *Let $\varphi_p(x) : (a,b) \longrightarrow \mathbb{R}$ be at least m times differentiable function, for some $m \geq 2$, $m \in \mathbb{N}$, which satisfies the following conditions:*

(a) $f^{(m)} > 0$ for $x \in (0,c)$;

(b) there is a right neighbourhood of zero in which the following inequalities hold:

$$f < 0, f' < 0, \ldots, f^{(m-1)} < 0;$$

(c) there is a left neighbourhood of the point c in which the following inequalities hold:

$$f > 0, f' > 0, \ldots, f^{(m-1)} > 0\,.$$

Then the function f has exactly one root $x_0 \in (0,c)$ and $f(x) < 0$ for $x \in (0,x_0)$ and $f(x) > 0$ for $x \in (x_0,c)$. Additionally, the function f has exactly one local minimum on the interval $(0,c)$. More precisely, there is exactly one point $t \in (0,x_0) \subset (0,c)$ such that $f(t) < 0$ is the smallest value of the function f on the interval $(0,x_0) \subset (0,c)$.

Theorem 2.3 (Nike theorem, II form, [9]). *Let $\varphi_p(x) : (a,b) \longrightarrow \mathbb{R}$ be at least m times differentiable function, for some $m \geq 2$, $m \in \mathbb{N}$, which satisfies the following conditions:*

(a) $f^{(m)}$ has exactly one root x_m on $(0,c)$ such that $f^{(m)} > 0$ on $(0,x_m)$ and $f^{(m)} < 0$ on (x_m,c);

(b) there is a right neighbourhood of zero in which the following inequalities hold:

$$f < 0, f' < 0, \dots, f^{(m-1)} < 0;$$

(c) there is a left neighbourhood of the point c in which the following inequalities hold:

$$f > 0, f' > 0, \dots, f^{(m-1)} > 0.$$

Then the function f has exactly one root $x_0 \in (0,c)$ and $f(x) < 0$ for $x \in (0,x_0)$ and $f(x) > 0$ for $x \in (x_0,c)$. Additionally, the function f has exactly one local minimum on the interval $(0,c)$. More precisely, there is exactly one point $t \in (0,x_0) \subset (0,c)$ such that $f(t) < 0$ is the smallest value of the function f on the interval $(0,x_0) \subset (0,c)$.

2.2 A method for proving MTP inequalities

In this subsection we present relevant assertions from the paper [8] for proving inequalities of the form

$$f(x) = \sum_{i=1}^{n} \alpha_i x^{p_i} \cos^{q_i} x \sin^{r_i} x > 0\,, \tag{2.3}$$

where $x \in (\delta_1,\delta_2)$, $\delta_1 \leq 0 \leq \delta_2$ and $\delta_1 < \delta_2$, where $\alpha_i \in \mathbb{R} \setminus \{0\}$, $p_i, q_i, r_i \in \mathbb{N}_0$ and $n \in \mathbb{N}$. The function $f(x)$ we denote as MTP - Mixed Trigonometric Polynomial [4], and the corresponding inequality (2.3) we denote as MTP inequality.

Let the function $f(x)$ be approximated by Taylor polynomial $T_k(x)$ of degree k in the neighbourhood of some point a. If there is $\eta > 0$ such that on the interval $x \in (a-\eta, a+\eta)$, it holds that $T_k(x) \geq f(x)$, then $T_k(x)$ denotes *the upward approximation* of the function $f(x)$ in the neighbourhood of the point a. In this case, the polynomial $T_k(x)$ is denoted by $\overline{T}_k^{f,a}(x)$, or short $\overline{T}_k(x)$. Analogously, if there is $\eta > 0$ such that on the interval $x \in (a-\eta, a+\eta)$, it holds that $T_k(x) \leq f(x)$, then $T_k(x)$ denotes *the downward approximation* of the function $f(x)$ in the neighbourhood of the point a. In this case, the polynomial $T_k(x)$ we also denote by $\underline{T}_k^{f,a}(x)$, or short $\underline{T}_k(x)$.

The following assertions are proved in [8].

Lemma 2.4. *(a) For the polynomial*

$$T_n(t) = \sum_{i=0}^{(n-1)/2} \frac{(-1)^i t^{2i+1}}{(2i+1)!}\,,$$

where $n = 4k+1$, $k \in \mathbb{N}_0$, *it holds:*

$$\overline{T}_n(t) \geq \overline{T}_{n+4}(t) \geq \sin t, \quad \forall t \in \left[0, \sqrt{(n+3)(n+4)}\right]$$

$$\underline{T}_n(t) \leq \underline{T}_{n+4}(t) \leq \sin t, \quad \forall t \in \left[-\sqrt{(n+3)(n+4)}, 0\right].$$

For $t = 0$ *the inequalities turn into equalities. For* $t = \pm\sqrt{(n+3)(n+4)}$ *the equalities* $\overline{T}_n(t) = \overline{T}_{n+4}(t)$ *and* $\underline{T}_n(t) = \underline{T}_{n+4}(t)$ *hold, respectively.*

(b) For the polynomial

$$T_n(t) = \sum_{i=0}^{(n-1)/2} \frac{(-1)^i t^{2i+1}}{(2i+1)!}\,,$$

where $n = 4k+3$, $k \in \mathbb{N}_0$, *it holds:*

$$\underline{T}_n(t) \leq \underline{T}_{n+4}(t) \leq \sin t, \quad \forall t \in \left[0, \sqrt{(n+3)(n+4)}\right],$$

$$\overline{T}_n(t) \geq \overline{T}_{n+4}(t) \geq \sin t, \quad \forall t \in \left[-\sqrt{(n+3)(n+4)}, 0\right].$$

For $t = 0$ *the inequalities turn into equalities. For* $t = \pm\sqrt{(n+3)(n+4)}$ *the equalities* $\underline{T}_n(t) = \underline{T}_{n+4}(t)$ *and* $\overline{T}_n(t) = \overline{T}_{n+4}(t)$ *hold, respectively.*

(c) For the polynomial

$$T_n(t) = \sum_{i=0}^{n/2} \frac{(-1)^i t^{2i}}{(2i)!}\,,$$

where $n = 4k$, $k \in \mathbb{N}_0$, *it holds:*

$$\overline{T}_n(t) \geq \overline{T}_{n+4}(t) \geq \cos t, \quad \forall t \in \left[-\sqrt{(n+3)(n+4)}, \sqrt{(n+3)(n+4)}\right].$$

For $t = 0$ *the inequalities turn into equalities. For* $t = \pm\sqrt{(n+3)(n+4)}$ *the equality* $\overline{T}_n(t) = \overline{T}_{n+4}(t)$ *holds.*

(d) For the polynomial

$$T_n(t) = \sum_{i=0}^{n/2} \frac{(-1)^i t^{2i}}{(2i)!},$$

where $n = 4k + 2$, $k \in \mathbb{N}_0$, *it holds:*

$$\underline{T}_n(t) \leq \underline{T}_{n+4}(t) \leq \cos t, \quad \forall t \in \left[-\sqrt{(n+3)(n+4)}, \sqrt{(n+3)(n+4)}\right].$$

For $t = 0$ *the inequalities turn into equalities. For* $t = \pm\sqrt{(n+3)(n+4)}$ *the equality* $\underline{T}_n(t) = \underline{T}_{n+4}(t)$ *holds.*

The main idea of the method described in [8] is to, for a given MTP function $f(x)$ defined on $(0, \pi/2)$, find a polynomial $P(x)$ using Lemma 1, such that $f(x) > P(x)$ and $P(x) > 0$ when $x \in (0, \pi/2)$. If such polynomial exists, then $f(x) > 0$ for $x \in (0, \pi/2)$.

For example, all results from the paper [20] can be proved by reduction to the appropriate MTP inequalities with the application of this method.

3 Main results

3.1 Improved results for inequality (1.1)

In this subsection we prove the results regarding the family of functions

$$\varphi_p(x) = x - \frac{3\sin x}{2+\cos x} - px^5, \quad \big(x \in (0, \pi/2) \quad \text{and} \quad p \in \mathbb{R}^+\big),$$

with the aim of improving the results for Frame's inequality (1.1) on the interval $(0, \pi/2)$. The following assertions are true.

Lemma 3.1. *The family of functions*

$$\varphi_p(x) = x - \frac{3\sin x}{2+\cos x} - p\,x^5, \quad \textit{for} \quad x \in (0, \pi/2)$$

is decreasingly stratified with respect to parameter $p \in \mathbb{R}^+$.

Proof. It holds that $\dfrac{\partial \varphi_p(x)}{\partial p} = -x^5 < 0$, for each $x \in (0, \pi/2)$. □

Proposition 3.2. *Let*

$$A = \frac{1}{180} = 0.00\overline{5} \quad \textit{and} \quad B = \frac{16\,(\pi-3)}{\pi^5} = 0.00740306\ldots$$

Then for $x \in (0, \pi/2)$, *it holds:*

$$\varphi_A(x) > 0 \quad \textit{and} \quad \varphi_B(x) < 0.$$

Proof. Let us write $\varphi_A(x)$ in the form:

$$\varphi_A(x) = x - \frac{3\sin x}{2+\cos x} - \frac{x^5}{180} = \frac{f_A(x)}{180(2+\cos x)},$$

where

$$f_A(x) = -540\sin x + (-x^5 + 180x)\cos x + 2(-x^5 + 180x)$$

is a MTP function defined on $[0, \pi/2]$.

Since $180(2+\cos x) > 0$ for each $x \in (0, \pi/2)$, it is sufficient to prove that $f_A(x) > 0$ for $x \in (0, \pi/2)$. We will use a method given in subsection 2.2.

The following inequalities are true based on Lemma 2.4:

$$\sin t < \overline{T}_5^{\sin,0}(t) \quad \text{for} \quad t \in (0, \sqrt{72}) = (0, 8.485\ldots)$$

and

$$\cos t > \underline{T}_6^{\cos,0}(t) \quad \text{for} \quad t \in (0, \sqrt{90}) = (0, 9.4868\ldots).$$

For each $x \in (0, \pi/2)$ it holds:

$$f_A(x) > P_{11}(x) = \underbrace{-540}_{<0}\,\overline{T}_5^{\sin,0}(x) + \underbrace{(-x^5 + 180x)}_{>0}\,\underline{T}_6^{\cos,0}(x) + \underbrace{2(-x^5 + 180x)}_{>0}.$$

The polynomial $P_{11}(x)$ can be written in the following way:

$$P_{11}(x) = \frac{1}{720}\,x^{11} - \frac{1}{24}\,x^9 + \frac{1}{4}\,x^7 = \frac{x^7}{720}\left(x^4 - 30x^2 + 180\right) = \frac{x^7}{720}P_4(x).$$

The first positive root of the biquadratic equation $P_4(x) = 0$ is $x_1 = \sqrt{15 - 3\sqrt{5}} = 2.879\ldots > \pi/2$. Since $P_4(x_1/2) = P_4(1.439) = 122.108 > 0$, it follows that $P_4(x) > 0$ for $x \in (0, \pi/2)$. Furthermore, $f_A(x) > P_{11}(x) > 0$ for $x \in (0, \pi/2)$. Therefore, $\varphi_A(x) > 0$ for each $x \in (0, \pi/2)$.

We prove $\varphi_B(x) < 0$ in a similar way. Let us write $\varphi_B(x)$ in the form:

$$\varphi_B(x) = x - \frac{3\sin x}{2+\cos x} - \frac{16(\pi-3)x^5}{\pi^5} = \frac{f_B(x)}{\pi^5(2+\cos x)}.$$

Since $\pi^5(2+\cos x) > 0$ for each $x \in (0, \pi/2)$, the requested inequality is equivalent to $f_B(x) < 0$ for $x \in (0, \pi/2)$, where

$$f_B(x) = -3\pi^5\sin x + (16(3-\pi)x^5 + \pi^5 x)\cos x + 2(16(3-\pi)x^5 + \pi^5 x)$$

is a MTP function defined on $[0, \pi/2]$. Let us notice that $f_B(0) = f_B(\pi/2) = 0$. For that reason, we consider two cases:

(1) $x \in (0, 1.199)$: We have $16(3-\pi)x^5 + \pi^5 x = x\left(16(3-\pi)x^4 + \pi^5\right) > 0$ on $(0, 1.199)$. The following inequalities are true based on Lemma 2.4:

$$\sin t > \underline{T}_7^{\sin,0}(t) \quad \text{for} \quad t \in (0, \sqrt{110}) = (0, 10.488\ldots)$$

and

$$\cos t < \overline{T}_4^{\cos,0}(t) \quad \text{for} \quad t \in (0, \sqrt{56}) = (0, 7.483\ldots).$$

For each $x \in (0, 1.199)$ it holds:

$$f_B(x) < Q_9(x) = \underbrace{-3\pi^5}_{<\,0}\underline{T}_7^{\sin,0}(x) + \underbrace{(16(3-\pi)x^5 + \pi^5 x)}_{>\,0}\overline{T}_4^{\cos,0}(x) + \underbrace{2(16(3-\pi)x^5 + \pi^5 x)}_{>\,0}\,.$$

The polynomial $Q_9(x)$ can be written in the following way:

$$\begin{aligned} Q_9(x) &= \frac{x^5}{1680}\Big(-1120(\pi-3)x^4 + (\pi^5 + 13440\pi - 40320)x^2 + 28\pi^5 - 80640\pi + 241920\Big) \\ &= \frac{x^5}{1680}Q_4(x)\,. \end{aligned}$$

The first positive root of the biquadratic equation $Q_4(x) = 0$ is $x_1 = 1.1993\ldots > 1.199$. Since $Q_4(x_1/2) = Q_4(0.599\ldots) = -2075.583\ldots < 0$, it follows that $Q_4(x) < 0$ on $(0, 1.199)$. Furthermore, $f_B(x) < Q_9(x) < 0$. Therefore, $\varphi_B(x) < 0$ for $x \in (0, 1.199)$.

(2) $x \in [1.199, \pi/2)$: Let us define a function

$$g_B(x) = f_B\left(\frac{\pi}{2} - x\right) = -3\pi^5\,\cos x + r(x)\,\sin x + 2\,r(x)\,,$$

where $r(x)$ is the polynomial

$$r(x) = \left(\frac{\pi}{2} - x\right)\left(16(3-\pi)\left(\frac{\pi}{2} - x\right)^4 + \pi^5\right),$$

for $x \in [1.199, \pi/2)$. It is easy to show that $r(x) > 0$ for each $x \in [1.199, \pi/2)$.

Here we prove the inequality $f_B(x) < 0$ for $x \in [1.199, \pi/2)$, which is equivalent to the MTP inequality $g_B(x) < 0$ for $x \in (0, c]$, where $c = \pi/2 - 1.199 = 0.371796\ldots$

The following inequalities are true based on Lemma 2.4:

$$\sin t < \overline{T}_5^{\sin,0}(t) \quad \text{for} \quad t \in (0, \sqrt{72}) = (0, 8.485\ldots)$$

and

$$\cos t > \underline{T}_2^{\cos,0}(t) \quad \text{for} \quad t \in (0, \sqrt{30}) = (0, 5.477\ldots).$$

For each $x \in (0, c]$, it holds:

$$g_B(x) < \underbrace{-3\pi^5}_{<\,0}\underline{T}_2^{\cos,0}(x) + \underbrace{r(x)}_{>\,0}\overline{T}_5^{\sin,0}(x) + \underbrace{2r(x)}_{>\,0} = x\,R(x)\,,$$

where $R(x)$ is the polynomial

$$\begin{aligned} R(x) = {} & \left(\frac{2\pi}{15} - \frac{2}{5}\right)x^9 + \left(-\frac{\pi^2}{3} + \pi\right)x^8 + \left(\frac{\pi^3}{3} - \pi^2 - \frac{8\pi}{3} + 8\right)x^7 \\ & + \left(-\frac{\pi^4}{6} + \frac{\pi^3}{2} + \frac{20\pi^2}{3} - 20\pi\right)x^6 + \left(\frac{\pi^5}{30} - \frac{\pi^4}{8} - \frac{20\pi^3}{3} + 20\pi^2 + 16\pi - 48\right)x^5 \\ & + \left(\frac{\pi^5}{80} + \frac{10\pi^4}{3} - 10\pi^3 - 40\pi^2 + 152\pi - 96\right)x^4 \\ & + \left(-\frac{2\pi^5}{3} + \frac{5\pi^4}{2} + 40\pi^3 - 200\pi^2 + 240\pi\right)x^3 \\ & + \left(-\frac{\pi^5}{4} - 20\pi^4 + 140\pi^3 - 240\pi^2\right)x^2 + \left(\frac{11\pi^5}{2} - 55\pi^4 + 120\pi^3\right)x + \left(\frac{19\pi^5}{2} - 30\pi^4\right). \end{aligned}$$

It is sufficient to prove that $R(x) < 0$ for $x \in (0, c]$. Let us denote the coefficients of the polynomial $R(x)$ respectively by $a_9, \ldots, a_0$:

$$\begin{aligned} R(x) &= a_9x^9 + a_8x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \\ &= (a_9x + a_8)x^8 + (a_7x^2 + a_6x + a_5)x^5 + (a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0) \\ &= (a_9x + a_8)x^8 + (a_7x^2 + a_6x + a_5)x^5 + S(x). \end{aligned}$$

It holds:

$$a_9x + a_8 = \left(\frac{2\pi}{15} - \frac{2}{5}\right)x + \left(-\frac{\pi^2}{3} + \pi\right) < 0$$

and

$$\begin{aligned} a_7x^2 + a_6x + a_5 = -\left(\pi^2 + \frac{8\pi}{3}\right)x^2 - \left(\frac{\pi^4}{6} - \frac{\pi^3}{2} - \frac{20\pi^2}{3} + 20\pi\right)x \\ + \left(\frac{11\pi^5}{2} - 55\pi^4 + 120\pi^3\right) < 0, \end{aligned}$$

for each $x \in (0, c]$. Let us prove that

$$\begin{aligned} S(x) &= a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \\ &= \Big(\frac{\pi^5}{80} + \frac{10\pi^4}{3} - 10\pi^3 - 40\pi^2 + 152\pi - 96\Big)x^4 \end{aligned}$$

$$+\Big(-\frac{2\pi^5}{3}+\frac{5\pi^4}{2}+40\pi^3-200\pi^2+240\pi\Big)x^3+\Big(-\frac{\pi^5}{4}-20\pi^4+140\pi^3-240\pi^2\Big)x^2$$
$$+\Big(\frac{11\pi^5}{2}-55\pi^4+120\pi^3\Big)x+\Big(\frac{19\pi^5}{2}-30\pi^4\Big)<0$$

for each $x\in(0,c]$. The third derivative of the polynomial $S(x)$ is

$$S'''(x)=\left(\frac{3}{10}\pi^5+80\pi^4-240\pi^3-960\pi^2+3648\pi-2304\right)x-4\pi^5+15\pi^4$$
$$+240\pi^3-1200\pi^2+1440\pi.$$

It holds that $S'''(x)>0$ for $x\in(0,c]$. Thus, $S''(x)$ is a monotonically increasing function for $x\in(0,c]$. Furthermore, $S''(x)$ is a quadratic function with roots $x_1=-6.034\ldots$ and $x_2=0.279\ldots$ This implies that $S'(x)$ has exactly one extremum on $(0,c]$ which is minimum at the point x_2. Since we have $S'(x_2)=31.480\ldots>0$ at the point of minimum, it follows that $S'(x)>0$ for each $x\in(0,c]$. Thus, the function $S(x)$ is monotonically increasing for each $x\in(0,c]$. Since $S(c)=-1.933\ldots<0$, it follows that $S(x)<0$ for each $x\in(0,c]$.

Therefore:

$$\begin{aligned} R(x)<0\,,\quad \text{for}\quad x\in(0,c] &\Longrightarrow g_B(x)<0\,,\quad \text{for}\quad x\in(0,c]\\ &\Longrightarrow f_B(x)<0,\quad \text{for}\quad x\in[1.199,\pi/2)\\ &\Longrightarrow \varphi_B(x)<0\,,\quad \text{for}\quad x\in[1.199,\pi/2). \end{aligned}$$

This completes the proof that $\varphi_B(x)<0$ for each $x\in(0,\pi/2)$. □

Proposition 3.3. *Let*

$$A=\frac{1}{180}=0.00\overline{5}\quad \textit{and}\quad B=\frac{16\,(\pi-3)}{\pi^5}=0.00740306\ldots$$

(i) If $p\in(0,A]$, then

$$x\in(0,\pi/2)\implies x-\frac{3\sin x}{2+\cos x}>A\,x^5\geq p\,x^5\,.$$

(ii) If $p\in(A,B)$, then $\varphi_p(x)=x-\dfrac{3\sin x}{2+\cos x}-p\,x^5$ has a unique root $x_0^{(p)}$ on $(0,\pi/2)$. Also,

$$x\in(0,x_0^{(p)})\implies x-\frac{3\sin x}{2+\cos x}<p\,x^5$$

and

$$x\in(x_0^{(p)},\pi/2)\implies x-\frac{3\sin x}{2+\cos x}>p\,x^5\,.$$

Every function $\varphi_p(x)$ has exactly one minimum $t_0^{(p)}\in(0,x_0^{(p)})$, for $p\in(A,B)$.

(iii) If $p \in [B, \infty)$*, then*

$$x \in (0, \pi/2) \implies x - \frac{3 \sin x}{2 + \cos x} < B\, x^5 \leq p\, x^5\,.$$

(iv) There is exactly one solution to the equation

$$|\varphi_p(t_0^{(p)})| = \varphi_p\,(\pi/2-)$$

with respect to parameter $p \in (A, B)$*, determined numerically as*

$$p_0 = 0.0072274\ldots$$

For the value

$$d_0 = \varphi_{p_0}\,(\pi/2-) = 0.0016797\ldots$$

it holds:

$$d_0 = \min_{p \in [0,\infty)} \max_{x \in [0,\pi/2]} |\varphi_p(x)|.$$

(v) For the value $p_0 = 0.0072274\ldots$ *the minimax approximant of the family* $\varphi_p(x)$ *is*

$$\varphi_{p_0}(x) = x - \frac{3 \sin x}{2 + \cos x} - p_0\, x^5,$$

which determines the appropriate minimax approximation

$$x - \frac{3 \sin x}{2 + \cos x} \approx 0.0072274\, x^5.$$

Proof. It has been shown in Proposition 3.2 that the inequalities $\varphi_A(x) > 0$ and $\varphi_B(x) < 0$ hold for each $x \in (0, \pi/2)$. Since the family of functions $\varphi_p(x)$ is decreasingly stratified, it follows that $\varphi_p(x) \geq \varphi_A(x) > 0$ for $p \in (0, A)$ and $\varphi_p(x) \leq \varphi_B(x) < 0$ for $p \in (B, \infty)$, for each $x \in (0, \pi/2)$. That proves the assertions (i) and (iii).

In order to prove the assertion (ii), we will use the Theorem 2.3 (Nike theorem, II form). Namely, for $p \in (A, B)$, the functions $\varphi_p(x)$ satisfy the conditions of Theorem 2.3:

(a) For $m = 6$, we have

$$\varphi_p^{(vi)}(x) = \frac{d^6 \varphi_p}{d\, x^6} = \frac{6 \sin x}{(2 + \cos x)^7} h(x)\,, \tag{3.1}$$

where $h(x)$ is the following MTP function:

$$h(x) = -(\cos^5 x - 98 \cos^4 x + 886 \cos^3 x - 892 \cos^2 x - 1216 \cos x + 104)\,.$$

Since $\dfrac{6\sin x}{(2+\cos x)^7} > 0$ for each $x \in (0, \pi/2)$, functions $\varphi_p^{(vi)}(x)$ and $h(x)$ have the same roots and sign on $(0, \pi/2)$.

By introducing the substitute $t = \cos x$, we get

$$H(t) = h(\arccos t) = -(t^5 - 98t^4 + 886t^3 - 892t^2 - 1216t + 104)\,.$$

It can be shown by numerical methods that $H(t)$ has a root $t_1 = 0.081088\ldots$ Since $H(t)$ is a polynomial with rational coefficients on the interval with rational endpoints $(0,1)$, using Sturm's algorithm [3, 14], we can conclude that $H(t)$ has exactly one root $t_1 = 0.081088\ldots$ on the interval $(0,1)$. Thus, $h(x)$ also has exactly one root $x_1 = \arccos t_1 = 1.489619\ldots$ on the interval $(0, \pi/2)$.

Let us notice again that $h(x)$ has only one root $x_1 = 1.489619\ldots$ on $(0, \pi/2)$. Since $h(1) = 681.964\ldots > 0$ and $h(1.5) = -13.831\ldots < 0$, it follows that

$$h(x) > 0 \quad \text{on} \quad (0, x_1) \quad \text{and} \quad h(x) < 0 \quad \text{on} \quad (x_1, \pi/2).$$

Considering (3.1), the previous conclusion is equivalent to

$$\varphi_p^{(vi)}(x) > 0 \quad \text{on} \quad (0, x_1) \quad \text{and} \quad \varphi_p^{(vi)}(x) < 0 \quad \text{on} \quad (x_1, \pi/2),$$

which satisfies the first condition of Theorem 2.3.

(b) Taylor approximations of functions $\varphi_p(x)$ around $x = 0$ are:

$$\varphi_p(x) = \left(\frac{1}{180} - p\right)x^5 + \frac{1}{1512}x^7 + O(x^9)\,.$$

Since we consider $p \in (A, B) = \left(\frac{1}{180}, \frac{16(\pi-3)}{\pi^5}\right)$, the coefficient next to x^5 in the approximation is negative, so we conclude that there is a right neighbourhood $\mathcal{U}_0$ of the point 0 such that

$$\varphi_p(x), \varphi_p'(x), \varphi_p''(x), \varphi_p'''(x), \varphi_p^{(iv)}(x), \varphi_p^{(v)}(x) < 0, \quad x \in \mathcal{U}_0.$$

(c) Taylor approximations of functions $\varphi_p(x)$ around $x = \dfrac{\pi}{2}$ are:

$$\begin{aligned}
\varphi_p(x) \;=\; & \left(-\frac{\pi^5 p}{32} + \frac{\pi-3}{2}\right) + \left(-\frac{5\pi^4 p}{16} + \frac{1}{4}\right)\left(x - \frac{\pi}{2}\right) + \left(-\frac{5\pi^3 p}{4} + \frac{3}{8}\right)\left(x - \frac{\pi}{2}\right)^2 \\
& + \left(-\frac{5\pi^2 p}{2} + \frac{5}{16}\right)\left(x - \frac{\pi}{2}\right)^3 + \left(-\frac{5\pi p}{2} + \frac{5}{32}\right)\left(x - \frac{\pi}{2}\right)^4 \\
& + \left(-p + \frac{13}{320}\right)\left(x - \frac{\pi}{2}\right)^5 - \frac{13}{1920}\left(x - \frac{\pi}{2}\right)^6 + O\left(\left(x - \frac{\pi}{2}\right)^7\right).
\end{aligned}$$

Since we consider $p \in (A, B)$, it is easy to show that in the approximation all coefficients next to $\left(x - \frac{\pi}{2}\right)^n$, $0 \leq n \leq 5$, are positive, so we conclude that there is a left neighbourhood $\mathcal{U}_{\pi/2}$ of the point $\frac{\pi}{2}$ such that

$$\varphi_p(x), \varphi_p'(x), \varphi_p''(x), \varphi_p'''(x), \varphi_p^{(iv)}(x), \varphi_p^{(v)}(x) > 0, \quad x \in \mathcal{U}_{\pi/2}.$$

Since the conditions of Theorem 2.3 are satisfied, the function $\varphi_p(x)$ has exactly one extremum $t^{(p)}$, which is minimum, on $(0, \pi/2)$ $\big($and one root $x_0^{(p)}$ on $(0, \pi/2)\big)$, and it holds that $\varphi_p(x) < 0$ for $x \in (0, x_0^{(p)})$ and $\varphi_p(x) > 0$ for $x \in (x_0^{(p)}, \pi/2)$. That proves the assertion (ii).

(iv), (v): The family of functions $\varphi_p(x)$, for values $p \in (A, B)$, satisfies the conditions of Theorem 2.1, which means that the minimax approximant exists. The minimax approximant and its error (infimum of the approximation error) can be numerically determined using Maple software. Let $f(x, p) := \varphi_p(x)$. Based on Maple code

```
fsolve({diff(f(x,p),x)=0,abs(f(x,p)=f(Pi/2,p)},{x=0..Pi/2,p=A..B});
```

we get numerical values

$$\{p = 0.007227413, x = 1.272430755\}.$$

For the value $p_0 = 0.0072274\ldots$ we obtain the minimax approximant of the family

$$\varphi_{p_0}(x) = x - \frac{3 \sin x}{2 + \cos x} - p_0\, x^5$$

and numerical value of the minimax error

$$d_0 = f(\pi/2, p_0) = 0.0016797\ldots.$$

This completes the proof. □

The following statement holds based on previous conclusions.

Proposition 3.4. *For each $0 < x < \pi/2$, it holds:*

$$\frac{1}{180}\, x^5 < x - \frac{3 \sin x}{2 + \cos x} < \frac{16\,(\pi - 3)}{\pi^5}\, x^5\,, \tag{3.2}$$

where the constants $A = \dfrac{1}{180} = 0.00\overline{5}$ and $B = \dfrac{16\,(\pi - 3)}{\pi^5} = 0.00740306\ldots$ are the best possible.

3.2 Improved results for inequality (1.2)

In this subsection we present the appropriate results for the family of functions

$$\varphi_p(x) = x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) - p\,x^7, \quad x \in (0,\pi/2) \quad \text{and} \quad p \in \mathbb{R}^+,$$

with the aim of improving the results for the Frame's inequality (1.2) on the interval $(0,\pi/2)$. The following statements are proved analogously to statements from the previous subsection.

Lemma 3.5. *The family of functions:*

$$\varphi_p(x) = x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) - p\,x^7, \quad \textit{for} \quad x \in (0,\pi/2)$$

is decreasingly stratified with respect to parameter $p \in \mathbb{R}^+$.

Proposition 3.6. *Let:*

$$A = \frac{1}{2100} = 0.000\overline{476190} \quad \textit{and} \quad B = \frac{64(9\pi-28)}{9\pi^7} = 0.0006459\ldots$$

Then for $x \in (0,\pi/2)$, it holds:

$$\varphi_A(x) > 0 \quad \textit{and} \quad \varphi_B(x) < 0\,.$$

Proposition 3.7. *Let:*

$$A = \frac{1}{2100} = 0.000\overline{476190} \quad \textit{and} \quad B = \frac{64(9\pi-28)}{9\pi^7} = 0.0006459\ldots$$

(i) If $p \in (0,A]$, then

$$x \in (0,\pi/2) \implies x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) > A\,x^7 \geq p\,x^7\,.$$

(ii) If $p \in (A,B)$, then $\varphi_p(x) = x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) - p\,x^7$ has a unique root $x_0^{(p)}$ on $(0,\pi/2)$. Also:

$$x \in (0, x_0^{(p)}) \implies x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) < p\,x^7$$

and

$$x \in (x_0^{(p)}, \pi/2) \implies x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) > p\,x^7\,.$$

Every function $\varphi_p(x)$ has exactly one minimum $t_0^{(p)} \in (0, x_0^{(p)})$, for $p \in (A,B)$.

(iii) *If* $p \in [B, \infty)$*, then:*

$$x \in (0, \pi/2) \implies x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) < B\,x^7 \leq p\,x^7\,.$$

(iv) *There is exactly one solution to the equation:*

$$|\varphi_p(t_0^{(p)})| = \varphi_p\,(\pi/2-)$$

with respect to parameter $p \in (A, B)$*, determined numerically as:*

$$p_0 = 0.000632762\ldots$$

For the value:

$$d_0 = \varphi_{p_0}\,(\pi/2-) = 0.000310091\ldots$$

it holds:

$$d_0 = \min_{p\in[0,\infty)}\ \max_{x\in[0,\pi/2]} |\varphi_p(x)|.$$

(v) *For the value* $p_0 = 0.000632762\ldots$ *the minimax approximant of the family* $\varphi_p(x)$ *is:*

$$\varphi_{p_0}(x) = x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) - p_0\,x^7,$$

which determines the appropriate minimax approximation:

$$x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) \approx 0.000632762x^7.$$

The following statement holds based on previous conclusions.

Proposition 3.8. *For each* $0 < x < \pi/2$*, it holds:*

$$\frac{1}{2100}\,x^7 < x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right) < \frac{64(9\pi-28)}{9\pi^7}\,x^7, \tag{3.3}$$

where the constants $A = \dfrac{1}{2100} = 0.000\overline{476190}$ *and* $B = \dfrac{64(9\pi-28)}{9\pi^7} = 0.0006459\ldots$ *are the best possible.*

4 Conclusion

Inequalities that we study in this paper are mainly used to estimate the precision of the Cusa-Huygens approximation. The Cusa-Huygens inequality and the estimate of the quality of approximation may be relevant to concrete applications such as [5, 12], see also the monograph [2]. The known results related to Frame's inequalities are obtained for special cases of parameters only. In this paper, we achieve the previous results based on the concept of stratification, and also expand the conclusions for unconsidered values of parameters. In analogy with this approach over families of stratified functions, it is possible to examine other types of inequalities and get new results in the Theory of Analytic Inequalities.

It should be noted that one part of the given method is limited to MTP inequalities (subsection 2.2). The aim of future research is to consider other classes of inequalities in a similar way, by combining different methods with the concept of stratification. In that regard, we refer the reader to papers [6, 15–18] for understanding the latest progress in the field.

Acknowledgments

This work was financially supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia under contract number: 451-03-65/2024-03/200103.

References

[1] "Queries—replies," *Mathematical Tables and other Aids to Computation*, vol. 3, no. 28, pp. 561–563, 1949.

[2] M. J. Cloud, B. C. Drachman, and L. P. Lebedev, *Inequalities. With applications to engineering*, 2nd ed. Springer, Cham, 2014, doi: 10.1007/978-3-319-05311-0.

[3] N. Cutland, *Computability. An introduction to recursive function theory.* Cambridge University Press, Cambridge-New York, 1980.

[4] B. Dong, B. Yu, and Y. Yu, "A symmetric homotopy and hybrid polynomial system solving method for mixed trigonometric polynomial systems," *Math. Comp.*, vol. 83, no. 288, pp. 1847–1868, 2014, doi: 10.1090/S0025-5718-2013-02763-9.

[5] G. T. Freitas De Abreu, "Jensen-Cotes upper and lower bounds on the Gaussian Q-function and related functions," *IEEE Transactions on Communications*, vol. 57, no. 11, pp. 3328–3338, 2009, doi: 10.1109/TCOMM.2009.11.080479.

[6] Y. Lv, G. Wang, and Y. Chu, "A note on Jordan type inequalities for hyperbolic functions," *Appl. Math. Lett.*, vol. 25, no. 3, pp. 505–508, 2012, doi: 10.1016/j.aml.2011.09.046.

[7] B. Malešević, T. Lutovac, and B. Banjac, "One method for proving some classes of exponential analytical inequalities," *Filomat*, vol. 32, no. 20, pp. 6921–6925, 2018, doi: 10.2298/fil1820921m.

[8] B. Malešević and M. Makragić, "A method for proving some inequalities on mixed trigonometric polynomial functions," *J. Math. Inequal.*, vol. 10, no. 3, pp. 849–876, 2016, doi: 10.7153/jmi-10-69.

[9] B. Malešević and B. Mihailović, "A minimax approximant in the theory of analytic inequalities," *Appl. Anal. Discrete Math.*, vol. 15, no. 2, pp. 486–509, 2021, doi: 10.2298/aadm210511032m.

[10] B. Malešević, M. Nenezić, L. Zhu, B. Banjac, and M. Petrović, "Some new estimates of precision of Cusa-Huygens and Huygens approximations," *Appl. Anal. Discrete Math.*, vol. 15, no. 1, pp. 243–259, 2021, doi: 10.2298/aadm190904055m.

[11] D. S. Mitrinović, *Analytic inequalities*, ser. Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag, New York-Berlin, 1970, vol. 165.

[12] G. Rahmatollahi and G. Abreu, "Closed-form hop-count distributions in random networks with arbitrary routing," *IEEE Transactions on Communications*, vol. 60, no. 2, pp. 429–444, 2012, doi: 10.1109/TCOMM.2012.010512.110125.

[13] T. Sch., “Literaturberichte: Konstruktionen und Approximationen,” *Monatsh. Math. Phys.*, vol. 23, no. 1, p. A61, 1912, doi: 10.1007/BF01707814.

[14] P. C. Sturm, *Mémoire sur la résolution des équations numériques.* Basel: Birkhäuser Basel, 2009, pp. 345–390, doi: 10.1007/978-3-7643-7990-2_29.

[15] Z.-H. Yang, Y.-M. Chu, Y.-Q. Song, and Y.-M. Li, “A sharp double inequality for trigonometric functions and its applications,” *Abstr. Appl. Anal.*, 2014, Art. ID 592085, doi: 10.1155/2014/592085.

[16] Z.-H. Yang, Y.-M. Chu, and X.-H. Zhang, “Sharp Cusa type inequalities with two parameters and their applications,” *Appl. Math. Comput.*, vol. 268, pp. 1177–1198, 2015, doi: 10.1016/j.amc.2015.07.025.

[17] Z.-H. Yang, Y.-L. Jiang, Y.-Q. Song, and Y.-M. Chu, “Sharp inequalities for trigonometric functions,” *Abstr. Appl. Anal.*, 2014, Art. ID 601839, doi: 10.1155/2014/601839.

[18] T.-H. Zhao and Y.-M. Chu, “Some general Wilker-Huygens inequalities,” *Appl. Anal. Discrete Math.*, vol. 16, no. 2, pp. 400–426, 2022.

[19] L. Zhu, “On Frame's inequalities,” *J. Inequal. Appl.*, 2018, Art. ID 94, doi: 10.1186/s13660-018-1687-x.

[20] L. Zhu and B. Malešević, “New inequalities of Huygens-type involving tangent and sine functions,” *Hacet. J. Math. Stat.*, vol. 52, no. 1, pp. 36–61, 2023, doi: 10.1007/s00182-022-00809-0.

Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 26, no. 01, pp. 21–32, April 2024
DOI: 10.56754/0719-0646.2601.021

Double asymptotic inequalities for the generalized Wallis ratio

Vito Lampret[1,✉]

[1] *University of Ljubljana*
1000 Ljubljana, 386 Slovenia.
vito.lampret@guest.arnes.si✉

ABSTRACT

Asymptotic estimates for the generalized Wallis ratio $W^*(x) := \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}$ are presented for $x \in \mathbb{R}^+$ on the basis of Stirling's approximation formula for the Γ function. For example, for an integer $p \geq 2$ and a real $x > -\frac{1}{2}$ we have the following double asymptotic inequality

$$A(p,x) < W^*(x) < B(p,x),$$

where

$$A(p,x) := W_p(x)\left(1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{379(x+p)^3}\right),$$

$$B(p,x) := W_p(x)\left(1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{191(x+p)^3}\right),$$

$$W_p(x) := \frac{1}{\sqrt{\pi\,(x+p)}} \cdot \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}},$$

with $y^{(p)} \equiv y(y+1)\cdots(y+p-1)$, the Pochhammer rising (upper) factorial of order p.

RESUMEN

Se presentan estimaciones asintóticas para la razón generalizada de Wallis $W^*(x) := \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}$ para $x \in \mathbb{R}^+$ sobre la base de la fórmula de aproximación de Stirling para la función Γ. Por ejemplo, para un entero $p \geq 2$ y un real $x > -\frac{1}{2}$, tenemos la siguiente desigualdad doble asintótica

$$A(p,x) < W^*(x) < B(p,x),$$

donde

$$A(p,x) := W_p(x)\left(1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{379(x+p)^3}\right),$$

$$B(p,x) := W_p(x)\left(1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{191(x+p)^3}\right),$$

$$W_p(x) := \frac{1}{\sqrt{\pi\,(x+p)}} \cdot \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}},$$

con $y^{(p)} \equiv y(y+1)\cdots(y+p-1)$, el factorial ascendiente de Pochhammer (superior) de orden p.

Keywords and Phrases: Approximation, asymptotic, estimate, generalized Wallis' ratio, double inequality.

2020 AMS Mathematics Subject Classification: 26D20, 41A60, 11Y99, 33E99, 33F05, 33B99

Published: 22 March, 2024
Accepted: 06 January, 2024
Received: 23 March, 2023

1 Introduction

In pure and applied mathematics, *e.g* in number theory, probability, combinatorics, statistics, and also in several exact sciences as, for example in statistical physics and quantum mechanics, we often encounter the Wallis ratios w_n,

$$\begin{aligned} w_n &:= \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} = 4^{-n}\frac{(2n)!}{(n!)^2} = 4^{-n}\binom{2n}{n} \\ &= \frac{2^n\prod_{k=1}^{n}(k-\frac{1}{2})}{2^n\cdot n!} = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(n+1)} \qquad (n\in\mathbb{N}). \end{aligned} \tag{1.1}$$

The sequence $n\mapsto W_n := \frac{1}{2n+1}\left(\prod_{k=1}^{n}\frac{2k}{2k-1}\right)^2$, called the Wallis sequence, is closely connected to the sequence of the Wallis ratios w_n by the identity $W_n = w_n^{-2}/(2n+1)$. The Wallis sequence was intensively studied by several mathematicians, see *e.g.* [9–11, 14, 19].

According to (1.1), the continuous version $W^*(x)$ of the Wallis ratio is defined as

$$W^*(x) := \frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)} \qquad \left(x>-\tfrac{1}{2}\right). \tag{1.2}$$

Thus, we have $W^*(0)=1$ and, referring to [11, 19], we have also

$$W^*(x) = \frac{2}{\pi}\cdot H(2x), \tag{1.3}$$

where $H(x)$ is the "Wallis-cos-sin" function, defined as

$$H(x) := \int_0^{\pi/2}\big(\cos t\big)^x\,\mathrm{d}t = \int_0^{\pi/2}\big(\sin t\big)^x\,\mathrm{d}t \qquad (x\geq -1). \tag{1.4}$$

Here, for $x>-1$, we have the derivatives

$$H'(x) = \int_0^{\pi/2}\big(\ln\cos t\big)\big(\cos t\big)^x\,\mathrm{d}t < 0\,, \quad H''(x) = \int_0^{\pi/2}\big(\ln\cos t\big)^2\big(\cos t\big)^x\,\mathrm{d}t > 0.$$

Consequently, using (1.3), we conclude that $W^*(x)$ is strictly decreasing and convex on the open interval $(-\frac{1}{2},\infty)$.

Referring to (1.2), we have

$$W^*(x) = \frac{1}{\sqrt{\pi}}\cdot Q_\Gamma(x,\tfrac{1}{2},1) \qquad \left(x>-\tfrac{1}{2}\right), \tag{1.5}$$

where the ratio $Q_\Gamma(x,a,b)$ is defined as

$$Q_\Gamma(x,a,b) := \frac{\Gamma(x+a)}{\Gamma(x+b)}\,, \qquad \text{for } x > -\max\{a,b\}. \tag{1.6}$$

The ratio[1] $Q_\Gamma(x,a,b)$ was studied by many researchers, see *e.g.* the papers [2, 3, 5–7, 12, 13, 15–18, 20–27, 29, 30, 32]. Just recently several accurate estimates of $Q_\Gamma(x,a,b)$ were presented in [16], as for example in the following proposition.

Proposition 1 ([16, Theorem 1]). *For $a,b\in[0,1]$, $r\in\mathbb{N}\cup\{0\}$ and $x\in\mathbb{R}^+$ we have*[2]

$$Q_\Gamma(x,a,b)=\Big(1+\frac{a}{x}\Big)^x\Big(1+\frac{b}{x}\Big)^{-x}\frac{(x+a)^{a-1/2}}{(x+b)^{b-1/2}}\exp(b-a) \cdot\exp\left(\sum_{i=1}^{r}\frac{B_{2i}}{2i(2i-1)}\Big((x+a)^{1-2i}-(x+b)^{1-2i}\Big)+\delta_r(x,a,b)\right), \tag{1.7}$$

where

$$\big|\delta_r(x,a,b)\big|<\Delta_r(x,a,b):=\frac{\big|B_{2r+2}\big|}{(2r+1)(2r+2)\big(x+\min\{a,b\}\big)^{2r+1}} \tag{1.8}$$

and the symbol B_k denotes the k-th Bernoulli coefficient [1, 23.1.2].

Thus, for $a=\frac12$ and $b=1$, the Proposition produces the formula

$$W^*(x)=\frac{1}{\sqrt{\pi(x+1)}}\cdot\Big(1+\frac{1}{2x}\Big)^x\Big(1+\frac{1}{x}\Big)^{-x}\sqrt{e} \cdot\exp\left(\sum_{i=1}^{r}\frac{B_{2i}}{2i(2i-1)}\Big((x+1/2)^{1-2i}-(x+1)^{1-2i}\Big)\right)\cdot\exp\Big(\delta_r(x,\tfrac12,1)\Big), \tag{1.9}$$

where

$$\big|\delta_r(x,\tfrac12,1)\big|<\frac{\big|B_{2r+2}\big|}{(2r+1)(2r+2)\big(x+\frac12\big)^{2r+1}}\,, \tag{1.10}$$

for integers $r\ge0$ and $x>0$ with r being a parameter that affects the magnitude of the error term $\delta_r(x,1/2,1)$.

In this paper we will introduce a formula that is more compact than that given by (1.9)–(1.10). Our results are close to some formulas given in [4] and [31], where the main role is played by complete monotonicity of suitable functions. Unfortunately, using these articles, our results cannot be achieved easily/quickly. In this paper, we offer a simple and fast derivation using the Stirling approximation formula for the gamma function.

Remark 1.1. *In 2011, the Wallis quotient function $W(x,s,t):=\frac{\Gamma(x+t)}{\Gamma(x+s)}$ was introduced*[3] *in [2]. In this paper and also in the subsequent articles [3, 7], the authors investigate the qualitative profile of $W(x,s,t)$ using asymptotic expansions. Quantitative estimates were mostly not given there. However, for us, the quantitative estimates are essential.*

[1] Instead of the symbol Q_Γ there was used in [2, 3, 7] the letter W: $Q_\Gamma(x,a,b)=W(x,a,b)$.
[2] Consider that $\sum_{i=1}^{0}x_i=0$, by definition.
[3] Clearly, $W^*(x)=W(x,1,\frac12)$.

2 Background

Using the definition (1.2) and the equality $\Gamma(y+1) = y\Gamma(y)$, valid for $y \in \mathbb{R}^+$, by induction we note the identity

$$W^*(x) = \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}}\, W^*(x+p), \tag{2.1}$$

valid for an integer $p \geq 0$ and real $x > -\frac{1}{2}$, where $y^{(p)}$ denotes the Pochhammer rising (upper) factorial, defined as

$$y^{(0)} := 1, \quad y^{(p)} := \prod_{i=0}^{p-1}(y+i) = y(y+1)\cdots(y+p-1) \qquad (\text{for } p \geq 1).$$

Using the duplication formula [1, 6.1.18], we have, for $x > 0$,

$$2x\Gamma(2x) = 2x \cdot (2\pi)^{-1/2}\, 2^{2x-1/2}\, \Gamma(x)\Gamma(x+\tfrac{1}{2}) = \pi^{-1/2} 2^{2x}\Gamma(x+1)\Gamma(x+\tfrac{1}{2}).$$

Hence, using (1.2), we obtain, for $x > 0$,

$$W^*(x) = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)} = 2^{-2x}\frac{2x\,\Gamma(2x)}{\big(\Gamma(x+1)\big)^2} = 2^{-2x}\frac{2x\,\Gamma(2x)}{\big(x\,\Gamma(x)\big)^2}\,. \tag{2.2}$$

The continuous version of Stirling's factorial formula of order $r \geq 0$, for $x \in \mathbb{R}^+$, can be given in the following way [8, Sect. 9.5]

$$x\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \cdot \exp\big(s_r(x) + d_r(x)\big)\,, \tag{2.3}$$

where

$$s_0(x) \equiv 0 \quad \text{and} \quad s_r(x) = \sum_{i=1}^{r} \frac{c_i}{x^{2i-1}} \qquad \text{for } r \geq 1, \tag{2.4}$$

$$c_i = \frac{B_{2i}}{2i(2i-1)} \qquad \text{for } i \geq 1, \tag{2.5}$$

and, for some $\vartheta_r(x) \in (0,1)$,

$$d_r(x) = \vartheta_r(x) \cdot \frac{c_{r+1}}{x^{2r+1}}\,. \tag{2.6}$$

Here $B_2, B_4, B_6, \ldots$ are the Bernoulli coefficients, alternating in sign as

$$B_{2i} = (-1)^{i+1}\big|B_{2i}\big| \qquad \text{for } i \geq 1, \tag{2.7}$$

thanks to [1, 23.1.15, p. 805]. For example, using Mathematica [28],

$$B_2 = \frac{1}{6}, \quad B_4 = B_8 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2\,730}, \quad B_{14} = \frac{7}{6},$$

with the estimates $|B_{12}| < \frac{1}{3}$, $|B_{16}| < 7$, $B_{18} < 55$, $|B_{20}| < 530$, $B_{22} < 6200$.

3 Result

According to (2.2) and (2.3), we calculate, for $x > 0$,

$$\begin{aligned} W^*(x) &= 2^{-2x} \frac{2x\,\Gamma(2x)}{x\,\big(\Gamma(x)\big)^2} \\ &= 2^{-2x} \left(\frac{2x}{e}\right)^{2x} \sqrt{2\pi \cdot 2x} \cdot \exp\big(s_r(2x) + d_r(2x)\big) \cdot \left[\left(\frac{e}{x}\right)^x \frac{1}{\sqrt{2\pi x}} \cdot \exp\big(-s_r(x) - d_r(x)\big)\right]^2 \\ &= \frac{1}{\sqrt{\pi\,x}} \exp\Big(\underbrace{s_r(2x) - 2s_r(x)} + \underbrace{d_r(2x) - 2d_r(x)}\Big). \end{aligned} \tag{3.1}$$

Referring to (3.1) and (2.3)–(2.6), we derive the following lemma.

Lemma 3.1. *For any $r \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}^+$ we have*[4]

$$W^*(x) = \frac{1}{\sqrt{\pi\,x}} \cdot \exp\left(-\sum_{i=1}^{r} \frac{(1-4^{-i})B_{2i}}{i(2i-1)x^{2i-1}}\right) \cdot \exp\big(\delta_r(x)\big), \tag{3.2}$$

where

$$\big|\delta_r(x)\big| < \frac{\big|B_{2r+2}\big|}{(r+1)(2r+1)x^{2r+1}}\,. \tag{3.3}$$

Proof. According to (2.4)–(2.5), we have

$$s_r(2x) - 2s_r(x) = \sum_{i=1}^{r} \frac{c_i}{(2x)^{2i-1}} - 2\sum_{i=1}^{r} \frac{c_i}{x^{2i-1}} = -\sum_{i=1}^{r} \frac{c_i}{x^{2i-1}}\big(2 - 2\cdot 4^{-i}\big) = -\sum_{i=1}^{r} \frac{(1-4^{-i})B_{2i}}{i(2i-1)x^{2i-1}}.$$

Similarly, referring to (2.5)–(2.6), we have the error

$$\delta_r(x) := d_r(2x) - 2d_r(x) = \vartheta_r^*(x) \cdot \frac{c_{r+1}}{(2x)^{2r+1}} - 2\vartheta_r(x) \cdot \frac{c_{r+1}}{x^{2r+1}} = \frac{c_{r+1}}{x^{2r+1}}\left(\frac{\vartheta_r^*(x)}{2^{2r+1}} - 2\vartheta_r(x)\right),$$

for some $\vartheta_r(x), \vartheta_r^*(x) \in (0,1)$. Thus, using (2.5), we get, for $x > 0$,

$$\big|\delta_r(x)\big| < \frac{\big|B_{2r+2}\big|}{(2r+2)(2r+1)x^{2r+1}} \cdot 2\,. \qquad \square$$

Remark 3.2. *The formula for $W^*(x)$, given in* (3.2)–(3.3)*, is more compact, but slightly less accurate, than the formula, given in* (1.9)–(1.10)*, where $x = 0$ is a regular point as opposed to* (3.2)–(3.3)*, where this point is seemingly singular.*

[4] Consider that $\sum_{i=1}^{0} x_i = 0$, by definition.

Thanks to (3.3), the absolute value of $\delta_r(x)$ is small for large x and any $r \geq 0$. But, for small $x > 0$, the formula in Lemma 3.1 becomes useless. This problem can be avoided by replacing x in Lemma 3.1 by $x+p$, for p large, $p \in \mathbb{N}$. In fact, using (2.1) and replacing x by $x' = x+p$ in Lemma 3.1, immediately follows the next theorem, with $\delta^*_{p,r}(x) = \delta_r(x+p,a,b)$.

Theorem 3.3. *For integers $p, r \geq 1$ and for $x > -\frac{1}{2}$, the ratio $W^*(x)$ can be expressed in the form*

$$W^*(x) = W^*_{p,r}(x) \cdot \exp\left(\delta^*_{p,r}(x)\right), \tag{3.4}$$

where

$$W^*_{p,r}(x) := \frac{1}{\sqrt{\pi\,(x+p)}} \cdot \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}}\ \exp\left(-\sum_{i=1}^{r} \frac{(1-4^{-i})B_{2i}}{i(2i-1)(x+p)^{2i-1}}\right) \tag{3.5}$$

and

$$\left|\delta^*_{p,r}(x)\right| < \frac{\left|B_{2r+2}\right|}{(r+1)(2r+1)(x+p)^{2r+1}}\,. \tag{3.6}$$

Here, p and r are parameters that affect the magnitude of the error term $\delta^*_{p,r}(x)$.

Example 3.4. *Setting $p=3$ and $r=5$ in Theorem 3.3, we obtain*

$$\begin{aligned} W^*(x) := &\frac{(x+1)(x+2)}{(x+\frac{1}{2})(x+\frac{3}{2})(x+\frac{5}{2})} \cdot \sqrt{\frac{x+3}{\pi}} \cdot \exp\left(-\frac{1}{8(x+3)} + \frac{1}{192(x+3)^3}\right. \\ &\left. - \frac{1}{640(x+3)^5} + \frac{17}{14\,336(x+3)^7} - \frac{31}{18\,432(x+3)^9}\right) \cdot \exp\left(\delta^*_{3,5}(x)\right), \end{aligned}$$

where $\left|\delta^*_{3,5}(x)\right| < \frac{1}{260(x+3)^{11}}$, for all $x > -\frac{1}{2}$. Consequently, $\left|\delta^*_{3,5}(x)\right| < 2 \cdot 10^{-7}$ for $x \in (-\frac{1}{2}, 0]$, $\left|\delta^*_{3,5}(x)\right| < 3 \cdot 10^{-8}$, for $x \in [0,1]$, and $\left|\delta^*_{3,5}(x)\right| < 10^{-9}$, for $x \geq 1$.

A direct, immediate consequence of Theorem 3.3 is the sequence of asymptotic expansions given in the following corollary.

Corollary 3.5. *For any integer $p \geq 1$ we have the asymptotic expansion*

$$\ln\left(W^*(x)\right) \sim \ln \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}} - \frac{1}{2}\ln\left(\pi(x+p)\right) - \sum_{i=1}^{\infty} \frac{(1-4^{-i})B_{2i}}{i(2i-1)(x+p)^{2i-1}}\,,$$

as $x \to \infty$.

Theorem 3.6. *For an integer $p \geq 2$ and real $x > -\frac{1}{2}$ there holds the following double asymptotic inequality*

$$A(p,x) \ < \ W^*(x) \ < \ B(p,x), \tag{3.7}$$

where

$$A(p,x) := W_p^*(x)\left(1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{379(x+p)^3}\right), \tag{3.8}$$

$$B(p,x) := W_p^*(x)\left(1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} + \frac{1}{191(x+p)^3}\right), \tag{3.9}$$

$$W_p^*(x) := W_{p,0}^*(x) = \frac{1}{\sqrt{\pi\,(x+p)}} \cdot \frac{(x+1)^{(p)}}{(x+\frac{1}{2})^{(p)}}\,. \tag{3.10}$$

Proof. We use Theorem 3.3 with $r=2$, when $\left|\delta_{p,2}^*(x)\right| < \frac{1}{630(x+p)^5}$ and thus we estimate

$$y_-(p,x) < -\sum_{i=1}^{2} \frac{(1-4^{-i})B_{2i}}{i(2i-1)(x+p)^{2i-1}} + \delta_{p,2}^*(x) < y_+(p,x) < 0, \tag{3.11}$$

for $p \in \mathbb{N}$ and $x \in \mathbb{R}^+$, where

$$y_-(p,x) := -\frac{1}{8(x+p)} + \frac{1}{192(x+p)^3} - \frac{1}{630(x+p)^5}\,, \tag{3.12}$$

$$y_+(p,x) := -\frac{1}{8(x+p)} + \frac{1}{192(x+p)^3} + \frac{1}{630(x+p)^5}\,. \tag{3.13}$$

Furthermore, by Taylor's formula of orders 3 and 2 we have, for $y<0$,

$$1 + y + \frac{y^2}{2} + \frac{y^3}{6} < e^y < 1 + y + \frac{y^2}{2}\,.$$

Thus, referring to (3.11)–(3.13), we have, for $p \in \mathbb{N}$,

$$\exp\big(y_-(p,x)\big) > 1 + y_-(p,x) + \frac{1}{2}y_-^2(p,x) + \frac{1}{6}y_-(p,x))^3, \tag{3.14}$$

$$\exp\big(y_+(p,x)\big) < 1 + y_+(p,x) + \frac{1}{2}y_+^2(p,x). \tag{3.15}$$

Now, due to (3.12), we estimate, for $x > -\frac{1}{2}$ and $x+p > -\frac{1}{2} + 2 > 1$, as follows:

$$\begin{aligned}
&1 + y_-(p,x) + \frac{1}{2}y_-^2(p,x) + \frac{1}{6}y_-(p,x))^3 \\
&= 1 - \tfrac{1}{8(x+p)} + \tfrac{1}{128(x+p)^2} + \tfrac{5}{1\,024(x+p)^3} - \tfrac{1}{1\,536(x+p)^4} + \cancel{\tfrac{1}{24\,576(x+p)^5}} \\
&+ \cancel{\tfrac{1}{73\,728(x+p)^6}} - \tfrac{1}{589\,824(x+p)^7} + \cancel{\tfrac{1}{42\,467\,328(x+p)^9}} - \tfrac{1}{630(x+p)^5} \\
&> 1 - \tfrac{1}{8(x+p)} + \tfrac{1}{128(x+p)^2} + \tfrac{5}{1\,024(x+p)^3} - \tfrac{1}{1\,536(x+p)^3} - \tfrac{1}{589\,824(x+p)^3} \\
&- \tfrac{1}{630(x+p)^3} > 1 - \tfrac{1}{8(x+p)} + \tfrac{1}{128(x+p)^2} + \tfrac{1}{379(x+p)^3}\,,
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&1+y_+(p,x)+\frac{1}{2}y_+^2(p,x)\\
&=1-\tfrac{1}{8(x+p)}+\tfrac{1}{128(x+p)^2}+\tfrac{1}{192(x+p)^3}-\tfrac{1}{1\,536(x+p)^4}+\tfrac{1}{73\,728(x+p)^6}\\
&<1-\tfrac{1}{8(x+p)}+\tfrac{1}{128(x+p)^2}+\tfrac{1}{191(x+p)^3}\,. \qquad (3.17)
\end{aligned}$$

Using Theorem 3.3, (3.11), (3.14)–(3.15) and (3.16)–(3.17) we note the double inequality (3.7). □

Example 3.7. *We have* $A(2,-\frac{49}{100})=32.25\ldots<W(-\frac{49}{100})=32.27\ldots<B(2,-\frac{49}{100})=32.28\ldots$ *However,* $A(1,-\frac{49}{100})=32.42\ldots>W(-\frac{49}{100})=32.27\ldots$

Example 3.8. *We have* $B(2,\frac{49}{100})-A(2,\frac{49}{100})<3\cdot10^{-2}$, $B(2,0)-A(2,0)<4\cdot10^{-4}$ *and* $B(2,\pi)-A(2,\pi)<6\cdot10^{-6}$.

Example 3.9. *We have exactly* $W(3)=w_3=\frac{5}{16}=0.3125$ *and, thanks to Theorem 3.6, we estimate* $0.312\,499\,6<A(9,3)<\underline{W(3)}<B(9,3)<0.312\,500\,1$.

Figure 1 illustrates the estimate (3.7) *by plotting*[5] *the graphs of the functions* $x\mapsto A(2,x)$, $x\mapsto W(x)$ *and* $x\mapsto B(2,x)$, *where all graphs practically coincide.*

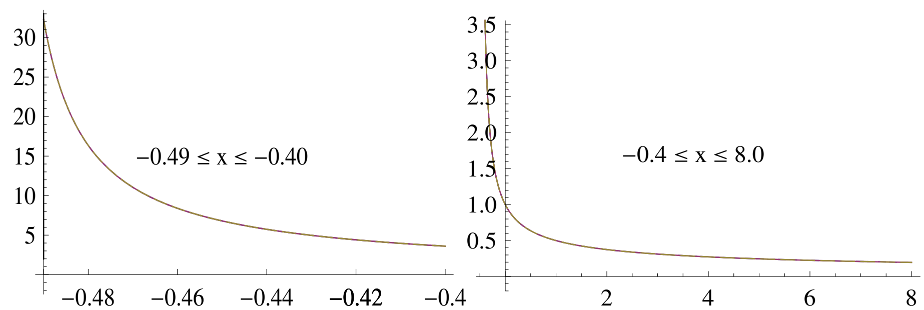


Figure 1: The graphs of the functions $x\mapsto A(2,x)$, $x\mapsto W(x)$ and $x\mapsto B(2,x)$.

Corollary 3.10. *For an integer* $p\geq2$ *and* $x>-\frac{1}{2}$ *the approximation* $W^*(x)\approx A(p,x)$ *has the relative error*

$$\rho(p,x):=\frac{W^*(x)-A(p,x)}{W^*(x)}$$

estimated as

$$0<\rho(p,x)<\frac{B(p,x)-A(p,x)}{A(p,x)}<\frac{1}{330(x+p)^3}\,.$$

Proof. Thanks to Theorem 3.6 we have

$$0<\rho(p,x)<\frac{B(p,x)-A(p,x)}{A(p,x)}=\frac{B(p,x)}{A(p,x)}-1=\frac{S+\Delta_2}{S+\Delta_1}-1,$$

[5] All figures and more demanding computations made in this paper were produced using Mathematica [28].

where

$$S = 1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)^2} \tag{3.18}$$

and

$$\Delta_1 = \frac{1}{379}(x+p)^{-3}, \quad \Delta_2 = \frac{1}{191}(x+p)^{-3}. \tag{3.19}$$

Thus,

$$0 < \rho(p,x) < \left(1 + \frac{\Delta_2 - \Delta_1}{S + \Delta_1}\right) - 1 < \frac{\Delta_2 - \Delta_1}{S},$$

where the assumptions $x > -\frac{1}{2}$ and $p \geq 2$ imply the estimate $x + p > 1$, which, due to (3.18), implies the inequalities

$$S \geq 1 - \frac{1}{8(x+p)} + \frac{1}{128(x+p)} > 1 - \frac{1}{7(x+p)} \geq \frac{6}{7}\,.$$

Consequently, thanks to (3.19),

$$\frac{\Delta_2 - \Delta_1}{S} < \frac{7}{6}\left(\frac{1}{191} - \frac{1}{379}\right)\frac{1}{(x+p)^3} < \frac{1}{330(x+p)^3}\,. \qquad \square$$

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, ser. National Bureau of Standards Applied Mathematics Series. U. S. Government Printing Office, Washington, DC, 1964, vol. 55.

