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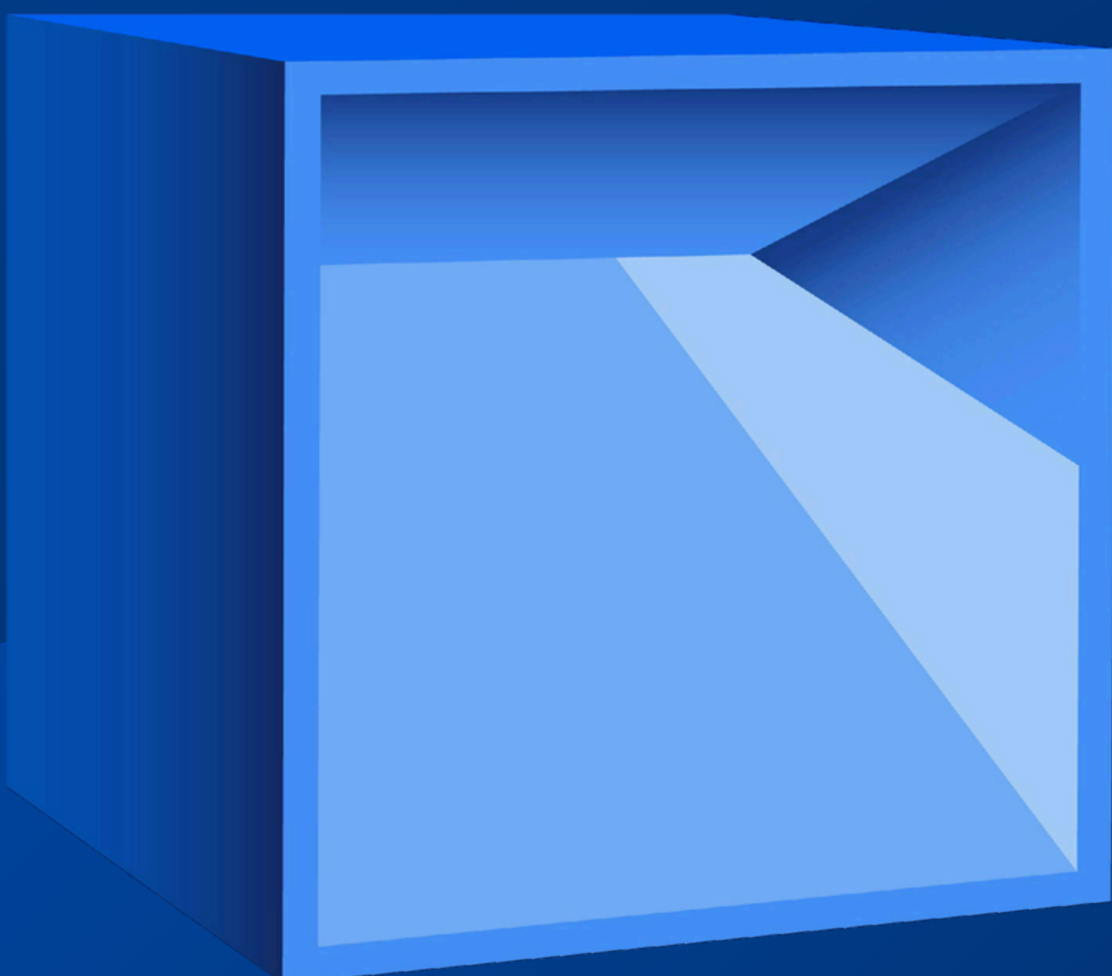

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SUMMARY

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Some properties of solutions of a linear set-valued differential equation with conformable fractional derivative

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ABSTRACT

The article explores a linear set-valued differential equation featuring both conformable fractional and generalized conformable fractional derivatives. It presents conditions for the existence of solutions and provides analytical expressions for the shape of solution sections at different time points. Model examples are employed to illustrate the results.

RESUMEN

Este artículo explora una ecuación diferencial lineal con valores en conjuntos que exhibe a la vez derivadas fraccionales conformables y conformables generalizadas. Se presentan condiciones para la existencia de soluciones y se proveen expresiones analíticas para la forma de secciones solución en diferentes puntos de tiempo. Se emplean ejemplos modelo para ilustrar los resultados.

Keywords and Phrases: Conformable fractional derivative, set-valued differential equation, Hukuhara derivative, generalized derivative.

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1 Introduction

Set-valued differential equations have recently been studied within the framework of an independent theory - set-valued equations, but they are widely used for ordinary differential inclusions and fuzzy differential equations and inclusions [7, 26, 29, 30, 36, 37, 46, 48, 53].

In 1967, M. Hukuhara introduced integral and derivative concepts for set-valued mappings and explored their relationship [20]. The proposed derivative and integral extend the conventional single-valued function derivative and integral to the set-valued context. However, the Hukuhara derivative has a notable limitation: if a mapping is Hukuhara differentiable, its cross-section diameter behaves as a non-decreasing function. To overcome this drawback, alternative derivative concepts were proposed: T. F. Bridgland introduced the Huygens derivative [6], while Yu. N. Tyurin [54] and H. T. Banks, M. Q. Jacobs [5] proposed the π-derivative using Radstrom's embedding theorem [52], and A. V. Plotnikov introduced the T-derivative [39, 48]. Additionally, Ş. E. Amrahov, A. Khastan, N. Gasilov, A. G. Fatullayev [3, 28] and A. V. Plotnikov, N. V. Skripnik [28, 44, 45] introduced generalized derivatives for set-valued mappings. Each of these derivatives has its own set of advantages and disadvantages [8, 12, 32, 33, 46, 48]. In 2003, A. N. Vityuk introduced an analogue of the fractional Riemann-Liouville derivative [23, 31] for set-valued mappings and established its properties [55, 56]. Subsequently, in 2019, A. A. Martyniuk introduced an analogue of the conformable fractional derivative [22] for set-valued mappings and proved its properties [34, 35]. The conformable fractional derivative for single-valued functions serves as a generalization of the ordinary derivative and, unlike fractional derivatives, adheres to all classical properties of the ordinary derivative [22]. Consequently, the Hukuhara conformable fractional derivative for set-valued mappings, introduced by A. A. Martyniuk, serves as a generalization of the Hukuhara derivative while preserving its properties [34, 35].

In 1969, F. S. de Blasi and F. Iervolino explored differential equations involving the Hukuhara derivative [12]. Subsequently, many authors investigated the properties of solutions to such equations [26, 29, 30, 36, 43, 46, 48], integral and integro-differential equations [41, 42], higher-order equations [38], as well as differential inclusions [11, 24, 48]. Furthermore, differential equations with the π-derivative [8, 37, 49], T-derivative [39, 48], set-valued equations with a generalized derivative [28, 40, 44, 45, 47], nonlinear equations with the fractional Riemann-Liouville derivative [55, 56], and conformable fractional derivative [34, 35, 57] have been explored. At first glance, such equations resemble their corresponding ordinary analogues; however, when studying and solving them, it is imperative to consider their set-valued nature. Consequently, traditional methods and approaches employed in studying and solving of single-valued systems may not always be applicable to set-valued systems, necessitating novel or alternative methods and approaches. It is also worth noting that due to set-valued nature, new properties emerge that warrant investigation.

This article delves into the Cauchy problem for a linear differential equation with the Hukuhara conformable fractional derivative, yielding analytical solutions in certain cases. Subsequently, we introduce a generalized conformable fractional derivative based on the generalized derivative for set-valued mappings [28, 44, 45], that allows us to expand the class of differentiable mappings. We then explore the Cauchy problem for a linear differential equation with the generalized conformable fractional derivative. Such a Cauchy problem boasts infinitely many solutions - two of which are termed basic [28, 44, 45], and we provide analytical forms for these solutions in selected cases. In conclusion, we demonstrate the feasibility of introducing conformable fractional derivatives akin to known conformable fractional derivatives for single-valued functions [1, 2, 4, 15, 17–19, 21, 22], alongside presenting analytical solutions for the corresponding Cauchy problems with these derivatives. The theoretical results are exemplified through model examples.

2 Preliminaries

In this section we recall some results from the publications that are of interest for our paper.

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ be the n-dimensional Euclidean space ($n \geq 2$). Denote by $conv(\mathbb{R}^n)$ the set of nonempty compact and convex subsets of $\mathbb{R}^n$ with the Hausdorff metric

$$h(X, Y) = \min\{r \geq 0 \,:\, X \subset Y + B_r(\mathbf{0}),\, Y \subset X + B_r(\mathbf{0})\},$$

where $X, Y \in conv(\mathbb{R}^n)$, $B_r(\mathbf{c}) = \{x \in \mathbb{R}^n \,:\, \|x - \mathbf{c}\| \leq r\}$ is the closed ball with radius $r > 0$ centered at the point $\mathbf{c} \in \mathbb{R}^n$ ($\|\cdot\|$ denotes the Euclidean norm), $\mathbf{0} = (0, \ldots, 0)^T$ is the zero vector.

In addition to the usual set-theoretic operations, the following operations in the space $conv(\mathbb{R}^n)$ are introduced: the sum of the sets, the product of the scalar on the set and the operation of the product of the matrix on the set:

$$X + Y = \bigcup_{x \in X,\, y \in Y} \{x + y\} \qquad \lambda X = \bigcup_{x \in X} \{\lambda x\}, \qquad AX = \bigcup_{x \in X} \{Ax\},$$

where $X, Y \in conv(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$, $A \in \mathbb{R}^{n \times n}$.

Lemma 2.1 ([51])**.** *The following properties hold:*

1) $(conv(\mathbb{R}^n), h)$ *is a complete metric space,*

2) $h(X + Z, Y + Z) = h(X, Y)$,

3) $h(\lambda X, \lambda Y) = |\lambda| h(X, Y)$ *for all* $X, Y, Z \in conv(\mathbb{R}^n)$ *and* $\lambda \in \mathbb{R}$.

However, $conv(\mathbb{R}^n)$ is not a linear space because it does not contain inverse elements for the addition, and therefore the difference is not well defined, *i.e.* if $X \in conv(\mathbb{R}^n)$ and $X \neq \{\mathbf{x}\}$, then

$X + (-1)X \neq \{\mathbf{0}\}$. As a consequence, alternative formulations for difference have been suggested [3, 5, 20, 39, 45, 51]. One of these alternatives is the Hukuhara difference [20].

Definition 2.2 ([20]). *Let $X, Y \in conv(\mathbb{R}^n)$. A set $Z \in conv(\mathbb{R}^n)$ such that $X = Y + Z$ is called a Hukuhara difference (H-difference) of the sets X and Y and is denoted by $X \overset{H}{-} Y$.*

In this case $X \overset{H}{-} X = \{\mathbf{0}\}$ and $(X + Y) \overset{H}{-} Y = X$ for any $X, Y \in conv(\mathbb{R}^n)$, but obviously, $X \overset{H}{-} Y \neq X + (-1)Y$. The properties of this difference are studied in detail in [37, 46, 48, 51]:

Lemma 2.3 ([27]). *If $X + Y = B_1(\mathbf{0})$, then $X = B_\mu(\mathbf{z_1})$ and $Y = B_\lambda(\mathbf{z_2})$, where $\mu + \lambda = 1$ and $\mathbf{z_1} + \mathbf{z_2} = \mathbf{0}$.*

Remark 2.4. *If the set X is subtracted from the ball $B_R(\mathbf{a})$ in the sense of Hukuhara and the difference $B_R(\mathbf{a}) \overset{H}{-} X$ exists, then the set X is the ball $B_r(\mathbf{b})$ and radius r does not exceed R.*

Theorem 2.5 ([14, 16]). *For any real $(n \times n)$-matrix A there exist two orthogonal $(n \times n)$-matrices U and V such that $U^T A V = \Sigma$, where Σ is the diagonal matrix. We can also choose matrices U and V such that the diagonal elements of the matrix Σ satisfy the condition*

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0,$$

where r is the rank of the matrix A. That is, if A is a nondegenerate matrix, then $\sigma_1 \geq \cdots \geq \sigma_n > 0$.

Therefore, this matrix A can be represented as $A = U\Sigma V^T$. This decomposition is called **singular decomposition**. Columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of matrix U are called the **left singular vectors**, columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of matrix V are called the **right singular vectors**, and the numbers $\sigma_1, \ldots, \sigma_n$ are called the **singular numbers** of the matrix A.

By [14], the set $Y = \{Ax : x \in B_1(\mathbf{0}), A \in \mathbb{R}^{n \times n}\}$ is r-dimensional ellipsoid, its axis lengths are equal to the corresponding singular numbers of the matrix A, where $r = rank(A)$. Also, if $rank(A) = n$, then

$$B_{\sigma_n}(\mathbf{0}) \subset Y \subset B_{\sigma_1}(\mathbf{0}),$$

where $B_{\sigma_n}(\mathbf{0})$ is the inscribed ball in the set Y (*i.e.* the largest ball $B_r(\mathbf{0})$ that can fit inside the set Y), $B_{\sigma_1}(\mathbf{0})$ is the circumscribed ball of the set Y (*i.e.* the smallest ball $B_r(\mathbf{0})$, such that $Y \subseteq B_r(\mathbf{0})$).

It is also easy to see that if A is an orthogonal matrix, then $AB_r(\mathbf{0}) \equiv B_r(\mathbf{0})$ for all $r > 0$.

Let $X : [0, T] \to conv(\mathbb{R}^n)$ be a set-valued mapping.

Definition 2.6 ([34])**.** *Let* $t \in (0,T)$ *and* $\alpha \in (0,1]$. *If the Hukuhara differences* $X\left(t+\varepsilon t^{1-\alpha}\right) \overset{H}{-} X\big(t\big)$ *and* $X\big(t\big) \overset{H}{-} X\left(t-\varepsilon t^{1-\alpha}\right)$ *exist for all sufficiently small* $\varepsilon > 0$ *and there exists* $Z \in conv(\mathbb{R}^n)$ *such that the following equality holds:*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\left(X\left(t+\varepsilon t^{1-\alpha}\right) \overset{H}{-} X\big(t\big)\right) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\left(X\big(t\big) \overset{H}{-} X\left(t-\varepsilon t^{1-\alpha}\right)\right) = Z, \tag{2.1}$$

then we say that the set-valued mapping $X(\cdot)$ *has a* ***Hukuhara conformable fractional derivative of order*** $\boldsymbol{\alpha}$ *at the point* $t \in (0,T)$ *and* $D^\alpha X(t) = Z$.

If $D^\alpha X(t)$ exists for all $t \in (0,T)$ and $\lim\limits_{t\to 0} D^\alpha X(t)$ exists, then we will assume that $D^\alpha X(0) = \lim\limits_{t\to 0} D^\alpha X(t)$.

Definition 2.7. *If the Hukuhara conformable fractional derivative* $D^\alpha X(t)$ *of order* α *exists for all* $t \geq 0$, *then we say that the set-valued mapping* $X(\cdot)$ *is* $\boldsymbol{\alpha}$***-differentiable*** *on* $\mathbb{R}_+$.

Next, we give some properties of the Hukuhara conformable fractional derivative of order α.

Lemma 2.8 ([34])**.** *If the set-valued mapping* $X(\cdot)$ *is* α*-differentiable on* $\mathbb{R}_+$, *then the set-valued mapping* $X(\cdot)$ *is continuous on* $\mathbb{R}_+$.

Lemma 2.9 ([34])**.** *If the set-valued mapping* $X(\cdot)$ *is* α*-differentiable on* $\mathbb{R}_+$, *then the function* $diam(X(\cdot))$ *is a nondecreasing function on* $\mathbb{R}_+$, *where* $diam(X) = \max\limits_{\psi \in S_1(\mathbf{0})} |c(X,\psi) + c(X,-\psi)|$, $S_1(\mathbf{0}) = \{\psi \in \mathbb{R}^n \,:\, \|\psi\| = 1\}$, $c(X,\psi) = \max\limits_{x\in X} \{x_1\psi_1 + \cdots + x_n\psi_n\}$.

Lemma 2.10 ([34])**.** *If the set-valued mapping* $X(t) \equiv X$ *for all* $t \geq 0$, *then*

$$D^\alpha X(t) \equiv \{\mathbf{0}\},$$

and vice versa, if $D^\alpha X(t) \equiv \{\mathbf{0}\}$ *for all* $t \geq 0$ *and* $X(t') = X$, *then* $X(t) \equiv X$ *for all* $t \geq 0$, *where* $t' \geq 0$ *is an arbitrary value.*

Lemma 2.11 ([34])**.** *If the set-valued mappings* $X(\cdot)$ *and* $Y(\cdot)$ *are* α*-differentiable at* $t > 0$, *then*

$$D^\alpha(aX(t) + bY(t)) = aD^\alpha X(t) + bD^\alpha Y(t),$$

where $a, b \in \mathbb{R}_+$.

Lemma 2.12 ([34])**.** *If the set-valued mapping* $X(\cdot)$ *is* α*-differentiable at* $t > 0$, *then*

$$D^\alpha X(t) = t^{1-\alpha} D_H X(t),$$

where $D_H X(t)$ *is the Hukuhara derivative [20].*

Remark 2.13. *From Lemma 2.12 we have that the necessary and sufficient condition for the existence of a Hukuhara conformable fractional derivative $D^{\alpha}X(t)$ of order α for the set-valued mapping $X(\cdot)$ is the existence of the Hukuhara derivative $D_H X(t)$.*

Remark 2.14. *From Definition 2.6 and Lemma 2.12, we have that $D^1 X(t)$ coincides with the Hukuhara derivative $D_H X(t)$.*

Definition 2.15 ([34])**.** *The fractional integral associated with the Hukuhara conformable fractional derivative of order α is defined by*

$$I^{\alpha}X(t) = \int\limits_0^t t^{\alpha-1} X(s)ds, \quad t \geq 0,$$

where the integral on the right-hand side is understood in the sense of the Hukuhara integral [20].

Lemma 2.16 ([34])**.** *If the set-valued mapping $X(\cdot)$ is continuous on $\mathbb{R}_+$, then*

$$D^{\alpha}I^{\alpha}X(t) = X(t), \quad t > 0.$$

Lemma 2.17 ([34])**.** *If the set-valued mapping $X(\cdot)$ is α-differentiable on $\mathbb{R}_+$, then*

$$I^{\alpha}D^{\alpha}X(t) = X(t) \overset{H}{-} X(0), \quad t > 0.$$

3 A linear set-valued differential equation with a Hukuhara conformable fractional derivative.

Consider the following Cauchy problem for linear set-valued differential equation with a Hukuhara conformable fractional derivative of order α

$$D^{\alpha}X(t) = AX(t), \quad X(0) = B_1(\mathbf{0}), \tag{3.1}$$

where $X : \mathbb{R}_+ \to conv(\mathbb{R}^2)$ is a set-valued mapping, $A \in \mathbb{R}^{2\times 2}$ is a nondegenerate matrix.

Definition 3.1. *A set-valued mapping $X : \mathbb{R}_+ \to conv(\mathbb{R}^2)$ is called a solution of Cauchy problem* (3.1) *if it is continuous and satisfies differential equation* (3.1) *for all $t \geq 0$ and $X(0) = B_1(\mathbf{0})$.*

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$ such that $ad - bc \neq 0$.

It is easy to obtain that the singular numbers of the matrix A have the form

$$\sigma_1 = \sqrt{\frac{a^2+b^2+c^2+d^2+\sqrt{\delta}}{2}}, \quad \sigma_2 = \sqrt{\frac{a^2+b^2+c^2+d^2-\sqrt{\delta}}{2}},$$

where $\delta = (a^2+b^2+c^2+d^2)^2 - 4(ad-bc)^2$.

It is obvious that

$$\delta = (a^2+b^2+c^2+d^2)^2 - 4(ad-bc)^2 = (a^2-d^2)^2 + (b^2-c^2)^2 + 2(ab+cd)^2 + 2(ac+bd)^2,$$

i.e. $\delta \geq 0$.

Accordingly, if $d = a$ and $c = -b$ or $d = -a$ and $b = c$, *i.e.* if

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

then $\delta = 0$ and $\sigma_1 = \sigma_2 = \sigma = \sqrt{a^2+b^2}$. In other cases $\delta \neq 0$.

Theorem 3.2. *If matrix A satisfies the condition $\delta = 0$, then Cauchy problem* (3.1) *has the following solution*

$$X(t) = e^{\beta t^\alpha} B_1(\mathbf{0}),$$

where $t \geq 0$, $\beta = \dfrac{\sqrt{a^2+b^2}}{\alpha}$.

Proof. Let us prove that $X(\cdot)$ is a solution of Cauchy problem (3.1) by the direct substitution of the set-valued mapping $X(t) = e^{\beta t^\alpha} B_1(\mathbf{0})$ into differential equation (3.1) and by checking that the identity is satisfied:

$$D^\alpha \left(e^{\beta t^\alpha} B_1(\mathbf{0}) \right) \equiv A e^{\beta t^\alpha} B_1(\mathbf{0}).$$

Since $\beta > 0$, then $e^{\beta t^\alpha}$ is an increasing function and as

$$e^{\beta t^\alpha} B_1(\mathbf{0}) = B_{e^{\beta t^\alpha}}(\mathbf{0}),$$

then accordingly $diam(X(\cdot))$ is an increasing function. Then, according to Definition 2.6, it follows that $B_1(\mathbf{0})$ is a centrally symmetric body and $(-1)B_1(\mathbf{0}) = B_1(\mathbf{0})$, we have

$$\begin{aligned}\lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(X\left(t + \varepsilon t^{1-\alpha}\right) \overset{H}{-} X(t) \right) &= \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{\beta\left(t+\varepsilon t^{1-\alpha}\right)^\alpha} B_1(\mathbf{0}) \overset{H}{-} e^{\beta t^\alpha} B_1(\mathbf{0}) \right) \\ &= \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{\beta\left(t+\varepsilon t^{1-\alpha}\right)^\alpha} - e^{\beta t^\alpha} \right) B_1(\mathbf{0}) = \alpha\beta e^{\beta t^\alpha} B_1(\mathbf{0})\end{aligned}$$

and

$$\lim_{\varepsilon\to 0_+} \varepsilon^{-1}\left(X(t) \overset{H}{-} X\left(t-\varepsilon t^{1-\alpha}\right)\right) = \lim_{\varepsilon\to 0_+} \varepsilon^{-1}\left(e^{\beta t^\alpha}B_1(\mathbf{0}) \overset{H}{-} e^{\beta\left(t-\varepsilon t^{1-\alpha}\right)^\alpha}B_1(\mathbf{0})\right)$$
$$= \lim_{\varepsilon\to 0_+} \varepsilon^{-1}\left(e^{\beta t^\alpha} - e^{\beta\left(t-\varepsilon t^{1-\alpha}\right)^\alpha}\right)B_1(\mathbf{0}) = -\alpha\beta e^{\beta t^\alpha}B_1(\mathbf{0}) = \alpha\beta e^{\beta t^\alpha}B_1(\mathbf{0}).$$

That is,

$$D^\alpha X(t) = D^\alpha\left(e^{\beta t^\alpha}B_1(\mathbf{0})\right) = \alpha\beta e^{\beta t^\alpha}B_1(\mathbf{0}).$$

Since the singular numbers of the matrix A are equal ($\sigma_1=\sigma_2=\sigma$), then the singular decomposition of the matrix A has the form $A=U\Sigma V^T$, where U, V are orthogonal matrices and $\Sigma=\sigma I$, I is the identity matrix. Since $V^T B_r(\mathbf{0}) = B_r(\mathbf{0})$ and $UB_r(\mathbf{0})=B_r(\mathbf{0})$ for all $r>0$, then

$$Ae^{\beta t^\alpha}B_1(\mathbf{0}) = U\Sigma V^T e^{\beta t^\alpha}B_1(\mathbf{0}) = U\sigma IV^T e^{\beta t^\alpha}B_1(\mathbf{0})$$
$$= \sigma UIV^T e^{\beta t^\alpha}B_1(\mathbf{0}) = \sigma e^{\beta t^\alpha}UIV^T B_1(\mathbf{0}) = \sigma e^{\beta t^\alpha}B_1(\mathbf{0}).$$

As $\alpha\beta = \alpha\frac{\sqrt{a^2+b^2}}{\alpha} = \sqrt{a^2+b^2}=\sigma$, then we have

$$D^\alpha X(t) = D^\alpha\left(e^{\beta t^\alpha}B_1(\mathbf{0})\right) = \alpha\beta e^{\beta t^\alpha}B_1(\mathbf{0}) = \sigma e^{\beta t^\alpha}B_1(\mathbf{0}) \equiv \sigma e^{\beta t^\alpha}B_1(\mathbf{0}) = Ae^{\beta t^\alpha}B_1(\mathbf{0}) = AX(t),$$

i.e. $X(\cdot)$ is a solution of differential equation (3.1). The theorem is proved. □

Example 3.3. *Let* $A=\begin{pmatrix}\sqrt{3} & 1\\ 1 & -\sqrt{3}\end{pmatrix}$. *Then the singular numbers* σ_1 *and* σ_2 *of the matrix* A *are* $\sigma_1=\sigma_2=2$. *Accordingly, Cauchy problem* (3.1) *has a solution* $X(t)=e^{2\alpha^{-1}t^\alpha}B_1(\mathbf{0})$. *That is,*

1) *if* $\alpha=0.25$, *then at every moment of time* $t\ge 0$ *the cross section* $X(t)$ *is a circle of radius* $e^{8\sqrt[4]{t}}$ *(Figure 1);*

2) *if* $\alpha=0.5$, *then at every moment of time* $t\ge 0$ *the cross section* $X(t)$ *is a circle of radius* $e^{4\sqrt{t}}$ *(Figure 2);*

3) *if* $\alpha=0.75$, *then at every moment of time* $t\ge 0$ *the cross section* $X(t)$ *is a circle of radius* $e^{\frac{8}{3}\sqrt[4]{t^3}}$ *(Figure 3);*

4) *if* $\alpha=1$, *then at every moment of time* $t\ge 0$ *the cross section* $X(t)$ *is a circle of radius* e^{2t} *(Figure 4).*

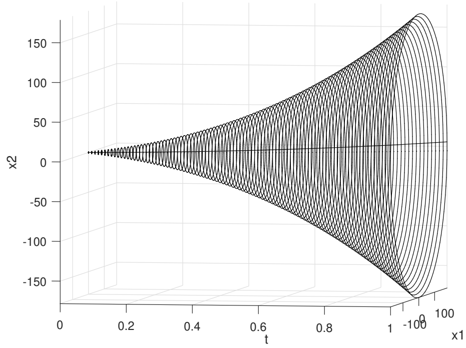

Figure 1: $\alpha = 0.25,\ X(t) = e^{8\sqrt[4]{t}} B_1(\mathbf{0})$.

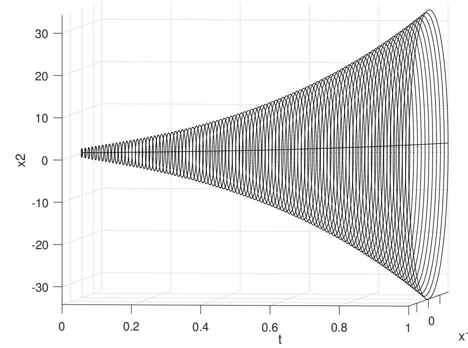

Figure 2: $\alpha = 0.5,\ X(t) = e^{4\sqrt{t}} B_1(\mathbf{0})$.

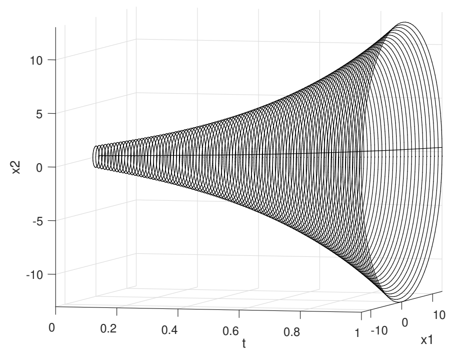

Figure 3: $\alpha = 0.75,\ X(t) = e^{\frac{8}{3}\sqrt[4]{t^3}} B_1(\mathbf{0})$.

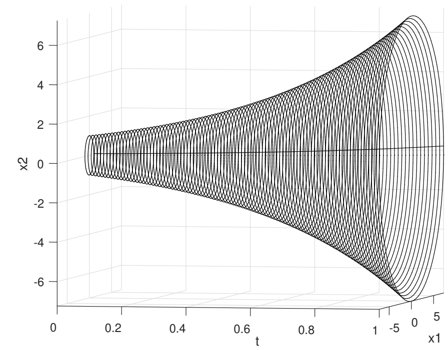

Figure 4: $\alpha = 1,\ X(t) = e^{2t} B_1(\mathbf{0})$.

Next, we consider the case when the matrix A satisfies the condition $\delta \neq 0$.

Theorem 3.4. *If matrix A is symmetric and $d \neq -a$, then Cauchy problem* (3.1) *has the following solution*

$$X(t) = U e^{\alpha^{-1} t^{\alpha} \Sigma} B_1(\mathbf{0}), \quad t \geq 0,$$

where $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$, $\sigma_{1,2} = |\lambda_{1,2}| = \left| \frac{a+d \pm \sqrt{(a-d)^2+4b^2}}{2} \right|$, $U = \begin{pmatrix} \frac{b}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{\lambda_2-d}{\sqrt{(\lambda_2-d)^2+b^2}} \\ \frac{\lambda_1-a}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{b}{\sqrt{(\lambda_2-d)^2+b^2}} \end{pmatrix}$.

Proof. Since the matrix A is symmetric and $d \neq -a$, it has the following form

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

It is known that the eigenvalues $\lambda_{1,2}$ of the symmetric matrix A are real, so in our case ($\delta \neq 0$), they will be different and not equal to zero. Let us consider all possible cases related to the eigenvalues of the matrix A, that is, three different cases are possible:

1) the eigenvalues $\lambda_{1,2} = \frac{a+d\pm\sqrt{D}}{2}$ of matrix A are positive, where $D = (a-d)^2 + 4b^2$, *i.e.* matrix A is a positive-definite matrix. In this case, the singular decomposition coincides with the spectral decomposition, *i.e.* $\sigma_1 = \lambda_1$, $\sigma_2 = \lambda_2$ and $U\Lambda U^T = U\Sigma U^T$, where

$$\Lambda = \Sigma = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad U = \begin{pmatrix} \frac{b}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{\lambda_2-d}{\sqrt{(\lambda_2-d)^2+b^2}} \\ \frac{\lambda_1-a}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{b}{\sqrt{(\lambda_2-d)^2+b^2}} \end{pmatrix}.$$

2) the eigenvalues $\lambda_{1,2}$ of matrix A are of different signs and $|\lambda_1| > |\lambda_2|$, *i.e.* matrix A is an indeterminate matrix. In this case, the singular decomposition is the following: $\sigma_1 = |\lambda_1|$, $\sigma_2 = |\lambda_2|$ and

$$U\Sigma W^T = U|\Lambda|DU^T,$$

where $W^T = DU^T$, $D = \begin{pmatrix} \frac{\lambda_1}{|\lambda_1|} & 0 \\ 0 & \frac{\lambda_2}{|\lambda_2|} \end{pmatrix}$.

3) the eigenvalues $\lambda_{1,2}$ of matrix A are negative and $|\lambda_1| > |\lambda_2|$, *i.e.* matrix A is a negative-definite matrix. In this case, the singular decomposition is $\sigma_1 = |\lambda_1|$, $\sigma_2 = |\lambda_2|$ and

$$U\Sigma W^T = U|\Lambda|DU^T,$$

where $W^T = DU^T$, $D = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

That is, in general, the singular decomposition of the matrix A has the form $A = U\Sigma W^T$, where $\Sigma = |\Lambda|$, $W = UD$.

We will prove that $X(\cdot)$ is a solution of Cauchy problem (3.1) by the direct substitution of the set-valued mapping $X(t) = Ue^{\alpha^{-1}t^\alpha\Sigma}B_1(0)$ into differential equation (3.1) and by checking that the identity is satisfied:

$$D^\alpha\left(Ue^{\alpha^{-1}t^\alpha\Sigma}B_1(\mathbf{0})\right) \equiv AUe^{\alpha^{-1}t^\alpha\Sigma}B_1(\mathbf{0}). \tag{3.2}$$

Since $\sigma_{1,2} > 0$, then $e^{\alpha^{-1}\sigma_1 t^\alpha}$ and $e^{\alpha^{-1}\sigma_2 t^\alpha}$ are the increasing functions and as

$$e^{\alpha^{-1}\sigma_1 t^\alpha} > e^{\alpha^{-1}\sigma_2 t^\alpha},$$

then accordingly $diam(X(t)) = 2e^{\alpha^{-1}\sigma_1 t^\alpha}$ is an increasing function. Then, according to Definition 2.6, it follows that $B_1(\mathbf{0})$ is a centrally symmetric body and, accordingly, $(-1)B_1(\mathbf{0}) = B_1(\mathbf{0})$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0_+}\varepsilon^{-1}\left(X\left(t+\varepsilon t^{1-\alpha}\right)\stackrel{H}{-}X(t)\right) &= \lim_{\varepsilon\to 0_+}\varepsilon^{-1}\left(Ue^{\alpha^{-1}\left(t+\varepsilon t^{1-\alpha}\right)^{\alpha}\Sigma}B_1(\mathbf{0})\stackrel{H}{-}Ue^{\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0})\right)\\ &= U\lim_{\varepsilon\to 0_+}\varepsilon^{-1}\left(e^{\alpha^{-1}\left(t+\varepsilon t^{1-\alpha}\right)^{\alpha}\Sigma}-e^{\alpha^{-1}t^{\alpha}\Sigma}\right)B_1(\mathbf{0})\\ &= U\lim_{\varepsilon\to 0_+}\varepsilon^{-1}\begin{pmatrix} e^{\alpha^{-1}\sigma_1\left(t+\varepsilon t^{1-\alpha}\right)^{\alpha}}-e^{\alpha^{-1}\sigma_1 t^{\alpha}} & 0\\ 0 & e^{\alpha^{-1}\sigma_2\left(t+\varepsilon t^{1-\alpha}\right)^{\alpha}}-e^{\alpha^{-1}\sigma_2 t^{\alpha}}\end{pmatrix}B_1(\mathbf{0})\\ &= U\begin{pmatrix}\sigma_1 e^{\alpha^{-1}\sigma_1 t^{\alpha}} & 0\\ 0 & \sigma_2 e^{\alpha^{-1}\sigma_2 t^{\alpha}}\end{pmatrix}B_1(\mathbf{0}) = U\Sigma e^{\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0})\end{aligned}$$

and similarly

$$\begin{aligned}\lim_{\varepsilon\to 0_+}\varepsilon^{-1}\left(X(t)\stackrel{H}{-}X\left(t-\varepsilon t^{1-\alpha}\right)\right) &= U\lim_{\varepsilon\to 0_+}\varepsilon^{-1}\left(e^{\alpha^{-1}t^{\alpha}\Sigma}-e^{\alpha^{-1}\left(t-\varepsilon t^{1-\alpha}\right)^{\alpha}\Sigma}\right)B_1(\mathbf{0})\\ &= U\Sigma e^{\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0}).\end{aligned}$$

That is,

$$D^{\alpha}X(t) = D^{\alpha}\left(e^{\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0})\right) = U\Sigma e^{\,\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0}).$$

Since the singular matrix decomposition of the symmetric matrix A has the form $A = U\Sigma DU^T$, then

$$AUe^{\,\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0}) = U\Sigma DU^T Ue^{\,\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0}) = U\Sigma e^{\,\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0}).$$

It is obvious that identity (3.2) holds and, accordingly, $X(\cdot)$ is a solution of Cauchy problem (3.1). The theorem is proved. ☐

Example 3.5. *Let* $A = \begin{pmatrix}0.8 & 0.5\\ 0.5 & 0.3\end{pmatrix}$. *Then the singular decomposition of the matrix A has the following form* $U\Sigma U^T = \begin{pmatrix}0.8507 & -0.5257\\ 0.5257 & 0.8507\end{pmatrix}\begin{pmatrix}1.1090 & 0\\ 0 & 0.0090\end{pmatrix}\begin{pmatrix}0.8507 & 0.5257\\ -0.5257 & 0.8507\end{pmatrix}$. *Accordingly, Cauchy problem* (3.1) *has a solution* $X(t) = Ue^{\,\alpha^{-1}t^{\alpha}\Sigma}B_1(\mathbf{0})$. *That is,*

1) *if $\alpha = 0.25$, then at every moment of time $t \geq 0$ the cross section $X(t)$ is an ellipse with semi-axes $e^{4.4361\sqrt[4]{t}}$ and $e^{0.0361\sqrt[4]{t}}$, rotated at an angle $\theta \approx 33^{\circ}$, which is determined by the matrix U (Figure 5);*

2) *if $\alpha = 0.5$, then at every moment of time $t \geq 0$ the cross section $X(t)$ is an ellipse with semi-axes $e^{2.2368\sqrt{t}}$ and $e^{0.0298\sqrt{t}}$, rotated at an angle $\theta \approx 33^{\circ}$ (Figure 6);*

3) *if $\alpha = 0.75$, then at every moment of time $t \geq 0$ the cross section $X(t)$ is an ellipse with semi-axes $e^{1.4787\sqrt[4]{t^3}}$ and $e^{0.0120\sqrt[4]{t^3}}$, rotated at an angle $\theta \approx 33^{\circ}$ (Figure 7);*

4) *if* $\alpha = 1$, *then at every moment of time* $t \geq 0$ *the cross section* $X(t)$ *is an ellipse with semi-axes* $e^{1.1090t}$ *and* $e^{0.009t}$, *rotated at an angle* $\theta \approx 33^\circ$ *(Figure 8).*

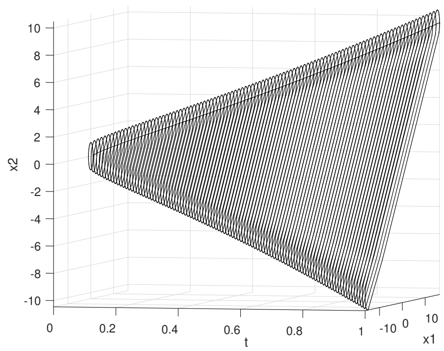

Figure 5: $\alpha = 0.25,\ X(t) = e^{4\sqrt[4]{t}\Sigma}B_1(\mathbf{0})$.

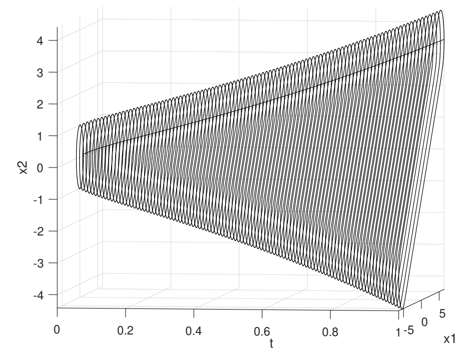

Figure 6: $\alpha = 0.5,\ X(t) = e^{2\sqrt{t}\Sigma}B_1(\mathbf{0})$.

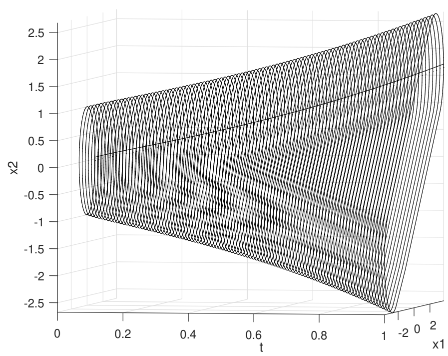

Figure 7: $\alpha = 0.75,\ X(t) = e^{\frac{4}{3}\sqrt[4]{t^3}\Sigma}B_1(\mathbf{0})$.

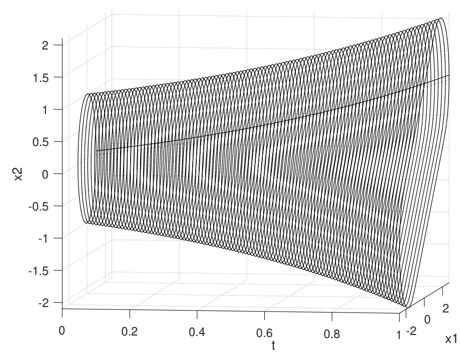

Figure 8: $\alpha = 1,\ X(t) = e^{t\Sigma}B_1(\mathbf{0})$.

4 A linear set-valued differential equation with a generalized conformable fractional derivative.

Let $X : [0, T] \to conv(\mathbb{R}^n)$ be a set-valued mapping.

Definition 4.1. *We say that a set-valued mapping* $X(\cdot)$ *has a* ***generalized conformable fractional derivative of order*** $\boldsymbol{\alpha}$ $D_g^\alpha X(t) \in conv(\mathbb{R}^n)$ *at* $t \in (0, T)$, *if for all sufficiently small* $\varepsilon > 0$ *the Hukuhara differences and the limits exist in at least one of the following cases:*

i) $\lim\limits_{\varepsilon \to 0} \varepsilon^{-1}\left(X\left(t + \varepsilon t^{1-\alpha}\right) \overset{H}{-} X(t)\right) = \lim\limits_{\varepsilon \to 0} \varepsilon^{-1}\left(X(t) \overset{H}{-} X\left(t - \varepsilon t^{1-\alpha}\right)\right) = D_g^\alpha X(t)$,

ii) $\lim\limits_{\varepsilon \to 0} \varepsilon^{-1}\left(X(t) \overset{H}{-} X\left(t + \varepsilon t^{1-\alpha}\right)\right) = \lim\limits_{\varepsilon \to 0} \varepsilon^{-1}\left(X\left(t - \varepsilon t^{1-\alpha}\right) \overset{H}{-} X(t)\right) = D_g^\alpha X(t)$, *iii)* $\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon t^{1-\alpha}\right)\overset{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t-\varepsilon t^{1-\alpha}\right)\overset{H}{-}X(t)\right)=D_g^{\alpha}X(t)$,

iv) $\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t+\varepsilon t^{1-\alpha}\right)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t-\varepsilon t^{1-\alpha}\right)\right)=D_g^{\alpha}X(t)$.

Definition 4.2. *If a generalized conformable fractional derivative of order α $D_g^{\alpha}X(t)$ exists for all $t\geq 0$, then we will say that the set-valued mapping $X(\cdot)$ is **generalized α-differentiable** on $\mathbb{R}_+$.*

Remark 4.3. *Obviously, if the set-valued mapping $X(\cdot)$ is α-differentiable at a point $t>0$, then the set-valued mapping $X(\cdot)$ is generalized α-differentiable at a point $t>0$.*

Lemma 4.4. *If the set-valued mapping $X(\cdot)$ is generalized α-differentiable at a point $t>0$, then*

$$D_g^{\alpha}X(t)=t^{1-\alpha}D_gX(t),$$

where $D_gX(t)$ is the generalized derivative [25, 28, 45].

Proof. If the set-valued mapping $X(\cdot)$ is generalized α-differentiable at a point $t>0$, then at least one of the conditions of Definition 4.1 must be fulfilled. We will assume that the first condition is fulfilled, *i.e.*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon t^{1-\alpha}\right)\overset{H}{-}X\big(t\big)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\big(t\big)\overset{H}{-}X\left(t-\varepsilon t^{1-\alpha}\right)\right)=D_g^{\alpha}X(t).$$

Let $\theta=\varepsilon t^{1-\alpha}$. Then

$$\begin{aligned}D_g^{\alpha}X(t)&=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon t^{1-\alpha}\right)\overset{H}{-}X\big(t\big)\right)=\lim_{\theta\to 0}t^{1-\alpha}\theta^{-1}\left(X\left(t+\theta\right)\overset{H}{-}X\big(t\big)\right)\\&=t^{1-\alpha}\lim_{\theta\to 0}\theta^{-1}\left(X\left(t+\theta\right)\overset{H}{-}X\big(t\big)\right)=t^{1-\alpha}D_gX(t).\end{aligned}$$

Likewise,

$$\begin{aligned}D_g^{\alpha}X(t)&=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\big(t\big)\overset{H}{-}X\left(t-\varepsilon t^{1-\alpha}\right)\right)=\lim_{\theta\to 0}t^{1-\alpha}\theta^{-1}\left(X\big(t\big)\overset{H}{-}X\left(t-\theta\right)\right)\\&=t^{1-\alpha}\lim_{\theta\to 0}\theta^{-1}\left(X\big(t\big)\overset{H}{-}X\left(t-\theta\right)\right)=t^{1-\alpha}D_gX(t).\end{aligned}$$

It is similarly proved if the second, third or fourth conditions are fulfilled. The lemma is proved. □

Remark 4.5. *It follows from Lemma 4.4 that a necessary and sufficient condition for the existence of a generalized conformable fractional derivative $D_g^{\alpha}X(t)$ is the existence of a generalized derivative $D_gX(t)$.*

Also, it is easy to see that if $\alpha=1$, then $D_g^{1}X(t)=D_gX(t)$.

Consider the following Cauchy problem for linear set-valued differential equation with a generalized conformable fractional derivative of order α

$$D_g^{\alpha} X(t) = AX(t), \quad X(0) = B_1(\mathbf{0}), \tag{4.1}$$

where $X : \mathbb{R}_+ \to conv(\mathbb{R}^2)$ is a set-valued mapping, $A \in \mathbb{R}^{2\times 2}$ is a nondegenerate matrix.

Definition 4.6. *A set-valued mapping $X : \mathbb{R}_+ \to conv(\mathbb{R}^2)$ is called a solution of Cauchy problem* (4.1) *if it is continuous and satisfies differential equation* (4.1) *for all $t \geq 0$ and $X(0) = B_1(\mathbf{0})$.*

Remark 4.7. *It follows from Remark 4.3 that if the set-valued mapping $X(t)$ is a solution of equation* (3.1)*, then it is a solution of equation* (4.1).

Remark 4.8. *In [25, 27, 28] a Cauchy problem for linear set-valued differential equation with a generalized derivative*

$$D_g X(t) = AX(t), \quad X(0) = B_1(\mathbf{0}) \tag{4.2}$$

was considered and the following results were obtained:

1) *Cauchy problem* (4.2) *has an infinite number of solutions, some (one or two) of which are called basic (their diameter are monotone functions), and others are mixed (their diameter are non-monotone functions). We also note that the first basic solution $X_1(\cdot)$ is the solution of Cauchy problem* (4.2)*, that satisfies the condition that $diam(X_1(t))$ is a nondecreasing function and is also the solution of the corresponding differential equation with the Hukuhara derivative. The second basic solution $X_2(\cdot)$ is called the solution of Cauchy problem* (4.2)*, that satisfies the condition that $diam(X_2(t))$ is a decreasing function;*

2) *if the singular numbers of the matrix A are such that $\sigma_1 = \sigma_2 = \sigma$, then Cauchy problem* (4.2) *has two basic solutions $X_1(t)$ and $X_2(t)$, whose cross-sections at each moment of time t are circles $B_{e^{\sigma t}}(0)$ and $B_{e^{-\sigma t}}(0)$, and if the singular numbers of the matrix A are such that $\sigma_1 \neq \sigma_2$, then Cauchy problem* (4.2) *has only the first basic solution $X_1(t)$, whose cross-section at each moment of time t is an ellipse with semiaxes equal to $e^{\sigma_1 t}$ and $e^{\sigma_2 t}$.*

Next, we obtain the results similar to Theorems 3.2 and 3.4.

Theorem 4.9. *If the matrix A satisfies the condition $\delta = 0$, then Cauchy problem* (4.1) *has two basic solutions $X_1(\cdot)$ and $X_2(\cdot)$ such that*

$$X_1(t) = e^{\beta t^{\alpha}} B_1(\mathbf{0}) \quad \textit{and} \quad X_2(t) = e^{-\beta t^{\alpha}} B_1(\mathbf{0}),$$

where $t \geq 0$, $\beta = \dfrac{\sqrt{a^2 + b^2}}{\alpha}$.

Proof. From Theorem 3.2, we have that the set-valued mapping $X_1(t)$ is a solution of Cauchy problem (3.1) and the function $diam(X(t))$ is non-decreasing. Then, taking into account Remark 4.3, $X_1(t)$ is the first basic solution of equation (4.1).

We will prove that $X_2(\cdot)$ is a solution of Cauchy problem (4.1) by the direct substitution of the set-valued mapping $X_2(t) = e^{-\beta t^\alpha} B_1(\mathbf{0})$ into differential equation (4.1) and by checking that the identity is satisfied:

$$D_g^\alpha \left(e^{-\beta t^\alpha} B_1(\mathbf{0}) \right) \equiv A e^{-\beta t^\alpha} B_1(\mathbf{0}).$$

Since $\beta > 0$, then $e^{-\beta t^\alpha}$ is a decreasing function, and as

$$e^{-\beta t^\alpha} B_1(\mathbf{0}) = B_{e^{-\beta t^\alpha}}(\mathbf{0}),$$

then, accordingly, the function $diam(X_2(\cdot))$ is a decreasing function. Then according to Definition 4.1 ii) and that the ball $B_1(\mathbf{0})$ is a centrally symmetric body and $(-1)B_1(\mathbf{0}) = B_1(\mathbf{0})$, we have

$$\begin{aligned}\lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(X_2(t) \stackrel{H}{-} X_2\left(t + \varepsilon t^{1-\alpha}\right)\right) &= \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{-\beta t^\alpha} B_1(\mathbf{0}) \stackrel{H}{-} e^{-\beta\left(t+\varepsilon t^{1-\alpha}\right)^\alpha} B_1(\mathbf{0}) \right) \\ &= \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{-\beta t^\alpha} - e^{-\beta\left(t+\varepsilon t^{1-\alpha}\right)^\alpha} \right) B_1(\mathbf{0}) = -\alpha\beta e^{-\beta t^\alpha} B_1(\mathbf{0}) = \alpha\beta e^{-\beta t^\alpha} B_1(\mathbf{0})\end{aligned}$$

and

$$\begin{aligned}\lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(X_2\left(t - \varepsilon t^{1-\alpha}\right) \stackrel{H}{-} X_2(t)\right) &= \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{-\beta\left(t-\varepsilon t^{1-\alpha}\right)^\alpha} B_1(\mathbf{0}) \stackrel{H}{-} e^{-\beta t^\alpha} B_1(\mathbf{0}) \right) \\ &= \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{-\beta\left(t-\varepsilon t^{1-\alpha}\right)^\alpha} - e^{-\beta t^\alpha} \right) B_1(\mathbf{0}) = \alpha\beta e^{-\beta t^\alpha} B_1(\mathbf{0}).\end{aligned}$$

That is,

$$D_g^\alpha X_2(t) = D^\alpha \left(e^{-\beta t^\alpha} B_1(\mathbf{0}) \right) = \alpha\beta e^{-\beta t^\alpha} B_1(\mathbf{0}).$$

Since the matrix A satisfies the condition $\delta = 0$, the singular decomposition of the matrix A has the form $A = U\Sigma V^T$, where U, V are orthogonal matrices, $\Sigma = \sigma I$, $\sigma = \sqrt{a^2 + b^2}$. As $V^T B_r(\mathbf{0}) = B_r(\mathbf{0})$ and $UB_r(\mathbf{0}) = B_r(\mathbf{0})$ for all $r > 0$, then

$$\begin{aligned}Ae^{-\beta t^\alpha} B_1(\mathbf{0}) &= U\Sigma V^T e^{-\beta t^\alpha} B_1(\mathbf{0}) = U\sigma EV^T e^{-\beta t^\alpha} B_1(\mathbf{0}) = \sigma UEV^T e^{-\beta t^\alpha} B_1(\mathbf{0}) \\ &= \sigma e^{-\beta t^\alpha} UEV^T B_1(\mathbf{0}) = \sigma e^{-\beta t^\alpha} B_1(\mathbf{0}).\end{aligned}$$

Since $\alpha\beta = \sigma$, we have

$$D_g^\alpha X_2(t) = \sigma e^{-\beta t^\alpha} B_1(\mathbf{0}) \equiv \sigma e^{-\beta t^\alpha} B_1(\mathbf{0}) = AX_2(t),$$

i.e. $X_2(\cdot)$ is the second basic solution of Cauchy problem (4.1). Thus the theorem is proved. □

Example 4.10. *Let* $A = \begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}$. *Then the singular numbers* σ_1 *and* σ_2 *of the matrix* A *are equal:* $\sigma_1 = \sigma_2 = \sigma = 2$.

Accordingly, Cauchy problem (4.1) *has solutions* $X_1(t) = e^{\,2\alpha^{-1}t^{\alpha}} B_1(\mathbf{0})$ *and* $X_2(t) = e^{\,-2\alpha^{-1}t^{\alpha}} B_1(\mathbf{0})$. *Below are the solutions for cases* $\alpha = 1$ *(Fig. 9, 10) and* $\alpha = 0.5$ *(Fig. 11, 12).*

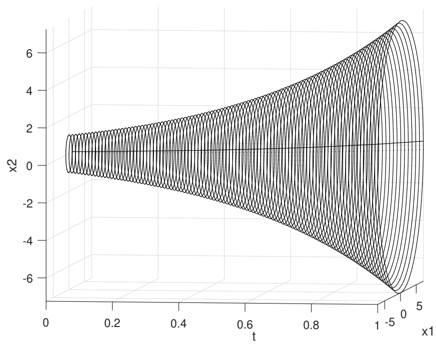

Figure 9: If $\alpha = 1$, then $X_1(t) = e^{2t} B_1(\mathbf{0})$.

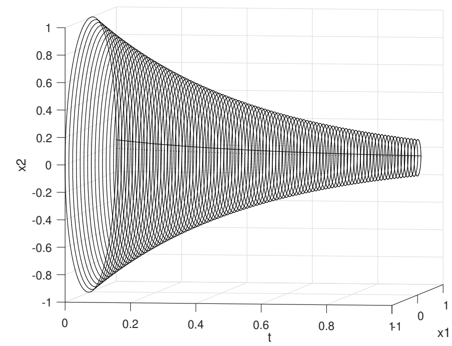

Figure 10: If $\alpha = 1$, then $X_2(t) = e^{-2t} B_1(\mathbf{0})$.

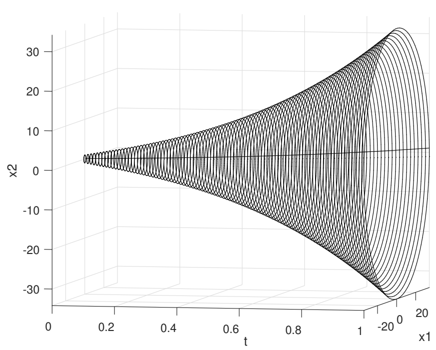

Figure 11: If $\alpha = 0.5$, then $X_1(t) = e^{4\sqrt{t}} B_1(\mathbf{0})$.

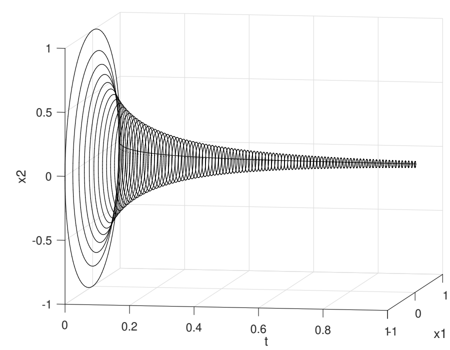

Figure 12: If $\alpha = 0.5$, then $X_2(t) = e^{-4\sqrt{t}} B_1(\mathbf{0})$.

Theorem 4.11. *If matrix* A *is symmetric and* $d \neq -a$, *then Cauchy problem* (4.1) *has only the first basic solution* $X_1(\cdot)$ *such that*

$$X_1(t) = U e^{\,\alpha^{-1}t^{\alpha}\Sigma} B_1(\mathbf{0}), \quad t \geq 0,$$

where $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$, $\sigma_{1,2} = |\lambda_{1,2}| = \left| \frac{a+d \pm \sqrt{(a-d)^2+4b^2}}{2} \right|$, $U = \begin{pmatrix} \frac{b}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{\lambda_2-d}{\sqrt{(\lambda_2-d)^2+b^2}} \\ \frac{\lambda_1-a}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{b}{\sqrt{(\lambda_2-d)^2+b^2}} \end{pmatrix}$.

Proof. According to Remark 4.7, the first basic solution of Cauchy problem (4.1) is also a solution of equation (3.1). Then, according to Theorem 3.4, the set-valued mapping $X_1(t) = Ue^{\alpha^{-1}t^\alpha\Sigma}B_1(\mathbf{0})$ is the first basic solution of Cauchy problem (4.1).

Now we will prove that the second basic solution $X_2(\cdot)$ of Cauchy problem (4.1) does not exist. We will prove it by contradiction. Let Cauchy problem (4.1) have the second basic solution $X_2(\cdot)$. Then $X_2(\cdot)$ satisfies the following integral equation

$$X_2(t) + A\int_0^t s^{\alpha-1}X_2(s)ds = B_1(\mathbf{0}).$$

Let us fix an arbitrary $T > 0$. Then $X_2(T) + A\int_0^T s^{\alpha-1}X_2(s)ds = B_1(\mathbf{0})$. From here,

$$B_1(0) \overset{H}{-} X_2(T) = A\int_0^T s^{\alpha-1}X_2(s)ds.$$

From Lemma 2.3, as $B_1(\mathbf{0})$ is a ball and Hukuhara difference $B_1(\mathbf{0}) \overset{H}{-} X_2(T)$ exists, then $X_2(T)$ is a ball, *i.e.* $X_2(T) \equiv B_{r(T)}(\mathbf{0})$, where $0 \le r(T) \le 1$. As T is arbitrary, then $X_2(t) \equiv B_{r(t)}(\mathbf{0})$ for all $t \ge 0$. Hence,

$$\int_0^T s^{\alpha-1}X_2(s)ds = \int_0^T s^{\alpha-1}B_{r(s)}(\mathbf{0})ds = \int_0^T s^{\alpha-1}r(s)dsB_1(\mathbf{0}) = R(T)B_1(0) = B_{R(T)}(\mathbf{0}),$$

where $R(T) = \int_0^T s^{\alpha-1}r(s)ds.$

That is, we have

$$B_{r(T)}(\mathbf{0}) + AB_{R(T)}(\mathbf{0}) = B_1(\mathbf{0}). \tag{4.3}$$

Since the matrix A has two different singular numbers, then $AB_{R(T)}(0)$ is an ellipse. So, the set $B_{r(T)}(\mathbf{0}) + AB_{R(T)}(\mathbf{0})$ is not a ball. That is, equality (4.3) is not fulfilled and we have obtained a contradiction. The theorem is proved. □

Conclusion

In conclusion, we present some remarks.

Remark 4.12. *If in Definition 2.6 we replace equality* (2.1) *by the equality*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t\,e^{\varepsilon t^{-\alpha}}\right)\overset{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t\,e^{-\varepsilon t^{-\alpha}}\right)\right)=Z, \tag{4.4}$$

or

$$\lim_{\varepsilon\to 0}e^{\varepsilon^{-1}}\left(X\left(t+e^{-\varepsilon^{-1}}t^{1-\alpha}\right)\overset{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}e^{\varepsilon^{-1}}\left(X(t)\overset{H}{-}X\left(t-e^{-\varepsilon^{-1}}t^{1-\alpha}\right)\right)=Z, \tag{4.5}$$

then we obtain a generalization of the conformable fractional derivative of order α of a single-valued function [19] or [21] for set-valued mappings. Similarly, as it was done in [34], it is possible to prove the validity of Lemmas 2.8–2.17, which makes it possible to introduce the corresponding generalized conformable fractional derivative of order α, consider the corresponding differential equations, and prove theorems similar to Theorems 3.2–4.11, and since in this case $D^\alpha X(t)=t^{1-\alpha}D_H X(t)$, then the analytical formulas of the solutions will also be the same.

Remark 4.13. *If in Definition 2.6 we replace equality (2.1) by the equality*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon e^{(\alpha-1)t}\right)\overset{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t-\varepsilon e^{(\alpha-1)t}\right)\right)=Z, \tag{4.6}$$

then we obtain a generalization of the conformable fractional derivative of order α of a single-valued function [18] for set-valued mappings. Similarly, as it was done in [34], it is possible to prove the validity of Lemmas 2.8–2.17, which makes it possible to introduce the corresponding generalized conformable fractional derivative of order α, consider the corresponding differential equations, and prove theorems similar to Theorems 3.2–4.11. However, since in this case $D^\alpha X(t)=e^{(\alpha-1)t}D_H X(t)$, then the analytical formulas of solutions will have the following form:

Theorem 3.2: $X(t)=e^{\frac{\sigma}{1-\alpha}e^{(1-\alpha)t}}B_1(\mathbf{0})$;

Theorem 3.4: $X(t)=Ue^{\frac{1}{1-\alpha}e^{(1-\alpha)t}\Sigma}B_1(\mathbf{0})$;

Theorem 4.9: $X_1(t)=e^{\frac{\sigma}{1-\alpha}e^{(1-\alpha)t}}B_1(\mathbf{0})$, $X_2(t)=e^{\frac{\sigma}{\alpha-1}e^{(1-\alpha)t}}B_1(\mathbf{0})$;

Theorem 4.11: $X_1(t)=Ue^{\frac{1}{1-\alpha}e^{(1-\alpha)t}\Sigma}B_1(\mathbf{0})$.

Remark 4.14. *If in Definition 2.6 we replace equality* (2.1) *by the equality*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right)\overset{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t-\varepsilon\left(t+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right)\right)=Z, \quad (4.7)$$

where $\Gamma(\alpha)$ *is gamma function, then we obtain a generalization of the conformable fractional derivative of order* α *of a single-valued function [4] for set-valued mappings. Similarly, as it was done in [34], it is possible to prove the validity of Lemmas 2.8–2.17, which makes it possible to introduce the corresponding generalized conformable fractional derivative of order* α*, consider the corresponding differential equations, and prove theorems similar to Theorems 3.2–4.11. However, since in this case* $D^\alpha X(t) = \left(t+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha} D_H X(t)$*, then the analytical formulas of solutions will have the following form:*

Theorem 3.2: $X(t) = e^{\frac{\sigma}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}} B_1(\mathbf{0})$;

Theorem 3.4: $X(t) = Ue^{\frac{1}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\Sigma} B_1(\mathbf{0})$;

Theorem 4.9: $X_1(t) = e^{\frac{\sigma}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}} B_1(\mathbf{0})$, $X_2(t) = e^{-\frac{\sigma}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}} B_1(\mathbf{0})$;

Theorem 4.11: $X_1(t) = Ue^{\frac{1}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\Sigma} B_1(\mathbf{0})$.

Remark 4.15. *If in Definition 2.6 we replace equality* (2.1) *by the equality*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon k(t)^{1-\alpha}\right)\overset{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t-\varepsilon k(t)^{1-\alpha}\right)\right)=Z, \quad (4.8)$$

where $k(t)$ *is a continuous positive function for all* $t \geq 0$*, then we obtain a generalization of the conformable fractional derivative of order* α *of a single-valued function [2, 15] for set-valued mappings. Similarly, as it was done in [34], it is possible to prove the validity of Lemmas 2.8–2.17, which makes it possible to introduce the corresponding generalized conformable fractional derivative of order* α*, consider the corresponding differential equations, and prove theorems similar to Theorems 3.2–4.11. However, since in this case* $D^\alpha X(t) = k(t)^{1-\alpha} D_H X(t)$*, then the analytical formulas of solutions will have the following form:*

Theorem 3.2: $X(t) = e^{\sigma\int_0^t (k(s))^{\alpha-1}ds} B_1(\mathbf{0})$;

Theorem 3.4: $X(t) = Ue^{\int_0^t (k(s))^{\alpha-1}ds\Sigma} B_1(\mathbf{0})$

Theorem 4.9: $X_1(t) = e^{\sigma\int_0^t (k(s))^{\alpha-1}ds} B_1(\mathbf{0})$, $X_2(t) = e^{-\sigma\int_0^t (k(s))^{\alpha-1}ds} B_1(\mathbf{0})$;

Theorem 4.11: $X_1(t) = Ue^{\int_0^t (k(s))^{\alpha-1}ds\Sigma} B_1(\mathbf{0})$.

Remark 4.16. *If in Definition 2.6 we replace equality* (2.1) *by the equality*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+k(t)-k(t)e^{\varepsilon\frac{k(t)^{-\alpha}}{|k'(t)|}}\right)\overset{H}{-}X\left(t\right)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t\right)\overset{H}{-}X\left(t+k(t)-k(t)e^{-\varepsilon\frac{k(t)^{-\alpha}}{|k'(t)|}}\right)\right)=Z, \tag{4.9}$$

where $k(t)$ is a differentiable function for all $t\geq 0$ such that $k(t)>0$ and $k'(t)\neq 0$ for all $t\geq 0$, then we obtain a generalization of the conformable fractional derivative of order α of a single-valued function [1] for set-valued mappings. Similarly, as it was done in [34], it is possible to prove the validity of Lemmas 2.8–2.17, which makes it possible to also introduce the corresponding generalized conformable fractional derivative of order α, consider the corresponding differential equations, and prove theorems similar to Theorems 3.2–4.11. However, since in this case $D^\alpha X(t)=\frac{k(t)^{1-\alpha}}{k'(t)}D_HX(t)$, then the analytical formulas of solutions will have the following form:

Theorem 3.2: $X(t)=e^{\sigma\alpha^{-1}(k(t)^\alpha-k(0)^\alpha)}B_1(\mathbf{0})$;

Theorem 3.4: $X(t)=Ue^{\alpha^{-1}(k(t)^\alpha-k(0)^\alpha)\Sigma}B_1(\mathbf{0})$;

Theorem 4.9: $X_1(t)=e^{\sigma\alpha^{-1}(k(t)^\alpha-k(0)^\alpha)}B_1(\mathbf{0})$, $X_2(t)=e^{\sigma\alpha^{-1}(k(0)^\alpha-k(t)^\alpha)}B_1(\mathbf{0})$;

Theorem 4.11: $X_1(t)=Ue^{\sigma\alpha^{-1}(k(t)^\alpha-k(0)^\alpha)\Sigma}B_1(\mathbf{0})$.

Remark 4.17. *We also note that if in Definition 2.6 we replace equality* (2.1) *by the equality*

$$\lim_{\varepsilon\to 0}((t+\varepsilon)^\alpha-t^\alpha)^{-1}\left(X\left(t+\varepsilon\right)\overset{H}{-}X\left(t\right)\right)=\lim_{\varepsilon\to 0}(t^\alpha-(t-\varepsilon)^\alpha)^{-1}\left(X\left(t\right)\overset{H}{-}X\left(t-\varepsilon\right)\right)=Z, \tag{4.10}$$

then we obtain a generalization of the ***Chen-Hausdorff fractal derivative of order $\boldsymbol{\alpha}$*** *of a single-valued function [9, 10] for set-valued mappings. Similarly, as it was done in [34], it is possible to prove the validity of Lemmas 2.8–2.17, which makes it possible to introduce the corresponding generalized Chen-Hausdorff fractal derivative of order α, consider the corresponding differential equations, and prove theorems similar to Theorems 3.2–4.11. However, since in this case $D^\alpha X(t)=\alpha^{-1}t^{1-\alpha}D_HX(t)$, then the analytical formulas of solutions will have the following form:*

Theorem 3.2: $X(t)=e^{\sigma t^\alpha}B_1(\mathbf{0})$;

Theorem 3.4: $X(t)=Ue^{t^\alpha\Sigma}B_1(\mathbf{0})$;

Theorem 4.9: $X_1(t)=e^{\sigma t^\alpha}B_1(\mathbf{0})$, $X_2(t)=e^{-\sigma t^\alpha}B_1(\mathbf{0})$;

Theorem 4.11: $X_1(t)=Ue^{t^\alpha\Sigma}B_1(\mathbf{0})$.

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Minimum-sized generating sets of the direct powers of free distributive lattices

Dedicated to the memory of George F. McNulty

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ABSTRACT

For a finite lattice L, let $\mathrm{Gm}(L)$ denote the least n such that L can be generated by n elements. For integers $r > 2$ and $k > 1$, denote by $\mathrm{FD}(r)^k$ the k-th direct power of the free distributive lattice $\mathrm{FD}(r)$ on r generators. We determine $\mathrm{Gm}(\mathrm{FD}(r)^k)$ for many pairs (r, k) either exactly or with good accuracy by giving a lower estimate that becomes an upper estimate if we increase it by 1. For example, for $(r, k) = (5, 25\,000)$ and $(r, k) = (20,\ 1.489 \cdot 10^{1789})$, $\mathrm{Gm}(\mathrm{FD}(r)^k)$ is 22 and 6 000, respectively. To reach our goal, we give estimates for the maximum number of pairwise unrelated copies of some specific posets (called full segment posets) in the subset lattice of an n-element set. In addition to analogous earlier results in lattice theory, a connection with cryptology is also mentioned among the motivations.

RESUMEN

Para un reticulado finito L, se denota por $\mathrm{Gm}(L)$ el menor n tal que L puede ser generado por n elementos. Para enteros $r > 2$ y $k > 1$, se denota por $\mathrm{FD}(r)^k$ la k-ésima potencia directa del reticulado distributivo libre $\mathrm{FD}(r)$ en r generadores. Determinamos $\mathrm{Gm}(\mathrm{FD}(r)^k)$ para muchos pares (r, k) ya sea exactamente o con buena precisión, dando una estimación inferior que se convierte en una estimación superior sumando 1. Por ejemplo, para $(r, k) = (5, 25\,000)$ y $(r, k) = (20,\ 1.489 \cdot 10^{1789})$, $\mathrm{Gm}(\mathrm{FD}(r)^k)$ es 22 y 6 000, respectivamente. Para alcanzar nuestro objetivo, damos estimaciones para el número máximo de copias no-relacionadas dos a dos de algunos posets específicos (llamados posets de segmento completo) en el reticulado de subconjuntos de un conjunto de n elementos. Adicionalmente a resultados análogos anteriores en teoría de reticulados, se menciona también entre las motivaciones una conexión con criptología.

Keywords and Phrases: Free distributive lattice, minimum-sized generating set, small generating set, direct power, Sperner theorem, 3-crown poset, cryptography.

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1 Introduction

This work belongs mainly to *lattice theory* but it also belongs to *extremal combinatorics*. The paper is more or less self-contained; those familiar with M.Sc. level mathematics and the concept of free distributive lattices can read it easily. We are interested in the smallest positive integer $n = n(k, r)$ such that the k-th direct power of the r-generated free distributive lattice is n-generated. In many cases, our estimates give a good approximation or even the exact value of n.

The search for small generating sets has belonged to lattice theory for long; for example, in chronological order, see Gelfand and Ponomarev [9], Strietz [17], Zádori [19, 20], Chajda and Czédli [2], Takách [18], Kulin [13], Czédli and Oluoch [7], and Ahmed and Czédli [1]. See also the surveying parts and the bibliographic sections in [1] and Czédli [5] for further references. If a large lattice L can be generated by few elements, then this lattice has many small generating sets. Czédli [5] and [3] have recently observed that these lattices can be used for cryptography; for a further note on this topic, see Remark 5.3. This fact and the results on small generating sets of lattices in the above-mentioned and some additional papers constitute the *lattice theoretic motivation* of the paper.

There is a *motivation* coming from *extremal combinatorics*, too. The first result on the maximum number $\mathrm{Sp}(U, n)$ of pairwise unrelated (in other words, incomparable) copies of a poset U in the powerset lattice of an n-element finite set was published by Sperner [16] ninety-six years ago. While U is the singleton poset in Sperner's theorem, the *Sperner theorem* (that is, the Sperner *type* theorem) in Griggs, Stahl, and Trotter [11] determines $\mathrm{Sp}(U, n)$ for any finite chain U. For some other finite posets, similar results were obtained by Katona and Nagy [12] and Czédli [4]. In general, the exact value of $\mathrm{Sp}(U, n)$ is rarely known. On the other hand, Katona and Nagy [12] and, independently from them, Dove and Griggs [8] determined the *asymptotic* value of $\mathrm{Sp}(U, n)$. Their celebrated result asserts that for any finite poset U,

$$\mathrm{Sp}(U, n) \sim \frac{1}{|U|}\binom{n}{\lfloor n/2 \rfloor}, \text{ that is, } \lim_{n\to\infty} \frac{1}{|U|}\binom{n}{\lfloor n/2 \rfloor} \cdot \mathrm{Sp}(U, n)^{-1} = 1. \tag{1.1}$$

By the main result of [4], the lattice theoretic motivation and the combinatorial one are strongly connected; see (2.4) later, which we are going to quote from [4]. Here we only mention that in order to get closer to what the title of the paper promises, we need to determine $\mathrm{Sp}(U, n)$ for some rather special posets U.

The asymptotic result (1.1) may suggest that for our special posets U, we can obtain $\mathrm{Sp}(U, n)$ or at least some of its estimates simply by copying what Dove and Griggs [8] or Katona and Nagy [12] did. However, we have three reasons not to follow this plan. First, while several constructions and considerations can lead to the asymptotically same result, we cannot expect a similar experience when dealing with small values of n. Furthermore, concrete (non-asymptotic) calculations and considerations are often harder and their asymptotic counterparts do not offer too much help. For example, while we know for any fixed $a, b \in \mathbb{Z}$ (the set of integers) that, with our vertical-space-saving permanent notation $f_{\mathrm{Sp}}(n) := \binom{n}{\lfloor n/2 \rfloor}$,

$$\binom{n+a}{\lfloor n/2 \rfloor + b} \sim 2^a \cdot \binom{n}{\lfloor n/2 \rfloor} = 2^a f_{\mathrm{Sp}}(n) \quad \text{as } n \to \infty \tag{1.2}$$

and so we can simply work with $2^a f_{\mathrm{Sp}}(n)$ in asymptotic considerations, we have to work with $\binom{n+a}{\lfloor n/2 \rfloor + b}$ in concrete calculations, which is more difficult. (Note at this point that both Dove and Griggs [8] and Katona and Nagy [12] use (1.2).) Second, even though a general construction could be specialized to our particular posets U, we cannot expect to exploit the peculiarities of our U's in this way. Third, an easy-to-read construction with a short and easy argument will hopefully be interesting for the reader, partially because these details are necessary to explain and perform the computations.

Hence, the construction we are going to give for lower estimates is different from those in Dove and Griggs [8] and Katona and Nagy [12]. At some places in the proofs, we are going to point out the difference from [8]; the difference from [12] is clearer. Note that our construction gives better lower estimates for our particular posets U than any of the Dove-Griggs and the Katona-Nagy construction would give, at least for small values of n. (For $n \to \infty$, that is, asymptotically, all the three constructions yield the same lower estimate.) On the other hand, let us emphasize the similarities. While many calculations in this paper are new, most of the ideas in our construction occur in Dove and Griggs [8] and Katona and Nagy [12]; more details will be mentioned right after the proof of Proposition 3.2.

Even though our result allows a big gap between the lower estimate and the upper estimate of $\mathrm{Sp}(U, n)$, this result will suffice to determine the least number n of elements that generate the direct powers $\mathrm{FD}(3)^k$ of $\mathrm{FD}(3)$ with quite a good accuracy, and we can give reasonable estimates on n in case of $\mathrm{FD}(r)^k$.