[2] T. Burić and N. Elezović, "Bernoulli polynomials and asymptotic expansions of the quotient of gamma functions," *J. Comput. Appl. Math.*, vol. 235, no. 11, pp. 3315–3331, 2011, doi: 10.1016/j.cam.2011.01.045.

[3] T. Burić and N. Elezović, "New asymptotic expansions of the quotient of gamma functions," *Integral Transforms Spec. Funct.*, vol. 23, no. 5, pp. 355–368, 2012, doi: 10.1080/10652469.2011.591393.

[4] C.-P. Chen and R. B. Paris, "Inequalities, asymptotic expansions and completely monotonic functions related to the gamma function," *Appl. Math. Comput.*, vol. 250, pp. 514–529, 2015, doi: 10.1016/j.amc.2014.11.010.

[5] V. G. Cristea, "A direct approach for proving Wallis ratio estimates and an improvement of Zhang-Xu-Situ inequality," *Stud. Univ. Babeş-Bolyai Math.*, vol. 60, no. 2, pp. 201–209, 2015.

[6] S. Dumitrescu, "Estimates for the ratio of gamma functions by using higher order roots," *Stud. Univ. Babeş-Bolyai Math.*, vol. 60, no. 2, pp. 173–181, 2015.

[7] N. Elezović, "Asymptotic expansions of gamma and related functions, binomial coefficients, inequalities and means," *J. Math. Inequal.*, vol. 9, no. 4, pp. 1001–1054, 2015, doi: 10.7153/jmi-09-81.

[8] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete mathematics: A Foundation for Computer Science*, 2nd ed. Addison-Wesley Publishing Company, Reading, MA, 1994.

[9] S. Guo, J.-G. Xu, and F. Qi, "Some exact constants for the approximation of the quantity in the Wallis' formula," *J. Inequal. Appl.*, 2013, Art. ID 67, doi: 10.1186/1029-242X-2013-67.

[10] M. D. Hirschhorn, "Comments on the paper: "Wallis sequence …" by Lampret," *Austral. Math. Soc. Gaz.*, vol. 32, no. 3, p. 194, 2005.

[11] D. K. Kazarinoff, "On Wallis' formula," *Edinburgh Math. Notes*, vol. 1956, no. 40, pp. 19–21, 1956.

[12] D. Kershaw, "Upper and lower bounds for a ratio involving the gamma function," *Anal. Appl. (Singap.)*, vol. 3, no. 3, pp. 293–295, 2005, doi: 10.1142/S0219530505000583.

[13] A. Laforgia and P. Natalini, "On the asymptotic expansion of a ratio of gamma functions," *J. Math. Anal. Appl.*, vol. 389, no. 2, pp. 833–837, 2012, doi: 10.1016/j.jmaa.2011.12.025.

[14] V. Lampret, "Wallis' sequence estimated accurately using an alternating series," *J. Number Theory*, vol. 172, pp. 256–269, 2017, doi: 10.1016/j.jnt.2016.08.014.

[15] V. Lampret, "A simple asymptotic estimate of Wallis' ratio using Stirling's factorial formula," *Bull. Malays. Math. Sci. Soc.*, vol. 42, no. 6, pp. 3213–3221, 2019, doi: 10.1007/s40840-018-0654-5.

[16] V. Lampret, "Simple, accurate, asymptotic estimates for the ratio of two gamma functions," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, vol. 115, no. 2, 2021, Art. ID 40, doi: 10.1007/s13398-020-00962-9.

[17] A.-J. Li, W.-Z. Zhao, and C.-P. Chen, "Logarithmically complete monotonicity properties for the ratio of gamma function," *Adv. Stud. Contemp. Math. (Kyungshang)*, vol. 13, no. 2, pp. 183–191, 2006.

[18] C. Mortici, "New approximation formulas for evaluating the ratio of gamma functions," *Math. Comput. Modelling*, vol. 52, no. 1-2, pp. 425–433, 2010, doi: 10.1016/j.mcm.2010.03.013.

[19] K. Nantomah, "Some inequalities for derivatives of the generalized Wallis' cosine formula," *Int. J. Open Problems Comput. Sci. and Math.*, vol. 11, no. 4, pp. 16–24, 2018, doi: 10.1016/j.mcm.2010.03.013.

[20] F. Qi, "Bounds for the ratio of two gamma functions," *RGMIA Res. Rep. Coll.*, vol. 11, no. 3, 2008, Art. ID 1, *https://rgmia.org/papers/v11n3/bounds-two-gammas.pdf.*

[21] F. Qi, "Bounds for the ratio of two gamma functions–From Gautschi's and Kershaw's inequalities to completely monotonic functions," 2009, *arXiv:0904.1049.*

[22] F. Qi, "Bounds for the ratio of two gamma functions," *J. Inequal. Appl.*, 2010, Art. ID 493058, doi: 10.1155/2010/493058.

[23] F. Qi, "Bounds for the ratio of two gamma functions: from Gautschi's and Kershaw's inequalities to complete monotonicity," *Turkish Journal of Analysis and Number Theory*, vol. 2, pp. 152–164, 2014, doi: 10.12691/tjant-2-5-1.

[24] F. Qi and Q.-M. Luo, "Bounds for the ratio of two gamma functions—from Wendel's and related inequalities to logarithmically completely monotonic functions," *Banach J. Math. Anal.*, vol. 6, no. 2, pp. 132–158, 2012, doi: 10.15352/bjma/1342210165.

[25] F. Qi and Q.-M. Luo, "Bounds for the ratio of two gamma functions: from Wendel's asymptotic relation to Elezović-Giordano-Pečarić's theorem," *J. Inequal. Appl.*, 2013, Art. ID 542, doi: 10.1186/1029-242x-2013-542.

[26] D. V. Slavić, "On inequalities for $\Gamma(x+1)/\Gamma(x+1/2)$," *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, no. 498-541, pp. 17–20, 1975.

[27] I. V. Tikhonov, V. B. Sherstyukov, and D. G. Tsvetkovich, "Comparative analysis of two-sided estimates of the central binomial coefficient," *Chelyab. Fiz.-Mat. Zh.*, vol. 5, no. 1, pp. 70–95, 2020, doi: 10.24411/2500-0101-2020-15106.

[28] S. Wolfram, *Mathematica 7.0*, (2008). Wolfram Research, Inc.

[29] Z.-H. Yang and J.-F. Tian, "On Burnside type approximation for the gamma function," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, vol. 113, no. 3, pp. 2665–2677, 2019, doi: 10.1007/s13398-019-00651-2.

[30] Z.-H. Yang and J.-F. Tian, "Monotonicity, convexity, and complete monotonicity of two functions related to the gamma function," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, vol. 113, no. 4, pp. 3603–3617, 2019, doi: 10.1007/s13398-019-00719-z.

[31] Z.-H. Yang, J.-F. Tian, and M.-H. Ha, "A new asymptotic expansion of a ratio of two gamma functions and complete monotonicity for its remainder," *Proc. Amer. Math. Soc.*, vol. 148, no. 5, pp. 2163–2178, 2020, doi: 10.1090/proc/14917.

[32] X. You, "Approximation and bounds for the wallis ratio," 2017, *arXiv:1712.02107*.

CUBO, A Mathematical Journal
Vol. 26, no. 01, pp. 33–51, April 2024
DOI: 10.56754/0719-0646.2601.033

Multiplicative maps on generalized n-matrix rings

Aisha Jabeen[1]
Bruno L. M. Ferreira[2,3,4,✉]

[1] *Department of Physics, Jamia Millia Islamia, New Delhi-110025, India.*
ajabeen329@gmail.com

[2] *Federal University of Technology-Paraná-Brazil, Av. Professora Laura Pacheco Bastos, 800, 85053-510, Guarapuava, Brazil.*
brunolmfalg@gmail.com

[3] *Federal University of ABC, Av. dos Estados, 5001, 09210-580, Santo André - SP-Brazil.*

[4] *São Paulo University, Rua Matão, 1010, CEP 05508-090, São Paulo, Brazil.*

ABSTRACT

Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two associative rings (not necessarily with identity elements). A bijective map φ of $\mathfrak{R}$ onto $\mathfrak{R}'$ is called an *m-multiplicative isomorphism* if $\varphi(x_1 \cdots x_m) = \varphi(x_1) \cdots \varphi(x_m)$ for all $x_1, \ldots, x_m \in \mathfrak{R}$. In this article, we establish a condition on generalized matrix rings, that assures that multiplicative maps are additive. And then, we apply our result for study of m-multiplicative isomorphisms and m-multiplicative derivations on generalized matrix rings.

RESUMEN

Sean $\mathfrak{R}$ y $\mathfrak{R}'$ dos anillos asociativos (no necesariamente con elementos identidad). Una aplicación biyectiva φ de $\mathfrak{R}$ en $\mathfrak{R}'$ se llama un *isomorfismo m-multiplicativo* si $\varphi(x_1 \cdots x_m) = \varphi(x_1) \cdots \varphi(x_m)$ para todos $x_1, \ldots, x_m \in \mathfrak{R}$. En este artículo, establecemos una condición en anillos de matrices generalizadas que asegura que las aplicaciones multiplicativas sean aditivas. Luego aplicamos nuestro resultado para estudiar isomorfismos m-multiplicativos y derivaciones m-multiplicativas de anillos de matrices generalizadas.

Keywords and Phrases: m-multiplicative maps, m-multiplicative derivations, generalized n-matrix rings, additivity.

2020 AMS Mathematics Subject Classification: 16W99, 47B47, 47L35.

Published: 25 March, 2024
Accepted: 09 January, 2024
Received: 01 February, 2023

1 Introduction

Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two associative rings (not necessarily with identity elements). We denote by $\mathfrak{Z}(\mathfrak{R})$ the center of $\mathfrak{R}$. A bijective map φ of $\mathfrak{R}$ onto $\mathfrak{R}'$ is called an *m-multiplicative isomorphism* if

$$\varphi(x_1 \cdots x_m) = \varphi(x_1) \cdots \varphi(x_m)$$

for all $x_1, \ldots, x_m \in \mathfrak{R}$. In particular, if $m = 2$ then φ is called a *multiplicative isomorphism*. Similarly, a map d of $\mathfrak{R}$ is called an *m-multiplicative derivation* if

$$d(x_1 \cdots x_m) = \sum_{i=1}^{m} x_1 \cdots d(x_i) \cdots x_m$$

for all $x_1, \ldots, x_m \in \mathfrak{R}$. If $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathfrak{R}$, we just say that d is a *multiplicative derivation* of $\mathfrak{R}$.

In last few decades, the multiplicative mappings on rings and algebras have been studied by many authors [1, 4–7, 10]. Martindale [7] established a condition on a ring such that multiplicative isomorphisms on this ring are all additive. In particular, every multiplicative isomorphism from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive. Lu [6] studied multiplicative isomorphisms of subalgebras of nest algebras which contain all finite rank operators but might contain no idempotents and proved that these multiplicative mappings are automatically additive and linear or conjugate linear. Further, Wang in [9, 10] considered the additivity of multiplicative maps on rings with idempotents and triangular rings respectively. Recently, in order to generalize the result in [10] the second author [3], defined a class of ring called triangular n-matrix ring and studied the additivity of multiplicative maps on that class of rings. In view of above discussed literature, in this article we discuss the additivity of multiplicative maps on a more general class of rings called generalized n-matrix rings.

We adopt and follow the same structure and demonstration presented in [3], in order to preserve the author ideas and to highlight the generalization of the triangular n-matrix results to the generalized n-matrix results.

Definition 1.1. *Let $\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_n$ be rings and $\mathfrak{M}_{ij}$ be an $(\mathfrak{R}_i, \mathfrak{R}_j)$-bimodule with $\mathfrak{M}_{ii} = \mathfrak{R}_i$ for all $i, j \in \{1, \ldots, n\}$. Let $\varphi_{ijk} : \mathfrak{M}_{ij} \otimes_{\mathfrak{R}_j} \mathfrak{M}_{jk} \longrightarrow \mathfrak{M}_{ik}$ be $(\mathfrak{R}_i, \mathfrak{R}_k)$-bimodule homomorphisms with $\varphi_{iij} : \mathfrak{R}_i \otimes_{\mathfrak{R}_i} \mathfrak{M}_{ij} \longrightarrow \mathfrak{M}_{ij}$ and $\varphi_{ijj} : \mathfrak{M}_{ij} \otimes_{\mathfrak{R}_j} \mathfrak{R}_j \longrightarrow \mathfrak{M}_{ij}$ the canonical isomorphisms for all $i, j, k \in \{1, \ldots, n\}$. Write $a \circ b = \varphi_{ijk}(a \otimes b)$ for $a \in \mathfrak{M}_{ij}$, $b \in \mathfrak{M}_{jk}$. Let*

$$\mathfrak{G} = \left\{ \begin{pmatrix} r_{11} & m_{12} & \ldots & m_{1n} \\ m_{21} & r_{22} & \ldots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \ldots & r_{nn} \end{pmatrix}_{n \times n} : \underbrace{r_{ii} \in \mathfrak{R}_i \ (= \mathfrak{M}_{ii}), \ m_{ij} \in \mathfrak{M}_{ij}}_{(i,j \in \{1,\ldots,n\})} \right\}$$

be the set of all $n \times n$ matrices (m_{ij}) with (i,j)-entry $m_{ij} \in \mathfrak{M}_{ij}$ for all $i,j \in \{1,\dots,n\}$. Observe that, with the obvious matrix operations of addition and multiplication, $\mathfrak{G}$ is a ring iff $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a \in \mathfrak{M}_{ik}$, $b \in \mathfrak{M}_{kl}$ and $c \in \mathfrak{M}_{lj}$ for all $i,j,k,l \in \{1,\dots,n\}$. When $\mathfrak{G}$ is a ring, it is called a generalized n-matrix ring.

Note that if $n = 2$, we get the definition of generalized matrix ring. We denote by $\bigoplus_{i=1}^{n} r_{ii}$ the element

$$\begin{pmatrix} r_{11} & & & \\ & r_{22} & & \\ & & \ddots & \\ & & & r_{nn} \end{pmatrix}$$

in $\mathfrak{G}$.

Set

$$\mathfrak{G}_{ij} = \left\{ (m_{kt}) : \; m_{kt} = \begin{cases} m_{ij}, & \text{if} \quad (k,t) = (i,j) \\ 0, & \text{if} \quad (k,t) \neq (i,j) \end{cases}, \quad i,j \in \{1,\dots,n\} \right\}.$$

Then we can write $\mathfrak{G} = \bigoplus_{i,j \in \{1,\dots,n\}} \mathfrak{G}_{ij}$. Note that, this special structure allows us to use the argument given in [7] even if non-trivial idempotents exist. Henceforth the element a_{ij} belongs to $\mathfrak{G}_{ij}$ and the corresponding elements are in $\mathfrak{R}_1, \dots, \mathfrak{R}_n$ or $\mathfrak{M}_{ij}$. By a direct calculation $a_{ij}a_{kl} = 0$ if $j \neq k$. We define natural projections $\pi_i : \mathfrak{G} \longrightarrow \mathfrak{R}_i$ $(1 \leq i \leq n)$ by

$$\begin{pmatrix} r_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & r_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & r_{nn} \end{pmatrix} \longmapsto r_{ii}.$$

The following result is a characterization of the center of a generalized n-matrix ring. Henceforth, we will consider

(i) $\mathfrak{M}_{ij}$ is faithful as a left $\mathfrak{R}_i$-module and faithful as a right $\mathfrak{R}_j$-module with $i \neq j$,

(ii) if $m_{ij} \in \mathfrak{M}_{ij}$ is such that $\mathfrak{R}_i m_{ij} \mathfrak{R}_j = 0$ then $m_{ij} = 0$ with $i \neq j$.

We will call them *special conditions.*

Proposition 1.2. *Let $\mathfrak{G}$ be a generalized n-matrix ring. The center of $\mathfrak{G}$ is*

$$\mathfrak{Z}(\mathfrak{G}) = \left\{ \bigoplus_{i=1}^{n} r_{ii} \;\middle|\; r_{ii}m_{ij} = m_{ij}r_{jj} \text{ for all } m_{ij} \in \mathfrak{M}_{ij}, \; i \neq j \right\}.$$

Furthermore, $\mathfrak{Z}(\mathfrak{G})_{ii} \cong \pi_i(\mathfrak{Z}(\mathfrak{G})) \subseteq \mathfrak{Z}(\mathfrak{R}_i)$, and there exists a unique ring isomorphism τ_i^j from $\pi_i(\mathfrak{Z}(\mathfrak{G}))$ to $\pi_{i\mathfrak{R}_j}(\mathfrak{Z}(\mathfrak{G}))$, $i \neq j$, such that $r_{ii}m_{ij} = m_{ij}\tau_i^j(r_{ii})$ for all $m_{ij} \in \mathfrak{M}_{ij}$.

Proof. Let $S = \left\{ \bigoplus_{i=1}^{n} r_{ii} \;\middle|\; r_{ii}m_{ij} = m_{ij}r_{jj} \text{ for all } m_{ij} \in \mathfrak{M}_{ij}, \; i \neq j \right\}$. By a direct calculation we have that if $r_{ii} \in \mathfrak{Z}(\mathfrak{R}_i)$ and $r_{ii}m_{ij} = m_{ij}r_{jj}$ for every $m_{ij} \in \mathfrak{M}_{ij}$ with $i \neq j$, then $\bigoplus_{i=1}^n r_{ii} \in \mathfrak{Z}(\mathfrak{G})$; that is, $(\bigoplus_{i=1}^n \mathfrak{Z}(\mathfrak{R}_i)) \cap S \subseteq \mathfrak{Z}(\mathfrak{G})$. To prove that $S = \mathfrak{Z}(\mathfrak{G})$, we only need to show that $\mathfrak{Z}(\mathfrak{G}) \subseteq S$ and $S \subseteq \bigoplus_{i=1}^n \mathfrak{Z}(\mathfrak{R}_i)$.

Suppose that $x = \begin{pmatrix} r_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & r_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & r_{nn} \end{pmatrix} \in \mathfrak{Z}(\mathfrak{G})$. Since $x\big(\bigoplus_{i=1}^n a_{ii}\big) = \big(\bigoplus_{i=1}^n a_{ii}\big)x$ for all $a_{ii} \in \mathfrak{R}_i$, we have $a_{ii}m_{ij} = m_{ij}a_{jj}$ for $i \neq j$. Making $a_{jj} = 0$ we conclude $a_{ii}m_{ij} = 0$ for all $a_{ii} \in \mathfrak{R}_i$ and so $m_{ij} = 0$ for all $i \neq j$ which implies that $x = \bigoplus_{i=1}^n r_{ii}$. Moreover, for any $m_{ij} \in \mathfrak{M}_{ij}$ as

$$x \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & m_{ij} & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & m_{ij} & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{pmatrix} x,$$

then $r_{ii}m_{ij} = m_{ij}r_{jj}$ for all $i \neq j$ which results in $\mathfrak{Z}(\mathfrak{G}) \subseteq S$. Now suppose $x = \bigoplus_{i=1}^n r_{ii} \in S$. Then for any $a_{ii} \in \mathfrak{R}_i$ $(i = 1, \dots, n-1)$, we have $(r_{ii}a_{ii} - a_{ii}r_{ii})m_{ij} = r_{ii}(a_{ii}m_{ij}) - a_{ii}(r_{ii}m_{ij}) = (a_{ii}m_{ij})r_{jj} - a_{ii}(m_{ij}r_{jj}) = 0$ for all $m_{ij} \in \mathfrak{M}_{ij}$ $(i \neq j)$ and hence $r_{ii}a_{ii} - a_{ii}r_{ii} = 0$ as $\mathfrak{M}_{ij}$ is a left faithful $\mathfrak{R}_i$-module.

The fact that $\pi_i(\mathfrak{Z}(\mathfrak{G})) \subseteq \mathfrak{Z}(\mathfrak{R}_i)$ for $i = 1, \dots, n$ are direct consequences of $\mathfrak{Z}(\mathfrak{G}) = S \subseteq \bigoplus_{i=1}^n \mathfrak{Z}(\mathfrak{R}_i)$. Now we prove the existence of the ring isomorphism $\tau_i^j : \pi_{\mathfrak{R}_i}(\mathfrak{Z}(\mathfrak{G})) \longrightarrow \pi_{\mathfrak{R}_j}(\mathfrak{Z}(\mathfrak{G}))$ for $i \neq j$. For this, let us consider a pair of indices (i, j) such that $i \neq j$. For any $r = \bigoplus_{k=1}^n r_{kk} \in \mathfrak{Z}(\mathfrak{G})$ let us define $\tau_i^j(r_{ii}) = r_{jj}$. The map is well defined because if $s = \bigoplus_{k=1}^n s_{kk} \in \mathfrak{Z}(\mathfrak{G})$ is such that $s_{ii} = r_{ii}$, then we have $m_{ij}r_{jj} = r_{ii}m_{ij} = s_{ii}m_{ij} = m_{ij}s_{jj}$ for all $m_{ij} \in \mathfrak{M}_{ij}$. Since $\mathfrak{M}_{ij}$ is a right faithful $\mathfrak{R}_j$-module, we conclude that $r_{jj} = s_{jj}$. Therefore, for any $r_{ii} \in \pi_{\mathfrak{R}_i}(\mathfrak{Z}(\mathfrak{G}))$, there exists a unique $r_{jj} \in \pi_{\mathfrak{R}_j}(\mathfrak{Z}(\mathfrak{G}))$, denoted by $\tau_i^j(r_{ii})$. It is easy to see that τ_i^j is bijective. Moreover, for any $r_{ii}, s_{ii} \in \pi_{\mathfrak{R}_i}(\mathfrak{Z}(\mathfrak{G}))$ we have $m_{ij}\tau_i^j(r_{ii}+s_{ii}) = (r_{ii}+s_{ii})m_{ij} = m_{ij}(r_{jj}+s_{jj}) = m_{ij}\big(\tau_i^j(r_{ii}) + \tau_i^j(s_{ii})\big)$ and $m_{ij}\tau_i^j(r_{ii}s_{ii}) = (r_{ii}s_{ii})m_{ij} = r_{ii}(s_{ii}m_{ij}) = (s_{ii}m_{ij})\tau_i^j(r_{ii}) = s_{ii}\big(m_{ij}\tau_i^j(r_{ii})\big) = m_{ij}\big(\tau_i^j(r_{ii})\tau_i^j(s_{ii})\big)$. Thus $\tau_i^j(r_{ii} + s_{ii}) = \tau_i^j(r_{ii}) + \tau_i^j(s_{ii})$ and $\tau_i^j(r_{ii}s_{ii}) = \tau_i^j(r_{ii})\tau_i^j(s_{ii})$ and so τ_i^j is a ring isomorphism. □

Proposition 1.3. *Let $\mathfrak{G}$ be a generalized n-matrix ring such that:*

(i) $a_{ii}\mathfrak{R}_i = 0$ *implies* $a_{ii} = 0$ *for* $a_{ii} \in \mathfrak{R}_i$*;*

(ii) $\mathfrak{R}_j b_{jj} = 0$ *implies* $b_{jj} = 0$ *for* $b_{jj} \in \mathfrak{R}_j$*.*

Then $u\mathfrak{G} = 0$ *or* $\mathfrak{G}u = 0$ *implies* $u = 0$ *for* $u \in \mathfrak{G}$*.*

Proof. First, let us observe that if $i \neq j$ and $\mathfrak{R}_i a_{ii} = 0$, then we have $\mathfrak{R}_i a_{ii} m_{ij} \mathfrak{R}_j = 0$, for all $m_{ij} \in \mathfrak{M}_{ij}$, which implies $a_{ii}m_{ij} = 0$ by condition *(ii)* of the special conditions. It follows that $a_{ii}\mathfrak{M}_{ij} = 0$ resulting in $a_{ii} = 0$. Hence, suppose $u = \bigoplus_{i,j\in\{1,\dots,n\}} u_{ij}$, with $u_{ij} \in \mathfrak{G}_{ij}$, satisfying $u\mathfrak{G} = 0$. Then $u_{kk}\mathfrak{R}_k = 0$ which yields $u_{kk} = 0$ for $k = 1, \dots, n-1$, by condition *(i)*. Now for $k = n$, $u_{nn}\mathfrak{R}_n = 0$, we have $\mathfrak{R}_i m_{in} u_{nn} \mathfrak{R}_n = 0$, for all $m_{in} \in \mathfrak{M}_{in}$, which implies $m_{in}u_{nn} = 0$ by condition *(ii)* of the special conditions. It follows that $\mathfrak{M}_{in}u_{nn} = 0$ which implies $u_{nn} = 0$. Thus $u_{ij}\mathfrak{R}_j = 0$ and then $u_{ij} = 0$ by condition *(ii)* of special conditions. Therefore $u = 0$. Similarly, we prove that if $\mathfrak{G}u = 0$ then $u = 0$. □

2 The main theorem

Follows our main result, where we are suppose that the special conditions hold. This generalizes the Theorem 2.1 in [3]. Our main result reads as follows.

Theorem 2.1. *Let* $B : \mathfrak{G} \times \mathfrak{G} \longrightarrow \mathfrak{G}$ *be a biadditive map such that:*

(i) $B(\mathfrak{G}_{pp}, \mathfrak{G}_{qq}) \subseteq \mathfrak{G}_{pp} \cap \mathfrak{G}_{qq}$*;* $B(\mathfrak{G}_{pp}, \mathfrak{G}_{rs}) \in \mathfrak{G}_{rs}$*,* $B(\mathfrak{G}_{ip}, \mathfrak{G}_{pq}) \in \mathfrak{G}_{iq}$ *and* $B(\mathfrak{G}_{rs}, \mathfrak{G}_{pp}) \in \mathfrak{G}_{rs}$*;* $B(\mathfrak{G}_{pq}, \mathfrak{G}_{rs}) = 0$*;*

(ii) if $B(\bigoplus_{1\le p\neq q\le n} c_{pq}, \mathfrak{G}_{nn}) = 0$ *or* $B(\bigoplus_{1\le r<n} \mathfrak{G}_{rr}, \bigoplus_{1\le p\neq q\le n} c_{pq}) = 0$*, then* $\bigoplus_{1\le p\neq q\le n} c_{pq} = 0$*;*

(iii) $B(\mathfrak{G}_{nn}, a_{nn}) = 0$ *implies* $a_{nn} = 0$ *and* $B(\bigoplus_{i=1}^{n} \mathfrak{G}_{ip}, a_{pq}) = 0$ *implies* $a_{pq} = 0$*;*

(iv) if $B(\bigoplus_{p=1}^{n} c_{pp}, \mathfrak{G}_{rs}) = B(\mathfrak{G}_{rs}, \bigoplus_{p=1}^{n} c_{pp}) = 0$ *for all* $1 \le r \neq s \le n$*, then* $\bigoplus_{p=1}^{n-1} c_{pp} \oplus (-c_{nn}) \in \mathfrak{Z}(\mathfrak{G})$*;*

(v) $B(c_{pp}, d_{pp}) = B(d_{pp}, c_{pp})$ *and* $B(c_{pp}, d_{pp})d_{pn}d_{nn} = d_{pp}d_{pn}B(c_{nn}, d_{nn})$ *for all* $c = \bigoplus_{p=1}^{n} c_{pp} \in \mathfrak{Z}(\mathfrak{G})$*;*

(vi) $B\big(c_{rr}, B(c_{kl}, c_{nn})\big) = B\big(B(c_{rr}, c_{kl}), c_{nn}\big)$*.*

Suppose $f : \mathfrak{G} \times \mathfrak{G} \longrightarrow \mathfrak{G}$ *is a map satisfying the following conditions:*

(vii) $f(\mathfrak{G}, 0) = f(0, \mathfrak{G}) = 0$*;*

(viii) $B\big(f(x,y),z\big) = f\big(B(x,z),B(y,z)\big)$;

(ix) $B\big(z,f(x,y)\big) = f\big(B(z,x),B(z,y)\big)$

for all $x,y,z \in \mathfrak{G}$ *. Then* $f = 0$.

Proof. Following the ideas of Ferreira in [3] we divide the proof into four cases. Then, let us consider arbitrary elements $x_{kl}, u_{kl}, a_{kl} \in \mathfrak{G}_{kl}$ $(k,l \in \{1,\ldots,n\})$.

First case. In this first case the reader should keep in mind that we want to show

$$f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right) = 0.$$

From the hypotheses of the theorem, we have

$$\begin{aligned} B\left(f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right), a_{nn}\right) &= f\left(B\left(\sum_{1\leq i<n} x_{ii}, a_{nn}\right), B\left(\sum_{1\leq j\neq k\leq n} x_{jk}, a_{nn}\right)\right) \\ &= f\left(0, B\left(\sum_{1\leq j\neq k\leq n} x_{jk}, a_{nn}\right)\right) \\ &= 0. \end{aligned}$$

In other words,

$$B\left(\sum_{1\leq p,q\leq n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pq}, a_{nn}\right) = 0.$$

Since by condition (i),

$$B\left(\sum_{1\leq p<n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pp}, a_{nn}\right) = 0,$$

$$B\left(\sum_{1\leq p\neq q\leq n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pq}, a_{nn}\right) \in \bigoplus_{1\leq p\neq q\leq n} \mathfrak{G}_{pq}$$

and

$$B\left(f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{nn}, a_{nn}\right) \in \mathfrak{G}_{nn},$$

then

$$\sum_{1\leq p\neq q\leq n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pq} = 0 \quad \text{by condition } (ii).$$

Next, we have

$$\begin{aligned} B\left(a_{nn}, f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right)\right) &= f\left(B\left(a_{nn}, \sum_{1\le i<n} x_{ii}\right), B\left(a_{nn}, \sum_{1\le j\ne k\le n} x_{jk}\right)\right) \\ &= f\left(0, B\left(a_{nn}, \sum_{1\le j\ne k\le n} x_{jk}\right)\right) \\ &= 0 \end{aligned}$$

which implies

$$\sum_{1\le p,q\le n} B\left(a_{nn}, f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right)_{pq}\right) = 0.$$

It follows that

$$B\left(a_{nn}, \sum_{1\le p<n} f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right)_{pp}\right) = 0,$$

$$B\left(a_{nn}, \sum_{1\le p\ne q\le n} f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right)_{pq}\right) \in \bigoplus_{1\le p\ne q\le n} \mathfrak{G}_{pq}$$

and

$$B\left(a_{nn}, f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right)_{nn}\right) \in \mathfrak{G}_{nn}.$$

Hence,

$$B\left(a_{nn}, f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right)_{nn}\right) = 0$$

which yields

$$f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right)_{nn} = 0$$

by condition (iii). Yet, we have

$$\begin{aligned} B\left(\sum_{1\le p<n} f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right)_{pp}, a_{rs}\right) &= B\left(f\left(\sum_{1\le i<n} x_{ii}, \sum_{1\le j\ne k\le n} x_{jk}\right), a_{rs}\right) \\ &= f\left(B\left(\sum_{1\le i<n} x_{ii}, a_{rs}\right), B\left(\sum_{1\le j\ne k\le n} x_{jk}, a_{rs}\right)\right) \\ &= f\left(B\left(\sum_{1\le i<n} x_{ii}, a_{rs}\right), 0\right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}
B\left(a_{rs}, \sum_{1\leq p<n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pp}\right) &= B\left(a_{rs}, f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)\right) \\
&= f\left(B\left(a_{rs}, \sum_{1\leq i<n} x_{ii}\right), B\left(a_{rs}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)\right) \\
&= f\left(B\left(a_{rs}, \sum_{1\leq i<n} x_{ii}\right), 0\right) \\
&= 0.
\end{aligned}$$

It follows the condition (iv) that $\sum_{1\leq p<n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pp} + 0 \in \mathfrak{Z}(\mathfrak{G})$ and so

$$\sum_{1\leq p<n} f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right)_{pp} = 0$$

by Proposition 1.2. Consequently, we have $f\left(\sum_{1\leq i<n} x_{ii}, \sum_{1\leq j\neq k\leq n} x_{jk}\right) = 0$.

Second case. In the second case it must be borne in mind that we want to show

$$f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right) = 0.$$

From the hypotheses of the theorem, we have

$$\begin{aligned}
B\left(\sum_{1\leq p,q\leq n} f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pq}, a_{rs}\right) &= B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{rs}\right) \\
&= f\left(B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{rs}\right), B\left(\sum_{1\leq k\neq l\leq n} y_{kl}, a_{rs}\right)\right) \\
&= f(0,0) \\
&= 0.
\end{aligned}$$

Since

$$B\left(\sum_{1\leq p\neq q\leq n} f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pq}, a_{rs}\right) = 0,$$

then

$$B\left(\sum_{1\leq p\leq n} f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pp}, a_{rs}\right) = 0.$$

Smilarly, we prove that

$$B\left(a_{rs}, \sum_{1\leq p\leq n} f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pp}\right) = 0.$$

By condition (iv), it follows that

$$\sum_{1\leq p<n} f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pp} + \left(-f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{nn}\right) \in \mathfrak{Z}(\mathfrak{G}). \quad (2.1)$$

Now, we observe that

$$\begin{aligned} B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right) &= f\left(B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right), B\left(\sum_{1\leq k\neq l\leq n} y_{kl}, a_{nn}\right)\right) \\ &= f\left(\sum_{1\leq i\neq j\leq n} B(x_{ij}, a_{nn}), \sum_{1\leq k\neq l\leq n} B(y_{kl}, a_{nn})\right). \end{aligned}$$

With (2.1), this implies that

$$\sum_{1\leq p<n} B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right)_{pp} + \left(-B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right)_{nn}\right) \in \mathfrak{Z}(\mathfrak{G}).$$

Since $B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right) \in \bigoplus_{1\leq p\neq q\leq n} \mathfrak{G}_{pq} \bigoplus \mathfrak{G}_{nn}$ then

$$\sum_{1\leq p<n} B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right)_{pp} = 0$$

which results in

$$eB\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right)_{nn} = 0$$

by Proposition 1.2. Hence $B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right) \in \bigoplus_{1\leq p\neq q\leq n} \mathfrak{G}_{pq}$.

It follows that

$$\begin{aligned}
&B\left(a_{rr}, B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right)\right)\\
&= B\left(a_{rr}, f\left(B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right), B\left(\sum_{1\leq k\neq l\leq n} y_{kl}, a_{nn}\right)\right)\right)\\
&= f\left(B\left(a_{rr}, B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right)\right), B\left(a_{rr}, B\left(\sum_{1\leq k\neq l\leq n} y_{kl}, a_{nn}\right)\right)\right)\\
&= f\left(B\left(a_{rr}, B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right)\right), B\left(B\left(a_{rr}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right)\right)\\
&= f\left(B\left(a_{rr}, a_{nn} + B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right)\right),\right.\\
&\qquad \left. B\left(B\left(a_{rr}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn} + B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right)\right)\right)\\
&= B\left(f\left(a_{rr}, B\left(a_{rr}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)\right), a_{nn} + B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right)\right)\\
&= B\left(0, a_{nn} + B\left(\sum_{1\leq i\neq j\leq n} x_{ij}, a_{nn}\right)\right)\\
&= 0
\end{aligned}$$

by first case, for all $1 \leq r < n$.

So $B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right), a_{nn}\right) = 0$, by condition (ii). It follows that

$$\sum_{1\leq p\leq n} B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pp}, a_{nn}\right) + \sum_{1\leq p\neq q\leq n} B\left(f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pq}, a_{nn}\right) = 0$$

which yields

$$B\left(\sum_{1\leq p\neq q\leq n} f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pq}, a_{nn}\right) = 0$$

and so

$$\sum_{1\leq p\neq q\leq n} f\left(\sum_{1\leq i\neq j\leq n} x_{ij}, \sum_{1\leq k\neq l\leq n} y_{kl}\right)_{pq} = 0 \quad \text{by condition } (ii).$$

Hence,

$$B\left(a_{nn}, f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)_{nn}\right) = B\left(a_{nn}, f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)\right)$$
$$= f\left(B\left(a_{nn}, \sum_{1\le i\ne j\le n} x_{ij}\right), B\left(a_{nn}, \sum_{1\le k\ne l\le n} y_{kl}\right)\right)$$

and by (2.1) above we have

$$\sum_{1\le p<n} B\left(a_{nn}, f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)_{nn}\right)_{pp} + \left(-B\left(a_{nn}, f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)_{nn}\right)_{nn}\right) \in \mathfrak{Z}(\mathfrak{G}).$$

Since

$$B\left(a_{nn}, f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)_{nn}\right) \in \mathfrak{G}_{nn}$$

then we have

$$\sum_{1\le p<n} B\left(a_{nn}, f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)_{nn}\right)_{pp} = 0$$

and so

$$B\left(a_{nn}, f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)_{nn}\right) = B\left(a_{nn}, f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)_{nn}\right)_{nn} = 0,$$

by Proposition 1.2. It follows that $f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)_{nn} = 0$, by condition (iii), which implies

$$\sum_{1\le p<n} f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right)_{pp} = 0,$$

by (2.1). Consequently, we have

$$f\left(\sum_{1\le i\ne j\le n} x_{ij}, \sum_{1\le k\ne l\le n} y_{kl}\right) = 0.$$

Third case. Here, in the third case, we are interested in checking

$$f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl}\right) = 0.$$

In view of second case, we observe that

$$\begin{aligned} & B\left(f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl}\right), a_{rs}\right) \\ = \quad & f\left(B\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq}, a_{rs}\right), B\left(\sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl}, a_{rs}\right)\right) \\ = \quad & f\left(\sum_{1\leq p<n} B(x_{pp}, a_{rs}), \sum_{1\leq k<n} B(u_{kk}, a_{rs})\right) \\ = \quad & 0. \end{aligned}$$

It follows that

$$\sum_{1\leq t\leq n} B\left(f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl}\right)_{tt}, a_{rs}\right) = 0.$$

Similarly, we have

$$\sum_{1\leq t\leq n} B\left(a_{rs}, f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl}\right)_{tt}\right) = 0.$$

It follows that

$$\begin{aligned} & \sum_{1\leq t<n} f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl}\right)_{tt} \\ & \qquad + \left(-f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl}\right)_{nn}\right) \in \mathfrak{Z}(\mathfrak{G}) \end{aligned}$$

by condition (iv). But

$$\begin{aligned} & B\left(f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl}\right), a_{nn}\right) \\ & = f\left(B\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq}, a_{nn}\right), B\left(\sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl}, a_{nn}\right)\right) \end{aligned}$$

$$= f\left(B\left(\sum_{1\le p\ne q\le n} x_{pq}, a_{nn}\right), B\left(\sum_{1\le k\ne l\le n} u_{kl}, a_{nn}\right)\right)$$
$$= f\left(\sum_{1\le p\ne q\le n} B\left(x_{pq}, a_{nn}\right), \sum_{1\le k\ne l\le n} B\left(u_{kl}, a_{nn}\right)\right)$$
$$= 0$$

by second case. As a result, we have

$$\sum_{1\le r\ne s\le n} f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)_{rs} = 0 \quad \text{by condition } (ii).$$

Hence from the second case

$$B\left(a_{nn}, f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)\right)$$
$$= f\left(B\left(a_{nn}, \sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}\right), B\left(a_{nn}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)\right)$$
$$= f\left(B\left(a_{nn}, \sum_{1\le p\ne q\le n} x_{pq}\right), B\left(a_{nn}, \sum_{1\le k\ne l\le n} u_{kl}\right)\right)$$
$$= f\left(\sum_{1\le p\ne q\le n} B\left(a_{nn}, x_{pq}\right), \sum_{1\le k\ne l\le n} B\left(a_{nn}, u_{kl}\right)\right)$$
$$= 0.$$

This implies

$$B\left(a_{nn}, f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)_{nn}\right) = 0.$$

Thus

$$f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)_{nn} = 0$$

implying

$$\sum_{1\le t<n} f\left(\sum_{1\le p<n} x_{pp} + \sum_{1\le p\ne q\le n} x_{pq}, \sum_{1\le k<n} u_{kk} + \sum_{1\le k\ne l\le n} u_{kl}\right)_{tt} = 0$$

by Proposition 1.2. Therefore,

$$f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl}\right) = 0.$$

Now we are interested in checking

$$f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq} + \sum_{1\leq q<n} x_{qq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl} + \sum_{1\leq l<n} u_{ll}\right) = 0.$$

In view of second case, we Observe that

$$\begin{aligned}
&B\left(f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq} + + \sum_{1\leq q<n} x_{qq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl} + \sum_{1\leq l<n} u_{ll}\right), a_{rs}\right)\\
&= f\left(B\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq} + \sum_{1\leq q<n} x_{qq}, a_{rs}\right), B\left(\sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl} + \sum_{1\leq l<n} u_{ll}, a_{rs}\right)\right)\\
&= 0.
\end{aligned}$$

It follows that

$$\sum_{1\leq t\leq n} B\left(f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq} + \sum_{1\leq q<n} x_{qq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl} + \sum_{1\leq l<n} u_{ll}\right)_{tt}, a_{rs}\right) = 0.$$

Similarly, we have

$$\sum_{1\leq t\leq n} B\left(a_{rs}, f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq} + \sum_{1\leq q<n} x_{qq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl} + \sum_{1\leq l<n} u_{ll}\right)_{tt}\right) = 0.$$

It follows that

$$\begin{aligned}
&\sum_{1\leq t<n} f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq} + \sum_{1\leq q<n} x_{qq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl} + \sum_{1\leq l<n} u_{ll}\right)_{tt}\\
&+\left(-f\left(\sum_{1\leq p<n} x_{pp} + \sum_{1\leq p\neq q\leq n} x_{pq} + \sum_{1\leq q<n} x_{qq}, \sum_{1\leq k<n} u_{kk} + \sum_{1\leq k\neq l\leq n} u_{kl} + \sum_{1\leq l<n} u_{ll}\right)_{nn}\right) \in \mathfrak{Z}(\mathfrak{G})
\end{aligned}$$

by condition (iv). But

$$
\begin{aligned}
&B\left(f\left(\sum_{1\le p<n} x_{pp}+\sum_{1\le p\neq q\le n} x_{pq}+\sum_{1\le q<n} x_{qq},\sum_{1\le k<n} u_{kk}+\sum_{1\le k\neq l\le n} u_{kl}+\sum_{1\le l<n} u_{ll}\right),a_{nn}\right)\\
&=f\left(B\left(\sum_{1\le p<n} x_{pp}+\sum_{1\le p\neq q\le n} x_{pq}+\sum_{1\le q<n} x_{qq},a_{nn}\right),B\left(\sum_{1\le k<n} u_{kk}+\sum_{1\le k\neq l\le n} u_{kl}+\sum_{1\le l<n} u_{ll},a_{nn}\right)\right)\\
&=0
\end{aligned}
$$

by second case. As a result, we have

$$\sum_{1\le r\neq s\le n} f\left(\sum_{1\le p<n} x_{pp}+\sum_{1\le p\neq q\le n} x_{pq},\sum_{1\le k<n} u_{kk}+\sum_{1\le k\neq l\le n} u_{kl}\right)_{rs}=0 \quad \text{by condition } (ii).$$

Hence from the second case

$$
\begin{aligned}
&B\left(a_{nn},f\left(\sum_{1\le p<n} x_{pp}+\sum_{1\le p\neq q\le n} x_{pq}+\sum_{1\le q<n} x_{qq},\sum_{1\le k<n} u_{kk}+\sum_{1\le k\neq l\le n} u_{kl}+\sum_{1\le l<n} u_{ll}\right)\right)\\
&=f\left(B\left(a_{nn},\sum_{1\le p<n} x_{pp}+\sum_{1\le p\neq q\le n} x_{pq}+\sum_{1\le q<n} x_{qq}\right),B\left(a_{nn},\sum_{1\le k<n} u_{kk}+\sum_{1\le k\neq l\le n} u_{kl}+\sum_{1\le l<n} u_{ll}\right)\right)\\
&=0.
\end{aligned}
$$

This implies

$$B\left(a_{nn},f\left(\sum_{1\le p<n} x_{pp}+\sum_{1\le p\neq q\le n} x_{pq}+\sum_{1\le q<n} x_{qq},\sum_{1\le k<n} u_{kk}+\sum_{1\le k\neq l\le n} u_{kl}+\sum_{1\le l<n} u_{ll}\right)_{nn}\right)=0.$$

Thus

$$f\left(\sum_{1\le p<n} x_{pp}+\sum_{1\le p\neq q\le n} x_{pq}+\sum_{1\le q<n} x_{qq},\sum_{1\le k<n} u_{kk}+\sum_{1\le k\neq l\le n} u_{kl}+\sum_{1\le l<n} u_{ll}\right)_{nn}=0$$

implying

$$\sum_{1\le t<n} f\left(\sum_{1\le p<n} x_{pp}+\sum_{1\le p\neq q\le n} x_{pq}+\sum_{1\le q<n} x_{qq},\sum_{1\le k<n} u_{kk}+\sum_{1\le k\neq l\le n} u_{kl}+\sum_{1\le l<n} u_{ll}\right)_{tt}=0$$

by Proposition 1.2. Therefore,

$$f\left(\sum_{1\le p<n} x_{pp}+\sum_{1\le p\neq q\le n} x_{pq}+\sum_{1\le q<n} x_{qq},\sum_{1\le k<n} u_{kk}+\sum_{1\le k\neq l\le n} u_{kl}+\sum_{1\le l<n} u_{ll}\right)=0.$$

Fourth case. Finally in the last case we show that $f = 0$.

Since $B\left(\sum_{1\le p,q\le n} x_{pq}, y_{rs}\right) \subseteq \mathfrak{G}_{rs}$ we have $B(f(x,u), a_{rs}) = f(B(x,a_{rs}), B(u,a_{rs})) = 0$. Then by second case, we obtain

$$B\left(\sum_{1\le p\le n} f(x,u)_{pp}, a_{rs}\right) = 0.$$

Similarly, we have

$$B\left(a_{rs}, \sum_{1\le p\le n} f(x,u)_{pp}\right) = 0.$$

It follows from condition (iv) that $\sum_{1\le p<n} f(x,u)_{pp} + (-f(x,u)_{nn}) \in \mathfrak{Z}(\mathfrak{G})$.

Now as $B\left(\sum_{1\le r<n} y_{rr}, y\right) \subseteq \sum_{1\le r<n} \mathfrak{G}_{rr} + \sum_{1\le r\neq s\le n} \mathfrak{G}_{rs}$ then by third case, we have

$$B\left(\sum_{1\le r<n} a_{rr}, f(x,u)\right) = f\left(B\left(\sum_{1\le r<n} a_{rr}, x\right), B\left(\sum_{1\le r<n} a_{rr}, u\right)\right) = 0.$$

It follows that $B\left(\sum_{1\le r<n} a_{rr}, \sum_{1\le r<n} f(x,u)_{rr} + \sum_{1\le r\neq s\le n} f(x,u)_{rs}\right) = 0$ implying

(1) $B\left(\sum_{1\le r<n} a_{rr}, \sum_{1\le r<n} f(x,u)_{rr}\right) = 0,$

(2) $B\left(\sum_{1\le r<n} a_{rr}, \sum_{1\le r\neq s\le n} f(x,u)_{rs}\right) = 0.$

By identity (1) above we have $\sum_{1\le r<n} B\big(a_{rr}, f(x,u)_{rr}\big) = 0$ resulting $B\big(a_{rr}, f(x,u)_{rr}\big) = 0$ for all $1 \le r < n$. We deduce

$$\begin{aligned} 0 &= B\big(a_{rr}, f(x,u)_{rr}\big)a_{rn}a_{nn} = B\big(f(x,u)_{rr}, a_{rr}\big)a_{rn}a_{nn} \\ &= a_{rr}a_{rn}B\big(-f(x,u)_{nn}, a_{nn}\big) = a_{rr}a_{rn}B\big(a_{nn}, -f(x,u)_{nn}\big) \end{aligned}$$

for all $r < n$, by condition (v). It follows that $B\big(a_{nn}, f(x,u)_{nn}\big) = 0$ which implies $f(x,u)_{nn} = 0$, by condition (iii). Thus, we have $\sum_{1\le p<n} f(x,u)_{pp} = 0$. Now, by identity (2), we have $\sum_{1\le r\neq s\le n} f(x,u)_{rs} = 0$ by condition (ii). Hence, we conclude that $f = 0$. □

As a consequence, we can apply our result to a particular case, *i.e.* the n-generalized matrix ring that satisfy the special conditions and $\mathfrak{G}_{pq}\mathfrak{G}_{qs} = 0$ as follows:

Corollary 2.2. *Let $\mathfrak{G}$ be a n-generalized matrix ring such that*

(i) for $a_{ii} \in \mathfrak{R}_i$, if $a_{ii}\mathfrak{R}_i = 0$, then $a_{ii} = 0$;

(ii) for $b_{jj} \in \mathfrak{R}_j$, if $\mathfrak{R}_j b_{jj} = 0$, then $b_{jj} = 0$.

Let k be a positive integer. If a map $f : \mathfrak{G} \times \mathfrak{G} \longrightarrow \mathfrak{G}$ satisfies

(i) $f(\mathfrak{G}, 0) = f(0, \mathfrak{G}) = 0$;

(ii) $f(x, y)z_1z_2 \cdots z_k = f(xz_1z_2 \cdots z_k, yz_1z_2 \cdots z_k)$;

(iii) $z_1z_2 \cdots z_k f(x, y) = f(z_1z_2 \cdots z_k x, z_1z_2 \cdots z_k y)$,

for all $x, y, z_1, z_2, \ldots, z_k \in \mathfrak{G}$, then $f = 0$.

Proof. We first claim that $f(x, y)z = f(xz, yz)$ and $zf(x, y) = f(zx, zy)$ for all $x, y, z \in \mathfrak{G}$. Indeed, since

$$f(x, y)(zz_1)z_2 \cdots z_k = f(xzz_1z_2 \cdots z_k, yzz_1z_2 \cdots z_k) = f(xz, yz)z_1z_2 \cdots z_k,$$

that is, $(f(x, y)z - f(xz, yz))\mathfrak{G}^k = 0$. Hence $f(x, y)z = f(xz, yz)$ by Proposition 1.3. Analogously, $zf(x, y) = f(zx, zy)$. Define $B : \mathfrak{G} \times \mathfrak{G} \longrightarrow \mathfrak{G}$ by $B(x, y) = xy$. It is easy to check that B and f satisfy the all conditions of Theorem 2.1. Hence $f = 0$. □

3 Applications

In this section we apply our main result to the case of n-generalized matrix ring satisfying the special conditions and $\mathfrak{G}_{pq}\mathfrak{G}_{qs} = 0$.

Theorem 3.1. *Let $\mathfrak{G}$ be a n-generalized matrix ring such that*

(i) For $a_{ii} \in \mathfrak{R}_i$, if $a_{ii}\mathfrak{R}_i = 0$, then $a_{ii} = 0$;

(ii) For $b_{jj} \in \mathfrak{R}_j$, if $\mathfrak{R}_j b_{jj} = 0$, then $b_{jj} = 0$.

Then every m-multiplicative isomorphism from $\mathfrak{G}$ onto a ring $\mathfrak{R}$ is additive.

Proof. Suppose that φ is a m-multiplicative isomorphism from $\mathfrak{G}$ onto a ring $\mathfrak{R}$. Since φ is onto, $\varphi(x) = 0$ for some $x \in \mathfrak{G}$. Then $\varphi(0) = \varphi(0 \cdots 0x) = \varphi(0) \cdots \varphi(0)\varphi(x) = \varphi(0) \cdots \varphi(0)0 = 0$ and so $\varphi^{-1}(0) = 0$. Let us check that the conditions of the Corollary 2.2 are satisfied. For every $x, y \in \mathfrak{G}$ we define $f(x, y) = \varphi^{-1}(\varphi(x + y) - \varphi(x) - \varphi(y))$, we see that $f(x, 0) = f(0, x) = 0$ for all $x \in \mathfrak{G}$. It is easy to check that φ^{-1} is also a m-multiplicative isomorphism. Thus, for any $u_1, \ldots, u_{m-1} \in \mathfrak{G}$, we have

$$\begin{aligned} f(x,y)u_1 \cdots u_{m-1} &= \varphi^{-1}(\varphi(x+y) - \varphi(x) - \varphi(y))\varphi^{-1}(\varphi(u_1)) \cdots \varphi^{-1}(\varphi(u_{m-1})) \\ &= \varphi^{-1}((\varphi(x+y) - \varphi(x) - \varphi(y))\varphi(u_1) \cdots \varphi(u_{m-1})) \\ &= f(xu_1 \cdots u_{m-1}, yu_1 \cdots u_{m-1}). \end{aligned}$$

Similarly we have $u_1 \cdots u_{m-1}f(x,y) = f(u_1 \cdots u_{m-1}x, u_1 \cdots u_{m-1}y)$. Therefore by Corollary 2.2, $f = 0$. That is, $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x, y \in \mathfrak{G}$. □

Theorem 3.2. *Let $\mathfrak{G}$ be a n-generalized matrix ring such that*

(i) For $a_{ii} \in \mathfrak{R}_i$, if $a_{ii}\mathfrak{R}_i = 0$, then $a_{ii} = 0$;

(ii) For $b_{jj} \in \mathfrak{R}_j$, if $\mathfrak{R}_j b_{jj} = 0$, then $b_{jj} = 0$.

Then any m-multiplicative derivation d of $\mathfrak{G}$ is additive.

Proof. We define $f(x,y) = d(x+y) - d(x) - d(y)$, for any $x, y \in \mathfrak{G}$. Hence f defined in this way satisfy the conditions of Corollary 2.2. Therefore $f = 0$ and so $d(x+y) = d(x) + d(y)$. □

It is worth noting that the technique used to prove the main result of this article is still not enough to answer the result obtained in Corollary 2.2, without the $\mathfrak{G}_{pq}\mathfrak{G}_{qs} = 0$ condition.

We therefore end our work with two open questions:

(a) When are m-multiplicative isomorphism additive?

(b) When are m-multiplicative derivation additive?

References

[1] X. Cheng and W. Jing, "Additivity of maps on triangular algebras," *Electron. J. Linear Algebra*, vol. 17, pp. 597–615, 2008, doi: 10.13001/1081-3810.1285.

[2] M. N. Daif, "When is a multiplicative derivation additive?" *Internat. J. Math. Math. Sci.*, vol. 14, no. 3, pp. 615–618, 1991, doi: 10.1155/S0161171291000844.

[3] B. L. M. Ferreira, "Multiplicative maps on triangular n-matrix rings," *Internat. J. Math., Game Theory and Algebra*, vol. 23, no. 2, pp. 1–14, 2014.

[4] Y. Li and Z. Xiao, "Additivity of maps on generalized matrix algebras," *Electron. J. Linear Algebra*, vol. 22, pp. 743–757, 2011, doi: 10.13001/1081-3810.1471.

[5] F. Y. Lu and J. H. Xie, "Multiplicative mappings of rings," *Acta Math. Sin. (Engl. Ser.)*, vol. 22, no. 4, pp. 1017–1020, 2006, doi: 10.1007/s10114-005-0620-7.

[6] F. Lu, "Multiplicative mappings of operator algebras," *Linear Algebra Appl.*, vol. 347, pp. 283–291, 2002, doi: 10.1016/S0024-3795(01)00560-2.

[7] W. S. Martindale, III, "When are multiplicative mappings additive?" *Proc. Amer. Math. Soc.*, vol. 21, pp. 695–698, 1969, doi: 10.2307/2036449.

[8] G. Tang and Y. Zhou, "A class of formal matrix rings," *Linear Algebra Appl.*, vol. 438, no. 12, pp. 4672–4688, 2013, doi: 10.1016/j.laa.2013.02.019.

[9] Y. Wang, "The additivity of multiplicative maps on rings," *Comm. Algebra*, vol. 37, no. 7, pp. 2351–2356, 2009, doi: 10.1080/00927870802623369.

[10] Y. Wang, "Additivity of multiplicative maps on triangular rings," *Linear Algebra Appl.*, vol. 434, no. 3, pp. 625–635, 2011, doi: 10.1016/j.laa.2010.09.015.

CUBO, A Mathematical Journal
Vol. 26, no. 01, pp. 53–73, April 2024
DOI: 10.56754/0719-0646.2601.053

On a class of fractional $\Gamma(.)$-Kirchhoff-Schrödinger system type

Hamza El-Houari[1,✉]
Lalla Saádia Chadli[1]
Hicham Moussa[1]

[1] *Faculty of Sciences and Techniques, Research Laboratory "Applied Mathematics and Scientific Computing", Beni Mellal, Morocco.*
h.elhouari94@gmail.com
sa.chadli@yahoo.fr
hichammoussa23@gmail.com

ABSTRACT

This paper focuses on the investigation of a Kirchhoff-Schrödinger type elliptic system involving a fractional $\gamma(.)$-Laplacian operator. The primary objective is to establish the existence of weak solutions for this system within the framework of fractional Orlicz-Sobolev Spaces. To achieve this, we employ the critical point approach and direct variational principle, which allow us to demonstrate the existence of such solutions. The utilization of fractional Orlicz-Sobolev spaces is essential for handling the nonlinearity of the problem, making it a powerful tool for the analysis. The results presented herein contribute to a deeper understanding of the behavior of this type of elliptic system and provide a foundation for further research in related areas.

RESUMEN

Este artículo se enfoca en la investigación de sistemas elípticos de tipo Kirchhoff-Schrödinger que involucran un operador fraccionario $\gamma(.)$-Laplaciano. El objetivo principal es establecer la existencia de soluciones débiles para este sistema en el marco de espacios de Orlicz-Sobolev fraccionarios. Para lograrlo, empleamos el enfoque de punto crítico y el principio variacional directo, que nos permiten demostrar la existencia de dichas soluciones. El uso de espacios de Orlicz-Sobolev fraccionarios es esencial para lidiar con la nolinealidad del problema, convirtiéndolo en una herramienta poderosa para el análisis. Los resultados presentados contribuyen a una comprensión más profunda del comportamiento de este tipo de sistemas elípticos y entregan una base para investigación futura en áreas relacionadas.

Keywords and Phrases: Fractional Orlicz-Sobolev spaces, Kirchhoff-Schrödinger system, Critical point theorem.

2020 AMS Mathematics Subject Classification: 35J50, 35J67, 35S15

Published: 04 April, 2024
Accepted: 12 January, 2024
Received: 07 August, 2023

1 Introduction

The objective of this paper is to establish the existence of weak solutions for a non-local elliptic systems, as described below:

$$\begin{cases} K_1\big[\mathcal{F}_1(u)+\Upsilon_1(u)\big]\Big((-\Delta)^s_{\gamma_1}u+a_1(x)\gamma_1(u)u\Big)=F_u(x,u,v) & \text{in } \Omega,\\ K_2\big[\mathcal{F}_2(v)+\Upsilon_2(v)\big]\Big((-\Delta)^s_{\gamma_2}v+a_2(x)\gamma_2(v)v\Big)=F_v(x,u,v) & \text{in } \Omega,\\ u=v=0 & \text{on } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $N\geq 2$, $s\in(0,1)$, $\mathcal{F}_{i=1,2},\Upsilon_{i=1,2}:E_i\to\mathbb{R}$ are two functionals, respectively defined by

$$\mathcal{F}_i(w)=\int_{\Omega^2}\Gamma_i\Big(\frac{|w(x)-w(y)|}{|x-y|^s}\Big)\frac{dx\,dy}{|x-y|^N},\quad \Upsilon_i(w)=\int_\Omega a_i(x)\Gamma_i(|w|)dx,$$

and $K_{i=1,2}$ are two bounded continuous Kirchhoff functions, F belongs to $C^1(\Omega\times\mathbb{R}^2)$ and satisfies certain suitable growth assumptions, and F_u (respectively, F_v) is the partial derivative of F with respect to u (respectively, v). Additionally, a_i with $i=1,2$, are two continuous functions that satisfy the following conditions:

(A_1): $a_i\in C(\Omega,\mathbb{R})$ and $\inf\limits_{x\in\Omega}a_i(x)\geq a_0>0$.

(A_2): $meas(x\in\Omega:a_i(x)\leq H)<\infty$, for all $H>0$, where $meas(.)$ denotes the Lebesgue measure in Ω.

The stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\right)\frac{\partial^2u}{\partial x^2}=0, \tag{1.2}$$

presented by Kirchhoff [19] in 1883. Later (1.2) was developed to form

$$u_{tt}-K\left(\int_\Omega|\nabla u|^2dx\right)\Delta u=f(x,u),\quad x\in\Omega. \tag{1.3}$$

After that, many authors studied the following nonlocal elliptic boundary value problem

$$-K\left(\int_\Omega|\nabla u|^2dx\right)\Delta u=f(x,u),\quad x\in\Omega. \tag{1.4}$$

In recent years, considerable research attention has been dedicated to investigating the existence of solutions for elliptic problems within the fractional Sobolev space. This growing interest is evident in the works of various researchers, such as those referenced in [8,9,18,24]. In a similar vein, Azroul *et al.* explored the existence of a solution for the following fractional (p,q)-Schrödinger-Kirchhoff system type, as documented in [2].

$$\begin{cases} K_1\big[I_{M_p}\big]((-\Delta)^s_p u + a(x)|u|^{p-2}u) = \lambda F_u(x,u,v) + \mu G_u(x,u,v) & \text{in } \mathbb{R}^N, \\ K_2\big[I_{M_q}\big]((-\Delta)^s_q v + a(x)|v|^{q-2}v) = \lambda F_v(x,u,v) + \mu G_v(x,u,v) & \text{in } \mathbb{R}^N, \\ (u,v) \in W^p \times W^q, \end{cases} \tag{1.5}$$

where

$$I_{M_r}(w) = \int_{\mathbb{R}^N\times\mathbb{R}^N} |w(x)-w(y)|^r M_r(x-y)\,dx\,dy + \int_{\mathbb{R}^N} a(x)|w|^r dx,$$

when we take $M_r(x) = |x|^{-N-sr}$. In this case, problem (1.5) become

$$\begin{cases} K_1\big[I^s_r(u)\big]((-\Delta)^s_p u + |u|^{p-2}u) = \lambda F_u(x,u,v) + \mu G_u(x,u,v) & \text{in } \mathbb{R}^N, \\ K_2\big[I^s_r(v)\big]((-\Delta)^s_q v + |v|^{q-2}v) = \lambda F_v(x,u,v) + \mu G_v(x,u,v) & \text{in } \mathbb{R}^N, \\ u = v = 0 & \text{on } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.6}$$

where

$$I^s_r(w) = \int_{\Omega^2} \frac{|w(x)-w(y)|^r}{|x-y|^{sr+N}}dx\,dy + \int_{\mathbb{R}^N} a(x)|w|^r dx.$$

In 2017, Bonder *et al.* in [17] made a significant advancement by introducing an extension of the fractional Sobolev space, known as the fractional Orlicz-Sobolev space. This extension involved the generalization of the conventional fractional Laplacian operator to the fractional $\gamma(.)$-Laplacian operator, which is defined as follows:

$$(-\Delta)^s_{\gamma(.)}u(x) = p.v. \int_{\mathbb{R}^N} \gamma\bigg(\frac{|u(x)-u(y)|}{|x-y|^s}\bigg)\frac{u(x)-u(y)}{|x-y|^{s+N}}dy, \quad \text{for all} \quad x \in \mathbb{R}^N, \tag{1.7}$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing and right continuous function, with

$$\gamma(0) = 0, \quad \gamma(t) > 0 \quad \text{for } t > 0 \quad \text{and} \quad \lim_{t\to\infty}\gamma(t) = \infty. \tag{1.8}$$

The replacement of the $\gamma(.)$-Laplace operator with a fractional $\gamma(.)$-Laplacian operator raises the question of what results can be achieved. Currently, there are only a few results available regarding the fractional Orlicz-Sobolev spaces. For instance, in [9], we studied a nonlocal Kirchhoff type problem within this space.

$$\begin{cases} K_1\big[\mathcal{F}_1(u)\big](-\Delta)^s_{\gamma_1}u = F_u(x,u,v) & \text{in } \Omega, \\ K_2\big[\mathcal{F}_2(v)\big](-\Delta)^s_{\gamma_2}v = F_u(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^N\backslash\Omega, \end{cases}$$

where K_i is the Kirchhoff function. In our problem (1.1), the function F is presumed to be a member of $C^1(\Omega \times \mathbb{R}^2)$ and complies with appropriate growth conditions, but notably does not satisfy the well-known Ambrosetti-Rabinowitz condition. For further problems related to the fractional Orlicz-Sobolev spaces, we refer to [6, 7, 10–16]. By setting $\Gamma_i(t) = \frac{|t|^r}{r}$, our problem (1.1) can be reduced to the fractional (p,q)-Schrödinger-Kirchhoff elliptic system given in (1.6). In this paper, preceding works of the Kirchhoff-Schrödinger system are extended in fractional Orlicz-Sobolev spaces.