Figure 1: FD(3) and the 3-crown $W_3 = \mathrm{FSgP}(3, 0, 3) \cong \mathrm{J}(\mathrm{FD}(3))$

2 Basic facts and notations

For $s \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$, let $\mathbb{N}^{\geq s}$ stand for $\{s, s+1, s+2, \dots\}$. Except for $\mathbb{N}^+$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}^+$, $\mathbb{N}^{\geq s}$, and their infinite subsets, all sets and structures in the paper will be assumed to be *finite*. (Sometimes, we repeat this convention for those who read only a part of the paper.) For $r \in \mathbb{N}^{\geq 3}$, the *free distributive lattice on r generators* is denoted by $\mathrm{FD}(r)$; for $r = 3$, it is drawn on the left of Figure 1. A lattice element with exactly one lower cover is called *join-irreducible*. For a lattice L, the *poset* (that is, the *partially ordered set*) of the join-irreducible elements of L is denoted by $\mathrm{J}(L)$. For $L = \mathrm{FD}(3)$, $\mathrm{J}(L)$ consists of the black-filled elements and it is also drawn separately on the right of the figure. For a set H, the *powerset lattice* of H is $(\{Y : Y \subseteq H\}; \cup, \cap)$; it (or its support set) is denoted by $\mathrm{Pow}(H)$. For $n \in \mathbb{N}_0$, the set $\{1, 2, \dots, n\}$ is denoted by $[n]$; note that $[0] = \emptyset$. For x, y in a poset, in particular, for $x, y \in \mathrm{Pow}([n])$, we write $x \parallel y$ to denote that neither $x \leq y$ nor $y \leq x$ holds; in $\mathrm{Pow}([n])$, "$\leq$" is "$\subseteq$". For a poset U, a *copy* of U in $\mathrm{Pow}([n])$ is a subset of $\mathrm{Pow}([n])$ that, equipped with "$\subseteq$", is order isomorphic to U. Two copies of U in $\mathrm{Pow}([n])$ are *unrelated* if for all X in the first copy and all Y in the second copy, $X \parallel Y$. Let us repeat that for $n \in \mathbb{N}_0$ and a poset U, we let

$$\mathrm{Sp}(U, n) := \max\{k : \text{there exist } k \text{ pairwise unrelated copies of } U \text{ in } \mathrm{Pow}([n])\}. \tag{2.1}$$

We often write $\mathrm{C}(n, k)$ instead of $\binom{n}{k}$; especially in text environment and if n or k is a complicated or subscripted expression. The notation "$\mathrm{Sp}(-,-)$" and "$\mathrm{C}(-,-)$" come from Sperner and binomial coefficient, respectively. As usual, $\lfloor\ \rfloor$ and $\lceil\ \rceil$ denote the lower and upper *integer part* functions; for example, $\lfloor 5/3 \rfloor = 1$ and $\lceil 5/3 \rceil = 2$. With our notations, Sperner's theorem [16] asserts that for every $n \in \mathbb{N}_0$,

$$\text{if } U \text{ is the 1-element poset, then } \mathrm{Sp}(U, n) = \binom{n}{\lfloor n/2 \rfloor} =: f_{\mathrm{Sp}}(n). \tag{2.2}$$

Recall that a subset X of a lattice $L = (L; \vee, \wedge)$ is a *generating set* of L if for every Y such that $X \subseteq Y \subseteq L$ and Y is closed with respect to $\vee$ and $\wedge$, we have that $Y = L$. We denote the *size of a minimum-sized generating set* of L by

$$\mathrm{Gm}(L) := \min\{|X| : X \text{ is a generating set of } L\}. \tag{2.3}$$

For $k \in \mathbb{N}^+$, the k-th *direct power* L^k of L consists of the k-tuples of elements of L and the lattice operations are performed componentwise. With our notations, the main result of Czédli [4] asserts that

$$\begin{gathered}\text{for } 2 \leq k \in \mathbb{N}^+ \text{ and a finite distributive lattice } L,\ \mathrm{Gm}(L^k) \\ \text{is the smallest } n \in \mathbb{N}^+ \text{ such that } k \leq \mathrm{Sp}(\mathrm{J}(L), n).\end{gathered} \tag{2.4}$$

It is also clear from [4] that for each finite distributive lattice L, the functions $k \mapsto \mathrm{Gm}(L^k)$ and $n \mapsto \mathrm{Sp}(\mathrm{J}(L), n)$ mutually determine each other, but we do not need this fact in the present paper. The following definition is crucial in the paper.

Definition 2.1. *For $0 \le a < b \le r \in \mathbb{N}_0$ such that $a + 2 \le b$, the* full segment poset $\mathrm{FSgP}(r, a, b)$ *is the poset U defined (up to isomorphism) by the conjunction of the following two rules.*

(A) *r is the smallest integer such that U is embeddable into* $\mathrm{Pow}([r])$*;*

(B) *the subposet $\{X \in \mathrm{Pow}([r]) : a < |X| < b\}$ of $\mathrm{Pow}([r])$ is order isomorphic to U.*

Even though $0 \le a$ in Definition 2.1 could be replaced by $-1 \le a$, we do not do so since the case $a = -1$ would need a different (in fact, easier) treatment; see [4]. Let U be a finite poset, and let $s \in \mathbb{N}^+$. If $f_1, f_2 \colon \mathbb{N}^{\ge s} \to \mathbb{N}_0$ are functions such that $f_1(n) \le \mathrm{Sp}(U, n) \le f_2(n)$ for all $n \in \mathbb{N}^{\ge s}$, then (f_1, f_2) is a *pair of estimates* of the function $\mathrm{Sp}(U, -)$ on $\mathbb{N}^{\ge s}$; in particular, f_1 is a *lower estimate* while f_2 is an *upper estimate* of $\mathrm{Sp}(U, -)$. A reasonably good property of pairs of estimates of $\mathrm{Sp}(U, -)$ is defined as follows:

$$\text{for } s \in \mathbb{N}^+, \text{ a pair } (f_1, f_2) \text{ of estimates is } \textit{separated} \text{ on } \mathbb{N}^{\ge s} \text{ if } f_2(n) \le f_1(n+1) \text{ for all } n \in \mathbb{N}^{\ge s}. \tag{2.5}$$

The following fact is a trivial consequence of (2.4) and for $k \ge 2$, it is implicit in Czédli [4]; see around (4.23) and (4.24) in [4].

Observation 2.2. *Let D be a finite distributive lattice. Denote the poset $\mathrm{J}(D)$ by U, and let $s \in \mathbb{N}^+$. Let (f_1, f_2) be a separated pair of estimates of $\mathrm{Sp}(U, -)$ on $\mathbb{N}^{\ge s}$ such that f_1 (the lower estimate) is strictly increasing on $\mathbb{N}^{\ge s}$. Then, for each $k \in \mathbb{N}^+$ such that $f_1(s) < k$, (f_1, f_2) determines $\mathrm{Gm}(D^k)$ "*with accuracy 1/2*" as follows: Letting n be the unique $n \in \mathbb{N}^+$ such that $f_1(n) < k \le f_1(n+1)$, either $k \le f_2(n)$ and $\mathrm{Gm}(D^k) \in \{n, n+1\}$ or $f_2(n) < k$ and $\mathrm{Gm}(D^k) = n+1$.*

The term "accuracy 1/2" comes from the fact that the distance between the never exact estimate $n + 1/2$ and $\mathrm{Gm}(D^k)$ is always $1/2$.

3 Lower estimates

The easy proof of the following lemma raises the possibility that the lemma might belong to the folklore even though the author has never met it.

Lemma 3.1. *For* $2 \leq r \in \mathbb{N}^+$, $\mathrm{J}(\mathrm{FD}(r))$ *is isomorphic to the poset* $\mathrm{FSgP}(r,0,r)$, *which is defined in Definition 2.1.*

Proof. The smallest element and the largest element of $\mathrm{FD}(r)$ will be denoted by 0_r and 1_r, respectively. Let $S_2 := \{0_r, 1_r\}$; it is a two-element sublattice of $\mathrm{FD}(r)$. Denote by $\{x_1, \ldots, x_r\}$ the set of free generators of $\mathrm{FD}(r)$. Let $\vec{x} := (x_1, \ldots, x_r)$, and let $\vec{\xi} = (\xi_1, \ldots, \xi_r)$ be a vector of variables. Call a subset J of $[r]$ *nontrivial* if $\emptyset \neq J \neq [r]$, and let $\mathrm{Pow}_{\mathrm{nt}}([r]) = \big(\mathrm{Pow}_{\mathrm{nt}}([r]); \subseteq\big)$ stand for the poset formed by the nontrivial subsets of $[r]$. For $J \in \mathrm{Pow}_{\mathrm{nt}}([r])$, let m_J stand for the r-ary lattice term defined by $m_J(\vec{\xi}) := \bigwedge_{i \in J} \xi_i$. Let $X := \{m_J(\vec{x}) : J \in \mathrm{Pow}_{\mathrm{nt}}([r])\}$. As $X \subseteq \mathrm{FD}(r)$, $X = (X; \leq)$ is a subposet of $\mathrm{FD}(r)$.

First, we show that the map $\varphi\colon \mathrm{Pow}_{\mathrm{nt}}([r]) \to X$ defined by $J \to m_J(\vec{x})$ is a dual order isomorphism. The tool we need is simple: Since $\mathrm{FD}(r)$ is free, it follows that whenever $J, K \in \mathrm{Pow}_{\mathrm{nt}}([r])$ and $m_J(\vec{x}) = m_K(\vec{x})$, then $m_J(\vec{y}) = m_K(\vec{y})$ for all $\vec{y} = (y_1, \ldots, y_r) \in S_2^r$, and similarly for "$\geq$" instead of "$=$".

The implication $J \subseteq K \Rightarrow m_J(\vec{x}) \geq m_K(\vec{x})$ is obvious. For the sake of contradiction, suppose that $m_J(\vec{x}) \geq m_K(\vec{x})$ for some $J, K \in \mathrm{Pow}_{\mathrm{nt}}([r])$ but $J \nsubseteq K$. Pick a $j \in J \setminus K$, and let $\vec{y} \in S_2^r$ be the vector for which $y_j = 0_r$ but $y_i = 1_r$ for all $i \in [r] \setminus \{j\}$. Then $m_K(\vec{y}) = 1_r$ but $m_J(\vec{y}) = 0_r$, whereby $m_J(\vec{y}) \ngeq m_K(\vec{y})$. By the tool mentioned above, this contradicts $m_J(\vec{x}) \geq m_K(\vec{x})$ and proves that "$\geq$" in X and "$\subseteq$" in $\mathrm{Pow}_{\mathrm{nt}}([r])$ correspond to each other. In particular, φ is a bijective map as the equality of two elements or subsets can be expressed by these relations. Thus, φ is a dual order isomorphism. The composite of φ and the selfdual automorphism of $\mathrm{Pow}_{\mathrm{nt}}([r])$ defined by $J \mapsto [r] \setminus J$ is an order isomorphism. Hence, $X \cong \mathrm{Pow}_{\mathrm{nt}}([r])$. Since $\mathrm{Pow}_{\mathrm{nt}}([r]) \cong \mathrm{FSgP}(r,0,r)$, we have shown that $X \cong \mathrm{FSgP}(r,0,r)$.

To complete the proof, it suffices to show that $\mathrm{J}(\mathrm{FD}(r)) = X$. Using the tool mentioned earlier and S_2, we obtain that $1_r = x_1 \vee \cdots \vee x_r \notin \mathrm{J}(\mathrm{FD}(r))$ and for every $J \in \mathrm{Pow}_{\mathrm{nt}}([r])$, $m_J(\vec{x}) \notin S_2$. By distributivity, each element of $\mathrm{FD}(r) \setminus S_2$ is the join of meets of some generators or, in other words, a disjunctive normal form of the generators. Clearly, neither the empty meet, nor the empty join, nor the meet of all generators is needed here, whereby there is at least one joinand and each of the joinands is of the form $m_J(\vec{x})$ with $J \in \mathrm{Pow}_{\mathrm{nt}}([r])$. As one joinand is sufficient for the elements of $\mathrm{J}(\mathrm{FD}(r))$, we obtain that $\mathrm{J}(\mathrm{FD}(r)) \subseteq X$.

To show the converse inclusion by way of contradiction, suppose that $m_J(\vec{x}) \in X \setminus \mathrm{J}(\mathrm{FD}(r))$. Then $m_J(\vec{x})$ is the join of some elements of $\mathrm{J}(\mathrm{FD}(r))$ that are smaller than $m_J(\vec{x})$. These elements are of the form $m_{I_j}(\vec{x})$ as $\mathrm{J}(\mathrm{FD}(r)) \subseteq X$. This fact and the dual isomorphism proved in the previous paragraph imply that there are $I_1, \ldots, I_t \in \mathrm{Pow}_{\mathrm{nt}}([r])$ such that $J \subset I_1, \ldots, J \subset I_t$ and $m_J(\vec{x}) = m_{I_1}(\vec{x}) \vee \cdots \vee m_{I_t}(\vec{x})$. As this equality holds for the free generators, it holds as an identity in S_2. However, if we define $\vec{y} \in S_2^r$ by $y_s := 1_r$ if $s \in J$ and $y_s = 0_r$ otherwise, then $m_J(\vec{y}) = 1_r$

but each of the joinands and so the join are 0_r. This contradiction completes the proof of Lemma 3.1. ☐

For $1 \le a < b \le r \in \mathbb{N}^+$ such that $a+2 \le b$ and $n \in \mathbb{N}^{\ge r}$, $\vec{v}$ will denote a vector $(v_0, \dots, v_a; v_b, \dots, v_r)$, so there is gap in the index set of the components. Let $p \in \{-r, -r+1, \dots, r\}$ be a parameter, and note that a binomial coefficient $\mathrm{C}(x_1, x_2)$ is 0 unless $x_1, x_2 \in \mathbb{N}_0$ and $0 \le x_2 \le x_1$. Define

$$\begin{aligned} f^{(p)}_{r,a,b}(n) := \sum_{i=0}^{\lfloor n/r\rfloor -1} \sum_{\substack{\vec{v}\in\{0,\dots,i\}^{r+a-b+2}\\ v_0+\dots+v_a+v_b+\dots+v_r=i}} \frac{i!}{v_0!\cdots v_a!\cdot v_b!\cdots v_r!} \times \\ \times \binom{n-(i+1)r}{p+\lfloor (n-r)/2\rfloor - 0v_0 - 1v_1 - \dots - av_a - bv_b - \dots - rv_r} \times \\ \times \binom{r}{0}^{v_0} \cdots \binom{r}{a}^{v_a} \cdot \binom{r}{b}^{v_b} \cdots \binom{r}{r}^{v_r}, \qquad \text{and} \end{aligned} \tag{3.1}$$

$$f^{(\max)}_{r,a,b}(n) := \max\left\{ f^{(p)}_{r,a,b}(n) : p \in \{-r, -r+1, \dots, r-1, r\}\right\}. \tag{3.2}$$

Proposition 3.2. *For $r \in \mathbb{N}^{\ge 3}$ and $0 \le a < b \le r \in \mathbb{N}^+$ such that $a+2 \le b$, $f^{(\max)}_{r,a,b}(n)$ is a lower estimate of* $\mathrm{Sp}(\mathrm{FSgP}(r,a,b), n)$ *on $\mathbb{N}^{\ge r}$.*

The proof below shows that Proposition 3.2 would still hold if we replaced $\{-r, -r+1, \dots, r-1, r\}$ in (3.2) with $\mathbb{Z}$ but we do not have any example where $\mathbb{Z}$, which would make practical computations longer, is better than $\{-r, -r+1, \dots, r-1, r\}$.

Proof. It suffices to show that for any $p \in \mathbb{Z}$, $f^{(p)}_{r,a,b}(n) \le \mathrm{Sp}(\mathrm{FSgP}(r,a,b), n)$. Take an n-element set M, and denote the quotient $\lfloor n/r \rfloor$ by q. Fix q pairwise disjoint r-element subsets $M_0, \dots, M_{q-1}$ of M, we call them *blocks*, and define $M_q := M \setminus (M_0 \cup \dots \cup M_{q-1})$. Let $h := p + \lfloor (n-r)/2 \rfloor$. For $j \in \{0, \dots, q-1\}$, a subset X of the block M_j is called *small* if $|X| \le a$. Similarly, if $|X| \ge b$, then X is *large* while in the remainder case when $a < |X| < b$, we say that X is *medium-sized*. By an *extremal subset* of M_j we mean a subset that is large or small; so "extremal" is the opposite of "medium-sized". For a subset B of M, $B \cap M_i$ is often denoted by B_i. We say that $(i, B) \in \{0, \dots, q-1\} \times \mathrm{Pow}(M)$ is a *fundamental pair* if

(F1) $|B| = h$, and

(F2) $B_i = \emptyset$ and for each $j \in \{0, \dots, i-1\}$, B_j is extremal (that is, small or large).

Four examples are given in Figure 2, where $n = 54$, $r = 8$, $a = 3$, $b = 6$, $p = 3$, $q = 6$, and $h = 26$. In each of the four parts of this figure, the green-filled solid ovals[1] represent extremal subsets of the appropriate M_j's, $j \in \{0, \dots, i-1\}$, the red dotted oval is a medium-sized subset of M_i, and

[1]Note for a grayscale version: the green-filled ovals contain black numbers in their interiors while the ovals with white numbers are magenta-filled.

(F2) imposes no condition on the subsets represented by magenta-filled solid ovals. Hence, in each of the four examples, the *set component* (that is, the second component, which was denoted by B) of the fundamental pair is the union of the color-filled solid ovals. The *index component* (that is, the first component) is indicated at the top of the figure. Each color-filled solid oval contains the number of elements of the subset B_j that this oval represents. Note, however, that a red dotted oval (regardless the number it contains) in the picture of (i, B) means that $B_i = \emptyset$. (The red dotted ovals will be explained right after (3.3).) Note also that, witnessed by $i = 5$ and $i = 4$ in the figure, the set component does not determine the index component.

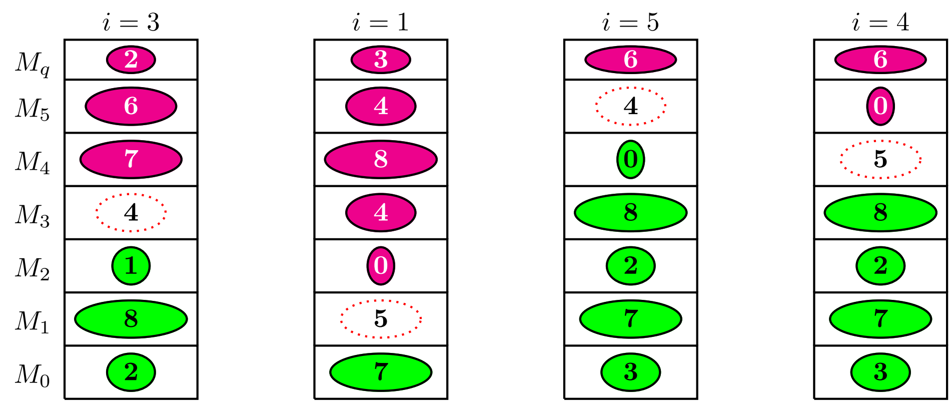

Figure 2: Illustrating the proof of Proposition 3.2 with FSgP$(8, 3, 6)$; $h = 26$, $n = 54$; in each fundamental pair, the set component is the union of the color-filled solid ovals.

For a fundamental pair (i, B), let

$$U(i, B) := \{B \cup X : X \subseteq M_i \text{ and } a < |X| < b\}. \tag{3.3}$$

Clearly, $U(i, B)$ is a copy of FSgP(r, a, b). The role of a red dotted oval in Figure 2 is to represent one of the sets X in (3.3). Now that we have defined our construction, we have to prove that the number of fundamental pairs is $f^{(p)}_{r,a,b}(n)$ and for different fundamental pairs (i, B) and (i', B'), $U(i, B)$ and $U(i', B')$ are unrelated.

To obtain a fundamental pair (i, B), first we choose $i \in \{0, \dots, q-1\}$; this explains the outer summation sign in (3.1). Then for each $j \in \{0, \dots, a, b, \dots, r\}$ we choose the number v_j of the j-element green-filled solid ovals. As there are i green-filled solid ovals, the choice of the vector formed from these v_j's is not quite arbitrary; this explains the subscript of the inner summation sign in (3.1). For example, on the right (that is, in the $i = 4$ part) of Figure 2, $\vec{v} = (v_0, \dots, v_3; v_6, v_7, v_8) = (0, 0, 1, 1; 0, 1, 1)$. The fraction in (3.1) is the multinomial coefficient showing how many ways v_0 zeros, v_1 1's, $\dots$, v_a a's, v_b b's, $\dots$, v_r r's can be ordered. On the right of the figure, this is how many ways the numbers 3, 7, 2, 8 can be written below the red dotted oval (the figure shows only one of these ways). As there is no stipulation on the magenta-filled solid ovals, the binomial coefficient in the middle of (3.1) gives the number of possible unions of the magenta-filled solid ovals, that is, it shows how many ways the system of these ovals can be chosen.

For $j \in \{0, \dots, a, b, \dots, r\}$, a j-element subset (green-filled solid oval) of an r-element block M_t can be chosen in $\mathrm{C}(r, j)$ ways. As there are v_j such subsets and there are several values of j, the product in the last row of (3.1) is the number how many ways the systems of the green-filled solid ovals can be chosen. Therefore, $f^{(p)}_{r,a,b}(n)$ is the number of fundamental pairs as required.

Next, let $(i, B) \neq (i', B')$ be distinct fundamental pairs, $Y = B \cup X \in U(i, B)$, and $Y' = B' \cup X' \in U(i', B')$. For the sake of contradiction, suppose that $Y \subseteq Y'$. If we had that $i = i'$, then $B = (M \setminus M_i) \cap Y \subseteq (M \setminus M_i) \cap Y' = (M \setminus M_{i'}) \cap Y' = B'$, which together with $|B| = h = |B'|$ would give that $B = B'$ and so $(i, B) = (i', B')$, a contradiction. Hence, $i \neq i'$. Observe that $Y \subseteq Y'$ gives that $M_j \cap Y \subseteq M_j \cap Y'$ for all $j \in \{0, \dots, q\}$. Furthermore, $M_j \cap Y = B_j$ for $j \neq i$ while $M_i \cap Y = X$. Similarly, $M_j \cap Y' = B'_j$ for $j \neq i'$ while $M_{i'} \cap Y' = X'$. Hence, $B_j \subseteq B'_j$ and so $|B_j| \leq |B'_j|$ for $j \in \{0, \dots, q\} \setminus \{i, i'\}$, implying that

$$z := \sum_{j \in \{0,\dots,q\} \setminus \{i,i'\}} |B_j| \leq \sum_{j \in \{0,\dots,q\} \setminus \{i,i'\}} |B'_j| =: z'. \tag{3.4}$$

As X is medium-sized, B'_i is extremal, and $X = M_i \cap Y \subseteq M_i \cap Y' = B'_i$, we have that B'_i is large, that is, $b \leq |B'_i|$. Hence, (3.4) gives that $z' + b \leq z' + |B'_i| = |B'|$. Similarly, X' is medium-sized, $B_{i'}$ is extremal, and $B_{i'} = M_{i'} \cap Y \subseteq M_{i'} \cap Y' = X'$, whence $B_{i'}$ is small, that is, $|B_{i'}| \leq a$. Thus, $|B| = z + |B_{i'}| \leq z + a$. Combining the inequalities $a < b$, $|B| \leq z + a$, $z' + b \leq |B'|$, and (3.4), we obtain that

$$|B| \leq z + a < z + b \leq z' + b \leq |B'|.$$

This strict inequality contradicts (F1), completing the proof of Proposition 3.2. ☐

Several ideas and ingredients of the proof above, like the way of partitioning the base set into blocks, are contained in Dove and Griggs [8] and Katona and Nagy [12]. However, even if the construction given in [8] were tailored to our particular posets U, (F1) would fail. The following assertion says that the lower estimate given in Proposition 3.2 is *asymptotically* as good as possible.

Proposition 3.3. *For $r \in \mathbb{N}^{\geq 3}$ and $0 \leq a < b \leq r \in \mathbb{N}^+$ such that $a + 2 \leq b$, $f^{(\max)}_{r,a,b}(n)$ and, for any fixed $p \in \mathbb{Z}$, $f^{(p)}_{r,a,b}(n)$ are asymptotically* $\mathrm{Sp}(\mathrm{FSgP}(r, a, b), n)$ *as $n \to \infty$.*

Proof. With $s := |\mathrm{FSgP}(r, a, b)|$, $s = 2^r - \binom{r}{0} - \cdots - \binom{r}{a} - \binom{r}{b} - \cdots - \binom{r}{r}$. Let κ be a real number such that $\kappa < 1$ but $1 - \kappa$ is very little. As we have that $\sum_{i=0}^{\infty} ((2^r - s)/2^r)^i = 2^r/s$, we can pick an $n_0 \in \mathbb{N}^+$ such that

$$\kappa \cdot 2^r/s \leq \sum_{i=0}^{\lfloor n/r \rfloor - 1} ((2^r - s)/2^r)^i \leq \frac{1}{\kappa} 2^r/s \quad \text{for all } n \text{ such that } n \geq n_0. \tag{3.5}$$

It suffices to deal with $f^{(p)}_{r,a,b}$ for a fixed $p \in \mathbb{Z}$. Using (1.2), we can pick an $n_1 \ge n_0$ such that

$$\begin{aligned} \kappa \cdot f_{\mathrm{Sp}}(n) \cdot 2^{-(i+1)r} &\le \binom{n-(i+1)r}{p + \lfloor (n-r)/2 \rfloor - 0v_0 - 1v_1 - \cdots - av_a - bv_b - \cdots - rv_r} \\ &\le \frac{1}{\kappa} \cdot f_{\mathrm{Sp}}(n) \cdot 2^{-(i+1)r} \end{aligned} \tag{3.6}$$

for all $n \ge n_1$. Let us define an auxiliary function for $n \ge n_1$ and apply the multinomial theorem to it as follows.

$$\begin{aligned} f_{\mathrm{aux}}(n) := \sum_{i=0}^{\lfloor n/r \rfloor -1} & \sum_{\substack{\vec{v} \in \{0,\dots,i\}^{r+a-b+2} \\ v_0 + \cdots + v_a + v_b + \cdots + v_r = i}} \frac{i!}{v_0! \cdots v_a! \cdot v_b! \cdots v_r!} \times \\ & \times f_{\mathrm{Sp}}(n) \cdot 2^{-(i+1)r} \binom{r}{0}^{v_0} \cdots \binom{r}{a}^{v_a} \cdot \binom{r}{b}^{v_b} \cdots \binom{r}{r}^{v_r} \end{aligned} \tag{3.7}$$

$$= \frac{f_{\mathrm{Sp}}(n)}{2^r} \sum_{i=0}^{\lfloor n/r \rfloor -1} (2^r)^{-i} \left(\binom{r}{0} + \cdots + \binom{r}{a} + \binom{r}{b} + \cdots + \binom{r}{r} \right)^i$$

$$= \frac{f_{\mathrm{Sp}}(n)}{2^r} \sum_{i=0}^{\lfloor n/r \rfloor -1} \left(\frac{2^r - s}{2^r} \right)^i . \tag{3.8}$$

Comparing (3.1), (3.6), and (3.7), we obtain that $\kappa f_{\mathrm{aux}}(n) \le f^{(p)}_{r,a,b}(n) \le \kappa^{-1} f_{\mathrm{aux}}(n)$ holds for all $n \ge n_1$. Applying (3.5) to the sum in (3.8), it follows that $\kappa f_{\mathrm{Sp}}(n)/s \le f_{\mathrm{aux}}(n) \le \frac{1}{\kappa} f_{\mathrm{Sp}}(n)/s$. Substituting this pair of inequalities into the previous one, we have that $\kappa^2 f_{\mathrm{Sp}}(n)/s \le f^{(p)}_{r,a,b}(n) \le \kappa^{-2} f_{\mathrm{Sp}}(n)/s$ for all $n \ge n_0$. Letting $\kappa \to 1$, it follows that $f^{(p)}_{r,a,b}(n)$ is asymptotically $f_{\mathrm{Sp}}(n)/s$. So is $\mathrm{Sp}(\mathrm{FSgP}(r,a,b), n)$ by Dove and Griggs [8] and Katona and Nagy [12]. By transitivity, we obtain the required asymptotic equality. The proof of Proposition 3.3 is complete. □

4 Pairs of estimates

For $n \in \mathbb{N}^{\ge 3}$, take the following "discrete 4-dimensional simplex"

$$H_4(n) := \{(t, x_1, x_2, x_3) \in \mathbb{N}_0^4 : x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ t + x_1 + x_2 + x_3 \le n\}. \tag{4.1}$$

Remembering that $[3] := \{1,2,3\}$, define the function $f_{3,4} \colon H_4(n) \to \mathbb{N}_0$ by

$$\begin{aligned} f_{3,4}(t, x_1, x_2, x_3) = \sum_{j \in [3]} (t + x_j)! \cdot (n - t - x_j)! + \sum_{\{j,u\} \subseteq [3],\ j \ne u} (t + x_j + x_u)! \cdot (n - t - x_j - x_u)! \\ - \sum_{(j,u) \in [3] \times [3],\ j \ne u} (t + x_j)! \cdot x_u! \cdot (n - t - x_j - x_u)!\,, \end{aligned} \tag{4.2}$$

and let

$$M_n := \min\{f_{3,4}(t, x_1, x_2, x_3) : (t, x_1, x_2, x_3) \in H_4(n)\}. \tag{4.3}$$

We also define the following three functions:

$$g_r(n) := \left\lfloor \frac{1}{2} f_{\mathrm{Sp}}(n + 2 - r) \right\rfloor, \tag{4.4}$$

$$g_3^*(n) := \lfloor n!/M_n \rfloor, \text{ where } M_n \text{ is given in (4.3), and} \tag{4.5}$$

$$g_3^{**}(n) := \Big\lfloor n! \cdot \Big(3 \cdot \lfloor n/2 \rfloor! \cdot \lceil n/2 \rceil! + 3 \cdot \lfloor (n+2)/2 \rfloor! \cdot \lceil (n-2)/2 \rceil! - 6 \cdot \lfloor n/2 \rfloor! \cdot \lceil (n-2)/2 \rceil! \Big)^{-1} \Big\rfloor. \tag{4.6}$$

Next, based on the notations and concepts given in (2.1), (2.5), Definition 2.1, (4.4), (4.5), and (4.6), we can formulate the main result of the paper.

Theorem 4.1. *For $3 \le r \le n \in \mathbb{N}^+$ and $p \in \{-r, -r+1, \ldots, r-1, r\}$, $g_r(n)$ is an upper estimate while*

$$f^{(p)}_{r,0,r}(n) := \sum_{i=0}^{\lfloor n/r \rfloor - 1} \sum_{j=0}^{i} \binom{i}{j} \binom{n - (i+1)r}{p + \lfloor (n-r)/2 \rfloor - jr} \quad \textit{and, in particular,} \tag{4.7}$$

$$f^{\blacktriangle}_{r,0,r}(n) := f^{(0)}_{r,0,r}(n) \tag{4.8}$$

are lower estimates of $\mathrm{Sp}(\mathrm{FSgP}(r,0,r), n) = \mathrm{Sp}(\mathrm{J}(\mathrm{FD}(r)), n)$ *on* $\mathbb{N}^{\ge r}$. *In particular,*

$$\textit{for all } n \in \mathbb{N}^{\ge r}, \quad f^{\blacktriangle}_{r,0,r}(n) \le f^{(\max)}_{r,0,r}(n) \le \mathrm{Sp}(\mathrm{J}(\mathrm{FD}(r)), n) \le g_r(n). \tag{4.9}$$

For $r = 3$, in addition to the satisfaction of (4.9), $g_3^*(n)$ is also an upper estimate of $\mathrm{Sp}(\mathrm{J}(\mathrm{FD}(3)), n)$ *on $\mathbb{N}^{\ge 3}$. For $n \in \{3, 4, \ldots, 300\}$, $g_3^*(n) = g_3^{**}(n) \le g_3(n)$; in fact, $g_3^{**}(n) < g_3(n)$ for $n \in \{5, 6, \ldots, 300\}$. The pair $(f^{\blacktriangle}_{3,0,3}, g_3)$ is separated for $n \in \mathbb{N}^{\ge 3}$, and so are the pairs $(f^{\blacktriangle}_{3,0,3}, g_3^{**})$ and $(f^{\blacktriangle}_{3,0,3}, g_3^*)$ for $n \in \{3, 4, \ldots, 300\}$. Finally, for $r \in \{3, 4, \ldots, 100\}$, the pair $(f^{\blacktriangle}_{r,0,r}, g_r)$ is separated on the set $\{r, r+1, \ldots, 300\}$.*

It took 952 seconds $\approx$ 16 minutes for a computer, see Footnote 2 later, to show that for $r \in \{3, \ldots, 200\}$ and $n \in \{r, \ldots, 300\}$, $f^{\blacktriangle}_{r,0,r}(n)$ is the same as $f^{(\max)}_{r,0,r}(n)$; the latter is defined in (3.2). Since $f^{\blacktriangle}_{r,0,r}(n)$ is easier to define and much easier to compute than $f^{(\max)}_{r,0,r}(n)$, it is the former that occurs in Theorem 4.1. However, it will be clear from the proof that the theorem holds with $f^{(\max)}_{r,0,r}$ in place of $f^{\blacktriangle}_{r,0,r}$.

Conjecture 4.2. *We guess that $g_3^*(n) = g_3^{**}(n)$ for all $n \in \mathbb{N}^{\ge 3}$ and $g_3^{**}(n) < g_3(n)$ for all $\mathbb{N}^{\ge 5}$.*

Example 5.4 in Section 5 will show that, combining Theorem 4.1 with Observation 2.2, we can determine $\mathrm{Gm}(\mathrm{FD}(3)^k)$ exactly in many cases and we can give a good approximation for $\mathrm{Gm}(\mathrm{FD}(r)^k)$

quite often.

Proof of Theorem 4.1. Substituting $(i-j,j)$ for (v_0,v_r) and observing that the multinomial coefficient becomes a binomial one, it is clear that $f^{(p)}_{r,0,r}$ in (4.7) is a particular case of (3.1). Hence, Lemma 3.1, (3.2), Proposition 3.2, and (4.8) yield the first inequality in (4.9).

Clearly, $\mathrm{FSgP}(r,0,r)\cong \mathrm{Pow}_{\mathrm{nt}}([r])$. Combining this with Lemma 3.1, we obtain that $\mathrm{J}(\mathrm{FD}(r))\cong \mathrm{Pow}_{\mathrm{nt}}([r])$. Take a maximal chain in each of the intervals $[\{1\}, \{1,3,4,\dots,r\}]$ and $[\{2\}, \{2,3,4,\dots,r\}]$ of $\mathrm{Pow}_{\mathrm{nt}}([r])$. These two chains are unrelated and each of them consists of $r-1$ elements. Let $n\in\mathbb{N}^{\geq r}$. With $k:=\mathrm{Sp}(\mathrm{Pow}_{\mathrm{nt}}([r]),n)=\mathrm{Sp}(\mathrm{J}(\mathrm{FD}(r)),n)$, we can take k pairwise unrelated copies of $\mathrm{Pow}_{\mathrm{nt}}([r])$ in $\mathrm{Pow}([n])$. Therefore, there $2k$ pairwise unrelated $(r-1)$-element chains in $\mathrm{Pow}([n])$. By Griggs, Stahl, and Trotter [11], the maximum number of chains with this property is $f_{\mathrm{Sp}}(n+2-r)$. Hence, $2k\leq f_{\mathrm{Sp}}(n+2-r)$, implying the second inequality in (4.9).

In the rest of the proof, $r:=3$. Let $\mathrm{Sym}(n)$ stand for the set of all permutations of $[n]$. For $\vec\sigma=(\sigma_1,\dots,\sigma_n)\in\mathrm{Sym}(n)$ and $i\in\{0,1\dots,n\}$, the i's *initial segment* of $\vec\sigma$ is $\mathrm{Is}(\vec\sigma,i):=\{\sigma_j : j\leq i\}$. For $X\in\mathrm{Pow}([n])$, the *permutation set* associated with X is $\mathrm{Ps}(X):=\{\vec\sigma\in\mathrm{Sym}(n) : X=\mathrm{Is}(\vec\sigma,|X|)\}$. The trivial fact that

$$\text{if } X,Y\in\mathrm{Pow}([n]) \text{ are incomparable (in notation, } X\parallel Y\text{), then } \mathrm{Ps}(X)\cap\mathrm{Ps}(Y)=\emptyset \tag{4.10}$$

was used first by Lubell [14], and then by Griggs, Stahl, and Trotter [11] and some other papers listed in the bibliographic section. To ease the notation, let $W_3:=\mathrm{FSgP}(3,0,3)$ and denote its elements by A,B,C,X,Y,Z according to Figure 1. Let $k:=\mathrm{Sp}(W_3,n)$, and let $W_3^{(1)},\dots,W_3^{(k)}$ be pairwise unrelated copies of W_3 in $\mathrm{Pow}([n])$. For $W_3^{(i)}$, we use the notation $W_3^{(i)}=\{A_i,B_i,C_i,X_i,Y_i,Z_i\}$ in harmony with Figure 1; for example, $A_i\subset X_i$ and $A_i\parallel Z_i$, etc. We claim that $W_3^{(1)},\dots,W_3^{(k)}$ can be chosen so that, for all $i\in[k]$,

$$X_i=A_i\cup B_i,\quad Y_i=A_i\cup C_i,\quad Z_i=B_i\cup C_i, \tag{4.11}$$

$$A_i=X_i\cap Y_i,\quad B_i=X_i\cap Z_i,\quad C_i=Y_i\cap Z_i. \tag{4.12}$$

Assume that the first equality in (4.11) fails. Let $X_i':=A_i\cup B_i$ and define $W_3^{(i)\prime}:=(W_3^{(i)}\setminus\{X_i\})\cup\{X_i'\}$. If we had that $X_i'\subseteq Y_i$, then $B_i\subseteq X_i'\subseteq Y_i$ would be a contradiction. As $Y_i\subseteq X_i'$ would lead to $Y_i\subseteq X_i$ since $X_i'\subseteq X_i$, we conclude that $X_i'\parallel Y_i$. We obtain similarly that $X_i'\parallel Z_i$. So $\{X_i',Y_i,Z_i\}$ is an antichain, and now it follows easily that $W_3^{(i)\prime}$ is a copy of W_3. For $j\in[k]\setminus\{i\}$ and $E\in W_3^{(j)}$, $E\subseteq X_i'$ would lead to $E\subseteq X_i$ while $X_i'\subseteq E$ to $A_i\subseteq E$. So $E\nparallel X_i'$ would lead to contradiction. Hence, $W_3^{(i)\prime}$ and $W_3^{(j)}$ are unrelated, showing that we can change $W_3^{(i)}$ to $W_3^{(i)\prime}$. As there is an analogous treatment for Y_i and Z_i, and we can take $i=1$, $i=2$, …, $i=k$ one by one, (4.11) can be assumed.