This article is divided into four sections. In the second section, we offer a brief review of the fractional Orlicz-Sobolev spaces, outlining their essential properties and results. Following that, the third section presents the specific assumptions made on the data. In the fourth section, we present our primary result concerning the existence of a weak solution and its proof, which relies on a contradiction argument.

2 Some preliminary results and hypotheses

In this section, we will briefly introduce the definitions and fundamental properties of FOSS. For detailed information and proofs, interested readers can refer to [1, 17, 20].

We take notice of $\mathbf{N}$ the set of all N-functions. The function $\Gamma \in \mathbf{N}$ is defined for $z \in \mathbb{R}$ by setting $\Gamma(z) = \displaystyle\int_0^{|z|} t\gamma(t)dt.$

We point out that $\Gamma \in \Delta_2$ if for a certain constant $k > 0$,

$$\Gamma(2z) \leq k\,\Gamma(z), \quad \text{for every } z > 0. \tag{2.1}$$

We observe that Γ and $\overline{\Gamma}$ satisfy the following Young's inequality:

$$rz \leq \Gamma(r) + \overline{\Gamma}(z) \quad \text{for all } z,\, r \geq 0 \text{ and } x \in \Omega. \tag{2.2}$$

In the Orlicz space $L_\Gamma(\Omega)$ is well-know, the Hölder inequality

$$\int_\Omega |u(z)v(z)|\,\mathrm{d}z \leq ||u||_\Gamma ||v||_{(\overline{\Gamma})} \quad \text{for all } u \in L_\Gamma(\Omega) \text{ and } v \in L_{\overline{\Gamma}}(\Omega), \tag{2.3}$$

where $L_\Gamma(\Omega)$ is defined as the set of equivalence classes of measurable functions $u : \Omega \to \mathbb{R}$ such that:

$$\int_\Omega \Gamma\Big(\frac{u(z)}{\tau}\Big)\,\mathrm{d}z < +\infty \quad \text{for certain } \tau > 0.$$

where $||.||_{(\Gamma)}$ is the Orlicz norm defined by

$$||u||_{(\Gamma)} := \sup_{||v||_{\overline{\Gamma} \leq 1}} \int_\Omega u(z)v(z)\,\mathrm{d}z.$$

$L_\Gamma(\Omega)$ is a Banach space under the following norm,

$$\|u\|_\Gamma = \inf\Big\{\lambda > 0 \,/\, \int_\Omega \Gamma\Big(\frac{|u(x)|}{\lambda}\Big)\,\mathrm{d}x \le 1\Big\}.$$

We assume that:

$$(A_0)\ \int_0^1 \frac{\Gamma^{-1}(t)}{t^{1+\frac{s}{N}}}\,\mathrm{d}t < \infty \quad \text{and} \quad (A_\infty) \int_1^{+\infty} \frac{\Gamma^{-1}(t)}{t^{1+\frac{s}{N}}}\,\mathrm{d}t = +\infty \quad \text{for } s \in (0,1).$$

Under the hypotheses (A_0) and (A_∞), we can insert an N-function Γ^*, given by the following expression of its inverse in $\mathbb{R}^+$:

$$(\Gamma^*)^{-1}(t) = \int_0^t \frac{\Gamma^{-1}(r)}{r^{\frac{N+s}{N}}}dr \quad \text{for } t \ge 0. \tag{2.4}$$

The fact that $\Gamma \in \Delta_2$-condition globally implies that:

$$u_k \to u \quad \text{in} \quad L_\Gamma(\Omega) \iff \int_\Omega \Gamma(|u_k - u|)dx \to 0. \tag{2.5}$$

Now we set an useful lemma which we need in the proof.

Lemma 2.1 ([4]). *Let $\overline{\Gamma}$ be the complementary of the N-functions Γ. Then we have*

$$\overline{\Gamma}(\gamma(t)) \le (n-1)\Gamma(t), \quad \textit{for all } t > 0, \tag{2.6}$$

where $n = \sup_{t>0} \frac{t^2\gamma(t)}{\Gamma(t)}$.

We define the fractional Orlicz-Sobolev spaces as follows

$$W^{s,\Gamma}(\Omega) = \Big\{u \in L_\Gamma(\Omega) : \int_\Omega\int_\Omega \Gamma\Big(\frac{\lambda|u(x)-u(y)|}{|x-y|^s}\Big)|x-y|^{-N}dx\,dy < \infty \quad \text{for some } \lambda > 0\Big\}.$$

This space is equipped with the norm,

$$||u||_{s,\Gamma} = ||u||_\Gamma + [u]_{s,\Gamma}, \tag{2.7}$$

where $[.]_{s,\Gamma}$ defined by

$$[u]_{s,\Gamma} = \inf\Big\{\lambda > 0 : \int_\Omega\int_\Omega \Gamma\Big(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\Big)|x-y|^{-N}dx\,dy \le 1\Big\}.$$

To deal with this problem, we choose

$$W_0^{s,\Gamma}(\Omega) = \big\{u \in W^{s,\Gamma}(\mathbb{R}^N) : u = 0 \quad \text{a.e.} \quad \mathbb{R}^N \setminus \Omega\big\},$$

which can be equivalently renormed by setting $||.|| = [.]_{s,\Gamma}$ and

$$E_i = \Big\{u \in W^{s,\Gamma_i}(\mathbb{R}^N) : \int_\Omega a_i(x)\Gamma_i(|u|)dx < \infty; \quad u = 0 \quad \text{a.e.} \quad \mathbb{R}^N \setminus \Omega\Big\},$$

equipped with the following norm $||.||_{E_i,\Gamma_i} = [.]_{s,\Gamma_i} + ||.||_{a_i,\Gamma_i}$, where

$$||u||_{a_i,\Gamma_i} = \inf\Big\{\lambda > 0, \int_\Omega a_i(x)\Gamma_i\Big(\frac{|u(x)|}{\lambda}\Big)dx \leq 1\Big\}.$$

Throughout this paper Ω is a bounded open subset of $\mathbb{R}^N$ and $s \in (0,1)$.

In $W_0^{s,\Gamma}(\Omega)$ we have the following Poincaré inequality

$$||u||_\Gamma \leq \tau[u]_{s,\Gamma}, \quad \forall u \in W_0^{s,\Gamma}(\Omega). \tag{2.8}$$

where τ is a positive constant.

Remark 2.2. $[.]_{s,\Gamma}$ *is a norm of* $W_0^{s,\Gamma}(\Omega)$ *equivalent to* $||.||_{s,\Gamma}$.

Lemma 2.3 ([7])**.** *The representation given by*

$$\Gamma_{i=1;2}(t) := \int_0^{|t|} r\gamma_i(r)dr \quad \textit{for all } t \in \mathbb{R}, \tag{2.9}$$

exists and it is an N*-function where* $\gamma_{i=1;2}$ *verified* (1.8).

3 Hypotheses

We use through our paper that $\Gamma_i \in \mathbf{N}$ defined in (2.9) and we suppose that $\Gamma_i \in \Delta_2$. Then by lemma 2.1 in [23] we have for all $t > 0$ that

$$1 < l_i := \inf_{t>0} \frac{t^2\gamma_i(t)}{\Gamma_i(t)} \leq \sup_{t>0} \frac{t^2\gamma_i(t)}{\Gamma_i(t)} := n_i < N. \tag{3.1}$$

Related to functions Γ_i, K_i and F our hypotheses are the following:

(ϕ_1)**:** The function $t \to \Gamma_i(\sqrt{t})$ where $t \in [0, +\infty)$ is convex.

(ϕ_2)**:** There exists $1 < \eta_i < l_i$, such that

$$\lim_{t\to+\infty} \frac{|t|^{\eta_i}}{\Gamma_i(t)} = 0,$$

and the Kirchhoff function $K_i : [0,\infty) \to (0,\infty)$ is a nondecreasing continuous function such that:

(A_3): There exist $\alpha_1, \alpha_2 > 0$ such that:

$$\alpha_2 \geq K_i(t) \geq \alpha_1 \quad \text{for all } t \in [0, \infty).$$

And F satisfies:

(F_1): $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $F(x, 0, 0) = 0$ for all $x \in \Omega$

$$\begin{cases} |F_u(x,u,v)| \leq c_1|u|^{r_1-1} + c_2|v|^{\frac{r_2(r_1-1)}{r_1}}, \\ |F_v(x,u,v)| \leq c_1|u|^{\frac{r_1(r_2-1)}{r_2}} + c_2|v|^{r_2-1}, \end{cases} \tag{3.2}$$

where $r_i \in (1, l_i)$.

(F_2): There exist an open set $\Omega \subset \mathbb{R}^N$ with $|\Omega| > 0$, and positive constants $\alpha_0 \in [1, l_1)$, $\beta_0 \in [1, l_2)$, $c > 0$ and $\rho, \sigma \in \mathbb{R}$ with $\rho + \sigma \neq 0$ such that

$$F(x, \rho t, \sigma t) \geq c(|\rho t|^{\alpha_0} + |\sigma t|^{\beta_0}), \quad \text{for all } (x,t) \in \Omega \times [0, 1).$$

Remark 3.1 ([17, Proposition 2.11])**.** *$W_0^{s,\Gamma}(\Omega)$ is a separable and reflexive Banach space.*

Lemma 3.2 ([5, Lemma 4.3])**.** *The following properties hold true:*

1) $\mathcal{F}_i\left(\frac{u}{[u]_{s,\Gamma_i}}\right) \leq 1$, *for all* $u \in E_i \setminus \{0\}$.

2) $\zeta_0([u]_{s,\Gamma_i}) \leq \mathcal{F}_i(u) \leq \zeta_1([u]_{s,\Gamma_i})$, *for all* $u \in E_i$.

3) $\zeta_0(||u||_{a_i,\Gamma_i}) \leq \Upsilon_i(u) \leq \zeta_1(||u||_{a_i,\Gamma_i})$, *for all* $u \in E_i$.

Lemma 3.3 ([5, Lemma 4.7])**.** *$\mathcal{F}_i$ and Υ_i are two weak lower semi-continuous functions.*

Lemma 3.4 ([7, Lemma 3.3])**.** *Under assumption (ϕ_1) we have that $(E, ||.||_{E_i})$ is a real uniformly convex Banach space.*

Now we state the embedding compactness result.

Theorem 3.5 ([5, Theorem 1.2.])**.** *Let Γ be an N-function.*

i) If (A_0), (A_∞) and (3.1) *hold, then the embedding $W^{s,\Gamma_i}(\Omega) \hookrightarrow L_{\Gamma_i^*}(\Omega)$ is continuous, and the embedding $W^{s,\Gamma_i}(\Omega) \hookrightarrow L_\Phi(\Omega)$ is compact for any N-function $\Phi \ll \Gamma_i$.*

ii) If (A_1), (A_2) and (3.1) *hold, then the embedding $E_i \hookrightarrow L_{\Gamma_i}(\Omega)$ is continuous, and the embedding $E \hookrightarrow L_\Phi(\Omega)$ is compact for any N-function $\Phi \ll \Gamma_i$.*

Remark 3.6. *The assumption (ϕ_2) implies that $|t|^{\eta_i} \ll \Gamma_i$, then by Theorem 3.5 the following embeddings $E_i \to L^{\eta_i}(\Omega)$ are compact, i.e., there exist constants $C_{\eta_i} > 0$ such that*

$$||u||_{\eta_i} \leq C_{\eta_i} ||u||_{E_i,\Gamma_i} \quad \textit{for all } u \in E_i. \tag{3.3}$$

4 Main results

In this section, we present the existence result.

Theorem 4.1. *Assume that* (A_1)-(A_3), (F_1)-(F_3), (3.1) *and* (ϕ_2) *hold true. Then system* (1.1) *possesses a nontrivial weak solution.*

In order to prove Theorem 4.1, we will use the following Lemma:

Lemma 4.2 ([22])**.** *Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$ satisfies (PS)-condition. If J is bounded from below, then $c = \inf_X J$ is a critical value of J.*

In fact, since

$$F(x,u,v) = \int_0^u F_p(x,p,v)dp + \int_0^v F_t(x,0,t)dt + F(x,0,0), \quad \forall (x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}.$$

By (3.2) and the fact that $F(x,0,0) = 0$, we show that:

$$\begin{aligned} |F(x,u,v)| \leq & \int_0^{|u|} |F_p(x,p,v)|dp + \int_0^{|v|} |F_t(x,0,t)|dt \\ \leq & \; c_1 \int_0^{|u|} |p|^{r_1-1}dp + c_2 \int_0^{|u|} |v|^{\frac{r_2(r_1-1)}{r_1}} dp + c_2 \int_0^{|v|} |t|^{r_2-1}dt \\ = & \; \frac{c_1}{r_1}|u|^{r_1} + c_2|u||v|^{\frac{r_2(r_1-1)}{r_1}} + \frac{c_2}{r_2}|v|^{r_2} \\ \leq & \; \frac{c_1}{r_1}|u|^{r_1} + \frac{c_2}{r_1}|u|^{r_1} + c_2\frac{r_1-1}{r_1}|v|^{r_2} + \frac{c_2}{r_2}|v|^{r_2} \\ = & \; c_3|u|^{r_1} + c_4|v|^{r_2}, \end{aligned} \tag{4.1}$$

where $c_3 = \dfrac{c_1+c_2}{r_1}$ and $c_4 = \dfrac{c_2 r_2(r_1-1)+r_1}{r_1 r_2}$.

Now we have all tools to study our problem (1.1). For that we shall define our working space $W := E_1 \times E_2$ with the norm

$$||(u,v)|| := ||u||_{E_1,\Gamma_1} + ||v||_{E_2,\Gamma_2}.$$

We can show that W is a separable and reflexive Banach space. We observe that the energy functional I on W corresponding to system (1.1) is

$$I(u,v) := \tilde{K}_1\Big[\mathcal{F}_1(u) + \Upsilon_1(u)\Big] + \tilde{K}_2\Big[\mathcal{F}_2(v) + \Upsilon_2(v)\Big] - \int_\Omega F(x,u,v)dx, \quad \forall (u,v) \in W.$$

Where

$$\tilde{K}(t) := \int_0^t K(\tau)d\tau.$$

Denote by $I_i : W \to \mathbb{R}$, $i = 1, 2$, the functionals $I_1(u, v) = (\tilde{K}o\mathcal{H})_1(u) + (\tilde{K}o\mathcal{H})_2(v)$ where

$$(\tilde{K}o\mathcal{H})_i(w) := \tilde{K}_i\Big[\int_{\Omega\times\Omega}\Gamma_i\Big(\frac{w(x)-w(y)}{|x-y|^s}\Big)\frac{dx\,dy}{|x-y|^N} + \int_\Omega a_i(x)\Gamma_i(w)dx\Big]$$

and

$$I_2(u, v) = \int_\Omega F(x, u, v)dx.$$

Then

$$I(u, v) = I_1(u, v) - I_2(u, v).$$

Lemma 4.3. *The function I is well define and it is $C^1(E_i, \mathbb{R})$ and we have*

$$\begin{aligned}\langle I'(u,v),(\overline{u},\overline{v})\rangle &= K_1\Big[\mathcal{F}_1(u)+\Upsilon_1(u)\Big]\Big(\int_{\Omega\times\Omega}\gamma_1(h_u)h_uh_{\overline{u}}\,d\mu+\int_\Omega a_1(x)\gamma_1(u)u\overline{u}\,dx\Big)\\ &+K_2\Big[\mathcal{F}_2(v)+\Upsilon_2(v)\Big]\Big(\int_{\Omega\times\Omega}\gamma_2(h_v)h_vh_{\overline{v}}\,d\mu+\int_\Omega a_2(x)\gamma_2(v)v\overline{v}\,dx\Big)\\ &-\int_\Omega(F_u(x,u,v)\overline{u}+F_v(x,u,v)\overline{v})dx,\end{aligned}$$

for all $\overline{u}, \overline{v} \in E_i$, where $h_u = \dfrac{u(x)-u(y)}{|x-y|^s}$ and $d\mu = \dfrac{dx\,dy}{|x-y|^N}$ (i.e, regular Borel measure on the set $\Omega \times \Omega$).

Lemma 4.4. *The function I is well define and it is $C^1(E_i, \mathbb{R})$ and we have*

$$\begin{aligned}\langle I'(u,v),(\overline{u},\overline{v})\rangle &= K_1\Big[\mathcal{F}_1(u)+\Upsilon_1(u)\Big]\Big(\int_{\Omega\times\Omega}\gamma_1(h_u)h_uh_{\overline{u}}\,d\mu+\int_\Omega a_1(x)\gamma_1(u)u\overline{u}\,dx\Big)\\ &+K_2\Big[\mathcal{F}_2(v)+\Upsilon_2(v)\Big]\Big(\int_{\Omega\times\Omega}\gamma_2(h_v)h_vh_{\overline{v}}\,d\mu+\int_\Omega a_2(x)\gamma_2(v)v\overline{v}\,dx\Big)\\ &-\int_\Omega(F_u(x,u,v)\overline{u}+F_v(x,u,v)\overline{v})dx,\end{aligned}$$

for all $\overline{u}, \overline{v} \in E_i$, where $h_u = \dfrac{u(x)-u(y)}{|x-y|^s}$ and $d\mu = \dfrac{dx\,dy}{|x-y|^N}$ (i.e, regular Borel measure on the set $\Omega \times \Omega$).

Proof. First, we can see that

$$\begin{aligned}\langle(\tilde{K}o\mathcal{H})_i'(u),v\rangle &= K_i\Big[\int_{\Omega\times\Omega}\Gamma_i(h_u)d\mu+\int_\Omega a_i(x)\Gamma_i(u)dx\Big]\\ &\times\Big(\int_{\Omega\times\Omega}\gamma_i(h_u)h_uh_v\,d\mu+\int_\Omega a_i(x)\gamma_i(u)uv\,dx\Big),\end{aligned}\tag{4.2}$$

for all $\overline{u}, \overline{v} \in E_i$. It follows from (4.2) that for each $u \in E_i$, $(\tilde{K}o\mathcal{H})_i'(u) \in (E_i)^*$.

Next we prove that $(\tilde{K}o\mathcal{H})_i \in C^1(E_i, \mathbb{R})$. Let $\{u_k\} \subset E_i$ with $u_k \to u$ strongly in E_i, for $v \in E_i$

we have $h_v \in L_{\Gamma_i}(\Omega\times\Omega, d\mu)$ and by Hölder inequality

$$\begin{aligned}&\left|\int_{\Omega\times\Omega}(\gamma_i(h_{u_k})h_{u_k}-\gamma_i(h_u)h_u)h_v\,d\mu+\int_\Omega(a_i(x)\gamma_i(u_k)u_k-a_i(x)\gamma_i(u)u)v\right|\\&\le 2\|\gamma_i(h_{u_k})h_{u_k}-\gamma_i(h_u)h_u\|_{L_{\overline{\Gamma}_i}}||h_v||_{L_{\Gamma_i}}+2||a||_\infty\|\gamma_i(u_k)u_k-\gamma_i(u)u\|_{L_{\overline{\Gamma}_i}}||v||_{L_{\Gamma_i}}.\end{aligned} \tag{4.3}$$

On the other hand, $u_k\to u$ in E_i, then $h_{u_k}\to h_u$ in $L_{\Gamma_i}(\Omega\times\Omega)$, so by dominated convergence theorem, there exists a subsequence $\{h_{u_{n_k}}\}$ and a function h in $L_{\Gamma_i}(\Omega\times\Omega)$ such that

$$|\gamma_i(h_{u_{n_k}})h_{u_{n_k}}|\le|\gamma_i(h)h|\in L_{\overline{\Gamma}_i}(\Omega\times\Omega)\quad\text{a.e. in }\Omega\times\Omega.$$

And

$$\gamma_i(h_{u_{n_k}})h_{u_{n_k}}\to\gamma_i(h_u)h_u\quad\text{a.e. in }\Omega\times\Omega. \tag{4.4}$$

Then by dominated convergence theorem we obtain that

$$\sup_{||v||_{s,\Gamma_i}\le 1}\left|\int_{\Omega\times\Omega}(\gamma_i(h_{u_k})h_{u_k}-\gamma_i(h_u)h_u)h_v\,d\mu\right|\to 0. \tag{4.5}$$

By same techniques we obtain that

$$\sup_{||v||_{\Gamma_i}\le 1}\left|\int_\Omega(\gamma_i(u_k)u_k-\gamma_i(u)u)v\,dx\right|\to 0. \tag{4.6}$$

By (A_1), (A_2), (2.5), ii) in Theorem 3.5 and boundedness of sequence $\{u_k, v_k\}$ and using similar argument in the proof of Lemma 3.3 in [7] we show that

$$\lim_{n\to\infty}\int_\Omega a_i(x)M_i\big(u_n\big)dx=\int_\Omega a_i(x)M_i\big(u\big)dx. \tag{4.7}$$

According to the last equation and the continuity of K_i, we have

$$K_i\bigg(\int_{\Omega\times\Omega}\Gamma_i(h_{u_k})d\mu+\int_\Omega a_i(x)\Gamma_i(u_k)dx\bigg)\to K_i\bigg(\int_{\Omega\times\Omega}\Gamma_i(h_u)d\mu+\int_\Omega a_i(x)\Gamma_i(u)dx\bigg). \tag{4.8}$$

Combining (4.5), (4.6) in (4.3) and with the fact (4.8) we get $(\tilde{K}o\mathcal{H})_i'$ is continuous. Now we turn to prove that

$$\langle I_2'(u,v),(\overline{u},\overline{v})\rangle=\int_\Omega(F_u(x,u,v)\overline{u}+F_v(x,u,v)\overline{v})dx\quad\text{for all }(u,v),\ (\overline{u},\overline{v})\in W. \tag{4.9}$$

By (4.1) and 3.3 we have

$$\begin{aligned}\int_\Omega F(x,u,v)dx \leq & \int_\Omega |F(x,u,v|dx \leq c_3\int_\Omega |u|^{r_1}dx + c_4\int_\Omega |v|^{r_2}dx \\ = & c_3||u||_{r_1}^{r_1} + c_4||v||_{r_2}^{r_2} \\ \leq & c_3C_{r_1}||u||_{E_1,\Gamma_1}^{r_1} + c_4C_{r_2}||v||_{E_2,\Gamma_2}^{r_2}.\end{aligned} \tag{4.10}$$

Then I_2 is well defined in W. Now by (3.2), (3.3) and the similar argument in [23, Lemma 3.1] we see that (4.9) holds. □

Lemma 4.5. *Suppose that (ϕ_1) is fulfilled. Moreover, we assume that the sequence (u_k) converges weakly to u in E_1 and*

$$\lim_{k\to\infty}\sup\langle(\tilde{K}o\mathcal{H})'_1(u_k), u_k - u\rangle \leq 0. \tag{4.11}$$

Then (u_k) converge strongly to $u \in E_1$.

Proof. Since (u_k) converges weakly to u in E_1, then $([u_k]_{s,\Gamma_1})$ and $(||u_k||_{a_1,\Gamma_1})$ are bounded sequences of real numbers. That fact and relations 2) and 3) from Lemma 3.2, imply that the sequences $(\mathcal{F}_i(u_k))$ and $(\Upsilon_i(u_k))$ are bounded. This means that the sequence $(\tilde{K}o\mathcal{H})_1(u_k)$ is bounded. Then, up to a subsequence, $(\tilde{K}o\mathcal{H})_1(u_k) \to c$. Furthermore, Lemma 3.3 implies

$$(\tilde{K}o\mathcal{H})_1(u) \leq \lim_{k\to\infty}\inf(\tilde{K}o\mathcal{H})_1(u_k) = c. \tag{4.12}$$

Since $(\tilde{K}o\mathcal{H})_1$ is convex, we have

$$(\tilde{K}o\mathcal{H})_1(u) \geq (\tilde{K}o\mathcal{H})_1(u_k) + \langle(\tilde{K}o\mathcal{H})'_1(u_k), u_k - u\rangle. \tag{4.13}$$

Therefore, combining (4.11), (4.12) and (4.13), we conclude that $(\tilde{K}o\mathcal{H})_1(u) = c$.

Taking into account that $\frac{u_k+u}{2}$ converges weakly to u in E_1 and using again the weak lower semicontinuity of $(\tilde{K}o\mathcal{H})_1$, we get

$$c = (\tilde{K}o\mathcal{H})_1(u) \leq \lim_{k\to\infty}\inf(\tilde{K}o\mathcal{H})_1\Big(\frac{u_k+u}{2}\Big). \tag{4.14}$$

We argue by contradiction, and suppose that (u_k) does not converge to u in E_1. Then, there exists $\beta > 0$ and a subsequence (u_{k_r}) of (u_k) such that

$$\Big\|\frac{u_{k_r} - u}{2}\Big\|_{a_1,\Gamma_1} \geq \beta.$$

By 2) and 3) in Lemma 3.2 we infer that

$$
\begin{aligned}
(\tilde{K}o\mathcal{H})_1\Big(\frac{u_k+u}{2}\Big) &\geq \zeta_0\Big(\Big\|\frac{u_{k_r}-u}{2}\Big\|_{a_1,\Gamma_1}\Big)+\zeta_0\Big(\Big[\frac{u_{k_r}-u}{2}\Big]_{s,\Gamma_1}\Big)\geq \zeta_0\Big(\Big\|\frac{u_{k_r}-u}{2}\Big\|_{a_1,\Gamma_1}\Big)\\
&\geq \zeta_0(\beta).
\end{aligned}
$$

On the other hand, the Δ_2-condition and relation (ϕ_1) enable us to apply Theorem 1.2 in [21] to obtain

$$
\frac{1}{2}(\tilde{K}o\mathcal{H})_1(u)+\frac{1}{2}(\tilde{K}o\mathcal{H})_1(u_{k_r})-(\tilde{K}o\mathcal{H})_1\Big(\frac{u_{k_r}+u}{2}\Big)\geq (\tilde{K}o\mathcal{H})_1\Big(\frac{u_{k_r}-u}{2}\Big)\geq \zeta_0(\beta), \tag{4.15}
$$

for all $r\in\mathbb{N}$.

Letting $r\to\infty$ in the above inequality, we get

$$
c-\zeta_0(\beta)\geq \lim_{r\to\infty}\sup(\tilde{K}o\mathcal{H})_1\Big(\frac{u_{k_r}+u}{2}\Big)\geq c. \tag{4.16}
$$

That is a contradiction. It follows that (u_k) converges strongly to u in E_1.
Similary we can obtain that, $v_k\to v$ in E_2. Therefore $\{(u_k,v_k)\}\to(u,v)$ in W. □

Lemma 4.6. *If a sequence (u_k,v_k) converges to (u_0,v_0) in W weakly, then*

$$
\int_\Omega\left(\frac{1}{\alpha_1}F_u(x,u_k,v_k)-\frac{1}{\alpha_2}F_u(x,u_0,v_0)\right)(u_k-u_0)dx\to 0 \tag{4.17}
$$

and

$$
\int_\Omega\left(\frac{1}{\alpha_1}F_v(x,u_k,v_k)-\frac{1}{\alpha_2}F_v(x,u_0,v_0)\right)(v_k-v_0)dx\to 0. \tag{4.18}
$$

Proof. By (3.2), remark 3.6 and Hölder inequality we have:

$$
\begin{aligned}
&\int_\Omega\left(\frac{1}{\alpha_1}F_u(x,u_k,v_k)-\frac{1}{\alpha_2}F_u(x,u_0,v_0)\right)(u_k-u_0)dx\\
&\leq \int_\Omega\left|\frac{1}{\alpha_1}F_u(x,u_k,v_k)-\frac{1}{\alpha_2}F_u(x,u_0,v_0)\right||u_k-u_0|dx\\
&\leq \frac{1}{\alpha_1}\int_\Omega|F_u(x,u_k,v_k)||u_k|dx+\frac{1}{\alpha_1}\int_\Omega|F_u(x,u_k,v_k)||u_0|dx\\
&+\frac{1}{\alpha_2}\int_\Omega|F_u(x,u_0,v_0)||u_k|dx+\frac{1}{\alpha_2}\int_\Omega|F_u(x,u_0,v_0)||u_0|dx\\
&\leq \frac{c_1}{\alpha_1}\int_\Omega|u_k|^{r_1}dx+\frac{c_2}{\alpha_1}\int_\Omega|v_k|^{\frac{r_2(r_1-1)}{r_1}}|u_k|dx+\frac{c_1}{\alpha_1}\int_\Omega|u_k|^{r_1-1}|u_0|dx\\
&+\frac{c_2}{\alpha_1}\int_\Omega|v_k|^{\frac{r_2(r_1-1)}{r_1}}|u_0|dx+\frac{c_1}{\alpha_2}\int_\Omega|u_0|^{r_1-1}|u_k|dx+\frac{c_2}{\alpha_2}\int_\Omega|v_0|^{\frac{r_2(r_1-1)}{r_1}}|u_k|dx\\
&+\frac{c_1}{\alpha_2}\int_\Omega|u_0|^{r_1}dx+\frac{c_2}{\alpha_2}\int_\Omega|v_0|^{\frac{r_2(r_1-1)}{r_1}}|u_0|dx
\end{aligned}
$$

$$
\begin{aligned}
&\le c_1'\Big(\int_\Omega |u_k|^{r_1}dx\Big)+c_2'\Big(\int_\Omega |u_k|^{r_1}dx\Big)^{\frac{1}{r_1}}\Big(\int_\Omega |v_k|^{r_2}dx\Big)^{\frac{r_1-1}{r_1}}\\
&+c_1'\Big(\int_\Omega |u_k|^{r_1}dx\Big)^{\frac{r_1-1}{r_1}}\Big(\int_\Omega |u_0|^{r_1}dx\Big)^{\frac{1}{r_1}}+c_2'\Big(\int_\Omega |u_0|^{r_1}dx\Big)^{\frac{1}{r_1}}\\
&\times\Big(\int_\Omega |v_k|^{r_2}dx\Big)^{\frac{r_1-1}{r_1}}+c_1'\Big(\int_\Omega |u_0|^{r_1}dx\Big)^{\frac{r_1-1}{r_1}}\Big(\int_\Omega |u_k|^{r_1}dx\Big)^{\frac{1}{r_1}}\\
&+c_2'\Big(\int_\Omega |u_k|^{r_1}dx\Big)^{\frac{1}{r_1}}\Big(\int_\Omega |v_0|^{r_2}dx\Big)^{\frac{r_1-1}{r_1}}+c_1'\Big(\int_\Omega |u_0|^{r_1}dx\Big)\\
&+c_2'\Big(\int_\Omega |u_0|^{r_1}dx\Big)^{\frac{1}{r_1}}\Big(\int_\Omega |v_0|^{r_2}dx\Big)^{\frac{r_1-1}{r_1}}. \qquad (4.19)
\end{aligned}
$$

Since $(u_k, v_k) \rightharpoonup (u_0, v_0)$ in W, it is clear that $u_k \rightharpoonup u_0$ in E_1 and $v_k \rightharpoonup v_0$ in E_2, then $\{||u_k||_{E_1}\}$ and $\{||v_k||_{E_2}\}$ are bounded. By remark 3.6, there exists $G > 0$ such that

$$
||u_k||_{E_1}, ||v_k||_{E_2}, ||u_k||_{r_1}, ||v_k||_{r_2} \le G, \quad \text{for } n \in \mathbb{N}. \qquad (4.20)
$$

Combining (4.19) and (4.20), then we have

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{\alpha_1}F_u(x,u_k,v_k)-\frac{1}{\alpha_2}F_u(x,u_0,v_0)\right)(u_k-u_0)dx \le c_1'||u_k||_{r_1}^{r_1}+c_2'||u_k||_{r_1}||v_k||_{r_2}^{\frac{r_2(r_1-1)}{r_1}}\\
&+c_1'||u_k||_{r_1}^{\frac{r_2(r_1-1)}{r_1}}||u_0||_{r_1}+c_2'||u_0||_{r_1}||v_k||_{r_2}^{\frac{r_2(r_1-1)}{r_1}}+c_1'||u_0||_{r_1}^{\frac{r_2(r_1-1)}{r_1}}||u_k||_{r_1}\\
&+c_2'||u_k||_{r_1}||v_0||_{r_2}^{\frac{r_2(r_1-1)}{r_1}}+c_1'||u_0||_{r_1}^{r_1}+c_2'||u_0||_{r_1}||v_0||_{r_2}^{\frac{r_2(r_1-1)}{r_1}}\\
&\le c_1'G^{r_1}+c_2'G^{\frac{r_2(r_1-1)}{r_1}+1}+c_1'G^{\frac{r_2(r_1-1)}{r_1}+1}+c_2'G^{\frac{r_2(r_1-1)}{r_1}+1}+c_1'c_2'G^{\frac{r_2(r_1-1)}{r_1}+1}\\
&+c_2'c_1'G^{\frac{r_2(r_1-1)}{r_1}+1}+c_1'G^{r_1}+c_2'c_1'G^{\frac{r_2(r_1-1)}{r_1}+1}. \qquad (4.21)
\end{aligned}
$$

Using again Remark 3.6, then for any positive ϵ, we can choose $n_0 \in \mathbb{N}$ such that

$$
\Big(\int_\Omega |u_k-u_0|^{r_1}dx\Big)^{\frac{1}{r_1}} < \epsilon \quad \text{for all } n > n_0.
$$

Furthermore,

$$
\begin{aligned}
&\int_\Omega \left|\frac{1}{\alpha_1}F_u(x,u_k,v_k)-\frac{1}{\alpha_2}F_u(x,u_0,v_0)\right||u_k-u_0|dx\\
&\le \Big(\int_\Omega \left|\frac{1}{\alpha_1}F_u(x,u_k,v_k)-\frac{1}{\alpha_2}F_u(x,u_0,v_0)\right|^{\frac{r_1}{r_1-1}}dx\Big)^{\frac{r_1-1}{r_1}}\Big(\int_\Omega |u_k-u_0|^{r_1}dx\Big)^{\frac{1}{r_1}}\\
&\le C\Big(\int_\Omega |F_u(x,u_k,v_k)|^{\frac{r_1}{r_1-1}}+|F_u(x,u_0,v_0)|^{\frac{r_1}{r_1-1}}dx\Big)^{\frac{r_1-1}{r_1}}\epsilon\\
&\le \epsilon C\Big(\int_\Omega \Big((c_1|u_k|^{r_1-1}+c_2|v_k|^{\frac{r_2(r_1-1)}{r_1}})^{\frac{r_1}{r_1-1}}+(c_1|u_0|^{r_1-1}+c_2|v_0|^{\frac{r_2(r_1-1)}{r_1}})^{\frac{r_1}{r_1-1}}\Big)dx\Big)^{\frac{r_1-1}{r_1}}
\end{aligned}
$$

$$\leq \epsilon C\Big(\int_{\Omega}\Big(c_1^{\frac{r_1}{r_1-1}}|u_k|^{r_1}+c_2^{\frac{r_1}{r_1-1}}|v_k|^{r_2}+c_1^{\frac{r_1}{r_1-1}}|u_0|^{r_1}+c_2^{\frac{r_1}{r_1-1}}|v_0|^{r_2}\Big)dx\Big)^{\frac{r_1-1}{r_1}}$$
$$\leq \epsilon C\Big(c_1^{\frac{r_1}{r_1-1}}G^{r_1}+c_2^{\frac{r_1}{r_1-1}}G^{r_2}+c_1^{\frac{r_1}{r_1-1}}G^{r_1}+c_2^{\frac{r_1}{r_1-1}}G^{r_2}\Big)^{\frac{r_1-1}{r_1}}. \tag{4.22}$$

for all $n > n_0$. As ϵ is arbitrary, combining (4.21) with (4.22), we conclude that (4.17) holds. With a similar discussion as above, we can prove that (4.18) holds. □

Lemma 4.7. *$I(u,v)$ is coercive on W, that is, $I(u,v) \to +\infty$ as $||(u,v)|| \to +\infty$.*

Proof. Using (A_3), (4.1), Remark 3.6, and Lemma 3.2 we obtain

$$\begin{aligned}
I(u,v) &= (\tilde{K}o\mathcal{H})_1(u) + (\tilde{K}o\mathcal{H})_2(v) - \int_{\Omega} F(x,u,v)dx \\
&\geq \alpha_1(\mathcal{F}_1+\Upsilon_1)(u) + \alpha_1(\mathcal{F}_2+\Upsilon_2)(v) - \int_{\Omega} F(x,u,v)dx \\
&\geq \alpha_1 \min\{[u]_{s,\Gamma_1}^{l_1}, [u]_{s,\Gamma_1}^{n_1}\} + \alpha_1 \min\{\|u\|_{a,\Gamma_1}^{l_1}, \|u\|_{a,\Gamma_1}^{n_1}\} \\
&+ \alpha_1 \min\{[v]_{s,\Gamma_2}^{l_2}, [v]_{s,\Gamma_2}^{n_2}\} + \alpha_1 \min\{\|v\|_{a,\Gamma_2}^{l_2}, \|v\|_{a,\Gamma_2}^{n_2}\} - c_1\int_{\Omega}|u|^{r_1}dx - c_2\int_{\Omega}|v|^{r_2}dx \\
&= \alpha_1\big([u]_{s,\Gamma_1}^{l_1} + \|u\|_{a,\Gamma_1}^{l_1}\big) + \alpha_1\big([v]_{s,\Gamma_2}^{l_2} + \|v\|_{a,\Gamma_2}^{l_2}\big) - c_3||u||_{E_1,\Gamma_1}^{r_1} - c_4||u||_{E_2,\Gamma_2}^{r_2} \\
&\geq \frac{\alpha_1}{2^{l_1-1}}\|u\|_{E_1,\Gamma_1}^{l_1} + \frac{\alpha_1}{2^{l_2-1}}\|v\|_{E_2,\Gamma_2}^{l_2} - c_3||u||_{E_1,\Gamma_1}^{r_1} - c_4||v||_{E_2,\Gamma_2}^{r_2}.
\end{aligned}$$

Since $r_i \in (1, l_i)$, then the last inequality implies that $I(u,v) \to +\infty$ as $||(u,v)|| = ||u||_{E_1,\Gamma_1} + ||v||_{E_2,\Gamma_2} \to +\infty$. □

Lemma 4.8. *Assume that $l_i = n_i$, (3.1), (4.11) (ϕ_1)-(ϕ_2) and (F_1) hold. Then the energy functional I satisfies (PS)-condition.*

Proof. Let $\{(u_k, v_k)\}$ be any (PS)-sequence in W for I. It follows from Lemma 4.7 that sequence $\{(u_k, v_k)\}$ is bounded in W. Therefore, going if necessary to a subsequence, we can assume that $(u_k, v_k) \rightharpoonup (u_0, v_0)$ in W. Then $u_k \rightharpoonup u_0$ in E_1 and $v_k \rightharpoonup v_0$ in E_2 respectively. Since (u_k, v_k) is a (PS)-sequence, there is $c \in \mathbb{R}$ such that

$$I(u_k, v_k) \to c \quad \text{as} \quad j \to \infty \quad \text{and} \quad \langle I'(u_k, v_k), (\phi, \psi)\rangle = 0 \quad \text{for all } \phi, \psi \in C_c^{\infty}(\mathbb{R}^n), \tag{4.23}$$

which follows that,

$$\begin{aligned}
o_k(1) &= \Big\langle \frac{1}{\alpha_1} I'(u_k, v_k) - \frac{1}{\alpha_2} I'(u_0, v_0), (u_k - u_0, v_k - v_0)\Big\rangle \\
&= \frac{1}{\alpha_1} K_1\big(\mathcal{F}_1(u_k) + \Upsilon_1(u_k)\big)\Big(\int_{\Omega\times\Omega} \gamma_1(h_{u_k})h_{u_k}h_{u_k-u_0}d\mu + \int_{\Omega} a_1(x)\gamma_1(u_k)u_k(u_k-u_0)dx\Big) \\
&+ \frac{1}{\alpha_1} K_2\big(\mathcal{F}_2(v_k) + \Upsilon_2(v_k)\big)\Big(\int_{\Omega\times\Omega} \gamma_2(h_{v_k})h_{v_k}h_{v_k-v_0}d\mu + \int_{\Omega} a_2(x)\gamma_2(v_k)v_k(v_k-v_0)dx\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\alpha_1} \int_\Omega \left(F_{u_k}(x,u_k,v_k)(u_k-u_0) + F_{v_k}(x,u_k,v_k)(v_k-v_0)\right) dx \\
&- \frac{1}{\alpha_2} K_1\big(\mathcal{F}_1(u_0)+\Upsilon_1(u_0)\big)\Big(\int_{\Omega\times\Omega} \gamma_1(h_{u_0})h_{u_0}h_{u_k-u_0}d\mu + \int_\Omega a_1(x)\gamma_1(u_0)u_0(u_k-u_0)dx\Big) \\
&- \frac{1}{\alpha_2} K_1\big(\mathcal{F}_1(u_0)+\Upsilon_1(u_0)\big)\Big(\int_{\Omega\times\Omega} \gamma_1(h_{u_0})h_{u_0}h_{u_k-u_0}d\mu + \int_\Omega a_1(x)\gamma_1(u_0)u_0(u_k-u_0)dx\Big) \\
&+ \frac{1}{\alpha_2} \int_\Omega \left(F_{u_0}(x,u_0,v_0)(u_k-u_0) + F_{v_0}(x,u_0,v_0)(v_k-v_0)\right) dx
\end{aligned}
$$

Using (A_3) we infer that

$$
\begin{aligned}
o_k(1) \geq &\Big(\int_{\Omega\times\Omega} \gamma_1(h_{u_k})h_{u_k}h_{u_k-u_0}d\mu + \int_\Omega a_1(x)\gamma_1(u_k)u_k(u_k-u_0)dx\Big) \\
&+ \Big(\int_{\Omega\times\Omega} \gamma_2(h_{v_k})h_{v_k}h_{v_k-v_0}d\mu + \int_\Omega a_2(x)\gamma_2(v_k)v_k(v_k-v_0)dx\Big) \\
&- \frac{1}{\alpha_1} \int_\Omega \left(F_{u_k}(x,u_k,v_k)(u_k-u_0) + F_{v_k}(x,u_k,v_k)(v_k-v_0)\right) dx \\
&- \Big(\int_{\Omega\times\Omega} (\gamma_1(h_{u_0})h_{u_0}h_{u_k-u_0}d\mu + \int_\Omega a_1(x)\gamma_1(u_0)u_0(u_k-u_0)dx\Big) \\
&- \Big(\int_{\Omega\times\Omega} \gamma_1(h_{u_0})h_{u_0}h_{u_k-u_0}d\mu + \int_\Omega a_1(x)\gamma_1(u_0)u_0(u_k-u_0)dx\Big) \\
&+ \frac{1}{\alpha_2} \int_\Omega \left(F_{u_0}(x,u_0,v_0)(u_k-u_0) + F_{v_0}(x,u_0,v_0)(v_k-v_0)\right) dx \\
= &\Big(\int_{\Omega\times\Omega} (\gamma_1(h_{u_k})h_{u_k} - \gamma_1(h_{u_0})h_{u_0})h_{u_k-u_0}d\mu + \int_\Omega a_1(x)(\gamma_1(u_k)u_k - \gamma_1(u_0)u_0)(u_k-u_0)dx\Big) \\
&+ \Big(\int_{\Omega\times\Omega} (\gamma_2(h_{v_k})h_{v_k} - \gamma_2(h_{v_0})h_{v_0})h_{v_k-v_0}d\mu + \int_\Omega a_2(x)(\gamma_2(v_k)v_k - \gamma_2(v_0)v_0)(v_k-v_0)dx\Big) \\
&- \int_\Omega \left(\frac{1}{\alpha_1}F_u(x,u_k,v_k) - \frac{1}{\alpha_2}F_u(x,u_0,v_0)\right)(u_k-u_0)dx \\
&- \int_\Omega \left(\frac{1}{\alpha_1}F_v(x,u_k,v_k) - \frac{1}{\alpha_2}F_v(x,u_0,v_0)\right)(v_k-v_0)dx.
\end{aligned}
$$

Furthermore, last inequality, Lemma 4.6 and strictly monotone of the operator I_1' implies that

$$
\int_{\Omega\times\Omega} (\gamma_1(h_{u_k})h_{u_k} - \gamma_1(h_{u_0})h_{u_0})h_{u_k-u_0}d\mu = \langle \mathcal{F}_1'(u_k) - \mathcal{F}_1'(u_0), u_k-u_0\rangle \to 0 \tag{4.24}
$$

$$
\int_{\Omega\times\Omega} (\gamma_2(h_{v_k})h_{v_k} - \gamma_2(h_{v_0})h_{v_0})h_{v_k-v_0}d\mu = \langle \mathcal{F}_2'(v_k) - \mathcal{F}_2'(v_0), v_k-v_0\rangle \to 0 \tag{4.25}
$$

$$
\int_\Omega a_1(x)(\gamma_1(u_k)u_k - \gamma_1(u_0)u_0)(u_k-u_0)dx = \langle \Upsilon_1'(u_k) - \Upsilon_1'(u_0), u_k-u_0\rangle \to 0 \tag{4.26}
$$

and

$$
\int_\Omega a_2(x)(\gamma_2(v_k)v_k - \gamma_2(v_0)v_0)(v_k-v_0)dx = \langle \Upsilon_2'(v_k) - \Upsilon_2'(v_0), v_k-v_0\rangle \to 0. \tag{4.27}
$$

By Remark 3.6, (4.24) and (4.25) we have

$$u_k(x) \to u_0(x) \quad \text{a.e. } x \in \Omega,$$

and

$$v_k(x) \to v_0(x) \quad \text{a.e. } x \in \Omega.$$

Using Fatou's Lemma we have

$$\int_{\Omega\times\Omega} \Gamma_1(h_{u_0})d\mu \le \lim_{k\to\infty} \inf \int_{\Omega\times\Omega} \Gamma_1(h_{u_k})d\mu, \tag{4.28}$$

$$\int_{\Omega\times\Omega} \Gamma_1(h_u)d\mu \le \lim_{k\to\infty} \inf \int_{\Omega\times\Omega} \Gamma_2(h_{v_k})d\mu, \tag{4.29}$$

$$\int_{\Omega} a_1(x)\Gamma_1(u_k(x))dx \le \lim_{k\to\infty} \inf \int_{\Omega} a_1(x)\Gamma_1(u_0(x))dx, \tag{4.30}$$

and

$$\int_{\Omega} a_2(x)\Gamma_2(v_k(x))dx \le \lim_{k\to\infty} \inf \int_{\Omega} a_2(x)\Gamma_2(v_0(x))dx. \tag{4.31}$$

Moreover,

$$\lim_{k\to\infty} \langle \mathcal{F}_1'(u_k), u_k - u_0\rangle = \lim_{k\to\infty} \langle \mathcal{F}_1'(u_k) - \mathcal{F}_1'(u_0), u_k - u_0\rangle = 0, \tag{4.32}$$

$$\lim_{k\to\infty} \langle \mathcal{F}_2'(v_k), v_k - v_0\rangle = \lim_{k\to\infty} \langle \mathcal{F}_2'(v_k) - \mathcal{F}_2'(v_0), v_k - v_0\rangle = 0. \tag{4.33}$$

By using Hölder inequality and (2.6) we have

$$\begin{aligned} \langle \mathcal{F}_1'(u_k), u_k - u_0\rangle &= \int_{\Omega\times\Omega} \gamma_1(h_{u_k})h_{u_k}h_{u_k-u_0}d\mu \\ &\ge l_1\int_{\Omega\times\Omega} \Gamma_1(h_{u_k})d\mu - \int_{\Omega\times\Omega} \overline{\Gamma}_1(\gamma_1(h_{u_k})h_{u_k})d\mu - \int_{\Omega\times\Omega} \Gamma_1(h_{u_0})d\mu \\ &\ge l_1\int_{\Omega\times\Omega} \Gamma_1(h_{u_k})d\mu - (n_1-1)\int_{\Omega\times\Omega} \Gamma_1(h_{u_k})d\mu - \int_{\Omega\times\Omega} \Gamma_1(h_{u_0})d\mu \\ &= \int_{\Omega\times\Omega} \Gamma_1(h_{u_k})d\mu - \int_{\Omega\times\Omega} \Gamma_1(h_{u_0})d\mu. \end{aligned} \tag{4.34}$$

Again using Hölder inequality and (2.6) we have

$$\begin{aligned} \langle \mathcal{F}_2'(v_k), v_k - v_0\rangle &= \int_{\Omega\times\Omega} \gamma_v(h_{v_k})h_{v_k}h_{v_k-v_0}d\mu \\ &\ge l_2\int_{\Omega\times\Omega} \Gamma_2(h_{v_k})d\mu - \int_{\Omega\times\Omega} \overline{\Gamma}_2(\gamma_2(h_{v_k})h_{v_k})d\mu - \int_{\Omega\times\Omega} \Gamma_2(h_{v_0})d\mu \\ &\ge l_2\int_{\Omega\times\Omega} \Gamma_2(h_{v_k})d\mu - (n_2-1)\int_{\Omega\times\Omega} \Gamma_2(h_{v_k})d\mu - \int_{\Omega\times\Omega} \Gamma_2(h_{2_0})d\mu \\ &= \int_{\Omega\times\Omega} \Gamma_2(h_{v_k})d\mu - \int_{\Omega\times\Omega} \Gamma_2(h_{v_0})d\mu. \end{aligned} \tag{4.35}$$

According to (4.28), (4.32) and (4.34) we infer that

$$\lim_{k\to\infty}\int_{\Omega\times\Omega}\Gamma_1(h_{u_k})d\mu=\int_{\Omega\times\Omega}\Gamma_1(h_{u_0})d\mu. \tag{4.36}$$

And by (4.29), (4.33) and (4.35) we have that

$$\lim_{k\to\infty}\int_{\Omega\times\Omega}\Gamma_2(h_{v_k})d\mu=\int_{\Omega\times\Omega}\Gamma_2(h_{v_0})d\mu. \tag{4.37}$$

Also, by (4.7) we have

$$\lim_{k\to\infty}\int_{\Omega}a_1(x)\Gamma_1(u_k)dx=\int_{\Omega}a_1(x)\Gamma_1(u_0)dx, \tag{4.38}$$

$$\lim_{k\to\infty}\int_{\Omega}a_2(x)\Gamma_2(v_k)dx=\int_{\Omega}a_2(x)\Gamma_2(v_0)dx. \tag{4.39}$$

In conclusion, estimates (4.36), (4.37), (4.38) and (4.39) get the result. □

Lemma 4.9. *There is a point $(u,v)\in W$ such that $I(u,v)<0$.*

Proof. Let $u_0=\rho w_0$ and $v_0=\sigma w_0$ where $w_0\in C_0^\infty(\Omega)\backslash\{0\}$ with $w_0(x)\geq 0$, $supp(w_0)\subset\Omega$ and $||w_0||\leq 1$. Without loss of generality, we can assume that $\rho\neq 0$. It is clear that $(u_0,v_0)\in W$ (see [3, Theorem 3.7]). When t is small enough by (A_3), (F_2) and Lemma 3.2 we have

$$\begin{aligned}
I(tu_0,tv_0)&=(\tilde{K}o\mathcal{H})_1(tu_0)+(\tilde{K}o\mathcal{H})_2(tv_0)-\int_\Omega F(x,tu_0,tv_0)dx\\
&\leq\alpha_2\Big(\int_{\Omega\times\Omega}\Gamma_1(h_{tu_0})d\mu+\int_\Omega a_1(x)\Gamma_1(tu_0)dx\Big)\\
&+\alpha_2\Big(\int_{\Omega\times\Omega}\Gamma_2(h_{tv_0})d\mu+\int_\Omega a_2(x)\Gamma_2(tv_0)dx\Big)-\int_\Omega F(x,t\rho w_0,t\sigma w_0)dx\\
&\leq\alpha_2\big(\max\{[tu_0]^{l_1}_{s,\Gamma_1};[tu_0]^{n_1}_{s,\Gamma_1}\}+\max\{\,||tu_0||^{l_1}_{a_1,\Gamma_1};||tu_0||^{n_1}_{a_1,\Gamma_1}\}\big)\\
&+\alpha_2\big(\max\{[tv_0]^{l_2}_{s,\Gamma_2};[tu_0]^{n_2}_{s,\Gamma_2}\}+\max\{\,||tv_0||^{l_2}_{a_2,\Gamma_2};||tv_0||^{n_2}_{a_2,\Gamma_2}\}\big)\\
&-\int_\Omega c(|t\rho w_0|^{\alpha_0}+|t\sigma w_0|^{\beta_0})dx\\
&\leq\alpha_2[tu_0]^{l_1}_{s,\Gamma_1}+\alpha_2[tu_0]^{n_1}_{s,\Gamma_1}+\alpha_2||tu_0||^{l_1}_{a_1,\Gamma_1}+\alpha_2||tu_0||^{n_1}_{a_1,\Gamma_1}\\
&+\alpha_2[tv_0]^{l_2}_{s,\Gamma_2}+\alpha_2[tv_0]^{n_2}_{s,\Gamma_2}+\alpha_2||tv_0||^{l_2}_{a_2,\Gamma_2}+\alpha_2||tv_0||^{n_2}_{a_2,\Gamma_2}\\
&-ct^{\alpha_0}\int_\Omega|\rho w_0|^{\alpha_0}dx-ct^{\beta_0}\int_\Omega|\sigma w_0|^{\beta_0}dx\\
&\leq\alpha_2t^{l_1}||u_0||^{l_1}_{E_1,\Gamma_1}+\alpha_2t^{n_1}||u_0||^{n_1}_{E_1,\Gamma_1}+\alpha_2t^{l_2}||v_0||^{l_2}_{E_2,\Gamma_2}+\alpha_2t^{n_2}||v_0||^{n_2}_{E_2,\Gamma_2}\\
&-ct^{\alpha_0}\int_\Omega|u_0|^{\alpha_0}dx-ct^{\beta_0}\int_\Omega|v_0|^{\beta_0}dx.
\end{aligned}$$

Hence $\alpha_0\in[1,l_1)$ and $\beta_0\in[1,l_2)$, we can choose $t_0>0$ small enough such that $I(t_0u_0,t_0v_0)<0$. □

Proof of Theorem 4.1. Let $X = W$ and $J = I$. By Lemma 4.4, Lemma 4.7, Lemma 4.8 all conditions of Lemma 4.2 hold. Then system (1.1) possesses a critical point $(u, v) \in W$ which is a weak solution of system (1.1) satisfying $I(u, v) = \inf_W I$. Lemma 4.9 implies that $(u, v) \neq 0$. Thus system (1.1) possesses at least one nontrivial weak solution. □

Authors contributions: All authors of this manuscript contributed equally to this work.

Funding: Not applicable.

Data Availability: No data were used to support this study.

Conflicts of Interest: The authors declare that there are no conflicts of interest.

References

[1] K. B. Ali, M. Hsini, K. Kefi, and N. T. Chung, "On a nonlocal fractional $p(\cdot,\cdot)$-Laplacian problem with competing nonlinearities," *Complex Anal. Oper. Theory*, vol. 13, no. 3, pp. 1377–1399, 2019, doi: 10.1007/s11785-018-00885-9.

[2] E. Azroul, A. Benkirane, A. Boumazourh, and M. Srati, "Multiple solutions for a nonlocal fractional (p,q)-Schrödinger-Kirchhoff type system," *Nonlinear Stud.*, vol. 27, no. 4, pp. 915–933, 2020.

[3] E. Azroul, A. Benkirane, and M. Srati, "Existence of solutions for a nonlocal type problem in fractional Orlicz Sobolev spaces," *Adv. Oper. Theory*, vol. 5, no. 4, pp. 1350–1375, 2020, doi: 10.1007/s43036-020-00042-0.

[4] A. Bahrouni, S. Bahrouni, and M. Xiang, "On a class of nonvariational problems in fractional Orlicz-Sobolev spaces," *Nonlinear Anal.*, vol. 190, 2020, Art. ID 111595, doi: 10.1016/j.na.2019.111595.

[5] S. Bahrouni and H. Ounaies, "Embedding theorems in the fractional Orlicz-Sobolev space and applications to non-local problems," *Discrete Contin. Dyn. Syst.*, vol. 40, no. 5, pp. 2917–2944, 2020, doi: 10.3934/dcds.2020155.

[6] L. S. Chadli, H. El-Houari, and H. Moussa, "Multiplicity of solutions for nonlocal parametric elliptic systems in fractional Orlicz-Sobolev spaces," *J. Elliptic Parabol. Equ.*, vol. 9, no. 2, pp. 1131–1164, 2023, doi: 10.1007/s41808-023-00238-4.

[7] H. El-Houari, L. S. Chadli, and H. Moussa, "Existence of a solution to a nonlocal Schrödinger system problem in fractional modular spaces," *Adv. Oper. Theory*, vol. 7, no. 1, 2022, Art. ID 6, doi: 10.1007/s43036-021-00166-x.

[8] H. El-Houari, L. S. Chadli, and H. Moussa, "Nehari manifold and fibering map approach for fractional $p(.)$-Laplacian Schrödinger system," *SeMA Journal*, vol. 2023, 2023, doi: 10.1007/s40324-023-00343-3.

[9] H. El-Houari, L. S. Chadli, and H. Moussa, "Existence of solution to M-Kirchhoff system type," in *2021 7th International Conference on Optimization and Applications (ICOA)*, 2021, pp. 1–6, doi: 10.1109/ICOA51614.2021.9442669.

[10] H. El-Houari, L. S. Chadli, and H. Moussa, "Existence of ground state solutions of elliptic system in fractional Orlicz-Sobolev spaces," *Results in Nonlinear Analysis*, vol. 5, no. 2, p. 112–130, 2022.

[11] H. El-Houari, L. S. Chadli, and H. Moussa, "A class of non-local elliptic system in non-reflexive fractional Orlicz-Sobolev spaces," *Asian-Eur. J. Math.*, vol. 16, no. 7, 2023, Art. ID 2350114, doi: 10.1142/S1793557123501140.

[12] H. El-Houari, L. S. Chadli, and H. Moussa, "Multiple solutions in fractional Orlicz-Sobolev spaces for a class of nonlocal kirchhoff systems," *Filomat*, vol. 38, no. 8, pp. 2857–2875, 2024, doi: 10.2298/FIL2408857E.

[13] H. El-Houari, H. Moussa, and L. S. Chadli, "Ground state solutions for a nonlocal system in fractional Orlicz-Sobolev spaces," *Int. J. Differ. Equ.*, 2022, Art. ID 3849217, doi: 10.1155/2022/3849217.

[14] H. El-Houari, H. Moussa, and L. S. Chadli, "A weak solution to a non-local problem in fractional Orlicz-Sobolev spaces," *Asia Pac. J. Math.*, vol. 10, no. 2, 2023, doi: 10.1155/2022/3849217.

[15] H. El-Houari, H. Moussa, and L. S. Chadli, "A class of elliptic inclusion in fractional Orlicz–Sobolev spaces," *Complex Variables and Elliptic Equations*, pp. 1–18, 2022, doi: 10.1080/17476933.2022.2159955.

[16] H. El-Houari, H. Moussa, S. Kassimi, and H. Sabiki, "Fractional Musielak spaces: a class of non-local problem involving concave–convex nonlinearity," *Journal of Elliptic and Parabolic Equations*, 2023, doi: 10.1007/s41808-023-00252-6.

[17] J. Fernández Bonder and A. M. Salort, "Fractional order Orlicz-Sobolev spaces," *J. Funct. Anal.*, vol. 277, no. 2, pp. 333–367, 2019, doi: 10.1016/j.jfa.2019.04.003.

[18] E. M. Hssini, N. Tsouli, and M. Haddaoui, "Existence and multiplicity solutions for $(p(x), q(x))$-Kirchhoff type systems," *Matematiche (Catania)*, vol. 71, no. 1, pp. 75–88, 2016, doi: 10.4418/2016.71.1.6.

[19] G. Kirchhoff, *Vorlesungen über Mechanik*. B.G. Teubner, Leipzig, Germany, 1883.

[20] M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ, *Convex functions and Orlicz Spaces*. Noordhoff, Groningen, 1969.

[21] J. Lamperti, "On the isometries of certain function-spaces," *Pacific J. Math.*, vol. 8, pp. 459–466, 1958.

[22] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, ser. Applied Mathematical Sciences. Springer New York, NY, 1989, vol. 74, doi: 10.1007/978-1-4757-2061-7.

[23] L. Wang, X. Zhang, and H. Fang, "Existence and multiplicity of solutions for a class of (ϕ_1, ϕ_2)-Laplacian elliptic system in $\mathbb{R}^N$ via genus theory," *Comput. Math. Appl.*, vol. 72, no. 1, pp. 110–130, 2016, doi: 10.1016/j.camwa.2016.04.034.

[24] Y. Wu, Z. Qiao, M. K. Hamdani, B. Kou, and L. Yang, "A class of variable-order fractional $p(\cdot)$-Kirchhoff-type systems," *J. Funct. Spaces*, 2021, Art. ID 5558074, doi: 10.1155/2021/5558074.

Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 26, no. 01, pp. 75–89, April 2024
DOI: 10.56754/0719-0646.2601.075

Families of skew linear harmonic Euler sums involving some parameters

Anthony Sofo[1,✉]

[1] *College of Engineering and Science, Victoria University, Australia.*
`anthony.sofo@vu.edu.au`

ABSTRACT

In this study we investigate a family of skew linear harmonic Euler sums involving some free parameters. Our analysis involves using the properties of the polylogarithm function, commonly referred to as the Bose-Einstein integral. A reciprocity property is utilized to highlight an explicit representation for a particular skew harmonic linear Euler sum. A number of examples are also given which highlight the theorems. This work generalizes some results in the published literature and introduces some new results.

RESUMEN

En este estudio, investigamos una familia de sumas de Euler lineales alternantes armónicas involucrando algunos parámetros libres. Nuestro análisis involucra el uso de propiedades de la función polilogaritmo, comúnmente referida como la integral de Bose-Einstein. Se utiliza una propiedad de reciprocidad para destacar una representación explícita para una suma de Euler lineal alternante armónica. Se entregan ejemplos que dan luces de los teoremas. Este trabajo generaliza algunos resultados publicados en la literatura e introduce algunos nuevos resultados.

Keywords and Phrases: Skew linear harmonic Euler sum, Polygamma function, harmonic number, polylogarithm function, Bernoulli number.

2020 AMS Mathematics Subject Classification: 11M06, 11M35, 26B15, 33B15, 42A70, 65B10.

Published: 05 April, 2024
Accepted: 16 January, 2024
Received: 16 August, 2023

1 Introduction and preliminaries

In this paper we will investigate the closed form representation of skew linear harmonic Euler sums of the form:

$$\sum_{n\geqslant 1}(-1)^{n+1}A_n^{(t)}\left\{\frac{(-1)^p}{(2n+1+a)^{p+1}}+\frac{(-1)^{t+1}}{(2n+1-a)^{p+1}}\right\} \tag{1.1}$$

for the parameter $-1\leq a<1$ and integers p and t, and by reciprocity, also give a closed form expression for

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{A_n^{(p+1)}\left(\frac{1}{2}\right)}{n^t}. \tag{1.2}$$

The alternating, or skew, harmonic numbers $A_n^{(t)}$ of order t, in (1.1), are defined by

$$A_n^{(t)}:=\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j^t}\qquad (t\in\mathbb{C},\, n\in\mathbb{N}) \tag{1.3}$$

and $A_n:=A_n^{(1)}$. The polylogarithm

$$\mathrm{Li}_s(z)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}}{z^{-1}\exp(t)-1}\,\mathrm{d}t \tag{1.4}$$

in this context, is sometimes referred to as a Bose-Einstein integral [15]. The skew harmonic Euler sum (1.1) under investigation in this paper can be thought of as belonging to an extended family which has its origin in the early investigations of Goldbach and Euler in which they initiated the study of sums of the type, see Flajolet and Salvy [6]

$$\mathbb{S}_{p,t}=\sum_{n\geq 1}\frac{H_n^{(p)}}{n^t}$$

known as linear harmonic Euler sums of weight $p+t$. Nielsen [9] and many others, see [1,2,6,11,12], expanded this work and it is now known that $\mathbb{S}_{p,t}$ can be explicitly evaluated, in terms of special functions such as the Riemann zeta function, in the cases when $p=t\in\mathbb{N}$, $p+t$ of odd weight, $p+t$ of even weight in only the pair $\{(4,2),(2,4)\}$ with $p\neq t$. A reciprocity (or shuffle) relation

$$\mathbb{S}_{p,t}+\mathbb{S}_{t,p}=\zeta(p)\,\zeta(t)+\zeta(p+t)$$

exists to evaluate $\mathbb{S}_{t,p}$ in the case $\mathbb{S}_{p,t}$ is known (or vice-versa). The subsequent notion of four distinct classes of linear harmonic Euler sums, of the kind

$$\mathbb{S}^{++}_{p,t}(a,b;q) := \sum_{n=1}^{\infty} \frac{H^{(p)}_{qn}(a)}{(n+b)^t}, \qquad \mathbb{S}^{+-}_{p,t}(a,b;q) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H^{(p)}_{qn}(a)}{(n+b)^t},$$
$$\mathbb{S}^{-+}_{p,t}(a,b;q) := \sum_{n=1}^{\infty} \frac{A^{(p)}_{qn}(a)}{(n+b)^t}, \qquad \mathbb{S}^{--}_{p,t}(a,b;q) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{A^{(p)}_{qn}(a)}{(n+b)^t}, \tag{1.5}$$

here $q \in \mathbb{Z}_{\geq 1}$, $a, b \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$ and $p, t \in \mathbb{C} \setminus \mathbb{Z}^-$ was identified by Flajolet and Salvy [6], in the case $a = 0$, $b = 0$ and $q = 1$. The case $a \in \mathbb{Z}_{\geq 1}$, $b \in \mathbb{Z}_{\geq 1}$ and $q = 1$ was examined by Alzer and Choi [1], and finally the case $a \in \mathbb{Z}_{\geq 1}$, $b \in \mathbb{Z}_{\geq 1}$ and $q \in \mathbb{Z}_{\geq 1}$ was examined by Sofo and Choi [13]. The skew linear harmonic Euler sums (1.1) for certain values of the parameters a, p and t belong to the family

$$\mathbb{S}^{--}_{t,p+1}\left(0, \frac{1-a}{2}\right) := \mathbb{S}^{--}_{t,p+1}\left(0, \frac{1-a}{2}; 1\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{A^{(t)}_n}{(n + \frac{1-a}{2})^{p+1}} \tag{1.6}$$

and will be explicitly represented in terms of special functions as described in (1.1). Using a reciprocity theorem due to Alzer and Choi [1], we also represent

$$\mathbb{S}^{--}_{p+1,t}\left(\frac{1-a}{2}, 0\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{A^{(p+1)}_n\left(\frac{1-a}{2}\right)}{n^t}$$

explicitly in terms of special functions. Interest in Euler sums and multiple zeta values has recently been intense (see, for example, [3–5,7,8,10,18]). Likewise various Euler sums with parameters have been researched (see, for example, [14,16,17]).

2 A parameterized integral

In the next Theorem we evaluate a particular integral which forms the basis in evaluating a family of skew linear harmonic Euler sums.

Theorem 2.1. *Let $t \in \mathbb{N}$ and let $-1 \leq a < 1$. The following integral formulas hold true:*

$$X(a,t,\infty) = \int_0^{\infty} \frac{x^a \operatorname{Li}_t(x^2)}{1+x^2}\, \mathrm{d}x \tag{2.1}$$

$$= -\frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right) \left(\eta(t) + \frac{i \exp\left(-\frac{ia\pi}{2}\right)}{2^t} \left(\zeta\left(t, \frac{1-a}{4}\right) - \zeta\left(t, \frac{3-a}{4}\right)\right)\right), \tag{2.2}$$

where $\operatorname{Li}_t(x^2)$ is the polylogarithm function, $\eta(t)$ is the Dirichlet eta function and $\zeta(t,\mu)$ is the generalized zeta function described in [5].

Proof. From (2.1), we apply the change of variable $x^2 = y$, (then rename $y = x$), so that

$$X(a, t, \infty) = \frac{1}{2}\int_0^\infty \frac{x^{\frac{a-1}{2}}\ \mathrm{Li}_t(x)}{1+x}\,\mathrm{d}x.$$

we now utilize the Bose-Einstein integral (1.4), described in [15] so that

$$X(a, t, \infty) = \frac{1}{2}\int_0^\infty \frac{x^{\frac{a-1}{2}}\ \mathrm{Li}_t(x)}{1+x}\,\mathrm{d}x = \frac{1}{2\Gamma(t)}\int_0^\infty y^{t-1}\int_0^\infty \frac{x^{\frac{a-1}{2}}}{(1+x)\left(x^{-1}\exp(y)-1\right)}\,\mathrm{d}x\,\mathrm{d}y$$

which yields

$$\begin{aligned}
&= \frac{\pi\sec\left(\frac{a\pi}{2}\right)}{2\Gamma(t)}\int_0^\infty \frac{y^{t-1}}{\exp(y)+1}\left(-1+(\sinh(y)-\cosh(y))^{-\frac{a+1}{2}}\right)\mathrm{d}y\\
&= \frac{\pi\sec\left(\frac{a\pi}{2}\right)}{2\Gamma(t)}\int_0^\infty \frac{y^{t-1}}{\exp(y)+1}\left(-1+(-\exp(-y))^{-\frac{a+1}{2}}\right)\mathrm{d}y\\
&= \frac{\pi\sec\left(\frac{a\pi}{2}\right)}{2\Gamma(t)}\int_0^\infty \frac{y^{t-1}}{\exp(y)+1}\left(-1+\exp\left(\frac{1}{2}(y-i\pi)(a+1)\right)\right)\mathrm{d}y\\
&= \pi\sec\left(\frac{a\pi}{2}\right)\left(-\left(1-2^{1-t}\right)\zeta(t)-\frac{i}{2^t}\exp\left(-\frac{ia\pi}{2}\right)\left(\zeta\left(t,\frac{1-a}{4}\right)-\zeta\left(t,\frac{3-a}{4}\right)\right)\right)\\
&= -\frac{\pi}{2}\sec\left(\frac{a\pi}{2}\right)\left(\eta(t)+\frac{i\exp\left(-\frac{ia\pi}{2}\right)}{2^t}\left(\zeta\left(t,\frac{1-a}{4}\right)-\zeta\left(t,\frac{3-a}{4}\right)\right)\right),
\end{aligned}$$

this completes the proof and (2.2) is achieved. □

In the next Lemma we will demonstrate a series representation for the integral (2.3) on the unit interval $x \in (0, 1)$.

Lemma 2.2. *Let $t \in \mathbb{N}$ and let $-1 \le a < 1$. The following integral formula holds true:*

$$X(a, t, 1) = \int_0^1 \frac{x^a\ \mathrm{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x = \sum_{n\ge 1}\frac{(-1)^{n+1}A_n^{(t)}}{2n+1+a}, \tag{2.3}$$

where $A_n^{(t)}$ are the skew harmonic numbers of order t.

Proof. A Taylor series expansion in the domain, $x \in (0, 1)$ gives,

$$\mathrm{Li}_t(x^2) = \sum_{n\ge 1}\frac{x^{2n}}{n^t}, \qquad \frac{1}{1+x^2} = \sum_{n\ge 0}(-1)^n x^{2n}.$$

By the Cauchy product of two convergent series, then it follows that

$$\frac{x^a\mathrm{Li}_t(x^2)}{1+x^2} = \sum_{n\ge 1}(-1)^{n+1}A_n^{(t)}x^{2n+a}$$

and therefore

$$\int_0^1 \frac{x^a\ \mathrm{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x = \sum_{n\geq 1}(-1)^{n+1} A_n^{(t)} \int_0^1 x^{2n+a}\,\mathrm{d}x = \sum_{n\geq 1}\frac{(-1)^{n+1} A_n^{(t)}}{2n+1+a}.$$

In a similar fashion, it follows that

$$X(-a,t,1) = \int_0^1 \frac{x^{-a}\mathrm{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x = \sum_{n\geq 1}\frac{(-1)^{n+1} A_n^{(t)}}{2n+1-a}.$$

In subsequent evaluations we also require the following result, which may be evaluated as a standard integral.

$$\int_0^1 \frac{x^{-a}\ \log^j(x)}{1+x^2}\,\mathrm{d}x = \frac{(-1)^j\, j!}{4^j}\left(\zeta\left(j+1, \frac{1-a}{4}\right) - \zeta\left(j+1, \frac{3-a}{4}\right)\right). \qquad \square$$

In the next Theorem we establish an identity for a linear skew harmonic Euler sum of weight $(t+1)$, $t\in\mathbb{N}$, for the parameter $a\neq 0$.

Theorem 2.3. *Let $t\in\mathbb{N}$, $-1\leq a<1$ with $a\neq 0$. The following formulas hold true:*

$$S(a,t) = \sum_{n\geqslant 1}(-1)^{n+1} A_n^{(t)}\left\{\frac{1}{2n+1+a} + \frac{(-1)^{t+1}}{2n+1-a}\right\} \tag{2.4}$$

$$= \frac{1}{4}\sum_{n\geqslant 1}\frac{1}{n^t}\left\{H_{\frac{n}{2}-\frac{1-a}{4}} - H_{\frac{n}{2}-\frac{3-a}{4}} + (-1)^{t+1}\left(H_{\frac{n}{2}-\frac{1+a}{4}} - H_{\frac{n}{2}-\frac{3+a}{4}}\right)\right\} \tag{2.5}$$

$$\begin{aligned}
&= -\frac{\pi}{2}\sec\left(\frac{a\pi}{2}\right)\left(\eta(t) + \frac{i\exp\left(-\frac{ia\pi}{2}\right)}{2^t}\left(\zeta\left(t,\frac{1-a}{4}\right) - \zeta\left(t,\frac{3-a}{4}\right)\right)\right)\\
&+ \frac{(-1)^{t+1}\,2^{t-2}\,(\pi i)^t\,B_t}{t!}\left(\psi\left(\frac{1-a}{4}\right) - \psi\left(\frac{3-a}{4}\right)\right)\\
&+ \frac{(-1)^{t+1}\,2^{t-2}}{t!}\sum_{j=0}^{t}\binom{t}{j}\frac{(-1)^j\, j! B_{t-j}\,(\pi i)^{t-j}}{4^j}\left(\zeta\left(j+1,\frac{1-a}{4}\right) - \zeta\left(j+1,\frac{3-a}{4}\right)\right).
\end{aligned} \tag{2.6}$$

Proof. Consider the following integral on the real half line $x\geq 0$

$$X(a,t,\infty) = \int_0^\infty \frac{x^a\ \mathrm{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x.$$

Putting

$$\Delta(a,t;x) := \frac{x^a\ \mathrm{Li}_t(x^2)}{1+x^2}$$

it may be seen that $\lim_{x\downarrow 0}\Delta(a,t;x)$, $\lim_{x\uparrow\infty}\Delta(a,t;x)$ and $\lim_{x\to 1}\Delta(a,t;x)$ exist, in fact $\lim_{x\to 1}\Delta(a,t;x) = \frac{1}{2}\mathrm{Li}_t(1) = \frac{1}{2}\zeta(t)$ and it may be expressed as

$$X(a,t,\infty) = \int_0^1 \Delta(a,t;x)\,\mathrm{d}x + \int_1^\infty \Delta(a,t;x)\,\mathrm{d}x. \tag{2.7}$$

Using the transformation $xy = 1$ in the last integral in (2.7) and recovering the variable x instead of y in the resultant integral, we obtain

$$X(a,t,\infty) = \int_0^1 \Delta(a,t;x)\,\mathrm{d}x + \int_0^1 \frac{x^{-a}\ \mathrm{Li}_t(\frac{1}{x^2})}{1+x^2}\,\mathrm{d}x.$$

From the properties of $\mathrm{Li}_t(\frac{1}{x^2})$, see [5], the last integral is expressed as

$$\begin{aligned} X(a,t,\infty) &= \int_0^1 \Delta(a,t;x)\,\mathrm{d}x + (-1)^{t+1}\int_0^1 \frac{x^{-a}}{1+x^2}\left(\mathrm{Li}_t(x^2) + \frac{(2\pi i)^t}{t!}B_t\left(\frac{\log x}{\pi i}\right)\right)\mathrm{d}x \\ &= \int_0^1 \Delta(a,t;x)\,\mathrm{d}x + (-1)^{t+1}\int_0^1 \Delta(-a,t;x)\,\mathrm{d}x + (-1)^{t+1}\frac{(2\pi i)^t}{t!}\int_0^1 \frac{x^{-a}}{1+x^2}B_t\left(\frac{\log x}{\pi i}\right)\mathrm{d}x \end{aligned}$$

where $B_t\left(\frac{\log x}{\pi i}\right)$ are the Bernoulli polynomials. From the recurrence relation of the Bernoulli polynomials,

$$B_j(t) = \sum_{k=0}^{j}\binom{j}{k}B_k\ t^{j-k} = \sum_{k=0}^{j}\binom{j}{k}B_{j-k}t^k,$$

we can express the last integral in the form

$$X(a,t,\infty) = \int_0^1 \Delta(a,t;x)\,\mathrm{d}x + (-1)^{t+1}\int_0^1 \Delta(-a,t;x)\,\mathrm{d}x + (-1)^{t+1}\frac{(2\pi i)^t}{t!}\sum_{j=0}^{t}\binom{t}{j}\frac{B_{t-j}}{(\pi i)^j}\int_0^1 \frac{x^{-a}\ \log^j(x)}{1+x^2}\mathrm{d}x.$$

We now rearrange the above relation and use the results of Theorem 2.1 and Lemma 2.2 to obtain

$$\begin{aligned} S(a,t) &= \sum_{n\geqslant 1}(-1)^{n+1}A_n^{(t)}\left\{\frac{1}{2n+1+a} + \frac{(-1)^{t+1}}{2n+1-a}\right\} \\ &= X(a,t,\infty) + \frac{(-1)^{t+1}2^{t-2}}{t!}\sum_{j=0}^{t}\binom{t}{j}\frac{(-1)^j j! B_{t-j}(\pi i)^{t-j}}{4^j}\left(\zeta\left(j+1,\frac{1-a}{4}\right) - \zeta\left(j+1,\frac{3-a}{4}\right)\right). \end{aligned}$$

If we isolate the $j = 0$ term and put (2.2) for $X(a,t,\infty)$, we obtain

$$\begin{aligned} S(a,t) &= \sum_{n\geqslant 1}(-1)^{n+1}A_n^{(t)}\left\{\frac{1}{2n+1+a} + \frac{(-1)^{t+1}}{2n+1-a}\right\} \\ &= X(a,t,\infty) + \frac{(-1)^{t+1}2^{t-2}(\pi i)^t B_t}{t!}\left(\psi\left(\frac{1-a}{4}\right) - \psi\left(\frac{3-a}{4}\right)\right) \\ &\quad + \frac{(-1)^{t+1}2^{t-2}}{t!}\sum_{j=0}^{t}\binom{t}{j}\frac{(-1)^j j! B_{t-j}(\pi i)^{t-j}}{4^j}\left(\zeta\left(j+1,\frac{1-a}{4}\right) - \zeta\left(j+1,\frac{3-a}{4}\right)\right), \end{aligned}$$

and (2.6) follows. The representation (2.5) is achieved in the following way.

$$\begin{aligned} X(a,t,1) &= \int_0^1 \frac{x^a\ \mathrm{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x = \sum_{n\geqslant 1}\frac{1}{n^t}\sum_{r\geqslant 0}(-1)^r\int_0^1 x^{2n+2r+a}\,\mathrm{d}x \\ &= \sum_{n\geqslant 1}\frac{1}{n^t}\sum_{r\geqslant 0}\frac{(-1)^r}{2n+2r+a+1} = \sum_{n\geqslant 1}\frac{1}{2n^t}\zeta\left(-1,1,\frac{1}{2}(2n+a+1)\right) \\ &= \sum_{n\geqslant 1}\frac{1}{4n^t}\left(\psi\left(\frac{2n+a+1}{4}\right)-\psi\left(\frac{2n+a+3}{4}\right)\right) \qquad (2.8) \\ &= \frac{1}{4}\sum_{n\geqslant 1}\frac{1}{n^t}\left(H_{\frac{n}{2}-\frac{1-a}{4}}-H_{\frac{n}{2}-\frac{3-a}{4}}\right). \end{aligned}$$

Following the same pattern we have,

$$X(-a,t,1) = \int_0^1 \frac{x^{-a}\ \mathrm{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x = \frac{1}{4}\sum_{n\geqslant 1}\frac{1}{n^t}\left(H_{\frac{n}{2}-\frac{1+a}{4}}-H_{\frac{n}{2}-\frac{3+a}{4}}\right)$$

and therefore $X(a,t,1)+(-1)^{t+1}X(-a,t,1)$ implies the result (2.5). □

In the next section we extend the results of the previous section by considering a more general version of the integral (2.1) thereby allowing an extension of the result (2.4).

3 The logarithmic case

This section establishes a number of general skew linear harmonic Euler sum identities.

Theorem 3.1. *Let $p,t\in\mathbb{N}$, $-1\le a<1$ with $a\neq 0$. The following formulas hold true:*

$$S(a,t,p) := p!\sum_{n\geqslant 1}(-1)^{n+1}A_n^{(t)}\left\{\frac{(-1)^p}{(2n+1+a)^{p+1}}+\frac{(-1)^{t+1}}{(2n+1-a)^{p+1}}\right\} \qquad (3.1)$$

$$= \sum_{n\geqslant 1}\frac{(-1)^p\,p!}{4^{p+1}n^t}\left(H^{(p+1)}_{\frac{n}{2}-\frac{1-a}{4}}-H^{(p+1)}_{\frac{n}{2}-\frac{3-a}{4}}+(-1)^{t+1}\left(H^{(p+1)}_{\frac{n}{2}-\frac{1+a}{4}}-H^{(p+1)}_{\frac{n}{2}-\frac{3+a}{4}}\right)\right) \qquad (3.2)$$

$$= \frac{\partial^p}{\partial a^p}\left(X(a,t,\infty)\right)+\frac{(-1)^{t+1}2^{t-2p-2}}{t!}\sum_{j=0}^{t}\frac{1}{4^j}\binom{t}{j}\binom{p+j}{j}(-1)^j j!B_{t-j}(\pi i)^{t-j}$$
$$\times\left(\zeta\left(p+j+1,\frac{1-a}{4}\right)-\zeta\left(p+j+1,\frac{3-a}{4}\right)\right), \qquad (3.3)$$

where $X(a,t,\infty)$ is given by (2.2), and $H_\alpha^{(m)}$ are harmonic numbers.

Proof. From Theorem 2.1, we note that

$$\frac{\partial^p}{\partial a^p}X(a,t,\infty) = \int_0^\infty \frac{x^a\ \ln^p(x)\,\mathrm{Li}_t(x^2)}{1+x^2}\,\mathrm{d}x. \qquad (3.4)$$

From Theorem 2.3 consider (2.4) and differentiate p times with respect to the parameter a so that

$$\frac{\partial^p}{\partial a^p}\left(S\left(a,t\right)\right)=S\left(a,p,t\right)=p!\sum_{n\geqslant 1}\left(-1\right)^{n+1}A_n^{(t)}\left\{\frac{\left(-1\right)^p}{\left(2n+1+a\right)^{p+1}}+\frac{\left(-1\right)^{t+1}}{\left(2n+1-a\right)^{p+1}}\right\}$$
$$=-\frac{\pi}{2}\frac{\partial^p}{\partial a^p}\left(\sec\left(\frac{a\pi}{2}\right)\left(\eta\left(t\right)+\frac{i\exp\left(-\frac{ia\pi}{2}\right)}{2^t}\left(\zeta\left(t,\frac{1-a}{4}\right)-\zeta\left(t,\frac{3-a}{4}\right)\right)\right)\right)$$
$$+\frac{\partial^p}{\partial a^p}\left(\frac{\left(-1\right)^{t+1}2^{t-2}\left(\pi i\right)^tB_t}{t!}\left(\psi\left(\frac{1-a}{4}\right)-\psi\left(\frac{3-a}{4}\right)\right)\right)$$
$$+\frac{\partial^p}{\partial a^p}\left(\frac{\left(-1\right)^{t+1}2^{t-2}}{t!}\sum_{j=0}^{t}\binom{t}{j}\frac{\left(-1\right)^j j!B_{t-j}\left(\pi i\right)^{t-j}}{4^j}\left(\zeta\left(j+1,\frac{1-a}{4}\right)-\zeta\left(j+1,\frac{3-a}{4}\right)\right)\right).$$

After some simplification and rearrangement we obtain the identity (3.3). The representation (3.2) can be attained from the representation (2.8),

$$\frac{\partial^p}{\partial a^p}\left(X\left(a,t,1\right)+\left(-1\right)^{t+1}X\left(-a,t,1\right)\right)=\frac{\partial^p}{\partial a^p}\left(\sum_{n\geqslant 1}\frac{1}{4n^t}\left(\psi\left(\frac{2n+a+1}{4}\right)-\psi\left(\frac{2n+a+3}{4}\right)\right)\right)$$
$$+\left(-1\right)^{t+1}\frac{\partial^p}{\partial a^p}\left(\sum_{n\geqslant 1}\frac{1}{4n^t}\left(\psi\left(\frac{2n-a+1}{4}\right)-\psi\left(\frac{2n-a+3}{4}\right)\right)\right)$$
$$+\sum_{n\geqslant 1}\frac{1}{4^{p+1}n^t}\left(\psi^{(p)}\left(\frac{2n+a+1}{4}\right)-\psi^{(p)}\left(\frac{2n+a+3}{4}\right)\right)$$
$$+\left(-1\right)^{p+t+1}\sum_{n\geqslant 1}\frac{1}{4^{p+1}n^t}\left(\psi^{(p)}\left(\frac{2n-a+1}{4}\right)-\psi^{(p)}\left(\frac{2n-a+3}{4}\right)\right),$$

where $H_\alpha^{(m)}$ are harmonic numbers of order $m\in\mathbb{N}$ with index $\alpha\in\mathbb{C}\backslash\mathbb{Z}_{\leqslant -1}$, and upon simplification of the above expression we obtain (3.2). □

There are some cases of the value of the parameter a of Theorem 3.1 which are worthy of investigation and these are given in the next Corollaries. In particular we examine the three cases of (1). $a=0$, (2). $t=1$ and p is an even integer, and (3). $p+1=t$, for $t\in\mathbb{N}_0$.

Corollary 3.2. *Let $p,t\in\mathbb{N}$, with $p+t$ of odd weight, and put $a=0$. The following formula holds true:*

$$S\left(0,t,p\right):=2\left(-1\right)^p p!\sum_{n\geqslant 1}\left(-1\right)^{n+1}\frac{A_n^{(t)}}{\left(2n+1\right)^{p+1}} \tag{3.5}$$
$$=-\left(\frac{\pi}{2}\right)^{p+1}\left|E_p\right|\eta\left(t\right)+2^t p!\binom{p+t}{t}\beta\left(p+t+1\right) \tag{3.6}$$
$$-2^t\sum_{j=0}^{t-2}\frac{\left(-1\right)^{p+j}p!}{\left(t-j\right)!}\left(i\pi\right)^{t-j}\binom{p+j}{j}B_{t-j}\beta\left(p+j+1\right)$$
$$-2^t\sum_{r=0}^{p-1}\frac{\pi^{p+1-r}\,r!\left(2^{p+1-r}-1\right)}{p+1-r}\binom{p}{r}\binom{t+r-1}{r}\left|B_{p+1-r}\right|\beta\left(t+r\right).$$

where B_z are the Bernoulli numbers, E_z are the Euler numbers and $\beta(z)$ is the Dirichlet beta function.

Proof. For $a=0$ and for odd weight $p+t$ Theorem 3.1 provides

$$\begin{aligned} S(0,t,p) &:= 2(-1)^p p! \sum_{n\geqslant 1} (-1)^{n+1} \frac{A_n^{(t)}}{(2n+1)^{p+1}} \\ &= -\frac{\pi}{2}\lim_{a\to 0}\frac{\partial^p}{\partial a^p}\left(\sec\left(\frac{a\pi}{2}\right)\left(\eta(t)+\frac{i\exp\left(-\frac{ia\pi}{2}\right)}{2^t}\left(\zeta\left(t,\frac{1-a}{4}\right)-\zeta\left(t,\frac{3-a}{4}\right)\right)\right)\right) \qquad (3.7)\\ &+\lim_{a\to 0}\frac{\partial^p}{\partial a^p}\left(\frac{(-1)^{t+1}2^{t-2}}{t!}\sum_{j=0}^{t}\binom{t}{j}\frac{(-1)^j j! B_{t-j}(\pi i)^{t-j}}{4^j}\left(\zeta\left(j+1,\frac{1-a}{4}\right)-\zeta\left(j+1,\frac{3-a}{4}\right)\right)\right). \end{aligned}$$

Consider

$$-\frac{\pi}{2}\eta(t)\lim_{a\to 0}\frac{\partial^p}{\partial a^p}\sec\left(\frac{a\pi}{2}\right)-\frac{\pi}{2}\lim_{a\to 0}\frac{\partial^p}{\partial a^p}\left(\sec\left(\frac{a\pi}{2}\right)\frac{i\exp\left(-\frac{ia\pi}{2}\right)}{2^t}\left(\zeta\left(t,\frac{1-a}{4}\right)-\zeta\left(t,\frac{3-a}{4}\right)\right)\right), \qquad (3.8)$$

now simplify the second term so that (3.8) can be written as

$$\begin{aligned} &= -\left(\frac{\pi}{2}\right)^{p+1}|E_p|\,\eta(t)-\frac{i\pi}{2^{t+1}}\lim_{a\to 0}\frac{\partial^p}{\partial a^p}\left(\left(i+\tan\left(\frac{a\pi}{2}\right)\right)\left(\zeta\left(t,\frac{1-a}{4}\right)-\zeta\left(t,\frac{3-a}{4}\right)\right)\right)\\ &= -\left(\frac{\pi}{2}\right)^{p+1}|E_p|\,\eta(t)-\frac{i\pi}{2^{t+1}}\lim_{a\to 0}\sum_{r=0}^{p}\binom{p}{r}\left(i+\tan\left(\frac{a\pi}{2}\right)\right)^{(r)}\left(\zeta\left(t,\frac{1-a}{4}\right)-\zeta\left(t,\frac{3-a}{4}\right)\right)^{(p-r)} \qquad (3.9) \end{aligned}$$

where $F^{(r)}$ indicates the r^{th} derivative of F with respect to the parameter a. The term

$$\lim_{a\to 0}\frac{\partial^r}{\partial a^r}\left(i+\tan\left(\frac{a\pi}{2}\right)\right)=\begin{cases} 0, & \text{for } r \text{ even}\\ i, & \text{for } r=0\\ \frac{2^{r+1}\left(2^{r+1}-1\right)}{r+1}\left(\frac{\pi}{2}\right)^r|B_{r+1}|, & \text{for } r \text{ odd}\end{cases}$$

and

$$\lim_{a\to 0}\frac{\partial^m}{\partial a^m}\left(\zeta\left(t,\frac{1-a}{4}\right)-\zeta\left(t,\frac{3-a}{4}\right)\right)=4^t\;(t)_m\;\beta(m+t), \quad \text{for } m\in\mathbb{N},$$

where $(t)_m$ is Pochhammer's symbol, see [5]. Now, substituting into (3.9) and simplifying we have

$$\begin{aligned} S(0,t,p) &:= 2(-1)^p p! \sum_{n\geqslant 1}(-1)^{n+1}\frac{A_n^{(t)}}{(2n+1)^{p+1}}\\ &= -\left(\frac{\pi}{2}\right)^{p+1}|E_p|\,\eta(t)-i\pi p!2^{t-1}\binom{p+t-1}{p}\beta(p+t) \qquad (3.10)\\ &- 2^t\sum_{j=0}^{t}\frac{(-1)^{p+j}\;p!}{(t-j)!}(i\pi)^{t-j}\binom{p+j}{j}B_{t-j}\beta(p+j+1)\\ &- 2^t\sum_{r=0}^{p-1}\frac{\pi^{p+1-r}\;r!\left(2^{p+1-r}-1\right)}{p+1-r}\binom{p}{r}\binom{t+r-1}{r}|B_{p+1-r}|\,\beta(t+r). \end{aligned}$$

In the second sum we isolate the $j = t$ term, with the value $B_0 = 1$ and the $j = t - 1$ term, with the value $B_1 = -\frac{1}{2}$, so that in simplifying we produce the result (3.6). □

Corollary 3.3. *For $t = 1$, and p an even integer, the following formula is valid:*

$$S(0,1,p) := 2p! \sum_{n\geqslant 1} (-1)^{n+1} \frac{A_n}{(2n+1)^{p+1}} = 2(p+1)!\beta(p+2) - \left(\frac{\pi}{2}\right)^{p+1} |E_p| \ln(2) - 2\sum_{r=0}^{p-1} \frac{\pi^{p+1-r} r! \left(2^{p+1-r}-1\right)}{p+1-r} \binom{p}{r} |B_{p+1-r}| \beta(r+1). \tag{3.11}$$

where B_z are the Bernoulli numbers, E_z are the Euler numbers and $\beta(\cdot)$ is the Dirichlet beta function.

Proof. The proof follows directly from Corollary 3.2. We remark that this case has been examined in the paper [18], but in slightly different form than Corollary 3.3. An equivalent expression, in less compact form, for (3.11) has been given by Stewart [18]. □

Another special case worthy of mention is for the situation when $p + 1 = t$, this is detailed in the next Corollary.

Corollary 3.4. *Let $p + 1 = t \in \mathbb{N}$. The following relation is valid:*

$$S(0,t,t-1) := 2(-1)^{t-1}(t-1)! \sum_{n\geqslant 1} (-1)^{n+1} \frac{A_n^{(t)}}{(2n+1)^t} \tag{3.12}$$

$$= -\left(\frac{\pi}{2}\right)^t |E_{t-1}| \eta(t) + 2^t (t-1)! \binom{2t-1}{t} \beta(2t) \tag{3.13}$$

$$+ 2^t \sum_{j=0}^{t-2} \frac{(-1)^{t+j}(t-1)!}{(t-j)!} (i\pi)^{t-j} \binom{t+j-1}{j} B_{t-j} \beta(t+j)$$

$$- 2^t \sum_{r=0}^{t-2} \frac{\pi^{t-r}\, r! \left(2^{t-r}-1\right)}{t-r} \binom{t-1}{r} \binom{t+r-1}{r} |B_{t-r}| \beta(t+r),$$

where B_z are the Bernoulli numbers, E_z are the Euler numbers and $\beta(z)$ is the Dirichlet beta function.

Proof. Follows directly from (3.2). □

4 Reciprocity identity

The following Theorem is enunciated by Alzer and Choi [1] regarding a general shuffle relation, and we shall utilize this result in the upcoming Corollary.

Theorem 4.1. *The following formula is given by Alzer and Choi [1, p.14]. Let $p, q \in N$, $\alpha, b \in \mathbb{C}\backslash\mathbb{Z}^-$, with $\alpha \neq b$, then,*

$$\mathbb{S}_{p,q}^{--}(\alpha, b) + \mathbb{S}_{q,p}^{--}(b, \alpha) = \eta(p, \alpha+1)\,\eta(q, b+1) + \sum_{k\geq 1} \frac{1}{(k+\alpha)^p (k+b)^q}. \tag{4.1}$$

The infinite sum can be expressed as finite linear combination of polygamma functions. Here $\eta(p, \alpha+1)$ is the generalized eta function.

Let us recall, from (3.1) and using the notation of (1.6), that

$$S(a,t,p) = \frac{(-1)^p\, p!}{2^{p+1}} \mathbb{S}_{t,p+1}^{--}\left(0, \frac{1+a}{2}\right) + \frac{(-1)^{t+1}\, p!}{2^{p+1}} \mathbb{S}_{t,p+1}^{--}\left(0, \frac{1-a}{2}\right).$$

The case $a = 0$ is described as

$$S(0,t,p) = \frac{(-1)^p\, p!}{2^p} \mathbb{S}_{t,p+1}^{--}\left(0, \frac{1}{2}\right)$$

and its closed form representation given by (3.3). We can now apply Theorem 4.1 to obtain reciprocity relations for some identities of Section 3. Consider the following Corollary.

Corollary 4.2. *Let $p, t \in \mathbb{N}$, with $p+t$ of odd weight. The following identity holds true:*

$$\begin{aligned}
\mathbb{S}_{p+1,t}^{--}\left(\frac{1}{2}, 0\right) &= \sum_{n\geqslant 1} \frac{(-1)^{n+1} A_n^{(p+1)}\left(\frac{1}{2}\right)}{n^t} \\
&= (-1)^{p+1}\, 2^{p+t-1} \binom{p+t-1}{t-1} (\ln 2 - 1) - \frac{(-1)^p\, 2^p}{p!} S(0,t,p+1) \\
&\quad + 2^{p+1}(1 - \beta(p+1))\,\eta(t) + (-1)^{p+1} \sum_{j=1}^{p} \frac{(-1)^{j+1}\, 2^{p+t-j}}{j} \binom{p+t-j-1}{t-1} \zeta(j+1) \\
&\quad + (-1)^{p+1} \sum_{j=1}^{p} j! 2^{p+t+1} \binom{p+t-j-1}{p} (1 - \lambda(j+1)),
\end{aligned} \tag{4.2}$$

where $S(0,t,p+1)$ is the expression (3.6), $\eta\left(p+1, \frac{3}{2}\right)$ is the generalized eta function and $\lambda(j+1)$ is the Dirichlet lambda function, see [5].

Proof. Applying Theorem 4.1, we are able to express

$$\mathbb{S}^{--}_{p+1,t}\left(\frac{1}{2},0\right)=\sum_{n\geqslant 1}\frac{(-1)^{n+1}A_n^{(p+1)}\left(\frac{1}{2}\right)}{n^t}=-\frac{(-1)^p\,2^p}{p!}S\left(0,t,p+1\right)+\eta\left(t,1\right)\eta\left(p+1,\frac{3}{2}\right)+\sum_{j\geq 1}\frac{1}{j^{p+1}\left(j+\frac{1}{2}\right)^t}. \tag{4.3}$$

The sum

$$\begin{aligned}\sum_{j\geq 1}\frac{1}{j^{p+1}\left(j+\frac{1}{2}\right)^t}&=(-1)^{p+1}\,2^{p+t}\binom{p+t-1}{t-1}\left(\psi(1)-\psi\left(\frac{3}{2}\right)\right)\\&=+(-1)^{p+1}\sum_{j=1}^{p}\frac{2^{p+t-j}}{j!}\binom{p+t-j-1}{t-1}\psi^{(j)}(1)\\&+(-1)^t\sum_{j=1}^{t-1}\frac{(-1)^{p+t-j}\,2^{p+t-j}}{j!}\binom{p+t-j-1}{p}\psi^{(j)}\left(\frac{3}{2}\right).\end{aligned} \tag{4.4}$$

The following relations apply, in simplifying the above expression,

$$\begin{aligned}&\psi(1)=-\gamma,\quad \psi\left(\frac{3}{2}\right)=-2\ln 2-\gamma,\quad \psi^{(j)}(1)=(-1)^{j-1}\,(j-1)!\zeta\left(j+1\right),\\&\psi^{(j)}\left(\frac{3}{2}\right)=\psi^{(j)}\left(\frac{1}{2}\right)+(-1)^j\,j!2^{j+1}=(-1)^j\,j!2^{j+1}\left(1-\lambda\left(j+1\right)\right),\\&\eta\left(t,1\right)=\eta\left(t\right),\quad \eta\left(p+1,\frac{3}{2}\right)=2^{p+1}\left(1-\beta\left(p+1\right)\right),\end{aligned}$$

where γ is the familiar Euler-Mascheroni constant (see, *e.g.*, [17, Section 1.2]), and therefore, using (4.4)

$$\begin{aligned}\sum_{j\geq 1}\frac{1}{j^{p+1}\left(j+\frac{1}{2}\right)^t}&=(-1)^{p+1}\,2^{p+t}\binom{p+t-1}{t-1}\left(2\ln 2-2\right)\\&=+(-1)^{p+1}\sum_{j=1}^{p}\frac{(-1)^{j+1}\,2^{p+t-j}}{j}\binom{p+t-j-1}{t-1}\zeta(j+1)\\&+(-1)^{p+1}\sum_{j=1}^{t-1}j!2^{p+t+1}\binom{p+t-j-1}{p}\left(1-\lambda\left(j+1\right)\right).\end{aligned}$$

Substituting this expression in (4.3) we arrive at the the expression (4.2) and the proof is finalized.

□

5 Some examples

Example 5.1. *Let $(a,t,p)=(0,2,3)$, the following identity holds,*

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}A_n^{(2)}}{(2n+1)^4}=\frac{1}{16}\mathbb{S}^{--}_{2,4}\left(0,\frac{1}{2}\right)=\frac{\pi^4 G}{24}+\frac{5\pi^2\beta(4)}{3}-20\beta(6).$$

Example 5.2. *Let $(a,t,p)=(0,4,3)$, the following identity holds,*

$$\frac{1}{16}\mathbb{S}^{--}_{4,4}\left(0,\frac{1}{2}\right)=\sum_{n\geqslant 1}\frac{(-1)^{n+1}A_n^{(4)}}{(2n+1)^4}=\frac{8\pi^4\beta(4)}{45}+\frac{80\pi^2\beta(6)}{3}+280\beta(8).$$

Example 5.3. *Let $(a,t,p)=(0,5,0)$, the following identity holds,*

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}A_n^{(5)}}{2n+1}=\frac{1}{2}\mathbb{S}^{--}_{5,1}\left(0,\frac{1}{2}\right)=-2G\beta(4)-8\zeta(2)\beta(4)+16\beta(6)-\frac{\pi\eta(5)}{4}.$$

Example 5.4. *Let $(a,t,p)=(0,2t,1)$, the following identity holds,*

$$\sum_{n\geqslant 1}(-1)^{n+1}\frac{A_n^{(2t)}}{(2n+1)^2}=\frac{1}{4}\mathbb{S}^{--}_{2t,2}\left(0,\frac{1}{2}\right)=\pi^2 2^{2t-3}\beta(2t)-(2t+1)2^{2t-1}\beta(2t+2)$$
$$+2^{2t-1}\sum_{j=0}^{2t-2}\frac{(-1)^{t+j}(j+1)}{(2t-j)!}(i\pi)^{2t-j}B_{2t-j}\beta(j+2).$$

Example 5.5. *As a last example we utilize the results of Example 5.4, so that the following identity holds,*

$$\mathbb{S}^{--}_{2,2t}\left(\frac{1}{2},0\right)=\sum_{n\geqslant 1}\frac{(-1)^{n+1}A_n^{(2)}\left(\frac{1}{2}\right)}{n^{2t}}=\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^{2t}}\sum_{j=1}^{n}\frac{(-1)^{j+1}}{\left(j+\frac{1}{2}\right)^2}$$
$$=\eta\left(2t,\frac{3}{2}\right)\eta(2,1)+\sum_{k\geq 1}\frac{1}{k^2\left(k+\frac{1}{2}\right)^{2t}}-\mathbb{S}^{--}_{2t,2}\left(0,\frac{1}{2}\right).$$

Simplifying the algebra we arrive at:

$$\mathbb{S}^{--}_{2,2t}\left(\frac{1}{2},0\right)=2^{2t+3}\ t\ \ln(2)+2^{2t}\zeta(2)-\mathbb{S}^{--}_{2t,2}\left(0,\frac{1}{2}\right)-\sum_{j=1}^{2t-1}(2t-j)\ j!\ 2^{2t+3}(1-\lambda(j+1)).$$

6 Concluding remarks

In this paper we have offered an explicit representation for integrals with log-polylog integrand both in the unit domain and on the positive real half line $x\geq 0$, see (2.1) and (3.4). These explicit evaluations enabled the representation, in closed form, of families of skew linear harmonic Euler sums of the form (2.4) and (3.1) which are new in the literature. An application of a reciprocity theorem allowed further explicit evaluations of families of skew linear harmonic Euler sums. A number of pertinent examples were also given to highlight the theorems and corollaries. It is expected that further work will follow examining variant linear Euler sums incorporating parameters.

References

[1] H. Alzer and J. Choi, "Four parametric linear Euler sums," *J. Math. Anal. Appl.*, vol. 484, no. 1, 2020, Art. ID 123661, doi: 10.1016/j.jmaa.2019.123661.

[2] D. Borwein, J. M. Borwein, and R. Girgensohn, "Explicit evaluation of Euler sums," *Proc. Edinburgh Math. Soc. (2)*, vol. 38, no. 2, pp. 277–294, 1995, doi: 10.1017/S0013091500019088.

[3] J. M. Borwein, D. J. Broadhurst, and J. Kamnitzer, "Central binomial sums, multiple Clausen values, and zeta values," *Experiment. Math.*, vol. 10, no. 1, pp. 25–34, 2001, doi: 10.1080/10586458.2001.10504426.

[4] W. Chu, "Infinite series on quadratic skew harmonic numbers," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, vol. 117, no. 2, 2023, Art. ID 75, doi: 10.1007/s13398-023-01407-9.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions. Vols. I, II.* McGraw-Hill Book Co., Inc., New York-Toronto-London, 1953.

[6] P. Flajolet and B. Salvy, "Euler sums and contour integral representations," *Experiment. Math.*, vol. 7, no. 1, pp. 15–35, 1998, doi: 10.1080/10586458.1998.10504356.

[7] L. Lewin, *Polylogarithms and associated functions.* North-Holland Publishing Co., New York-Amsterdam, 1981.

[8] L. A. Medina and V. H. Moll, "The integrals in Gradshteyn and Ryzhik part 27: More logarithmic examples," *Scientia*, vol. 26, pp. 31–47, 2015.

[9] N. Nielsen, *Die Gammafunktion. Band I. Handbuch der Theorie der Gammafunktion. Band II. Theorie des Integrallogarithmus und verwandter Transzendenten.* Chelsea Publishing Co., New York, 1965.

[10] A. S. Nimbran, P. Levrie, and A. Sofo, "Harmonic-binomial Euler-like sums via expansions of $(\arcsin x)^p$," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, vol. 116, no. 1, 2022, Art. ID 23, doi: 10.1007/s13398-021-01156-7.

[11] A. Sofo, "General order Euler sums with multiple argument," *J. Number Theory*, vol. 189, pp. 255–271, 2018, doi: 10.1016/j.jnt.2017.12.006.

[12] A. Sofo, "General order Euler sums with rational argument," *Integral Transforms Spec. Funct.*, vol. 30, no. 12, pp. 978–991, 2019, doi: 10.1080/10652469.2019.1643851.

[13] A. Sofo and J. Choi, "Extension of the four Euler sums being linear with parameters and series involving the zeta functions," *J. Math. Anal. Appl.*, vol. 515, no. 1, 2022, Art. ID 126370, doi: 10.1016/j.jmaa.2022.126370.

[14] A. Sofo and A. S. Nimbran, "Euler-like sums via powers of log, arctan and arctanh functions," *Integral Transforms Spec. Funct.*, vol. 31, no. 12, pp. 966–981, 2020, doi: 10.1080/10652469.2020.1765775.

[15] H. M. Srivastava, M. A. Chaudhry, A. Qadir, and A. Tassaddiq, "Some extensions of the Fermi-Dirac and Bose-Einstein functions with applications to the family of the zeta and related functions," *Russ. J. Math. Phys.*, vol. 18, no. 1, pp. 107–121, 2011, doi: 10.1134/S1061920811010110.

[16] H. M. Srivastava and J. Choi, *Series associated with the zeta and related functions.* Kluwer Academic Publishers, Dordrecht, 2001, doi: 10.1007/978-94-015-9672-5.

[17] H. M. Srivastava and J. Choi, *Zeta and q-Zeta functions and associated series and integrals.* Elsevier, Inc., Amsterdam, 2012, doi: 10.1016/B978-0-12-385218-2.00001-3.

[18] S. M. Stewart, "Explicit expressions for some linear Euler-type sums containing harmonic and skew-harmonic numbers," *J. Class. Anal.*, vol. 20, no. 2, pp. 79–101, 2022, doi: 10.7153/jca-2022-20-07.

CUBO, A Mathematical Journal
Vol. 26, no. 01, pp. 91–105, April 2024
DOI: 10.56754/0719-0646.2601.091

Curvature properties of α-cosymplectic manifolds with $*$-η-Ricci-Yamabe solitons

Vandana[1]
Rajeev Budhiraja[1]
Aliya Naaz Siddiqui Diop[2,✉]

[1] *Department of Mathematics and Humanities, Maharishi Markandeshwar (Deemed to be University), Mullana, 133207, Ambala-Haryana, India.*
chandelvandana93@gmail.com
rajeevkumarbudhiraja@gmail.com

[2] *Division of Mathematics, School of Basic Sciences, Galgotias University, Greater Noida, Uttar Pradesh 203201, India.*
aliya.siddiqui@galgotiasuniversity.edu.in

ABSTRACT

In this research article, we study $*$-η-Ricci-Yamabe solitons on an α-cosymplectic manifold by giving an example in the support and also prove that it is an η-Einstein manifold. In addition, we investigate an α-cosymplectic manifold admitting $*$-η-Ricci-Yamabe solitons under some conditions. Lastly, we discuss the concircular, conformal, conharmonic, and W_2-curvatures on the said manifold admitting $*$-η-Ricci-Yamabe solitons.

RESUMEN

En el presente artículo, estudiamos solitones $*$-η-Ricci-Yamabe en una variedad α-cosimpléctica dando un ejemplo que lo soporta y también probamos que es una variedad η-Einstein. Adicionalmente, investigamos una variedad α-cosimpléctica que admite solitones $*$-η-Ricci-Yamabe bajo ciertas condiciones. Finalmente, discutimos las curvaturas concircular, conforme, con-armónica y W_2 en dicha variedad admitiendo solitones $*$-η-Ricci-Yamabe.

Keywords and Phrases: $*$-η-Ricci-Yamabe soliton, α-cosymplectic manifold, curvature, η-Einstein manifold.

2020 AMS Mathematics Subject Classification: 53B20, 53C21, 53C44, 53C25, 53C50, 53D35

Published: 06 April, 2024
Accepted: 21 January, 2024
Received: 02 August, 2023

1 Introduction

In the year 1982, R. S. Hamilton [9] investigated the concept of Ricci flow on a smooth Riemannian manifold (shortly, RM). A self-similar solution to the Ricci flow is nothing but a Ricci soliton if it moves only by a one parameter family of diffeomorphism and scaling. After introducing the idea of Ricci flow, the theory of Yamabe flow was also initiated by Hamilton in [10] to construct Yamabe metrics on a compact RM. A Yamabe soliton is again corresponded to a self-similar solution of the Yamabe flow.

S. Guler and M. Crasmareanu gave a new class of geometric flow of type (ρ, q), known as Ricci-Yamabe flow in [7]. They proposed the idea of Ricci-Yamabe soliton (shortly, RYS) if it moves only by one parameter group of diffeomorphism and scaling. The metric of the RM (M^n, h), $n > 2$, is said to be RYS (h, V, Λ, ρ, q) if it satisfies the following [20]:

$$\pounds_V h + 2\rho Ric = [2\Lambda - qr]h, \tag{1.1}$$

where Lie derivative operator of the metric h along the vector field V represented by $\pounds_V h$, the Ricci curvature tensor by Ric (the Ricci operator Q defined by $Ric(A, B) = h(QA, B)$ for $A, B \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M), the scalar curvature by r and the real scalars by Λ, ρ, q. According to Λ, RYS will be expanding, steady or shrinking if Λ is negative, zero or positive, respectively.

The concept of η-Ricci-Yamabe solitons (η-RYS) was defined by M. D. Siddiqi, *et al.* [20] in 2020 as a new generalization of RYS and it is defined as

$$\pounds_V h + 2Ric + [2\Lambda - qr]\, h + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where μ is a constant and η is a 1-form on M.

On the other hand, S. Dey and S. Roy [5] inaugurated a new generalization of η-Ricci soliton (η-RS) [3], namely $*$-η-Ricci soliton ($*$-η-RS), defined below:

$$\pounds_V h + 2Ric^* + 2\Lambda h + 2\mu\eta \otimes \eta = 0, \tag{1.3}$$

where $*$-Ricci tensor (shortly, $*$-RT) is denoted by Ric^*.

Tachibana [22] brought up the concept of $*$-RT on almost Hermitian manifolds and afterwards Hamada [8] studied $*$-RT on real hypersurfaces of non-flat complex space forms. Such geometrical works inspired S. Roy, *et al.* to come up with new idea $*$-η-Ricci-Yamabe soliton (shortly, $*$-η-RYS) of type (ρ, q), which is RM and fulfilling [18]

$$\pounds_V h + 2\rho Ric^* + [2\Lambda - qr^*]h + 2\mu\eta \otimes \eta = 0, \tag{1.4}$$

where $r^*(= trace(Ric^*))$ is the $*$-scalar curvature and Λ, ρ, q, μ are real scalars. The $*$-η-RYS is shrinking, steady or expanding if Λ is negative, zero or positive respectively. And they discussed $*$-η-RYS on α-cosymplectic manifolds with a quarter-symmetric metric (shortly, QSM) connection.

Further, A. Haseeb, R. Prasad and F. Mofarreh [12] obtained some interesting results on an α-Sasakian manifold admitting $*$-η-RYS with the potential vector field ζ satisfying conditions $Rim(\zeta, X).Ric = 0$, $Q(h, Ric) = 0$ and pseudo-Ricci symmetric and also showed that α-Sasakian admitting $*$-η-RYS is an η-EM.

In last few years, numerous authors have worked on the characterizations of Ricci, Ricci-Yamabe, η-Ricci-Yamabe and $*$-η-Ricci-Yamabe solitons (respectively, RS, RYS, η-RYS and $*$-η-RYS) in contact geometry. First, the study of RS in contact geometry was proposed by Sharma in [19]. After the initial work on Ricci solitons, some notable classes of contact geometry explored by H. I. Yoldas in [25, 26] where Ricci solitons have been investigated. Later on, D. Dey [2] provided the idea of an almost Kenmotsu metric as RYS. Also, P. Zhang *et al.* [27] have studied conformal RYS on perfect fluid space-time. New type of soliton namely $*$-RYS on contact geometry introduced by M. D. Siddiqi and Akyol in [20] and they have discussed the notion of η-RYS for geometrical structure on Riemannian submersions admitting η-RYS with the potential field. In recent years, a Kenmotsu metric in terms of η-RYS was measured by Yoldas in [23]. Next, the notion of $*$-η-RYS was studied by many authors on different odd dimensional Riemannian manifolds. It should be noted that the geometry of $*$-k-RYS and gradient $*$-k-RYS on Kenmotsu manifolds were given by S. Dey and S. Roy in [4].

We organize this paper as follows: In section 2, we review some basic definitions and tools of an α-cosymplectic manifold M. The main results are stated in section 3. In fact, we prove that an n-dimensional M admitting a $*$-η-RYS is an η-Einstein manifold. Then some curvature tensor conditions are studied on M with $*$-η-RYS. Finally, in section 4, we discuss some results on M when it is ζ-concircularly flat, ζ-conharmonically flat, ζ-W_2 flat and ζ-conformal.

2 Preliminaries

On an $n(= 2m + 1)$-dimensional RM M, if an almost contact metric structure (Φ, ζ, η, h) satisfies the following relations, then M is called an almost contact metric manifold:

$$\Phi^2 A = A - \eta(A)\zeta \tag{2.1}$$

$$\eta(\zeta) = 1, \quad \Phi(\zeta) = 0, \quad \eta(\Phi\zeta) = 0 \tag{2.2}$$

$$h(A, \Phi B) = -h(\Phi A, B), \tag{2.3}$$

$$h(A, \zeta) = \eta(A), \quad h(\Phi A, \Phi B) = h(A, B) - \eta(A)\eta(B), \tag{2.4}$$

for all $A, B \in \chi(M)$, where Φ denotes a $(1,1)$ tensor field, ζ is a vector field, η is a 1-form and h is the compatible Riemannian metric.

The fundamental form ϕ on M is defined as [1]

$$\phi(A, B) = h(\Phi A, B), \tag{2.5}$$

for all $A, B \in \chi(M)$.

If the Nijenhuis tensor field of Φ on M satisfies $N_\Phi(A, B) + 2d\eta(A, B)\zeta = 0$, then M is called a normal almost contact metric manifold. Here

$$N_\Phi(A, B) = \Phi^2[A, B] + [\Phi A, \Phi B] - \Phi[A, \Phi B] - \Phi[\Phi A, B],$$

for any $A, B \in \chi(M)$.

Under the following conditions, a normal almost contact metric manifold M is known as an α-cosymplectic manifold (shortly, α-CM):

(1) $d\eta = 0$,

(2) $d\phi = 2\alpha\eta \wedge \phi$,

for $\alpha \in \mathbb{R}$.

We note that an α-CM can be

(1) a cosymplectic manifold provided that $\alpha = 0$,

(2) an α-Kenmotsu manifold provided that $\alpha \neq 0$.

For an α-CM M, we have

$$(\nabla_A \Phi)B = \alpha(h(\Phi A, B)\zeta - \eta(B)\Phi A) \tag{2.6}$$

and

$$\nabla_A \zeta = -\alpha\Phi^2 A = \alpha[A - \eta(A)\zeta], \tag{2.7}$$

where ∇ is the Levi-Civita connection associated with h.

The main examples and curvature characteristics of α-CM were firstly obtained in [11,14,15]. Also, we have the following relations for the Riemannian curvature tensor Rim and the Ricci curvature tensor Ric of M:

$$Rim(A,B)\zeta = \alpha^2 \left[\eta(A)B - \eta(B)A\right], \tag{2.8}$$

$$Rim(\zeta,A)B = \alpha^2 \left[\eta(B)A - h(A,B)\zeta\right], \tag{2.9}$$

$$Rim(\zeta,A)\zeta = \alpha^2 \left[A - \eta(A)\zeta\right], \tag{2.10}$$

$$\eta(Rim(A,B)C) = \alpha^2 \left[\eta(B)h(A,C) - \eta(A)h(B,C)\right], \tag{2.11}$$

$$Ric(A,\zeta) = -\alpha^2(n-1)\eta(A), \tag{2.12}$$

for all $A, B, C \in \chi(M)$.

In [11], the $*$-RT Ric^* of type $(0,2)$ on an n-dimensional α-CM M is obtained as

$$Ric^*(B,C) = Ric(B,C) + \alpha^2(n-2)h(B,C) + \alpha^2\eta(B)\eta(C), \tag{2.13}$$

for any $B, C \in \chi(M)$.

Let $\{E_i | i = 1, 2, \ldots, n\}$ be an orthonormal basis of $T_p(M)$, $p \in M$. We set $B = C = E_i$ and it is easy to derive the $*$-scalar curvature $r^* = trace(Ric^*)$ as

$$r^* = r + \alpha^2(n-1)^2. \tag{2.14}$$

On the other hand, α-CM M is said to be an η-EM if the Ricci curvature tensor has the following form [24]:

$$Ric(A,B) = uh(A,B) + v\eta(A)\eta(B), \tag{2.15}$$

for $A, B \in \chi(M)$, where u and v being constants.

For this paper, we need some curvature tensors on a RM (M^n, h), which are given below [17]:

$$\overline{\mathbb{C}}(A,B)C = Rim(A,B)C - \frac{r}{n(n-1)}[h(B,C)A - h(A,C)B], \tag{2.16}$$

$$H(A,B)C = Rim(A,B)C - \frac{1}{n-2}[h(B,C)QA - h(A,C)QB + Ric(B,C)A - Ric(A,C)B], \tag{2.17}$$

$$W_2(A,B)C = Rim(A,B)C + \frac{1}{n-1}[h(A,C)QB - h(B,C)QA], \tag{2.18}$$

$$\begin{aligned}\mathbb{C}^*(A,B)C = {} & Rim(A,B)C - \frac{1}{n-2}[Ric(B,C)A - Ric(A,C)B + h(B,C)QA \\ & - h(A,C)QB] + \frac{r}{(n-1)(n-2)}[h(B,C)A - h(A,C)B],\end{aligned} \tag{2.19}$$

where $\overline{\mathbb{C}}$, H, W_2 and $\mathbb{C}^*$ represent the concircular curvature tensor [16], the conharmonic curvature tensor [13], the W_2-curvature tensor [16] and the conformal curvature tensor [6].

3 On α-CM admitting $*$-η-RYS

Let us take a $*$-η-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$ on an n-dimensional α-CM M, which is given by

$$(\pounds_\zeta h)(A,B) + 2\rho Ric^*(A,B) + [2\Lambda - qr^*]\, h(A,B) + 2\mu\eta(A)\eta(B) = 0, \tag{3.1}$$

for any $A, B \in \chi(M)$.

Theorem 3.1. *An n-dimensional α-CM M admitting $*$-η-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$ is an η-EM of the constant scalar curvature r. Moreover, the scalars Λ and μ are related by*

$$\Lambda + \mu = \frac{qr}{2} + \frac{q\alpha^2(n-1)^2}{2}. \tag{3.2}$$

Proof. From (2.4) and (2.7), we arrive at

$$(\pounds_\zeta h)(A,B) = h(\nabla_A \zeta, B) + h(A, \nabla_B \zeta) = 2\alpha\Big(h(A,B) - \eta(A)\eta(B)\Big). \tag{3.3}$$

Substitute (3.3)) into (3.1) to get

$$Ric^*(A,B) = -\frac{1}{\rho}\left(\Lambda - \frac{qr^*}{2} + \alpha\right) h(A,B) - \frac{(\mu-\alpha)}{\rho}\eta(A)\eta(B). \tag{3.4}$$

By using (2.13) and (2.14) in (3.4), we obtain

$$Ric(A,B) = \left[-\frac{1}{\rho}\left(\Lambda - \frac{qr}{2} - \frac{q\alpha^2(n-1)^2}{2} + \alpha\right) - \alpha^2(n-2)\right] h(A,B) - \left(\frac{(\mu-\alpha)}{\rho} + \alpha^2\right)\eta(A)\eta(B), \tag{3.5}$$

that is,

$$Ric(A,B) = \sigma_1 h(A,B) + \sigma_2 \eta(A)\eta(B), \tag{3.6}$$

where

$$\sigma_1 = -\frac{1}{\rho}\left(\Lambda - \frac{qr}{2} - \frac{q\alpha^2(n-1)^2}{2} + \alpha\right) - \alpha^2(n-2), \quad \sigma_2 = -\left(\frac{(\mu-\alpha)}{\rho} + \alpha^2\right).$$

Now, if we fix $B = \zeta$ in (3.6), then we can easily get the following relation:

$$Ric(A,\zeta) = \left[-\frac{1}{\rho}\left(\Lambda - \frac{qr}{2} - \frac{q\alpha^2(n-1)^2}{2} + \mu\right) - \alpha^2(n-1)\right]\eta(A). \tag{3.7}$$

Using (2.12) and values of σ_1 and σ_2 in (3.7), we can have (3.2). Also, on contracting (3.6) and using the values of σ_1 and σ_2, we find

$$r = (n-1)\left(\frac{\mu}{\rho} - \frac{\alpha}{\rho} - \alpha^2(n-1)\right), \tag{3.8}$$

where μ and $\rho(\neq 0)$ are constant.

Thus, (3.6) together with (3.2) and (3.8) give the relation of Λ and μ, which shows that $*$-η-RYS on α-CM is an η-EM. □

Remark 3.2. *For the particular value of $\rho = 0$ in (3.1), an n-dimensional α-CM M endowed with $*$-η-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$ furnishes the scalar quantities as $\Lambda = -\alpha + \frac{qr^*}{2}$ and $\mu = \alpha$.*

First we give the more general construction of α-cosymplectic manifold:

Example 3.3. *Let $(N, J, \tilde{h})$ be a Kähler manifold. Denote by $\mathbb{R}\times_\sigma N$ the manifold $(\mathbb{R}\times N, \Phi, \zeta, \eta, h)$, where Φ is the tensor field such that*

$$\Phi\left(\frac{d}{dt}\right) = 0, \quad \Phi(A) = J(A), \quad A \in TN,$$
$$\zeta = \frac{d}{dt}, \quad \eta = dt, \quad h = dt \otimes dt + \exp(2\alpha t)\tilde{h}, \quad \alpha \in \mathbb{R}.$$

Putting $\sigma = \exp(\alpha t)$, h is the warped product metric of the Euclidean metric and $\tilde{h}$ by means of the function σ. Then $\mathbb{R} \times_\sigma N$ is α-cosymplectic and $(N, \tilde{h})$ is a totally umbilical submanifold with mean curvature vector $-\alpha\zeta$. Assume that $\alpha \neq 0$. Applying well-known curvature formulas, one relates the Ricci tensors of N and $\mathbb{R} \times_\sigma N$. But here we consider the flat Kähler manifold $\mathbb{R}^4$ endowed with the canonical complex structure and then the α-cosymplectic manifold $\mathbb{R} \times_\sigma \mathbb{R}^4$. If $\alpha = 0$, one has $\sigma = 1$, $\mathbb{R} \times_\sigma N = \mathbb{R} \times N$ is cosymplectic and N is totally geodetic. In this case the Ricci tensors are related by:

$$Ric(A, B) = \tilde{Ric}(A - \eta(A)\zeta, B - \eta(B)\zeta). \tag{3.9}$$

It follows that if N is an Einstein manifold, then $\mathbb{R} \times N$ is η-Einstein.

Next, by giving the following example we can show the existence of this soliton in α-cosymplectic manifold:

Example 3.4. *Recall an example of 5-dimensional α-CM in [11], that is,*

$$M = \{(x_1, x_2, y_1, y_2, z) \in \mathbb{R}^5, \Phi, \zeta, \eta, h\},$$

where (x_1, x_2, y_1, y_2, z) are the standard coordinates in $\mathbb{R}^5$.