Recall that Grätzer [10, Lemma 73], which is a well known easy statement, asserts that whenever a, b, c are elements of a lattice such that $\{a \vee b, a \vee c, b \vee c\}$ is a 3-element antichain, then this antichain generates an 8-element Boolean sublattice in which $\{a \vee b, a \vee c, b \vee c\}$ is the set of coatoms. Therefore, if we apply the dual of the procedure above (that is, if we replace A_i by $X_i \cap Y_i$, etc.), then we reach (4.12) without destroying the validity of (4.11). We have shown that both (4.11) and (4.12) can be assumed; so we assume them in the rest of the proof.

Let $T_i := X_i \cap Y_i \cap Z_i$. By (4.12), T_i is also the intersection of any two of A_i, B_i, and C_i. Hence, letting $A_i^\bullet := A_i \setminus T_i$, $B_i^\bullet := B_i \setminus T_i$, and $C_i^\bullet := C_i \setminus T_i$, it follows from (4.11), (4.12), and $W_3^{(i)} \cong W_3$ that $A_i^\bullet$, $B_i^\bullet$, and $C_i^\bullet$ are pairwise disjoint subsets of $[n]$, none of them is empty, they are disjoint from T_i, and

$$\begin{aligned} &A_i = T_i \cup A_i^\bullet, \quad B_i = T_i \cup B_i^\bullet, \quad C_i = T_i \cup C_i^\bullet, \\ &X_i = T_i \cup A_i^\bullet \cup B_i^\bullet, \quad Y_i = T_i \cup A_i^\bullet \cup C_i^\bullet, \quad Z_i = T_i \cup B_i^\bullet \cup C_i^\bullet. \end{aligned} \tag{4.13}$$

For $i \in [k]$, we let

$$G_i := \mathrm{Ps}(A_i) \cup \mathrm{Ps}(B_i) \cup \mathrm{Ps}(C_i) \cup \mathrm{Ps}(X_i) \cup \mathrm{Ps}(Y_i) \cup \mathrm{Ps}(Z_i). \tag{4.14}$$

As each of $A_i, \ldots, Z_i$ is incomparable with each of $A_j, \ldots, Z_j$ provided that $i \neq j$, (4.10) together with (4.14) imply that

$$\text{for } i, j \in [k], \quad \text{if } i \neq j \text{ then } G_i \cap G_j = \emptyset. \tag{4.15}$$

It follows from (4.15), $G_1 \cup \cdots \cup G_k \subseteq \mathrm{Sym}(n)$, and $|\mathrm{Sym}(n)| = n!$ that

$$\sum_{i \in [k]} |G_i| \leq n! \ . \tag{4.16}$$

Next, for $i \in [k]$, we focus on $|G_i|$. Denote $|T_i|$, $|A_i^\bullet|$, $|B_i^\bullet|$, and $|C_i^\bullet|$ by t_i, a_i, b_i, and c_i, respectively. By (4.13), $|A_i| = t_i + a_i$, $|B_i| = t_i + b_i$, $|C_i| = t_i + c_i$, $|X_i| = t_i + a_i + b_i$, $|Y_i| = t_i + a_i + c_i$, and $|Z_i| = t_i + b_i + c_i$. For any $\vec{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathrm{Ps}(A_i)$, A_i is the set of the first $|A_i| = t_i + a_i$ components of $\vec{\sigma}$; we can choose these components in $(t_i + a_i)!$ ways. To obtain the rest of the components, we can arrange the elements of $[n] \setminus A_i$ in $(n - (t_i + a_i))!$ ways. Hence, $|\mathrm{Ps}(A_i)| = (t_i + a_i)! \cdot (n - t_i - a_i)!$. We obtain similarly that $|\mathrm{Ps}(B_i)| = (t_i + b_i)! \cdot (n - t_i - b_i)!$, $|\mathrm{Ps}(C_i)| = (t_i + c_i)! \cdot (n - t_i - c_i)!$, $|\mathrm{Ps}(X_i)| = (t_i + a_i + b_i)! \cdot (n - t_i - a_i - b_i)!$, $|\mathrm{Ps}(Y_i)| = (t_i + a_i + c_i)! \cdot (n - t_i - a_i - c_i)!$, and $|\mathrm{Ps}(Z_i)| = (t_i + b_i + c_i)! \cdot (n - t_i - b_i - c_i)!$. It follows from (4.10) that the intersection of any three of the six permutation sets considered above is empty since there is no 3-element chain in $W_3^{(i)}$. By (4.10) again, we need to take care of the intersections of two permutation sets associated with comparable members of $W_3^{(i)}$; there are six such intersections as the diagram of W_3 has exactly six edges; see Figure 1. One of the just-mentioned six intersections is $\mathrm{Ps}(A_i) \cap \mathrm{Ps}(X_i)$. For a permutation $\vec{\sigma} \in \mathrm{Ps}(A_i) \cap \mathrm{Ps}(X_i)$, (4.13) yields that there are $|A_i|! = (t_i + a_i)!$ possibilities to arrange the elements of A_i in the first $|A_i|$ places, $b_i!$ many possibilities to arrange the elements of $X_i \setminus A_i = B_i^\bullet$ in the next b_i places, and $(n - t_i - a_i - b_i)!$ possibilities for the rest of entries of $\vec{\sigma}$. Hence, $|\mathrm{Ps}(A_i) \cap \mathrm{Ps}(X_i)| = (t_i + a_i)! \cdot b_i! \cdot (n - t_i - a_i - b_i)!$, and analogously for the other five intersections of two permutation sets.

The considerations above imply that for $i \in [k]$, $|G_i| = f_{3,4}(t_i, a_i, b_i, c_i)$; the function $f_{3,4}$ is defined (4.2). As (t_i, a_i, b_i, c_i) is clearly in $H_4(n)$, (4.3) yields that $M_n \le |G_i|$. This fact and (4.16) imply that $kM_n \le \sum_{i\in[k]} |G_i| \le n!$. Dividing by M_n and taking into account that $k \in \mathbb{N}^+$, we obtain that $\mathrm{Sp}(W_3, n) = k \le \lfloor n!/M_n \rfloor = g_3^*(n)$, as required.

We only guess but could not prove that for all $n \in \mathbb{N}^{\ge 3}$, $f_{3,4}$ takes its minimum on $H_4(n)$ at $(\lfloor (n-2)/2 \rfloor, 1, 1, 1)$; see also Conjecture 4.2. However, we can reduce the computational difficulties by considering the auxiliary function

$$
\begin{aligned}
f_{3,3}(t, x, y) &= (t + x)! \cdot (n - t - x)! + (t + y)! \cdot (n - t - y)! + 2(t + x + y)! \cdot (n - t - x - y)! \\
&\quad - 2(t + x)! \cdot y! \cdot (n - t - x - y)! - 2(t + y)! \cdot x! \cdot (n - t - y - x)!\,. \qquad (4.17)
\end{aligned}
$$

The definition of $H_4(n)$, given in (4.1), and

$$
2f_{3,4}(t, x_1, x_2, x_3) = f_{3,3}(t, x_1, x_2) + f_{3,3}(t, x_2, x_3) + f_{3,3}(t, x_1, x_3)\,, \qquad (4.18)
$$

explain that we are interested in $f_{3,3}$ on the first one of the following two sets,

$$
H_3(n) := \{(t, x, y) \in \mathbb{N}_0^3 : x > 0,\ y > 0,\ t + x + y \le n - 1\} \quad \text{and} \qquad (4.19)
$$

$$
H_3'(n) := \{(t, x, y) \in \mathbb{N}_0^3 : x > 0,\ y \ge x,\ t + x + y \le n - 1\}. \qquad (4.20)
$$

In (4.19), the sum is only at most $n - 1$ since the fourth variable of $f_{3,4}$, which does not occur in $f_{3,3}$, is at least 1. The progress is that $H_3(n)$ has significantly fewer elements than $H_4(n)$, and $H_3'(n)$ has even fewer; this is why we could reach 300 in Theorem 4.1. (Note that a priori, it was not clear that when $2f_{3,4}(t, x_1, x_2, x_3)$ takes its minimum value, then so do all of its summands in (4.18).) Observe that since $f_{3,3}$ is symmetric in its last two variables,

$$
\min\{f_{3,3}(t, x, y) : (t, x, y) \in H_3(n)\} = \min\{f_{3,3}(t, x, y) : (t, x, y) \in H_3'(n)\}. \qquad (4.21)
$$

A straightforward Maple program[2], which benefits from (4.21), shows that

$$\begin{aligned}&\text{for } 3 \le n \le 300,\ f_{3,3} \text{ takes its minimum on the discrete}\\&\text{tetrahedron } H_3(n) \text{ at } (t,x,y) = (\lfloor (n-2)/2 \rfloor, 1, 1).\end{aligned} \tag{4.22}$$

(Note that $f_{3,3}$ takes its minimum at two triples if n is even but only at a unique triple if n is odd.) If $n \in \{3,4,\dots,300\}$ and $(\lfloor (n-2)/2 \rfloor, 1, 1, 1)$ is substituted for (t,x,y,z), then each of the three summands in (4.18) takes its minimal value by (4.22). This allows us to conclude that at $(t,x,y,z) = (\lfloor (n-2)/2 \rfloor, 1, 1, 1)$, $f_{3,4}$ takes its minimum on $H_4(n)$. Thus, for $n \in \{3,4,\dots,300\}$ and for M_n from (4.3),

$$\begin{aligned}M_n &= f_{3,4}(\lfloor (n-2)/2 \rfloor, 1, 1, 1)\\&= 3 \cdot \lfloor n/2 \rfloor! \cdot \lceil n/2 \rceil! + 3 \cdot \lfloor (n+2)/2 \rfloor! \cdot \lceil (n-2)/2 \rceil! - 6 \cdot \lfloor n/2 \rfloor! \cdot \lceil (n-2)/2 \rceil!\ .\end{aligned} \tag{4.23}$$

Combining (4.5), (4.23), and (4.6), we obtain that $g_3^*(n) = g_3^{**}(n)$ for n belonging to the set $\{3,4,\dots,300\}$, as required.

Next, to show that the pair $(f^{\blacktriangle}_{3,0,3}, g_3) = (f^{(0)}_{3,0,3}, g_3)$ is separating, we need to show that $f^{(0)}_{3,0,3}(n+1) - g_3(n) \ge 0$ for all $n \in \mathbb{N}^{\ge 3}$. Depending on the parity of n, there are two cases. If n is of the form $n = 2m+2$ then, reducing the sum in (4.7) to its summands corresponding to $(i,j) = (0,0)$ and $(i,j) = (1,0)$,

$$\begin{aligned}2f^{(0)}_{3,0,3}(n+1) - 2g_3(n) &\ge 2\binom{2m}{m} + 2\binom{2m-3}{m} - \binom{2m+1}{m}\\&= \frac{2\cdot(2m)!}{m!\cdot m!} + \frac{2\cdot(2m-3)!}{m!(m-3)!} - \frac{(2m+1)!}{m!(m+1)!}\\&= \frac{(2m-3)!}{m!(m+1)!}\cdot\alpha, \quad \text{where}\end{aligned} \tag{4.24}$$

$$\begin{aligned}\alpha &= 2(m+1)2m(2m-1)(2m-2) + 2(m+1)m(m-1)(m-2)\\&\quad - (2m+1)2m(2m-1)(2m-2)\\&= 2m^4 + 4m^3 - 14m^2 + 8m = 2m(m+4)(m-1)^2.\end{aligned} \tag{4.25}$$

Hence, both α and the fraction multiplied by α are non-negative for $m \in \mathbb{N}^+$. Thus, $f^{(0)}_{3,0,3}(n+1) - g_3(n) \ge 0$ for $n \ge 4$ even. Similarly, for $n = 2m+1$ odd,

$$2f^{(0)}_{3,0,3}(n+1) - 2g_3(n) \ge 2\binom{2m-1}{m-1} + 2\binom{2m-4}{m-1} - \binom{2m}{m} = \frac{(2m-4)!}{m!m!}\cdot 2m^2(m-1)(m-2).$$

[2] Maple V Release 5 (1997); this computer algebraic program ran on a desktop computer (AMD Ryzen 7 2700X Eight-Core Processor 3.70 GHz) in Windows XP environment simulated by Oracle VM VirtualBox 6.0 (2019) under Windows 10 Pro. The whole computation for (4.21) and the data in Section 5 took 7 hours and 16 minutes; (4.21) in itself needed about 7 hours. The program is available from the (Appendix) Section 6 of the extended arXiv:2309.13783 version of the paper and, at the time of writing, from the author's website.

Therefore, $f^{(0)}_{3,0,3}(n+1) - g_3(n) \geq 0$ for $2 \leq m \in \mathbb{N}^+$, that is, for $n \geq 5$ odd. For $n = 3$, $f^{(0)}_{3,0,3}(n+1) - g_3(n) \geq 0$ is trivial; see also 5.2. We have shown that $(f^{\blacktriangle}_{3,0,3}, g_3)$ is separated.

The already mentioned Maple program has computed $g_3(n)$, $g_3^*(n)$, and $g_3^{**}(n)$ for all $n \in \{3, 4, \dots, 300\}$. This computation proves that $g_3^{**}(n) = g_3^*(n) \leq g_3(n)$ for all these n and $g_3^{**}(n) = g_3^*(n) < g_3(n)$ for $n \in \{5, 6, \dots, 300\}$. These inequalities and that $(f^{\blacktriangle}_{3,0,3}, g_3)$ is separated imply that $(f^{\blacktriangle}_{3,0,3}, g_3^*)$ and $(f^{\blacktriangle}_{3,0,3}, g_3^{**})$ are separated on $\{3, 4, \dots, 300\}$. The same Maple program has computed all the relevant $f^{\blacktriangle}_{r,0,r}(n+1)$ and $g_r(n)$, from which we conclude that for $r \in \{3, 4, \dots, 100\}$, the pair $(f^{\blacktriangle}_{r,0,r}, g_r)$ is separated on the set $\{r, r+1, \dots, 300\}$. The proof of Theorem 4.1 is complete. □

Some comments on this proof are appropriate here. While we could use quite a rough estimation in (4.24) when proving that $(f^{\blacktriangle}_{3,0,3}, g_3)$ is separating on the set $\mathbb{N}^{\geq 3}$, there is no similar possibility for $(f^{\blacktriangle}_{r,0,r}, g_r)$. Indeed, since $f^{\blacktriangle}_{r,0,r}(n+1) = g_r(n)$ for, say, $(r, n) = (20, 56)$ when $f^{\blacktriangle}_{20,0,20}(56+1) = 17\,672\,631\,900 = g_{20}(56)$, no estimation would be possible. As $g_r(n)$ is far from being asymptotically good, it is not worth putting more work into its investigation. While we could use Grätzer [10, Lemma 73] to reach a pleasant situation for $r = 3$, see (4.11) and (4.12), we have no similar tool for $r > 3$; this explains that Theorem 4.1 does not tell too much about upper estimates in case of $r > 3$. Finally, note that even though $f_{3,3}$ in (4.17) is simpler than $f_{3,4}$ in (4.2), the three-variate function $f_{3,3}$ is still too complicated. In particular, we know from computer-assisted calculations that $f_{3,3}$ has several "local minima" on the discrete tetrahedron $H_3(n)$ defined in (4.19); this is our excuse that we could verify Conjecture 4.2 only for $n \leq 300$ and only with a computer.

5 Odds and ends, including some computational results

Theorem 4.1 pays no attention to the case $r = 2$, which is trivial by the following remark. As in (4.4), $g_2(n) := \lfloor f_{\mathrm{Sp}}(n)/2 \rfloor = \lfloor \mathrm{C}(n, \lfloor n/2 \rfloor)/2 \rfloor$.

Remark 5.1. *For $n \in \mathbb{N}^{\geq 2}$, $\mathrm{Sp}(\mathrm{J}(\mathrm{FD}(2)), n) = g_2(n)$.*

Proof. By Lemma 3.1 or trivially, J(FD(2)) is the two-element antichain. Hence, Remark 5.1 follows from Sperner's theorem, which we quoted in (2.2). □

Corollary 5.2. *For $r \in \mathbb{N}^{\geq 3}$ and $k \in \mathbb{N}^{\geq 2}$, let $n \in \mathbb{N}^+$ be the smallest integer such that $k \leq f^{\blacktriangle}_{r,0,r}(n)$; see (4.8). Then for every distributive lattice D generated by r elements, the direct power D^k has an at most n-element generating set.*

Proof. Let k, D, and n be as in the corollary. Since $k \leq f^{\blacktriangle}_{r,0,r}(n)$ is included in the assumption and $f^{\blacktriangle}_{r,0,r}(n) \leq \mathrm{Sp}(\mathrm{J}(\mathrm{FD}(r)), n)$ by Theorem 4.1, it follows from (2.4) that $\mathrm{FD}(r)^k$ can be generated by an at most n element subset Y. Using that $\mathrm{FD}(r)$ is the *free* r-generated distributive lattice, we can pick a surjective (in other words, onto) homomorphism $\varphi\colon \mathrm{FD}(r) \to D$. Then $\varphi^k\colon \mathrm{FD}(r)^k \to D^k$, defined by $(x_1, \dots, x_k) \mapsto (\varphi(x_1), \dots, \varphi(x_k))$, is also a surjective homomorphism. Thus, $\varphi^k(Y)$ generates D^k and $|\varphi^k(Y)| \le |Y| \le n$ proves Corollary 5.2. □

The just-proved corollary and the abundance of large lattices that are easy to describe and easy to work with motivate the following extension of the cryptographic "protocol" outlined in Czédli [5] and, mainly, in [3]. The purpose of the quotation marks here is *to warn the reader*: none of our protocols is fully elaborated and, thus, it does not meet the requirements of nowadays' cryptology. In particular, neither a concrete method of choosing the master key according to some probabilistic distribution is given nor we have proved that the average case withstands attacks; we do not even say that we are close to meeting these requirements. On the other hand, no rigorous average case analysis supports some widely used and, according to experience, safe cryptographic protocols like RSA and AES and, furthermore, many others rely ultimately on the *conjecture* that the complexity class **P** is different from **NP**. This is our excuse to tell a bit more about one of our motivations in Remark 5.3 below. For a lattice L and $\vec{h} = (h_1, \dots, h_k) \in L^k$, $\vec{h}$ is a (k-dimensional) *generating vector* of L if $\{h_1, \dots, h_k\}$ is a generating set of L.

Remark 5.3. *In the* session key exchange protocol *given in Czédli [3][3], the secret master key known only by the communicating parties was a k-dimensional generating vector $\vec{h}$ of the* 2^n*-element Boolean lattice B_n. The point was that* $\mathrm{Gm}(B_n)$*, defined in* (2.3)*, is small, and so there are very many k-dimensional generating vectors $\vec{h}$ if k is a few times, say, seven times larger than* $\mathrm{Gm}(B_n)$. *Here we suggest to add* (A) *or* (B) *to the protocol outlined in [3] and to work in a lattice different from B_n.*

(A) *Choose a medium-sized finite random poset U and an exponent $n \in \mathbb{N}^+$; for example, a 20-element random poset U and $n = 500$ are sufficient. (There are very many 20-element posets; see A000112 in Sloan [15]; the direct link is* `https://oeis.org/A000112`*.) By the well-known structure theorem of finite distributive lattices, see Grätzer [10, Theorem 107], U determines a finite distributive lattice D. Then replace B_n with D^n in the [3]-protocol so that, in addition to $\vec{h}$, U and n also belong to the secret master key.*

(B) *Choose a random poset U of size 100 or so. As in [6], this U determines the huge lattice* $(\mathrm{Quo}^{\le}(U); \subseteq)$ *of quasiorders extending $\le_U$; this lattice can be generated by few elements. Use this lattice instead of B_n. The poset U and a k-dimensional generating vector of* $(\mathrm{Quo}^{\le}(U); \subseteq)$ *constitute the secret master key; otherwise the protocol is the same as in [3].*

[3]At the time of writing, see (4.3) in `https://arxiv.org/abs/2303.10790v3` .

Next, we present some computational data, see Footnote 2; at the "≈" rows, the last decimals are correctly rounded.

(5.1)

n	298	299	300
$f^{\blacktriangle}_{3,0,3}(n) \approx$	$3.919\,720 \cdot 10^{87}$	$7.839\,440 \cdot 10^{87}$	$1.562\,662 \cdot 10^{88}$
$g_3^{**}(n) \approx$	$3.932\,918 \cdot 10^{87}$	$7.865\,747 \cdot 10^{87}$	$1.567\,888 \cdot 10^{88}$
$\frac{g_3^{**}(n)}{f^{\blacktriangle}_{3,0,3}(n)} \approx$	1.003 367 003	1.003 355 705	1.003 344 482

(5.2)

$n =$	3	4	5	6	7	8
$f^{\blacktriangle}_{3,0,3}(n)$	1	1	2	3	6	11
$g_3^{*}(n) = g_3^{**}(n)$	1	1	2	4	7	13
$g_3(n)$	1	1	3	5	10	17
$n =$	9	10	11	12	13	14
$f^{\blacktriangle}_{3,0,3}(n)$	24	42	84	153	306	570
$g_3^{*}(n) = g_3^{**}(n)$	26	46	92	168	333	616
$g_3(n)$	35	63	126	231	462	858
$n =$	15	16	17	18	19	20
$f^{\blacktriangle}_{3,0,3}(n)$	1146	2145	4290	8100	16200	30786
$g_3^{*}(n) = g_3^{**}(n)$	1225	2288	4558	8580	17107	32413
$g_3(n)$	1716	3217	6435	12155	24310	46189

(5.3)

$n =$	4	5	6	7	8	9	10	11	12
$f^{\blacktriangle}_{4,0,4}(n)$	1	1	2	3	6	10	20	36	74
$g_4(n)$	1	1	3	5	10	17	35	63	126
$n =$	13	14	15	16	17	18	19	20	21
$f^{\blacktriangle}_{4,0,4}(n)$	134	268	496	992	1856	3712	7004	14014	26598
$g_4(n)$	231	462	858	1716	3217	6435	12155	24310	46189

(5.4)

$n =$	5	6	7	8	9	10	11	12	13
$f^{\blacktriangle}_{5,0,5}(n)$	1	1	2	3	6	10	20	35	70
$g_5(n)$	1	1	3	5	10	17	35	63	126
$n =$	14	15	16	17	18	19	20	21	22
$f^{\blacktriangle}_{5,0,5}(n)$	127	256	471	942	1758	3516	6620	13240	25095
$g_5(n)$	231	462	858	1716	3217	6435	12155	24310	46189

The computation for the following table took 306 seconds.

n	5 999	6 000
$f^{\blacktriangle}_{20,0,20}(n) \approx$	$7.445\,882\,708\,069 \cdot 10^{1797}$	$1.489\,176\,541\,614 \cdot 10^{1798}$
$g_{20}(n) \approx$	$1.488\,924\,847\,889 \cdot 10^{1798}$	$2.977\,849\,695\,779 \cdot 10^{1798}$

(5.5)

Next, we give some examples; each of them is based on (2.4), Observation 2.2, and one of the computational tables that will be specified.

Example 5.4. (A) *By* (5.2), $\mathrm{Gm}(\mathrm{FD}(3)^{30\,000}) = 20$. *That is, the direct power* $\mathrm{FD}(3)^{30\,000}$ *can be generated by 20 elements but not by 19.*

(B) *By* (5.3), $\mathrm{Gm}(\mathrm{FD}(4)^{20\,000})$ *is either 20 or 21 but we do not know which one.*

(C) *By* (5.4), $\mathrm{Gm}(\mathrm{FD}(5)^{25\,000}) = 22$.

(D) *By* (5.1), $\mathrm{Gm}(\mathrm{FD}(3)^{10^{88}}) = 300$ *(the exponent in the direct power is* 10^{88}*).*

(E) *By* (5.5), $\mathrm{Gm}(\mathrm{FD}(20)^{1.489 \cdot 10^{1798}}) = 6\,000$ *(the exponent is* $1.489 \cdot 10^{1\,798}$*).*

At the time of writing, we know from Sloan [15] (`https://oeis.org/A000372`) that in spite of lots of work by many contributors, the largest integer r for which $|\mathrm{FD}(r)|$ is known is $r = 9$. We mention the following well-known folkloric lower estimate:

$$2^{1024} = 2^{2^{10}} \leq |\mathrm{FD}(20)|. \tag{5.6}$$

Indeed, the free Boolean lattice FB(10) on 10 generators consists of $2^{2^{10}}$ elements and it is lattice-generated by the free generators of FB(10) and their complements. So FB(10) as a distributive lattice is generated by 20 elements, implying (5.6).

Based on (5.6) and the paragraph above, the direct power in part (E) of Example 5.4 consists of an unknown but very large number of elements. However, only 306 seconds were needed to determine the least possible size of its generating sets.

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The Levi-Civita connections of Lorentzian manifolds with prescribed optical geometries

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ABSTRACT

We explicitly derive the Christoffel symbols in terms of adapted frame fields for the Levi-Civita connection of a Lorentzian n-manifold (M, g), equipped with a prescribed optical geometry of Kähler-Sasaki type. The formulas found in this paper have several important applications, such as determining the geometric invariants of Lorentzian manifolds with prescribed optical geometries or solving curvature constraints.

RESUMEN

Derivamos explícitamente los símbolos de Christoffel en términos de los campos de marcos adaptados para la conexión de Levi-Civita de una n-variedad Lorentziana (M, g), equipada con una geometría óptica prescrita de tipo Kähler-Sasaki. Las fórmulas halladas en este artículo tienen diversas aplicaciones importantes, tales como determinar los invariantes geométricos de variedades Lorentzianas con geometrías ópticas prescritas o resolver restricciones sobre la curvatura.

Keywords and Phrases: Levi-Civita connection, optical geometry, congruence of shearfree geodesics, Sasaki manifolds.

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1 Introduction

An *optical geometry*, a notion introduced in the late eighties by Robinson and Trautman, is a geometrical structure that encodes the existence of an electromagnetic plane wave – or an appropriate higher dimensional generalisation [2] – propagating along a prescribed foliation by curves of a Lorentzian manifold. Let us recall the relevant definitions. A *null congruence* on a Lorentzian n-manifold (M, g), $n \geq 3$, is a foliation by curves, which are tangent to some nowhere vanishing null vector field. Given a Lorentzian n-manifold (M, g), $n \geq 3$, a null congruence is called *geodesic shearfree*, or *shearfree* for short, if there is a choice for a nowhere vanishing tangent null vector field p, whose local flow preserves both the codimension one distribution $\mathscr{W} := \mathrm{p}^{\perp_g}$ and the conformal class of the induced degenerate metric $h := g|_{\mathscr{W}\times\mathscr{W}}$ on the spaces $\mathscr{W}_x = \mathrm{p}^{\perp_g}|_x$, $x \in M$. These conditions are equivalent to requiring that the Lie derivative $\mathscr{L}_{\mathrm{p}} g$ has the form

$$\mathscr{L}_{\mathrm{p}} g = fg + \mathrm{p}^\flat \vee \eta \qquad \text{for some function } f \text{ and some 1-form } \eta\ . \tag{1.1}$$

If this holds, the vector field p is also geodesic, *i.e.* $\nabla_{\mathrm{p}}\mathrm{p} = \lambda \mathrm{p}$, and the curves of the congruence are geodesics (see *e.g.* [1, 2, 5, 14]). A quadruple $\mathscr{Q} := (\mathrm{p}, \mathscr{W}, [h], \{g\})$, given by

(a) a nowhere vanishing vector field p, determined up to multiplication by a nowhere zero smooth function f,

(b) a codimension one distribution $\mathscr{W}$,

(c) a conformal class $[h]$ of semi-positive metrics on $\mathscr{W}$,

(d) a non-empty set of Lorentzian metrics $\{g\}$, which are exactly all metrics g with respect to which p is a null vector field with $\mathscr{W} = \mathrm{p}^{\perp_g}$ and $[h] = [g|_{\mathscr{W}\times\mathscr{W}}]$ and both $\mathscr{W}$ and $[h]$ are preserved by the local flow of p,

is an *optical geometry* in the sense of Robinson and Trautman [2, 5, 14] [1]. The Lorentzian metrics g in the set $\{g\}$ are called *compatible with the prescribed optical geometry* $\mathscr{Q}$.

By Robinson's Theorem [8, 13], the shearfree null congruences of a real analytic four dimensional Lorentzian manifold are exactly the foliations by the lines of propagation of electromagnetic plane waves.

Many interesting examples of optical geometries $\mathscr{Q} = (\mathrm{p}, \mathscr{W}, [h], \{g\})$ are provided by connections on principal A-bundles $\pi : M \to S = M/A$ with one-dimensional structure groups $A = \mathbb{R}$ or S^1. On each bundle of this kind, one may consider an optical geometry in which p is the generator of the

[1] As a matter of fact, all four elements of $\mathscr{Q}$ can be recovered just by (i) the 1-dimensional distribution $\mathscr{K}$, which is generated by p and (ii) the set of metrics $\{g\}$, provided that they satisfy appropriate conditions. Thus, the optical geometries can be also defined as such pairs $(\mathscr{K}, \{g\})$ – see the original definition in [14].

action of the group A along the fibers, and $\mathscr{W}$ and $[h]$ are the appropriate A-invariant distribution and conformal class. In this case, the quadruple $\mathscr{M} := (\pi : M \to S, \mathrm{p}, \mathscr{W}, [h])$ is called a *regular shearfree manifold* and a metric $g \in \{g\}$ of the corresponding optical geometry $\mathscr{Q} = (\mathrm{p}, \mathscr{W}, [h], \{g\})$ is said to be a *compatible metric* of $\mathscr{M}$.

The regular shearfree manifolds are important geometric objects not only for their role in Lorentzian geometry, but also for their relations with CR geometry. Indeed, for any regular shearfree manifold $\mathscr{M} := (\pi : M \to S, \mathrm{p}, \mathscr{W}, [h])$, the base manifold $S = M/A$ is naturally equipped with the codimension one distribution $\mathscr{W}^S \subset TS$ and the positive definite conformal metric $[h^S]$ that are obtained by projecting the A-invariant distribution $\mathscr{W} := \mathrm{p}^\perp$ and conformal class $[h]$ onto $S = M/A$. If M is even dimensional and the projected distribution $\mathscr{W}^S \subset TS$ is contact then the regular shearfree manifold $\mathscr{M}$ is called *(maximally) twisting*. For any such $\mathscr{M}$, the corresponding optical geometry $\mathscr{Q} = (\mathrm{p}, \mathscr{W}, [h], \{g\})$ determines a family J^S of complex structures $J^S_x : \mathscr{W}^S_x \to \mathscr{W}^S_x$ on the projected distribution of S, that make S a *strongly pseudoconvex almost CR manifold* (see, *e.g.* [1, 2, 5, 7] and references therein).

Celebrated examples of twisting regular shearfree manifolds are given by the 4-dimensional spacetimes with Taub-NUT metrics and the 4-dimensional Kerr black holes. For such Lorentzian manifolds, the base manifold of the A-bundle $\pi : M \to S$ has an additional remarkable geometric feature: it is a principal bundle $\pi^S = S \to N$ with one dimensional structure group $A' = \mathbb{R}$ or $A' = S^1$, and the base manifold $N = S/A' = M/(A \cdot A')$ has a natural structure of a Kähler manifold. Moreover, the strongly pseudoconvex almost CR manifold $(S, \mathscr{W}^S, J^S)$ is a *regular Sasaki manifold* and the structure group A' of S preserves

(i) the CR structure $(\mathscr{W}^S, J^S)$,

(ii) a contact 1-form θ_o for $\mathscr{W}^S$, *i.e.*, $\mathscr{W}^S = \ker \theta_o$, such that $d\theta_o = \pi^{S*}\omega_o$ for some Kähler form $\omega_o = g_o(J\cdot, \cdot)$ on (N, J);

(iii) the conformal class $[h]$ on $\mathscr{W}$ contains the degenerate metric $h_o = \big((\pi^S \circ \pi)^* g_o\big)|_{\mathscr{W}}$.

The fact that the Taub-NUT and Kerr metrics have these properties is one of the reasons of the interest in twisting regular shearfree manifolds, in which the almost CR manifold $(S, \mathscr{W}^S, J^S)$ is a Sasaki manifold projecting onto a Kähler manifold. Such manifolds are called *of Kähler-Sasaki type* [2].

We recall that, according to classical results in the theory of G-structures, any local isometric invariant of a pseudo-Riemannian manifold is fully determined by the components in orthonormal bases of the Riemann tensor R and its covariant derivatives up to an appropriate order (see, for instance, [9–12, 16] and references therein). Such components are in turn given by the components of g and the Christoffel symbols of ∇^g in a frame field. This observation indicates that the explicit expressions of the Christoffel symbols in appropriate frame fields represent a fundamental tool for studying the compatible metrics of a given regular shearfree manifold of Kähler-Sasaki type and possibly finding solutions of the Einstein (or other physically relevant) equations in this class of metrics.

In this paper, we discuss in great detail the Christoffel symbols of the Levi-Civita connection ∇^g of a compatible metric g of a regular shearfree manifold $\mathscr{M} := (\pi : M \to S, \mathrm{p}, \mathscr{W}, [h])$ of Kähler-Sasaki type. More precisely, we fix a special (locally defined) frame field $(e_1, \ldots, e_n)$, which is well adapted to the optical geometry and is determined only up to a choice of a local frame field on the underlying Kähler manifold $N = M/(A \cdot A')$. Such a frame field has the following two useful properties:

(1) the last two vector fields e_{n-1}, e_n are the generators of the actions of the groups A and A', respectively, and are therefore canonically associated with the considered manifold;

(2) the vector fields e_i, $1 \leq i \leq n-2$, are tangent to the distribution $\mathscr{W}$ at all points and are $A \cdot A'$-invariant, thus projecting onto a frame field $(\widetilde{e}_1, \ldots, \widetilde{e}_{n-2})$ on N.

Note that (1) and (2) allow to minimise the number of parameters that are necessary to determine the components of a compatible metric g. Notice also that, due to the fact that $\mathscr{M}$ is twisting, a frame field satisfying (1) and (2) cannot coincide with a coordinate frame field. This forces us to avoid the use of coordinates in all subsequent computations.

After choosing an adapted frame field of this kind, we write down the general expression of a compatible metric g in terms of its dual frame field and we determine the Christoffel symbols of ∇^g in such frame and coframe fields, using just Koszul's formula and classical results on transformations of Levi-Civita connections under conformal transformations.

The expressions for the Christoffel symbols given in this paper have been originally determined during the preparation of [2] and have been successfully used to derive a coordinate-free characterisation of the generalised Taub-NUT metrics on even dimensional manifolds (see *e.g.* [3] and references therein for other characterisations of the metrics of such a kind). However, the details of the actual computations did not appear in [2] and some formulas of that paper had some minor sign errors – very few indeed and with no effect on any statement and proof. The same explicit (and amended) expressions have been later used in [6] for determining explicit expressions for the components of the Ricci tensor of compatible metrics of a shearfree manifold $\mathscr{M}$ of Kähler-Sasaki type satisfying conditions that generalise Kerr's ansatz for the 4-dimensional rotating black holes. These expressions for the Ricci tensor allowed us to translate the Einstein equations for a compatible metric into equations on its parameters in an adapted frame and to find a large class of exact solutions that naturally includes the classical Kerr black holes. We anticipate a number of further applications of the explicit expressions of these Christoffel symbols and believe that the detailed computations we present in this paper will be a helpful tool for other researchers who are interested in the developments of this field.

The paper is structured into two sections: In section 2, we define the adapted frame fields of a compatible metric, that is the frame fields in which all computations of this paper are performed; In section 3 we derive the explicit list of Christoffel symbols and provide the details of the computations.

2 The general form of a compatible metric on a shearfree manifold of Kähler-Sasaki type

2.1 Notational issues

Consider a shearfree manifold $\mathscr{M} := (\pi : M \to S,\, \mathrm{p}, \mathscr{W},\, [h])$ of Kähler-Sasaki type. We use the following notation:

(1) (N, J, g_o) is the Kähler manifold onto which S projects and $\omega_o = g_o(J\cdot, \cdot)$ is the Kähler form of N [2];

(2) A and A' are the 1-dimensional structure groups of the principal bundles $\pi : M \to S$ and $\pi^S : S \to N$, respectively;

(3) p_o and q_o^S are the fundamental vector fields of the principal bundles $\pi : M \to S$ and $\pi^S : S \to N$, corresponding to the element of the standard basis of $Lie(A) = Lie(A') = \mathbb{R}$. This means that $\Phi_s^{\mathrm{p}_o}(x) = e^s(x)$, $x \in M$, and $\Phi_s^{\mathrm{q}_o^S}(y) = e^s(y)$, $y \in S$;

(4) θ_o is the contact A'-invariant 1-form on S satisfying the conditions

$$d\theta_o = \pi^{S*}\omega_o \ , \qquad \theta_o(\mathrm{q}_o) = 1 \ , \qquad \ker \theta_o|_x = \mathscr{W}_x^S \ \ , \ x \in S \ ; \tag{2.1}$$

and ϑ_o is the pull-back $\vartheta_o = \pi^*(\theta_o)$ of θ_o on M.

It is important to note that $\mathscr{W}^S$ is an A'-invariant horizontal distribution on the principal bundle $\pi^S : S \to N$, and it is therefore a connection for this bundle. The associated connection 1-form is θ_o and its curvature 2-form is $d\theta_o = \pi^{S*}\omega_o$.

For what concerns the A-bundle $\pi : M \to S$, throughout the paper *we assume that it is trivial and equipped with the natural* flat *connection of a Cartesian product.* This apparently restrictive condition can be always locally satisfied replacing S by an open subset $\mathscr{V} \subset S$, on which the bundle

[2]Note that there is a sign difference in the definition of ω_o w.r.t. [2]. There it is defined as $\omega_o := g_o(\cdot, J\cdot)$.

is trivialisable, and identifying $\pi : M \to S$ with the trivial bundle $\pi : \pi^{-1}(\mathscr{V}) \simeq \mathscr{V} \times A \to \mathscr{V}$ equipped with the standard flat connection.

We denote by $\mathscr{H}_o$ the horizontal distribution of the flat connection of $\pi : M \to S$.

For any given vector field X on the Kähler manifold N, we denote by

– $X^{(S)}$ the unique A'-invariant horizontal vector field in $\mathscr{W}^S \subset S$ projecting onto X;

– $\widehat{X}$ the unique A-invariant horizontal vector field in $\mathscr{H}$ projecting onto $X^{(S)}$ and thus also onto X; note that, by definition of $\mathscr{W}^S$, the vector field $\widehat{X}$ takes values in $\mathscr{H}_o \cap \mathscr{W}$.

The unique A-invariant horizontal vector field in $\mathscr{H}_o$ projecting onto q_o^S is denoted by q_o.

Owing to the A- and A'-invariance of the connections of $\pi : M \to S$ and $\pi^S : S \to N$ and the properties of the connection 1-form θ_o, for any pair of vector fields X, Y on N the following Lie bracket relations hold [3]:

$$[\widehat{X}, \widehat{Y}] - \widehat{[X, Y]} = -g_o(JX, Y)\mathrm{q}_o \ , \qquad [\widehat{X}, \mathrm{p}_o] = [\widehat{X}, \mathrm{q}_o] = [\mathrm{p}_o, \mathrm{q}_o] = 0 \ . \tag{2.2}$$

2.2 The adapted frame fields

Consider a frame field $(E_1, \ldots, E_{n-2})$ on an open set $\mathscr{V} \subset N$ of the Kähler manifold and the corresponding lifted vector fields $(\widehat{E}_1, \ldots, \widehat{E}_{n-2})$ on M, taking values in the distribution $\mathscr{W}' = \mathscr{H} \cap \mathscr{W}$. The vector fields of the $(n-1)$-tuple $(\widehat{E}_1, \ldots, \widehat{E}_{n-2}, \mathrm{p}_o)$ are pointwise linearly independent and hence give linear frames for the spaces $\mathscr{W}_x \subset T_xM$, $x \in \mathscr{U} = (\pi^S \circ \pi)^{-1}(\mathscr{V})$. Since q_o projects onto q_o^S and q_o^S is transversal to $\mathscr{W}^S = \pi_*(\mathscr{W})$, the vector fields of the n-tuple

$$(\widehat{E}_1, \ldots, \widehat{E}_{n-2}, \mathrm{p}_o, \mathrm{q}_o) \tag{2.3}$$

are pointwise linearly independent and determine a frame field on $\mathscr{U}$. We call (2.3) the *adapted frame field of $\mathscr{M}$ determined by the frame field (E_i) on N.*

Note that, due to (2.2), the Lie brackets between any two vector fields of an adapted frame have the form

$$[\widehat{E}_i, \widehat{E}_j] = c_{ij}^k \widehat{E}_k - g_o(JE_i, E_j)\mathrm{q}_o \ , \qquad [\widehat{E}_i, \mathrm{p}_o] = [\widehat{E}_i, \mathrm{q}_o] = [\mathrm{p}_o, \mathrm{q}_o] = 0 \ , \tag{2.4}$$

where the c_{ij}^k are the functions such that $[E_i, E_j] = c_{ij}^k E_k$.

The dual coframe field of $(\widehat{E}_1, \ldots, \widehat{E}_{n-2}, \mathrm{p}_o, \mathrm{q}_o)$ is denoted by $(\widehat{E}^1, \ldots, \widehat{E}^{n-2}, \mathrm{p}_o^*, \mathrm{q}_o^*)$. Since the dual 1-form q_o^* satisfies $\mathrm{q}_o^*(\mathrm{q}_o) = 1$ and vanishes identically on $\mathscr{W}$ (because $\mathscr{W}$ is spanned by the

[3] The Lie bracket $[\widehat{X}, \widehat{Y}]$ differs by a sign from the one used in [2]. Since in both papers, it is assumed $d\theta_o = \omega_o$, the sign difference is a consequence of the different definitions of the Kähler form ω_o.

$\widehat{E}_i$ and p_o), it has the same kernel and takes the same value on q_o as the 1-form ϑ_o. Thus

$$\mathrm{q}_o^* = \vartheta_o \tag{2.5}$$

for any *choice of the adapted frame* $(\widehat{E}_i, \mathrm{p}_o, \mathrm{q}_o)$.

2.3 Parameterisation of the compatible metrics

Let (E_i) be a (local) frame field on N and denote by $(\widehat{E}_1, \ldots, \widehat{E}_{n-2}, \mathrm{p}_o, \mathrm{q}_o)$ the corresponding adapted frame field for $\mathscr{M}$. Since we are assuming that $\mathscr{M}$ is of Kähler-Sasaki type, the conformal class $[h]$ consists of the degenerate metrics on $\mathscr{W}$ having the form

$$h = \sigma(\pi^S \circ \pi)^*(g_o)|_{\mathscr{W}}, \qquad \sigma = \text{conformal scaling factor} . \tag{2.6}$$

By the results in [2, Section 2.5] (see also [6]), the compatible Lorentzian metrics on $\mathscr{M}$ are locally in one-to-one correspondence with the pairs (h, q) given by

- a degenerated metric h on $\mathscr{W}$ as in (2.6):
- a vector field q, which is transversal to the distribution $\mathscr{W} = \mathscr{W}' + \mathbb{R}\mathrm{p}_o$, *i.e.*, of the form

$$\mathrm{q} := a\mathrm{q}_o + b\mathrm{p}_o + c^i\widehat{E}_i , \qquad a \neq 0 . \tag{2.7}$$

More precisely, given the conformal factor σ and the vector field q, the corresponding compatible metric $g = g^{(\sigma,\mathrm{q})}$ is the unique Lorentzian metric satisfying conditions

$$\begin{aligned} &g(\widehat{X}, \widehat{Y}) = \sigma g_o(X, Y) , \qquad g(\widehat{X}, \mathrm{p}_o) = g(\mathrm{p}_o, \mathrm{p}_o) = 0 , \\ &g(\widehat{X}, \mathrm{q}) = 0 , \qquad g(\mathrm{p}_o, \mathrm{q}) = 1 , \qquad g(\mathrm{q}, \mathrm{q}) = 0 . \end{aligned} \tag{2.8}$$

From (2.7) and the first line of (2.8), the second line in (2.8) is equivalent to

$$\begin{aligned} &g(\widehat{X}, \mathrm{q}_o) = -\frac{c^i\sigma}{a} g_o(X, E_i) , \qquad g(\mathrm{p}_o, \mathrm{q}_o) = \frac{1}{a} , \\ &g(\mathrm{q}_o, \mathrm{q}_o) = -2\frac{b}{a^2} + \frac{1}{a^2} c^i c^j \sigma g_o(E_j, E_i) . \end{aligned} \tag{2.9}$$

Introducing the shorter notation

$$\alpha := \frac{2}{a\sigma} , \qquad \beta := \frac{2}{\sigma}\left(-2\frac{b}{a^2} + \frac{1}{a^2} c^i c^j \sigma g_o(E_j, E_i)\right) , \qquad \gamma^i := -2\frac{c^i}{a} , \tag{2.10}$$

we get that $g = g^{(\sigma,\mathrm{q})}$ is the unique Lorentzian metric satisfying the condition

$$\begin{gathered} g(\widehat{X},\widehat{Y}) = \sigma g_o(X,Y)\ , \qquad g(\widehat{X},\mathrm{p}_o) = g(\mathrm{p}_o,\mathrm{p}_o) = 0\ , \qquad g(\mathrm{p}_o,\mathrm{q}_o) = \frac{\sigma\alpha}{2}\ , \\ g(\mathrm{q}_o,\widehat{X}) = \frac{\sigma\gamma^i}{2} g_o(X,E_i)\ , \qquad g(\mathrm{q}_o,\mathrm{q}_o) = \frac{\sigma}{2}\beta\ . \end{gathered} \tag{2.11}$$

This means that g has the form

$$\begin{aligned} g &= \sigma g_o(E_i,E_j)\widehat{E}^i \vee \widehat{E}^j + \mathrm{q}_o^* \vee \left(\sigma\alpha \mathrm{p}_o^* + \sigma\gamma^i g_o(E_i,E_k)\widehat{E}^k + \frac{\sigma\beta}{2}\mathrm{q}_o^*\right) \\ &= \sigma\left\{(\pi^S\circ\pi)^*(g_o)\big|_{\mathscr{W}'} + \vartheta_o \vee \left(\alpha \mathrm{p}_o^* + \gamma^i g_o(E_k,E_i)\widehat{E}^k + \frac{\beta}{2}\vartheta_o\right)\right\}. \end{aligned} \tag{2.12}$$

The expression (2.12) gives a convenient parameterisation in terms of the $(n+1)$-tuple of smooth functions $(\sigma,\alpha,\beta,\gamma^i)$ for the compatible metrics of $\mathscr{M} = (\pi : M \to S, \mathrm{p}, \mathscr{W}, [h])$. We emphasise that, conversely, any metric having the form (2.12), for some $\sigma > 0$ and $\alpha \neq 0$, is a compatible metric. Indeed, it is associated with the conformal factor σ and with $\mathrm{q} = a\mathrm{q}_o + b\mathrm{p}_o + c^i\widehat{E}_i$ where a, b, c^j are solutions to (2.10) for given α, β, γ^i. They are

$$a = \frac{2}{\alpha\sigma}\ , \qquad b := -\frac{\beta}{\alpha^2\sigma} + \frac{1}{2\alpha^2\sigma}\gamma^i\gamma^j g_o(E_j,E_i)\ , \qquad c^i = -\frac{\gamma^i}{\alpha\sigma}\ .$$

3 The Christoffel symbols in an adapted frame field of the Levi-Civita connection of a compatible Lorentzian metric

3.1 The complete list of the Christoffel symbols

Let $\mathscr{M} = (\pi : M \to S, \mathrm{p}, \mathscr{W}, [h])$ be a twisting regular shearfree manifold of Kähler-Sasaki type, with S projecting onto the Kähler manifold (N, J, g_o). Let also (E_i) be a frame field on an open set $\mathscr{V} \subset N$ and $(X_A) = (\widehat{E}_1,\ldots,\widehat{E}_{n-2},\mathrm{p}_o,\mathrm{q}_o)$ the corresponding adapted frame field on $\mathscr{U} = (\pi^S\circ\pi)^{-1}(\mathscr{V}) \subset M$. We use the notation g_{ij}, ω_{ij}, J_i^j, c_{ij}^k for the functions defined by

$$g_{ij} := g_o(E_i,E_j)\ , \qquad \omega_{ij} := g_o(JE_i,E_j)\ , \qquad JE_i = J_i^j E_j\ , \qquad [E_i,E_j] = c_{ij}^k E_k\ .$$

For what concerns the Christoffel symbols $\mathbf{\Gamma}_{AB}^C$ (*i.e.*, the functions defined by $\nabla_{X_A}X_B = \mathbf{\Gamma}_{AB}^C X_C$), we are going to use the convention that $\mathbf{\Gamma}_{ij}^m$ denotes the function which gives the component of $\nabla_{\widehat{E}_i}\widehat{E}_j$ in the direction of $\widehat{E}_m$, $\mathbf{\Gamma}_{ij}^{\mathrm{p}_o}$ is the function that gives the component of $\nabla_{\widehat{E}_i}\widehat{E}_j$ in the direction of p_o, $\mathbf{\Gamma}_{ij}^{\mathrm{q}_o}$ is the function giving the component of $\nabla_{\widehat{E}_i}\widehat{E}_j$ in the direction of q_o, and so on.

Our main result is the following:

Proposition 3.1. *Let g be a compatible metric for $\mathscr{M}$, hence of the form* (2.12) *for an* $(n+1)$*-tuple of smooth functions* $(\sigma, \alpha, \beta, \gamma^i)$ *on* $\mathscr{U}$*, with* $\sigma > 0$ *and* $\alpha \neq 0$ *at all points. The Christoffel symbols* $\mathbf{\Gamma}_{AB}^{\ C}$ *of the Levi-Civita connection of g in the frame field* $(X_A) = (\widehat{E}_i, \mathrm{p}_o, \mathrm{q}_o)$ *are given by*

$$\begin{aligned}\mathbf{\Gamma}_{ij}^{\ m} &= g^{mk} g_o(\nabla^o_{E_i} E_j, E_k) + g^{mk} S_{ij|k} + \frac{\gamma^m \omega_{ij}}{4} + \frac{1}{2\sigma}\widehat{E}_i(\sigma)\delta^m_j + \frac{1}{2\sigma}\widehat{E}_j(\sigma)\delta^m_i \\ &\quad - \frac{g_{ij}}{2\sigma}\left(g^{mk}\widehat{E}_k(\sigma) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\sigma)\right) ,\end{aligned} \tag{3.1}$$

where $S_{ij|k}$ *is defined by*

$$S_{ij|k} := \frac{\gamma^\ell}{4} g_o(JE_i, E_k) g_o(E_\ell, E_j) + \frac{\gamma^\ell}{4} g_o(JE_j, E_k) g_o(E_\ell, E_i) - \frac{\gamma^\ell}{4} g_o(JE_i, E_j) g_o(E_\ell, E_k) ,$$

$$\begin{aligned}\mathbf{\Gamma}_{ij}^{\ \mathrm{p}_o} &= \frac{1}{2\alpha}\widehat{E}_i(\gamma^k g_{jk}) + \frac{1}{2\alpha}\widehat{E}_j(\gamma^k g_{ik}) - \frac{1}{4\alpha}\gamma^m\gamma^k g_{mk}\omega_{ij} - \frac{\gamma^m}{\alpha} g_o(\nabla^o_{E_i}E_j, E_m) - \frac{\gamma^m}{\alpha} S_{ij|m} \\ &\quad - \frac{g_{ij}}{2\sigma}\left(\frac{2}{\alpha}\mathrm{q}_o(\sigma) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\sigma) - \frac{\gamma^m}{\alpha}\widehat{E}_m(\sigma)\right) ,\end{aligned} \tag{3.2}$$

$$\mathbf{\Gamma}_{ij}^{\ \mathrm{q}_o} = -\frac{\omega_{ij}}{2} - \frac{g_{ij}}{\alpha\sigma}\mathrm{p}_o(\sigma) , \tag{3.3}$$

$$\mathbf{\Gamma}_{i\mathrm{p}_o}^{\ m} = \mathbf{\Gamma}_{\mathrm{p}_o i}^{\ m} = \frac{\alpha g^{mk}\omega_{ik}}{4} + \frac{1}{2\sigma}\mathrm{p}_o(\sigma)\delta^m_i , \tag{3.4}$$

$$\mathbf{\Gamma}_{i\mathrm{p}_o}^{\ \mathrm{p}_o} = \mathbf{\Gamma}_{\mathrm{p}_o i}^{\ \mathrm{p}_o} = \frac{1}{2\alpha}\widehat{E}_i(\alpha) + \frac{1}{2\alpha}\mathrm{p}_o(\gamma^k)g_{ik} - \frac{\gamma^m\omega_{im}}{4} + \frac{1}{2\sigma}\widehat{E}_i(\sigma) , \tag{3.5}$$

$$\mathbf{\Gamma}_{i\mathrm{p}_o}^{\ \mathrm{q}_o} = \mathbf{\Gamma}_{\mathrm{p}_o i}^{\ \mathrm{q}_o} = 0 , \tag{3.6}$$

$$\begin{aligned}\mathbf{\Gamma}_{i\mathrm{q}_o}^{\ m} = \mathbf{\Gamma}_{\mathrm{q}_o i}^{\ m} &= \frac{g^{mk}}{4}\widehat{E}_i(\gamma^t g_{tk}) - \frac{g^{mk}}{4}\widehat{E}_k(\gamma^t g_{ti}) - \frac{\gamma^\ell}{4}c^t_{ir}g_{t\ell}g^{mr} + \frac{g^{mk}\omega_{ik}}{4}\beta - \\ &\quad - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^t)g_{ti} + \frac{1}{2\sigma}\mathrm{q}_o(\sigma)\delta^m_i - \frac{\gamma^t}{4\sigma}g_{ti}\left(g^{mk}\widehat{E}_k(\sigma) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\sigma)\right) ,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\mathbf{\Gamma}_{i\mathrm{q}_o}^{\ \mathrm{p}_o} = \mathbf{\Gamma}_{\mathrm{q}_o i}^{\ \mathrm{p}_o} &= \frac{1}{2\alpha}\widehat{E}_i(\beta) + \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\widehat{E}_i(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\gamma^t)g_{it} - \frac{1}{2\alpha^2}\beta\widehat{E}_i(\alpha) + \\ &\quad + \frac{1}{2\alpha^2}\beta\mathrm{p}_o(\gamma^t)g_{it} - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\gamma^t g_{tm}) + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\gamma^t g_{it}) + \frac{\gamma^m\gamma^t}{4\alpha}g_{t\ell}c^\ell_{im} + \frac{\gamma^m}{4\alpha}\omega_{im}\beta - \\ &\quad - \frac{\gamma^t}{4\sigma}g_{ti}\left(\frac{2}{\alpha}\mathrm{q}_o(\sigma) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\sigma) - \frac{\gamma^m}{\alpha}\widehat{E}_m(\sigma)\right) ,\end{aligned} \tag{3.8}$$

$$\mathbf{\Gamma}_{i\mathrm{q}_o}^{\ \mathrm{q}_o} = \mathbf{\Gamma}_{\mathrm{q}_o i}^{\ \mathrm{q}_o} = \frac{1}{2\alpha}\widehat{E}_i(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\gamma^t)g_{it} + \frac{1}{2\sigma}\widehat{E}_i(\sigma) - \frac{\gamma^t g_{ti}}{2\alpha\sigma}\mathrm{p}_o(\sigma) , \tag{3.9}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ m} = 0 , \tag{3.10}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \mathrm{p}_o} = \mathrm{p}_o(\log(\alpha\sigma)) , \tag{3.11}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \mathrm{q}_o} = 0 , \tag{3.12}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ m} = \mathbf{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ m} = \frac{1}{4}\mathrm{p}_o(\gamma^m) - \frac{g^{mk}}{4}\widehat{E}_k(\alpha) - \frac{\alpha}{4\sigma}\left(g^{mk}\widehat{E}_k(\sigma) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\sigma)\right) , \tag{3.13}$$

$$\begin{aligned}\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ \mathrm{p}_o} = \mathbf{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ \mathrm{p}_o} &= \frac{1}{2\alpha}\mathrm{p}_o(\beta) - \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\alpha) + \frac{1}{2\sigma}\mathrm{q}_o(\sigma) - \\ &\quad - \frac{1}{2\sigma}\left(\mathrm{q}_o(\sigma) + \frac{1}{2\alpha}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\sigma) - \frac{\gamma^m}{2}\widehat{E}_m(\sigma)\right) ,\end{aligned} \tag{3.14}$$

$$\boldsymbol{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \boldsymbol{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ \ \mathrm{q}_o} = 0 \ , \tag{3.15}$$

$$\boldsymbol{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\ \ m} = \frac{g^{mk}}{2}\mathrm{q}_o(\gamma^i)g_{ik} - \frac{g^{mk}}{4}\widehat{E}_k(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\beta) - \frac{\beta}{4\sigma}\left(g^{mk}\widehat{E}_k(\sigma) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\sigma)\right) \ , \tag{3.16}$$

$$\boldsymbol{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \mathrm{p}_o} = \frac{1}{2\alpha}\mathrm{q}_o(\beta) + \frac{1}{2\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{q}_o(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\beta) - \frac{1}{\alpha^2}\beta\mathrm{q}_o(\alpha) + \frac{\beta}{2\alpha^2}\mathrm{p}_o(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\beta) - \frac{\beta}{2\sigma}\left(\frac{1}{\alpha}\mathrm{q}_o(\sigma) + \frac{1}{2\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\sigma) - \frac{\gamma^m}{2\alpha}\widehat{E}_m(\sigma)\right) \ , \tag{3.17}$$

$$\boldsymbol{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \frac{1}{\alpha}\mathrm{q}_o(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\beta) + \frac{1}{\sigma}\mathrm{q}_o(\sigma) - \frac{\beta}{2\alpha\sigma}\mathrm{p}_o(\sigma) \ . \tag{3.18}$$

The proof will be carried out in three steps, which we provide in the next subsections. In the first step we compute all covariant derivatives $\nabla_{X_A}X_B$ determined by two vector fields of the adapted frame field $(X_A) = (\widehat{E}_i, \mathrm{p}_o, \mathrm{q}_o)$ *under the assumption* $\sigma \equiv 1$. In the second step, the determined covariant derivatives are used to compute the Christoffel symbols $\boldsymbol{\Gamma}_{AB}^{\ \ C}$, *still under the condition* $\sigma \equiv 1$. In the concluding third step, the Christoffel symbols $\boldsymbol{\Gamma}_{AB}^{\ \ C}$ are determined with no restriction on σ by using classical transformation formulas for the Levi-Civita covariant derivatives under conformal changes of the metric.

3.2 The first step

By Koszul's formula, for any triple of vector fields X_1, X_2, X_3,

$$g(\nabla_{X_1}X_2, X_3) = \frac{1}{2}\Big(X_1(g(X_2,X_3)) + X_2(g(X_1,X_3)) - X_3(g(X_1,X_2)) - g([X_1,X_3],X_2) - g([X_2,X_3],X_1) + g([X_1,X_2],X_3)\Big) \ . \tag{3.19}$$

Using this formula, we may determine the functions $g(\nabla_{X_1}X_2, X_3)$, for a compatible metric g *with* $\sigma \equiv 1$, for with any choice of X_1, X_2, X_3 in a set of vector fields of the form

$$\{ \ \widehat{X}, \mathrm{p}_o, \mathrm{q}_o, \ \text{where } \widehat{X} \text{ is the lift of a vector field } X \text{ on } N \ \} \ .$$

We get the following expressions:

$$\nabla_{\widehat{X}}\widehat{Y}: \qquad g(\nabla_{\widehat{X}}\widehat{Y}, \widehat{Z}) = g_o(\nabla^o_X Y, Z) + g(S_{XY}, Z) \ , \tag{3.20}$$

$$g(\nabla_{\widehat{X}}\widehat{Y}, \mathrm{p}_o) = -\frac{\alpha}{4}g_o(JX, Y) \ , \tag{3.21}$$

$$g(\nabla_{\widehat{X}}\widehat{Y}, \mathrm{q}_o) = \frac{1}{4}\widehat{X}(\gamma^k g_o(Y, E_k)) + \frac{1}{4}\widehat{Y}(\gamma^k g_o(X, E_k)) - \frac{1}{4}\beta g_o(JX, Y) \ , \tag{3.22}$$

where S is the tensor field of type $(0,3)$ on N, defined by

$$g(S_{XY},Z) := \frac{\gamma^j}{4} g_o(JX,Z) g_o(E_j,Y) + \frac{\gamma^j}{4} g_o(JY,Z) g_o(E_j,X) - \frac{\gamma^j}{4} g_o(JX,Y) g_o(E_j,Z) \ ;$$

$$\nabla_{\widehat{X}}\mathrm{p}_o : \quad g(\nabla_{\widehat{X}}\mathrm{p}_o, \widehat{Z}) = \frac{\alpha}{4} g_o(JX,Z) \ , \tag{3.23}$$

$$g(\nabla_{\widehat{X}}\mathrm{p}_o, \mathrm{p}_o) = 0 \ , \tag{3.24}$$

$$g(\nabla_{\widehat{X}}\mathrm{p}_o, \mathrm{q}_o) = \frac{1}{4}\widehat{X}(\alpha) + \frac{1}{4}\mathrm{p}_o(\gamma^i) g_o(X,E_i) \ ; \tag{3.25}$$

$$\nabla_{\widehat{X}}\mathrm{q}_o : \quad g(\nabla_{\widehat{X}}\mathrm{q}_o, \widehat{Z}) = \frac{1}{4}\widehat{X}(\gamma^t g_o(E_t,Z)) - \frac{1}{4}\widehat{Z}(\gamma^i g_o(X,E_i)) - \\ - \frac{1}{4}\gamma^t g_o([X,Z],E_t) + \frac{1}{4}\beta g_o(JX,Z) \ , \tag{3.26}$$

$$g(\nabla_{\widehat{X}}\mathrm{q}_o, \mathrm{p}_o) = \frac{1}{4}\widehat{X}(\alpha) - \frac{1}{4}\mathrm{p}_o(\gamma^i) g_o(X,E_i) \ , \tag{3.27}$$

$$g(\nabla_{\widehat{X}}\mathrm{q}_o, \mathrm{q}_o) = \frac{1}{4}\widehat{X}(\beta) \ ; \tag{3.28}$$

$$\nabla_{\mathrm{p}_o}\widehat{Y} : \quad g(\nabla_{\mathrm{p}_o}\widehat{Y}, \widehat{Z}) = \frac{\alpha}{4} g_o(JY,Z) \ , \tag{3.29}$$

$$g(\nabla_{\mathrm{p}_o}\widehat{Y}, \mathrm{p}_o) = 0 \ , \tag{3.30}$$

$$g(\nabla_{\mathrm{p}_o}\widehat{Y}, \mathrm{q}_o) = \frac{1}{4}\mathrm{p}_o(\gamma^i) g_o(Y,E_i) + \frac{1}{4}\widehat{Y}(\alpha) \ ; \tag{3.31}$$

$$\nabla_{\mathrm{p}_o}\mathrm{p}_o : \quad g(\nabla_{\mathrm{p}_o}\mathrm{p}_o, \widehat{Z}) = 0 \ , \tag{3.32}$$

$$g(\nabla_{\mathrm{p}_o}\mathrm{p}_o, \mathrm{p}_o) = 0 \ , \tag{3.33}$$

$$g(\nabla_{\mathrm{p}_o}\mathrm{p}_o, \mathrm{q}_o) = \frac{1}{2}\mathrm{p}_o(\alpha) \ ; \tag{3.34}$$

$$\nabla_{\mathrm{p}_o}\mathrm{q}_o : \quad g(\nabla_{\mathrm{p}_o}\mathrm{q}_o, \widehat{Z}) = \frac{1}{4}\mathrm{p}_o(\gamma^i) g_o(E_i,Z) - \frac{1}{4}\widehat{Z}(\alpha) \ , \tag{3.35}$$

$$g(\nabla_{\mathrm{p}_o}\mathrm{q}_o, \mathrm{p}_o) = 0 \ , \tag{3.36}$$

$$g(\nabla_{\mathrm{p}_o}\mathrm{q}_o, \mathrm{q}_o) = \frac{\mathrm{p}_o(\beta)}{4} \ ; \tag{3.37}$$

$$\nabla_{\mathrm{q}_o}\widehat{Y}: \quad g(\nabla_{\mathrm{q}_o}\widehat{Y},\widehat{Z}) = \frac{1}{4}\widehat{Y}(\gamma^i g_o(E_i,Z)) - \frac{1}{4}\widehat{Z}(\gamma^t g_o(Y,E_t)) - \\ - \frac{1}{4}\gamma^t g_o([Y,Z],E_t) + \frac{1}{4}\beta g_o(JY,Z) \ , \tag{3.38}$$

$$g(\nabla_{\mathrm{q}_o}\widehat{Y},\mathrm{p}_o) = \frac{1}{4}\big(\widehat{Y}(\alpha) - \mathrm{p}_o(\gamma^i)g_o(E_i,Y)\big) \ , \tag{3.39}$$

$$g(\nabla_{\mathrm{q}_o}\widehat{Y},\mathrm{q}_o) = \frac{\widehat{Y}(\beta)}{4} \ ; \tag{3.40}$$

$$\nabla_{\mathrm{q}_o}\mathrm{p}_o: \quad g(\nabla_{\mathrm{q}_o}\mathrm{p}_o,\widehat{Z}) = \frac{1}{4}\mathrm{p}_o(\gamma^i)g_o(E_i,\widehat{Z}) - \frac{1}{4}\widehat{Z}(\alpha) \ , \tag{3.41}$$

$$g(\nabla_{\mathrm{q}_o}\mathrm{p}_o,\mathrm{p}_o) = 0 \ , \tag{3.42}$$

$$g(\nabla_{\mathrm{q}_o}\mathrm{p}_o,\mathrm{q}_o) = \frac{\mathrm{p}_o(\beta)}{4} \ ; \tag{3.43}$$

$$\nabla_{\mathrm{q}_o}\mathrm{q}_o: \quad g(\nabla_{\mathrm{q}_o}\mathrm{q}_o,\widehat{Z}) = \frac{1}{2}\mathrm{q}_o(\gamma^i)g_o(E_i,Z) - \frac{1}{4}\widehat{Z}(\beta) \ , \tag{3.44}$$

$$g(\nabla_{\mathrm{q}_o}\mathrm{q}_o,\mathrm{p}_o) = \frac{1}{2}\mathrm{q}_o(\alpha) - \frac{\mathrm{p}_o(\beta)}{4} \ , \tag{3.45}$$

$$g(\nabla_{\mathrm{q}_o}\mathrm{q}_o,\mathrm{q}_o) = \frac{\mathrm{q}_o(\beta)}{4} \ . \tag{3.46}$$

From this list, we may recover the explicit expressions of the covariant derivatives of vector fields of the adapted frame field $(\widehat{E}_i,\mathrm{p}_o,\mathrm{q}_o)$ as follows. We claim that the dual coframe field $(\widehat{E}^i,\mathrm{p}_o^*,\mathrm{q}_o^*)$ is given by the following 1-forms (here, $(g^{\ell m}) := (g_{ij})^{-1} = \big(g_o(E_i,E_j)\big)^{-1}$)

$$\widehat{E}^i = g\left(g^{ik}\widehat{E}_k - \frac{\gamma^i}{\alpha}\mathrm{p}_o,\cdot\right) \ , \quad \mathrm{p}_o^* = g\left(\frac{2}{\alpha}\mathrm{q}_o + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o - \frac{\gamma^m}{\alpha}\widehat{E}_m,\cdot\right) \ , \\ \mathrm{q}_o^* = g\left(\frac{2}{\alpha}\mathrm{p}_o,\cdot\right) \ . \tag{3.47}$$

This claim can be checked using (2.11) and observing that the right hand sides in the above equalities are 1-forms that satisfy the equalities

$$\widehat{E}^i(\widehat{E}_j) = g^{ik}g_{kj} = \delta^i_j \ , \quad \widehat{E}^i(\mathrm{p}_o) = 0 \ , \quad \widehat{E}^i(\mathrm{q}_o) = g^{ik}\frac{\gamma^m}{2}g_{mk} - \frac{\gamma^i}{\alpha}\frac{\alpha}{2} = 0 \ ,$$
$$\mathrm{p}_o^*(\widehat{E}_j) = \frac{2}{\alpha}\frac{\gamma^m}{2}g_{jm} - \frac{\gamma^m}{\alpha}g_{mj} = 0 \ , \quad \mathrm{p}_o^*(\mathrm{p}_o) = \frac{2}{\alpha}\frac{\alpha}{2} = 1 \ ,$$
$$\mathrm{p}_o^*(\mathrm{q}_o) = \frac{2}{\alpha}\frac{\beta}{2} + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\frac{\alpha}{2} - \frac{\gamma^m}{\alpha}\frac{\gamma^k}{2}g_{mk} = 0 \ ,$$
$$\mathrm{q}_o^*(\widehat{E}_j) = 0 \ , \quad \mathrm{q}_o^*(\mathrm{p}_o) = 0 \ , \quad \mathrm{q}_o^*(\mathrm{q}_o) = \frac{2}{\alpha}\frac{\alpha}{2} = 1 \ .$$

Since any local vector field Z on M can be written in terms of the frame field $(\widehat{E}_i, \mathrm{p}_o, \mathrm{q}_o)$ as

$$Z = \widehat{E}^i(Z)\widehat{E}_i + \mathrm{p}_o^*(Z)\mathrm{p}_o + \mathrm{q}_o^*(Z)\mathrm{q}_o \ ,$$

from the above expressions for the 1-forms $\widehat{E}^i, \mathrm{p}_o^*$, and q_o^*, we get that for any pair of vector fields X, Y on M, the Levi-Civita covariant derivative $\nabla_X Y$ is equal to

$$\begin{aligned}\nabla_X Y = &\left(g^{mk} g(\nabla_X Y, \widehat{E}_k) - \frac{\gamma^m}{\alpha} g(\nabla_X Y, \mathrm{p}_o)\right)\widehat{E}_m + \\ &+ \left(\frac{2}{\alpha} g(\nabla_X Y, \mathrm{q}_o) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right) g(\nabla_X Y, \mathrm{p}_o) - \frac{\gamma^m}{\alpha} g(\nabla_X Y, \widehat{E}_m)\right)\mathrm{p}_o + \\ &+ \left(\frac{2}{\alpha} g(\nabla_X Y, \mathrm{p}_o)\right)\mathrm{q}_o \ . \qquad (3.48)\end{aligned}$$

Combining (3.20) – (3.46) with (3.48), we get the covariant derivatives we are looking for. We list them in (3.49) – (3.57) (here, we denote by $S_{ij|m}$ the components of the tensor field S in terms of the frame field (E_i) on N):

$$\begin{aligned}\nabla_{\widehat{E}_i}\widehat{E}_j = &\left(g^{mk} g_o(\nabla^o_{E_i} E_j, E_k) + g^{mk} S_{ij|k} + \frac{\gamma^m\omega_{ij}}{4}\right)\widehat{E}_m + \\ &+ \left(\frac{1}{2\alpha}\widehat{E}_i(\gamma^k g_{jk}) + \frac{1}{2\alpha}\widehat{E}_j(\gamma^k g_{ik}) - \right. \\ &\left. - \frac{1}{4\alpha}\gamma^m\gamma^k g_{mk}\omega_{ij} - \frac{\gamma^m}{\alpha} g_o(\nabla^o_{E_i} E_j, E_m) - \frac{\gamma^m}{\alpha} S_{ij|m}\right)\mathrm{p}_o - \frac{\omega_{ij}}{2}\mathrm{q}_o \ , \qquad (3.49)\end{aligned}$$

$$\nabla_{\widehat{E}_i}\mathrm{p}_o = \frac{\alpha g^{mk}\omega_{ik}}{4}\widehat{E}_m + \left(\frac{1}{2\alpha}\widehat{E}_i(\alpha) + \frac{1}{2\alpha}\mathrm{p}_o(\gamma^k) g_{ik} - \frac{\gamma^m\omega_{im}}{4}\right)\mathrm{p}_o \ , \qquad (3.50)$$

$$\begin{aligned}\nabla_{\widehat{E}_i}\mathrm{q}_o = &\left(\frac{g^{mk}}{4}\widehat{E}_i(\gamma^t g_{tk}) - \frac{g^{mk}}{4}\widehat{E}_k(\gamma^t g_{ti}) - \frac{\gamma^\ell}{4} c^t_{ir} g_{t\ell} g^{mr} + \frac{g^{mk}}{4}\beta\omega_{ik} - \right. \\ &\left. - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^t) g_{it}\right)\widehat{E}_m + \\ &+ \left(\frac{1}{2\alpha}\widehat{E}_i(\beta) + \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\widehat{E}_i(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\gamma^t) g_{it} - \frac{\beta}{2\alpha^2}\widehat{E}_i(\alpha) + \frac{\beta}{2\alpha^2}\mathrm{p}_o(\gamma^t) g_{it} - \right. \\ &\left. - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\gamma^t g_{tm}) + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\gamma^t g_{it}) + \frac{\gamma^m\gamma^t}{4\alpha} g_{t\ell} c^\ell_{im} - \frac{\gamma^m}{4\alpha}\beta\omega_{im}\right)\mathrm{p}_o + \\ &+ \left(\frac{1}{2\alpha}\widehat{E}_i(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\gamma^t) g_{it}\right)\mathrm{q}_o \ , \qquad (3.51)\end{aligned}$$

$$\nabla_{\mathrm{p}_o}\widehat{E}_i = \frac{\alpha g^{mk}}{4}\omega_{ik}\widehat{E}_m + \left(\frac{1}{2\alpha}\widehat{E}_i(\alpha) + \frac{1}{2\alpha}\mathrm{p}_o(\gamma^t) g_{it} - \frac{\gamma^m}{4}\omega_{im}\right)\mathrm{p}_o \ , \qquad (3.52)$$

$$\nabla_{\mathrm{p}_o}\mathrm{p}_o = \mathrm{p}_o(\log\alpha)\mathrm{p}_o \ , \tag{3.53}$$

$$\nabla_{\mathrm{p}_o}\mathrm{q}_o = \left(\frac{1}{4}\mathrm{p}_o(\gamma^m) - \frac{g^{mk}}{4}\widehat{E}_k(\alpha)\right)\widehat{E}_m + \left(\frac{1}{2\alpha}\mathrm{p}_o(\beta) - \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\alpha)\right)\mathrm{p}_o \ , \tag{3.54}$$

$$\begin{aligned}\nabla_{\mathrm{q}_o}\widehat{E}_i = \Bigg(&\frac{g^{mk}}{4}\widehat{E}_i(\gamma^t g_{tk}) - \frac{g^{mk}}{4}\widehat{E}_k(\gamma^t g_{ti}) - \frac{\gamma^\ell}{4}c^t_{ir}g_{t\ell}g^{mr} + \frac{g^{mk}}{4}\beta\omega_{ik} - \\ &- \frac{\gamma^m}{4\alpha}\widehat{E}_i(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^t)g_{ti}\Bigg)\widehat{E}_m + \\ &+ \Bigg(\frac{1}{2\alpha}\widehat{E}_i(\beta) + \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\widehat{E}_i(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\gamma^t)g_{it} - \frac{1}{2\alpha^2}\beta\widehat{E}_i(\alpha) + \\ &+ \frac{1}{2\alpha^2}\beta\mathrm{p}_o(\gamma^t)g_{it} - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\gamma^t g_{tm}) + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\gamma^t g_{it}) + \frac{\gamma^m\gamma^t}{4\alpha}g_{t\ell}c^\ell_{im} - \frac{\gamma^m}{4\alpha}\beta\omega_{im}\Bigg)\mathrm{p}_o + \\ &+ \left(\frac{1}{2\alpha}\widehat{E}_i(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\gamma^t)g_{it}\right)\mathrm{q}_o \ ,\end{aligned} \tag{3.55}$$

$$\begin{aligned}\nabla_{\mathrm{q}_o}\mathrm{p}_o = &\left(\frac{g^{mk}}{4}\mathrm{p}_o(\gamma^i)g_{ik} - \frac{g^{mk}}{4}\widehat{E}_k(\alpha)\right)\widehat{E}_m + \\ &+ \left(\frac{1}{2\alpha}\mathrm{p}_o(\beta) - \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^t)g_{tm} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\alpha)\right)\mathrm{p}_o \ ,\end{aligned} \tag{3.56}$$

$$\begin{aligned}\nabla_{\mathrm{q}_o}\mathrm{q}_o = &\left(\frac{g^{mk}}{2}\mathrm{q}_o(\gamma^i)g_{ik} - \frac{g^{mk}}{4}\widehat{E}_k(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\beta)\right)\widehat{E}_m + \\ &+ \Bigg(\frac{1}{2\alpha}\mathrm{q}_o(\beta) + \frac{1}{2\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{q}_o(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\beta) - \frac{\beta}{\alpha^2}\mathrm{q}_o(\alpha) + \frac{\beta}{2\alpha^2}\mathrm{p}_o(\beta) - \\ &- \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\beta)\Bigg)\mathrm{p}_o + \left(\frac{1}{\alpha}\mathrm{q}_o(\alpha) - \frac{\mathrm{p}_o(\beta)}{2\alpha}\right)\mathrm{q}_o \ .\end{aligned} \tag{3.57}$$