The linearly independent vector fields on M are denoted by $E_1 = \exp^{\alpha z} \partial x_1$, $E_2 = \exp^{\alpha z} \partial x_2$, $E_3 = \exp^{\alpha z} \partial y_1$, $E_4 = \exp^{\alpha z} \partial y_2$ and $E_5 = \zeta = -\partial z$ for $i = \{1, 2\}$. Thus, h and Φ are respectively defined as

$$h(E_i, E_i) = 1, \quad h(E_i, E_j) = 0, \quad i \neq j = \{1, 2, 3, 4, 5\}$$

and

$$\Phi E_1 = -E_2, \quad \Phi E_2 = E_1, \quad \Phi E_3 = -E_4, \quad \Phi E_4 = E_3, \quad \Phi E_5 = \Phi\zeta = 0.$$

By the linearity of these tensors, it is quite easy to compute (2.1)-(2.4)*. Also,* (2.6) *and* (2.7) *are verified in [11].*

By applying Koszul's formula, Rim of M (see [11]) can be obtained easily and hence the components Ric of Ricci tensor of M are: $Ric(E_i, E_i) = -4\alpha^2$ for $i = \{1,2,3,4,5\}$. Since $r = \sum_{i=1}^{5} Ric(E_i, E_i)$, so we have $r = -20\alpha^2$.

Now, we use (3.7) *and find*

$$Ric(E_5, E_5) = Ric(\zeta, \zeta) = \left[-\frac{1}{\rho}\left(\Lambda + 2q\alpha^2 + \mu\right) - 4\alpha^2 \right].$$

By equating the values of $Ric(\zeta,\zeta)$, we arrive at a relation: $\Lambda + \mu = -2q\alpha^2$. We also verify this relation for $n = 5$ by using (3.2)*. Thus, h gives an $*$-η-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$ on an α-cosymplectic manifold M of dimension* 5*.*

On the other hand, suppose that an n-dimensional α-CM M admitting $*$-η-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$ satisfies

$$Q(h, Ric)(A, B, C, D) = 0, \tag{3.10}$$

where $Q(h, Ric)(A, B, C, D) = (h(A, B).Ric)(C, D)$, for all vector fields A, B, C, D on M. This can be expressed as

$$\begin{aligned} Q(h, Ric)(A, B, C, D) =& h(B, C)Ric(A, D) - h(A, C)Ric(B, D) \\ &+ h(B, D)Ric(A, C) - h(A, D)Ric(B, C). \end{aligned} \tag{3.11}$$

Theorem 3.5. *If $*$-η-RYS on an α-CM M satisfies $Q(h, Ric) = 0$, then*

$$\Lambda = \frac{q}{2}\left(r + \alpha^2(n-1)^2\right) - \alpha\left(1 - \alpha\rho\right), \tag{3.12}$$

$$\mu = \alpha(1 - \alpha\rho). \tag{3.13}$$

Proof. From the expressions (3.6), (3.10) and (3.11), we have

$$\sigma_2[h(B, C)\eta(A)\eta(D) - h(A, C)\eta(B)\eta(D) + h(B, D)\eta(A)\eta(C) - h(A, D)\eta(B)\eta(C)] = 0. \tag{3.14}$$

Above equation follows that $\sigma_2 = 0$, which implies that

$$\mu = \alpha(1 - \alpha\rho).$$

We obtain the following from (3.2)

$$\Lambda = \frac{q}{2}\left(r + \alpha^2(n-1)^2\right) - \alpha\left(1 - \alpha\rho\right). \tag{3.15}$$

□

Now, by using these values of σ_1, σ_2 and Λ as well as μ in (3.6), we calculate

$$Ric(A, B) = -(\alpha^2(n-1))h(A, B). \tag{3.16}$$

Thus, from above we can state the following result:

Corollary 3.6. *If $*$-η-RYS on an α-CM M satisfies $Q(h, Ric) = 0$, then M is an EM.*

Next, we have

$$Rim(\zeta, A).Ric = 0, \tag{3.17}$$

then we have

$$Ric(Rim(\zeta, A)B, C) + Ric(B, Rim(\zeta, A)C) = 0, \tag{3.18}$$

for all vector fields A, B, C on M.

Theorem 3.7. *If $*$-η-RYS on an α-CM M satisfies $Rim(\zeta, A).Ric = 0$, then either M becomes CM or we have*

$$\Lambda = \frac{q}{2}\left(r + \alpha^2(n-1)^2\right) - \alpha(1 - \alpha\rho) \tag{3.19}$$

$$\mu = \alpha(1 - \alpha\rho). \tag{3.20}$$

Proof. In view of (3.6) and (3.18), we compute

$$\alpha^2\sigma_2\big(2\eta(A)\eta(B)\eta(C) - \eta(C)h(A, B) - \eta(B)h(A, C)\big) = 0. \tag{3.21}$$

Putting $C = \zeta$ into (3.21) and using (2.4), it is quite easy to see

$$\alpha^2\sigma_2 h(\Phi A, \Phi B) = 0, \tag{3.22}$$

which implies either $\alpha = 0$ or $\sigma_2 = 0$. Further, from later case we find $\mu = \alpha(1 - \alpha\rho)$ and hence from (3.2), we calculate the value of Λ. From the first case we can also say that M is CM. This is the desired result. □

Next, by using the values of Λ as well μ in (3.6), we have

$$Ric(A, B) = -(\alpha^2(n-1))h(A, B). \tag{3.23}$$

Thus, we can state the following:

Corollary 3.8. *If $* - \eta$-RYS on an α-CM M satisfies $Rim(\zeta, A).Ric = 0$ then M is either an EM or CM.*

The non-flat manifold M of n-dimension is named pseudo Ricci symmetric, if $Ric\,(\neq 0)$ of M satisfies the condition:

$$(\nabla_C Ric)\,(A, B) = 2\kappa(C)Ric(A, B) + \kappa(A)Ric(C, B) + \kappa(B)Ric(C, A), \tag{3.24}$$

where κ is a non-zero 1-form. In particular, M is said to be Ricci symmetric if $\kappa = 0$.

Theorem 3.9. *If an α-CM M admitting $*$-η-RYS is pseudo-Ricci-symmetric, then M is either Ricci symmetric or CM.*

Proof. The covariant derivative of (3.6) leads

$$(\nabla_C Ric)\,(A, B) = \nabla_C\left[\sigma_1 h(A, B) + \sigma_2\eta(A)\eta(B)\right] = \alpha\sigma_2\left[h(\Phi A, \Phi C)\eta(B) + \eta(A)h(\Phi B, \Phi C)\right]. \tag{3.25}$$

Further, we use the relations (3.6), (3.24), (3.25) and obtain

$$\begin{aligned} &2\kappa(C)\left[\sigma_1 h(A, B) + \sigma_2\eta(A)\eta(B)\right] + \kappa(A)\left[\sigma_1 h(C, B) + \sigma_2\eta(C)\eta(B)\right] \\ &+ \kappa(B)\left[\sigma_1 h(C, A) + \sigma_2\eta(C)\eta(A)\right] = \alpha\sigma_2\left[h(\Phi A, \Phi C)\eta(B) + \eta(A)h(\Phi B, \Phi C)\right]. \end{aligned} \tag{3.26}$$

Taking $C = B = \zeta$ in (3.26), we get

$$(\sigma_1 + \sigma_2)\big(\kappa(A) + 3\eta(A)\kappa(\zeta)\big) = 0,$$

which gives either

$$\kappa(A) = -3\eta(A)\kappa(\zeta) \tag{3.27}$$

or

$$\sigma_1 + \sigma_2 = 0. \tag{3.28}$$

Putting $A = \zeta$ in (3.27), we have $\kappa(\zeta) = 0$, which further implies that $\kappa(A) = 0$. Also, from (3.28) and (3.2), we can have $\alpha^2(n-1) = 0$. This implies that $\alpha = 0$ because $n \neq 1$. Thus, we arrive at our desired result. □

4 Some curvature properties on α-CM admitting $*$-η-RYS

This section deals with the curvature properties on M admitting $*$-η-RYS. We mainly discuss the conditions that M is ζ-concircularly flat, ζ-conharmonically flat, ζ-W_2 flat and ζ-conformal flat.

Theorem 4.1. *Let M be an n-dimensional α-CM admitting $*$-η-RYS $(h, \zeta, \Lambda, \mu, \rho, q)$, where ζ being the Reeb vector field on M. Then M is*

(1) *ζ-concircularly flat if and only if $\mu = \alpha - \rho\alpha^2$.*

(2) *ζ-conformal curvature flat.*

(3) *ζ-conharmonically flat if and only if $\mu = \alpha + (n-1)\alpha^2\rho$.*

(4) *$\zeta - W_2-$curvature flat if and only if $\mu = \alpha - \rho\alpha^2$.*

Proof. By using the property $h(QA, B) = Ric(A, B)$ in (3.6), we arrive at

$$QB = \sigma_1 B + \sigma_2 \eta(B)\zeta, \tag{4.1}$$

where $\sigma_1 = -\frac{1}{\rho}\left(\Lambda - \frac{qr}{2} - \frac{q\alpha^2(n-1)^2}{2} + \alpha\right) - \alpha^2(n-2)$ and $\sigma_2 = -\left(\frac{\mu-\alpha}{\rho} + \alpha^2\right)$.

Firstly, we put $C = \zeta$ into (2.16) and use the relations (2.4), (2.8) and (3.8), we have

$$\overline{\mathbb{C}}(A, B)\zeta = \frac{1}{n}\left(\frac{\mu}{\rho} - \frac{\alpha}{\rho} + \alpha^2\right)(\eta(A)B - \eta(B)A), \tag{4.2}$$

which gives $\overline{\mathbb{C}}(A, B)\zeta = 0$ if and only if $\mu = \alpha - \rho\alpha^2$.

Secondly, if we put $C = \zeta$ and use (2.8), (2.12), (4.1) in (2.19), then we have

$$\mathbb{C}^*(A, B)\zeta = \left(\frac{\alpha^2 - \sigma_1}{n-2} + \frac{r}{(n-1)(n-2)}\right)(\eta(B)A - \eta(A)B). \tag{4.3}$$

Again, using the value of σ_1, (3.2) and (3.8), we have

$$\mathbb{C}^*(A, B)\zeta = 0. \tag{4.4}$$

Thirdly, we take $C = \zeta$ in (2.17) and make use of (2.8), (4.1) and (2.12), we get

$$H(A, B)\zeta = \left(\frac{\sigma_1 - \alpha^2}{n-2}\right)(\eta(A)B - \eta(B)A). \tag{4.5}$$

This implies

$$H(A, B)\zeta = 0$$

if and only if

$$\sigma_1 = \alpha^2.$$

Thus,

$$H(A, B)\zeta = 0$$

if and only if

$$\mu = \alpha + (n-1)\alpha^2\rho.$$

Lastly, by taking $C = \zeta$ and using (2.8) and (4.1) in (2.18), we conclude

$$W_2(A, B)\zeta = \left(\alpha^2 + \frac{\sigma_1}{n-1}\right)(\eta(A)B - \eta(B)A). \tag{4.6}$$

From (4.6),

$$W_2(A, B)\zeta = 0$$

if and only if

$$\alpha^2 + \frac{\sigma_1}{n-1} = 0.$$

This further implies that

$$W_2(A, B)\zeta = 0$$

if and only if

$$\mu = \alpha - \rho\alpha^2. \qquad \square$$

Remark 4.2. *We observe that above results are true only for α-Kenmotsu manifolds because Λ and μ are depending on α. But for $\alpha = 0$, one puts in* (3.1) *$B = \xi$ obtains*

$$\Lambda + \mu = \frac{1}{2}qr.$$

Then (3.1) *implies*

$$Ric = \frac{\mu}{\rho}(h - \eta \otimes \eta).$$

So, according to the cases μ is zero or non-zero, M is Ricci-flat or η-Einstein for $\alpha = 0$. This is consistent with the formula (3.9), *when N is Einstein.*

Remark 4.3. *If M is a cosymplectic manifold, then we have*

$$\overline{\mathbb{C}}(A, B)\zeta = -\left(\frac{\mu}{n-\rho}\right)(\eta(B)A - \eta(A)B).$$

and similar relations can be obtained for $H(A, B)\zeta$ and $W_2(A, B)\zeta$, while

$$\mathbb{C}^*(A, B)\zeta = 0.$$

By the above formulas, one has $\mu = 0$ *if and only if* $\overline{\mathbb{C}}(A, B)\zeta = 0$ *if and only if* $H(A, B)\zeta = 0$ *if and only if* $W_2(A, B)\zeta = 0$.

Acknowledgments

The authors would like to thank the reviewer for the valuable comments and constructive suggestions.

Conflict of Interest: The authors declare no competing interests.

Funding: Not Applicable.

Data Availability: Not Applicable.

Ethical Conduct: The manuscript is not currently being submitted for publication elsewhere and has not been previously publish.

References

[1] D. E. Blair, *Contact manifolds in Riemannian geometry*, ser. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1976, vol. 509.

[2] J. T. Cho and M. Kimura, "Ricci solitons and real hypersurfaces in a complex space form," *Tohoku Math. J. (2)*, vol. 61, no. 2, pp. 205–212, 2009, doi: 10.2748/tmj/1245849443.

[3] D. Dey, "Almost Kenmotsu metric as Ricci-Yamabe soliton," 2020, *arXiv:2005.02322*.

[4] S. Dey, P. L.-i. Laurian-ıoan, and S. Roy, "Geometry of $*$-k-ricci-yamabe soliton and gradient $*$-k-ricci-yamabe soliton on kenmotsu manifolds," *Hacettepe Journal of Mathematics and Statistics*, vol. 52, no. 4, p. 907–922, 2023, doi: 10.15672/hujms.1074722.

[5] S. Dey and S. Roy, "$*$-η-Ricci soliton within the framework of Sasakian manifold," *J. Dyn. Syst. Geom. Theor.*, vol. 18, no. 2, pp. 163–181, 2020, doi: 10.1080/1726037X.2020.1856339.

[6] L. P. Eisenhart, *Riemannian Geometry.* Princeton University Press, Princeton, NJ, 1949, 2d printing.

[7] S. Güler and M. Crasmareanu, "Ricci-Yamabe maps for Riemannian flows and their volume variation and volume entropy," *Turkish J. Math.*, vol. 43, no. 5, pp. 2631–2641, 2019, doi: 10.3906/mat-1902-38.

[8] T. Hamada, "Real hypersurfaces of complex space forms in terms of Ricci $*$-tensor," *Tokyo J. Math.*, vol. 25, no. 2, pp. 473–483, 2002, doi: 10.3836/tjm/1244208866.

[9] R. S. Hamilton, "Three-manifolds with positive Ricci curvature," *J. Differential Geometry*, vol. 17, no. 2, pp. 255–306, 1982.

[10] R. S. Hamilton, "The Ricci flow on surfaces," in *Mathematics and general relativity (Santa Cruz, CA, 1986)*, ser. Contemp. Math. Amer. Math. Soc., Providence, RI, 1988, vol. 71, pp. 237–262, doi: 10.1090/conm/071/954419.

[11] A. Haseeb, D. G. Prakasha, and H. Harish, "$*$-conformal η-Ricci solitons on α-cosymplectic manifolds," *IJAA*, vol. 19, no. 2, pp. 165–179, 2021, Art. ID 5718736, doi: 10.28924/2291-8639.

[12] A. Haseeb, R. Prasad, and F. Mofarreh, "Sasakian manifolds admitting $*$-η-Ricci-Yamabe solitons," *Adv. Math. Phys.*, 2022, Art. ID 5718736, doi: 10.1155/2022/5718736.

[13] Y. Ishii, "On conharmonic transformations," *Tensor (N.S.)*, vol. 7, pp. 73–80, 1957.

[14] Z. Olszak, "Locally conformal almost cosymplectic manifolds," *Colloq. Math.*, vol. 57, no. 1, pp. 73–87, 1989, doi: 10.4064/cm-57-1-73-87.

[15] Z. Olszak and R. Roşca, "Normal locally conformal almost cosymplectic manifolds," *Publ. Math. Debrecen*, vol. 39, no. 3-4, pp. 315–323, 1991, doi: 10.5486/pmd.1991.39.3-4.12.

[16] G. P. Pokhariyal and R. S. Mishra, "Curvature tensors' and their relativistics significance," *Yokohama Math. J.*, vol. 18, pp. 105–108, 1970.

[17] S. Roy, S. Dey, and A. Bhattacharyya, "Some results on η-Yamabe solitons in 3-dimensional trans-Sasakian manifold," *Carpathian Math. Publ.*, vol. 14, no. 1, pp. 158–170, 2022, doi: 10.15330/cmp.14.1.158-170.

[18] S. Roy, S. Dey, A. Bhattacharyya, and M. D. Siddiqi, "$*$-η-Ricci-Yamabe solitons on α-cosymplectic manifolds with a quarter-symmetric metric connection," 2021, *arXiv:2109.04700.*

[19] R. Sharma, "Certain results on K-contact and (k, μ)-contact manifolds," *J. Geom.*, vol. 89, no. 1-2, pp. 138–147, 2008, doi: 10.1007/s00022-008-2004-5.

[20] M. D. Siddiqi and M. A. Akyol, "η-Ricci-Yamabe soliton on Riemannian submersions from Riemannian manifolds," 2021, *arXiv:2004.14124*.

[21] A. N. Siddiqui and M. D. Siddiqi, "Almost Ricci-Bourguignon solitons and geometrical structure in a relativistic perfect fluid spacetime," *Balkan J. Geom. Appl.*, vol. 26, no. 2, pp. 126–138, 2021.

[22] S.-i. Tachibana, "On almost-analytic vectors in almost-Kählerian manifolds," *Tohoku Math. J. (2)*, vol. 11, pp. 247–265, 1959, doi: 10.2748/tmj/1178244584.

[23] H. I. Yoldaş, "On Kenmotsu manifolds admitting η-Ricci-Yamabe solitons," *Int. J. Geom. Methods Mod. Phys.*, vol. 18, no. 12, 2021, Art. ID 2150189, doi: 10.1142/S0219887821501899.

[24] H. I. Yoldaş, "Some results on α-cosympletic manifolds," *Bull. Transilv. Univ. Braşov Ser. III. Math. Comput. Sci.*, vol. 1(63), no. 2, pp. 115–128, 2021, doi: 10.31926/but.mif.2021.1.63.2.10.

[25] H. I. Yoldaş, "Some soliton types on Riemannian manifolds," *Rom. J. Math. Comput. Sci.*, vol. 11, no. 2, pp. 13–20, 2021.

[26] H. I. Yoldaş, "Some classes of Ricci solitons on Lorentzian α-Sasakian manifolds," *Differ. Geom. Dyn. Syst.*, vol. 24, pp. 232–244, 2022, doi: 10.3390/e24020244.

[27] P. Zhang, Y. Li, S. Roy, S. Dey, and A. Bhattacharyya, "Geometrical structure in a perfect fluid spacetime with conformal ricci–yamabe soliton," *Symmetry*, vol. 14, no. 3, 2022, Art. ID 594, doi: 10.3390/sym14030594 .

Cubo A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 26, no. 01, pp. 107–122, April 2024
DOI: 10.56754/0719-0646.2601.107

On a class of fractional $p(x, y)-$Kirchhoff type problems with indefinite weight

Seyed Mostafa Sajjadi[1]
Ghasem Alizadeh Afrouzi[1,✉]

[1] *Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.*
sjadysydmstfy@gmail.com
afrouzi@umz.ac.ir✉

ABSTRACT

This paper is concerned with a class of fractional $p(x, y)-$Kirchhoff type problems with Dirichlet boundary data along with indefinite weight of the following form

$$\begin{cases} M\left(\int_Q \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\right) \\ \left(-\Delta_{p(x)}\right)^s u(x) + |u(x)|^{q(x)-2}u(x) \\ = \lambda V(x)|u(x)|^{r(x)-2}u(x) & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases}$$

By means of direct variational approach and Ekeland's variational principle, we investigate the existence of nontrivial weak solutions for the above problem in case of the competition between the growth rates of functions p and r involved in above problem, this fact is essential in describing the set of eigenvalues of this problem.

RESUMEN

Este artículo estudia una clase de problemas de tipo $p(x, y)-$Kirchhoff fraccionarios con data Dirichlet en el borde junto con un peso indefinido de la siguiente forma

$$\begin{cases} M\left(\int_Q \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\right) \\ \left(-\Delta_{p(x)}\right)^s u(x) + |u(x)|^{q(x)-2}u(x) \\ = \lambda V(x)|u(x)|^{r(x)-2}u(x) & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases}$$

A través del enfoque variacional directo y el principio variacional de Ekeland, investigamos la existencia de soluciones débiles no triviales para el problema anterior en el caso de competencia entre las tasas de crecimiento de las funciones p y r involucradas en el problema. Este hecho es esencial para describir el conjunto de valores propios de este problema.

Keywords and Phrases: Kirchhoff type problems, indefinite weight, Ekeland's variational principle, variable exponent, fractional $p(x, y)-$Laplacian problems.

2020 AMS Mathematics Subject Classification: 35R11, 35D30, 35J20, 46E35.

Published: 08 April, 2024
Accepted: 26 January, 2024
Received: 19 August, 2023

1 Introduction

Fractional differential equations have been an area of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. In [5], a non-Kirchhoff equation was investigated, which had an indefinite weight function.

In [3], a Kirchhoff-type equation was surveyed, which lacked an indefinite weight function. We combine these equations and, using the methods applied in [3] and [5], open a corridor to an equation that is both Kirchhoff-type and equipped with an indefinite weight function. In this paper, we aim to discuss the existence of a nontrivial solution for a fractional $p(x,y)-$Kirchhoff type eigenvalue problem

$$\begin{cases} M\left(\int_Q \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\right)\left(-\Delta_{p(x)}\right)^s u(x) + |u(x)|^{q(x)-2}u(x) = \lambda V(x)|u(x)|^{r(x)-2}u(x), \quad \text{in } \Omega, \\ u = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a Lipschitz bounded open domain and $Q := \mathbb{R}^{2N} \backslash (C\Omega \times C\Omega)$ with $C\Omega = \mathbb{R}^N \backslash \Omega$, $N \geq 3$, $p : \overline{Q} \to (1, +\infty)$ is continuous, $q,\ r \in C_+\left(\overline{\Omega}\right)$, $V : \Omega \to \mathbb{R}$ is an indefinite weight function in the sense that it is allowed to change sign in Ω, λ is a positive constant and $s \in (0,1)$ and $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function which satisfies the (polynomial growth condition)

($\mathbf{M}_1$): There exist $m_2 \geq m_1 > 0$ and $\alpha > 1$ such that

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1} \quad \text{for all} \quad t \in \mathbb{R}^+.$$

Here the operator $\left(-\Delta_{p(x)}\right)^s$ is the fractional $p(x)-$Laplacian operator defined as follows

$$\left(-\Delta_{p(x)}\right)^s u(x) = p \cdot v \cdot \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}\,dy, \quad \text{for all } x \in \mathbb{R}^N,$$

where $p \cdot v \cdot$ is a commonly used abbreviation in the principal value sense.

Throughout this paper, we assume that

$$\alpha p(x,x) < q(x) < p_s^*(x) := \frac{Np(x,x)}{N - sp(x,x)}, \quad p(x,y) < \frac{N}{s}, \quad \forall x, y \in \overline{\Omega}, \tag{1.2}$$

where $p_s^*(x)$ is the so-called critical exponent in fractional Sobolev space with variable exponent.

If $s = 1$ problem (1.1) becomes the $p(\cdot)-$Kirchhoff Laplacian problem.

Problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguishing feature of Eq. (1.3) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ which depends on the average $\frac{1}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$, and hence the equation is no longer a pointwise identity. The parameters in (1.3) have the following meanings: L is the length of the string, h is the area of the crosssection, E is the Young modulus of the material, ρ is the mass density and ρ_0 is the initial tension.

This paper is organised as follows. In Section 2, we give some definitions and fundamental properties of generalized Lebesgue spaces $L^{q(x)}(\Omega)$ and fractional Sobolev spaces with variable exponent $W^{s,q(x),p(x,y)}(\Omega)$, moreover, we compare the space $W^{s,q(x),p(x,y)}(\Omega)$ with the fractional Sobolev space X and we study the completeness, reflexivity and separability of these spaces. Furthermore, we establish a continuous and compact embedding theorem of these spaces into variable exponent Lebesgue spaces. In Section 3, we discuss the existence of nontrivial weak solutions of problem in sublinear case, when $1 < r(x) < p^-$ for all $x \in \overline{\Omega}$. We apply Ekeland's variational principle.

2 Preliminaries

Consider the set

$$C_+ \left(\overline{\Omega} \right) = \{ h \in C \left(\overline{\Omega} \right) : h(x) > 1 \quad \text{for all } x \in \overline{\Omega} \}.$$

For all $h \in C_+ \left(\overline{\Omega} \right)$, we define

$$h^+ = \sup_{x \in \overline{\Omega}} h(x) \quad \text{and} \quad h^- = \inf_{x \in \overline{\Omega}} h(x) \quad \text{such that,} \quad 1 < h^- \leq h(x) \leq h^+ < +\infty. \tag{2.1}$$

For any $h \in C_+ \left(\overline{\Omega} \right)$, we define the variable exponent Lebesgue space as

$$L^{h(x)}(\Omega) = \left\{ u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{h(x)} dx < +\infty \right\}.$$

This vector space endowed with the Luxemburg norm, which is defined by

$$\|u\|_{L^{h(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{h(x)} dx \leq 1 \right\}$$

is a separable reflexive Banach space.

Let $\hat{h} \in C_+\left(\overline{\Omega}\right)$ be the conjugate exponent of h, that is, $1/h(x) + 1/\hat{h}(x) = 1$.

Then we have the following Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{h^-} + \frac{1}{\hat{h}^-}\right)\|u\|_{L^{h(x)}(\Omega)}\|v\|_{L^{\hat{h}(x)}(\Omega)} \le 2\|u\|_{L^{h(x)}(\Omega)}\|v\|_{L^{\hat{h}(x)}(\Omega)}$$

Moreover, if $h_1, h_2, h_3 \in C_+\left(\overline{\Omega}\right)$ and $1/h_1 + 1/h_2 + 1/h_3 = 1$, then for any $u \in L^{h_1(x)}(\Omega)$, $v \in L^{h_2(x)}(\Omega)$ and $w \in L^{h_3(x)}(\Omega)$ we have

$$\left|\int_\Omega uvw\,dx\right| \le \left(\frac{1}{{h_1}^-} + \frac{1}{{h_2}^-} + \frac{1}{{h_3}^-}\right)\|u\|_{L^{h_1(x)}(\Omega)}\|v\|_{L^{h_2(x)}(\Omega)}\|w\|_{L^{h_3(x)}(\Omega)}. \tag{2.2}$$

Note that $L^{h_1(x)}(\Omega) \hookrightarrow L^{h_2(x)}(\Omega)$ for all functions h_1 and h_2 in $C_+\left(\overline{\Omega}\right)$ satisfying $h_1(x) \le h_2(x)$ for all $x \in \left(\overline{\Omega}\right)$. In addition this embedding is continuous.

The modular of the $L^{h(x)}(\Omega)$ space is the mapping $\rho_{h(\cdot)} : L^{h(x)}(\Omega) \to \mathbb{R}$ defined by

$$u \mapsto \rho_{h(\cdot)}(u) = \int_\Omega |u(x)|^{h(x)}dx.$$

Proposition 2.1. *Let $u \in L^{h(x)}(\Omega)$, then we have*

(i) $\|u\|_{L^{h(x)}(\Omega)} < 1$ *(resp.* $= 1, > 1$*)* $\iff \rho_{h(\cdot)}(u) < 1$ *(resp.* $= 1, > 1$*),*

(ii) $\|u\|_{L^{h(x)}(\Omega)} < 1 \implies \|u\|^{h^+}_{L^{h(x)}(\Omega)} \le \rho_{h(\cdot)}(u) \le \|u\|^{h^-}_{L^{h(x)}(\Omega)}$,

(iii) $\|u\|_{L^{h(x)}(\Omega)} > 1 \implies \|u\|^{h^-}_{L^{h(x)}(\Omega)} \le \rho_{h(\cdot)}(u) \le \|u\|^{h^+}_{L^{h(x)}(\Omega)}$.

Proposition 2.2. *If $u, u_k \in L^{h(x)}(\Omega)$ and $k \in \mathbb{N}$, then the following assertions are equivalent*

(i) $\lim_{k\to+\infty} \|u_k - u\|_{L^{h(x)}(\Omega)} = 0$,

(ii) $\lim_{k\to+\infty}\rho_{h(\cdot)}(u_k - u) = 0$,

(iii) $u_k \to u$ *in measure in* Ω *and* $\lim_{k\to+\infty}\rho_{h(\cdot)}(u_k) = \rho_{h(\cdot)}(u)$.

From [8, Theorems 1.6 and 1.10], we obtain the following proposition:

Proposition 2.3. *Suppose that* (2.1) *is satisfied. If Ω is a bounded open domain, then* $(L^{h(x)}(\Omega), \|u\|_{L^{h(x)}(\Omega)})$ *is a reflexive uniformly convex and separable Banach space.*

Proposition 2.4 (see [7])**.** *Let h_1 and h_2 be measurable functions such that $h_1 \in L^\infty(\mathbb{R}^N)$ and $1 \le h_1(x)h_2(x) \le +\infty$ for a.e. $x \in \mathbb{R}^N$. Let $u \in L^{h_2(x)}(\mathbb{R}^N)$, $u \neq 0$. Then we have the following assertions*

$$\|u\|_{L^{h_1(x)h_2(x))}(\mathbb{R}^N)} \le 1 \implies \|u\|^{h_1^+}_{L^{h_1(x)h_2(x))}(\mathbb{R}^N)} \le \||u|^{h_1(x)}\|_{L^{h_2(x))}(\mathbb{R}^N)} \le \|u\|^{h_1^-}_{L^{h_1(x)h_2(x))}(\mathbb{R}^N)},$$

$$\|u\|_{L^{h_1(x)h_2(x))}(\mathbb{R}^N)} \geq 1 \implies \|u\|^{h_1^-}_{L^{h_1(x)h_2(x))}(\mathbb{R}^N)} \leq \||u|^{h_1(x)}\|_{L^{h_2(x))}(\mathbb{R}^N)} \leq \|u\|^{h_1^+}_{L^{h_1(x)h_2(x))}(\mathbb{R}^N)}.$$

In particular, if $h_1(x) = h_1$ is a constant, then it holds that

$$\||u|^{h_1}\|_{L^{h_2(x)}(\mathbb{R}^N)} = \|u\|^{h_1}_{L^{h_1(x)h_2(x)}(\mathbb{R}^N)}.$$

Let Ω be a Lipschitz bounded open set in $\mathbb{R}^N$ and let $p : \overline{\Omega}\times\overline{\Omega} \to (1,+\infty)$ be a continuous bounded function. We assume that

$$1 < p^- := \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \leq p(x,y) \leq p^+ := \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < +\infty, \tag{2.3}$$

and

$$p \text{ is symmetric, that is, } p(x,y) = p(y,x) \quad \text{for all } (x,y) \in \overline{\Omega}\times\overline{\Omega}. \tag{2.4}$$

Set

$$\overline{p}(x) = p(x,x) \quad \text{for any } x \in \overline{\Omega}.$$

Throughout this paper s is a fixed real number such that $0 < s < 1$.

We define the fractional Sobolev space with variable exponent via Gagliardo approach as follows

$$W = W^{s,q(x),p(x,y)}(\Omega) = \left\{ u \in L^{q(x)}(\Omega), \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < +\infty \text{ for some } \lambda > 0 \right\}.$$

The space $W^{s,q(x),p(x,y)}(\Omega)$ is a Banach space if it is equipped with the norm

$$\|u\|_W = \|u\|_{L^{q(x)}(\Omega)} + [u]_{s,p(x,y)},$$

where $[\cdot]_{s,p(x,y)}$ is a Gagliardo seminorm with variable exponent, which is defined by

$$[u]_{s,p(x,y)} = [u]_{s,p(x,y)}(\Omega) := \inf\left\{ \lambda > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy \leq 1 \right\}.$$

Due to [9, Lemma 3.1], $(W, \|.\|_W)$ is a separable and reflexive Banach space.

Proposition 2.5 (see [9]). *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz bounded domain and $s \in (0,1)$. Let $q(x), p(x,y)$ be continuous variable exponents with $sp(x,y) < N$ for all $(x,y) \in \overline{\Omega}\times\overline{\Omega}$ and $q(x) > p(x,x)$ for all $x \in \overline{\Omega}$. Assume that $r : \overline{\Omega} \to (1,+\infty)$ is a continuous function such that*

$$p_s^*(x) := \frac{Np(x,x)}{N - sp(x,x)} > r(x) \geq r^- > 1$$

for all $x \in \overline{\Omega}$. Then, there exists a constant $c = c(N,s,p,q,r,\Omega)$ such that for every $u \in W =$

$W^{s,q(x),p(x,y)}(\Omega)$, *it holds that*

$$\|u\|_{L^{r(x)}(\Omega)} \leq c\|u\|_W.$$

That is, if $1 < r(x) < p_s^*(x)$ *for all* $x \in \overline{\Omega}$ *then the space* W *is continuously embedded in* $L^{r(x)}(\Omega)$. *Moreover, this embedding is compact.*

It is important to encode the boundary condition $u = 0$ in $\mathbb{R}^N \backslash \Omega$ in the weak formulation. For this purpose, we introduce the new fractional Sobolev space as follows

$$\left\{ \begin{array}{c} u : \mathbb{R}^N \to \mathbb{R} \text{ measurable, such that } u_{|\Omega} \in L^{q(x)}(\Omega) \text{ with} \\ \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dx\, dy < +\infty \text{ for some } \lambda > 0 \end{array} \right\},$$

where $p : \overline{Q} \to (1, +\infty)$ satisfies (2.3) and (2.4) on $\overline{Q}$. The space X is endowed with the following norm

$$\|u\|_X = \|u\|_{L^{q(x)}(\Omega)} + [u]_X,$$

where $[u]_X$ is a Gagliardo seminorm with variable exponent, defined by

$$[u]_X = [u]_{s,p(x,y)}(Q) := \inf \left\{ \lambda > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}}\, dx\, dy \leq 1 \right\}.$$

Similar to the space $(W, \|.\|_W)$ we have that $(X, \|.\|_X)$ is a separable reflexive Banach space.

Remark 2.6. *Note that the norms* $\|.\|_X$ *and* $\|.\|_W$ *are not the same, because* $\Omega \times \Omega$ *is strictly contained in* Q. *This makes the fractional Sobolev space with variable exponent* $W = W^{s,q(x),p(x,y)}(\Omega)$ *not sufficient for studying the nonlocal problems.*

Now let X_0 denote the following linear subspace of X

$$X_0 = \left\{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\},$$

with the norm

$$\|u\|_{X_0} = \|u\|_X = \inf \left\{ \lambda > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}}\, dx\, dy \leq 1 \right\}.$$

It is easy to check that $\|.\|_{X_0}$ is a norm on X_0.

Similar to [3, Theorem 2.2] we have

Theorem 2.7. *Let* Ω *be a Lipschitz bounded domain in* $\mathbb{R}^N$ *and let* $s \in (0, 1)$. *Let* $p : \overline{Q} \to (1, +\infty)$ *be a continuous function satisfying* (2.3) *and* (2.4) *on* $\overline{Q}$ *with* $sp^+ < N$. *Then the following assertions hold:*

(i) If $u \in X$, then $u \in W$. Moreover,

$$\|u\|_W \leq \|u\|_X,$$

(ii) If $u \in X_0$, then $u \in W^{s,q(x),p(x,y)}(\mathbb{R}^N)$. Moreover,

$$\|u\|_W \leq \|u\|_{W^{s,q(x),p(x,y)}(\mathbb{R}^N)} = \|u\|_X,$$

(iii) If $r : \overline{\Omega} \to (1, +\infty)$ be a continuous variable exponent such that

$$1 < r^- \leq r(x) < p_s^*(x) = \frac{N\overline{p}(x)}{N - s\overline{p}(x)} \quad \text{for all } x \in \overline{\Omega},$$

then, there exists a constant $C = C(N, s, p, q, r, \Omega) > 0$ such that, for any $u \in W$,

$$\|u\|_{L^{r(x)}(\Omega)} \leq C\|u\|_X.$$

That is, the space X is continuously embedded in $L^{r(x)}(\Omega)$. Moreover, this embedding is compact.

Remark 2.8. *(i) The assertion (iii) in Theorem 2.7 remains true if we replace X by X_0.*

(ii) Since by (1.2) we have $1 < q^- \leq q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. then by Theorem 2.7 (iii) we have that $\|\cdot\|_{X_0} = [\cdot]_X$ and $\|\cdot\|_X$ are equivalent on X_0.

Definition 2.9. *Let $p : \overline{Q} \to (1, +\infty)$ be a continuous variable exponent and let $s \in (0, 1)$, we define the modular $\rho_{p(.,.)} : X_0 \to \mathbb{R}$, by*

$$\rho_{p(.,.)}(u) = \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}}\, dx\, dy.$$

Then $\|u\|_{\rho_{p(.,.)}} = \inf \left\{ \lambda > 0 : \rho_{p(.,.)}\left(\frac{u}{\lambda}\right) \leq 1 \right\} = [u]_X$.

The modular $\rho_{p(.,.)}$ checks the following result, which is similar to [2, Proposition 2.1 and Lemma 2.2].

Lemma 2.10. *Let $p : \overline{Q} \to (1, +\infty)$ be a continuous variable exponent and let $s \in (0, 1)$, for any $u \in X_0$, we have*

(i) $1 \leq \|u\|_{X_0} \implies \|u\|_{X_0}^{p^-} \leq \rho_{p(.,.)}(u) \leq \|u\|_{X_0}^{p^+}$,

(ii) $\|u\|_{X_0} \leq 1 \implies \|u\|_{X_0}^{p^+} \leq \rho_{p(.,.)}(u) \leq \|u\|_{X_0}^{p^-}$.

Remark 2.11. *Note that $\rho_{p(.,.)}$ satisfies the results of Proposition 2.2.*

Similar to [3, Lemma 2.3] we have

Lemma 2.12. *$(X_0, \|.\|_{X_0})$ is a separable, reflexive, and uniformly convex Banach space.*

Let denote by $\mathcal{L}$ the operator associated to the $\left(-\Delta_{p(x)}\right)^s$ defined as

$$\mathcal{L} : X_0 \to X_0^*, \quad u \mapsto \mathcal{L}(u) : X_0 \to \mathbb{R}, \quad \varphi \mapsto \langle \mathcal{L}(u), \varphi \rangle$$

such that

$$\langle \mathcal{L}(u), \varphi \rangle = \int_Q \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}}\, dx\, dy,$$

where X_0^* is the dual space of X_0.

Lemma 2.13 (see [4]). *Under the conditions of Proposition 2.5, the following assertions hold true:*

(i) $\mathcal{L}$ *is a bounded and strictly monotone operator.*

(ii) $\mathcal{L}$ *is a mapping of type* (s_+), *that is, if* $u_k \rightharpoonup u$ *in* X_0 *and* $\limsup_{k\to+\infty} \langle \mathcal{L}(u_k) - \mathcal{L}(u), u_k - u \rangle \leq 0$, *then* $u_k \to u$ *in* X_0.

(iii) $\mathcal{L}$ *is a homeomorphism.*

Throughout this paper, for simplicity, we use c_i to denote the general nonnegative or positive constant (the exact value may change from line to line).

3 The main result and proof of the theorem

Definition 3.1. *We say that $u \in X_0$ is a weak solution of problem* (1.1) *if*

$$M(\sigma_{p(x,y)}(u)) \int_Q \frac{|u(x) - u(y)|^{p(x,y)-2}((u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}}\, dx\, dy \tag{3.1}$$

$$+ \int_\Omega |u(x)|^{q(x)-2} u(x)\varphi(x) dx - \lambda \int_\Omega V(x)|u(x)|^{r(x)-2} u(x)\varphi(x) dx = 0$$

for all $\varphi \in X_0$, where

$$\sigma_{p(x,y)}(u) = \int_Q \frac{1}{p(x,y)} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}}\, dx\, dy.$$

Let us consider the Euler-Lagrange functional associated to (1.1), defined by

$$\begin{aligned}\mathcal{J}_\lambda : X_0 \to \mathbb{R}, \quad \mathcal{J}_\lambda(u) &= \widehat{M}\left(\int_Q \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy\right) \\ &\quad + \int_\Omega \frac{1}{q(x)}|u(x)|^{q(x)}dx - \lambda \int_\Omega \frac{V(x)}{r(x)}|u(x)|^{r(x)}dx \\ &= \widehat{M}\left(\sigma_{p(x,y)}(u)\right) + \int_\Omega \frac{1}{q(x)}|u(x)|^{q(x)}dx - \lambda \int_\Omega \frac{V(x)}{r(x)}|u(x)|^{r(x)}dx,\end{aligned}$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$.

Theorem 3.2. *Under the same assumptions of Theorem 2.7, if we assume that* (M_1) *holds and* $\sigma, r \in C_+(\overline{\Omega})$ *satisfy the following conditions:*

(H_1) $1 < r^- \leq r(x) \leq r^+ < p^- \leq p^+ < \frac{N}{s} < \sigma(x)$ *for all* $x \in \overline{\Omega}$,

(H_2) $V \in L^{\sigma(x)}(\Omega)$ *and there exists a measurable set* $\Omega_0 \subset\subset \Omega$ *of positive measure such that* $V(x) > 0$ *for all* $x \in \Omega_0$.

Then there exists $\bar{\lambda} > 0$ *such that any* $\lambda \in (0, \bar{\lambda})$ *is an eigenvalue of problem* (1.1).

Proof. For each $\lambda > 0$, let us consider the functional $\mathcal{J}_\lambda : X_0 \to \mathbb{R}$ associated with problem (1.1) by the formula

$$\mathcal{J}_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where

$$\Phi(u) = \widehat{M}(\sigma_{p(x,y)}(u)) + \int_\Omega \frac{1}{q(x)}|u(x)|^{q(x)}dx, \quad \Psi(u) = \frac{V(x)}{r(x)}|u(x)|^{r(x)}dx.$$

From conditions $(H_1) - (H_2)$ and Proposition 2.4, for all $u \in X_0$, we get

$$|\Phi(u)| \leq \frac{2}{r^-}\|V\|_{L^{\sigma(x)}(\Omega)}\||u|^{r(x)}\|_{L^{\sigma(x)/(\sigma(x)-1)}(\Omega)}$$

$$\leq \begin{cases} \frac{2}{r^-}\|V\|_{L^{\sigma(x)}(\Omega)}\|u\|^{r^-}_{L^{\sigma(x)r(x)/(\sigma(x)-1)}(\Omega)} & \text{if } \|u\|_{L^{\sigma(x)r(x)/(\sigma(x)-1)}(\Omega)} \leq 1, \\ \frac{2}{r^-}\|V\|_{L^{\sigma(x)}(\Omega)}\|u\|^{r^+}_{L^{\sigma(x)r(x)/(\sigma(x)-1)}(\Omega)} & \text{if } \|u\|_{L^{\sigma(x)r(x)/(\sigma(x)-1)}(\Omega)} \geq 1. \end{cases} \tag{3.2}$$

We also deduce from (H_1) that $\beta(x) = \sigma(x)r(x)/(\sigma(x)-r(x)) < p_s^*(x)$ and $\gamma(x) = \sigma(x)r(x)/(\sigma(x)-1) < p_s^*(x)$ for all $x \in \overline{\Omega}$. In view of (Theorem 2.7 (iii) and Remark 2.8 (i)) the embeddings $X_0 \hookrightarrow L^{\beta(x)}(\Omega)$ and $X_0 \hookrightarrow L^{\gamma(x)}(\Omega)$ are continuous and compact. Thus, the functional $\mathcal{J}_\lambda$ is well-defined on X_0. The proof of Theorem 3.2 is divided into following four steps.

Step 1. We show that $\mathcal{J}_\lambda \in C^1(X_0, \mathbb{R})$ and its derivative is

$$\langle \mathcal{J}_\lambda'(u), \varphi\rangle = M(\sigma_{p(x,y)}(u)) \int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}((u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}\, dx\, dy$$
$$+ \int_\Omega |u(x)|^{q(x)-2} u(x)\varphi(x)dx - \lambda \int_\Omega V(x)|u(x)|^{r(x)-2}u(x)\varphi(x)dx$$

for all $u, \varphi \in X_0$. This means that weak solutions for problem (1.1) can be found as the critical points of the functional $\mathcal{J}_\lambda$ in the space X_0.

Using the same method as in the proof of [1, Lemma 4.1] and [6, Lemma 3.1] and the continuity of M, we can show that $\Phi \in C^1(X_0, \mathbb{R})$ and

$$\langle \Phi'(u), \varphi\rangle = M(\sigma_{p(x,y)}(u)) \int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}((u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}\, dx\, dy$$
$$+ \int_\Omega |u(x)|^{q(x)-2} u(x)\varphi(x)dx$$

for all $u, \varphi \in X_0$.

Also it has been proved by Chung in [5] that $\Psi \in C^1(X_0, \mathbb{R})$ and

$$\langle \Psi'(u), \varphi\rangle = \int_\Omega V(x)|u(x)|^{r(x)-2}u(x)\varphi(x)dx, \quad \forall u, \varphi \in X_0$$

and thus Step 1 is completed.

Step 2. We prove that there exists $\bar{\lambda} > 0$ such that for any $\lambda \in (0, \bar{\lambda})$, there exist constants $R, \rho > 0$ such that $\mathcal{J}_\lambda(u) \geq R$ for all $u \in X_0$ with $\|u\|_{X_0} = \rho$.

Indeed, since $\gamma(x) = \sigma(x)r(x)/(\sigma(x)-1) < p_s^*(x)$ for all $x \in \overline{\Omega}$, the embedding $X_0 \hookrightarrow L^{\gamma(x)}(\Omega)$ is continuous and there exists $c_2 > 0$ such that

$$\|u\|_{L^{\gamma(x)}}(\Omega) \leq c_2 \|u\|_{x_0}, \quad \forall u \in X_0.$$

Hence, by relation (3.2), for any $u \in X_0$ with $\|u\| = \rho$ small enough,

$$\begin{aligned}
\mathcal{J}_\lambda(u) &= \widehat{M}(\sigma_{p(x,y)}(u)) + \int_\Omega \frac{1}{q(x)} |u(x)|^{q(x)} dx - \lambda \int_\Omega \frac{V(x)}{r(x)} |u(x)|^{r(x)} dx \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha} \|u\|_{X_0}^{\alpha p^+} - \lambda \frac{2c_2^{r^-}}{r^-} \|V\|_{L^{\sigma(x)}(\Omega)} \|u\|_{X_0}^{r^-} = \frac{m_1}{\alpha(p^+)^\alpha} \rho^{\alpha p^+} - \lambda \frac{2c_2^{r^-}}{r^-} \|V\|_{L^{\sigma(x)}(\Omega)} \rho^{r^-} \\
&= \rho^{r^-} \left(\frac{m_1}{\alpha(p^+)^\alpha} \rho^{\alpha p^+ - r^-} - \lambda \frac{2c_2^{r^-}}{r^-} \|V\|_{L^{\sigma(x)}(\Omega)} \right).
\end{aligned}$$

Putting

$$\bar{\lambda} = \frac{m_1}{2\alpha(p^+)^\alpha} \rho^{\alpha p^+ - r^-} \cdot \frac{r^-}{2c_2^{r^-} \|V\|_{L^{\sigma(x)}(\Omega)}} > 0,$$

for any $\lambda \in (0,\bar{\lambda})$ and $u \in X_0$ with $\|u\| = \rho$, there exists $R = \frac{m_1 \rho^{\alpha p^+}}{2\alpha (p^+)^\alpha}$ such that $\mathcal{J}_\lambda(u) \geq R > 0$.

Step 3. We prove that there exists $\varphi_0 \in X_0$ such that $\varphi_0 \geq 0$, $\varphi_0 \neq 0$ and $\mathcal{J}_\lambda(t\varphi_0) < 0$ for all $t > 0$ small enough.

Indeed, condition (H_1) implies that $r(x) < \min\{p^-, q^-\} = p^-$ for all $x \in \overline{\Omega_0}$. In the sequel, we use the notation $r_0^- = \inf_{x\in\Omega_0} r(x)$. Let $\varepsilon_0 > 0$ be such that $r_0^- + \varepsilon_0 < p^-$. We also have since $r \in C(\overline{\Omega_0})$ that there exists an open subset $\Omega_1 \subset \Omega_0$ such that

$$|r(x) - r_0^-| < \varepsilon_0, \quad \forall x \in \Omega_1$$

and thus

$$r(x) \leq r_0^- + \varepsilon_0 < p^- < \alpha p^-, \quad \forall x \in \Omega_1.$$

Let $\varphi_0 \in C_0^\infty(\Omega_0)$ such that $\overline{\Omega_1} \subset \operatorname{supp}(\varphi_0)$, $\varphi_0(x) = 1$ for all $x \in \overline{\Omega_1}$ and $0 \leq \varphi_0 \leq 1$ in Ω_0. Then, using the above information and assumption(M_1), for any $t \in (0,1)$ we have

$$\begin{aligned}
\mathcal{J}_\lambda(t\varphi_0) &= \widehat{M}\left(\sigma_{p(x,y)}(t\varphi_0)\right) + \int_\Omega \frac{1}{q(x)}|t\varphi_0|^{q(x)}dx - \lambda \int_\Omega \frac{V(x)}{r(x)}|t\varphi_0|^{r(x)}dx \\
&\leq \frac{m_2}{\alpha}\left(\sigma_{p(x,y)}(t\varphi_0)\right)^\alpha + \frac{t^{q^-}}{q^-}\int_{\Omega_0}|\varphi_0|^{q(x)}dx - \lambda\int_{\Omega_0}\frac{V(x)}{r(x)}t^{r(x)}|\varphi_0|^{r(x)}dx \\
&\leq \frac{m_2}{\alpha(p^-)^\alpha}t^{\alpha p^-}\left(\rho_{p(.,.)}(\varphi_0)\right)^\alpha + \frac{t^{q^-}}{q^-}\int_{\Omega_0}|\varphi_0|^{q(x)}dx - \frac{\lambda t^{r_0^-+\varepsilon_0}}{r_0^+}\int_{\Omega_1}V(x)|\varphi_0|^{r(x)}dx \\
&\leq k t^{\alpha p^-}\left(\left(\rho_{p(.,.)}(\varphi_0)\right)^\alpha + \int_{\Omega_0}|\varphi_0|^{q(x)}dx\right) - \frac{\lambda t^{r_0^-+\varepsilon_0}}{r_0^+}\int_{\Omega_1}V(x)|\varphi_0|^{r(x)}dx
\end{aligned}$$

where

$$k = \max\left\{\frac{m_2}{\alpha(p^-)^\alpha}, \frac{1}{q^-}\right\}.$$

Therefore

$$\mathcal{J}_\lambda(t\varphi_0) < 0 \quad \text{for } 0 < t < \delta^{1/(\alpha p^- - r_0^- - \varepsilon_0)}$$

with

$$0 < \delta < \min\left\{1, \frac{\lambda}{k r_0^+}\cdot\frac{\int_{\Omega_1}V(x)|\varphi_0|^{r(x)}dx}{\left(\rho_{p(.,.)}(\varphi_0)\right)^\alpha + \int_\Omega|\varphi_0|^{q(x)}dx}\right\}.$$

The above fraction is meaningful if we can show that

$$\left(\rho_{p(.,.)}(\varphi_0)\right)^\alpha + \int_\Omega|\varphi_0|^{q(x)}dx > 0.$$

Since $\varphi_0(x) = 1$ for all $x \in \overline{\Omega_1}$, we have

$$\int_\Omega|\varphi_0|^{q(x)}dx > 0.$$

Thus, the above fraction is meaningful.

Indeed, it is clear that

$$\int_{\Omega_1} |\varphi_0|^{r(x)} dx \le \int_{\Omega} |\varphi_0|^{r(x)} dx \le \int_{\Omega} |\varphi_0|^{r^-} dx.$$

On the other hand, the space X_0 is continuously embedded in $L^{r^-}(\Omega)$ and thus, there exists $c_3 > 0$ such that $\|\varphi_0\|_{L^{r^-}(\Omega)} \le c_3 \|\varphi_0\|_{X_0}$, which implies that $\|\varphi_0\|_{X_0} > 0$. Thus, Step 3 is completed.

By Step 2 we have

$$\inf_{u \in \partial B_\rho(0)} \mathcal{J}_\lambda(u) > 0.$$

We also deduce from Step 2 that, the functional $\mathcal{J}_\lambda$ is bounded from below on $B_\rho(0)$. Moreover, by Step 3, there exists $\varphi \in X$ such that $\mathcal{J}_\lambda(t\varphi) < 0$ for all $t > 0$ small enough.

It follows from Step 2 that

$$\mathcal{J}_\lambda(u) \ge \frac{m_1}{\alpha(p^+)^\alpha} \|u\|_{X_0}^{\alpha p^+} - \lambda \frac{2c_2^{r^-}}{r^-} \|V\|_{L^{\sigma(x)}(\Omega)} \|u\|_{X_0}^{r^-},$$

which yields

$$-\infty < \underline{c}_\lambda = \inf_{u \in \overline{B_\rho(0)}} \mathcal{J}_\lambda(u) < 0.$$

Let us choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \inf_{u \in \partial B_\rho(0)} \mathcal{J}_\lambda(u) - \inf_{u \in \overline{B_\rho(0)}} \mathcal{J}_\lambda(u).$$

Applying the Ekeland variational principle [7] to the functional $\mathcal{J}_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, it follows that there exists $u_\varepsilon \in \overline{B_\rho(0)}$

$$\mathcal{J}_\lambda(u_\varepsilon) < \inf_{u \in \overline{B_\rho(0)}} \mathcal{J}_\lambda(u) + \varepsilon, \ \mathcal{J}_\lambda(u_\varepsilon) < \mathcal{J}_\lambda(u) + \varepsilon \|u - u_\varepsilon\|_{X_0}, \quad u \ne u_\varepsilon,$$

then we infer that

$$\mathcal{J}_\lambda(u_\varepsilon) < \inf_{u \in \partial B_\rho(0)} \mathcal{J}_\lambda(u)$$

and thus

$$u_\varepsilon \in B_\rho(0).$$

Let us consider the functional

$$I_\lambda : \overline{B_\rho(0)} \to \mathbb{R} \quad \text{by} \quad I_\lambda(u) = \mathcal{J}_\lambda(u) + \varepsilon \|u - u_\varepsilon\|_{X_0}.$$

Then u_ε is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\varepsilon+\tau\varphi)-I_\lambda(u_\varepsilon)}{\tau}\geq 0$$

for all $\tau>0$ small enough and $\varphi\in B_\rho(0)$. The above information shows that

$$\frac{\mathcal{J}_\lambda(u_\varepsilon+\tau\varphi)-\mathcal{J}_\lambda(u_\varepsilon)}{\tau}+\varepsilon\|\varphi\|_{X_0}\geq 0.$$

Letting $\tau\to 0^+$, we deduce that

$$\langle \mathcal{J}_\lambda'(u_\varepsilon),\varphi\rangle+\varepsilon\|\varphi\|_{X_0}\geq 0$$

and we infer that

$$\|\mathcal{J}_\lambda'(u_\varepsilon)\|_{X_0^*}\leq\varepsilon.$$

Therefore, there exists a sequence $\{u_n\}\subset B_\rho(0)$ such that

$$\mathcal{J}_\lambda(u_n)\to \underline{c}_\lambda=\inf_{u\in\overline{B_\rho(0)}}\mathcal{J}_\lambda(u)<0 \quad\text{and}\quad \mathcal{J}_\lambda'(u_n)\to 0\quad\text{in}\quad X_0^*\quad\text{as}\quad n\to\infty. \tag{3.3}$$

It is clear that the sequence $\{u_n\}$ is bounded in X_0. Now, since X_0 is a reflexive Banach space, there exists $u\in X_0$ such that passing to a subsequence, still denoted by $\{u_n\}$, it converges weakly to u in X_0.

Step 4. We prove that $\{u_n\}$ which is given by (3.3) converges strongly to u in X_0, *i.e.*, $\lim\limits_{n\to+\infty}\|u_n-u\|_{X_0}=0$.

By conditions $(H_1)-(H_2)$, using Hölder's inequality (2.2) and Propositions 2.4 and 2.5 we deduce that

$$\begin{aligned}\left|\int_\Omega |u_n|^{q(x)-2}u_n(u_n-u)dx\right| &\leq 2\||u_n|^{q(x)-2}u_n\|_{L^{q(x)/(q(x)-1)}(\Omega)}\|u_n-u\|_{L^{q(x)}(\Omega)}\\ &\leq 2\|u_n\|^{q^+-1}_{L^{q(x)}(\Omega)}\|u_n-u\|_{L^{q(x)}(\Omega)}\to 0\quad\text{as}\quad n\to\infty,\end{aligned}$$

and

$$\begin{aligned}\left|\int_\Omega V(x)|u_n|^{r(x)-2}u_n(u_n-u)dx\right| &\leq 3\|V\|_{L^{\sigma(x)}(\Omega)}\||u_n|^{r(x)-2}u_n\|_{L^{r(x)/(r(x)-1)}(\Omega)}\|u_n-u\|_{L^{\beta(x)}(\Omega)}\\ &\leq 3\|V\|_{L^{\sigma(x)}(\Omega)}\left(1+\|u_n\|^{r^+-1}_{L^{r(x)}(\Omega)}\right)\|u_n-u\|_{L^{\beta(x)}(\Omega)}\to 0\quad\text{as}\quad n\to\infty,\end{aligned}$$

where $\beta(x)=\sigma(x)r(x)/(\sigma(x)-r(x))$. Moreover, by (3.3) we have $\lim_{n\to\infty}\langle\mathcal{J}_\lambda'(u_n),u_n-u\rangle=0$, *i.e.*,

$$M(\sigma_{p(x,y)}(u_n))\mathcal{I}_Q(u_n) + \int_\Omega |u_n|^{q(x)-2}u_n(u_n-u)dx \\ - \lambda \int_\Omega V(x)|u_n|^{r(x)-2}u_n(u_n-u)dx \to 0 \quad \text{as} \quad n\to\infty,$$

which yields

$$M(\sigma_{p(x,y)}(u_n))\mathcal{I}_Q(u_n) \to 0 \tag{3.4}$$

where

$$\mathcal{I}_Q(u_n) = \int_Q \frac{|u_n(x)-u_n(y)|^{p(x,y)-2}((u_n(x)-u_n(y))((u_n(x)-u(x))-(u_n(y)-u(y)))}{|x-y|^{N+sp(x,y)}}\,dx\,dy$$

Since $\{u_n\}$ is bounded in X_0, passing to subsequence, if necessary, we may assume that

$$\sigma_{p(x,y)}(u_n) \overset{n\to+\infty}{\longrightarrow} t_1 \geq 0.$$

If $t_1 = 0$, then $\{u_n\}$ converge strongly to $u = 0$ in X_0, then by (3.3), we obtain

$$\lim_{n\to+\infty} \mathcal{J}_\lambda(u_n) = \mathcal{J}_\lambda(u) = \mathcal{J}_\lambda(0) = 0 = \underline{c}_\lambda < 0.$$

That is a contradiction, thus $t_1 > 0$.

Since the function M is continuous, we have

$$M\left(\sigma_{p(x,y)}(u_n)\right) \overset{n\to+\infty}{\longrightarrow} M\left(t_1\right) > 0.$$

Hence, by (M_1), for n large enough, we get

$$0 < c_4 < M\left(\sigma_{p(x,y)}(u_n)\right) < c_5. \tag{3.5}$$

Combining (3.4) and (3.5), we deduce

$$\lim_{n\to+\infty} \mathcal{I}_Q(u_n) = 0.$$

Using the above information, Lemma 2.13 (ii) and the fact that $u_n \rightharpoonup u$ in X_0, we get

$$\begin{cases} \limsup_{n\to+\infty} \langle \mathcal{L}(u_n), u_n - u\rangle \leq 0, \\ u_n \rightharpoonup u \quad \text{in} \quad X_0, & \Longrightarrow u_n \to u \quad \text{in} \quad X_0. \\ \mathcal{L} \quad \text{is a mapping of type } (S_+). \end{cases}$$

Thus, in view of (3.3), we obtain

$$\mathcal{J}_\lambda(u) = \underline{c}_\lambda < 0 \quad \text{and} \quad \mathcal{J}'_\lambda(u) = 0.$$

This means that u is a non-trivial weak solution of (1.1), *i.e.*, any $\lambda \in (0, +\infty)$ is an eigenvalue of problem (1.1). Theorem 3.2 is completely proved. □

Acknowledgments

We wish to express our gratitude to the referees for reading this paper carefully and making constructive comments and remarks.

References

[1] S. Antontsev, F. Miranda, and L. Santos, "Blow-up and finite time extinction for $p(x,t)$-curl systems arising in electromagnetism," *J. Math. Anal. Appl.*, vol. 440, no. 1, pp. 300–322, 2016, doi: 10.1016/j.jmaa.2016.03.045.

[2] E. Azroul, A. Benkirane, and M. Shimi, "Eigenvalue problems involving the fractional $p(x)$-Laplacian operator," *Adv. Oper. Theory*, vol. 4, no. 2, pp. 539–555, 2019, doi: 10.15352/aot.1809-1420.

[3] E. Azroul, A. Benkirane, M. Shimi, and M. Srati, "On a class of fractional $p(x)$-Kirchhoff type problems," *Appl. Anal.*, vol. 100, no. 2, pp. 383–402, 2021, doi: 10.1080/00036811.2019.1603372.

[4] A. Bahrouni and V. D. Rădulescu, "On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent," *Discrete Contin. Dyn. Syst. Ser. S*, vol. 11, no. 3, pp. 379–389, 2018, doi: 10.3934/dcdss.2018021.

[5] N. T. Chung, "Eigenvalue problems for fractional $p(x,y)$-Laplacian equations with indefinite weight," *Taiwanese J. Math.*, vol. 23, no. 5, pp. 1153–1173, 2019, doi: 10.11650/tjm/190404.

[6] F. J. S. A. Corrêa and G. M. Figueiredo, "On a p-Kirchhoff equation via Krasnoselskii's genus," *Appl. Math. Lett.*, vol. 22, no. 6, pp. 819–822, 2009, doi: 10.1016/j.aml.2008.06.042.

[7] I. Ekeland, "On the variational principle," *J. Math. Anal. Appl.*, vol. 47, pp. 324–353, 1974, doi: 10.1016/0022-247X(74)90025-0.

[8] X. Fan and D. Zhao, "On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$," *J. Math. Anal. Appl.*, vol. 263, no. 2, pp. 424–446, 2001, doi: 10.1006/jmaa.2000.7617.

[9] U. Kaufmann, J. D. Rossi, and R. Vidal, "Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians," *Electron. J. Qual. Theory Differ. Equ.*, 2017, Art. ID 76, doi: 10.14232/ejqtde.2017.1.76.

Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 26, no. 01, pp. 123–151, April 2024
DOI: 10.56754/0719-0646.2601.123

On a class of evolution problems driven by maximal monotone operators with integral perturbation

Fatima Fennour[1]
Soumia Saïdi [1,✉]

[1] *LMPA Laboratory, Department of Mathematics, Faculty of Exact Sciences and Informatics, University of Jijel, 18000 Jijel, Algeria.*
fennourfatima38@gmail.com
soumiasaidi44@gmail.com[✉]

ABSTRACT

The present paper is dedicated to the study of a first-order differential inclusion driven by time and state-dependent maximal monotone operators with integral perturbation, in the context of Hilbert spaces. Based on a fixed point method, we derive a new existence theorem for this class of differential inclusions. Then, we investigate an optimal control problem subject to such a class, by considering control maps acting in the state of the operators and the integral perturbation.

RESUMEN

El presente artículo está dedicado al estudio de una inclusión diferencial de primer orden impulsada por operadores monótonos maximales dependiendo del tiempo y del estado con una perturbación integral, en el contexto de espacios de Hilbert. En base a un método de punto fijo, derivamos un nuevo teorema de existencia para esta clase de inclusiones diferenciales. A continuación investigamos un problema de control óptimo sujeto a dicha clase, considerando funciones de control actuando en el estado de los operadores y de la perturbación integral.

Keywords and Phrases: Integro-differential inclusion, maximal monotone operator, integral perturbation, optimal solution.

2020 AMS Mathematics Subject Classification: 34A60, 34G25, 47H10, 47J35, 49J52, 49J53, 45J05.

Published: 09 April, 2024
Accepted: 31 January, 2024
Received: 28 March, 2023

1 Introduction

Sweeping processes with integral forcing term or integro-differential sweeping processes have been introduced in [8]. Later, the well-posedness result to the non-convex integro-differential sweeping process has been shown in [20]. Recent investigations on this topic have been developed in [5–7]. More recently, differential inclusions with integral perturbation involving m-accretive operators or subdifferentials or time-dependent maximal monotone operators have been studied in [4, 13, 14]. The aforementioned contributions find many areas of applications such as electrical circuits, nonlinear integro-differential complementarity systems, optimal control, fractional systems, etc.

We are concerned, in this paper, with the following Integro-Differential Problem with time and state-dependent maximal monotone operators $A(t, u)$

$$(IDP_{A(t,u)}) \qquad \begin{cases} -\dot{u}(t) \in A(t, u(t))u(t) + \displaystyle\int_{T_0}^{t} f(t, s, u(s))ds \quad \text{a.e. } t \in I := [T_0, T], \\ u(T_0) = u_0 \in D(A(T_0, u_0)), \end{cases}$$

where H stands for a real Hilbert space, $A(t, x) : D(A(t, x)) \subset H \rightrightarrows H$ is a maximal monotone operator whose domain is denoted $D(A(t, x))$, for each $(t, x) \in I \times H$, and $f : I \times I \times H \to H$ is a single-valued map.

Our problem generalizes the Integro-Differential Problem with time-dependent maximal monotone operators $A(t)$

$$(IDP_{A(t)}) \qquad \begin{cases} -\dot{u}(t) \in A(t)u(t) + \displaystyle\int_{T_0}^{t} f(t, s, u(s))ds \quad \text{a.e. } t \in I, \\ u(T_0) = u_0 \in D(A(T_0)), \end{cases}$$

stated in [14]. So, we aim to study a more general case, that is, when the operator depends on both time and state variables.

Note that the evolution problem when a single-valued map $f(\cdot, \cdot)$ instead of the integral perturbation in $(IDP_{A(t,u)})$ has been discussed in [1, 28, 34]. Here, we use Schauder's fixed point theorem (see also [1]) to establish our main existence result. For this purpose, we make use of the uniqueness of the solution to $(IDP_{A(t)})$ and an estimate of the velocity. However, the papers [28, 34] have followed a discretization method.

In the next part of the paper, we deal with the $\mathcal{O}$ptimal $\mathcal{C}$ontrol $\mathcal{P}$roblem

$$(\mathcal{OCP}) \qquad \min \phi[u, a, b] = \phi_1(u(T)) + \int_0^T \phi_2(t, u(t), a(t), b(t), \dot{u}(t), \dot{a}(t), \dot{b}(t))dt,$$

on the set of control maps $(a(\cdot), b(\cdot))$ and the associated solutions $u(\cdot)$ of the $\mathcal{C}$ontrolled $\mathcal{P}$roblem

$$(\mathcal{CP}_{a,b}) \qquad \begin{cases} -\dot{u}(t) \in A(t,a(t))u(t) + \displaystyle\int_0^t f(t,s,b(s),u(s))ds \quad \text{a.e. } t \in [0,T], \\ u(t) \in D(A(t,a(t))), \quad t \in [0,T], \\ (a(\cdot),b(\cdot)) \in W^{1,2}([0,T],\mathbb{R}^{n+m}), \\ a(0) = a_0, \quad u(0) = u_0 \in D(A(0,a_0)), \end{cases}$$

where the cost functional $\phi_1 : \mathbb{R}^n \to \overline{\mathbb{R}}$ and the running cost $\phi_2 : [0,T] \times \mathbb{R}^{4n+2m} \to \overline{\mathbb{R}}$ satisfy convenient conditions.

This investigation is inspired by the related one on the controlled integro-sweeping process in [5], see also [10–12, 17–19, 21–23, 29–32, 37], among others, for further contributions on optimization problems subject to controlled sweeping processes or control problems governed by maximal monotone operators.

Let us give the two following motivating examples: the first-one consists of minimizing a Bolza-type functional subject to the controlled differential inclusion of the form

$$(\mathcal{CP}_{x,a,b}) \qquad -\dot{u}(t) \in N_{C(x(t))}(u(t)) + f_1(a(t),u(t)) + \int_0^t f_2(b(s),u(s))ds \quad \text{a.e. } t \in [0,T],$$

where $A(t,x(t)) = N_{C(x(t))}$ is the normal cone of a moving set $C(x(t))$, $(x(\cdot),a(\cdot),b(\cdot))$ are controls acting in the moving sets, additive perturbations, and the integral part of the sweeping dynamics (see [5]). The second example concerns an optimization problem subject to the controlled differential inclusion described by

$$(\mathcal{CP}_{x,a}) \qquad -\dot{u}(t) \in N_{C(t)}(u(t)) + f(a(t),u(t)) \quad \text{a.e. } t \in [0,T],$$

where $C(t) = C + x(t)$ and $(x(\cdot),a(\cdot))$ are control maps (see [12]).

The considered problem $(\mathcal{OCP})$ is new, since we minimize over the solution set to the controlled integro-differential inclusion $(\mathcal{CP}_{a,b})$, where the controls act in both the state of the (time and state-dependent) operator and the integral perturbation. To the best of our knowledge, this topic is new in the scientific literature.

The rest of the paper is organized as follows. After recalling some preliminaries in Section 2, we handle $(IDP_{A(t)})$. Then, we develop the case $(IDP_{A(t,u)})$. Section 4 applies the obtained results to show the well-posedness of $(\mathcal{CP}_{a,b})$ and establishes the existence of optimal solutions to $(\mathcal{OCP})$.

2 Notation and preliminaries

Let $I := [T_0, T]$ be an interval of $\mathbb{R}$ and let H be a real separable Hilbert space whose inner product is denoted $\langle\cdot,\cdot\rangle$ and the associated norm by $\|\cdot\|$. Denote by $\overline{B}_H$ the closed unit ball of H and $\overline{B}_H[x,r]$ its closed ball of center $x \in H$ and radius $r > 0$.

On the space $\mathcal{C}_H(I)$ of continuous maps $x : I \to H$, we consider the norm of uniform convergence on I, $\|x\|_\infty = \sup_{t\in I} \|x(t)\|$.

By $L^p_H(I)$, for $p \in [1, +\infty[$ (resp. $p = +\infty$), we denote the space of measurable maps $x : I \to H$ such that $\int_I \|x(t)\|^p dt < +\infty$ (resp. which are essentially bounded) endowed with the usual norm $\|x\|_{L^p_H(I)} = (\int_I \|x(t)\|^p dt)^{\frac{1}{p}}$, $1 \le p < +\infty$ (resp. endowed with the usual essential supremum norm $\|\cdot\|_{L^\infty_H(I)}$). Denote by $W^{1,2}(I,H)$, the space of absolutely continuous functions from I to H with derivatives in $L^2_H(I)$.

Recall the definition and some properties of maximal monotone operators, see [3, 9, 36].

Let $A : D(A) \subset H \rightrightarrows H$ be a set-valued operator whose domain, range and graph are defined by

$$\begin{aligned} D(A) &= \{x \in H : \ Ax \neq \emptyset\}, \\ R(A) &= \{y \in H : \ \exists\, x \in D(A),\ y \in Ax\} = \cup\{Ax : \ x \in D(A)\}, \\ Gr(A) &= \{(x,y) \in H \times H : \ x \in D(A),\ y \in Ax\}. \end{aligned}$$

The operator $A : D(A) \subset H \rightrightarrows H$ is monotone, if $\langle y_1 - y_2, x_1 - x_2\rangle \ge 0$ whenever $(x_i, y_i) \in Gr(A)$, $i = 1, 2$. It is maximal monotone, if its graph could not be contained strictly in the graph of any other monotone operator, in this case, for all $\mu > 0$, $R(I_H + \mu A) = H$, where I_H stands for the identity map of H. If A is a maximal monotone operator then, for every $x \in D(A)$, Ax is non-empty, closed and convex. Then, the projection of the origin onto Ax, $A^0(x)$, exists and is unique.

Associated with any maximal monotone operator A is the so-called resolvent $J^A_\mu = (I_H + \mu A)^{-1}$, $\mu > 0$, which turns out to be a nice firmly non-expansive operator with full domain. Resolvents not only provide an alternative view on monotone operators because one can recover the underlying maximal monotone operator via $(J^A_\mu)^{-1} - I_H$ but they also are crucial for the formulation of algorithms for finding zeros of A (*e.g.*, the celebrated proximal point algorithm).

Recall that the Yosida approximation of A of index $\mu > 0$ is defined by $A_\mu = \frac{1}{\mu}\left(I_H - J^A_\mu\right)$. Yosida approximations are powerful tools to study monotone operators. They can be viewed as regularizations and approximations of A because A_μ is a single-valued Lipschitz-continuous operator on H and A_μ approximates A in the sense that $A_\mu x \to A^0(x) \in Ax$ as $\mu \to 0^+$.

Let us summarize the following properties of these operators:

$$J_\mu^A x \in D(A) \text{ and } A_\mu(x) \in A(J_\mu^A x) \quad \text{for every } x \in H, \tag{2.1}$$
$$\|A_\mu(x)\| \leq \|A^0(x)\| \quad \text{for every } x \in D(A).$$

The normal cone to a non-empty closed convex set S at $x \in H$ denoted $N_S(x)$ defined by

$$N_S(x) = \{y \in H : \langle y, z - x\rangle \leq 0 \quad \forall z \in S\}, \tag{2.2}$$

is a maximal monotone operator.

Let $A : D(A) \subset H \rightrightarrows H$ and $B : D(B) \subset H \rightrightarrows H$ be two maximal monotone operators, then, we denote by $\operatorname{dis}(A, B)$ (see [35]) the pseudo-distance between A and B defined by

$$\operatorname{dis}(A, B) = \sup\left\{\frac{\langle y - y', x' - x\rangle}{1 + \|y\| + \|y'\|} : (x, y) \in Gr(A),\ (x', y') \in Gr(B)\right\}.$$

Clearly, $\operatorname{dis}(A, B) \in [0, +\infty]$, $\operatorname{dis}(A, B) = \operatorname{dis}(B, A)$ and $\operatorname{dis}(A, B) = 0$ iff $A = B$.

Let us first recall some useful lemmas that will be used in what follows (see [27]).

The first one permits to prove some inclusions using a convergence in the sense of the pseudo-distance.

Lemma 2.1. *Let A_n $(n \in \mathbb{N})$, A be maximal monotone operators of H such that* $\operatorname{dis}(A_n, A) \to 0$. *Suppose also that $x_n \in D(A_n)$ with $x_n \to x$ and $y_n \in A(x_n)$ with $y_n \to y$ weakly for some $x, y \in H$. Then, $x \in D(A)$ and $y \in A(x)$.*

The next lemma deals with some modes of convergence in the sense of the pseudo-distance and the element of minimal norm.

Lemma 2.2. *Let A_n $(n \in \mathbb{N})$, A be maximal monotone operators of H such that* $\operatorname{dis}(A_n, A) \to 0$ *and $\|A_n^0(x)\| \leq c(1 + \|x\|)$ for some $c > 0$, all $n \in \mathbb{N}$ and $x \in D(A_n)$. Then, for every $\zeta \in D(A)$, there exists a sequence (ζ_n) such that*

$$\zeta_n \in D(A_n),\ \zeta_n \to \zeta \ \text{ and } \ A_n^0(\zeta_n) \to A^0(\zeta).$$

Another approach on how to prove some inclusions using an estimate involving the element of minimal norm is provided by the following lemma.

Lemma 2.3. *Let A be a maximal monotone operator. If $x, y \in H$ are such that*

$$\langle A^0(z) - y, z - x\rangle \geq 0 \quad \forall z \in D(A),$$

then, $x \in D(A)$ *and* $y \in A(x)$.

In the last lemma, we provide an estimate by means of the pseudo-distance, the element of minimal norm, and the resolvent.

Lemma 2.4. *Let* A, B *be maximal monotone operators of* H. *Then, for* $\mu > 0$ *and* $x \in D(A)$ *one has*

$$\|x - J_\mu^B(x)\| \le \mu\|A^0(x)\| + \operatorname{dis}(A,B) + \sqrt{\mu\big(1+\|A^0(x)\|\big)\operatorname{dis}(A,B)}.$$

Recall the classical definition of Komlós convergence (see [16, p. 128]).

Definition 2.5. *A sequence* (u_n) *in* $L^1_H(I)$ *Komlós converges to a function* $u \in L^1_H(I)$ *if for any subsequence* (v_n) *of* (u_n), *one has*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} v_j(t) = u(t) \quad \text{a.e.}$$

We also need the following theorem about the relationship between Komlós convergence and bounded sequences in $L^1_H(I)$ (see [25, Theorem 3.1]).

Proposition 2.6. *Let* (u_n) *be a bounded sequence in* $L^1_H(I)$. *Then, there exists a subsequence* (v_n) *of* (u_n) *and* $u \in L^1_H(I)$ *such that*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} w_j(t) = u(t) \quad \text{a.e.}$$

for any subsequence (w_n) *of* (v_n).

Let us recall the Schauder's fixed point theorem (see [24]).

Theorem 2.7. *Let* C *be a non-empty closed bounded convex subset of a Banach space* E *and let* $f : C \to C$ *be a continuous map. If* $f(C)$ *is relatively compact, then,* f *has a fixed point.*

The discrete version of Gronwall's lemma (see [27]) is given as follows:

Lemma 2.8. *Let* $(\alpha_i), (\beta_i), (\gamma_i)$ *and* (η_i) *be sequences of non-negative real numbers such that*

$$\eta_{i+1} \le \alpha_i + \beta_i(\eta_0 + \eta_1 + \cdots + \eta_{i-1}) + (1+\gamma_i)\eta_i \quad \text{for } i \in \mathbb{N}.$$

Then,

$$\eta_k \le \left(\eta_0 + \sum_{j=0}^{k-1}\alpha_j\right)\exp\left(\sum_{j=0}^{k-1}(j\beta_j + \gamma_j)\right) \quad \text{for } k \in \mathbb{N}^*.$$

We end this section by recalling the Gronwall-like differential inequality proved in [6].

Lemma 2.9. *Let $y : I \to \mathbb{R}$ be a non-negative absolutely continuous function and let $h_1, h_2, g : I \to \mathbb{R}_+$ be non-negative integrable functions. Suppose for some $\varepsilon > 0$*

$$\dot{y}(t) \leq g(t) + \varepsilon + h_1(t)y(t) + h_2(t)(y(t))^{\frac{1}{2}} \int_0^t (y(s))^{\frac{1}{2}} ds \quad a.e.\ t \in I.$$

Then, for all $t \in I$, one has

$$\begin{aligned}(y(t))^{\frac{1}{2}} \leq{}& (y(0)+\varepsilon)^{\frac{1}{2}} \exp\left(\int_0^t (h(s)+1)ds\right) + \frac{\varepsilon^{\frac{1}{2}}}{2} \int_0^t \exp\left(\int_s^t (h(r)+1)dr\right) ds \\ &+ 2\left[\left(\int_0^t g(s)ds + \varepsilon\right)^{\frac{1}{2}} - \varepsilon^{\frac{1}{2}} \exp\left(\int_0^t (h(r)+1)dr\right)\right] \\ &+ 2\int_0^t \Big(h(s)+1\Big) \exp\left(\int_s^t (h(r)+1)dr\right)\left(\int_0^s g(r)dr + \varepsilon\right)^{\frac{1}{2}} ds,\end{aligned}$$

where $h(t) = \max\left(\frac{h_1(t)}{2}, \frac{h_2(t)}{2}\right)$ a.e. $t \in I$.

3 Main result

We start this section by giving some important details to [14, Proposition 4.4] which asserts the existence result to $(IDP_{A(t)})$. We succeed further to obtain the uniqueness of the solution and an estimate of its derivative.

Theorem 3.1. *Let $A(t) : D(A(t)) \subset H \rightrightarrows H$ be a maximal monotone operator for each $t \in I$, satisfying*

- (h_1) *there exists a function $\beta(\cdot) \in W^{1,2}(I, \mathbb{R})$ which is non-negative on $[T_0, T[$ and non-decreasing with $\beta(T_0) = 0$ and $\beta(T) < +\infty$ such that*

$$\operatorname{dis}(A(t), A(s)) \leq |\beta(t) - \beta(s)| \quad \text{for all } t, s \in I;$$

- (h_2) *there exists a non-negative real constant c such that*

$$\|A^0(t)x\| \leq c(1 + \|x\|) \quad \text{for all } t \in I,\ x \in D(A(t));$$

- (h_3) *the set $D(A(t))$ is relatively ball-compact for any $t \in I$.*

Let $f : I \times I \times H \longrightarrow H$ be a map such that

- (i) *the map $f(\cdot, \cdot, x)$ is measurable on $I \times I$ for each $x \in H$;*

(ii) *the map $f(t,s,\cdot)$ is continuous on H for each $(t,s) \in I \times I$, and for every $\eta > 0$, there exists a non-negative function $\xi_\eta(\cdot) \in L^1_{\mathbb{R}}(I)$ such that for all $t, s \in I$ and for any $x, y \in \overline{B}_H[0,\eta]$*

$$\|f(t,s,x) - f(t,s,y)\| \leq \xi_\eta(t)\|x - y\|;$$

(iii) *there exists a non-negative real constant m such that for all $(t,s,x) \in I \times I \times H$, one has*

$$\|f(t,s,x)\| \leq m(1 + \|x\|).$$

Then, for all $u_0 \in D(A(T_0))$, the Integro-Differential Problem $(IDP_{A(t)})$ has a unique absolutely continuous solution $u(\cdot)$ that satisfies

$$\|\dot{u}(t)\| \leq K(1 + \dot{\beta}(t)) \quad a.e.\ t \in I, \tag{3.1}$$

for the non-negative real constant $K = (2(T - T_0)m + \frac{3}{2}c)(K_1 + 1) + 2$ where

$$K_1 = \left(\|u_0\| + \left(2(T-T_0)m + \frac{3}{2}c + 2\right)(T + \beta(T))\right) \exp\left(\left((T-T_0)m + \frac{3}{2}c\right)(T - T_0) + m(T + \beta(T))^2\right).$$

Proof. [14, Proposition 4.4] ensures the existence of a solution $u(\cdot)$. Our main concern is to find a suitable estimate of $\dot{u}(\cdot)$, then, to prove that $u(\cdot)$ is unique.

For any $n \geq 1$, define a subdivision of I by $T_0 = t_0^n < t_1^n < \cdots < t_n^n = T$.

Set for any $n \geq 1$ and $i = 0, 1, \ldots, n-1$,

$$h_{i+1}^n = t_{i+1}^n - t_i^n, \quad \beta_{i+1}^n = \beta(t_{i+1}^n) - \beta(t_i^n).$$

Suppose that

$$h_i^n \leq h_{i+1}^n, \quad \beta_i^n \leq \beta_{i+1}^n.$$

Define the function $\gamma(t) = t + \beta(t)$, $t \in I$. Choose the subdivision such that for all $i = 0, \ldots, n-1$ and $n \geq 1$,

$$\gamma_{i+1}^n = \beta_{i+1}^n + h_{i+1}^n \leq \frac{\gamma(T)}{n} =: \eta_n. \tag{3.2}$$

Fix any integer $n \geq 1$. Let us start by setting $u_0^n := u_0$, for $i = 0, \ldots, n-1$ and $\tau \in]t_i^n, t_{i+1}^n]$,

$$u_{i+1}^n = J_{i+1}^n\left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} \left\{\sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau, s, u_j^n)ds + \int_{t_i^n}^{\tau} f(\tau, s, u_i^n)ds\right\} d\tau\right), \tag{3.3}$$

where

$$J_{i+1}^n := J_{h_{i+1}^n}^{A(t_{i+1}^n)} = (I_H + h_{i+1}^n A(t_{i+1}^n))^{-1}.$$

In view of (2.1) and (3.3), observe that

$$u_{i+1}^n \in D(A(t_{i+1}^n)), \tag{3.4}$$

and

$$u_i^n - \int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau,s,u_j^n)ds + \int_{t_i^n}^{\tau} f(\tau,s,u_i^n)ds \right\} d\tau \in u_{i+1}^n + h_{i+1}^n A(t_{i+1}^n)u_{i+1}^n.$$

Then, one writes

$$-\frac{u_{i+1}^n - u_i^n}{h_{i+1}^n} \in A(t_{i+1}^n)u_{i+1}^n + \frac{1}{h_{i+1}^n}\int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau,s,u_j^n)ds + \int_{t_i^n}^{\tau} f(\tau,s,u_i^n)ds \right\} d\tau. \tag{3.5}$$

Thanks to Lemma 2.4 and (3.3), one has

$$\begin{aligned}
&\|u_{i+1}^n - u_i^n\| \\
&= \left\| J_{i+1}^n\left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau,s,u_j^n)ds + \int_{t_i^n}^{\tau} f(\tau,s,u_i^n)ds \right\} d\tau \right) - u_i^n \right\| \\
&\le \left\| J_{i+1}^n\left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau,s,u_j^n)ds + \int_{t_i^n}^{\tau} f(\tau,s,u_i^n)ds \right\} d\tau \right) - J_{i+1}^n(u_i^n) \right\| \\
&+ \|J_{i+1}^n(u_i^n) - u_i^n\| \\
&\le \int_{t_i^n}^{t_{i+1}^n} \left\| \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau,s,u_j^n)ds + \int_{t_i^n}^{\tau} f(\tau,s,u_i^n)ds \right\| d\tau + h_{i+1}^n \|A^0(t_i^n)u_i^n\| \\
&+ \operatorname{dis}(A(t_i^n), A(t_{i+1}^n)) + \sqrt{h_{i+1}^n(1 + \|A^0(t_i^n)u_i^n\|)\operatorname{dis}(A(t_i^n), A(t_{i+1}^n))}.
\end{aligned}$$

Using the fact that $\sqrt{ab} \le \frac{1}{2}(a+b)$ for all $a, b \in \mathbb{R}_+$, one has

$$\begin{aligned}
\|u_{i+1}^n - u_i^n\| \le &\int_{t_i^n}^{t_{i+1}^n} \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} \|f(\tau,s,u_j^n)\| ds\, d\tau + \int_{t_i^n}^{t_{i+1}^n} \int_{t_i^n}^{\tau} \|f(\tau,s,u_i^n)\| ds\, d\tau \\
&+ \frac{3}{2} h_{i+1}^n \|A^0(t_i^n)u_i^n\| + \frac{3}{2}\operatorname{dis}(A(t_{i+1}^n), A(t_i^n)) + \frac{1}{2} h_{i+1}^n.
\end{aligned}$$

Next, combining (h_1), (h_2) and (iii), one obtains

$$\begin{aligned}
\|u_{i+1}^n - u_i^n\| \le &\frac{3}{2} h_{i+1}^n c(1 + \|u_i^n\|) + \frac{3}{2}\beta_{i+1}^n + \frac{1}{2} h_{i+1}^n + h_{i+1}^n m \sum_{j=0}^{i-1} h_{j+1}^n (1 + \|u_j^n\|) \\
&+ \int_{t_i^n}^{t_{i+1}^n} (\tau - t_i^n) m (1 + \|u_i^n\|) d\tau,
\end{aligned}$$

along with (3.2) and the fact that $\tau - t_i^n \leq T - T_0$, one simplifies

$$\|u_{i+1}^n - u_i^n\| \leq h_{i+1}^n \left((T-T_0)m + \frac{3}{2}c\right)\|u_i^n\| + \gamma_{i+1}^n \left((T-T_0)m + \frac{3}{2}c + 2\right) + h_{i+1}^n m \sum_{j=0}^{i-1} h_{j+1}^n (1 + \|u_j^n\|). \tag{3.6}$$

Remember that $h_{i+1}^n \leq \eta_n$ for $i = 0, \ldots, n-1$, and $\sum_{j=0}^{i-1} h_{j+1}^n \leq T - T_0$, along with (3.2), one gets

$$\|u_{i+1}^n - u_i^n\| \leq h_{i+1}^n \left((T-T_0)m + \frac{3}{2}c\right)\|u_i^n\| + \gamma_{i+1}^n \left(2(T-T_0)m + \frac{3}{2}c + 2\right) + \eta_n m \sum_{j=0}^{i-1} h_{j+1}^n \|u_j^n\|.$$

This yields

$$\|u_{i+1}^n\| \leq \left(1 + h_{i+1}^n \left((T-T_0)m + \frac{3}{2}c\right)\right)\|u_i^n\| + \gamma_{i+1}^n \left(2(T-T_0)m + \frac{3}{2}c + 2\right) + \eta_n^2 m \sum_{j=0}^{i-1} \|u_j^n\|.$$

An application of Lemma 2.8, it follows that for all $n \geq 1$ and $i = 1, \ldots, n$

$$\|u_i^n\| \leq K_1, \tag{3.7}$$

with

$$K_1 := \left(\|u_0\| + \left(2(T-T_0)m + \frac{3}{2}c + 2\right)\gamma(T)\right)\exp\left(\left((T-T_0)m + \frac{3}{2}c\right)(T-T_0) + m\gamma^2(T)\right).$$

Coming back to (3.6) with the help of (3.2), one gets

$$\|u_{i+1}^n - u_i^n\| \leq \gamma_{i+1}^n K, \tag{3.8}$$

with

$$K := \left(2(T-T_0)m + \frac{3}{2}c\right)(K_1 + 1) + 2.$$

For each $n \geq 1$, we define the map $u_n(\cdot) : I \to H$ by: for $t \in [t_i^n, t_{i+1}^n[, 0 \leq i \leq n-1$

$$\begin{aligned} u_n(t) = u_i^n + \frac{t - t_i^n}{h_{i+1}^n}\Bigg(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} \Bigg\{\sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau, s, u_j^n)ds + \int_{t_i^n}^{\tau} f(\tau, s, u_i^n)ds\Bigg\}d\tau\Bigg) \\ - \int_{t_i^n}^{t} \Bigg\{\sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(\tau, s, u_j^n)ds + \int_{t_i^n}^{\tau} f(\tau, s, u_i^n)ds\Bigg\}d\tau, \end{aligned} \tag{3.9}$$

$$u_n(T) = u_n^n, \quad u_n(T_0) = u_0^n.$$

It is clear that the function $u_n(\cdot) : I \to H$ is absolutely continuous for each $n \geq 1$, with $u_n(t_i^n) = u_i^n$ and $u_n(t_{i+1}^n) = u_{i+1}^n$. Moreover, for all $t \in]t_i^n, t_{i+1}^n[$

$$\dot{u}_n(t) = \frac{1}{h_{i+1}^n}\left(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n}\left\{\sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n} f(\tau,s,u_j^n)ds + \int_{t_i^n}^{\tau} f(\tau,s,u_i^n)ds\right\}d\tau\right)$$
$$-\sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n} f(t,s,u_j^n)ds - \int_{t_i^n}^{t} f(t,s,u_i^n)ds. \tag{3.10}$$

Combining (iii), (3.7), (3.8) and (3.9), it results

$$\|u_n(t) - u_i^n\| \leq \|u_{i+1}^n - u_i^n\| + 2(T-T_0)m(1+K_1)h_{i+1}^n \leq \gamma_{i+1}^n(K + 2(T-T_0)m(1+K_1)),$$

along with (3.2) yields

$$\|u_n(t) - u_i^n\| \leq L\eta_n \tag{3.11}$$

where

$$L := K + 2(T-T_0)m(1+K_1).$$

Fix $s \in [t_i^n, t_{i+1}^n[$ and $t \in [t_j^n, t_{j+1}^n[$ with $j > i$. Then, by (3.2), (3.8) and (3.11), one has

$$\begin{aligned}
\|u_n(t) - u_n(s)\| &\leq \|u_n(t) - u_j^n\| + \|u_j^n - u_i^n\| + \|u_i^n - u_n(s)\| \\
&\leq \|u_j^n - u_i^n\| + 2L\eta_n \leq \sum_{p=0}^{j-i-1}\|u_{i+p+1}^n - u_{i+p}^n\| + 2L\eta_n \\
&\leq K\sum_{p=0}^{j-i-1}\gamma_{i+p+1}^n + 2L\eta_n = K\left(\gamma(t_j^n) - \gamma(t_i^n)\right) + 2L\eta_n \\
&\leq K\left(\gamma(t) - \gamma(t_i^n)\right) + 2L\eta_n = K\left(\gamma(t) - \gamma(s) + \gamma(s) - \gamma(t_i^n)\right) + 2L\eta_n \\
&\leq K\left(\gamma(t) - \gamma(s) + \gamma(t_{i+1}^n) - \gamma(t_i^n)\right) + 2L\eta_n \\
&= K\left(\gamma(t) - \gamma(s)\right) + K\gamma_{i+1}^n + 2L\eta_n \\
&\leq K\left(\gamma(t) - \gamma(s)\right) + (K + 2L)\eta_n.
\end{aligned}$$

Then, for any $n \geq 1$ and $T_0 \leq s \leq t \leq T$, one gets

$$\|u_n(t) - u_n(s)\| \leq K\big(\gamma(t) - \gamma(s)\big) + (K+2L)\eta_n = K\big(t - s + \beta(t) - \beta(s)\big) + (K+2L)\eta_n. \tag{3.12}$$

Combining (3.4)-(3.5) and (3.9)-(3.10), it results that

$$-\dot{u}_n(t) \in A(\delta_n(t))u_n(\delta_n(t)) + g_n(t) \quad \text{a.e. } t \in I, \quad u_n(\delta_n(t)) \in D(A(\delta_n(t))),$$

where $g_n(t) = \int_{T_0}^t f(t,s,u_n(\theta_n(s)))ds$ and the maps $\theta_n, \delta_n : I \to I$ are defined by $\theta_n(T_0) = T_0$,

$\theta_n(t) = t_i^n$ if $t \in]t_i^n, t_{i+1}^n]$ and $\delta_n(T_0) = T_0$, $\delta_n(t) = t_{i+1}^n$ if $t \in]t_i^n, t_{i+1}^n]$ for some $i \in \{0, \ldots, n-1\}$.

By Arzelà-Ascoli theorem (with the help of (h_3)), it is easy to show that the constructed sequence $(u_n(\cdot))$ uniformly converges to some $u(\cdot) \in W^{1,2}(I, H)$. To verify that $u(\cdot)$ is a solution of the required integro-differential inclusion, we proceed as in Step 3 in the proof of [2, Theorem 3.2] with appropriate changes.

Finally, passing to the limit in (3.12) as $n \to \infty$ (noting that $\eta_n \to 0$) yields

$$\|\dot{u}(t)\| \leq K(1 + \dot{\beta}(t)) \quad \text{a.e. } t \in I.$$

Uniqueness. Let $u_1(\cdot)$ and $u_2(\cdot)$ be two solutions to $(IDP_{A(t)})$. Since $A(t)$ is monotone then, one has

$$\frac{1}{2}\frac{d}{dt}\|u_2(t) - u_1(t)\|^2 \leq \left\langle \int_{T_0}^{t} f(t, s, u_1(s))ds - \int_{T_0}^{t} f(t, s, u_2(s))ds, u_2(t) - u_1(t) \right\rangle. \tag{3.13}$$

By the estimate of the velocity above, there exists a non-negative real constant η such that $\|u_1(t)\| \leq \eta$ and $\|u_2(t)\| \leq \eta$, for each $t \in I$, along with (ii), there is $\xi_\eta(\cdot) \in L^1_{\mathbb{R}}(I)$ such that

$$\|f(t, s, u_1(s)) - f(t, s, u_2(s))\| \leq \xi_\eta(t)\|u_1(s) - u_2(s)\| \quad \text{for all } (t, s) \in I \times I,$$

so that coming back to (3.13), it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_2(t) - u_1(t)\|^2 \leq \xi_\eta(t)\|u_2(t) - u_1(t)\| \int_{T_0}^{t} \|u_2(s) - u_1(s)\|ds.$$

Hence, Lemma 2.9 with $\varepsilon > 0$ arbitrary yields $u_1 = u_2$ and guarantees the uniqueness of the solution to $(IDP_{A(t)})$. □

Now, we are able to prove our main result concerning $(IDP_{A(t,u)})$.

Theorem 3.2. *Let $A(t, x) : D(A(t, x)) \subset H \rightrightarrows H$ be a maximal monotone operator for each $(t, x) \in I \times H$ satisfying*

(H_1) *there exist a non-negative and non-decreasing real function $\alpha(\cdot) \in W^{1,2}(I, \mathbb{R})$ and a non-negative real constant $\lambda < 1$ such that*

$$\operatorname{dis}(A(t, x), A(s, y)) \leq |\alpha(t) - \alpha(s)| + \lambda\|x - y\| \quad \forall t, s \in I \text{ and } \forall x, y \in H;$$

(H_2) *there exists a non-negative real constant c such that*

$$\|A^0(t, x)y\| \leq c(1 + \|x\| + \|y\|) \quad \text{for all } (t, x) \in I \times H \text{ and } y \in D(A(t, x));$$

(H_3) for any bounded subset X of H, the set $D(A(I \times X))$ is relatively ball-compact.

Let $f : I \times I \times H \longrightarrow H$ be a map satisfying assumptions (i)-(ii)-(iii) of Theorem 3.1.

Put $d = c(2 + \|u_0\|)$, $S = \left(2(T - T_0)m + \frac{3}{2}d\right)(S_1 + 1) + 2$, where

$$S_1 = \left(\|u_0\| + \left(2(T - T_0)m + \frac{3}{2}d + 2\right)(T + \alpha(T) + 1)\right)$$
$$\exp\left(\left((T - T_0)m + \frac{3}{2}d\right)(T - T_0) + m(T + \alpha(T) + 1)^2\right).$$

If $\lambda S < 1$, then, the Integro-Differential Problem $(IDP_{A(t,u)})$ admits an absolutely continuous solution $u(\cdot)$ that satisfies

$$\|\dot{u}(t)\| \leq \dot{\varphi}(t) \quad a.e.\ t \in I, \tag{3.14}$$

where $\varphi : I \to \mathbb{R}_+$ is the absolutely continuous solution to

$$\dot{\varphi}(t) = \frac{L}{1 - \lambda L}(1 + \dot{\alpha}(t)), \quad \varphi(T_0) = 0,$$

for the non-negative real constant $L = \left(2(T - T_0)m + \frac{3}{2}d\right)(L_1 + 1) + 2$, where

$$L_1 = \left(\|u_0\| + \left(2(T - T_0)m + \frac{3}{2}d + 2\right)(T + \alpha(T) + \lambda)\right)$$
$$\exp\left(\left((T - T_0)m + \frac{3}{2}d\right)(T - T_0) + m(T + \alpha(T) + \lambda)^2\right).$$

Proof. Observe that $1 - \lambda L > 0$ (in the differential equation) noting that $\lambda S < 1$ by assumption and since $L < S$ then, $\lambda < \frac{1}{L}$.

Since $\varphi(\cdot)$ is absolutely continuous, then, there exists some non-negative real constant $\delta > 0$ such that

$$\int_{T_0}^{T} \dot{\varphi}(s)ds < \delta \quad \text{for all } t \in I.$$

Let us just take $\delta = 1$ (for simplicity) and suppose that

$$\int_{T_0}^{T} \dot{\varphi}(s)ds < 1 \quad \text{for all } t \in I. \tag{3.15}$$

Let us consider the convex bounded closed subset Y of the Banach space $\mathcal{C}_H(I)$ defined by

$$Y := \left\{u \in \mathcal{C}_H(I) : u(t) = u_0 + \int_{T_0}^{t} \dot{u}(s)ds,\ \|\dot{u}(t)\| \leq \dot{\varphi}(t),\ t \in I\right\}.$$

Let $h \in Y$, and define the time-dependent maximal monotone operator $B_h(t) = A(t, h(t))$, $t \in I$

(as in [15, Lemma 5]). For all $T_0 \leq \tau \leq t \leq T$, one has using (H_1)

$$\begin{aligned}\operatorname{dis}(B_h(t), B_h(\tau)) &= \operatorname{dis}(A(t,h(t)), A(\tau,h(\tau))) \leq \alpha(t)-\alpha(\tau)+\lambda\|h(t)-h(\tau)\| \\ &\leq \int_\tau^t \dot{\alpha}(s)ds + \lambda\int_\tau^t \|\dot{h}(s)\|ds \leq \int_\tau^t [\dot{\alpha}(s)+\lambda\dot{\varphi}(s)]\,ds = \beta(t)-\beta(\tau),\end{aligned}$$

where $\beta(\cdot) \in W^{1,2}(I,\mathbb{R})$ is given by

$$\beta(t) = \int_{T_0}^t [\dot{\alpha}(s)+\lambda\dot{\varphi}(s)]\,ds, \quad \forall t \in I.$$

Furthermore, one writes using (H_2) and (3.15)

$$\begin{aligned}\|B_h^0(t)x\| = \|A^0(t,h(t))x\| &\leq c(1+\|h(t)\|+\|x\|) \\ &\leq c\left(1+\|u_0\|+\int_{T_0}^t \dot{\varphi}(s)ds+\|x\|\right) \\ &\leq c(2+\|u_0\|+\|x\|) \leq d(1+\|x\|),\end{aligned}$$

for all $t \in I$ and $x \in D(A(t,h(t)))$, where $d = c(2+\|u_0\|)$.

In view of Theorem 3.1, there exists a unique absolutely continuous solution $u_h : I \to H$ to the integro-differential inclusion

$$(\mathcal{I}_h) \qquad \begin{cases} -\dot{u}_h(t) \in B_h(t)u_h(t) + \displaystyle\int_{T_0}^t f(t,s,u_h(s))ds \quad \text{a.e. } t \in I,\ h \in Y, \\ u_h(t) \in D(B_h(t)) = D(A(t,h(t))), \quad \forall t \in I \\ u_h(T_0) = u_0 \in D(B_h(T_0)) = D(A(T_0,u_0)), \end{cases}$$

with

$$\|\dot{u}_h(t)\| \leq \rho(1+\dot{\alpha}(t)+\lambda\dot{\varphi}(t)) \quad \text{a.e. } t \in I, \tag{3.16}$$

for the non-negative real constant $\rho = (2(T-T_0)m+\frac{3}{2}d)(\rho_1+1)+2$, where

$$\begin{aligned}\rho_1 = &\left(\|u_0\| + \left(2(T-T_0)m+\frac{3}{2}d+2\right)(T+\beta(T))\right) \\ &\exp\left(\left((T-T_0)m+\frac{3}{2}d\right)(T-T_0)+m(T+\beta(T))^2\right).\end{aligned}$$

Now, for each $h \in Y$, let us consider the map Φ defined on Y by

$$\Phi(h)(t) := u_h(t), \quad t \in I,$$

where $u_h(\cdot)$ is the unique absolutely continuous solution to the latter integro-differential inclusion, namely $(\mathcal{I}_h)$.

Observe that $\rho < L$. Indeed, note by (H_1) that $\alpha(\cdot)$ is a non-decreasing and non-negative function, along with the definition of $\beta(\cdot)$, one writes

$$\beta(T) = \int_{T_0}^{T} [\dot{\alpha}(s) + \lambda\dot{\varphi}(s)]\, ds \leq \alpha(T) + \lambda \int_{T_0}^{T} \dot{\varphi}(s)ds \leq \alpha(T) + \lambda,$$

using the fact that $\int_{T_0}^{T} \dot{\varphi}(s)ds < 1$ by (3.15). Then, from the definition of ρ_1 and L_1, this just shows that $\rho_1 < L_1$. We return therefore to the expression of ρ and L to compare.

Thus, coming back to (3.16), one writes

$$\|\dot{u}_h(t)\| \leq L(1 + \dot{\alpha}(t) + \lambda\dot{\varphi}(t)) = \dot{\varphi}(t). \tag{3.17}$$

As a result, $\Phi(h) \in Y$.

Also, note that using (3.15) for any $h \in Y$, one gets

$$\|u_h(t)\| \leq \|u_0\| + \varphi(T) \quad \text{for all } t \in I. \tag{3.18}$$

Let us prove that $\Phi(Y)$ is relatively compact in $\mathcal{C}_H(I)$.

On the one hand, note by (3.18) that for any $h \in Y$

$$h(t) \in (\|u_0\| + \varphi(T))\overline{B}_H.$$

On the other hand, since $u_h(t) \in D(A(t, h(t)))$ for each $t \in I$ then,

$$u_h(t) \in D(A(I \times (\|u_0\| + \varphi(T))\overline{B}_H)) \cap (\|u_0\| + \varphi(T))\overline{B}_H.$$

Using the ball-compactness assumption in (H_3), one deduces that for each $t \in I$, $\{\Phi(h)(t),\ h \in Y\}$ is relatively compact in H, for any $t \in I$. Moreover, $(\Phi(h))$ is equi-continuous. By Arzelà-Ascoli theorem, $\Phi(Y)$ is relatively compact in $\mathcal{C}_H(I)$.

Now, we check that Φ is continuous. It is sufficient to show that: if (h_n) uniformly converges to h in Y, then, the sequence of absolutely continuous solutions u_{h_n} associated with h_n to the integro-differential inclusion

$$\begin{cases} -\dot{u}_{h_n}(t) \in A(t, h_n(t))u_{h_n}(t) + \displaystyle\int_{T_0}^{t} f(t, s, u_{h_n}(s))ds \quad \text{a.e. } t \in I,\ h_n \in Y, \\ u_{h_n}(T_0) = u_0 \in D(A(T_0, u_0)), \end{cases}$$

uniformly converges to the absolutely continuous solution u_h associated with h to the integro-differential inclusion $(\mathcal{I}_h)$.

As $(u_{h_n}(t))$ is relatively compact in H, for any $t \in I$ (from above) and (u_{h_n}) is equi-absolutely continuous, along with the estimate (3.16), we may assume that there exists some map $z \in W^{1,2}(I,H)$ such that

$$(u_{h_n}) \text{ uniformly converges to } z(\cdot), \tag{3.19}$$

and

$$(\dot{u}_{h_n})\ \sigma(L^1_H(I), L^\infty_H(I))\text{-converges to } w \in L^1_H(I) \text{ with } w = \dot{z} \text{ a.e.} \tag{3.20}$$

Put $\eta := \|u_0\| + \varphi(T)$. Then, by (ii), there exists a non-negative function $\xi_\eta(\cdot) \in L^1_{\mathbb{R}}(I)$ such that for all $t, s \in I$

$$\|f(t,s,u_{h_n}(s)) - f(t,s,z(s))\| \le \xi_\eta(t)\|u_{h_n}(s) - z(s)\|.$$

This along with the pointwise convergence of (u_{h_n}) to z gives

$$\lim_{n\to\infty} \|f(t,s,u_{h_n}(s)) - f(t,s,z(s))\| = 0. \tag{3.21}$$

Note by (3.18) and (iii) that for any n and any $t, s \in I$

$$\|f(t,s,u_{h_n}(s))\| \le m(1+\eta), \tag{3.22}$$

along with (3.21), it follows from the Lebesgue dominated convergence theorem that

$$\left\|\int_{T_0}^t f(t,s,u_{h_n}(s))ds - \int_{T_0}^t f(t,s,z(s))ds\right\| \le \int_{T_0}^t \|f(t,s,u_{h_n}(s)) - f(t,s,z(s))\|ds \to 0,$$

as $n \to \infty$.

Moreover, thanks to (3.22), we note that for any $t, s \in I$

$$\left\|\int_{T_0}^t f(t,s,u_{h_n}(s))ds\right\| \le m(T-T_0)(1+\eta). \tag{3.23}$$

This along with the convergence above, the Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty} \int_{T_0}^T \left\|\int_{T_0}^t f(t,s,u_{h_n}(s))ds - \int_{T_0}^t f(t,s,z(s))ds\right\| dt = 0. \tag{3.24}$$

Define for any $n \ge 1$, the functions g_n, g on I by

$$g_n(t) = \int_{T_0}^t f(t,s,u_{h_n}(s))ds, \quad g(t) = \int_{T_0}^t f(t,s,z(s))ds \quad \text{for any } t \in I.$$

As $u_{h_n}(t) \in D(A(t,h_n(t)))$ for all $t \in I$ and $u_{h_n}(t) \to z(t)$, $(A^0(t,h_n(t))u_{h_n}(t))$ is bounded by (H_2)

and the boundedness of the sequences (u_{h_n}) and (h_n) in $\mathcal{C}_H(I)$, for every $t \in I$

$$\operatorname{dis}(A(t,h_n(t)),A(t,h(t))) \leq \lambda \|h_n(t)-h(t)\| \to 0, \quad \text{as} \quad n \to \infty, \tag{3.25}$$

by (H_1) and the uniform convergence of (h_n) to h in $\mathcal{C}_H(I)$. Thus, from Lemma 2.1, one deduces that $z(t) \in D(A(t,h(t)))$, for each $t \in I$.

Now, let us verify that z satisfies the integro-differential inclusion

$$-\dot{z}(t) \in A(t,h(t))z(t) + \int_{T_0}^{t} f(t,s,z(s))ds \quad \text{a.e. } t \in I.$$

From (3.20) and (3.24), one deduces that $(\dot{u}_{h_n}(\cdot)+g_n(\cdot))$ $\sigma(L^1_H(I), L^\infty_H(I))$-converges to $\dot{z}(\cdot)+g(\cdot)$. Hence, $(\dot{u}_{h_n}(\cdot)+g_n(\cdot))$ Komlós-converges to $\dot{z}(\cdot)+g(\cdot)$, and there is a negligible set V such that for $t \in I \setminus V$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n}(\dot{u}_{h_j}(t)+g_j(t)) = \dot{z}(t)+g(t), \tag{3.26}$$

and

$$-\dot{u}_{h_n}(t) \in A(t,h_n(t))u_{h_n}(t)+g_n(t). \tag{3.27}$$

Let $x \in D(A(t,h(t)))$. From (H_2) and (3.25) along with Lemma 2.2, there is a sequence (x_n) such that $x_n \in D(A(t,h_n(t)))$,

$$x_n \to x \quad \text{and} \quad A^0(t,h_n(t))x_n \to A^0(t,h(t))x. \tag{3.28}$$

In view of (3.27), by the monotonicity of the operators $A(t,h_n(t))$ for each n and $t \in I$, one has

$$\langle \dot{u}_{h_n}(t)+g_n(t), u_{h_n}(t)-x_n\rangle \leq \left\langle A^0(t,h_n(t))x_n, x_n-u_{h_n}(t)\right\rangle. \tag{3.29}$$

Note that

$$\begin{aligned}\langle \dot{u}_{h_n}(t)+g_n(t), z(t)-x\rangle &= \langle \dot{u}_{h_n}(t)+g_n(t), u_{h_n}(t)-x_n\rangle \\ &\quad + \langle \dot{u}_{h_n}(t)+g_n(t), z(t)-u_{h_n}(t)\rangle + \langle \dot{u}_{h_n}(t)+g_n(t), x_n-x\rangle,\end{aligned}$$

then,

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\langle \dot{u}_{h_j}(t)+g_j(t), z(t)-x\rangle &= \frac{1}{n}\sum_{j=1}^{n}\langle \dot{u}_{h_j}(t)+g_j(t), u_{h_j}(t)-x_j\rangle \\ &\quad + \frac{1}{n}\sum_{j=1}^{n}\langle \dot{u}_{h_j}(t)+g_j(t), z(t)-u_{h_j}(t)\rangle + \frac{1}{n}\sum_{j=1}^{n}\langle \dot{u}_{h_j}(t)+g_j(t), x_j-x\rangle.\end{aligned}$$

Hence, combining (3.17), (3.23) and (3.29), one deduces that

$$\frac{1}{n}\sum_{j=1}^{n}\langle \dot{u}_{h_j}(t)+g_j(t), z(t)-x\rangle \leq \frac{1}{n}\sum_{j=1}^{n}\left\langle A^0(t,h_j(t))x_j, x_j-u_{h_j}(t)\right\rangle$$

$$+\left(\dot{\varphi}(t)+(T-T_0)m(1+\eta)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\|z(t)-u_{h_j}(t)\|+\frac{1}{n}\sum_{j=1}^{n}\|x_j-x\|\right).$$

Passing to the limit when $n\to\infty$, using (3.19), (3.26), (3.28), this last inequality yields

$$\langle \dot{z}(t)+g(t), z(t)-x\rangle \leq \langle A^0(t,h(t))x, x-z(t)\rangle \quad \text{a.e. } \forall x\in D(A(t,h(t))).$$

It results from Lemma 2.3 that

$$-\dot{z}(t)\in A(t,h(t))z(t)+g(t) \quad \text{a.e. } t\in I,$$

with $z(T_0)=u_0\in D(A(T_0,u_0))$ and by uniqueness $z=u_h$.

Therefore, one just checks that $\Phi(h_n)-\Phi(h)\to 0$ in $\mathcal{C}_H(I)$ as $n\to\infty$. Consequently, $\Phi: Y\to Y$ is continuous from the bounded convex closed subset Y of the Banach space $\mathcal{C}_H(I)$ with $\Phi(Y)$ is relatively compact. Applying Schauder's fixed point theorem (see Theorem 2.7) there exists $h\in Y$ such that $h=\Phi(h)$, that is, $h(t)=u_h(t)$. Furthermore, the estimation (3.14) holds true on I. The proof of the theorem is then complete. □

We derive from Theorem 3.2, the particular case of the sweeping process, that is, $A(t,x)=N_{C(t,x)}$, for $(t,x)\in I\times H$.

Corollary 3.3. *Let $C: I\times H \rightrightarrows H$ be a set-valued mapping satisfying:*

(H_1') *For each $(t,y)\in I\times H$, $C(t,y)$ is a non-empty closed convex subset of H.*

(H_2') *There exist a non-negative real constant $\lambda<1$, and a function $\alpha\in W^{1,2}(I,\mathbb{R})$ which is non-negative on $[T_0,T[$ and non-decreasing such that*

$$|d(x,C(t,u))-d(x,C(s,v))|\leq|\alpha(t)-\alpha(s)|+\lambda||v-u|| \quad \forall t,s\in I, \quad \forall x,v,u\in H.$$

(H_3') *For any bounded subset X of H, the set $C(I\times X)$ is relatively ball-compact.*

Let $f: I\times I\times H\longrightarrow H$ be a map satisfying assumptions of Theorem 3.2.

Choose any $d > 0$ *and put* $S = \left(2(T-T_0)m + \frac{3}{2}d\right)(S_1+1)+2$, *where*

$$S_1 = \left(\|u_0\| + \left(2(T-T_0)m + \frac{3}{2}d + 2\right)(T+\alpha(T)+1)\right)$$
$$\exp\left(\left((T-T_0)m + \frac{3}{2}d\right)(T-T_0) + m(T+\alpha(T)+1)^2\right).$$

If $\lambda S < 1$, *then, the integro-differential sweeping process*

$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}u(t) + \displaystyle\int_{T_0}^{t} f(t,s,u(s))ds \quad a.e.\ t \in I, \\ u(T_0) = u_0 \in C(T_0,u_0), \end{cases}$$

has an absolutely continuous solution $u(\cdot)$. *Moreover, an appropriate estimate of* $\dot{u}(\cdot)$ *holds true.*

Proof. We follow the arguments used in the proof of [33, Corollary 8].

Let $A(t,x) = N_{C(t,x)}$, for each $(t,x) \in I \times H$. Then, for any $(t,x) \in I \times H$, $A(t,x) : D(A(t,x)) \subset H \rightrightarrows H$ is a maximal monotone operator with $D(A(t,x)) = C(t,x)$ and since the projection of the origin onto $N_{C(t,x)}y$ equals 0 then $\|A^0(t,x)y\| = 0$ for any $(t,x) \in I \times H$ and any $y \in C(t,x)$ (keeping in mind (2.2) and (H_1')). So, (H_2) holds true for any non-negative real constant c. Moreover, it is easily seen that (H_3) is satisfied. Let us verify (H_1).

On the one hand, from [26], one has

$$d_H(C(t,u),C(s,v)) = \sup_{x \in H} |d(x,C(t,u)) - d(x,C(s,v))|, \tag{3.30}$$

where $d_H(\cdot,\cdot)$ denotes the Hausdorff distance between two closed subsets of H.

On the other hand, it is known from [35] that since $C(t,u)$, $C(s,v)$ are convex closed sets, then

$$\operatorname{dis}(N_{C(t,u)}, N_{C(s,v)}) = d_H(C(t,u),C(s,v)). \tag{3.31}$$

Combining (3.30) and (3.31) with (H_2'), then, (H_1) holds true.

Hence, all assumptions of Theorem 3.2 are satisfied. The latter ensures the existence of a solution to the integro-differential sweeping process under consideration.

Furthermore, in view of (3.14), an appropriate estimate of $\dot{u}$ is obtained. □

4 An optimal control problem

In this section, we focus on the $\mathcal{O}$ptimal $\mathcal{C}$ontrol $\mathcal{P}$roblem ($\mathcal{OCP}$).

First, let us prove the existence and uniqueness of the solution to problem ($\mathcal{CP}_{a,b}$).

Proposition 4.1. *Let $H = \mathbb{R}^n$ and $I := [0,T]$. Fix a couple $(a(\cdot), b(\cdot)) \in W^{1,2}(I, \mathbb{R}^{n+m})$. Assume that for any $(t,y) \in I \times \mathbb{R}^n$, $A(t,y) : D(A(t,y)) \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a maximal monotone operator satisfying assumptions (H_1)-(H_2). Let $f : I \times I \times \mathbb{R}^{m+n} \to \mathbb{R}^n$ be a map such that $f(\cdot,\cdot,x,y)$ is measurable on $I \times I$ for each $(x,y) \in \mathbb{R}^{m+n}$, $f(t,s,\cdot,\cdot)$ is continuous on $\mathbb{R}^{m+n}$ for each $(t,s) \in I \times I$ and satisfying the following assumptions*

($\underline{i}$) *there exists a non-negative real constant M, for any $b(\cdot) \in W^{1,2}(I, \mathbb{R}^m)$ such that*

$$\|f(t,s,b(s),x)\| \leq \|b(s)\| + M\|x\|, \quad \forall t,s \in I, \quad \forall x \in \mathbb{R}^n;$$

($\underline{ii}$) *for a non-negative real constant η and any $b(\cdot) \in W^{1,2}(I, \mathbb{R}^m)$, there exists a non-negative real constant l such that*

$$\|f(t,s,b(s),x_1) - f(t,s,b(s),x_2)\| \leq l\|x_1 - x_2\|, \quad \forall t,s \in I, \quad \forall x_1, x_2 \in \overline{B}_{\mathbb{R}^n}[0,\eta].$$

Then, this couple control generates a unique solution $u(\cdot) \in W^{1,2}(I, \mathbb{R}^n)$ to the $\mathcal{C}$ontrolled $\mathcal{P}$roblem ($\mathcal{CP}_{a,b}$). Moreover, one has for a.e. $t \in I$

$$\left\|\dot{u}(t) + \int_0^t f(t,s,b(s),u(s))ds\right\| \leq K(1 + \dot{\beta}(t)) + (1+L)\zeta, \tag{4.1}$$

$$\|\dot{u}(t)\| \leq K(1 + \dot{\beta}(t)), \tag{4.2}$$

where $\zeta = \max\left(\|b\|_{L^1_{\mathbb{R}^m}(I)}, TM\right)$, $L = \|u_0\| + K\int_0^T (1 + \dot{\beta}(s))ds$, and the function β is defined by

$$\beta(t) = \int_0^t [\dot{\alpha}(s) + \lambda\|\dot{a}(s)\|]ds, \quad t \in I,$$

and K is a non-negative real constant which depends on $\|u_0\|$, $\|a_0\|$, c, ζ, T, and β.

Proof. For any $t \in I$ and any fixed $a(\cdot) \in W^{1,2}(I, \mathbb{R}^n)$, define the time-dependent maximal monotone operators $B_a(t) := A(t, a(t))$ and proceed as in the first part of the proof of Theorem 3.2.

Let $\tau, t \in I$ such that $0 \leq \tau \leq t \leq T$. Then, one has by (H_1)

$$\operatorname{dis}(B_a(t), B_a(\tau)) = \operatorname{dis}(A(t,a(t)), A(\tau, a(\tau))) \leq \beta(t) - \beta(\tau),$$

and clearly $\beta(\cdot) \in W^{1,2}(I, \mathbb{R})$ is defined by

$$\beta(t) = \int_0^t [\dot{\alpha}(s) + \lambda \|\dot{a}(s)\|] ds, \quad t \in I.$$

Now, in view of (H_2), there exists a non-negative real number c such that for $t \in I$, $z \in D(A(t, a(t)))$

$$\|B_a^0(t)z\| = \|A^0(t, a(t))z\| \leq c(1 + \|a(t)\| + \|z\|) \leq c\left(1 + \left\|a_0 + \int_0^t \dot{a}(s)ds\right\| + \|z\|\right) \leq c_1(1 + \|z\|),$$

where $c_1 = c(1 + \|a_0\| + \int_0^T \|\dot{a}(s)\|ds)$.

Hence, the operator $B_a(t)$ satisfies (h_1)-(h_2) of Theorem 3.1.

Next, for $b(\cdot) \in W^{1,2}(I, \mathbb{R}^m)$ fixed, define the function f_b on $I \times I \times \mathbb{R}^n$ by

$$f_b(t, s, u) = f(t, s, b(s), u) \quad \text{for all } (t, s, u) \in I \times I \times \mathbb{R}^n.$$

It is clear that the function $f_b(\cdot, \cdot, u)$ is measurable on $I \times I$ for any fixed $u \in \mathbb{R}^n$, by assumption and by continuity of $b(\cdot)$. Moreover, from $(\underline{i})$ one gets

$$\|f_b(t, s, u)\| \leq \|b(s)\| + M\|u\| \leq \zeta(1 + \|u\|), \tag{4.3}$$

for all $(t, s, u) \in I \times I \times \mathbb{R}^n$, where $\zeta = \max(\|b\|_\infty, M)$.

Now, by $(\underline{ii})$ for a non-negative real constant η, there exists a non-negative real constant l such that

$$\|f_b(t, s, u_1) - f_b(t, s, u_2)\| \leq l\|u_1 - u_2\|, \quad \forall t \in I, \quad \forall u_1, u_2 \in \overline{B}_{\mathbb{R}^n}[0, \eta].$$

Thus, the map f_b satisfies assumptions of Theorem 3.1. Consequently, it follows the existence and uniqueness of the solution to the considered integro-differential inclusion.

Furthermore, in view of (3.1) and (4.3) along with the absolute continuity of $u(\cdot)$, estimates (4.1)-(4.2) hold true. The velocity $\dot{u}(\cdot)$ is clearly in $L^2_{\mathbb{R}^n}(I)$, and $u(\cdot) \in W^{1,2}(I, \mathbb{R}^n)$. The proof of the proposition is therefore finished. □

We are going to impose convenient assumptions that guarantee the existence of (global) optimal solutions to the $\mathcal{O}$ptimal $\mathcal{C}$ontrol $\mathcal{P}$roblem ($\mathcal{OCP}$) subject to the solution set of the $\mathcal{C}$ontrolled $\mathcal{P}$roblem ($\mathcal{CP}_{a,b}$).

Theorem 4.2 (Existence of optimal solutions)**.** *Assume that for any $(t, y) \in I \times \mathbb{R}^n$, $A(t, y) : D(A(t, y)) \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a maximal monotone operator satisfying assumptions (H_1)-(H_2). Let $f : I \times I \times \mathbb{R}^{m+n} \to \mathbb{R}^n$ be a continuous map satisfying assumptions of Proposition 4.1. Suppose that the terminal cost functional $\phi_1 : \mathbb{R}^n \to \overline{\mathbb{R}}$ is lower semi-continuous, while the running cost $\phi_2 : I \times \mathbb{R}^{4n+2m} \to \overline{\mathbb{R}}$ is lower semi-continuous with respect to t and is majorized by a summable* *function on I along reference curves. Moreover, assume that $\phi_2(t,\cdot)$ is bounded from below on bounded sets for a.e. $t \in I$. Let the running cost ϕ_2 be convex with respect to velocity variables $\dot{u}$, $\dot{a}$, $\dot{b}$, and that there is a minimizing sequence $(u^k(\cdot), a^k(\cdot), b^k(\cdot))$ of $(\mathcal{OCP})$, which $(a^k(\cdot), b^k(\cdot))$ is bounded in $W^{1,2}(I,\mathbb{R}^{n+m})$. Then, the $\mathcal{O}$ptimal $\mathcal{C}$ontrol $\mathcal{P}$roblem $(\mathcal{OCP})$ admits an optimal solution in the space $W^{1,2}(I,\mathbb{R}^{2n+m})$.*

Proof. From Proposition 4.1, one deduces that the set of feasible solutions to the $\mathcal{O}$ptimal $\mathcal{C}$ontrol $\mathcal{P}$roblem $(\mathcal{OCP})$ is non-empty. Let us fix the minimizing sequence of feasible solutions $(u^k(\cdot), a^k(\cdot), b^k(\cdot))$ for $(\mathcal{OCP})$ (from the statement of the theorem), which is bounded in $W^{1,2}(I,\mathbb{R}^{2n+m})$. This implies in particular that there exists a couple $(a_0, b_0) \in \mathbb{R}^{n+m}$ such that $(a^k(0), b^k(0)) \to (a_0, b_0)$ in this space as $k \to \infty$, while the triple $(u_0, a_0, b_0) = (u(0), a(0), b(0))$ clearly satisfies the initial conditions. It is readily seen that the sequence $(\dot{a}^k(\cdot), \dot{b}^k(\cdot))$ is bounded in $L^2_{\mathbb{R}^{n+m}}(I)$. Then, up to a subsequence that we do not relabel, there exists a couple $(v^a(\cdot), v^b(\cdot)) \in L^2_{\mathbb{R}^{n+m}}(I)$ such that

$$(\dot{a}^k(\cdot), \dot{b}^k(\cdot)) \text{ weakly converges in } L^2_{\mathbb{R}^{n+m}}(I) \text{ to } (v^a(\cdot), v^b(\cdot)).$$

Define now the functions

$$(\hat{a}(t), \hat{b}(t)) = (a_0, b_0) + \int_0^t (v^a(s), v^b(s))ds, \text{ for all } t \in I,$$

and observe that $(\dot{\hat{a}}(t), \dot{\hat{b}}(t)) = (v^a(t), v^b(t))$ for a.e. $t \in I$, and that the couple $(\hat{a}(\cdot), \hat{b}(\cdot))$ belongs to the space $W^{1,2}(I,\mathbb{R}^{n+m})$. It follows from above and the estimates of Proposition 4.1 that the sequence of the corresponding solutions $(u^k(\cdot))$ is uniformly bounded and equi-continuous on I. By Arzelà-Ascoli theorem, up to a subsequence that we do not relabel, $(u^k(\cdot))$ uniformly converges on I to some $\hat{u}(\cdot) \in \mathcal{C}_{\mathbb{R}^n}(I)$ which is absolutely continuous on this interval. It follows from (4.2) that $(\dot{u}^k(\cdot))$ is bounded in $L^2_{\mathbb{R}^n}(I)$ and hence it weakly converges in $L^2_{\mathbb{R}^n}(I)$ up to a subsequence, to some function $w(\cdot)$ with $\dot{\hat{u}}(t) = w(t)$ for a.e. $t \in I$, that is,

$$(\dot{u}^k(\cdot)) \text{ weakly converges in } L^2_{\mathbb{R}^n}(I) \text{ to } \dot{\hat{u}}(\cdot). \tag{4.4}$$

The next step is to check that the limiting triple $\hat{z}(\cdot) = (\hat{u}(\cdot), \hat{a}(\cdot), \hat{b}(\cdot))$ satisfies the differential inclusion $(\mathcal{CP}_{a,b})$.