3.3 The second step

Let us now denote by $\Gamma_{AB}^{\ C}$ the Christoffel symbols of the Levi-Civita connection of a compatible metric g as in (2.12) under the assumption that the function σ is identically equal to 1. Since the $\Gamma_{AB}^{\ C}$ are the functions that appear in the expansions $\nabla_{X_A}X_B = \Gamma_{AB}^{\ C}X_C$ of the covariant derivatives (3.49) – (3.57), all such Christoffel symbols can be determined by just looking at those formulas. For convenience of the reader, we provide the complete list in the next lines

$$\Gamma_{ij}^{\ m} = g^{mk}g_o(\nabla^o_{E_i}E_j, E_k) + g^{mk}S_{ij|k} + \frac{\gamma^m\omega_{ij}}{4} \ , \tag{3.58}$$

$$\Gamma_{ij}^{\ \mathrm{p}_o} = \frac{1}{2\alpha}\widehat{E}_i(\gamma^k g_{jk}) + \frac{1}{2\alpha}\widehat{E}_j(\gamma^k g_{ik}) + \\ - \frac{1}{4\alpha}\gamma^m\gamma^k g_{mk}\omega_{ij} - \frac{\gamma^m}{\alpha} g_o(\nabla^o_{E_i} E_j, E_m) - \frac{\gamma^m}{\alpha} S_{ij|m}\ , \tag{3.59}$$

$$\Gamma_{ij}^{\ \mathrm{q}_o} = -\frac{\omega_{ij}}{2}\ , \tag{3.60}$$

$$\Gamma_{i\mathrm{p}_o}^{\ m} = \Gamma_{\mathrm{p}_o i}^{\ \ m} = \frac{\alpha g^{mk}\omega_{ik}}{4}\ , \tag{3.61}$$

$$\Gamma_{i\mathrm{p}_o}^{\ \mathrm{p}_o} = \Gamma_{\mathrm{p}_o i}^{\ \ \mathrm{p}_o} = \frac{1}{2\alpha}\widehat{E}_i(\alpha) + \frac{1}{2\alpha}\mathrm{p}_o(\gamma^k) g_{ik} - \frac{\gamma^m\omega_{im}}{4}\ , \tag{3.62}$$

$$\Gamma_{i\mathrm{p}_o}^{\ \mathrm{q}_o} = \Gamma_{\mathrm{p}_o i}^{\ \ \mathrm{q}_o} = 0\ , \tag{3.63}$$

$$\Gamma_{i\mathrm{q}_o}^{\ m} = \Gamma_{\mathrm{q}_o i}^{\ \ m} = \frac{g^{mk}}{4}\widehat{E}_i(\gamma^t g_{tk}) - \frac{g^{mk}}{4}\widehat{E}_k(\gamma^t g_{ti}) - \frac{\gamma^\ell}{4} c^t_{ir} g_{t\ell} g^{mr} + \frac{g^{mk}}{4}\beta\omega_{ik} - \\ - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^t) g_{it}\ , \tag{3.64}$$

$$\Gamma_{i\mathrm{q}_o}^{\ \mathrm{p}_o} = \Gamma_{\mathrm{q}_o i}^{\ \ \mathrm{p}_o} = \frac{1}{2\alpha}\widehat{E}_i(\beta) + \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\widehat{E}_i(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\gamma^t) g_{it} - \frac{\beta}{2\alpha^2}\widehat{E}_i(\alpha) + \\ + \frac{1}{2\alpha^2}\beta\mathrm{p}_o(\gamma^t) g_{it} - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\gamma^t g_{tm}) + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\gamma^t g_{it}) + \frac{\gamma^m\gamma^t}{4\alpha} g_{t\ell} c^\ell_{im} - \frac{\gamma^m}{4\alpha}\beta\omega_{im}\ , \tag{3.65}$$

$$\Gamma_{i\mathrm{q}_o}^{\ \mathrm{q}_o} = \Gamma_{\mathrm{q}_o i}^{\ \ \mathrm{q}_o} = \frac{1}{2\alpha}\widehat{E}_i(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\gamma^t) g_{it}\ , \tag{3.66}$$

$$\Gamma_{\mathrm{p}_o\mathrm{p}_o}^{\ \ m} = 0\ , \tag{3.67}$$

$$\Gamma_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \mathrm{p}_o} = \mathrm{p}_o(\log\alpha)\ , \tag{3.68}$$

$$\Gamma_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \mathrm{q}_o} = 0\ , \tag{3.69}$$

$$\Gamma_{\mathrm{p}_o\mathrm{q}_o}^{\ \ m} = \Gamma_{\mathrm{q}_o\mathrm{p}_o}^{\ \ m} = \frac{1}{4}\mathrm{p}_o(\gamma^m) - \frac{g^{mk}}{4}\widehat{E}_k(\alpha)\ , \tag{3.70}$$

$$\Gamma_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \mathrm{p}_o} = \Gamma_{\mathrm{q}_o\mathrm{p}_o}^{\ \ \mathrm{p}_o} = \frac{1}{2\alpha}\mathrm{p}_o(\beta) - \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^i) g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\alpha)\ , \tag{3.71}$$

$$\Gamma_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \Gamma_{\mathrm{q}_o\mathrm{p}_o}^{\ \ \mathrm{q}_o} = 0\ , \tag{3.72}$$

$$\Gamma_{\mathrm{q}_o\mathrm{q}_o}^{\ \ m} = \frac{g^{mk}}{2}\mathrm{q}_o(\gamma^i) g_{ik} - \frac{g^{mk}}{4}\widehat{E}_k(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\beta)\ , \tag{3.73}$$

$$\Gamma_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \mathrm{p}_o} = \frac{1}{2\alpha}\mathrm{q}_o(\beta) + \frac{1}{2\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{q}_o(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\beta) - \frac{\beta}{\alpha^2}\mathrm{q}_o(\alpha) + \\ + \frac{\beta}{2\alpha^2}\mathrm{p}_o(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\gamma^i) g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\beta)\ m\ , \tag{3.74}$$

$$\Gamma_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \frac{1}{\alpha}\mathrm{q}_o(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\beta)\ . \tag{3.75}$$

Note that the equalities $\Gamma_{i\mathrm{p}_o}^{\ A} = \Gamma_{\mathrm{p}_o i}^{\ \ A}$, $\Gamma_{i\mathrm{q}_o}^{\ A} = \Gamma_{\mathrm{q}_o i}^{\ \ A}$, etc. are also consequences of the fact that the torsion of the Levi-Civita connection is 0 and that the pairs of vector fields $\{\widehat{E}_i, \mathrm{p}_o\}$, $\{\widehat{E}_i\ \mathrm{q}_o\}$, etc., commute.

3.4 The third step

Assume that g is one of the metrics considered in the previous two subsections (*i.e.*, compatible with $\sigma \equiv 1$) and denote by D the Levi-Civita connection of a conformally scaled metric $g^\varphi = e^{2\varphi} g$ for some smooth φ. It is well known that, for any pair of vector fields X, Y of M (see *e.g.* [4, Th. 1.159]),

$$D_X Y = \nabla_X Y + X(\varphi) Y + Y(\varphi) X - g(X,Y)\mathrm{grad}(\varphi) \ . \tag{3.76}$$

If we expand gradφ in terms of the frame field $(\widehat{E}_i, \mathrm{p}_o, \mathrm{q}_o)$ as

$$\mathrm{grad}\varphi = (\mathrm{grad}\varphi)^{\widehat{E}_i} \widehat{E}_i + (\mathrm{grad}\varphi)^{\mathrm{p}_o} \mathrm{p}_o + (\mathrm{grad}\varphi)^{\mathrm{q}_o} \mathrm{q}_o \ , \tag{3.77}$$

we see that the Christoffel symbols $\Gamma_{AB}^{\ \ C}$ for a compatible metric g with $\sigma \equiv 1$, as considered in the previous subsections, and the Christoffel symbols $\boldsymbol{\Gamma}_{AB}^{\ \ C}$ for the conformally scaled metric g^φ are related to each other by

$$\boldsymbol{\Gamma}_{ij}^{\ \ m} = \Gamma_{ij}^{\ \ m} + \widehat{E}_i(\varphi)\delta_j^m + \widehat{E}_j(\varphi)\delta_i^m - g_{ij}(\mathrm{grad}\varphi)^{\widehat{E}_m} \ , \tag{3.78}$$

$$\boldsymbol{\Gamma}_{ij}^{\ \ \mathrm{p}_o} = \Gamma_{ij}^{\ \ \mathrm{p}_o} - g_{ij}(\mathrm{grad}\varphi)^{\mathrm{p}_o} \ , \tag{3.79}$$

$$\boldsymbol{\Gamma}_{ij}^{\ \ \mathrm{q}_o} = \Gamma_{ij}^{\ \ \mathrm{q}_o} - g_{ij}(\mathrm{grad}\varphi)^{\mathrm{q}_o} \ , \tag{3.80}$$

$$\boldsymbol{\Gamma}_{i\mathrm{p}_o}^{\ \ m} = \boldsymbol{\Gamma}_{\mathrm{p}_o i}^{\ \ m} = \Gamma_{i\mathrm{p}_o}^{\ \ m} + \mathrm{p}_o(\varphi)\delta_i^m \ , \tag{3.81}$$

$$\boldsymbol{\Gamma}_{i\mathrm{p}_o}^{\ \ \mathrm{p}_o} = \boldsymbol{\Gamma}_{\mathrm{p}_o i}^{\ \ \mathrm{p}_o} = \Gamma_{i\mathrm{p}_o}^{\ \ \mathrm{p}_o} + \widehat{E}_i(\varphi) \ , \tag{3.82}$$

$$\boldsymbol{\Gamma}_{i\mathrm{p}_o}^{\ \ \mathrm{q}_o} = \boldsymbol{\Gamma}_{\mathrm{p}_o i}^{\ \ \mathrm{q}_o} = \Gamma_{i\mathrm{p}_o}^{\ \ \mathrm{q}_o} \ , \tag{3.83}$$

$$\boldsymbol{\Gamma}_{i\mathrm{q}_o}^{\ \ m} = \boldsymbol{\Gamma}_{\mathrm{q}_o i}^{\ \ m} = \Gamma_{i\mathrm{q}_o}^{\ \ m} + \mathrm{q}_o(\varphi)\delta_i^m - \frac{\gamma^t}{2} g_{ti}(\mathrm{grad}\varphi)^{\widehat{E}_m} \ , \tag{3.84}$$

$$\boldsymbol{\Gamma}_{i\mathrm{q}_o}^{\ \ \mathrm{p}_o} = \boldsymbol{\Gamma}_{\mathrm{q}_o i}^{\ \ \mathrm{p}_o} = \Gamma_{i\mathrm{q}_o}^{\ \ \mathrm{p}_o} - \frac{\gamma^t}{2} g_{ti}(\mathrm{grad}\varphi)^{\mathrm{p}_o} \ , \tag{3.85}$$

$$\boldsymbol{\Gamma}_{i\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \boldsymbol{\Gamma}_{\mathrm{q}_o i}^{\ \ \mathrm{q}_o} = \Gamma_{i\mathrm{q}_o}^{\ \ \mathrm{q}_o} + \widehat{E}_i(\varphi) - \frac{\gamma^t}{2} g_{ti}(\mathrm{grad}\varphi)^{\mathrm{q}_o} \ , \tag{3.86}$$

$$\boldsymbol{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \ \ m} = \Gamma_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \ \ m} \ , \tag{3.87}$$

$$\boldsymbol{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \ \ \mathrm{p}_o} = \Gamma_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \ \ \mathrm{p}_o} + 2\mathrm{p}_o(\varphi) \ , \tag{3.88}$$

$$\boldsymbol{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \ \ \mathrm{q}_o} = \Gamma_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \ \ \mathrm{q}_o} \ , \tag{3.89}$$

$$\boldsymbol{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \ \ m} = \boldsymbol{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ \ \ \ m} = \Gamma_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \ \ m} - \frac{\alpha}{2}(\mathrm{grad}\varphi)^{\widehat{E}_m} \ , \tag{3.90}$$

$$\boldsymbol{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \ \ \mathrm{p}_o} = \boldsymbol{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ \ \ \ \mathrm{p}_o} = \Gamma_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \ \ \mathrm{p}_o} + \mathrm{q}_o(\varphi) - \frac{\alpha}{2}(\mathrm{grad}\varphi)^{\mathrm{p}_o} \ , \tag{3.91}$$

$$\boldsymbol{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \ \ \mathrm{q}_o} = \boldsymbol{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ \ \ \ \mathrm{q}_o} = \Gamma_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \ \ \mathrm{q}_o} + \mathrm{p}_o(\varphi) - \frac{\alpha}{2}(\mathrm{grad}\varphi)^{\mathrm{q}_o} \ , \tag{3.92}$$

$$\boldsymbol{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \ \ m} = \Gamma_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \ \ m} - \frac{\beta}{2}(\mathrm{grad}\varphi)^{\widehat{E}_m} \ , \tag{3.93}$$

$$\boldsymbol{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \ \ \mathrm{p}_o} = \Gamma_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \ \ \mathrm{p}_o} - \frac{\beta}{2}(\mathrm{grad}\varphi)^{\mathrm{p}_o} \ , \tag{3.94}$$

$$\boldsymbol{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \ \ \mathrm{q}_o} = \Gamma_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \ \ \mathrm{q}_o} + 2\mathrm{q}_o(\varphi) - \frac{\beta}{2}(\mathrm{grad}\varphi)^{\mathrm{q}_o} \ . \tag{3.95}$$

We now recall that any vector field X on M decomposes into the sum

$$\begin{aligned} X = \widehat{E}^i(X)\widehat{E}_i + \mathrm{p}_o^*(X)\mathrm{p}_o + \mathrm{q}_o^*(X)\mathrm{q}_o = g\left(X, g^{ik}\widehat{E}_k - \frac{\gamma^i}{\alpha}\mathrm{p}_o\right)\widehat{E}_i + \\ + g\left(X, \frac{2}{\alpha}\mathrm{q}_o + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o - \frac{\gamma^m}{\alpha}\widehat{E}_m\right)\mathrm{p}_o + g\left(X, \frac{2}{\alpha}\mathrm{p}_o\right)\mathrm{q}_o\ . \end{aligned} \tag{3.96}$$

From this, we get that the components $(\mathrm{grad}\varphi)^A$ of the gradient of φ are equal to

$$\begin{aligned} (\mathrm{grad}\varphi)^{\widehat{E}_i} &:= g^{ik}\widehat{E}_k(\varphi) - \frac{\gamma^i}{\alpha}\mathrm{p}_o(\varphi)\ , \\ (\mathrm{grad}\varphi)^{\mathrm{p}_o} &:= \frac{2}{\alpha}\mathrm{q}_o(\varphi) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(f) - \frac{\gamma^m}{\alpha}\widehat{E}_m(f)\ , \\ (\mathrm{grad}\varphi)^{\mathrm{q}_o} &:= \frac{2}{\alpha}\mathrm{p}_o(\varphi). \end{aligned} \tag{3.97}$$

Inserting these expressions and (3.78) – (3.95) into (3.78) – (3.95), we get the explicit formulas for the Christoffel symbols $\mathbf{\Gamma}_{AB}^{\ \ C}$ of the scaled metric $g^\varphi = e^{2\varphi}g$. They are:

$$\begin{aligned} \mathbf{\Gamma}_{ij}^{\ \ m} &= g^{mk}g_o(\nabla^o_{E_i}E_j, E_k) + g^{mk}S_{ij|k} + \frac{\gamma^m\omega_{ij}}{4} + \widehat{E}_i(\varphi)\delta^m_j + \widehat{E}_j(\varphi)\delta^m_i \\ &\quad - g_{ij}\left(g^{mk}\widehat{E}_k(\varphi) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\varphi)\right)\ , \end{aligned} \tag{3.98}$$

$$\begin{aligned} \mathbf{\Gamma}_{ij}^{\ \ \mathrm{p}_o} &= \frac{1}{2\alpha}\widehat{E}_i(\gamma^k g_{jk}) + \frac{1}{2\alpha}\widehat{E}_j(\gamma^k g_{ik}) - \frac{1}{4\alpha}\gamma^m\gamma^k g_{mk}\omega_{ij} - \frac{\gamma^m}{\alpha}g_o(\nabla^o_{E_i}E_j, E_m) - \frac{\gamma^m}{\alpha}S_{ij|m} \\ &\quad - g_{ij}\left(\frac{2}{\alpha}\mathrm{q}_o(\varphi) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\varphi) - \frac{\gamma^m}{\alpha}\widehat{E}_m(\varphi)\right)\ , \end{aligned} \tag{3.99}$$

$$\mathbf{\Gamma}_{ij}^{\ \ \mathrm{q}_o} = -\frac{\omega_{ij}}{2} - \frac{2g_{ij}}{\alpha}\mathrm{p}_o(\varphi)\ , \tag{3.100}$$

$$\mathbf{\Gamma}_{i\mathrm{p}_o}^{\ \ m} = \mathbf{\Gamma}_{\mathrm{p}_o i}^{\ \ m} = \frac{\alpha g^{mk}\omega_{ik}}{4} + \mathrm{p}_o(\varphi)\delta^m_i\ , \tag{3.101}$$

$$\mathbf{\Gamma}_{i\mathrm{p}_o}^{\ \ \mathrm{p}_o} = \mathbf{\Gamma}_{\mathrm{p}_o i}^{\ \ \mathrm{p}_o} = \frac{1}{2\alpha}\widehat{E}_i(\alpha) + \frac{1}{2\alpha}\mathrm{p}_o(\gamma^k)g_{ik} - \frac{\gamma^m\omega_{im}}{4} + \widehat{E}_i(\varphi)\ , \tag{3.102}$$

$$\mathbf{\Gamma}_{i\mathrm{p}_o}^{\ \ \mathrm{q}_o} = \mathbf{\Gamma}_{i\mathrm{p}_o}^{\ \ \mathrm{q}_o} = 0\ , \tag{3.103}$$

$$\begin{aligned} \mathbf{\Gamma}_{i\mathrm{q}_o}^{\ \ m} = \mathbf{\Gamma}_{\mathrm{q}_o i}^{\ \ m} &= \frac{g^{mk}}{4}\widehat{E}_i(\gamma^t g_{tk}) - \frac{g^{mk}}{4}\widehat{E}_k(\gamma^t g_{ti}) - \frac{\gamma^\ell}{4}c^t_{ir}g_{t\ell}g^{mr} + \frac{g^{mk}}{4}\beta\omega_{ik} - \\ &\quad - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^t)g_{ti} + \mathrm{q}_o(\varphi)\delta^m_i - \frac{\gamma^t}{2}g_{ti}\left(g^{mk}\widehat{E}_k(\varphi) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\varphi)\right)\ , \end{aligned} \tag{3.104}$$

$$\begin{aligned} \mathbf{\Gamma}_{i\mathrm{q}_o}^{\ \ \mathrm{p}_o} = \mathbf{\Gamma}_{\mathrm{q}_o i}^{\ \ \mathrm{p}_o} &= \frac{1}{2\alpha}\widehat{E}_i(\beta) + \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\widehat{E}_i(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\gamma^t)g_{it} - \frac{1}{2\alpha^2}\beta\widehat{E}_i(\alpha) + \\ &\quad + \frac{1}{2\alpha^2}\beta\mathrm{p}_o(\gamma^t)g_{it} - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\gamma^t g_{tm}) + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\gamma^t g_{it}) + \frac{\gamma^m\gamma^t}{4\alpha}g_{t\ell}c^\ell_{im} - \frac{\gamma^m}{4\alpha}\beta\omega_{im} - \\ &\quad - \frac{\gamma^t}{2}g_{ti}\left(\frac{2}{\alpha}\mathrm{q}_o(\varphi) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\varphi) - \frac{\gamma^m}{\alpha}\widehat{E}_m(\varphi)\right)\ , \end{aligned} \tag{3.105}$$

$$\mathbf{\Gamma}_{i\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \mathbf{\Gamma}_{\mathrm{q}_o i}^{\ \ \mathrm{q}_o} = \frac{1}{2\alpha}\widehat{E}_i(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\gamma^t)g_{it} + \widehat{E}_i(\varphi) - \frac{\gamma^t}{2}g_{ti}\left(\frac{2}{\alpha}\mathrm{p}_o(\varphi)\right)\ , \tag{3.106}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \ m} = 0\ , \tag{3.107}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}{}^{\mathrm{p}_o} = \mathrm{p}_o(\log\alpha) + 2\mathrm{p}_o(\varphi)\ , \tag{3.108}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}{}^{\mathrm{q}_o} = 0\ , \tag{3.109}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}{}^{m} = \mathbf{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}{}^{m} = \frac{1}{4}\mathrm{p}_o(\gamma^m) - \frac{g^{mk}}{4}\widehat{E}_k(\alpha) - \frac{\alpha}{2}\left(g^{mk}\widehat{E}_k(\varphi) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\varphi)\right)\ , \tag{3.110}$$

$$\begin{aligned}\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}{}^{\mathrm{p}_o} = \mathbf{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}{}^{\mathrm{p}_o} = &\frac{1}{2\alpha}\mathrm{p}_o(\beta) - \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\alpha) + \mathrm{q}_o(\varphi) - \\ &- \frac{\alpha}{2}\left(\frac{2}{\alpha}\mathrm{q}_o(\varphi) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\varphi) - \frac{\gamma^m}{\alpha}\widehat{E}_m(\varphi)\right)\ ,\end{aligned} \tag{3.111}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}{}^{\mathrm{q}_o} = \mathbf{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}{}^{\mathrm{q}_o} = 0\ , \tag{3.112}$$

$$\begin{aligned}\mathbf{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}{}^{m} = &\frac{g^{mk}}{2}\mathrm{q}_o(\gamma^i)g_{ik} - \frac{g^{mk}}{4}\widehat{E}_k(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\beta) - \\ &- \frac{\beta}{2}\left(g^{mk}\widehat{E}_k(\varphi) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\varphi)\right)\ ,\end{aligned} \tag{3.113}$$

$$\begin{aligned}\mathbf{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}{}^{\mathrm{p}_o} = &\frac{1}{2\alpha}\mathrm{q}_o(\beta) + \frac{1}{2\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{q}_o(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\beta) - \frac{1}{\alpha^2}\beta\mathrm{q}_o(\alpha) + \\ &+ \frac{1}{2\alpha^2}\beta\mathrm{p}_o(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\beta) - \\ &- \frac{\beta}{2}\left(\frac{2}{\alpha}\mathrm{q}_o(\varphi) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\varphi) - \frac{\gamma^m}{\alpha}\widehat{E}_m(\varphi)\right)\ ,\end{aligned} \tag{3.114}$$

$$\mathbf{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}{}^{\mathrm{q}_o} = \frac{1}{\alpha}\mathrm{q}_o(\alpha) - \frac{\mathrm{p}_o(\beta)}{2\alpha} + 2\mathrm{q}_o(\varphi) - \frac{\beta}{2}\left(\frac{2}{\alpha}\mathrm{p}_o(\varphi)\right)\ . \tag{3.115}$$

In order to conclude, it is now sufficient to observe that the metric (2.12) with an arbitrary $\sigma > 0$ can be obtained from the metric considered in subsection 3.2 (*i.e.*, with $\sigma \equiv 1$) by applying the scaling factor $e^{2\varphi}$ with $\varphi := \frac{1}{2}\log\sigma$. Hence, the desired expressions for the Christoffel symbols are given by (3.98) – (3.115) with φ replaced by $\frac{1}{2}\log\sigma$ at all places. These substitutions yield (3.1) – (3.18).

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Lyapunov-type inequalities for higher-order Caputo fractional differential equations with general two-point boundary conditions

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ABSTRACT

In this paper the authors present three different Lyapunov-type inequalities for a higher-order Caputo fractional differential equation with identical boundary conditions marking the inaugural instance of such an approach in the existing literature. Their findings extend and complement certain prior results in the literature.

RESUMEN

En este artículo, los autores presentan tres desigualdades de tipo Lyapunov diferentes para una ecuación diferencial fraccionaria de Caputo de alto orden con condiciones de frontera idénticas, marcando la primera vez que este enfoque aparece en la literatura existente. Sus hallazgos extienden y complementan ciertos resultados anteriores en la literatura.

Keywords and Phrases: Fractional integral, Caputo fractional derivative, boundary value problem, existence of solution, Lyapunov inequality, Green's function.

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1 Introduction

In this paper we consider the fractional differential equation

$$\left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, \quad n-1 < \gamma \leq n, \quad n \geq 3, \tag{1.1}$$

where $q : \mathbb{R} \to \mathbb{R}$ is continuous and $q(t) \neq 0$, together with the boundary conditions

$$x^{(i)}(a) = 0,\ x^{(k)}(b) = 0, \quad 0 \leq i \leq n-1 \quad \text{and} \quad i \neq k, \tag{1.2}$$

where k is a natural number between 1 and $n-1$.

Over the course of more than a century, numerous Lyapunov-type inequalities have been derived, taking into account their applications in various areas, such as eigenvalue problems, stability theory, oscillation theory, and the estimation of intervals of disconjugacy. The paper by Lyapunov [14] in 1907 is considered to be the first work in this direction. In recent decades, especially with the development of fractional differential equations, significant advancements and further generalizations of Lyapunov inequalities have been obtained. To explore some of the research that has provided some of the motivation for studying the problem (1.1)–(1.2), first note that Cabrera *et al.* [7] derived Lyapunov-like inequalities and established a lower bound for the eigenvalues of the fractional problem

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, & a < t < b, \quad \gamma \in (n-1, n], \quad n \geq 4, \\ x^{(i)}(a) = x''(b) = 0, & 0 \leq i \leq n-1, \quad i \neq 2. \end{cases}$$

It can be observed that the boundary value problem discussed in [7] is a particular case of the problem considered here, that is, of (1.1)–(1.2) with the parameter k taken to be 2. Additional notable work for $k = 2$ can found in [1, 7, 23, 24]. Compared to the problems investigated in [1, 7, 23, 24], our boundary condition (1.2) is more comprehensive and inclusive.

In [6], Bohner *et al.* applied a Vallée-Poussin theorem to obtain explicit inequality criteria for the solvability of the problem consisting of the Caputo fractional functional differential equation

$$({}^{C}D^{\gamma}_{a+}x)(t) + \sum_{i=0}^{m}(T_i x^{(i)})(t) = f(t), \quad t \in [a, b],$$

and the boundary condition (1.2), where the operator $T_i : C \to L_\infty$ with $C = C([a, b], \mathbb{R})$ can include a delay or advanced argument, an integral operator, or various linear combinations of such things. In another work, Domonshnitsky *et al.* [10] obtained such criteria for fractional functional differential equations with Riemann-Liouville derivatives again based on the Vallée-Poussin theorem. Rong and Bai [21] obtained a Lyapunov inequality for the problem

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, & a < t < b, \quad 1 < \gamma \le 2, \\ x(a) = 0, \ \left({}^{\mathrm{C}}D^{\beta}_{a}x\right)(b) = 0, & 0 < \beta \le 1, \end{cases}$$

where $1 < \gamma \le 1+\beta$. Extensive research has been conducted on Lyapunov inequalities using different forms of fractional derivatives such as in [5,11,12,15,16,22]. For a comprehensive exploration of Lyapunov inequalities, a detailed study can be found in the recent monograph by Agarwal, Bohner and Özbekler [2].

Using estimates of the Green's function has been a common technique employed in the study of Lyapunov type inequalities. In cases where the Green's function possesses a fixed sign, estimating it becomes relatively straightforward compared to cases where the sign is unknown. Nevertheless, several researchers have successfully managed to find estimates and derive Lyapunov-type inequalities even if the sign constancy of the Green's function is not known; for example, see the recent papers [21, 22] and the book [2].

The present work is divided into six sections. Section 1 provides an introduction and background information pertaining to the problem. Preliminaries concepts are introduced in Section 2. In Section 3, we obtain a Lyapunov inequality that improves the results in [7]. In the process, we are able to obtain a new Lyapunov inequality for a third-order linear differential equation (see Corollary 3.6 below). In Section 4, we obtain a Lyapunov inequality under a restrictive condition (see (4.1)). A Lyapunov inequality for a general k with $1 \le k \le n-2$ is discussed in Section 5. We conclude this work in Section 6 with some applications and open problems.

2 Preliminaries

The monographs [13, 18] offer a thorough examination of the basics of fractional calculus. The recent publication [22] contains the required fundamental definitions and lemmas utilized here in this study. Next, we discuss the Green's function and its sign in order to enhance our comprehension of the primary outcomes.

Lemma 2.1. *Assume that $\gamma \in (n-1, n]$, $1 \le k \le n-1$, and $f \in L_\infty$. Then the unique solution of the fractional boundary value problem*

$$\begin{cases} ({}^{C}D^{\gamma}_{a+}x)(t) + f(t) = 0, & a < t < b, \\ x^{(i)}(a) = x^{(k)}(b) = 0, & 0 \le i \le n-1 \quad \textit{and} \quad i \ne k, \end{cases} \tag{2.1}$$

is given by

$$x(t) = \int_a^b G_k(t,s) f(s) ds, \tag{2.2}$$

where $G_k(t,s)$ *is the Green's function given by*

$$G_k(t,s) = \frac{1}{\Gamma(\gamma)} \begin{cases} \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^k(b-s)^{\gamma-k-1} - (t-s)^{\gamma-1}, & a \le s \le t \le b, \\ \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^k(b-s)^{\gamma-k-1}, & a \le t \le s \le b. \end{cases} \tag{2.3}$$

Proof. Consider the equation

$$({}^C D^{\gamma}_{a+} x)(t) = -f(t).$$

Then, using some fundamental concepts in the fractional calculus (see [13, 18]), we see that

$$(I^{\gamma}_{a+}({}^C D^{\gamma}_{a+} x))(t) = -(I^{\gamma}_{a+} f)(t),$$

which, in turn, implies that there are constants $b_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, such that

$$x(t) = b_0 + b_1(t-a) + b_2(t-a)^2 + \cdots + b_{n-1}(t-a)^{n-1} - \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds,$$

for $t \in [a,b]$. From the boundary condition $x^{(i)}(a) = 0$ for $0 \le i \le n-1$ and $i \ne k$, we obtain $b_i = 0$ for $0 \le i \le n-1$, $i \ne k$. Since $x^{(k)}(a) \ne 0$, we have $b_k \ne 0$. Therefore,

$$x(t) = b_k(t-a)^k - \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds, \tag{2.4}$$

and so

$$\begin{aligned} x'(t) &= k b_k (t-a)^{k-1} - \frac{\gamma-1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-2} f(s) ds, \\ x''(t) &= k(k-1) b_k (t-a)^{k-2} - \frac{(\gamma-1)(\gamma-2)}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-3} f(s) ds, \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ x^{(k)}(b) &= k!\, b_k - \frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}{\Gamma(\gamma)} \int_a^b (b-s)^{\gamma-k-1} f(s) ds. \end{aligned}$$

Applying the boundary condition $x^{(k)}(b) = 0$ gives

$$b_k = \frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}{k!\Gamma(\gamma)} \int_a^b (b-s)^{\gamma-k-1} f(s) ds.$$

Using this value of b_k in (2.4), we obtain

$$x(t) = \frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}{k!\Gamma(\gamma)} (t-a)^k \int_a^b (b-s)^{\gamma-k-1} f(s) ds - \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds, \tag{2.5}$$

or

$$x(t) = \frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}{k!\Gamma(\gamma)}\left[\int_a^t (t-a)^k(b-s)^{\gamma-k-1}f(s)ds + \int_t^b (t-a)^k(b-s)^{\gamma-k-1}f(s)ds\right] - \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}f(s)ds. \quad (2.6)$$

This proves the lemma. □

The following lemma provides some valuable information about the sign of the Green's function.

Lemma 2.2. *If $\gamma \in (n-1,n]$ and $\gamma > k+1$, then $G_k(t,s) > 0$ for all $t,s \in [a,b]$.*

Proof. Clearly,

$$G_k(t,s) = \frac{1}{\Gamma(\gamma)k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^k(b-s)^{\gamma-k-1} > 0,$$

for $a < t < s < b$. If $a < s \leq t < b$, we obtain

$$\begin{aligned}
G_k(t,s) &= \frac{1}{\Gamma(\gamma)k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^k(b-s)^{\gamma-k-1} - \frac{1}{\Gamma(\gamma)}(t-s)^{\gamma-1} \\
&\geq \frac{1}{k!\Gamma(\gamma)}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-s)^k(t-s)^{\gamma-k-1} - \frac{1}{\Gamma(\gamma)}(t-s)^{\gamma-1} \\
&= \frac{1}{k!\Gamma(\gamma)}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-s)^{\gamma-1} - \frac{1}{\Gamma(\gamma)}(t-s)^{\gamma-1} \\
&= \frac{1}{\Gamma(\gamma)}(t-s)^{\gamma-1}\left[\frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k) - 1\right] \\
&> \frac{1}{\Gamma(\gamma)}(t-s)^{\gamma-1}\left[\frac{1}{k!}k(k-1)\cdots(1) - 1\right] = 0,
\end{aligned}$$

where we have used the facts that $\gamma > k+1$, $t-a \geq t-s$, and $b-s \geq t-s$, so that $(t-a)^k(b-s)^{\gamma-k-1} \geq (t-s)^k(t-s)^{\gamma-k-1}$. This completes the proof. □

3 Main results: Lyapunov type inequalities–I

We begin this section with another lemma on the properties of $G_k(t,s)$.

Lemma 3.1. *If $\gamma \in (n-1,n]$ and $\gamma > k+1$, then Green's function $G_k(t,s)$ given in (2.3) has the property that $\frac{\partial G_k(t,s)}{\partial t} > 0$ for all $t,s \in [a,b]$. Furthermore,*

$$G_k(t,s) \leq G_k(b,s) \equiv \frac{1}{\Gamma(\gamma)}\left[\frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(b-a)^k(b-s)^{\gamma-k-1} - (b-s)^{\gamma-1}\right]. \quad (3.1)$$

Proof. For $a < t < s < b$, we have

$$\frac{\partial G_k(t,s)}{\partial t} = \frac{k}{k!\Gamma(\gamma)}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^{k-1}(b-s)^{\gamma-k-1} \geq 0.$$

For $a < s \leq t < b$,

$$\begin{aligned}\frac{\partial G_k(t,s)}{\partial t} &= \frac{k}{k!\Gamma(\gamma)}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^{k-1}(b-s)^{\gamma-k-1} - \frac{1}{\Gamma(\gamma)}(\gamma-1)(t-s)^{\gamma-2}\\ &\geq \frac{(\gamma-1)}{\Gamma(\gamma)}\left[\frac{k}{k!}(\gamma-2)\cdots(\gamma-k)(t-s)^{k-1}(t-s)^{\gamma-k-1} - (t-s)^{\gamma-2}\right]\\ &= \frac{(\gamma-1)}{\Gamma(\gamma)}\left[\frac{1}{(k-1)!}(\gamma-2)\cdots(\gamma-k)(t-s)^{\gamma-2} - (t-s)^{\gamma-2}\right]\\ &> \frac{(\gamma-1)}{\Gamma(\gamma)}\left[\frac{(k-1)!}{(k-1)!} - 1\right](t-s)^{\gamma-2} = 0,\end{aligned}$$

where we have used the fact that $\gamma > k+1$. Therefore, the function $G_k(t,s)$ is nondecreasing with respect to t, and this implies $G_k(t,s) \leq G_k(b,s)$ for all $t, s \in [0,1]$. This proves the lemma. □

The following theorem is the major result in this section.