Since f is continuous by assumption along with the preceding modes of convergence above, then, one has

$$f(t, s, b^k(s), u^k(s)) \to f(t, s, \hat{b}(s), \hat{u}(s)) \quad \text{as } k \to \infty, \quad t, s \in I.$$

By $(\underline{i})$, one has

$$\|f(t, s, b^k(s), u^k(s))\| \le \|b^k(s)\| + M\|u^k(s)\|, \quad t, s \in I.$$

which is uniformly bounded since $(b^k(\cdot))$ and $(u^k(\cdot))$ are bounded in $\mathcal{C}_H(I)$.

From the Lebesgue dominated convergence theorem, it results

$$\lim_{k\to\infty}\left\|\int_0^t f(t,s,b^k(s),u^k(s))ds-\int_0^t f(t,s,\hat{b}(s),\hat{u}(s))ds\right\|=0.$$

Moreover, note that

$$\left\|\int_0^t f(t,s,b^k(s),u^k(s))ds\right\|<\|b^k\|_{L^1_{\mathbb{R}^m}(I)}+M\|u^k\|_{L^1_{\mathbb{R}^n}(I)},$$

is uniformly bounded, then, the Lebesgue dominated convergence theorem yields

$$\lim_{k\to\infty}\int_0^T\left\|\int_0^t f(t,s,b^k(s),u^k(s))ds-\int_0^t f(t,s,\hat{b}(s),\hat{u}(s))ds\right\|^2dt=0. \tag{4.5}$$

Observe that $u^k(t)\in D(A(t,a^k(t)))$, $a^k(t)\to\hat{a}(t)$, $u^k(t)\to\hat{u}(t)$, for all $t\in I$, the sequence $(A^0(t,a^k(t))u^k(t))$ is bounded by (H_2) for all $t\in I$, and

$$\operatorname{dis}(A(t,a^k(t)),A(t,\hat{a}(t)))\leq\lambda\|a^k(t)-\hat{a}(t)\|\to 0,\quad\text{when } k\to\infty, \tag{4.6}$$

using (H_1). Then, from Lemma 2.1 one deduces that $\hat{u}(t)\in D(A(t,\hat{a}(t))),\forall t\in I$.

Now, we are going to verify that $\hat{u}(\cdot)$ satisfies the integro-differential inclusion

$$-\dot{\hat{u}}(t)\in A(t,\hat{a}(t))\hat{u}(t)+\int_0^t f(t,s,\hat{b}(s),\hat{u}(s))ds\quad\text{a.e. } t\in I.$$

Define the maps g^k and g on I by

$$g^k(t)=\int_0^t f(t,s,b^k(s),u^k(s))ds,\quad g(t)=\int_0^t f(t,s,\hat{b}(s),\hat{u}(s))ds,\quad\text{for any } t\in I.$$

In view of (4.4) and (4.5),

$$(\dot{u}^k(\cdot)+g^k(\cdot))\text{ weakly converges in } L^2_{\mathbb{R}^n}(I)\text{ to }\dot{\hat{u}}(\cdot)+g(\cdot).$$

Hence, $(\dot{u}^k(\cdot)+g^k(\cdot))$ Komlós-converges to $\dot{\hat{u}}(\cdot)+g(\cdot)$ (see Proposition 2.6). So, there is a negligible set Y such that for $t\in I\setminus Y$: $\dot{u}^k(\cdot)+g^k(\cdot)\to\dot{\hat{u}}(\cdot)+g(\cdot)$ Komlós, that is,

$$\lim_{k\to\infty}\frac{1}{k}\sum_{p=1}^k\left(\dot{u}^p(t)+\int_0^t f(t,s,b^p(s),u^p(s))ds\right)=\dot{\hat{u}}(t)+\int_0^t f(t,s,\hat{b}(s),\hat{u}(s))ds, \tag{4.7}$$

and

$$-\dot{u}^k(t)\in A(t,a^k(t))u^k(t)+\int_0^t f(t,s,b^k(s),u^k(s))ds.$$

Let $y \in D(A(t, \hat{a}(t)))$. Applying Lemma 2.2 to the maximal monotone operators $A(t, a^k(t))$ and $A(t, \hat{a}(t))$ that satisfy (4.6), ensures the existence of a sequence (y^k) such that $y^k \in D(A(t, a^k(t)))$

$$y^k \to y \text{ and } A^0(t, a^k(t))y^k \to A^0(t, \hat{a}(t))y. \tag{4.8}$$

Since

$$-\dot{u}^k(t) \in A(t, a^k(t))u^k(t) + \int_0^t f(t, s, b^k(s), u^k(s))ds \quad \text{a.e.},$$

and $A(t, a^k(t))$ is monotone, one has

$$\langle \dot{u}^k(t) + g^k(t), u^k(t) - y^k \rangle \leq \langle A^0(t, a^k(t))y^k, y^k - u^k(t) \rangle. \tag{4.9}$$

Note that

$$\langle \dot{u}^k(t) + g^k(t), \hat{u}(t) - y \rangle = \langle \dot{u}^k(t) + g^k(t), u^k(t) - y^k \rangle + \langle \dot{u}^k(t) + g^k(t), \hat{u}(t) - u^k(t) - (y - y^k) \rangle,$$

then,

$$\begin{aligned} \frac{1}{k}\sum_{p=1}^{k}\langle \dot{u}^p(t) + g^p(t), \hat{u}(t) - y \rangle &= \frac{1}{k}\sum_{p=1}^{k}\big\langle \dot{u}^p(t) + g^p(t), y^p - y \big\rangle + \frac{1}{k}\sum_{p=1}^{k}\big\langle \dot{u}^p(t) + g^p(t), u^p(t) - y^p \big\rangle \\ &\quad + \frac{1}{k}\sum_{p=1}^{k}\langle \dot{u}^p(t) + g^p(t), \hat{u}(t) - u^p(t) \rangle. \end{aligned}$$

Thus, one gets using (4.9)

$$\begin{aligned} \frac{1}{k}\sum_{p=1}^{k}\big\langle \dot{u}^p(t) + g^p(t), \hat{u}(t) - y \big\rangle &\leq \frac{1}{k}\sum_{p=1}^{k}\big\langle \dot{u}^p(t) + g^p(t), y^p - y \big\rangle + \frac{1}{k}\sum_{p=1}^{k}\big\langle A^0(t, a^p(t))y^p, y^p - u^p(t) \big\rangle \\ &\quad + \frac{1}{k}\sum_{p=1}^{k}\big\langle \dot{u}^p(t) + g^p(t), \hat{u}(t) - u^p(t) \big\rangle. \end{aligned}$$

A passage to the limit as $k \to \infty$ with the use of (4.7)-(4.8), the boundedness of $(\dot{u}^p(\cdot) + g^p(\cdot))$ in $\mathbb{R}^n$, and the preceding modes of convergence above, yields

$$\Big\langle \dot{\hat{u}}(t) + \int_0^t f(t, s, \hat{b}(s), \hat{u}(s))ds, \hat{u}(t) - y \Big\rangle \leq \Big\langle A^0(t, \hat{a}(t))y, y - \hat{u}(t) \Big\rangle \quad \text{a.e.}$$

Hence, Lemma 2.3 guarantees that

$$-\dot{\hat{u}}(t) \in A(t, \hat{a}(t))\hat{u}(t) + \int_0^t f(t, s, \hat{b}(s), \hat{u}(s))ds \quad \text{a.e. } t \in I,$$

with $\hat{u}(t) \in D(A(t, \hat{a}(t)))$ for all $t \in I$. By uniqueness, it follows that $\hat{u}$ is the unique solution to $(\mathcal{CP}_{\hat{a},\hat{b}})$ associated to the couple control maps $(\hat{a}(\cdot),\hat{b}(\cdot))$. To justify further the optimality of $(\hat{u}(\cdot),\hat{a}(\cdot),\hat{b}(\cdot))$ in $(\mathcal{OCP})$, it is sufficient to show that

$$\phi[\hat{u},\hat{a},\hat{b}] \leq \liminf_{k\to\infty} \phi[u^k,a^k,b^k] \tag{4.10}$$

for the Bolza-type functional in $(\mathcal{OCP})$. The latter (4.10) readily follows from the assumptions on the cost functions ϕ_1 and ϕ_2 due to the Mazur weak closure theorem and the Lebesgue dominated convergence theorem. Indeed, Mazur's theorem ensures that the weak convergence of $\{\dot{u}^k,\dot{a}^k,\dot{b}^k\}$ to $\{\dot{\hat{u}},\dot{\hat{a}},\dot{\hat{b}}\}$ in $L^2_{\mathbb{R}^{2n+m}}(I)$ yields the $L^2_{\mathbb{R}^{2n+m}}(I)$ strong convergence of convex combinations of $(\dot{u}^k,\dot{a}^k,\dot{b}^k)$ to $(\dot{\hat{u}},\dot{\hat{a}},\dot{\hat{b}})$, and thus the a.e. convergence of a subsequence of these convex combinations on I to the limiting triple.

Employing finally the assumed convexity of the running cost ϕ_2 with respect to the velocity variables verifies (4.10) and hence completes the proof of the theorem. □

We derive from Theorem 4.2, the particular case of the controlled sweeping process.

Corollary 4.3. *Let $C : I\times\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map with non-empty closed convex values. Suppose that there exist a non-negative real constant $\lambda < 1$, and a function $\beta \in W^{1,2}(I,\mathbb{R})$ which is non-negative on $[0,T[$ and non-decreasing with $\beta(T) < \infty$ and $\beta(0) = 0$ such that*

$$|d(u,C(t,y)) - d(u,C(s,z))| \leq |\beta(t)-\beta(s)| + \lambda\|y-z\| \quad \forall t,s\in I, \quad \forall u,y,z\in\mathbb{R}^n.$$

Let $f : I\times I\times\mathbb{R}^{m+n} \to \mathbb{R}^n$, $\phi_1 : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\phi_2 : I\times\mathbb{R}^{4n+2m} \to \overline{\mathbb{R}}$ be defined as in Theorem 4.2. The optimal control problem is

$$\min \phi[u,a,b] = \phi_1(u(T)) + \int_0^T \phi_2(t,u(t),a(t),b(t),\dot{u}(t),\dot{a}(t),\dot{b}(t))dt,$$

on the set of controls $(a(\cdot),b(\cdot))$ and the associated solutions $u(\cdot)$ of the controlled integro-sweeping process

$$\begin{cases} -\dot{u}(t) \in N_{C(t,a(t))}u(t) + \displaystyle\int_0^t f(t,s,b(s),u(s))ds \quad a.e.\ t\in I, \\ u(t)\in C(t,a(t)), \quad t\in I, \\ (a(\cdot),b(\cdot)) \in W^{1,2}(I,\mathbb{R}^{n+m}), \\ a(0)=a_0, \quad u(0)=u_0\in C(0,a_0). \end{cases}$$

Then, the minimizing problem above admits an optimal solution in the space $W^{1,2}(I,\mathbb{R}^{2n+m})$.

References

[1] F. Amiour, M. Sene, and T. Haddad, “Existence results for state-dependent maximal monotone differential inclusions: fixed point approach,” *Numer. Funct. Anal. Optim.*, vol. 43, no. 7, pp. 838–859, 2022, doi: 10.1080/01630563.2022.2059675.

[2] D. Azzam-Laouir, W. Belhoula, C. Castaing, and M. D. P. Monteiro Marques, “Perturbed evolution problems with absolutely continuous variation in time and applications,” *J. Fixed Point Theory Appl.*, vol. 21, no. 2, 2019, Art. ID. 40, doi: 10.1007/s11784-019-0666-2.

[3] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces.* Editura Academiei Republicii Socialiste România, Bucharest; Noordhoff International Publishing, Leiden, 1976, translated from the Romanian.

[4] A. Bouabsa and S. Saïdi, “Coupled systems of subdifferential type with integral perturbation and fractional differential equations,” *Advances in the Theory of Nonlinear Analysis and its Applications*, vol. 7, no. 1, pp. 253–271, 2023, doi: 10.31197/atnaa.1149751.

[5] A. Bouach, T. Haddad, and B. S. Mordukhovich, “Optimal control of nonconvex integro-differential sweeping processes,” *J. Differential Equations*, vol. 329, pp. 255–317, 2022, doi: 10.1016/j.jde.2022.05.004.

[6] A. Bouach, T. Haddad, and L. Thibault, “Nonconvex integro-differential sweeping process with applications,” *SIAM J. Control Optim.*, vol. 60, no. 5, pp. 2971–2995, 2022, doi: 10.1137/21M1397635.

[7] A. Bouach, T. Haddad, and L. Thibault, “On the discretization of truncated integro-differential sweeping process and optimal control,” *J. Optim. Theory Appl.*, vol. 193, no. 1-3, pp. 785–830, 2022, doi: 10.1007/s10957-021-01991-z.

[8] Y. Brenier, W. Gangbo, G. Savaré, and M. Westdickenberg, “Sticky particle dynamics with interactions,” *J. Math. Pures Appl.*, vol. 99, no. 5, pp. 577–617, 2013, doi: 10.1016/j.matpur.2012.09.013.

[9] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, ser. North-Holland Mathematics Studies. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973, vol. 5.

[10] L. M. Briceño Arias, N. D. Hoang, and J. Peypouquet, “Existence, stability and optimality for optimal control problems governed by maximal monotone operators,” *J. Differential Equations*, vol. 260, no. 1, pp. 733–757, 2016, doi: 10.1016/j.jde.2015.09.006.

[11] M. Brokate and P. Krejčí, "Optimal control of ODE systems involving a rate independent variational inequality," *Discrete Contin. Dyn. Syst. Ser. B*, vol. 18, no. 2, pp. 331–348, 2013, doi: 10.3934/dcdsb.2013.18.331.

[12] T. H. Cao and B. S. Mordukhovich, "Optimal control of a nonconvex perturbed sweeping process," *J. Differential Equations*, vol. 266, no. 2-3, pp. 1003–1050, 2019, doi: 10.1016/j.jde.2018.07.066.

[13] C. Castaing, C. Godet-Thobie, M. D. P. Monteiro Marques, and A. Salvadori, "Evolution problems with m-accretive operators and perturbations," *Mathematics*, vol. 10, no. 3, 2022, Art. ID 317, doi: 10.3390/math10030317.

[14] C. Castaing, C. Godet-Thobie, S. Saïdi, and M. D. P. Monteiro Marques, "Various perturbations of time dependent maximal monotone/accretive operators in evolution inclusions with applications," *Appl. Math. Optim.*, vol. 87, no. 2, 2023, Art. ID 24, doi: 10.1007/s00245-022-09898-5.

[15] C. Castaing, C. Godet-Thobie, and L. X. Truong, "Fractional order of evolution inclusion coupled with a time and state dependent maximal monotone operator," *Mathematics*, vol. 8, no. 9, 2020, Art. ID 1395, doi: 10.3390/math8091395.

[16] C. Castaing, P. Raynaud de Fitte, and M. Valadier, *Young measures on topological spaces*, ser. Mathematics and its Applications. Kluwer Academic Publishers, Dordrecht, 2004, vol. 571, doi: 10.1007/1-4020-1964-5.

[17] G. Colombo, R. Henrion, N. D. Hoang, and B. S. Mordukhovich, "Optimal control of the sweeping process," *Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms*, vol. 19, no. 1-2, pp. 117–159, 2012.

[18] G. Colombo, R. Henrion, N. D. Hoang, and B. S. Mordukhovich, "Discrete approximations of a controlled sweeping process," *Set-Valued Var. Anal.*, vol. 23, no. 1, pp. 69–86, 2015, doi: 10.1007/s11228-014-0299-y.

[19] G. Colombo, R. Henrion, N. D. Hoang, and B. S. Mordukhovich, "Optimal control of the sweeping process over polyhedral controlled sets," *J. Differential Equations*, vol. 260, no. 4, pp. 3397–3447, 2016, doi: 10.1016/j.jde.2015.10.039.

[20] G. Colombo and C. Kozaily, "Existence and uniqueness of solutions for an integral perturbation of Moreau's sweeping process," *J. Convex Anal.*, vol. 27, no. 1, pp. 229–238, 2020.

[21] M. d. R. de Pinho, M. M. A. Ferreira, and G. V. Smirnov, "Optimal control involving sweeping processes," *Set-Valued Var. Anal.*, vol. 27, no. 2, pp. 523–548, 2019, doi: 10.1007/s11228-018-0501-8.

[22] M. d. R. de Pinho, M. M. A. Ferreira, and G. Smirnov, "Necessary conditions for optimal control problems with sweeping systems and end point constraints," *Optimization*, vol. 71, no. 11, pp. 3363–3381, 2022, doi: 10.1080/02331934.2022.2101111.

[23] M. de Pinho, M. M. A. Ferreira, and G. Smirnov, "Optimal control with sweeping processes: numerical method," *J. Optim. Theory Appl.*, vol. 185, no. 3, pp. 845–858, 2020, doi: 10.1007/s10957-020-01670-5.

[24] K. Deimling, *Non Linear Functional Analysis.* Springer-Verlag, Berlin, 1985, doi: 10.1007/978-3-662-00547-7.

[25] M. Guessous, "An elementary proof of Komlós-Révész theorem in Hilbert spaces," *J. Convex Anal.*, vol. 4, no. 2, pp. 321–332, 1997.

[26] E. Klein and A. C. Thompson, *Theory of correspondences*, ser. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1984.

[27] M. Kunze and M. D. P. Monteiro Marques, "BV solutions to evolution problems with time-dependent domains," *Set-Valued Anal.*, vol. 5, no. 1, pp. 57–72, 1997, doi: 10.1023/A:1008621327851.

[28] B. K. Le, "Well-posedness and nonsmooth Lyapunov pairs for state-dependent maximal monotone differential inclusions," *Optimization*, vol. 69, no. 6, pp. 1187–1217, 2020, doi: 10.1080/02331934.2019.1686504.

[29] C. Nour and V. Zeidan, "Numerical solution for a controlled nonconvex sweeping process," *IEEE Control Syst. Lett.*, vol. 6, pp. 1190–1195, 2022, doi: 10.1109/lcsys.2021.3089977.

[30] C. Nour and V. Zeidan, "Optimal control of nonconvex sweeping processes with separable endpoints: nonsmooth maximum principle for local minimizers," *J. Differential Equations*, vol. 318, pp. 113–168, 2022, doi: 10.1016/j.jde.2022.02.021.

[31] C. Nour and V. Zeidan, "A control space ensuring the strong convergence of continuous approximation for a controlled sweeping process," *Set-Valued Var. Anal.*, vol. 31, no. 3, 2023, Art. ID 23, doi: 10.1007/s11228-023-00686-z.

[32] C. Nour and V. Zeidan, "Nonsmooth optimality criterion for a $W^{1,2}$-controlled sweeping process: Nonautonomous perturbation," *Appl. Set-Valued Anal. Optim.*, vol. 5, no. 2, pp. 193–212, 2023, doi: 10.23952/asvao.5.2023.2.06.

[33] S. Saïdi, "A perturbed second-order problem with time and state-dependent maximal monotone operators," *Discuss. Math. Differ. Incl. Control Optim.*, vol. 41, no. 1, pp. 61–86, 2021, doi: 10.7151/dmdico.

[34] F. Selamnia, D. Azzam-Laouir, and M. D. P. Monteiro Marques, "Evolution problems involving state-dependent maximal monotone operators," *Appl. Anal.*, vol. 101, no. 1, pp. 297–313, 2022, doi: 10.1080/00036811.2020.1738401.

[35] A. A. Vladimirov, "Nonstationary dissipative evolution equations in a Hilbert space," *Nonlinear Anal.*, vol. 17, no. 6, pp. 499–518, 1991, doi: 10.1016/0362-546X(91)90061-5.

[36] I. I. Vrabie, *Compactness methods for nonlinear evolutions*, ser. Pitman Monographs and Surveys in Pure and Applied Mathematics. Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1987, vol. 32.

[37] V. Zeidan, C. Nour, and H. Saoud, "A nonsmooth maximum principle for a controlled nonconvex sweeping process," *J. Differential Equations*, vol. 269, no. 11, pp. 9531–9582, 2020, doi: 10.1016/j.jde.2020.06.053.

CUBO, A Mathematical Journal
Vol. 26, no. 01, pp. 153–166, April 2024
DOI: 10.56754/0719-0646.2601.153

Quarter-symmetric metric connection on a p-Kenmotsu manifold

Bhawana Chaube[1,✉]
S. K. Chanyal[1]

[1] *Department of Mathematics, D.S.B. Campus, Kumaun University, Nainital-263001, India.*
bhawanachaube18@gmail.com
skchanyal.math@gmail.com

ABSTRACT

In the present paper we study para-Kenmotsu (p-Kenmotsu) manifold equipped with quarter-symmetric metric connection and discuss certain derivation conditions.

RESUMEN

En el presente artículo estudiamos variedades para-Kenmotsu (p-Kenmotsu) equipadas con conexiones métricas cuarto-simétricas y discutimos ciertas condiciones derivadas.

Keywords and Phrases: Para-Kenmotsu manifold, quarter-symmetric metric connection, curvature tensor, η-Einstein manifold.

2020 AMS Mathematics Subject Classification: 53C15, 53C25.

Published: 10 April, 2024
Accepted: 27 February, 2024
Received: 13 September, 2023

1 Introduction

Kenmotsu in 1971, introduced a class of almost contact Riemannian manifolds satisfying some special conditions, called Kenmotsu manifold [10]. Many researchers including U.C. De and R. N. Singh studied some properties of Kenmotsu manifolds endowed with various conditions [2, 3, 9, 15]. Sato [13] in 1976, introduced the notion of an almost para-contact structure on Riemannian manifolds which is similar to the almost contact structure on Riemannian manifolds. In 1995, B. B. Sinha and K. L. Sai Prasad [16] defined a class of almost para contact metric manifolds analogous to the class of Kenmotsu manifolds, known as para-Kenmotsu (p-Kenmotsu) manifolds. T. Satyanarayana *et al.* [14] studied curvature properties in a p-Kenmotsu manifold.

Friedmann and Schouten in 1924 [6], presented the idea of semi-symmetric connection on a differentiable manifold. Yano introduced semi-symmetric metric connection in 1970 using the idea of metric connection given by Hayden in 1932. M. M. Tripathi [19] and Tang *et al.* [18] studied semi-symmetric metric connection in a Kenmotsu manifold. A linear connection $\bar{\nabla}$ on a Riemannian manifold M is said to be a semi- symmetric connection if the torsion tensor T given by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]$$

satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form and $g(X,\xi) = \eta(X)$, ξ is a vector field and for all vector fields $X, Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

Gołąb [7] in 1975 studied quarter-symmetric metric connection in differentiable manifolds with affine connections. Further S. C. Biswas, U. C. De and many others [1, 4, 5, 17] studied quarter-symmetric metric connection in Riemannian manifolds equipped with various structures. A quarter-symmetric connection is considered as a generalisation of semi-symmetric connection since its torsion tensor T satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where ϕ is a $(1,1)$ tensor field. If quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0,$$

where $X, Y, Z \in \chi(M)$, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection. Let M be an n-dimensional Riemannian manifold and ∇ be its Levi-Civita connection. The Riemannian curvature tensor R, the concircular curvature tensor W, Weyl projective curvature tensor P of M

are defined by [11, 12]

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1}$$

$$W(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.2}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{1.3}$$

where $X, Y, Z \in \chi(M)$ and r is the scalar curvature.

The paper is organised as follows: In section 2, a brief introduction of p-Kenmotsu manifolds is given. In section 3, the relation between the curvature tensors of Riemannian connection and the quarter-symmetric metric connection in a p-Kenmotsu manifold is obtained. The study of a p-Kenmotsu manifold with respect to the quarter-symmetric metric connection satisfying the curvature condition $\bar{R} \cdot \bar{S}$ is contained in section 4. In section 5, we study ϕ-concircularly flat p-Kenmotsu manifold with respect to quarter-symmetric metric connection. The curvature condition $\bar{P} \cdot \bar{S} = 0$ and ϕ-Weyl projective flat p-Kenmotsu manifold with respect to quarter-symmetric metric connection are respectively studied in the sections 6 and 7. Finally we give an example of a 5-dimensional p-Kenmotsu manifold.

2 Preliminaries

Let M be a $(2n+1)$-dimensional differentiabe manifold endowed with an almost para-contact structure (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ is a vector field, and η is a 1-form on M, then

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1. \tag{2.1}$$

$$\phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad \text{rank } (\phi) = 2n. \tag{2.2}$$

where X is a vector field on M. The manifold M endowed with (ϕ, ξ, η) is called an almost para-contact manifold [13].

Let g be a Riemannian metric on M compatible to the structure (ϕ, ξ, η), *i.e.*, the following equations are satisfied

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2.3}$$

for all vector fields X and Y on M. Then the manifold M is said to admit an almost para-contact Riemannian structure (ϕ, ξ, η, g).

If moreover, (ϕ, ξ, η, g) satisfy the following conditions

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.4}$$

$$\nabla_X \xi = X - \eta(X)\xi = \phi^2(X), \tag{2.5}$$

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.6}$$

then M is called a para-Kenmotsu (p-Kenmotsu) manifold [16].

In a p-Kenmotsu manifold the following relations hold [16]:

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.7}$$

$$S(X,\xi) = -(n-1)\eta(X), \quad \text{where } g(QX,Y) = S(X,Y), \tag{2.8}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.9}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.10}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.11}$$

where S is the Ricci tensor and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor and R is the Riemannian curvature.

If the Ricci curvature tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.12}$$

then M is called η-Einstein manifold and if $b = 0$ then it is said to be Einstein manifold. M is called generalized η-Einstein manifold, if S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + cg(\phi X, Y), \tag{2.13}$$

where a, b, c are scalar functions on M.

In a p-Kenmotsu manifold M, the connection $\bar{\nabla}$ given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi \tag{2.14}$$

is a quarter-symmetric metric connection [8].

3 Curvature tensor of para-Kenmotsu manifold with respect to the quarter-symmetric metric connection

Let M be a p-Kenmotsu manifold. The curvature tensor $\bar{R}$ of a p-Kenmotsu manifold with respect to the quarter-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z.$$

Using equations (2.1)-(2.6) and (2.13) we get

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z + g(X,Z)\phi Y - g(Y,Z)\phi X + g(\phi X,Z)Y - g(\phi Y,Z)X \\ + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X,\end{aligned} \tag{3.1}$$

where R is the Riemannian curvature tensor of the connection ∇ given in (1.1).

Now from (3.1), we have

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0, \tag{3.2}$$

or equivalently

$$\bar{R}(X,Y,Z,W) + \bar{R}(Y,Z,X,W) + \bar{R}(Z,X,Y,W) = 0, \tag{3.3}$$

where $\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$. Thus the curvature tensor with respect to the quarter-symmetric metric connection satisfies the Bianchi first identity. Taking inner product of (3.1) with respect to W, we get

$$\begin{aligned}\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(X,Z)g(\phi Y,W) - g(Y,Z)g(\phi X,W) \\ + g(\phi X,Z)g(Y,W) - g(\phi Y,Z)g(X,W) \\ + g(\phi X,Z)g(\phi Y,W) - g(\phi Y,Z)g(\phi X,W).\end{aligned} \tag{3.4}$$

Contracting (3.4) over X and W, we get

$$\bar{S}(Y,Z) = S(Y,Z) + (1-2n-\psi)g(\phi Y,Z) + (1-\psi)g(Y,Z) - \eta(Y)\eta(Z), \tag{3.5}$$

where $\psi = trace\ \phi$, S and $\bar{S}$ are the Ricci tensors with respect to the connections ∇ and $\bar{\nabla}$ respectively on M. Now contracting (3.5), we have

$$\bar{r} = r + 2n(1-2\psi) - \psi^2, \tag{3.6}$$

where r and $\bar{r}$ denote the scalar curvatures with respect to the connections ∇ and $\bar{\nabla}$ respectively on M. Now we state the following theorem.

Theorem 3.1. *For a p-Kenmotsu manifold M with respect to the quarter-symmetric metric connection $\bar{\nabla}$*

(1) *The curvature tensor $\bar{R}$ satisfies the Bianchi first identity and is given by*

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + g(X,Z)\phi Y - g(Y,Z)\phi X \\ &\quad + g(\phi X,Z)Y - g(\phi Y,Z)X + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X.\end{aligned}$$

(2) *The Ricci tensor $\bar{S}$ is given by*

$$\bar{S}(Y,Z) = S(Y,Z) + (1-2n-\psi)g(\phi Y,Z) + (1-\psi)g(Y,Z) - \eta(Y)\eta(Z).$$

(3) *The relation between r and $\bar{r}$, respectively the scalar curvatures with respect to ∇ and $\bar{\nabla}$, is given by*

$$\bar{r} = r + 2n(1-2\psi) - \psi^2.$$

Proof. The proof follows from the equations (3.1), (3.2), (3.3), (3.5) and (3.6). □

Some properties of the curvature tensor with respect to the quarter- symmetric metric connection are given in the following lemma.

Lemma 3.2. *In a $(2n+1)$-dimensional p-Kenmotsu manifold with the structure (ϕ,ξ,η,g) with respect to the quarter-symmetric metric connection, the following hold*

$$\bar{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X + \eta(X)\phi Y - \eta(Y)\phi X, \tag{3.7}$$

$$\bar{R}(\xi,Y)Z = \eta(Z)Y + \eta(Z)\phi Y - g(Y,Z)\xi - g(\phi Y,Z)\xi, \tag{3.8}$$

$$\bar{R}(\xi,Y)\xi = Y + \phi Y - \eta(Y)\xi, \tag{3.9}$$

$$\bar{S}(Y,\xi) = (1-n-\psi)\eta(Y), \tag{3.10}$$

$$\bar{S}(\xi,\xi) = (1-n-\psi). \tag{3.11}$$

4 p-Kenmotsu manifold satisfying $\bar{R} \cdot \bar{S} = 0$.

In this section we consider a p-Kenmotsu manifold with respect to the quarter-symmetric metric connection $\bar{\nabla}$ satisfying

$$\bar{R}(X,Y) \cdot \bar{S} = 0.$$

This equation implies

$$\bar{S}(\bar{R}(X,Y)U,V) + \bar{S}(U,\bar{R}(X,Y)V) = 0 \tag{4.1}$$

where $X,Y,U,V \in \chi(M)$. Putting $X = \xi$ in (4.1), we have

$$\bar{S}(\bar{R}(\xi,Y)U,V) + \bar{S}(U,\bar{R}(\xi,Y)V) = 0 \tag{4.2}$$

By the equations (3.5), (3.8) and (3.10), equation (4.2) yields

$$\begin{aligned}&\eta(U)\bar{S}(Y,V) + \eta(U)\bar{S}(\phi Y,V) - (1-n-\psi)g(Y,U)\eta(V) - (1-n-\psi)g(\phi Y,U)\eta(V)\\ &+\eta(V)\bar{S}(U,Y) + \eta(V)\bar{S}(\phi Y,U) - (1-n-\psi)g(Y,V)\eta(U) - (1-n-\psi)g(\phi Y,V)\eta(U) = 0.\end{aligned}$$

Putting $U = \xi$ and using (2.1) and (2.2), it follows that

$$\bar{S}(Y,V) + \bar{S}(\phi Y,V) = (1-n-\psi)g(Y,V) + (1-n-\psi)g(\phi Y,V). \tag{4.3}$$

Making use of (3.5), (4.3) takes form

$$S(Y,V) + S(\phi Y,V) = (\psi+n-1)g(Y,V) + (2-2n-\psi)\eta(Y)\eta(V) + (\psi+n-1)g(\phi Y,V). \tag{4.4}$$

Therefore we have the following theorem:

Theorem 4.1. *If a p-Kenmotsu manifold with respect to the quarter-symmetric metric connection satisfying the condition $\bar{R} \cdot \bar{S} = 0$, then the Ricci tensor S of the manifold satisfies*

$$S(X,Y) + S(\phi X,Y) = (\psi+n-1)g(X,Y) + (2-2n-\psi)\eta(X)\eta(Y) + (\psi+n-1)g(\phi X,Y).$$

5 ϕ-concircularly flat p-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogous to the definition of the concircular curvature given in (1.2), $\bar{W}$, the concircular curvature with respect to quarter-symmetric metric connection is given by

$$\bar{W}(X,Y)Z = \bar{R}(X,Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{5.1}$$

A p-Kenmotsu manifold is said to be ϕ-concircularly flat with respect to the quarter-symmetric metric connection if

$$\bar{W}(\phi X, \phi Y, \phi Z, \phi W) = 0, \tag{5.2}$$

where $X, Y, Z, W \in \chi(M)$.

Taking inner-product of (5.1) with respect to U and replacing X by ϕX, Y by ϕY, Z by ϕZ and U by ϕU, we get

$$\bar{R}(\phi X, \phi Y, \phi Z, \phi W) = \frac{\bar{r}}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)].$$

In view of (3.1) and (3.6), (5.3) takes the form

$$\begin{aligned} R(\phi X, \phi Y, \phi Z, \phi W) &= g(\phi Y, \phi Z)g(X, \phi W) - g(\phi X, \phi Z)g(Y, \phi W) + g(Y, \phi Z)g(\phi X, \phi W) \\ &- g(X, \phi Z)g(\phi Y, \phi W) - g(X, \phi Z)g(Y, \phi W) + g(Y, \phi Z)g(X, \phi W) \\ &+ \frac{r + 2n(1-2\psi) - \psi^2}{n(n-1)}\Big[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)\Big]. \end{aligned} \tag{5.3}$$

Let $\{e_1, e_2, \ldots, e_{2n}, e_{2n+1} = \xi\}$ be a local orthonormal ϕ-basis of vector fields in M, so that $\{\phi e_1, \phi e_2, \ldots, \phi e_{2n}, \xi\}$ is also a local orthonormal basis in M. Putting $X = W = e_i$ in the last equation and summing over i, we get

$$\begin{aligned} S(Y,Z) &= \frac{(r + 2n(1-2\psi) - \psi^2)(2n-1) + n(n-1)\psi}{n(n-1)} g(Y,Z) \\ &- \frac{(r + 2n(1-2\psi) - \psi^2)(2n-1) + n(n-1)(n-1+\psi)}{n(n-1)} \eta(Y)\eta(Z) \\ &- (2 - 2n - \psi)g(\phi Y, Z). \end{aligned} \tag{5.4}$$

Thus we state the following theorem:

Theorem 5.1. *A ϕ-concircularly flat p-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized η-Einstein manifold with the scalar curvature r given by* (5.4).

6 p-Kenmotsu manifold satisfying $\bar{P}\cdot\bar{S}=0$ with respect to quarter-symmetric metric connection.

Analogous to (1.3), the Weyl projective curvature $\bar{P}$ with respect to quarter-symmetric metric connection is given by

$$\bar{P}(X,Y)Z=\bar{R}(X,Y)Z-\frac{1}{n-1}[\bar{S}(Y,Z)X-\bar{S}(X,Z)Y].$$

Using (3.1) and (3.5), this equation implies

$$\begin{aligned}\bar{P}(X,Y)Z=&\ R(X,Y)Z+g(X,Z)\phi Y-g(Y,Z)\phi X+g(\phi X,Z)Y\\&-g(\phi Y,Z)X+g(\phi X,Z)\phi Y-g(\phi Y,Z)\phi X-\frac{1}{n-1}\Big[S(Y,Z)X\\&+(1-2n-\psi)g(\phi Y,Z)X+(1-\psi)g(Y,Z)X-\eta(Y)\eta(Z)X\\&-S(X,Z)Y-(1-2n-\psi)g(\phi X,Z)Y-(1-\psi)g(X,Z)Y+\eta(X)\eta(Z)Y\Big].\end{aligned}\tag{6.1}$$

From the equation (6.1), we have the following properties of the Weyl projective curvature $\bar{P}$.

$$\begin{aligned}\bar{P}(\xi,Y)Z=&\ \eta(Y)Z-g(Y,Z)\xi+\eta(Z)\phi Y-g(\phi Y,Z)\xi-\frac{1}{n-1}\Big[S(Y,Z)\xi\\&+(1-2n-\psi)g(\phi Y,Z)\xi+(1-\psi)g(Y,Z)\xi-\eta(Y)\eta(Z)\xi-S(\xi,Z)Y\\&-(1-\psi)\eta(Z)Y+\eta(Z)Y\Big].\end{aligned}\tag{6.2}$$

and

$$\bar{P}(\xi,Y)\xi=Y-\eta(Y)\xi+\phi Y-\frac{1}{n-1}\Big[(1-\psi-n)\eta(Y)\xi+(\psi+n-1)Y\Big].\tag{6.3}$$

Now, we consider a p-Kenmotsu manifold satisfying the curvature condition

$$\bar{P}(X,Y)\cdot\bar{S}=0,$$

which is equivalent to

$$\bar{S}(\bar{P}(X,Y)U,V)+\bar{S}(U,\bar{P}(X,Y)V)=0.$$

The last equation implies

$$\bar{S}(\bar{P}(\xi,Y)\xi,V)+\bar{S}(\xi,\bar{P}(\xi,Y)V)=0.\tag{6.4}$$

Using equation (6.2) and (6.3) in (6.4), we once again get the equation (4.4). Therefore we have the following theorem:

Theorem 6.1. *For a $(2n+1)$-dimensional p-Kenmotsu manifold with respect to the quarter-symmetric metric connection satisfying the condition $\bar{P}\cdot\bar{S}=0$, the Ricci tensor S satisfies*

$$S(X,Y)+S(\phi X,Y)=(\psi+n-1)g(\phi X,Y)+(\psi+n-1)g(X,Y)+(2-2n-\psi)\eta(X)\eta(Y).$$

7 ϕ-Weyl projective flat p-Kenmotsu manifolds with respect to the quarter-symmetric metric connection.

A p-Kenmotsu manifold is said to be ϕ-Weyl projective flat with respect to the quarter-symmetric metric connection if

$$\bar{P}(\phi X,\phi Y,\phi Z,\phi U)=0, \tag{7.1}$$

where $X,Y,Z,U\in\chi(M)$. Taking inner-product of (6.1) with respect to U and replacing X by ϕX, Y by ϕY, Z by ϕZ and U by ϕU, we get

$$\begin{aligned}\bar{P}(\phi X,\phi Y,\phi Z,\phi U)=\bar{R}(\phi X,\phi Y,\phi Z,\phi U)-\frac{1}{n-1}\Big[&S(\phi Y,\phi Z)g(\phi X,\phi U)\\&+(1-2n-\psi)g(Y,\phi Z)g(\phi X,\phi U)+(1-\psi)g(\phi X,\phi Z)\\&-S(\phi X,\phi Z)g(\phi Y,\phi U)-(1-2n-\psi)g(X,\phi Z)g(\phi Y,\phi U)\\&+g(\phi X,\phi U)-(1-\psi)g(\phi X,\phi Z)g(\phi Y,\phi U)\Big].\end{aligned} \tag{7.2}$$

Using (3.1), (7.1) in (7.2), we obtain

$$\begin{aligned}R(\phi X,\phi Y,\phi Z,\phi W)=&-g(\phi X,\phi Z)g(Y,\phi W)+g(\phi Y,\phi Z)g(X,\phi W)-g(X,\phi Z)g(\phi Y,\phi W)\\&+g(Y,\phi Z)g(\phi X,\phi W)-g(X,\phi Z)g(Y,\phi W)+g(Y,\phi Z)g(X,\phi W)\\&+\frac{1}{n-1}\Big[S(\phi Y,\phi Z)g(\phi X,\phi U)+(1-2n-\psi)g(Y,\phi Z)g(\phi X,\phi U)\\&+(1-\psi)g(\phi X,\phi Z)g(\phi X,\phi U)-S(\phi X,\phi Z)g(\phi Y,\phi U)\\&-(1-2n-\psi)g(X,\phi Z)g(\phi Y,\phi U)-(1-\psi)g(\phi X,\phi Z)g(\phi Y,\phi U)\Big].\end{aligned}$$

Let $\{e_1,e_2,\ldots,e_{2n},\xi\}$ be a local orthonormal ϕ-basis of vector fields in M, putting $X=W=e_i$ in the last equation and summing over i, we get

$$S(Y,Z)=\frac{(2n-\psi n-1)}{2(1-n)}g(Y,Z)-\frac{(n^2-3n+n\psi+1)}{2(n-1)}\eta(Y)\eta(Z)+\frac{(2n^2+3n+2\psi-3)}{2(n-1)}g(\phi Y,Z).$$

Thus we state the following theorem:

Theorem 7.1. *If a p-Kenmotsu manifold is ϕ-Weyl projective flat with respect to the quarter-symmetric metric connection, it is a generalized η-Einstein manifold.*

8 Example

Example 8.1. *Consider the 5-dimensional manifold* $M = \{(u, v, x, y, z) \in R^5\}$ *with standard coordinates* (u, v, x, y, z) *in* R^5*. Then the following vector fields*

$$e_1 = z\frac{\partial}{\partial u}, \quad e_2 = z\frac{\partial}{\partial v}, \quad e_3 = z\frac{\partial}{\partial x}, \quad e_4 = z\frac{\partial}{\partial y}, \quad e_5 = -\frac{\partial}{\partial z}$$

are linearly independent at each point of M*. Suppose* g *be the Riemannian metric defined by,*

$$g(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j;\ i, j = 1, 2, 3, 4, 5. \end{cases}$$

Let ϕ *be the tensor field of type* $(1, 1)$ *defined by*

$$\phi(e_1) = e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = e_4, \quad \phi(e_4) = e_3, \quad \phi(e_5) = 0,$$

and η *be the* 1*-form defined by* $\eta(X) = g(X, e_5)$*. Using the linearity of* ϕ *and* g*, we have*

$$\eta(e_5) = 1, \quad \phi^2 X = X - \eta(X)e_5, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields $X, Y \in \chi(M)$*. If we take* $e_5 = \xi$*, the structure* (ϕ, ξ, η, g) *is an almost para-contact Riemannian structure on* M*. Then we have,*

$$\begin{aligned} &[e_1, e_2] = 0, \quad [e_1, e_3] = 0, \quad [e_1, e_4] = 0, \quad [e_1, e_5] = e_1, \quad [e_2, e_3] = 0, \\ &[e_2, e_4] = 0, \quad [e_2, e_5] = e_2, \quad [e_3, e_4] = 0, \quad [e_3, e_5] = e_3, \quad [e_4, e_5] = e_4. \end{aligned}$$

Using Koszul's formula, we obtain the Levi-Civita connection ∇ *of the metric tensor* g *as follows:*

$$\begin{array}{lllll} \nabla_{e_1} e_1 = -e_5, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_5, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = -e_5, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = e_3, \\ \nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = -e_5, & \nabla_{e_4} e_5 = e_4, \\ \nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 0, & \nabla_{e_5} e_4 = 0, & \nabla_{e_5} e_5 = 0. \end{array}$$

Above relations show that equations (2.4)*-*(2.6) *are satisfied. Therefore the manifold is a p-Kenmotsu manifold with the structure* (ϕ, ξ, η, g)*.*

Using (2.14), *we get the quarter symmetric metric connection*

$$\begin{array}{lllll}
\bar{\nabla}_{e_1}e_1 = -e_5, & \bar{\nabla}_{e_1}e_2 = -e_5, & \bar{\nabla}_{e_1}e_3 = 0, & \bar{\nabla}_{e_1}e_4 = 0, & \bar{\nabla}_{e_1}e_5 = e_1 + e_2, \\
\bar{\nabla}_{e_2}e_1 = -e_5, & \bar{\nabla}_{e_2}e_2 = -e_5, & \bar{\nabla}_{e_2}e_3 = 0, & \bar{\nabla}_{e_2}e_4 = 0, & \bar{\nabla}_{e_2}e_5 = e_1 + e_2, \\
\bar{\nabla}_{e_3}e_1 = 0, & \bar{\nabla}_{e_3}e_2 = 0, & \bar{\nabla}_{e_3}e_3 = -e_5, & \bar{\nabla}_{e_3}e_4 = -e_5, & \bar{\nabla}_{e_3}e_5 = e_3 + e_4, \\
\bar{\nabla}_{e_4}e_1 = 0, & \bar{\nabla}_{e_4}e_2 = 0, & \bar{\nabla}_{e_4}e_3 = -e_5, & \bar{\nabla}_{e_4}e_4 = -e_5, & \bar{\nabla}_{e_4}e_5 = e_3 + e_4, \\
\bar{\nabla}_{e_5}e_1 = 0, & \bar{\nabla}_{e_5}e_2 = 0, & \bar{\nabla}_{e_5}e_3 = 0, & \bar{\nabla}_{e_5}e_4 = 0, & \bar{\nabla}_{e_5}e_5 = 0.
\end{array}$$

Now we obtain non-zero components of their curvature tensors:

$$\begin{array}{llll}
R(e_1,e_2)e_1 = e_2, & R(e_1,e_3)e_1 = e_3, & R(e_1,e_4)e_1 = e_4, & R(e_1,e_5)e_1 = e_5, \\
R(e_2,e_1)e_2 = e_1, & R(e_2,e_3)e_2 = e_3, & R(e_2,e_4)e_2 = e_4, & R(e_2,e_5)e_2 = e_5, \\
R(e_3,e_1)e_3 = e_1, & R(e_3,e_2)e_3 = e_2, & R(e_3,e_4)e_3 = e_4, & R(e_3,e_5)e_3 = e_5, \\
R(e_4,e_1)e_4 = e_2, & R(e_4,e_2)e_4 = e_2, & R(e_4,e_3)e_4 = e_3, & R(e_4,e_5)e_4 = e_5.
\end{array}$$

and

$$\begin{array}{lll}
\bar{R}(e_1,e_3)e_1 = e_3 + e_4, & \bar{R}(e_1,e_4)e_1 = e_3 + e_4, & \bar{R}(e_1,e_5)e_1 = e_5, \\
\bar{R}(e_2,e_3)e_2 = e_3 + e_4, & \bar{R}(e_2,e_4)e_2 = e_3 + e_4, & \bar{R}(e_2,e_5)e_2 = e_5, \\
\bar{R}(e_3,e_1)e_3 = e_1 + e_2, & \bar{R}(e_3,e_2)e_3 = e_1 + e_2, & \bar{R}(e_3,e_5)e_3 = e_5, \\
\bar{R}(e_4,e_1)e_2 = e_1 + e_2, & \bar{R}(e_4,e_2)e_4 = e_1 + e_2, & \bar{R}(e_4,e_5)e_4 = e_5, \\
\bar{R}(e_5,e_1)e_5 = e_1 + e_2, & \bar{R}(e_5,e_2)e_5 = e_1 + e_2, & \bar{R}(e_5,e_3)e_5 = e_3 + e_4, \\
\bar{R}(e_5,e_4)e_5 = e_3 + e_4. & &
\end{array}$$

From the above results, it is easy to find the following non-zero components of Ricci tensors:

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = S(e_4,e_4) = S(e_5,e_5) = -4,$$

and

$$\bar{S}(e_1,e_1) = \bar{S}(e_1,e_2) = \bar{S}(e_2,e_2) = \bar{S}(e_3,e_3) = \bar{S}(e_3,e_4) = -3, \quad \bar{S}(e_4,e_4) = -3, \quad \bar{S}(e_5,e_5) = -4.$$

Therefore, we get $r = -20$ *and* $\bar{r} = -16$. *Hence the statement of Theorem 3.1 is verified. Also by the relations mentioned above, the results in sections 5 and 6 are easily verified.*

Acknowledgement

The first author is thankful to the Department of Science and Technology (DST), New Delhi for the financial support in the form of JRF (IF200486).

References

[1] S. C. Biswas and U. C. De, “Quarter-symmetric metric connection in an SP-Sasakian manifold,” *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, vol. 46, no. 1-2, pp. 49–56, 1997.

[2] A. De, “On Kenmotsu manifold,” *Bull. Math. Anal. Appl.*, vol. 2, no. 3, pp. 1–6, 2010.

[3] U. C. De and G. Pathak, “On 3-dimensional Kenmotsu manifolds,” *Indian J. Pure Appl. Math.*, vol. 35, no. 2, pp. 159–165, 2004.

[4] U. C. De and J. Sengupta, “Quater-symmetric metric connection on a Sasakian manifold,” *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, vol. 49, no. 1-2, pp. 7–13, 2000.

[5] U. C. De, D. Mandal, and K. Mandal, “Some characterizations of Kenmotsu manifolds admitting a quarter-symmetric metric connection,” *Bull. Transilv. Univ. Braşov Ser. III*, vol. 9(58), no. 1, pp. 39–52, 2016.

[6] A. Friedman and J. A. Schouten, “Über die Geometrie der halbsymmetrischen Übertragungen,” *Math Z*, vol. 21, pp. 211–223, 1924, doi: 10.1007/BF01187468.

[7] S. Gołąb, “On semi-symmetric and quarter-symmetric linear connections,” *Tensor (N.S.)*, vol. 29, no. 3, pp. 249–254, 1975.

[8] A. Haseeb and R. Prasad, “Certain results on Lorentzian para-Kenmotsu manifolds,” *Bol. Soc. Parana. Mat. (3)*, vol. 39, no. 3, pp. 201–220, 2021.

[9] J.-B. Jun, U. C. De, and G. Pathak, “On Kenmotsu manifolds,” *J. Korean Math. Soc.*, vol. 42, no. 3, pp. 435–445, 2005, doi: 10.4134/JKMS.2005.42.3.435.

[10] K. Kenmotsu, “A class of almost contact Riemannian manifolds,” *Tohoku Math. J. (2)*, vol. 24, pp. 93–103, 1972, doi: 10.2748/tmj/1178241594.

[11] M. Kon and K. Yano, *Structures on manifolds*, ser. Series in Pure Mathematics. Chandrama Prakashan, Allahabad, 1985, vol. 3, doi: 10.1142/0067.

[12] R. S. Mishra, *Structures on a differentiable manifold and their applications.* Chandrama Prakashan, Allahabad, 1984.

[13] I. Sato, “On a structure similar to the almost contact structure,” *Tensor (N.S.)*, vol. 30, no. 3, pp. 219–224, 1976.

[14] T. Satyanarayana and K. L. S. Prasad, “On a type of para-Kenmotsu manifold,” *Pure Mathematical Sciences*, vol. 2, no. 4, pp. 165–170, 2013.

[15] R. N. Singh, S. K. Pandey, and G. Pandey, “On a type of Kenmotsu manifold,” *Bull. Math. Anal. Appl.*, vol. 4, no. 1, pp. 117–132, 2012.

[16] B. B. Sinha and K. L. Sai Prasad, "A class of almost para contact metric manifold," *Bull. Calcutta Math. Soc.*, vol. 87, no. 4, pp. 307–312, 1995.

[17] S. Sular, C. Özgür, and U. C. De, "Quarter-symmetric metric connection in a Kenmotsu manifold," *SUT J. Math.*, vol. 44, no. 2, pp. 297–306, 2008.

[18] W. Tang, P. Majhi, P. Zhao, and U. C. De, "Legendre curves on 3-dimensional Kenmotsu manifolds admitting semisymmetric metric connection," *Filomat*, vol. 32, no. 10, pp. 3651–3656, 2018, doi: 10.2298/fil1810651t.

[19] M. M. Tripathi, "On a semi symmetric metric connection in a Kenmotsu manifold," *J. Pure Math.*, vol. 16, pp. 67–71, 1999.

CUBO, A Mathematical Journal
Vol. 26, no. 01, pp. 167–190, April 2024
DOI: 10.56754/0719-0646.2601.167

Global convergence analysis of Caputo fractional Whittaker method with real world applications

Sapan Kumar Nayak[1]
P. K. Parida[1,✉]

[1] *Department of Mathematics, Central University of Jharkhand, Ranchi-835222, India.*
sapannayak7@gmail.com
pkparida@cuj.ac.in

ABSTRACT

The present article deals with the effect of convexity in the study of the well-known Whittaker iterative method, because an iterative method converges to a unique solution t^* of the nonlinear equation $\psi(t) = 0$ faster when the function's convexity is smaller. Indeed, fractional iterative methods are a simple way to learn more about the dynamic properties of iterative methods, *i.e.*, for an initial guess, the sequence generated by the iterative method converges to a fixed point or diverges. Often, for a complex root search of nonlinear equations, the selective real initial guess fails to converge, which can be overcome by the fractional iterative methods. So, we have studied a Caputo fractional double convex acceleration Whittaker's method (CFDCAWM) of order at least $(1 + 2\zeta)$ and its global convergence in broad ways. Also, the faster convergent CFDCAWM method provides better results than the existing Caputo fractional Newton method (CFNM), which has $(1 + \zeta)$ order of convergence. Moreover, we have applied both fractional methods to solve the nonlinear equations that arise from different real-life problems.

Published: 11 April, 2024
Accepted: 07 March, 2024
Received: 13 November, 2023

RESUMEN

El presente artículo trata con el efecto de la convexidad en el estudio del bien conocido método iterativo de Whittaker, puesto que un método iterativo converge a una única solución t^* de una ecuación no-lineal $\psi(t) = 0$ más rápidamente cuando la convexidad de la función es más pequeña. De hecho, métodos iterativos fraccionarios son una manera simple de aprender más sobre las propiedades dinámicas de los métodos iterativos, *i.e.*, para una suposición inicial, la sucesión generada por el método iterativo converge a un punto fijo o diverge. A menudo, para búsquedas de raíces complejas de ecuaciones no-lineales, la suposición inicial real elegida no converge, lo que se puede superar usando métodos iterativos fraccionarios. Así, hemos estudiado un método de Whittaker con aceleración convexa doble Caputo fraccionario (CFD-CAWM) de orden al menos $(1+2\zeta)$ y su convergencia global de manera amplia. También el método convergente CFD-CAWM más rápido entrega mejores resultados que el método de Newton Caputo fraccionario (CFNM) existente, que tiene orden de convergencia $(1 + \zeta)$. Más aún, hemos aplicado ambos métodos fraccionarios para resolver ecuaciones no-lineales que aparecen en diferentes problemas de la vida real.

Keywords and Phrases: Fractional derivative, efficiency index, nonlinear equations, Newton's method, Whittaker's method, convergence plane, basin of attraction.

2020 AMS Mathematics Subject Classification: 65H105, 26A33.

1 Introduction

In 1695, two famous mathematicians changed the concept of calculus when they came up with the fractional derivative. Fortunately, L'Hospital had raised a question in a letter to Leibniz, and in the letter, both of them discussed their ideas about the possibilities of semi-derivative function. Since then, there have been vast changes in the theory of fractional calculus and its real-world applications. Thus, fractional calculus builds useful tools in many real-world applications such as science, engineering, economics, medicine, and other fields (see, [1, 3, 10, 18, 19, 22, 29]).

Generally, we know that the classical work in mathematics is to solve the nonlinear equation

$$\psi(t) = 0, \tag{1.1}$$

where ψ is a real-valued function of a real variable. This task becomes more difficult when the degree of polynomials is greater than or equal to five, or it is a transcendental equation. In general, as there are no analytical methods to handle the above equation, the demand for iterative methods has increased day by day in the last few decades. The most suitable method to solve nonlinear equations, as we know, is quadratic convergent Newton's method (NM):

$$\begin{cases} t_0 \quad \text{given}, \\ t_{n+1} = t_n - \frac{\psi(t_n)}{\psi'(t_n)}, \ n \geq 0. \end{cases} \tag{1.2}$$

Indeed, using iterative methods to solve (1.1) is more suitable and reliable, and it is also true that by using these methods, we can obtain many significant numerical results and related information about nonlinear equations. The effect of fractional derivative on NM was first deduced by Brambila *et al.* [30], who observed that the fractional Newton method (FNM) keeps the ability to search the complex roots of a polynomial even if we choose a real suitable initial guess. By deepening the fractional order, the complex roots of the polynomial are hidden. The nature of fractional iterative methods is that they can locate the positions of different polynomial roots in a different order of derivative. In the year 2019, Akgül *et al.* [2] , studied the FNM

$$t_{n+1} = t_n - \left(\Gamma(\zeta+1) \frac{\psi(t_n)}{{}^{\mathcal{C}}\mathcal{D}_a^{\zeta}\psi(t_n)} \right),$$

and proved its order of convergence as 2ζ. Later, Candelario *et al.* [7] modified the FNM to a better form

$$t_{n+1} = t_n - \left(\Gamma(\zeta+1) \frac{\psi(t_n)}{{}^{\mathcal{C}}\mathcal{D}_a^{\zeta}\psi(t_n)} \right)^{\frac{1}{\zeta}}$$

with order of convergence $(1+\zeta)$. They tested the FNM on some numerical examples and provided good results with its dynamics, too.

If we see some research papers (for example, [11,14,32]), we can see how the influence of convexity on a real function enhanced the order of convergence. Moreover, the smaller convexity of a nonlinear equation causes the faster convergence of (1.2) to a unique solution t^* of a nonlinear equation. The classical double convex acceleration of the Whittaker method [32] employing convexity is given below:

$$t_{n+1} = t_n - \frac{1}{4}\left(2 - L_\psi(t_n) + \frac{4 + 2L_\psi(t_n)}{2 - L_\psi(t_n)(2 - L_\psi(t_n))}\right)\frac{\psi(t_n)}{\psi'(t_n)} \tag{1.3}$$

where $L_\psi(t_n) = \frac{\psi(t)\psi''(t)}{(\psi'(t))^2}$. The cubic order convergence method developed by Whittaker is a simplified version of the method developed by Newton. It is also known as the parallel-chord method, which comes from its geometric interpretation of functions. It is known [12], that if we have an iterative process $t_{n+1} = F(t_n)$ with $t_{n+1} = t_n - \frac{\psi(t_n)}{\psi'(t_n)}H(L_\psi(t_n))$ and $H(0) = 1$, $H'(0) = \frac{1}{2}$ and $|H''(0)| < +\infty$, it has a third order convergence.

In this paper, we have introduced a new convex acceleration of the Whittaker method using the concept of the Caputo fractional derivative, that is, the Caputo fractional double convex acceleration of the Whittaker method (CFDCAWM). Hence, our main aim in the present article is to investigate further the global convergence analysis, stability, and reliability of CFDCAWM. A detailed comparison of the Caputo fractional Newton method (CFNM) and the CFDCAWM with some good numerical examples is provided, with the order of convergence of CFDCAWM being at least $(1 + 2\zeta)$.

The remaining part of the article is assembled in the following manner: Section 2 includes some primary results and information regarding our method. In Section 3, we provide the order of convergence of the proposed method, and its subsection contains details of the efficiency of our method. Section 4 is devoted to the numerical results of the proposed method with real-life applications and their corresponding convergence planes. Finally, the conclusion of the paper ends with Section 5.

2 Basic definitions and results

For centuries, the concept of a non-integer order type derivative has been crucial in many research areas. Also, there are so many definitions and formulas in fractional calculus. For our present work, we have just discussed some of them.

Definition 2.1 (Gamma function [20]). *The gamma function is a generalized idea of the factorial function, and is defined as follows:*

$$\Gamma(t) = \begin{cases} (t-1)!, & t \in \mathbb{N} \\ \int_0^{+\infty} s^{t-1}e^{-s}ds, & \textit{whenever } t > 0. \end{cases}$$

Definition 2.2 (Riemann-Liouville fractional derivative [16]). *Suppose the function $\psi : \mathbb{R} \to \mathbb{R}$ and $\psi \in \mathcal{L}^1([a,t])$ $(-\infty < a < t < +\infty)$ be integrable with $\zeta \geq 0$ and $k = [\zeta] + 1$. Then the Riemann-Liouville fractional derivative (RLFD) of $\psi(t)$ at ζth order is defined as below:*

$$(\mathcal{D}^{\zeta}_{a+})\psi(t) = \begin{cases} \dfrac{1}{\Gamma(k-\zeta)} \dfrac{d^k}{dt^k} \displaystyle\int_a^t \dfrac{\psi(x)}{(t-x)^{\zeta-k+1}} \, \mathrm{d}x, & \zeta \notin N \\ \dfrac{d^{k-1}\psi(t)}{dt^{k-1}}, & \zeta = k-1 \in N \cup \{0\}. \end{cases}$$

And the reverse process of RLFD is Caputo fractional derivative, which is shown below.

Definition 2.3 (Caputo fractional derivative [8]). *Consider the function $\psi : \mathbb{R} \to \mathbb{R}$, $\psi \in C^{+\infty}([a,t])$ $(-\infty < a < t < +\infty)$ with $\zeta \geq 0$ and $k = [\zeta] + 1$, where $[\zeta]$ is the integer part of ζ, then the Caputo fractional derivative (CFD) of $\psi(t)$ at ζth order can be given as:*

$$(\mathcal{C}^{\mathcal{D}^{\zeta}_a})\psi(t) = \begin{cases} \dfrac{1}{\Gamma(k-\zeta)} \displaystyle\int_a^t \dfrac{d^k\psi(x)}{dx^k} \dfrac{dx}{(t-x)^{\zeta-k+1}}, & \zeta \notin N \\ \dfrac{d^{k-1}\psi(t)}{dt^{k-1}}, & \zeta = k-1 \in N \cup \{0\}. \end{cases}$$

The main difference between RLFD and CFD is, the fractional derivative of a constant function is non-zero in RLFD. On the other hand, Caputo fractional derivative of a constant function is zero. Hence, the nature of the Caputo derivative is, it coincides with the classical derivative. So, our experiments use the CFD with the value $\zeta \in (0,1]$.

Theorem 2.4 ([24, Proposition 26]). *Let $\psi(t) = (t-a)^{\lambda}$, $\zeta \geq 0$, $k = [\zeta] + 1$, and $\lambda \in \mathbb{R}$. Then the RLFD of $\psi(t)$ of ζ^{th} order is:*

$$\mathcal{D}^{\zeta}_{a+}(t-a)^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\zeta)}(t-a)^{\lambda-\zeta}.$$

The next theorem discusses the relation between RLFD and CFD of a function.

Theorem 2.5 ([24, Proposition 31]). *Suppose $\psi(t)$ be a function whose CFD and RLFD exist of order $\zeta \notin \mathbb{N}$ such that $\zeta \geq 0$, $k = 1 + [\zeta]$. Then the following equality hold*

$$\mathcal{C}^{\mathcal{D}^{\zeta}_a \psi(t)} = \mathcal{D}^{\zeta}_{a+}\psi(t) - \sum_{j=0}^{k+1} \frac{\psi^{(j)}(a)}{\Gamma(j+1-\zeta)}(t-a)^{j-\zeta}, \quad t > a.$$

With preceding results, we can say $\mathcal{C}^{\mathcal{D}^{\zeta}_a}(t-a)^k = \mathcal{D}^{\zeta}_a(t-a)^k$, $k = 1, 2, \ldots$

Proof. A function $\psi(t)$ with a residual term near point 'a' has the following Taylor series:

$$\psi(t) = \sum_{j=0}^{\alpha-1} \frac{t^j}{\Gamma(j+1)} \psi^{(j)}(a) + R_{\alpha-1},$$

where

$$R_{\alpha-1} = \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{(\alpha)}(\theta)(t-\theta)^{\alpha-1} d\theta = I^{\alpha}\psi^{(\alpha)}(t).$$

Then, applying the linearity property of RLFD, we have

$$\begin{aligned}
\mathcal{D}^{\zeta}_{a+}\psi(t) &= \mathcal{D}^{\zeta}_{a+}\left(\sum_{j=0}^{\alpha-1} \frac{\psi^j}{\Gamma(j+1)}\psi^{(j)}(a) + R_{\alpha-1}\right) = \sum_{j=0}^{\alpha-1} \frac{\mathcal{D}^{\zeta}_{a+}t^j}{\Gamma(j+1)}\psi^{(j)}(a) + \mathcal{D}^{\zeta}_{a+}R_{\alpha-1} \\
&= \sum_{j=0}^{\alpha-1} \frac{\Gamma(j+1)t^{j-\zeta}}{\Gamma(j-\zeta+1)\Gamma(j+1)}\psi^{(j)}(a) + \mathcal{D}^{\zeta}_{a+}I^{\alpha}\psi^{(\alpha)}(t) \\
&= \sum_{j=0}^{\alpha-1} \frac{t^{j-\zeta}}{\Gamma(j-\zeta+1)}\psi^{(j)}(a) + I^{j-\zeta}\psi^{(\alpha)}(t) = \sum_{j=0}^{\alpha-1} \frac{t^{j-\zeta}}{\Gamma(j-\zeta+1)}\psi^{(j)}(a) + \mathcal{C}^{\mathcal{D}^{\zeta}_{a}\psi(t)}. \qquad \square
\end{aligned}$$

The following theorem represents the fractional-order Taylor series, the extended version of the classical Taylor's theorem.