Theorem 3.2. *Assume that* $\gamma \in (n-1, n]$ *and* $\gamma > k+1$. *If a nontrivial continuous solution of* (1.1)–(1.2) *exists, then*

$$\int_a^b \left[\frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(b-a)^k(b-s)^{\gamma-k-1} - (b-s)^{\gamma-1}\right]|q(s)|ds \geq \Gamma(\gamma). \tag{3.2}$$

Proof. Let $x(t)$ be a nonzero solution of (1.1)–(1.2) and let $X = C([a,b])$ be a Banach space endowed with the norm

$$\|x\| = \sup_{a\leq t\leq b}|x(t)|.$$

Then, for a solution x of (1.1)–(1.2), by Lemma 2.1,

$$x(t) = \int_a^b G_k(t,s)q(s)x(s)ds.$$

Since $q(t)$ cannot be zero,

$$|x(t)| \leq \frac{1}{\Gamma(\gamma)}\int_a^b \left[\frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(b-a)^k(b-s)^{\gamma-k-1} - (b-s)^{\gamma-1}\right]|q(s)|\,|x(s)|ds,$$

which yields (3.2). This proves the theorem. □

We have the following consequences of this result.

Corollary 3.3. *Under the conditions of Theorem 3.2, if* (1.1)–(1.2) *has a nontrivial continuous solution, then*

$$\int_a^b (b-s)^{\gamma-k-1}|q(s)|ds \geq \frac{k!\Gamma(\gamma-k)}{(b-a)^k}. \tag{3.3}$$

Corollary 3.4. *Under the conditions of Theorem 3.2, if* (1.1)–(1.2) *has a nontrivial continuous solution, then*

$$\int_a^b |q(s)|ds \geq \frac{k!\Gamma(\gamma-k)}{(b-a)^{\gamma-1}}. \tag{3.4}$$

If we set $n \equiv 3$, then $\gamma \in (2,3]$, and since $\gamma > k+1$, this means we take $k=1$. The problem (1.1)–(1.2) then reduces to

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a+}x\right)(t) + q(t)x(t) = 0, \qquad 2 < \gamma \leq 3, \\ x(a) = x''(a) = x'(b) = 0. \end{cases} \tag{3.5}$$

Fractional BVPs of the form (3.5) were studied by Qin and Bai [19, 20]. Applying Theorem 3.2, Corollary 3.3, and Corollary 3.4 to (3.5), we obtain the following corollary.

Corollary 3.5. *If* (3.5) *has a continuous nontrivial solution, then*

$$\int_a^b \left[\frac{1}{k!}(\gamma-1)(b-a)(b-s)^{\gamma-2} - (b-s)^{\gamma-1}\right]|q(s)|ds \geq \Gamma(\gamma), \tag{3.6}$$

$$\int_a^b (b-s)^{\gamma-2}|q(s)|ds \geq \frac{\Gamma(\gamma-1)}{(b-a)}, \tag{3.7}$$

and

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\gamma-1)}{(b-a)^{\gamma-1}}. \tag{3.8}$$

As discussed earlier, (3.6) implies (3.7), and (3.7) implies (3.8). In particular, applying inequality (3.8) of Corollary 3.5 to the third-order boundary value problem

$$\begin{cases} x'''(t) + q(t)x(t) = 0, \\ x(a) = x''(a) = x'(b) = 0, \end{cases} \tag{3.9}$$

we obtain the following result.

Corollary 3.6. *If* (3.9) *has a continuous nontrivial solution, then*

$$\int_a^b |q(s)|ds \geq \frac{1}{(b-a)^2}. \tag{3.10}$$

As far as our knowledge is concerned, Corollary 3.6 is new in the literature. The boundary conditions used in (3.9) are different from those of Aktaş and Çakmak [3,4] and Parhi and Panigrahi [17]. Our Corollary 3.6 can not be compared to the results in [5] because of the restrictive condition $x''(a) + x''(b) = 0$ (see the third condition of (1.7) in [5]) required there. Similarly, Corollary 3.5 can not be compared to Dhar and Kong [8, 9].

Next, suppose that $n \geq 4$. Our parameter k considered in (1.2) varies from 1 to $n-1$. In particular, if $k = 2$, we obtain the results of Cabrera, Lopez, and Sadarangani [7]. Our Green's function $G_k(t, s)$ extends the Green's functions obtained in [1, 23, 24] for $a = 0$, $b = 1$, and $k = 2$.

4 Main results: Lyapunov type inequalities–II

In this section, we derive a new Lyapunov type inequality, different from the ones presented in the previous section. We use the maximum of the Green's function $G_k(t, s)$ given in (2.3) to find a new inequality for (1.1)–(1.2) for a general k, $1 \leq k \leq n-1$, with the price being that the following restrictive inequality is imposed:

$$k! > (\gamma - 1) \cdots (\gamma - k)(\gamma - k - 1). \tag{4.1}$$

As prescribed by our boundary condition (1.2), we consider the following cases:

(A$_1$) $x(0) = x''(0) = \cdots = x^{(n-1)}(0) = 0,\ x'(1) = 0$

(A$_2$) $x(0) = x'(0) = x'''(0) = \cdots = x^{(n-1)}(0) = 0,\ x''(1) = 0$

(A$_3$) $x(0) = x'(0) = x''(0) = x''''(0) = \cdots = x^{(n-1)}(0) = 0,\ x'''(1) = 0$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

(A$_{n-1}$) $x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0,\ x^{(n-1)}(1) = 0$

Remark 4.1. *Observe that:*

(B$_1$) *For $k = 1$, that is, in the case (A$_1$), we can take $\gamma = 2.5 \in (2, 3]$. Then, condition (4.1) is satisfied, i.e.,*

$$1 = k! > (\gamma - 1) \cdots (\gamma - k)(\gamma - k - 1) = (2.5 - 1)(2.5 - 2) = 0.75.$$

(B$_2$) *For $k = 2$, that is, in the case (A$_2$), we can take $\gamma = 3.5 \in (3, 4]$, so that condition (4.1) becomes*

$$2 = k! > (\gamma - 1) \cdots (\gamma - k)(\gamma - k - 1) = (3.5 - 1)(3.5 - 2)(3.5 - 3) = 1.875.$$

(B$_3$) *For $k = 3$, that is, in the case (A$_3$), we can take $\gamma = 4.4 \in (4, 5]$, and* (4.1) *becomes*

$$6 = k! > (\gamma - 1)\cdots(\gamma - k)(\gamma - k - 1) = (4.4 - 1)(4.4 - 2)(4.4 - 3)(4.4 - 4) = 4.5696.$$

The following lemma gives an upper bound on $G_k(t, s)$.

Lemma 4.2. *Let $\gamma > k + 1$ and assume that* (4.1) *is satisfied. Then*

$$G_k(t,s) \le \frac{1}{\Gamma(\gamma)} \frac{k(b-a)^{\gamma-1}}{\gamma-k-1} \left(\frac{(\gamma-2)(\gamma-3)\cdots(\gamma-k)(\gamma-k-1)}{k!} \right)^{\frac{\gamma-1}{k}}. \tag{4.2}$$

Proof. By Lemma 3.1, we have $G_k(t, s) \le G_k(b, s)$. Set

$$F(s) = \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(b-a)^k(b-s)^{\gamma-k-1} - (b-s)^{\gamma-1}; \tag{4.3}$$

then $G_k(b, s) = \frac{1}{\Gamma(\gamma)} F(s)$. To obtain the maximum of $F(s)$, set $F'(s)$ equal to zero to obtain

$$F'(s) = -\frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}(b-a)^k(b-s)^{\gamma-k-2} + (\gamma-1)(b-s)^{\gamma-2} = 0,$$

which is true if and only if

$$s(:= s^*) = b - \left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!} \right)^{\frac{1}{k}} (b-a). \tag{4.4}$$

Clearly, $s^* < b$. Also, if $s^* < a$, then

$$k! < (\gamma-2)\cdots(\gamma-k)(\gamma-k-1),$$

which contradicts (4.1). Hence, $s^* \ge a$.

Now,

$$\begin{aligned} F''(s) &= \frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)(\gamma-k-2)}{k!}(b-s)^{\gamma-k-3}(b-a)^k \\ &\quad - (\gamma-1)(\gamma-2)(b-s)^{\gamma-3} \\ &= (\gamma-1)(\gamma-2)(b-s)^{\gamma-k-3} \left[\frac{(\gamma-3)\cdots(\gamma-k)(\gamma-k-1)(\gamma-k-2)}{k!}(b-a)^k - (b-s)^k \right]. \end{aligned}$$

If we set

$$g(s) = \frac{(\gamma-3)\cdots(\gamma-k)(\gamma-k-1)(\gamma-k-2)}{k!}(b-a)^k - (b-s)^k,$$

then

$$\begin{aligned} g(s^*) &= \frac{(\gamma-3)\cdots(\gamma-k)(\gamma-k-1)(\gamma-k-2)}{k!}(b-a)^k - \frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}(b-a)^k \\ &= \frac{(\gamma-3)\cdots(\gamma-k)(\gamma-k-1)}{k!}(b-a)^k(\gamma-k-2-\gamma+2) \\ &= -k\frac{(\gamma-3)\cdots(\gamma-k)(\gamma-k-1)}{k!}(b-a)^k < 0. \end{aligned}$$

Therefore, $F(s)$ attains its maximum at $s = s^*$, and the maximum of $F(s)$ is given by

$$\begin{aligned} F(s) &\le \max F(s) = F(s^*) = \\ &= \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(b-a)^k \left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right)^{\frac{\gamma-k-1}{k}} (b-a)^{\gamma-k-1} \\ &\quad - \left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right)^{\frac{\gamma-1}{k}} (b-a)^{\gamma-1} \\ &= (b-a)^{\gamma-1}\left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right)^{\frac{\gamma-k-1}{k}} \left[\frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}{k!}\right. \\ &\quad \left. -\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right] \\ &= (b-a)^{\gamma-1}\left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right)^{\frac{\gamma-k-1}{k}} \frac{(\gamma-2)\cdots(\gamma-k)}{k!}(\gamma-1-\gamma+k+1) \\ &= k(b-a)^{\gamma-1}\left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right)^{\frac{\gamma-k-1}{k}} \frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!(\gamma-k-1)} \\ &= \frac{k(b-a)^{\gamma-1}}{(\gamma-k-1)}\left(\frac{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}{k!}\right)^{\frac{\gamma-1}{k}}. \end{aligned}$$

Consequently, (4.2) holds, and this completes the proof of the lemma. □

Next, based on the above lemma, we present our main inequality in this section.

Theorem 4.3. *If $\gamma > k+1$, (4.1) is satisfied, and a nontrivial continuous solution of (1.1)–(1.2) exists, then*

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\gamma)(\gamma-k-1)}{k(b-a)^{\gamma-1}}\left(\frac{k!}{(\gamma-2)\cdots(\gamma-k)(\gamma-k-1)}\right)^{\frac{\gamma-1}{k}}. \tag{4.5}$$

As before, we obtain the following corollaries.

Corollary 4.4. *Let $\gamma \in (2,3)$ and $(\gamma-1)(\gamma-2) < 1$. If a nontrivial continuous solution of the fractional boundary value problem (3.5) exists, then*

$$\int_a^b |q(t)|\mathrm{d}t \ge \frac{\Gamma(\gamma)}{(b-a)^{\gamma-1}}\frac{1}{(\gamma-2)^{\gamma-2}}.$$

Proof. This can be proved by letting $k = 1$ in (4.1) and (4.5). □

Corollary 4.5. *Let $\gamma \in (3,4)$ and $(\gamma-1)(\gamma-2)(\gamma-3) < 2!$. If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, & 3 < \gamma \leq 4, \\ x(a) = x'(a) = x'''(a) = x''(b) = 0 \end{cases} \tag{4.6}$$

exists, then

$$\int_a^b |q(t)|\mathrm{d}t \geq \frac{\Gamma(\gamma)(\gamma-3)}{2(b-a)^{\gamma-1}} \left(\frac{2!}{(\gamma-2)(\gamma-3)}\right)^{\frac{\gamma-1}{2}}.$$

Proof. This can be proved by letting $k = 2$ in (4.1) and (4.5). □

Corollary 4.6. *Let $\gamma \in (4,5)$ and $(\gamma-1)(\gamma-2)(\gamma-3)(\gamma-4) < 3!$. If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, & 4 < \gamma \leq 5, \\ x(a) = x'(a) = x''(a) = x''''(a) = x'''(b) = 0 \end{cases} \tag{4.7}$$

exists, then

$$\int_a^b |q(t)|\mathrm{d}t \geq \frac{\Gamma(\gamma)(\gamma-4)}{3(b-a)^{\gamma-1}} \left(\frac{3!}{(\gamma-2)(\gamma-3)(\gamma-4)}\right)^{\frac{\gamma-1}{3}}.$$

Proof. This can be proved by letting $k = 3$ in (4.1) and (4.5). □

5 Main results: Lyapunov type inequalities–III

In Sections 3 and 4, we obtained two different Lyapunov-type inequalities. In this section, we obtain one more such inequality that is also different from the previous ones. Here we will have the same integrand that appeared in (3.2) in Section 3, whereas we only had q as the integrand in (4.5) in Section 4. Although the condition $\gamma > k+1$ is required in both of these sections, the inequality (4.1) prevents us from considering many types of boundary conditions. For example, from the observations (B_2)–(B_3) and condition (4.1), we see that we cannot ask that $k < n-2$.

In this section, we avoid condition (4.1) and find a general Lyapunov-type inequality for (1.1) together with the boundary condition (1.2), which is valid for the case $1 \leq k \leq n-2$.

Set

$$M = \max\left\{ \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)\frac{k^k(b-a)^{\gamma-1}(\gamma-k-1)^{\gamma-k-1}}{(\gamma-1)^{\gamma-1}}, \right.$$
$$\frac{k(b-a)^{\gamma-1}}{(\gamma-k-1)}\left(\frac{(\gamma-2)(\gamma-3)\cdots(\gamma-k-1)}{k!}\right)^{\frac{\gamma-1}{k}}, \tag{5.1}$$
$$\left.\frac{(b-a)^{\gamma-1}}{k!}((\gamma-1)(\gamma-2)\cdots(\gamma-k)-k!)\right\}.$$

Lemma 5.1. *Let $\gamma > k+1$. The inequality*

$$\max_{t,s\in[a,b]} G_k(t,s) \leq \frac{1}{\Gamma(\gamma)}M, \tag{5.2}$$

holds, where M is defined in (5.1).

Proof. We have $\Gamma(\gamma)G_k(t,s) = \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^k(b-s)^{\gamma-k-1}$ for $a \leq t \leq s \leq b$. Now,

$$\Gamma(\gamma)\frac{\partial G_k}{\partial t} = \frac{k}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(t-a)^{k-1}(b-s)^{\gamma-k-1} \geq 0$$

implies that $G_k(t,s)$ is non decreasing with respect to t. Hence, $\Gamma(\gamma)G_k(t,s) \leq G_k(s,s)\Gamma(\gamma)$. Set $\Gamma(\gamma)G_k(s,s) = g_1(s)$. Then,

$$g_1(s) = \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)(s-a)^k(b-s)^{\gamma-k-1},$$

and $\dfrac{dg_1}{ds} = 0$ if and only if

$$s =: s^* = a + \frac{k(b-a)}{\gamma-1}.$$

Clearly, $a < s^* < b$, and

$$\begin{aligned}
\frac{d^2g_1}{ds^2} &= \frac{1}{k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)[k(k-1)(s-a)^{k-2}(b-s)^{\gamma-k-1} \\
&\quad - k(\gamma-k-1)(s-a)^{k-1}(b-s)^{\gamma-k-2} - k(\gamma-k-1)(s-a)^{k-1}(b-s)^{\gamma-k-2} \\
&\quad + (\gamma-k-1)(\gamma-k-2)(s-a)^k(b-s)^{\gamma-k-3}] \\
&= \frac{1}{k!}(\gamma-1)\cdots(\gamma-k)(s-a)^{k-2}(b-s)^{\gamma-k-3}[k(k-1)(b-s)^2 \\
&\quad - 2k(\gamma-k-1)(s-a)(b-s) + (\gamma-k-1)(\gamma-k-2)(s-a)^2].
\end{aligned} \tag{5.3}$$

Now, $s^*-a = \frac{k(b-a)}{\gamma-1}$ and

$$(b-s^*) = (b-a) - \frac{k(b-a)}{(\gamma-1)} = \frac{(b-a)(\gamma-k-1)}{(\gamma-1)}.$$

Thus, from (5.3), we have

$$\frac{d^2 g_1}{ds^2}\Big|_{s=s^*} = -\frac{(\gamma-1)(\gamma-2)\cdots(\gamma-k)}{k!}\frac{k^{k-2}(b-a)^{\gamma-3}(\gamma-k-1)^{\gamma-k-2}}{(\gamma-1)^{\gamma-4}} < 0,$$

which shows that $g_1(s)$ attains its maximum at $s = s^*$. Hence,

$$\max_{a\le t\le s\le b} G_k(t,s) = \frac{1}{\Gamma(\gamma)k!}(\gamma-1)(\gamma-2)\cdots(\gamma-k)\frac{k^k(b-a)^{\gamma-1}(\gamma-k-1)^{\gamma-k-1}}{(\gamma-1)^{\gamma-1}}. \tag{5.4}$$

Next, suppose that $a \le s \le t \le b$. Since $\gamma > k+1$, $G_k(t,s)$ is nondecreasing with respect to t. Thus, for $a \le s \le t \le b$, we have

$$\max_{a\le s\le t\le b} G_k(t,s) = G_k(b,s) := \frac{1}{\Gamma(\gamma)}F(s), \tag{5.5}$$

where $F(s)$ is given in (4.3). Clearly $F'(s) = 0$ if and only if $s = s^*$, where s^* is given in (4.4). Moreover, $s^* < b$, $F(s)$ is nondecreasing for $s \le s^*$, nonincreasing for $s \ge s^*$, and attains its extreme (maximum) value at $s = s^*$.

First, suppose that $a \le s^*$. Then $F(s)$ attains its maximum at $s = s^*$, and the maximum value of $G_k(t,s)$ is given by

$$\max_{a\le s\le t\le b} G_k(t,s) = G_k(b,s^*) = \frac{1}{\Gamma(\gamma)}\frac{k(b-a)^{\gamma-1}}{(\gamma-k-1)}\left(\frac{(\gamma-2)(\gamma-3)\cdots(\gamma-k-1)}{k!}\right)^{\frac{\gamma-1}{k}}. \tag{5.6}$$

Finally, suppose that $s^* < a$. Then,

$$\begin{aligned}\max_{a\le s\le t\le b} G_k(t,s) &\le \max_{a\le s\le b} G_k(b,s) \le G_k(b,a)\\ &= \frac{(b-a)^{\gamma-1}}{k!\Gamma(\gamma)}((\gamma-1)(\gamma-2)\cdots(\gamma-k)-k!).\end{aligned} \tag{5.7}$$

Therefore, in view of (5.4), (5.6), and (5.7), the lemma is proved. □

Theorem 5.2. *Let $\gamma > k+1$. If $x(t)$ is a nonzero solution of* (1.1)–(1.2), *then*

$$\int_a^b |q(t)|\,dt > \frac{\Gamma(\gamma)}{M}. \tag{5.8}$$

6 Discussion and conclusions

In this section, we obtain Lyapunov-type inequalities for fractional differential equations of various orders and with different boundary conditions. We also compare our results with some existing ones in the literature.

6.1 The case $\gamma \in (2,3]$

Let $\gamma \in (2,3]$. Since $\gamma > k+1$ and $\gamma > 2$, we have $k = 1$. In this case,

$$M_1 = \max\left\{(b-a)^{\gamma-1}\left(\frac{\gamma-2}{\gamma-1}\right)^{\gamma-2},\ (b-a)^{\gamma-1}(\gamma-2)^{\gamma-2},\ (b-a)^{\gamma-1}(\gamma-2)\right\}, \tag{6.1}$$

where $M_1 = M|_{k=1}$ and M is given in (5.1). Now $2 < \gamma \leq 3$ implies $(\gamma-2)^{\gamma-2} \geq \gamma - 2$, so

$$M_1 = \left\{(b-a)^{\gamma-1}\left(\frac{\gamma-2}{\gamma-1}\right)^{\gamma-2},\ (b-a)^{\gamma-1}(\gamma-2)^{\gamma-2}\right\}. \tag{6.2}$$

We then have the following corollary.

Corollary 6.1. *Let $\gamma \in (2,3]$. If $x(t)$ is a nonzero solution of*

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+}x\right)(t) + q(t)x(t) = 0, \\ x(a) = x''(a) = x'(b) = 0, \end{cases} \tag{6.3}$$

then

$$\int_a^b |q(t)|\,dt > \frac{\Gamma(\gamma)}{M_1}. \tag{6.4}$$

Since $(b-a)^{\gamma-2} \geq \frac{1}{(\gamma-1)^{\gamma-2}}$ holds if and only if $b \geq a + \frac{1}{\gamma-1}$, we obtain the following corollary from Corollary 6.1.

Corollary 6.2. *Let $\gamma \in (2,3]$ and $b \geq a + \frac{1}{\gamma-1}$. If $x(t)$ is a nonzero solution of (6.3), then*

$$\int_a^b |q(t)|\,dt > \frac{\Gamma(\gamma)}{(b-a)^{\gamma-1}(\gamma-2)^{\gamma-2}}. \tag{6.5}$$

Now we consider the problem (3.9). Here $n = 3$, $\gamma = 3$, and $k = 1$. In this case, Corollary 3.6 shows that if (3.9) has a nontrivial solution, then (3.10) holds. Corollary 4.4 cannot be applied because $(\gamma-1)(\gamma-2) = 2 > 1$ and so (4.1) fails. By Corollary 6.1, if x is a nonzero solution of the problem (3.9), then

$$\int_a^b |q(t)|\,dt > \frac{2}{\max\{\frac{(b-a)}{2}, (b-a)^2\}} \tag{6.6}$$

holds. If $b \geq \frac{1}{2} + a$, then $\max\{\frac{(b-a)}{2}, (b-a)^2\} = (b-a)^2$. Consequently, (6.6) yields (3.10). On, the other hand, if $b < \frac{1}{2} + a$, then $\max\{\frac{(b-a)}{2}, (b-a)^2\} = \frac{(b-a)}{2}$. In this case, (6.6) yields

$$\int_a^b |q(t)|\,dt > \frac{4}{(b-a)}. \tag{6.7}$$

6.2 The case $\gamma \in (3, 4]$

Let $\gamma \in (3, 4]$. Since $\gamma > k + 1$ and $k \neq 0$, we consider the following two cases: $k = 1$ and $k = 2$. First, suppose that $k = 1$; then Theorem 5.2 yields the following corollary.

Corollary 6.3. *Let $\gamma \in (3, 4]$. If $x(t)$ is a nonzero solution of*

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+} x\right)(t) + q(t)x(t) = 0, \\ x(a) = x''(a) = x'''(a) = x'(b) = 0, \end{cases} \tag{6.8}$$

then

$$\int_a^b |q(t)|\, dt > \frac{\Gamma(\gamma)}{M_1} \tag{6.9}$$

where M_1 is given in (6.2).

Corollary 6.4. *Let $\gamma \in (3, 4]$ and $b \geq a + \frac{1}{\gamma-1}$. If $x(t)$ is a nonzero solution of* (6.8)*, then (6.5) holds.*

Finally, suppose that $k = 2$. Then Theorem 5.2 reduces to the following corollary.

Corollary 6.5. *Let $\gamma \in (3, 4]$. If $x(t)$ is a nonzero solution of*

$$\begin{cases} \left({}^{\mathrm{C}}D^{\gamma}_{a_+} x\right)(t) + q(t)x(t) = 0, \\ x(a) = x'(a) = x'''(a) = x''(b) = 0, \end{cases} \tag{6.10}$$

then

$$\int_a^b |q(t)|\, dt > \frac{\Gamma(\gamma)}{M_2}, \tag{6.11}$$

where

$$M_2 = \left\{ \frac{2(b-a)^{\gamma-1}(\gamma-2)(\gamma-3)^{\gamma-3}}{(\gamma-1)^{\gamma-2}}, \frac{2(b-a)^{\gamma-1}}{\gamma-3}\left(\frac{(\gamma-2)(\gamma-3)}{2}\right)^{\frac{\gamma-1}{2}}, \frac{\gamma(\gamma-3)(b-a)^{\gamma-1}}{2} \right\}. \tag{6.12}$$

In this paper, we obtained Lyapunov-type inequalities for higher-order fractional differential equations of Caputo-type with general two point boundary conditions. The assumption that $\gamma > k+1$ helped us to analyze the signs of the Green's function $G_k(t, s)$ and its derivatives with the price that $k \neq n-1$. Similarly, by our assumption, we have $k \neq 0$. Therefore, it would be interesting to discover a Lyapunov-type inequality for problem (1.1) for either of the boundary conditions

$$x^{(i)}(a) = x^{(n-1)}(b) = 0, \quad 0 \leq i \leq n-2$$

or

$$x^{(i)}(a) = x(b) = 0, \quad 0 \leq i \leq n-1.$$

This is left to the reader.

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Conflicts of interest

The authors have no conflicts of interest to report.

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Dual digraphs of finite meet-distributive and modular lattices

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ABSTRACT

We describe the digraphs that are dual representations of finite lattices satisfying conditions related to meet-distributivity and modularity. This is done using the dual digraph representation of finite lattices by Craig, Gouveia and Haviar (2015). These digraphs, known as TiRS digraphs, have their origins in the dual representations of lattices by Urquhart (1978) and Ploščica (1995). We describe two properties of finite lattices which are weakenings of (upper) semimodularity and lower semimodularity respectively, and then show how these properties have a simple description in the dual digraphs. Combined with previous work in this journal on dual digraphs of semidistributive lattices (2022), it leads to a dual representation of finite meet-distributive lattices. This provides a natural link to finite convex geometries. In addition, we present two sufficient conditions on a finite TiRS digraph for its dual lattice to be modular. We close by posing three open problems.

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RESUMEN

Describimos los digrafos que son representaciones duales de reticulados finitos satisfaciendo condiciones relacionadas con encuentro-distributividad y modularidad. Esto se obtiene usando la representación digrafo dual de reticulados finitos de Craig, Gouveia y Haviar (2015). Estos digrafos, conocidos como digrafos TiRS, tienen sus orígenes en las representaciones duales de reticulados de Urquhart (1978) y Ploščica (1995). Describimos dos propiedades de reticulados finitos que son debilitamientos de la semimodularidad (superior) y semimodularidad inferior respectivamente, y luego mostramos cómo estas propiedades tienen una descripción simple en los digrafos duales. Combinado con trabajo previo sobre digrafos duales de reticulados semidistributivos (2022) en esta revista, se tiene una representación dual de reticulados encuentro-distributivos. Esto entrega una conexión natural a geometrías convexas finitas. Adicionalmente, presentamos dos condiciones suficientes en un digrafo TiRS finito para que su reticulado dual sea modular. Concluimos presentando tres problemas abiertos.

Keywords and Phrases: Semimodular lattice, lower semimodular lattice, modular lattice, TiRS digraph, meet-distributive lattice, finite convex geometry.

2020 AMS Mathematics Subject Classification: 06B15, 06C10, 06C05, 05C20, 06A75.

1 Introduction

The first dual representation of arbitrary bounded lattices was given by Urquhart in 1978 [15]. Since then, many different authors have attempted to provide dualities and dual representations of classes of lattices that are not necessarily distributive (see the recent survey by the first author [4]).

In this paper we examine representations for finite lattices that satisfy conditions related to meet-distributivity and modularity. The dual structures of these finite lattices will be TiRS digraphs satisfying some additional conditions. It was shown by Craig, Gouveia and Haviar [6] that there is a one-to-one correspondence between the class of finite lattices and finite digraphs known as TiRS digraphs (see Definition 2.4 and Theorem 2.6). We remark that this correspondence generalises Birkhoff's one-to-one correspondence between finite distributive lattices and finite posets from the 1930s.

A goal of any representation is to use simple, familiar structures to represent the objects of interest. Finite TiRS digraphs provide a straightforward generalisation of finite posets. Moreover, digraphs are a well-studied class of mathematical structures and hence are well suited to be used as dual objects. In addition, the first-order description of TiRS digraphs can be used to study the finite ones with computational tools such as Prover9/Mace4 [11].

We introduce and study lattice-theoretic conditions which generalise both lower semimodularity and (upper) semimodularity for finite lattices and seem to be more natural and simpler than the conditions from [8]. We are also able to provide equivalent conditions to them on the dual TiRS digraph of a finite lattice. We can combine our lattice-theoretic conditions with our previous results in this journal [5] to characterise the dual digraphs of finite meet-distributive lattices, which correspond to finite convex geometries.

Currently, the only known dual characterisation of finite modular lattices is given by the theory of Formal Concept Analysis [8]. A rather complicated condition is available for the standard context dual to a finite semimodular lattice [8, Theorem 42]. We are able to provide conditions on the dual digraph of a finite lattice, which are sufficient though not necessary for modularity of the lattice.

The paper is laid out as follows. In Section 2 we provide some background definitions and results that will be needed later on in the paper. Section 3 defines two conditions which generalise, respectively, (upper) semimodularity and lower semimodularity. We focus on the generalisation of lower semimodularity—a condition we call (JM-LSM) (see Definition 3.6). We characterise the dual of (JM-LSM) on the dual digraphs of finite lattices. For completeness we state corresponding conditions and results related to upper semimodularity. In Section 4 we combine the results of Section 3 with results from a recent paper by Craig, Haviar and São João [5]. There, characterisations were given of the digraphs dual to finite join- and meet-semidistributive lattices (and hence also finite semidistributive lattices). The combination of these dual characterisations gives us a characterisation of the dual digraphs of finite meet-distributive lattices (also know as locally distributive lattices). Furthermore, this allows us to describe a new class of structures that is in a one-to-one correspondence with finite convex geometries. In Section 5 we give two sufficient conditions on a finite TiRS digraph for the dual lattice to be modular. In Section 6 we list three open problems and indicate why the task of describing digraphs dual to finite modular lattices is challenging.

2 Preliminaries

Central to the representation of a finite lattice that we will use is the notion of a maximal-disjoint filter-ideal pair. This can, equivalently, be viewed as a maximal partial homomorphism from a lattice L into the two-element lattice.

Definition 2.1 ([15, Section 3])**.** *Let L be a lattice. Then $\langle F, I\rangle$ is a* disjoint filter-ideal pair *of L if F is a filter of L and I is an ideal of L such that $F \cap I = \varnothing$. A disjoint filter-ideal pair $\langle F, I\rangle$ is said to be a* maximal disjoint filter-ideal pair *(MDFIP) if there is no disjoint filter-ideal pair $\langle G, J\rangle \neq \langle F, I\rangle$ such that $F \subseteq G$ and $I \subseteq J$.*

The following fact was noted by Urquhart. It is needed for our characterisation of MDFIPs in Theorem 3.2.

Proposition 2.2 ([15, p. 52])**.** *Let L be a finite lattice. If $\langle F, I\rangle$ is an MDFIP of L then $\bigwedge F$ is join-irreducible and $\bigvee I$ is meet-irreducible.*

The set of join-irreducible elements of L is denoted $\mathsf{J}(L)$ and the set of meet-irreducible elements is denoted $\mathsf{M}(L)$.

Given a lattice L, we will add a set of arcs to the set of MDFIPs of L. The use of such digraphs for lattice representation is due to Ploščica [12]. We point out that the original work using (topologised) digraphs used so-called *maximal partial homomorphisms* (see [12, Section 1]). It is easy to show that these are in a one-to-one correspondence with MDFIPs. For a lattice L, we now present its dual digraph $G_L = (X_L, E)$ where the vertices are the MDFIPs of L. Ploščica's relation E, when transferred to the set of MDFIPs, is defined below for two MDFIPs $\langle F, I\rangle$ and $\langle G, J\rangle$:

$$\text{(E)}\ \ \langle F, I\rangle E \langle G, J\rangle \quad \iff \quad F \cap J = \emptyset.$$

For finite lattices every filter is the up-set of a unique element and every ideal is the down-set of a unique element, so we can represent every disjoint filter-ideal pair $\langle F, I\rangle$ by an ordered pair $\langle {\uparrow}a, {\downarrow}b\rangle$ where $a = \bigwedge F$ and $b = \bigvee I$. Hence for finite lattices we have $\langle {\uparrow}a, {\downarrow}b\rangle E \langle {\uparrow}c, {\downarrow}d\rangle$ if and only if $a \nleqslant d$. For a digraph $G = (V, E)$ we let $xE = \{\, y \in V \mid xEy \,\}$ and $Ex = \{\, y \in V \mid yEx \,\}$. The next lemma is easy to prove and it will be useful later on.

Lemma 2.3. *Let $G_L = (X_L, E)$ be the dual digraph of a finite lattice L. If $x = \langle \uparrow a, \downarrow b\rangle$ and $y = \langle \uparrow c, \downarrow d\rangle$, then*

(i) $xE \subseteq yE$ if and only if $a \leqslant c$;

(ii) $Ex \subseteq Ey$ if and only if $d \leqslant b$.

Figure 1 shows three lattices and their dual digraphs. These three examples will be important throughout this paper. To make the labelling more succinct, we have denoted by ab the MDFIP $\langle \uparrow a, \downarrow b\rangle$. We have also left out the loop on each vertex to keep the display less cluttered. Observe that the directed edge set is not a transitive relation. The labels L_4 and L_4^∂ (as well as L_3^∂ which appears later) come from the paper by Davey *et al.* [7].

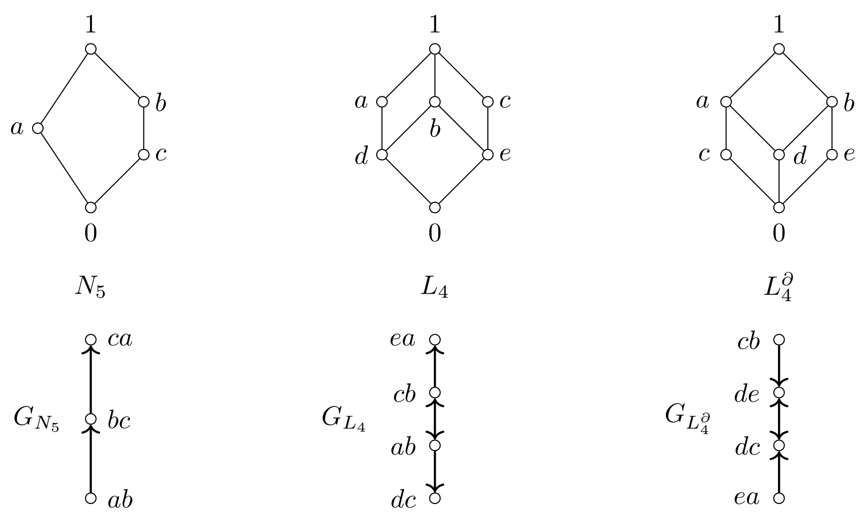

Figure 1: Finite lattices N_5, L_4, L_4^∂ and their dual digraphs.

The digraphs coming from lattices were described by Craig, Gouveia and Haviar [6]. The name "TiRS" comes from combining the conditions (Ti), (R), (S) below, where they are abbreviations for "transitive interval", "reduced" and "separated" respectively.

Definition 2.4 ([6, Definition 2.2])**.** *A TiRS digraph $G = (V, E)$ is a set V and a reflexive relation $E \subseteq V \times V$ such that:*

(S) If $x, y \in V$ and $x \neq y$ then $xE \neq yE$ or $Ex \neq Ey$.

(R) For all $x, y \in V$, if $xE \subset yE$ then $(x, y) \notin E$, and if $Ey \subset Ex$ then $(x, y) \notin E$.

(Ti) For all $x, y \in V$, if xEy then there exists $z \in V$ such that $zE \subseteq xE$ and $Ez \subseteq Ey$.

The result below gives a description of dual digraphs of lattices with least and greatest elements.

Proposition 2.5 ([6, Proposition 2.3]). *For any bounded lattice L, its dual digraph $G_L = (X_L, E)$ is a TiRS digraph.*

We recall from [12] a fact concerning general graphs $G = (X, E)$. Let $\underset{\sim}{2} = (\{0,1\}, \leqslant)$ denote the two-element graph. A partial map $\varphi\colon X \to \underset{\sim}{2}$ preserves the relation E if $x, y \in \operatorname{dom}\varphi$ and xEy imply $\varphi(x) \leqslant \varphi(y)$. The set of maximal partial E-preserving maps (*i.e.* those that cannot be properly extended) from G to $\underset{\sim}{2}$ is denoted by $\mathscr{G}^{\mathrm{mp}}(G, \underset{\sim}{2})$. We use the abbreviation MPEs for such partial maps.

For a graph $G = (X, E)$ and $\varphi, \psi \in \mathscr{G}^{\mathrm{mp}}(G, \underset{\sim}{2})$, it was shown by Ploščica [12, Lemma 1.3] that $\varphi^{-1}(1) \subseteq \psi^{-1}(1) \iff \psi^{-1}(0) \subseteq \varphi^{-1}(0)$. This implies that the reflexive and transitive binary relation $\leqslant$ defined on $\mathscr{G}^{\mathrm{mp}}(G, \underset{\sim}{2})$ by $\varphi \leqslant \psi \iff \varphi^{-1}(1) \subseteq \psi^{-1}(1)$ is a partial order. In fact, this is a lattice order [3, Theorem 2.3]. For a graph $G = (X, E)$, denote by $\mathbb{C}(G)$ the (complete) lattice of MPEs $(\mathscr{G}^{\mathrm{mp}}(G, \underset{\sim}{2}), \leqslant)$.