Theorem 2.6 ([23]). *Let us assume that mth order Caputo derivative $\mathcal{C}^{\mathcal{D}^{m\zeta}_{a}p(t)} \in C([a,b])$, for $m = 1, 2, \ldots, k+1$, where $0 < \zeta \leq 1$. Then, the generalized Taylor's formula is given as below:*

$$p(t) = \sum_{j=0}^{k} \mathcal{C}^{\mathcal{D}^{j\zeta}_{a}p(a)} \frac{(t-a))^{j\zeta}}{\Gamma(j\zeta+1)} + \mathcal{C}^{\mathcal{D}^{(k+1)\zeta}p(\eta)} \frac{(t-a)^{(k+1)\zeta}}{\Gamma((k+1)\zeta+1)},$$

for $a \leq \eta \leq t$, $\forall\ t \in (a,b]$, where $\mathcal{C}^{\mathcal{D}^{k\zeta}_{a}} = \mathcal{C}^{\mathcal{D}^{\zeta}_{a}} \cdots \mathcal{C}^{\mathcal{D}^{\zeta}_{a}}$ (k-times). Thus, we can conclude that the Taylor series of $\psi(t)$ around t^*, by using Caputo fractional derivative is given as follows:

$$\psi(t) = \frac{\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t^*)}}{\Gamma(\zeta+1)}\left[(t-t^*)^{\zeta} + B_2(t-t^*)^{2\zeta} + B_3(t-t^*)^{3\zeta}\right] + \mathcal{O}((t-t^*)^{3\zeta}),$$

where

$$B_j = \frac{\Gamma(\zeta+1)}{\Gamma(j\zeta+1)} \frac{\mathcal{C}^{\mathcal{D}^{j\zeta}_{t^*}\psi(t^*)}}{\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t^*)}}, \quad \text{for } j \geq 2.$$

3 Convergence analysis of CFDCAWM

In this paper section, we have generalized the double convex acceleration of Whittaker's method (DCAWM) to CFDCAWM using the Caputo fractional derivative. The following theorem shows the convergence of the proposed method CFDCAWM with its order of convergence. Based on the definition of the Caputo derivative, CFDCAWM can be derived as in the following theorem:

Theorem 3.1. *Suppose ψ: $\mathcal{D} \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function, and for any $\zeta \in (0,1]$ in the domain $\mathcal{D}$, it has m-order fractional derivatives, $m \in \mathbb{N}$. If t^* is a solution of the equation $\psi(t) = 0$*

and $\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t)}$ *is non-zero continuous function at* t^**, then the method*

$$t_{n+1} = t_n - \left(\Gamma(\zeta+1)\left(2 - 2\mathcal{T}\mathcal{C}^{L^{\zeta}_{\psi}t_n} + \frac{4+4\mathcal{T}\mathcal{C}^{L^{\zeta}_{\psi}t_n}}{2-\mathcal{C}^{L^{\zeta}_{\psi}t_n}(4\mathcal{T}-2\mathcal{T}\mathcal{C}^{L^{\zeta}_{\psi}t_n})}\right)\frac{\psi(t_n)}{4\mathcal{C}^{\mathcal{D}^{\zeta}_{a}\psi(t_n)}}\right)^{\frac{1}{\zeta}}$$

having at least $(1+2\zeta)$ *order of convergence only if* $\mathcal{T} = \frac{\Gamma(2\zeta+1)-\Gamma^2(\zeta+1)}{\Gamma(2\zeta+1)}$*. The desired error equation is mentioned as below:*

$$\begin{aligned} e_{n+1} = \frac{1}{\zeta}\Bigg[&- \Gamma(2\zeta+1)\left(1-\frac{\Gamma(2\zeta+1)}{\Gamma^4(\zeta+1)}\right){B_2}^2 \\ &+ \frac{\mathcal{T}\Gamma(2\zeta+1)}{\Gamma^3(\zeta+1)}\left(2-\frac{3\Gamma(2\zeta+1)}{\Gamma^2(\zeta+1)} - \frac{(3\mathcal{T}-\frac{1}{2})\Gamma(2\zeta+1)}{\Gamma^2(\zeta+1)}\right){B_2}^2 \\ &+ \frac{1}{\Gamma(\zeta+1)}\left(1+\frac{\mathcal{T}\Gamma(3\zeta+1)}{\Gamma^3(\zeta+1)} - \frac{\Gamma(3\zeta+1)}{\Gamma(\zeta+1)\Gamma(2\zeta+1)}\right)B_3\Bigg]e_n^{1+2\zeta} + \mathcal{O}(e_n^{1+3\zeta}). \end{aligned}$$

Proof. With the help of Theorems 2.4 and 2.6, the fractional Taylor's series expansion of the nonlinear function $\psi(t_n)$ using CFD around t^* is

$$\psi(t_n) = \frac{\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t^*)}}{\Gamma(\zeta+1)}\left[e_n^{\zeta} + B_2 e_n^{2\zeta} + B_3 e_n^{3\zeta}\right] + \mathcal{O}(e_n^{4\zeta}). \tag{3.1}$$

Also, the first and second Caputo derivatives can be given as:

$$\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t_n)} = \frac{\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t^*)}}{\Gamma(\zeta+1)}\left[\Gamma(\zeta+1) + \frac{\Gamma(2\zeta+1)}{\Gamma(\zeta+1)}B_2 e_n^{\zeta} + \frac{\Gamma(3\zeta+1)}{\Gamma(2\zeta+1)}B_3 e_n^{2\zeta}\right] + \mathcal{O}(e_n^{3\zeta}), \tag{3.2}$$

and

$$\mathcal{C}^{\mathcal{D}^{2\zeta}_{t^*}\psi(t_n)} = \frac{\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t^*)}}{\Gamma(\zeta+1)}\left[\Gamma(2\zeta+1)B_2 + \frac{\Gamma(3\zeta+1)}{\Gamma(\zeta+1)}B_3 e_n^{\zeta}\right] + \mathcal{O}(e_n^{2\zeta}). \tag{3.3}$$

Squaring the equation (3.2), we have

$$\begin{aligned} (\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t_n)})^2 = \left(\frac{\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t^*)}}{\Gamma(\zeta+1)}\right)^2 &\Bigg[\Gamma^2(\zeta+1) + 2\Gamma(2\zeta+1)B_2 e_n^{\zeta} \\ &+ \left(\frac{\Gamma^2(2\zeta+1)}{\Gamma^2(\zeta+1)}{B_2}^2 + \frac{2\Gamma(\zeta+1)\Gamma(3\zeta+1)}{\Gamma(2\zeta+1)}B_3\right)e_n^{2\zeta} + \frac{2\Gamma(2\zeta+1)\Gamma(3\zeta+1)}{\Gamma(\zeta+1)\Gamma(2\zeta+1)}B_2B_3e_n^{3\zeta}\Bigg] + \mathcal{O}(e_n^{4\zeta}). \end{aligned} \tag{3.4}$$

Also from the equations (3.1) and (3.2), we get

$$\begin{aligned} \frac{\psi(t_n)}{\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t_n)}} = \frac{1}{\Gamma(\zeta+1)}\Bigg[&e_n^{\zeta} + \frac{\Gamma^2(\zeta+1)-\Gamma(2\zeta+1)}{\Gamma^2(\zeta+1)}B_2e_n^{2\zeta} \\ &+ \left[\left(\frac{\Gamma^2(2\zeta+1)}{\Gamma^4(\zeta+1)} - \frac{\Gamma(2\zeta+1)}{\Gamma^2(\zeta+1)}\right){B_2}^2 + \left(\frac{\Gamma(\zeta+1)\Gamma(2\zeta+1)-\Gamma(3\zeta+1)}{\Gamma(\zeta+1)\Gamma(2\zeta+1)}\right)B_3\right]e_n^{3\zeta}\Bigg] + \mathcal{O}(e_n^{4\zeta}). \end{aligned}$$

Combining (3.1) and (3.3), we obtain

$$\psi(t_n)\mathcal{C}^{\mathcal{D}^{2\zeta}_{t^*}\psi(t_n)} = \left(\frac{\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t^*)}}{\Gamma(\zeta+1)}\right)^2\left[\Gamma(2\zeta+1)B_2{e_n}^{\zeta} + \left({B_2}^2\Gamma(2\zeta+1) + \frac{\Gamma(3\zeta+1)}{\Gamma(\zeta+1)}B_3\right)e_n^{2\zeta}\right] + \mathcal{O}(e_n^{3\zeta}).$$

Using (3.4) in the above equation, the Taylor expansion of $\mathcal{C}^{L^{\zeta}_{\psi}\psi(t_n)}$ around t^* can be given as:

$$\mathcal{C}^{L^{\zeta}_{\psi}t_n} = \frac{\Gamma(2\zeta+1)}{\Gamma^2(\zeta+1)}B_2e_n^{\zeta} + \frac{1}{\Gamma^2(\zeta+1)}\left[B_2^2\Gamma(2\zeta+1) + \frac{\Gamma(3\zeta+1)}{\Gamma(\zeta+1)}B_3 - 2\frac{1}{\Gamma^2(\zeta+1)}\Gamma^2(2\zeta+1)B_2^2\right]e_n^{2\zeta} + \mathcal{O}(e_n^{3\zeta}).$$

Squaring the above term, we get

$$\left(\mathcal{C}^{L^{\zeta}_{\psi}t_n}\right)^2 = \frac{\Gamma^2(2\zeta+1)}{\Gamma^4(\zeta+1)}B_2^2e_n^{2\zeta} + \mathcal{O}(e_n^{3\zeta}).$$

Thus

$$\begin{aligned}\mathcal{C}^{L^{\zeta}_{\psi}t_n}\frac{\psi(t_n)}{\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t_n)}} = \frac{1}{\Gamma^3(\zeta+1)}\Bigg[&\Gamma(2\zeta+1)B_2e_n^{2\zeta} + \Bigg\{2\left(\Gamma(2\zeta+1) - \Gamma^2(2\zeta+1) - \frac{\Gamma^2(2\zeta+1)}{2\Gamma^2(\zeta+1)}\right)B_2^2 \\ &+ \frac{\Gamma(3\zeta+1)}{\Gamma(\zeta+1)}B_3\Bigg\}e_n^{3\zeta}\Bigg] + \mathcal{O}(e_n^{4\zeta}).\end{aligned}$$

Now

$$\left(\mathcal{C}^{L^{\zeta}_{\psi}t_n}\right)^2\frac{\psi(t_n)}{\mathcal{C}^{\mathcal{D}^{\zeta}_{t^*}\psi(t_n)}} = \frac{\Gamma^2(2\zeta+1)}{\Gamma^5(\zeta+1)}B_2^2e_n^{3\zeta} + \mathcal{O}(e_n^{4\zeta}).$$

By using geometric series expansion, we obtain

$$\begin{aligned}\frac{4+4\mathcal{T}\mathcal{C}^{L^{\zeta}_{\psi}t_n}}{2-\mathcal{C}^{L^{\zeta}_{\psi}t_n}(4\mathcal{T}-2\mathcal{T}\mathcal{C}^{L^{\zeta}_{\psi}t_n})} &= (2+2\mathcal{T}\mathcal{C}^{L^{\zeta}_{\psi}t_n})\left[1-\left(2-\mathcal{T}\mathcal{C}^{L^{\zeta}_{\psi}t_n}-\mathcal{T}\left(\mathcal{C}^{L^{\zeta}_{\psi}t_n}\right)^2\right)\right]^{-1} \\ &= (2+2\mathcal{T}\mathcal{C}^{L^{\zeta}_{\psi}t_n})\left[1+E+E^2+\cdots\right],\end{aligned}$$

where, $E = \left[2-\mathcal{T}\mathcal{C}^{L^{\zeta}_{\psi}t_n}-\mathcal{T}\left(\mathcal{C}^{L^{\zeta}_{\psi}t_n}\right)^2\right]$.

Finally, we reach to the destination as the error equation is:

$$\begin{aligned}e_{n+1} = \frac{1}{\zeta}\Bigg[&-\Gamma(2\zeta+1)\left(1-\frac{\Gamma(2\zeta+1)}{\Gamma^4(\zeta+1)}\right){B_2}^2 \\ &+ \frac{\mathcal{T}\Gamma(2\zeta+1)}{\Gamma^3(\zeta+1)}\left(2-\frac{3\Gamma(2\zeta+1)}{\Gamma^2(\zeta+1)}-\frac{(3\mathcal{T}-\frac{1}{2})\Gamma(2\zeta+1)}{\Gamma^2(\zeta+1)}\right){B_2}^2 \\ &+ \frac{1}{\Gamma(\zeta+1)}\left(1+\frac{\mathcal{T}\Gamma(3\zeta+1)}{\Gamma^3(\zeta+1)}-\frac{\Gamma(3\zeta+1)}{\Gamma(\zeta+1)\Gamma(2\zeta+1)}\right)B_3\Bigg]e_n^{1+2\zeta} + \mathcal{O}(e_n^{1+3\zeta}).\end{aligned}$$

This ends the proof. □

3.1 Efficiency index

When studying iterative processes, it is important to consider both the speed of convergence (order of convergence) and the computational cost (number of functions and derivative evaluations) required to compute t_{n+1} from t_n. The efficiency index of the iterative method explained by Traub [15] is $\mathcal{E}^* = z^{1/k}$, where z plays the role of order of convergence of the method and k denotes total functional cost evaluations per each iteration. It can be seen from Figure 1 that both the fractional iterative method's efficiency index increases with increasing the order of derivative ζ. Moreover, the maximum value $\mathcal{E}^*$ found in CFNM and CFDCAWM are 1.414 and 1.442, respectively. So, as illustrated in the figure, the efficiency index curve of CFDCAWM always lies above the CFNM. Hence, the $(1+2\zeta)^{th}$ order method CFDCAWM provides better performance and is more efficient than the $(1+\zeta)^{th}$ order method CFNM.

In the next section, we have taken some nonlinear equations for the convergence test of the proposed method and provided more information about the stability and faster convergence of CFDCAWM with some good numerical results and convergence plane.

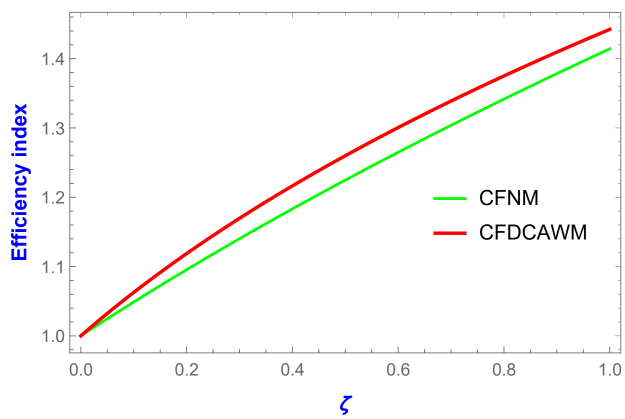


Figure 1: Efficiency indices of CFNM and CFDCAWM.

4 CFDCAWM with their numerical results and convergence plane.

To obtain the numerical results of iterative methods, we use Matlab R2018a with the arithmetic of the double-precision procedure to solve different kinds of nonlinear equations. The stopping criteria of the fractional iterative methods are frequently terminated when either $|t_{n+1} - t_n| < 10^{-6}$ or $|\psi(t_n)| < 10^{-6}$, with a maximum of 300 iterations. Using the program made by Paul Godfrey based on [20], we calculate the Gamma function, whose accuracy along the real axis is 15 significant digits and in the complex plane is 13 significant digits. Moreover, the graphical part of this paper, that is, a convergence plane of iterative methods, has been made by using modified algorithms based on [21] in Mathematica 11.1 and a laptop Lenovo Ideapad flex 5, 1.19 GHz Intel(R) Core™ i5-1035G1 CPU. Each convergence plane consists of a mesh of 400×400 real and complex points. Different colors (red, blue, green, yellow…) on convergence planes mean different roots, whereas black indicates the divergence of the method.

Example 4.1 ([13]). *A state equation links the gas constant to a gas's pressure, volume, and temperature. In the Beattie-Bridgman equation, experimental constants are employed to allow for the decrease in the effective number of molecules caused by various types of molecular aggregation. In the first test function, we used the Beattie-Bridgman equation, which is as follows:*

$$c = \frac{RT}{V} + \frac{\beta}{V^2} + \frac{\gamma}{V^3} + \frac{\delta}{V^4} - P = 0 \tag{4.1}$$

P is atmospheric pressure, R is gas constant, T is absolute temperature in K, and volume V in L/mol. For $T = 273.15K$, $\beta = -1.16584$, $\gamma = 0.0542254$, and $\delta = -0.0001251$. After inserting above values in (4.1), the equation convert to following quartic degree polynomial equation for a pressure of 100 atm:

$$\psi_1(t) = t^4 - 0.22411958\ t^3 + 0.011658361\ t^2 - 5.422539 \times 10^{-4}\ t - 1.251 \times 10^{-6}$$

with the roots $t_1 = -0.0022$, $t_2 = 0.1755$, $t_3 = 0.0254 + 0.0510i$, and $t_4 = 0.0254 - 0.0510i$.

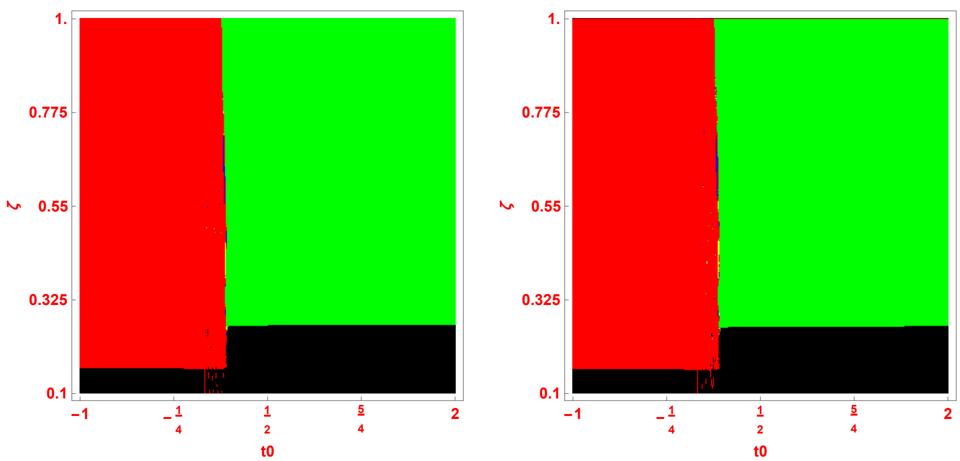
Table 1: Results of CFNM for $\psi_1(t)$ with initial guess $t_0 = 1.5$

ζ	t^*	$\|t_{n+1} - t_n\|$	$\|\psi(t_{n+1})\|$	Iterations
0.1	0.178610412640083	2.451188583441066e-06	1.450813559042032e-05	300
0.2	0.177091772811241	1.824477708772809e-06	7.157942621320157e-06	300
0.3	0.176208764373106	1.144633855865163e-06	3.056113886250757e-06	300
0.4	0.175822192040086	9.973031851462366e-07	1.299498966609209e-06	218
0.5	0.175754695139549	3.307545208941498e-06	9.952113269586370e-07	82
0.6	0.175744973843258	1.206591802094259e-05	9.514452435446740e-07	38
0.7	0.175750319786336	3.910969726900193e-05	9.755112797258072e-07	23
0.8	0.175722751968572	9.416356483973876e-05	8.514564575890538e-07	17
0.9	0.175637670869286	1.558852358621021e-04	4.693474071063554e-07	14
1	0.175715573286650	0.003163967768901	8.191721350991289e-07	11

Table 2: Results of CFDCAWM for $\psi_1(t)$ with initial guess $t_0 = 1.5$

ζ	t^*	$\|t_{n+1} - t_n\|$	$\|\psi(t_{n+1})\|$	Iterations
0.1	0.178561953323709	2.368122878099177e-06	1.426773613993988e-05	300
0.2	0.177049664573626	1.719411282186112e-06	6.959494002208448e-06	300
0.3	0.176189681978262	1.074871852274617e-06	2.968845997087551e-06	300
0.4	0.175820273163475	9.926277080019030e-07	1.290838413491345e-06	210
0.5	0.175752994062801	3.283711977664083e-06	9.875518627812638e-07	76
0.6	0.175749495618264	1.261041176645050e-05	9.718007971009330e-07	32
0.7	0.175740309046350	3.669505604023127e-05	9.304492295014291e-07	18
0.8	0.175714407451239	8.935095908427226e-05	8.139298489053516e-07	12
0.9	0.175645204608465	1.737789640035292e-04	5.031363206827803e-07	09
1	0.175532814939696	2.402266629901728e-04	6.317771224496375e-12	07

As we can see from Tables 1 and 2, for a real initial guess, the CFDCAWM performs faster with a lower error rate than the CFNM. The minimum number of iterations that reach the root is when ζ is close to 1. Furthermore, we have presented the convergence plane with its percentage of convergence for global convergence analysis. The convergence plane is painted with different colors, like t_1 (red), t_2 (green), t_3 (blue), and t_4 (yellow), where the black color represents the divergence. Using the CFNM and CFDCAWM methods, we obtain the percentages of convergence as 86.62% *and* 86.89%, *respectively.*



(a) CFNM, 86.62% convergence (b) CFDCAWM, 86.89% convergence

Figure 2: Convergence planes of $\psi_1(t)$ for real initial guess $t_0 = a$, $a \in \mathbb{R}$.

Example 4.2 ([9]). *Thermodynamics is an important tool for mechanical engineers and other types of engineers. The zero-pressure specific heat of dry air, $\mathfrak{C}_p$ kJ/(kg K), is related to temperature (K) by the following polynomial:*

$$\psi_2(t) = 1.9520 \times 10^{-14} t^4 - 9.5838 \times 10^{-11} t^3 + 9.7215 \times 10^{-8} t^2 + 1.671 \times 10^{-4} t + 0.99403$$

having the roots

$$t_1 = -1001.9347479801513 - 1506.1391327465992i,$$
$$t_2 = -1001.9347479801513 + 1506.1391327465992i,$$
$$t_3 = 3456.80155125884 - 1900.6392904677366i,$$
$$t_4 = 3456.80155125884 + 1900.6392904677366i.$$

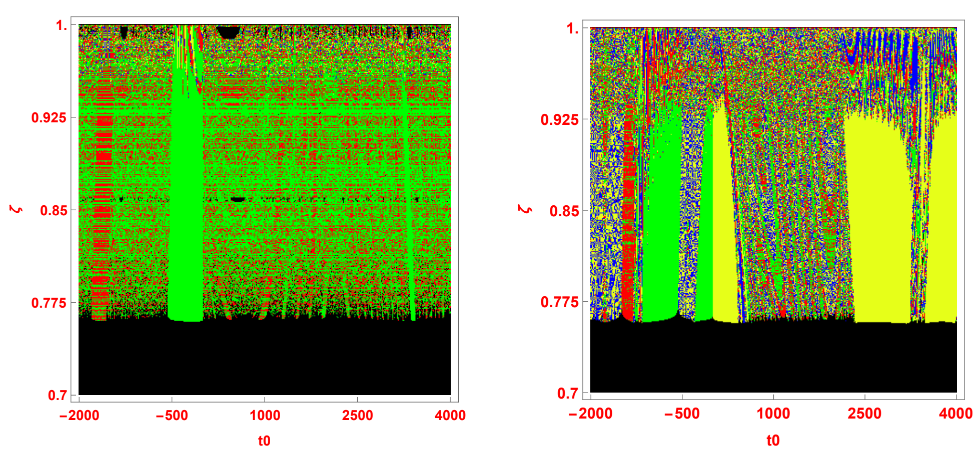
Table 3: Results of CFNMM for $\psi_2(t)$ with initial guess $t_0 = 1200$

ζ	t^*	$\|t_{n+1} - t_n\|$	$\|\psi(t_{n+1})\|$	Iterations
0.90	-1.0019351270e+03- 1.506139649e+03i	1.68009e-04	9.46029e-07	127
0.91	-1.001934949337910e+03+1.506139772e+03i	2.22485e-04	9.89906e-07	105
0.92	-1.001934982e+03+1.506139634e+03i	2.27315e-04	8.17029e-07	101
0.93	-1.001934996e+03+1.506139613e+03i	2.85064e-04	7.98920e-07	95
0.94	-1.001935115e+03+1.506139440e+03i	3.25671e-04	7.07102e-07	52
0.95	-1.001934691e+03-1.5061397413e+03i	5.72666e-04	9.02668e-07	83
0.96	-1.00193460826e+03+1.506139435e+03i	4.09812e-04	4.92409e-07	48
0.97	-1.001934870e+03+1.506139378e+03i	4.965333e-04	4.05183e-07	46
0.98	-1.001934981e+03+1.506139396e+03i	0.0011022	5.20419e-07	85
0.99	-1.001935145e+03+1.506139393e+03i	0.0035052	7.023826e-07	86
1	-2.278375918070995e+03	1.15764e+03	2.7774308	300

Table 4: Results of CFDCAWM for $\psi_2(t)$ with initial guess $t_0 = 1200$

ζ	t^*	$\|t_{n+1} - t_n\|$	$\|\psi(t_{n+1})\|$	Iterations
0.90	3.456801942e+03+1.90063953e+03i	1.165819e-04	8.63779e-07	57
0.91	-1.001934857e+03-1.506139637e+03i	1.65497e-04	7.629208e-07	58
0.92	-1.001934909e+03-1.506139589e+03i	1.95917e-04	7.161345e-07	37
0.93	-1.001934830e+03+1.506139574e+03i	2.31665e-04	6.64036e-07	32
0.94	3.456801839e+03+1.900639491e+03i	2.320012e-04	6.55686e-07	28
0.95	3.456801872e+03-1.900639586e+03i	3.96711e-04	8.14106e-07	36
0.96	3.456801709e+03+1.900639507e+03i	3.25045e-04	5.00443e-07	26
0.97	-1.001934720e+03-1.506139406e+03i	4.95736e-04	4.05118e-07	30
0.98	3.456801786e+03+1.9006393777e+03i	7.626349e-04	4.6759e-07	43
0.99	-1.00193501e+03-1.506139695e+03i	0.00469	9.17793e-07	54
1	1.601282223e+05	1.57591e+05	1.24427e+07	300

The CFDCAWM converges quicker than the CFNM, as seen in the Tables 3 and 4. In Figure 3, the convergence plane of the CFDCAWM (78.08%) provide better stability than CFNM (77.61%).



(a) CFNM, 77.61% convergence (b) CFDCAWM, 78.08% convergence

Figure 3: Convergence planes of $\psi_2(t)$ for real initial guess $t_0 = a$, $a \in \mathbb{R}$.

Example 4.3 ([27]). *Blood is represented as a "Casson fluid", a non-Newtonian fluid. A basic fluid, such as water or blood, will flow through a tube so that the fluid's central core travels as a plug with little distortion and a velocity gradient towards the tube's wall, according to the Casson fluid model. The following non-linear polynomial equation has been used to explain the plug flow of Casson fluids, where the change in flow rate is measured by*

$$R = 1 - \frac{16}{7}\sqrt{t} + \frac{4}{3}t - \frac{1}{21}t^4$$

where reduction in flow rate is measure by R. Take $R = 0.40$ in the above equation we have the third test function

$$\psi_3(t) = \frac{1}{441}t^8 - \frac{8}{63}t^5 - 0.05714285714t^4 + \frac{16}{9}t^2 - 3.624489796t + 0.36$$

which contains the following roots $t_1 = 3.82239$, $t_2 = 0.104699$, $t_3 = -2.27869 - 1.98748i$, $t_4 = -2.27869+1.98748i$, $t_5 = -1.23877-3.40852i$, $t_6 = -1.23877+3.40852i$, $t_7 = 1.55392-0.940415i$, and $t_8 = 1.55392 + 0.94041i$.

Table 5: Results of CFNM for $\psi_3(t)$ with initial guess $t_0 = -0.5 - 0.5i$

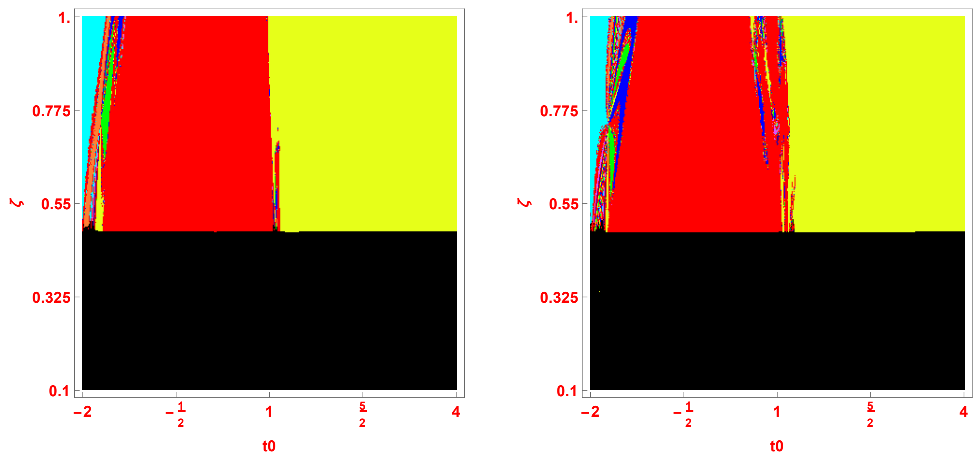
ζ	t^*	$\lvert t_{n+1} - t_n\rvert$	$\lvert\psi(t_{n+1})\rvert$	Iterations
0.1	0.138106293-0.000115792i	1.18425e-04	0.37383	300
0.2	0.138216126-0.000225384i	4.12661e-05	0.10668	300
0.3	0.115361954-0.000027866i	1.80432e-05	0.034481	300
0.4	0.107711638-0.000006849i	7.15017e-06	0.00978	300
0.5	0.105318676-0.000001208i	2.10397e-06	0.00201	300
0.6	0.104830090-0.000000277i	9.962501e-07	4.27483e-04	197
0.7	0.104737418-0.000001542i	9.83237e-07	1.26519e-04	89
0.8	0.104708969-0.000001038i	9.24479e-07	3.37291e-05	41
0.9	0.104700546-0.000000350i	7.36274e-07	6.26827e-06	18
1	0.104698651+0.0i	5.92960e-06	6.23236e-11	05

Table 6: Results of CFDCAWM for $\psi_3(t)$ with initial guess $t_0 = -0.5 - 0.5i$

ζ	t^*	$\lvert t_{n+1} - t_n\rvert$	$\lvert\psi(t_{n+1})\rvert$	Iterations
0.1	0.225782730-0.0009883555i	1.14521e-04	0.36795	300
0.2	0.137305522-0.0000924585i	3.93043e-05	0.10417	300
0.3	0.115118894-0.0000195106i	1.722201e-05	1.72220e-05	300
0.4	0.1076588596-0.0000038959i	6.90493e-06	0.00961	300
0.5	0.1053110124-0.0000004292i	2.05602e-06	0.00199	300
0.6	0.1048302280+0.0000000401i	9.98358e-07	4.279307e-04	195
0.7	0.1047374670+0.0000003244i	9.80324e-07	1.26251e-04	88
0.8	0.1047092613+0.0000011816i	9.59435e-07	3.47222e-05	40
0.9	0.1047006008-0.0000001524i	7.48819e-07	6.35958e-06	18
1	0.104698651+0.00000i	0.00103	4.40814e-11	04

The CFDCAWM converges better than the CFNM with complex starting estimate $t_0 = -0.5 - 0.5i$

and provides less error, as illustrated in Tables 6 and 5.



(a) CFNM, 57.45% convergence (b) CFDCAWM, 57.71% convergence

Figure 4: Convergence planes of $\psi_3(t)$ for complex initial guess $t_0 = a + ai$, $a \in \mathbb{R}$.

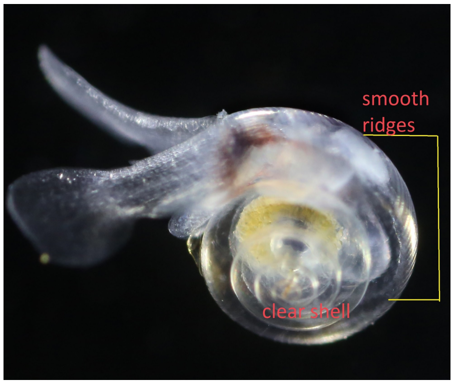
In the Figure 4 convergence planes of $\psi_3(t)$ *are illustrated where the horizontal axis of the graph contains all the complex initial guesses of the form* $t_0 = a + ai$*. The CFDCAWM contains* 57.71% *and CFNM contains* 57.45% *region of convergence. To find all most all root in CFNM, the best initial guess* $t_0 \in (-2, -\frac{1}{2})$ *but in case of CFDCAWM the best initial guesses lies in* $t_0 \in (-2, -\frac{1}{2})$ *and* $t_0 \in (0, 2)$*.*

Example 4.4 ([4])**.** *The increasing pH reduction of Earth's seas due to their absorption of anthropogenic carbon dioxide from the atmosphere is known as "ocean acidification". If alkalinity and temperature remain constant, a* 0.1*-unit decrease in ocean pH results in a* 30% *increase in hydrogen ion concentration. The concentration of hydrogen ions increases as a result of a series of chemical reactions that take place when* CO_2 *is absorbed by saltwater. So, the acidity increases in the seawater and causes carbonate ions to be relatively less abundant. Carbonate ions are vital components of many different kinds of organisms, including the skeletons of coral and seashells.*

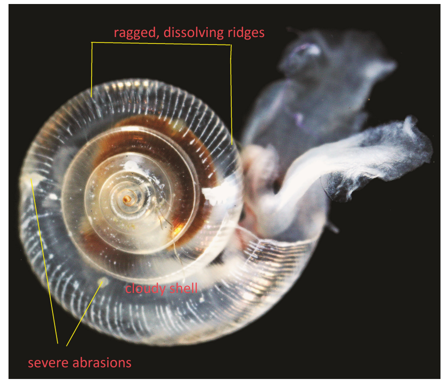
Lack of carbonate ions can make developing and maintaining shells and other calcium carbonate structures of organisms difficult for calcifying species such as oysters, clams, sea urchins, shallow water corals, deep sea corals, and calcareous plankton.

As CO_2 *dissolves in saltwater, the concentration of hydrogen ions* $[H^+]$ *rises, which lowers the pH of the ocean as follows:*

$$CO_{2(aq)} + H_2O \rightleftharpoons H_2CO_3 \rightleftharpoons HCO_3^- + H^+. \tag{4.2}$$



(a) Healthy shell (normal pH)



(b) Damaged shell (lower pH)

Figure 5: Healthy and damaged pteropod image taken from NOAA website [17].

Bicarbonate ions in turn dissociate into carbonate ions CO_3^{2-},

$$HCO_3^- \rightleftharpoons H^+ + CO_3^{2-}. \tag{4.3}$$

The chemical processes results in hydrogen ions, which add to the acidification. Also, H_2O *separates to form hydrogen ions is given as below*

$$H_2O \rightleftharpoons H^+ + OH^-. \tag{4.4}$$

Furthermore, the seawater's boron hydroxide dissociates to release hydrogen ions as

$$B(OH)_3 + H_2O \rightleftharpoons H^+ + B(OH)_4^-. \tag{4.5}$$

The partial pressure $\mathcal{P}_t$ *of the gas phase* CO_2 *is measured in ppm by the National Oceanic and Atmospheric Administration (NOAA) at the Mauna Loa Observatory in Hawaii [28], and according to Bacastow and Keeling [5], the equilibrium constants are measured in mol/ltr and the relationship between liquid and gaseous* CO_2 *is*

$$S_0 = \frac{[CO_2]}{\mathcal{P}_t} = 3.347e - 05, \tag{4.6}$$

being the $[CO_2]$ *represent as the sum of the dissolved* CO_2. *From the reaction* (4.2),

$$S_1 = \frac{[H^+][HCO_3^-]}{[CO_2]} = 9.747e - 07. \tag{4.7}$$

From the reaction (4.3)

$$S_2 = \frac{[H^+][CO_3^{2-}]}{[HCO_3^-]} = 8.501e - 10. \tag{4.8}$$

From the reaction (4.4)

$$S_w = \frac{[H^+]}{[OH^-]} = 6.46e - 15. \tag{4.9}$$

From the reaction (4.5)

$$S_B = \frac{[H^+][B(OH)_4^-]}{[B(OH)_3]} = 1.881e - 09. \tag{4.10}$$

Now the alkanity is

$$\begin{aligned} \mathcal{A} &= \sum(\textit{conservative cations}) - \sum(\textit{conservative anions}) \\ &= [HCO_3^-] + 2[CO_3^{2-}] + [B(OH)_4^-] + [OH^-] - [H^+]. \end{aligned} \tag{4.11}$$

We can suppose that the values of $\mathcal{A}$ *are independent with time as given in the article [4]. The concentrated* CO_2 *is evaluated from* (4.6) *as*

$$[CO_2] = S_0\mathcal{P}_t. \tag{4.12}$$

With the help of equations (4.7) *and* (4.12), *we get*

$$[HCO_3^-] = \frac{S_1[CO_2]}{H^+} = \frac{S_0S_1\mathcal{P}_t}{[H^+]}. \tag{4.13}$$

In the same way, we can find

$$[CO_3^-] = \frac{S_2[HCO_3^-]}{[H^+]} = \frac{S_0S_1S_2\mathcal{P}_t}{[H^+]^2}. \tag{4.14}$$

Now to find $[B(OH)_4^-]$ *with the help of equations* (4.8) *and* (4.13),

and $B = [B(OH)_3] + [B(OH)_4^-]$ *in* (4.10)

$$[B(OH)_4^-] = \frac{BS_B}{S_B + [H^+]}. \tag{4.15}$$

Next, substitute the equations (4.9) *and* (4.12)-(4.15), *we get the alkanity* $\mathcal{A}$ *as below:*

$$\mathcal{A} = \frac{S_0S_1\mathcal{P}_t}{[H^+]} + \frac{2S_0S_1S_2\mathcal{P}_t}{[H^+]^2} + \frac{BS_B}{S_B + [H^+]} + \frac{S_w}{[H^+]} - [H^+].$$

It reduces to the result of the following fourth-degree polynomial equation.

$$p([H^+]) = \sum_{k=0}^{4} \Delta_k [H^+]^k, \tag{4.16}$$

where

$$\Delta_0 = 2S_0S_1S_2\mathcal{P}_tS_B, \quad \Delta_1 = S_0S_1S_B\mathcal{P}_t + 2S_0S_1S_2\mathcal{P}_t + S_WS_B,$$
$$\Delta_2 = S_0S_1\mathcal{P}_t + BS_B + S_w - \mathcal{A}S_B, \quad \Delta_3 = -\mathcal{A} - S_B, \quad \Delta_4 = -1.$$

The value of $\mathcal{A} = 2.050$ *[5, p. 334],* $B = 0.409$ *[26, p. 131] and* $\mathcal{P}_t = 420.19$ *measured by NOAA on February 2023.*

The dynamic study of (4.16) *needs the variable change as* $t = \frac{1}{[H^+]}$, $t \in \mathbb{Z}$, *and* $pH = \log_{10} t$. *Hence, we need to find the solutions of new quartic order polynomial*

$$\psi_4(t) = \sum_{k=0}^{4} \Delta_{k-4}t^k = 4.3839 \times 10^{-26}t^4 + 4.9091 \times 10^{-17}t^3 + 1.0621 \times 10^{-8}t^2 - 2.05t - 1,$$

which contain the roots

$$t_1 = 1.1970408047866759 \times 10^8,$$
$$t_2 = -0.4878048768159488,$$
$$t_3 = -6.197530413868866 \times 10^8 + 8.095038662704764i \times 10^7,$$
$$t_4 = -6.197530413868866 \times 10^8 - 8.095038662704764i \times 10^7.$$

Table 7: Results of CFNM for $\psi_4(t)$ with initial guess $t_0 = -5 \times 10^8$

ζ	t^*	$\lvert t_{n+1} - t_n\rvert$	$\lvert\psi(t_{n+1})\rvert$	Iterations
0.90	-6.1975304138e+08+8.0950386627e+07i	9.53674e-07	7.87102e-05	208
0.91	-6.1975304138e+08+8.0950386627e+07i	9.68575e-07	5.94664e-05	163
0.92	-6.1975304138e+08+8.095038662706130e+07i	9.83476e-07	4.13854e-05	128
0.93	-6.197530413e+08+8.095038662e+07i	8.79168e-07	2.93874e-05	101
0.94	-6.197530413e+08+8.0950386627e+07i	8.67856e-07	1.72211e-05	80
0.95	-6.1975304138e+08+8.0950386627e+07i	8.39241e-07	1.13882e-05	64
0.96	-6.1975304138e+08+8.0950386627e+07i	4.80548e-07	6.864633e-06	51
0.97	-6.1975304138e+08+8.0950386627e+07i	6.85453e-07	5.50698e-06	41
0.98	-6.197530413868866e+08-8.0950386627e+07i	6.44722e-07	2.32458e-06	31
0.99	-6.1975304138e+08-8.0950386627e+07i	3.653064e-07	1.45543e-06	23
1	-0.487804876815949	4.16475e-05	0.00	37

In Tables 7 and 8, the solutions of $\psi_4(t)$ *are shown in different order of derivative and faster convergence can be observed in CFDCAWM with minimum error.*

Table 8: Results of CFDCAWM for $\psi_4(t)$ with initial guess $t_0 = -5 \times 10^8$

ζ	t^*	$\lvert t_{n+1} - t_n \rvert$	$\lvert \psi(t_{n+1}) \rvert$	Iterations
0.90	-6.1975304138e+08+8.0950386627e+07i	9.90675e-07	8.01199e-05	207
0.91	-6.1975304138e+08+8.0950386627e+07i	9.61096e-07	5.84519e-05	163
0.92	-6.1975304138e+08+8.0950386627e+07i	9.83476e-07	4.02239e-05	128
0.93	-6.19753e+08+8.0950386627e+07i	9.68575e-07	2.53841e-05	101
0.94	-6.1975304138e+08+8.0950386627e+07i	9.90675e-07	2.10638e-05	80
0.95	-6.1975304138e+08-8.0950386627e+07i	7.83976e-07	1.18761e-05	64
0.96	-6.1975304138e+08-8.0950386627e+07i	8.34465e-07	6.69969e-06	48
0.97	-6.1975304138e+08-8.0950386627e+07i	1.435547e-06	1.249140e-05	36
0.98	-6.197530413868e+08-8.0950386627e+07i	4.29815e-07	2.53319e-06	28
0.99	-6.1975304138e+08+8.0950386627e+07i	2.53319e-07	2.96409e-06	24
1	-0.487804876815949	0.00231	0.00	07

Table 9: The data $\mathcal{P}_t$ available from NOAA to calculate the pH of the ocean from 2012-2023 using Whittaker method.

Year	$\mathcal{P}_t$	pH	Year	$\mathcal{P}_t$	pH
2012	394.06	8.1013	2018	408.72	8.0881
2013	396.74	8.0988	2019	411.66	8.0855
2014	398.81	8.0969	2020	414.24	8.0833
2015	401.01	8.0950	2021	416.45	8.0813
2016	404.41	8.0919	2022	418.56	8.0560
2017	406.76	8.0898	2023	420.19	8.0781

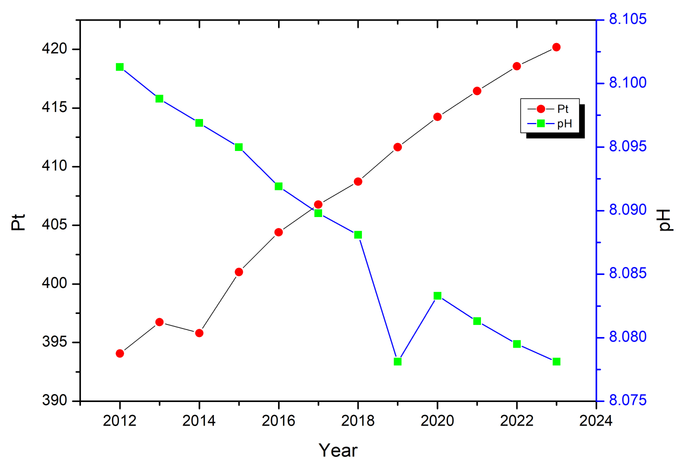
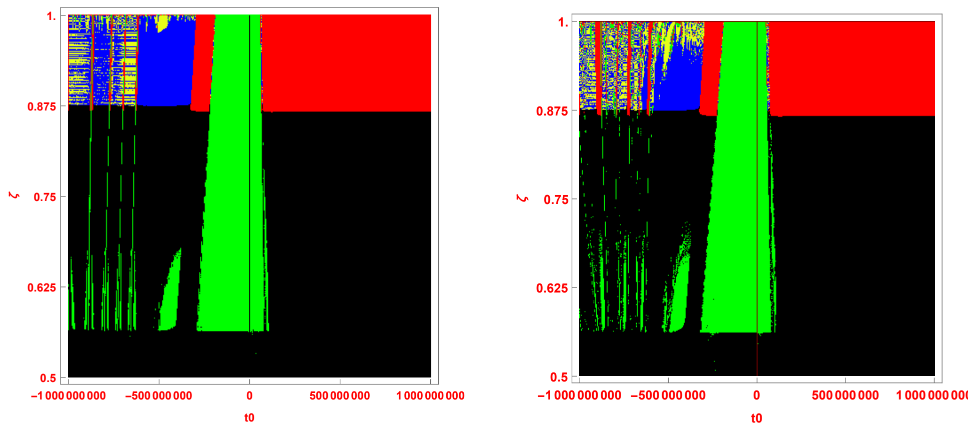


Figure 6: Relation between $\mathcal{P}_t$ and pH

The pH is calculated in the Table 9 for different values of $\mathcal{P}_t$ given by NOAA from the year 2012-2023 (February). The graph 6 says about the relation between pH and $\mathcal{P}_t$, and it can also be noticed that the pH is inversely proportional to $\mathcal{P}_t$.



(a) CFNM, 38.25% convergence (b) CFDCAWM, 40.24% convergence

Figure 7: Convergence planes of $\psi_4(t)$ for real initial guess $t_0 = a$, $a \in \mathbb{R}$.

In Figure 7, the convergence planes of CFNM and CFDCAWM are illustrated. Also, the real root t_1=*1.1970408047866759*$\times 10^8$ *corresponds to the solution* $[H^+]^* = 8.3539 \times 10^7$ *is painted in red color. Moreover, we have found that the* H^+ *ion concentration in CFDCAWM (*40.24%*) is more compared to CFNM (*38.25%*).*

Example 4.5 (Schrödinger wave equation for a hydrogen atom [25])**.** *The location of the electron relative to the core has a probability distribution in quantum mechanics, which is connected to the solution of the Schrödinger wave equation for a charged particle travelling in a Coulomb potential. The classic Schrödinger equation for a single particle of mass m moving in a central potential is as follows:*

$$-\frac{\hbar^2}{2m}\frac{\partial \Psi}{\partial t} - K\frac{e^2}{r}\Psi = E\Psi,$$

where r is the distance of the electron from the core and E is the energy. And the equation has the following representation in spherical coordinates:

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial(r^2\frac{\partial\Psi}{\partial r})}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial(\sin\theta\frac{\partial\Psi}{\partial\theta})}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right] + \frac{e^2\Psi}{r} = E\Psi.$$

The final equation can be divided into an angular equation and a radial equation by applying certain conventional techniques. The angular equation can alternatively be divided into two equations, one of which leads to the corresponding Legendre equation [6]

$$(1-x^2)\frac{d^2f}{dx^2} - 2x\frac{df}{dx} + \left(n(n+1) - \frac{m^2}{1-x^2}\right)f(x) = 0.$$

And for $m = 0$, i.e. the case of azimuthally symmetric, the equation reduced to Legendre polynomials. So, for our purpose we have taken the obtained form of Legendre equation in the following:

$$\psi_5(t) = 46189t^{10} - 109395t^8 + 90090t^6 - 30030t^4 + 3465t^2 - 63$$

with the roots $t_1 = -0.9739$, $t_2 = 0.9739$, $t_3 = -0.8651$, $t_4 = 0.8651$, $t_5 = -0.6794$, $t_6 = 0.6794$, $t_7 = -0.4334$, $t_8 = 0.4334$, $t_9 = -0.1489$, and $t_{10} = 0.1489$.

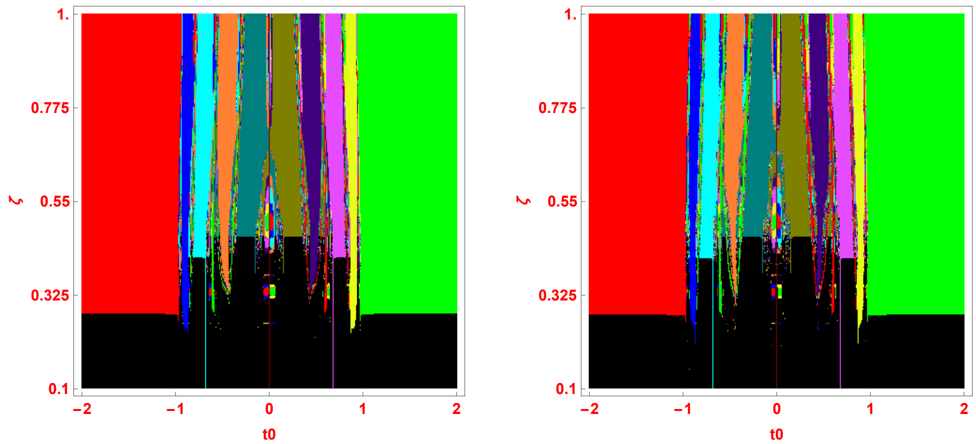
Table 10: Results of CFNM for $\psi_5(t)$ with initial guess $t_0 = -1.6$

ζ	t^*	$\|t_{n+1} - t_n\|$	$\|\psi(t_{n+1})\|$	Iterations
0.50	-0.974022883623179	9.934223703655931e-07	0.720438	131
0.55	-0.973979706177354+0.00i	9.767897406476322e-07	0.45272	103
0.60	-0.973952883699453+0.00i	9.894410853972246e-07	0.286638	80
0.65	-0.973934954067361+0.00i	9.675145213883240e-07	0.1757107	63
0.70	-0.973924328185314-0.00i	9.993784602091438e-07	0.1100054	49
0.75	-0.973916963198489-0.00i	9.656237642818866e-07	0.064479	39
0.80	-0.973912597976135+0.00i	9.649762192642797e-07	0.037502	31
0.85	-0.973909751305419-0.00i	9.191366918681609e-07	0.019911	25
0.90	-0.973908232217628+0.00i	9.887810010766884e-07	0.0105259	20
0.95	-0.973906845098482-0.00i	4.377130401467255e-07	0.001955	17
1	-0.973906528517171	8.214154911811988e-10	1.045918e-11	13

Table 11: Results of CFDCAWM for $\psi_5(t)$ with initial guess $t_0 = -1.6$

ζ	t^*	$\|t_{n+1} - t_n\|$	$\|\psi(t_{n+1})\|$	Iterations
0.50	-0.974022525194601	9.919475889574870e-07	0.71821	125
0.55	-0.973979997925276+0.00i	9.867753415493397e-07	0.45453	97
0.60	-0.973952593340544-0.00i	9.807070255885009e-07	0.284841	75
0.65	-0.973934897599979+0.00i	9.655019376220153e-07	0.175361	58
0.70	-0.973923495437265-0.00i	9.323057104104748e-07	0.104857	45
0.75	-0.973917138639778+0.00i	9.883953832057202e-07	0.065563	34
0.80	-0.973912010298633+0.00i	8.454325272078123e-07	0.033870	27
0.85	-0.973909290555717+0.00i	7.594809537936342e-07	0.017065	21
0.90	-0.973907816277493+0.00i	7.096176641852026e-07	0.007956	16
0.95	-0.973906956012490-0.00i	6.159379745129812e-07	0.002641	12
1	-0.973906528517169	3.798566096668843e-06	2.346630e-02	07

In Tables 10 and 11, the CFDCAWM provides faster convergence when ζ is close to 1. Both the CFNM and CFDCAWM converge to the root t_1 for the initial guess $t_0 = -1.6$, but it can be noticed that the CFDCAWM beats the CFNM in the speed of convergence.



(a) CFNM, 73.6% convergence (b) CFDCAWM, 74.26% convergence

Figure 8: Convergence planes of $\psi_5(t)$ for real initial guess $t_0 = a$, $a \in \mathbb{R}$.

The convergence plane in Figure 8 gives the CFNM (73.6%) and CFDCAWM (74.26%) percentage of convergence. Moreover, one can find all the roots of $\psi_5(t)$ by choosing an initial guess in the neighbourhood of zero and changing the order of the derivative in both CFDCAWM and CFNM.

5 Conclusion

This research aimed to introduce a new convex acceleration of the fractional Whittaker technique, namely CFDCAWM, in the sense of the Caputo fractional derivative. We have developed the speed of convergence of CFDCAWM to at least $(1+2\zeta)$, and we have studied the efficiency and stability of the proposed method. Then, for both CFNM and CFDCAWM, many real-world applications with numerical results are discussed. The convergence planes are illustrated with their convergence percentage for a more straightforward analysis. The results confirmed that CFDCAWM leads CFNM in terms of efficiency and performance.

Author Contributions: Both authors have equally contributed to the design and implementation of the research, to the analysis of the results and to the writing of the manuscript.

References

[1] P. Agarwal and A. A. El-Sayed, "Vieta-Lucas polynomials for solving a fractional-order mathematical physics model," *Adv. Difference Equ.*, 2020, Art. ID 626, doi: 10.1186/s13662-020-03085-y.

[2] A. Akgül, A. Cordero, and J. R. Torregrosa, "A fractional Newton method with 2αth-order of convergence and its stability," *Appl. Math. Lett.*, vol. 98, pp. 344–351, 2019, doi: 10.1016/j.aml.2019.06.028.

[3] T. M. Atanacković, S. Pilipović, B. Stanković, and D. Zorica, *Fractional calculus with applications in mechanics*, ser. Mechanical Engineering and Solid Mechanics Series. ISTE, London; John Wiley & Sons, Inc., Hoboken, NJ, 2014.

[4] D. Babajee and V. Jaunky, "Applications of higher-order optimal Newton secant iterative methods in ocean acidification and investigation of long-run implications of emissions on alkalinity of seawater," *International Scholarly Research Notices*, vol. 2013, 2013, Art. ID 785287, doi: 10.1155/2013/785287.

[5] R. Bacastow and C. D. Keeling, "Atmospheric carbon dioxide and radiocarbon in the natural carbon cycle: II. Changes from AD 1700 to 2070 as deduced from a geochemical model," in *Brookhaven Symposia in Biology*, vol. 24, 1973, pp. 86–135.

[6] I. N. Bronshtein, K. A. Semendyayev, G. Musiol, and H. Mühlig, *Handbook of mathematics*, 6th ed. Springer, Heidelberg, 2015.

[7] G. Candelario, A. Cordero, and J. R. Torregrosa, "Multipoint fractional iterative methods with $(2\alpha + 1)$th-order of convergence for solving nonlinear problems," *Mathematics*, vol. 8, no. 3, 2019, Art. ID 452, doi: 10.3390/math8030452.

[8] M. C. Caputo and D. F. Torres, "Duality for the left and right fractional derivatives," *Signal Processing*, vol. 107, pp. 265–271, 2015, doi: 10.1016/j.sigpro.2014.09.026.

[9] S. C. Chapra, *Applied numerical methods with MATLAB for engineers and scientists*, 3rd ed. McGraw-Hill, 2018.

[10] A. A. El-Sayed and P. Agarwal, "Numerical solution of multiterm variable-order fractional differential equations via shifted Legendre polynomials," *Math. Methods Appl. Sci.*, vol. 42, no. 11, pp. 3978–3991, 2019, doi: 10.1002/mma.5627.

[11] J. A. Ezquerro and M. A. Hernández, "Different acceleration procedures of Newton's method," *Novi Sad J. Math.*, vol. 27, no. 1, pp. 1–17, 1997.

[12] W. Gander, "On Halley's iteration method," *Amer. Math. Monthly*, vol. 92, no. 2, pp. 131–134, 1985, doi: 10.2307/2322644.

[13] K. S. Gritton, J. Seader, and W.-J. Lin, "Global homotopy continuation procedures for seeking all roots of a nonlinear equation," *Computers & Chemical Engineering*, vol. 25, no. 7-8, pp. 1003–1019, 2001.

[14] G. Honorato and S. Plaza, "Dynamical aspects of some convex acceleration methods as purely iterative algorithm for Newton's maps," *Appl. Math. Comput.*, vol. 251, pp. 507–520, 2015, doi: 10.1016/j.amc.2014.11.083.

[15] M. K. Jain, S. R. K. Iyengar, and R. K. Jain, *Numerical methods for scientific and engineering computation*, 6th ed. New Age International Publishers, 2012.

[16] G. Jumarie, "Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results," *Comput. Math. Appl.*, vol. 51, no. 9-10, pp. 1367–1376, 2006, doi: 10.1016/j.camwa.2006.02.001.

[17] C. Kennedy, "Ocean acidity dissolving tiny snails' protective shell," 2014, www.climate.gov/news-features/featured-images/ocean-acidity-dissolving-tiny-snails'-protective-shell [Accessed: April 5, 2024].

[18] M. A. Khan, S. Ullah, and M. Farhan, "The dynamics of Zika virus with Caputo fractional derivative," *AIMS Math.*, vol. 4, no. 1, pp. 134–146, 2019, doi: 10.3934/Math.2019.1.134.

[19] S. Kumar, D. Kumar, J. R. Sharma, C. Cesarano, P. Agarwal, and Y.-M. Chu, "An optimal fourth order derivative-free numerical algorithm for multiple roots," *Symmetry*, vol. 12, no. 6, 2020, Art. ID 1038, doi: 10.3390/sym12061038.

[20] C. Lanczos, "A precision approximation of the gamma function," *J. Soc. Indust. Appl. Math. Ser. B Numer. Anal.*, vol. 1, pp. 86–96, 1964.

[21] A. A. Magreñán, "A new tool to study real dynamics: the convergence plane," *Appl. Math. Comput.*, vol. 248, pp. 215–224, 2014, doi: 10.1016/j.amc.2014.09.061.

[22] V. Morales-Delgado, J. Gómez-Aguilar, K. M. Saad, M. A. Khan, and P. Agarwal, "Analytic solution for oxygen diffusion from capillary to tissues involving external force effects: A fractional calculus approach," *Physica A: Statistical Mechanics and its Applications*, vol. 523, pp. 48–65, 2019, doi: 10.1016/j.physa.2019.02.018.

[23] Z. M. Odibat and N. T. Shawagfeh, "Generalized Taylor's formula," *Appl. Math. Comput.*, vol. 186, no. 1, pp. 286–293, 2007, doi: 10.1016/j.amc.2006.07.102.

[24] A. L. Ozores, "Cálculo fraccionario y dinámica newtoniana," *Pensamiento Matemático*, vol. 4, no. 1, pp. 77–105, 2014.

[25] T. M. Pavkov, V. G. Kabadzhov, I. K. Ivanov, and S. I. Ivanov, "Local convergence analysis of a one parameter family of simultaneous methods with applications to real-world problems," *Algorithms*, vol. 16, no. 2, p. 103, 2023, Art. ID 103, doi: 10.3390/a16020103.

[26] J. L. Sarmiento, *Ocean biogeochemical dynamics.* Princeton university press, 2006.

[27] M. Shams, N. Kausar, C. Samaniego, P. Agarwal, S. F. Ahmed, and S. Momani, "On efficient fractional Caputo-type simultaneous scheme for finding all roots of polynomial equations with biomedical engineering applications," *Fractals*, vol. 31, no. 04, 2023, Art. ID 2340075, doi: 10.1142/S0218348X23400753.

[28] P. Tans and R. Keeling, "Trends in atmospheric carbon dioxide, national oceanic & atmospheric administration, earth system research laboratory (noaa/esrl) & scripps institution of oceanography," 2014, url: https://gml.noaa.gov/ccgg/trends/data.html.

[29] J. Tariboon, S. K. Ntouyas, and P. Agarwal, "New concepts of fractional quantum calculus and applications to impulsive fractional q-difference equations," *Adv. Difference Equ.*, 2015, Art. ID 2015:18, doi: 10.1186/s13662-014-0348-8.

[30] A. Torres-Hernández and F. Brambila-Paz, "Fractional Newton-Raphson method," *MathSJ*, vol. 8, no. 1, 2021, doi: 10.5121/mathsj.2021.8101.

[31] J. F. Traub, *Iterative methods for the solution of equations*, ser. Prentice-Hall Series in Automatic Computation. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964.

[32] M. A. H. Verón, "An acceleration procedure of the Whittaker method by means of convexity," *Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.*, vol. 20, no. 1, pp. 27–38, 1990.

CUBO
A Mathematical Journal

All papers submitted to CUBO are pre-evaluated by the Editorial Board, who can decide to reject those articles considered imprecise, unsuitable or lacking in mathematical soundness. These manuscripts will not continue the editorial process and will be returned to their author(s).

Those articles that fulfill CUBO's editorial criteria will proceed to an external evaluation. These referees will write a report with a recommendation to the editors regarding the possibility that the paper may be published. The referee report should be received within 120 days. If the paper is accepted, the authors will have 15 days to apply all modifications suggested by the editorial board.

The final acceptance of the manuscripts is decided by the Editor-in-chief and the Managing editor, based on the recommendations by the referees and the corresponding Associate editor. The author will be formally communicated of the acceptance or rejection of the manuscript by the Editor-in-chief.

All opinions and research results presented in the articles are of exclusive responsibility of the authors.

Submitting: By submitting a paper to this journal, authors certify that the manuscript has not been previously published nor is it under consideration for publication by another journal or similar publication. Work submitted to CUBO will be refereed by specialists appointed by the Editorial Board of the journal.

Manuscript: Manuscripts should be written in English and submitted in duplicate to cubo@ufrontera.cl. The first page should contain a short descriptive title, the name(s) of the author(s), and the institutional affiliation and complete address (including e-mail) of each author. Papers should be accompanied by a short abstract, keywords and the 2020 AMS Mathematical Subject Classification codes corresponding to the topic of the paper. References are indicated in the text by consecutive Arabic numerals enclosed in square brackets. The full list should be collected and typed at the end of the paper in numerical order.

Press requirement: The abstract should be no longer than 250 words. CUBO strongly encourages the use of LaTeX for manuscript preparation. References should be headed numerically, alphabetically organized and complete in all its details. Authors' initials should precede their names; journal title abbreviations should follow the style of Mathematical Reviews.

For technical questions about CUBO, please send an e-mail to cubo@ufrontera.cl.

Cubo

A Mathematical Journal

VOLUME 26 · ISSUE 1 · APRIL 2024

UNIVERSIDAD DE LA FRONTERA