The theorem below gives a one-to-one correspondence between finite lattices and finite TiRS digraphs. This result is essential to the work done in the rest of the current paper.

Theorem 2.6 ([6, Theorem 1.7 and p. 87]). *For any finite lattice L we have that L is isomorphic to $\mathbb{C}(G_L)$ and for any finite TiRS digraph $G = (V, E)$ we have that G is isomorphic to $G_{\mathbb{C}(G)}$.*

3 Generalising lower and upper semimodularity

For lattice elements a and b we write $a \prec b$ to denote that a is covered by b. A lattice is *upper semimodular* if whenever $a \wedge b \prec a$ then $b \prec a \vee b$. It is common to refer to such lattices as *semimodular*. A lattice is *lower semimodular* if whenever $a \prec a \vee b$ then $a \wedge b \prec b$. We use (USM) and (LSM) as abbreviations for these two conditions.

The lattices in Figure 1 provide useful examples: N_5 satisfies neither (USM) nor (LSM), L_4 satisfies (USM) but not (LSM), and L_4^∂ satisfies (LSM) but not (USM).

We will focus on lower semimodularity, rather than upper semimodularity, because of the connection between lower semimodularity and finite convex geometries (see Section 4). We note that modularity implies both semimodularity and lower semimodularity. If a lattice L has finite length and is semimodular and lower semimodular, then L is also modular (*cf.* [9, Corollary 376]). For further reading we refer to the book by Stern [14].

Figure 2 presents a number of different generalisations of distributivity and modularity (including those presented above) and the relationships between them. The 'B' denotes *bounded* in the sense of bounded homomorphic image of a free lattice (*cf.* [9, p. 504]). Observe that the conditions in the top left and top right, which are weakenings of (LSM) and (USM) respectively, are in fact conditions on the standard context dual to a finite lattice. For the necessary terms and notation, we refer to the book from where Figure 2 is taken [8, p. 234].

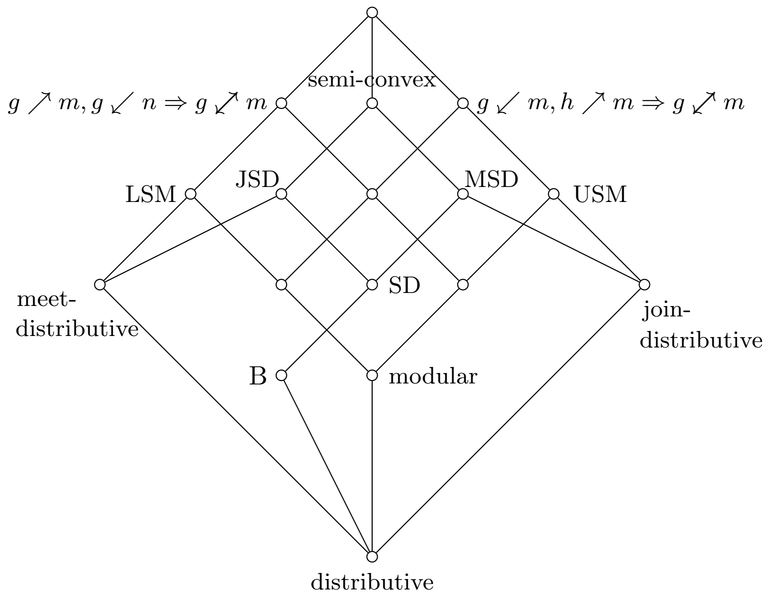

Figure 2: Relationships between generalisations of distributivity.

We begin by proving some new results about MDFIPs. These will be needed in the proofs of later results.

Lemma 3.1. *Let L be a finite lattice.*

(*i*) *If $b \in \mathsf{M}(L)$ and $b \prec a \vee b$, then $\downarrow b$ is maximal with respect to being disjoint from $\uparrow a$.*

(*ii*) *If $a \in \mathsf{J}(L)$ and $a \wedge b \prec a$, then $\uparrow a$ is maximal with respect to being disjoint from $\downarrow b$.*

Proof. Assume that $b \in \mathsf{M}(L)$ and $b \prec a \vee b$. This implies $b < a \vee b$ and hence $a \nleqslant b$ and so $\uparrow a \cap \downarrow b = \emptyset$. Suppose the ideal $\downarrow b$ were to be extended to $\downarrow c$ with $b < c$ and $\uparrow a \cap \downarrow c = \emptyset$. Since $b \in \mathsf{M}(L)$, the element $a \vee b$ is the unique upper cover of b and so $a \vee b \in \downarrow c$. This implies $a \vee b \in \uparrow a \cap \downarrow c$, a contradiction, showing the maximality of $\downarrow b$ with respect to being disjoint from $\uparrow a$.

The proof of (*ii*) follows by a dual argument. □

The next theorem gives a characterisation of MDFIPs.

Theorem 3.2. *A disjoint filter-ideal pair $\langle{\uparrow}a, {\downarrow}b\rangle$ is an MDFIP if and only if it satisfies the following conditions:*

(*i*) $a \in \mathsf{J}(L)$;

(*ii*) $b \in \mathsf{M}(L)$;

(*iii*) $b \prec a \vee b$;

(*iv*) $a \wedge b \prec a$.

Proof. If $\langle{\uparrow}a, {\downarrow}b\rangle$ is an MDFIP, by Proposition 2.2, $a \in \mathsf{J}(L)$ and $b \in \mathsf{M}(L)$. We also have $b < a \vee b$, since $b = a \vee b$ would imply $a \in {\downarrow}b$. Suppose there exists $c \in L$ such that $b < c < a \vee b$. If $a \leqslant c$ then c would be an upper bound for $\{a, b\}$ and then $a \vee b \leqslant c$. Therefore $a \nleqslant c$. This would make $\langle{\uparrow}a, {\downarrow}c\rangle$ a disjoint filter-ideal pair with ${\downarrow}b \subsetneq {\downarrow}c$, contradicting the maximality of the pair $\langle{\uparrow}a, {\downarrow}b\rangle$. A dual argument can be applied to show that $a \wedge b \prec a$.

Assume $\langle{\uparrow}a, {\downarrow}b\rangle$ satisfies $(i) - (iv)$. Lemma 3.1 says ${\downarrow}b$ is maximal with respect to being disjoint from ${\uparrow}a$ and vice versa. Hence $\langle{\uparrow}a, {\downarrow}b\rangle$ is an MDFIP. □

The lemmas below will be used in our later investigations.

Lemma 3.3. *Let L be a finite lattice, $a, b \in L$. The following are equivalent:*

(*i*) $a \nleqslant b$;

(*ii*) *there exists $j \in \mathsf{J}(L)$ such that $j \leqslant a$ and $j \nleqslant b$;*

(*iii*) *there exists $m \in \mathsf{M}(L)$ such that $b \leqslant m$ and $a \nleqslant m$.*

Proof. It is well-known that in a finite lattice the set $\mathsf{J}(L)$ is join-dense. Hence $a \leqslant b$ is equivalent to the condition that for all $j \in \mathsf{J}(L)$, $j \leqslant a$ implies $j \leqslant b$. This settles the equivalence of (i) and (ii). The equivalence of (i) and (iii) follows similarly from the meet-density of $\mathsf{M}(L)$ in L. □

For $a, b \in L$ we define the set $T_{ab} := \{\, m \in \mathsf{M}(L) \mid b \leqslant m, a \nleqslant m \,\}$. An important consequence of Lemma 3.3 is that T_{ab} is non-empty whenever $a \nleqslant b$. This is needed for our next result.

Lemma 3.4. *Let L be a finite lattice and $a, b \in L$, $a \nleqslant b$. Let d be a maximal element of T_{ab}. Then $d \prec d \vee a$.*

Proof. Firstly, we point out that T_{ab} is a non-empty finite poset and hence has a maximal element. Since $a \nleqslant d$, we have $a \vee d \neq d$, and so $d < d \vee a$. Suppose there exists $c \in L$ such that $d < c < d \vee a$. As $d \vee a \nleqslant c$, by Lemma 3.3 there exists $m \in \mathsf{M}(L)$ such that $c \leqslant m$ but $d \vee a \nleqslant m$. So $d < m$. If $a \leqslant m$ then $d \vee a \leqslant m$. It follows $a \nleqslant m$ and $b \leqslant d < m$, so $m \in T_{ab}$. Since d was maximal in T_{ab} and $d < m$, we get a contradiction. Hence $d \prec d \vee a$. □

From the previous lemmas one can derive the following result.

Proposition 3.5. *Let L be a finite lattice with $a \in \mathsf{J}(L)$ and $b \in \mathsf{M}(L)$. Then*

(*i*) *there exists $m \in \mathsf{M}(L)$ such that $\langle {\uparrow} a, {\downarrow} m\rangle$ is an MDFIP;*

(*ii*) *there exists $j \in \mathsf{J}(L)$ such that $\langle {\uparrow} j, {\downarrow} b\rangle$ is an MDFIP.*

Proof. We prove only (i), as then (ii) will follow by a dual argument. Since $a \in \mathsf{J}(L)$, it has a unique lower cover c. Clearly $a \nleq c$, so by Lemma 3.4, there exists a maximal element $m \in T_{ac}$ such that $m \prec m \vee a$. From Lemma 3.1(i) we know that ${\downarrow} m$ is maximal with respect to being disjoint from ${\uparrow} a$. If it were possible to extend ${\uparrow} a$ to ${\uparrow} d$ with $d < a$, then since c is the unique lower cover of a, we would get $c \in {\uparrow} d \cap {\downarrow} m$. Hence ${\uparrow} a$ is maximal with respect to being disjoint from ${\downarrow} m$. It follows that $\langle {\uparrow} a, {\downarrow} m\rangle$ is an MDFIP. □

We now define a new condition, (JM-LSM), which will be central to the results that follow. We believe it is a more natural weakening of (LSM) than the condition given in the top left of Figure 2. The name of the condition comes from the fact that it is almost identical to the condition (LSM), but the elements involved are quantified over $\mathsf{J}(L)$ and $\mathsf{M}(L)$.

Definition 3.6. *A finite lattice L satisfies* (JM-LSM) *if for any $a \in \mathsf{J}(L)$ and $b \in \mathsf{M}(L)$, if $b \prec a \vee b$ then $a \wedge b \prec a$.*

Example 3.7. *Condition* (JM-LSM) *is a proper weakening of the condition* (LSM). *Indeed, the lattice in Figure 3 satisfies* (JM-LSM) *but not* (LSM). *To see this, observe that $c \prec c \vee d$ and $c \wedge d \not\prec d$, yet $d \notin \mathsf{J}(L)$.*

We note that the lattice L_4 in Figure 1 does not satisfy (LSM)*, and also does not satisfy* (JM-LSM)*: $c \in \mathsf{J}(L)$, $a \in \mathsf{M}(L)$ and $a \prec c \vee a$, yet $c \wedge a \not\prec c$.*

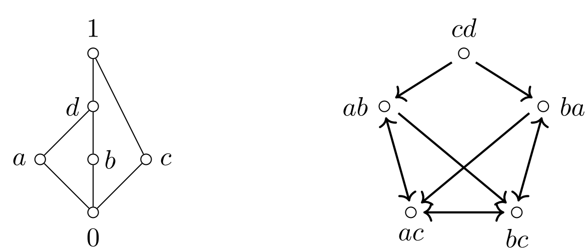

Figure 3: A finite lattice that satisfies (JM-LSM) but not (LSM). Its dual digraph (right) satisfies (LTi).

Below is a condition that we will prove is equivalent to (JM-LSM). It will assist us in proving that the digraph condition (LTi), given in Definition 3.11, can be used to characterise the dual digraphs of finite (JM-LSM) lattices.

Definition 3.8. *Condition* (L-abc)*: Let $a \in \mathsf{J}(L)$ and $b \in \mathsf{M}(L)$. If $a \nleqslant b$ then there exists $c \geqslant b$ such that $\langle {\uparrow}a, {\downarrow}c\rangle$ is an MDFIP.*

Notice that if $\langle {\uparrow}a, {\downarrow}c\rangle$ is an MDFIP, then Proposition 2.2 (*cf.* also Theorem 3.2) implies that for the element c in Definition 3.8 we have $c \in \mathsf{M}(L)$. Notice also that the finite lattice L_4 in Figure 1 does not satisfy (L-abc): we have $a \in \mathsf{J}(L)$, $c \in \mathsf{M}(L)$ and $a \nleqslant c$ and there is no $m \geqslant c$ such that $\langle {\uparrow}a, {\downarrow}m\rangle$ is an MDFIP.

The following theorem shows that for finite lattices the central property (JM-LSM) can be characterised exactly via the condition (L-abc).

Theorem 3.9. *A finite lattice satisfies* (JM-LSM) *iff it satisfies* (L-abc).

Proof. Assume (JM-LSM) and let $a \in \mathsf{J}(L)$, $b \in \mathsf{M}(L)$ and $a \nleqslant b$. Let $T_{ab} = \{m \in \mathsf{M}(L) \mid b \leqslant m \ \& \ a \nleqslant m\}$. Then T_{ab} is a non-empty finite poset. Hence it has a maximal element, say c. So $c \in \mathsf{M}(L)$, $b \leqslant c$ and $\langle {\uparrow}a, {\downarrow}c\rangle$ is a disjoint filter-ideal pair. To show that $\langle {\uparrow}a, {\downarrow}c\rangle$ is an MDFIP, by Theorem 3.2 we need to show that $c \wedge a \prec a$ and $c \prec c \vee a$. By (JM-LSM) we only need to prove $c \prec c \vee a$, which follows from Lemma 3.4. We have shown that (L-abc) holds.

Now assume (L-abc). To show (JM-LSM), let $a \in \mathsf{J}(L)$, $b \in \mathsf{M}(L)$ and $b \prec a \vee b$. We need to prove $a \wedge b \prec a$. From $b \prec a \vee b$ we have $a \nleqslant b$. By (L-abc) there exists $c \geqslant b$ such that $\langle {\uparrow}a, {\downarrow}c\rangle$ is an MDFIP. Hence $c \in \mathsf{M}(L)$ and by Theorem 3.2, $c \wedge a \prec a$. We claim that $c = b$. Suppose that $c > b$. Then, since $b \in \mathsf{M}(L)$, it has a unique upper cover $b^\star$. As $b \prec a \vee b$, we get $b^\star = a \vee b$. From $c > b$ we have $c \geqslant b^\star = a \vee b \geqslant a$. This contradicts the fact that $\langle {\uparrow}a, {\downarrow}c\rangle$ is an MDFIP. Hence $c = b$. This proves $a \wedge b = c \wedge a \prec a$. □

Remark 3.10. *We notice that if a finite lattice L satisfies* (L-abc)*, then in the situation $a \nleqslant b$ for $a \in \mathsf{J}(L)$, $b \in \mathsf{M}(L)$, an arbitrary maximal element of T_{ab} can be taken for the element $c \geqslant b$ such that $\langle {\uparrow}a, {\downarrow}c\rangle$ is an MDFIP. Indeed, if c is any maximal element of T_{ab}, then $c \in \mathsf{M}(L)$, $a \nleqslant c$, $b \leqslant c$ and so by the assumed condition* (L-abc) *there is $c' \geqslant c$ such that $\langle {\uparrow}a, {\downarrow}c'\rangle$ is an MDFIP. Hence $c' \in \mathsf{M}(L)$, $a \nleqslant c'$, $b \leqslant c'$, thus $c' \in T_{ab}$. From the maximality of c in T_{ab} we get $c = c'$ as required.*

Now we present a digraph condition dual to (JM-LSM). The condition is a strengthening of the (Ti) condition, and because of its connection to lower semimodularity, we have chosen the name (LTi). Later, in Definition 3.16, (UTi) is used for the dual condition related to upper semimodularity.

Definition 3.11. *Consider the condition below on a TiRS digraph $G = (V, E)$:*

$$\text{(LTi)} \qquad uEv \implies (\exists\, w \in V)(wE = uE \,\&\, Ew \subseteq Ev).$$

Note that (LTi) is not dual to (LSM) as Figure 3 shows. For each pair of E related vertices, there is some vertex making the consequent true. For example, if $u = ba$ and $v = ac$, we have $baEac$, and we can let $w = bc$, since $baE = \{bc, ba, ac\} = bcE$, and $Ebc = \{ab, ac, ba, bc\} \subseteq \{ab, ac, ba, bc\} = Eac$. The next two results prove that it is (JM-LSM) that is dual to (LTi).

Proposition 3.12. *A finite TiRS digraph satisfies* (LTi) *if and only if it is the dual digraph of a lattice that satisfies* (L-abc)*.*

Proof. Assume a finite lattice L satisfies (L-abc). To show that the dual digraph G_L satisfies (LTi), let $u = \langle\uparrow a, \downarrow m\rangle$, $v = \langle\uparrow j, \downarrow b\rangle$ be vertices of the digraph G and let uEv, whence $a \nleqslant b$. Then by (L-abc) there exists $c \in \mathsf{M}(L)$ such that $b \leqslant c$ and $\langle\uparrow a, \downarrow c\rangle$ is an MDFIP. If we denote $w = \langle\uparrow a, \downarrow c\rangle$ as a vertex of G, then by Lemma 2.3 we have $wE = uE$ and $Ew \subseteq Ev$ as required.

For the converse, assume that a finite TiRS digraph G satisfies (LTi). To show that its dual lattice L satisfies (L-abc), let $a \in \mathsf{J}(L)$, $b \in \mathsf{M}(L)$ and $a \nleqslant b$. Since $a \in \mathsf{J}(L)$ and L is finite, by Proposition 3.5(i), there exists an element $m \in M(L)$ such that $u = \langle\uparrow a, \downarrow m\rangle$ is an MDFIP. Similarly, since $b \in \mathsf{M}(L)$, by Proposition 3.5(ii) there exists $j \in J(L)$ such that $v = \langle\uparrow j, \downarrow b\rangle$ is an MDFIP. Since $a \nleqslant b$, we have uEv. Now, by (LTi), there is a vertex $w = \langle\uparrow c, \downarrow d\rangle \in V(G)$ satisfying $wE = uE$ and $Ew \subseteq Ev$. Since $wE = uE$, we get $\uparrow c = \uparrow a$, so $c = a$. Since $Ew \subseteq Ev$, Lemma 2.3(ii) tells us that $d \geqslant b$. This proves that d is the desired element such that $\langle\uparrow a, \downarrow d\rangle$ is an MDFIP. $\square$

The main theorem of this section follows directly from Theorem 3.9 and Proposition 3.12.

Theorem 3.13. *A finite TiRS digraph is the dual digraph of a finite lattice satisfying* (JM-LSM) *if and only if it satisfies* (LTi)*.*

For completeness, we now state the conditions and results related to finite upper semimodular lattices and their dual digraphs.

Definition 3.14. *Let L be a finite lattice. We say that L satisfies the condition* (JM-LSM) *if whenever $a \in \mathsf{J}(L)$, $b \in \mathsf{M}(L)$, and $a \wedge b \prec a$, then $b \prec a \vee b$. We say that L satisfies* (U-abc) *if whenever $a \in \mathsf{J}(L)$ and $b \in \mathsf{M}(L)$ and $a \nleqslant b$ then there exists $c \leqslant a$ such that $\langle\uparrow c, \downarrow b\rangle$ is an MDFIP.*

The proposition below connects the two conditions defined above.

Proposition 3.15. *A finite lattice satisfies* (U-abc) *iff it satisfies* (JM-USM)*.*

Our last definition is the condition (UTi) which is, like (LTi), a strengthening of the (Ti) condition from Definition 2.4.

Definition 3.16. *Consider the condition below on a TiRS digraph $G = (V, E)$:*

$$\text{(UTi)} \qquad uEv \implies (\exists\, w \in V)(wE \subseteq uE \;\&\; Ev = Ew).$$

Theorem 3.17. *A finite TiRS digraph satisfies* (UTi) *if and only if it is the dual digraph of a finite lattice that satisfies* (JM-USM)*.*

4 Dual digraphs of meet-distributive lattices

In this section we will combine the results from Section 3 with results about dual digraphs of finite join- and meet-semidistributive lattices from [5]. The goal is to give a description of the dual digraphs of finite meet-distributive lattices. This will give a description of a new class of structures that are in a one-to-one correspondence with the class of finite convex geometries. First, we recall some basic definitions.

A lattice L is *join-semidistributive* if it satisfies the following quasi-equation for all $a, b, c \in L$:

$$\text{(JSD)} \qquad a \vee b \approx a \vee c \quad \longrightarrow \quad a \vee b \approx a \vee (b \wedge c).$$

A lattice L is *meet-semidistributive* if it satisfies the following quasi-equation for all $a, b, c \in L$:

$$\text{(MSD)} \qquad a \wedge b \approx a \wedge c \quad \longrightarrow \quad a \wedge b \approx a \wedge (b \vee c).$$

A lattice is *semidistributive* if it satisfies both (JSD) and (MSD).

Considering the lattices in Figure 1 one can see that N_5 is semidistributive, L_4 is meet-semidistributive but not join-semidistributive, and L_4^∂ is join-semidistributive but not meet-semidistributive.

For a finite lattice L and $a \in L$, consider $\mu(a) = \bigwedge\{ b \in L \mid b \prec a \}$. A finite lattice is *meet-distributive* (also called *locally distributive*) if for any $a \in L$, the interval $[\mu(a), a]$ is a distributive lattice (*cf.* [1, Section 5 - 2]). The class of finite meet distributive lattices is an important class of lattices because of their link to finite convex geometries. The following results therefore lead us to a new characterisation of finite convex geometries, which we present in Theorem 4.13 (*iv*) and (*v*).

The following equivalence is extracted from [1, Theorem 5-2.1].

Theorem 4.1. *Let L be a finite lattice. Then the following are equivalent:*

(*i*) *L is meet-distributive;*

(*ii*) *L satisfies* (JSD) *and* (LSM)*.*

The results below use Theorem 4.1 to provide an additional characterisation of meet-distributive lattices using (JM-LSM), the condition that was central to Section 3. Later, we will use this to characterise their dual digraphs.

Theorem 4.2. *If a finite lattice L satisfies* (JM-LSM) *and* (JSD)*, then it is lower semimodular.*

Proof. Let L be a finite lattice satisfying (JM-LSM) and (JSD). Let $a, b \in L$ be arbitrary such that $a \prec a \vee b$. We are going to show that $a \wedge b \prec b$. We will proceed by contradiction.

Suppose that $a \wedge b \not\prec b$. Then there exists $c \in L$ such that $a \wedge b < c < b$. Then $b \nleqslant c$ and by Lemma 3.3 the set $S_{cb} = \{ j \in \mathsf{J}(L) \mid j \leqslant b, j \nleqslant c \}$ is non-empty. Let p be a minimal element of S_{cb}.

Suppose $p \leqslant a$, then since $p \leqslant b$, we get $p \leqslant a \wedge b \leqslant c$, which is a contradiction, so $p \nleqslant a$. Then by Lemma 3.3, the set $T_{pa} = \{m \in M(L) \mid a \leqslant m \text{ and } p \nleqslant m\}$ is non-empty. Let m be a maximal element of T_{pa}. By Lemma 3.4, $m \prec m \vee p$. Since $m \in M(L)$, $p \in J(L)$, and L satisifies (JM-LSM), we obtain $m \wedge p \prec p$.

The join irreducible element p has a unique lower cover p_*; likewise the meet irreducible element m has a unique upper cover m^*. Then $p_* \leqslant m$ as $p_* = m \wedge p$. Now $p \nleqslant c$ and $p_* \leqslant c$ imply $c \wedge p = p_*$. Analogously, $p \nleqslant m$ and $p \leqslant m^*$ imply $m \vee p = m^*$. It follows that $c \nleqslant m$ as otherwise we get $c \leqslant m \wedge (a \vee b) = a$, whence $c \leqslant a \wedge b$, which contradicts $a \wedge b < c$. But $c \leqslant m^*$ since $m^* \geqslant a \vee p = a \vee b \geqslant b$. Here we used that since $p \leqslant b$, we have $a \leqslant a \vee p \leqslant a \vee b$, and since $a \prec a \vee b$, we have $a = a \vee p$ or $a \vee p = a \vee b$. But $a \neq a \vee p$ since $p \nleqslant a$, so $a \vee p = a \vee b$.

Hence $m \vee c = m^*$. Combining the above,

$$m^* = m \vee p = m \vee c = m \vee (p \wedge c) = m \vee p_* = m$$

by (JSD), a contradiction. Hence c cannot exist showing that $a \wedge b \prec b$. □

Remark 4.3. *Notice in the proof we actually use a weaker form of* (JSD)*. We will say that a lattice L is* weakly join-semidistributive *if it satisfies the following quasi-equation for all $a \in \mathsf{M}(L)$, $b \in \mathsf{J}(L)$, $c \in L$:*

$$\text{(W-JSD)} \qquad a \vee b \approx a \vee c \quad \longrightarrow \quad a \vee b \approx a \vee (b \wedge c).$$

Hence in Theorem 4.2 we actually showed that (JM-LSM) *and* (W-JSD) *implies* (LSM)*.*

We notice the lattice in Figure 3 satisfies (JM-LSM) *but not* (W-JSD)*: indeed $c \in \mathsf{M}(L)$, $b \in \mathsf{J}(L)$ and $c \vee b = c \vee a$ but $c \vee (b \wedge a) \neq c \vee a$.*

The result below follows from Theorems 4.1 and 4.2.

Corollary 4.4. *A finite lattice is meet-distributive if and only if it satisfies both* (JM-LSM) *and* (JSD)*.*

The following theorem provides a characterisation of the dual digraphs of finite join- and meet-semidistributive lattices. Its proof (see [5]) relies on the well-known κ map used in the characterisation of semidistributivity. Notice that each of the conditions (i), (ii) and (iii) below is a strengthening of the (S) condition from the definition of TiRS digraphs (Definition 2.4).

Theorem 4.5 ([5, Theorem 3.6])**.** *Let $G = (V, E)$ be a finite TiRS digraph with $u, v \in V$. Then*

(i) *G is the dual digraph of a finite lattice satisfying* (JSD) *if and only if it satisfies the following condition:*

$$\text{(dJSD)} \qquad \textit{if } u \neq v \textit{ then } Eu \neq Ev.$$

(ii) *G is the dual digraph of a finite lattice satisfying* (MSD) *if and only if it satisfies the following condition:*

$$\text{(dMSD)} \qquad \textit{if } u \neq v \textit{ then } uE \neq vE.$$

(iii) *G is the dual digraph of a finite semidistributive lattice if and only if it satisfies the following condition:*

$$\text{(dSD)} \qquad \textit{if } u \neq v \textit{ then } Eu \neq Ev \textit{ and } uE \neq vE.$$

The next few results in this section link the properties discussed earlier to distributivity in lattices and transitivity in dual digraphs.

Theorem 4.6. *Let $G = (V, E)$ be a finite TiRS digraph that satisfies both* (dMSD) *and* (LTi)*. Then E is transitive.*

Proof. We first claim that if a finite TiRS digraph $G = (V, E)$ satisfies both (dMSD) and (LTi), then for any vertices $u, v \in V$, uEv implies $Eu \subseteq Ev$. Indeed, uEv by (LTi) implies the existence of $w \in V$ such that $wE = uE$ and $Ew \subseteq Ev$. By the property (dMSD), $wE = uE$ means $w = u$, whence $Eu \subseteq Ev$ as required.

Now to show the transitivity of E, if uEv and vEw for some vertices $u, v, w \in V$, then by the above claim, $Eu \subseteq Ev$ and $Ev \subseteq Ew$. Hence $Eu \subseteq Ew$, which means $u \in Ew$, whence uEw as required. ☐

Proposition 4.7. *If $G = (V, E)$ is TiRS digraph with transitive E, then G is a poset.*

Proof. As in a TiRS digraph $G = (V, E)$ the relation E is reflexive, it only remains to show the antisymmetry of E.

Assume for $x, y \in V$ that xEy and yEx. We firstly show that $xE \subseteq yE$: if $z \in V$ and $z \in xE$, then xEz and with yEx we get yEz by transitivity of E, hence $z \in yE$ as required. Now $xE \subset yE$ by the condition (R) from Definition 2.4 would give $(x, y) \notin E$, a contradiction. Hence $xE = yE$.

Analogously one can show that $Ey \subseteq Ex$ and since $Ey \subset Ex$ would by (R) give $(x, y) \notin E$, we have $Ey = Ex$. Using that G satisfies the separation property (S) from Definition 2.4, it follows that $x = y$ as required. ☐

The result below follows from Theorem 4.6, Proposition 4.7 and Birkhoff's one-to-one correspondence between finite distributive lattices and finite posets, which was in [6] generalised into a one-to-one correspondence between the class of finite lattices and finite TiRS digraphs (*cf.* Theorem 2.6 here).

Corollary 4.8. *If a finite lattice L satisfies* (MSD) *and* (JM-LSM)*, then L is distributive.*

We now return to focus on finite meet-distributive lattices, with the goal of describing a class of digraphs connected to finite convex geometries.

Using the TiRS conditions, our conditions for the dual digraphs of (JM-LSM) and (JSD), respectively, and Corollary 4.4, we get the following dual condition for meet-distributivity. Notice how (dJSD) is a strengthening of the (S) condition, and (LTi) is a strengthening of the (Ti) condition.

Theorem 4.9. *A finite digraph $G = (V, E)$ with a reflexive relation E is the dual digraph of some finite meet-distributive lattice if and only if G satisfies the following conditions:*

(dJSD) *If $x, y \in V$ and $x \neq y$ then $Ex \neq Ey$.*

(R) *For all $x, y \in V$, if $xE \subset yE$ then $(x, y) \notin E$, and if $Ey \subset Ex$ then $(x, y) \notin E$.*

(LTi) *For all $x, y \in V$, if xEy then there exists $z \in V$ such that $zE = xE$ and $Ez \subseteq Ey$.*

Proof. Let G be the dual digraph of some finite meet-distributive lattice L. Then by Theorem 2.6 the digraph G will satisfy (R). By Corollary 4.4, L satisfies (JSD) and (JM-LSM). Hence by Theorem 4.5(i), G satisfies (dJSD). Lastly, by Theorem 3.13, G will satisfy (LTi).

Conversely, assume G satisfies (dJSD), (R) and (LTi). Clearly G is a TiRS digraph, hence the dual of a finite lattice L. Theorem 4.5(i) shows that L satisfies (JSD) and Theorem 3.13 implies that L satisfies (JM-LSM). Hence by Corollary 4.4, L is meet-distributive. □

The theorem above establishes a one-to-one correspondence between finite meet-distributive lattices and finite digraphs satisfying the conditions (dJSD), (R) and (LTi). It is a restriction of Theorem 2.6, while still generalising Birkhoff's one-to-one correspondence between finite distributive lattices and finite posets.

Definition 4.10 ([9, Definition 30])**.** *Let X be a set and $\phi : \wp(X) \to \wp(X)$. Then ϕ is a* closure operator *on X if for all $Y, Z \in \wp(X)$*

(*i*) $Y \subseteq \phi(Y)$;

(*ii*) $Y \subseteq Z$ *implies* $\phi(Y) \subseteq \phi(Z)$;

(*iii*) $\phi(\phi(Y)) = \phi(Y)$.

If X is a set and ϕ a closure operator on X then the pair $\langle X, \phi\rangle$ is called a closure system. *For $Y \subseteq X$ we say that Y is* closed *if $\phi(Y) = Y$. The closed sets of a closure operator ϕ on X form a complete lattice, denoted by* $\mathrm{Cld}(X, \phi)$. *A* zero-closure *system is a closure system $\langle X, \phi\rangle$ such that $\phi(\emptyset) = \emptyset$.*

Now we turn our attention to convex geometries. The presentation here follows that of the book chapter by Adaricheva and Nation [1].

Definition 4.11 ([1, Definition 5-1.1])**.** *A closure system $\langle X, \phi\rangle$ satisfies the* anti-exchange property *if for all $x \neq y$ and all closed sets $A \subseteq X$,*

$$\text{(AEP)} \qquad x \in \phi(A \cup \{y\}) \text{ and } x \notin A \text{ imply that } y \notin \phi(A \cup \{x\}).$$

Definition 4.12 ([2, Definition 1.6])**.** *A zero-closure system that satisfies the anti-exchange property is called a* convex geometry.

We now combine Theorem 4.9 with known equivalences to obtain the following characterisation of finite convex geometries. There are other equivalent conditions [1, Theorem 5-2.1] that we have not included here.

Theorem 4.13. *Let L be a finite lattice. Then the following are equivalent:*

(*i*) *L is the closure lattice* $\mathrm{Cld}(X, \phi)$ *of a closure space $\langle X, \phi\rangle$ with the* (AEP).

(*ii*) *L is a meet-distributive lattice.*

(*iii*) *L satisfies* (JSD) *and* (LSM).

(*iv*) *L satisfies* (JSD) *and* (JM-LSM).

(*v*) *L is the lattice $\mathbb{C}(G)$ of a reflexive digraph G satisfying* (dJSD), (R) *and* (LTi).

Proof. The equivalences of (*i*), (*ii*) and (*iii*) are known [1, Theorem 5-2.1]. The equivalence of (*iii*) and (*iv*) is the result of Corollary 4.4, and the equivalence of (*iv*) and (*v*) is Theorem 4.9. □

5 Dual digraphs of finite modular lattices

In this section we provide two sufficient conditions for a finite TiRS digraph to be the dual digraph of a finite modular lattice.

For $i = 0, 1, 2$, let us denote by $G_i = (V_i, E_i)$ an induced subgraph of G_{N_5} (see Figure 1) with $V_i = \{x, y, z\}$ and with i of the arcs xEy and yEz missing compared to G_{N_5}. (For $i = 1$ we can, w.l.o.g., consider the arc yEz missing.) Hence $G_0 = G_{N_5}$, G_1 has one arc and an isolated vertex, and G_2 has no arc and consists of two isolated vertices. All three digraphs are reflexive, hence they have loops at each vertex.

We introduce the following condition for the dual digraph G_L of a finite lattice L in terms of "Forbidden Induced Subgraphs":

(FIS) G_L has neither $G_0 = G_{N_5}$ nor G_1 as an induced subgraph.

The next lemma and two propositions lead to showing that the condition (FIS) is sufficient for modularity of a finite lattice L. Note that by Lemma 3.3, for $a, b \in L$ with $a \nleqslant b$, there always exist elements $\underline{a} \leqslant a$ and $\bar{b} \geqslant b$ such that $\langle \uparrow\underline{a}, \downarrow\bar{b} \rangle$ is an MDFIP. Below we write $a||b$ to indicate that $a \nleqslant b$ and $b \nleqslant a$.

Lemma 5.1. *Let $a, b, c, 0, 1$ be any elements of the lattice that form a sublattice isomorphic to N_5 (where $0 < a, b, c < 1$, $c < b$ and $a||b, a||c$). (See the left side of Figure 4.) Let $x = \langle \uparrow\underline{a}, \downarrow\bar{b} \rangle$, $y = \langle \uparrow\underline{b}, \downarrow\bar{c} \rangle$ and $z = \langle \uparrow\underline{c}, \downarrow\bar{a} \rangle$ be any maximal disjoint extensions of $\langle \uparrow a, \downarrow b \rangle$, $\langle \uparrow b, \downarrow c \rangle$ and $\langle \uparrow c, \downarrow a \rangle$, respectively. Then the induced subgraph $\{x, y, z\}$ of G_L is isomorphic either to $G_0 = G_{N_5}$, G_1, or G_2.*

Proof. First we must confirm that x, y, z are distinct MDFIPs. If $x = y$ then $\uparrow\underline{a} = \uparrow\underline{b}$ which implies $\uparrow\underline{a} \cap \downarrow\bar{b} \neq \emptyset$, *i.e.* x would not be an MDFIP. If $x = z$ then $\uparrow\underline{a} = \uparrow\underline{c}$ which means z would not be an MDFIP. Lastly, if $y = z$ then $\downarrow\bar{c} = \downarrow\bar{a}$ and z would not be an MDFIP.

We claim that in the induced subgraph $\{x, y, z\}$ of G_L, the arcs xEy and yEz are possible, but the induced subgraph $\{x, y, z\}$ has none of the other four possible arcs between distinct vertices: indeed, the arcs yEx, zEy, xEz and zEx are not present in G_L because clearly $b \in \uparrow\underline{b} \cap \downarrow\bar{b}$, $c \in \uparrow\underline{c} \cap \downarrow\bar{c}$, $a \in \uparrow\underline{a} \cap \downarrow\bar{a}$ and $c \in \uparrow\underline{c} \cap \downarrow\bar{b}$, respectively.

Hence $\{x, y, z\}$ is isomorphic to G_i in case i of the arcs xEy and yEz are missing in the induced subgraph $\{x, y, z\}$ for $i = 0, 1, 2$. □

Proposition 5.2. *Let L be a finite lattice and assume that its dual digraph $G_L = (V, E)$ satisfies* (FIS)*. Then L is lower semimodular.*

Proof. Suppose to the contrary that L does not satisfy (LSM). Then there exist elements $a, b \in L$ such that $a \prec a \vee b$ but $a \wedge b \nprec b$. Then there exists an element $c \in L$ such that $a \wedge b < c < b$. Hence $a \vee c \leqslant a \vee b$. Since $a \prec a \vee b$, and $a \leqslant a \vee c \leqslant a \vee b$, we get $a \vee c = a$ or $a \vee c = a \vee b$. If $a \vee c = a$, then $c \leqslant a$, so $c \leqslant a \wedge b$, which contradicts $a \wedge b < c$. It follows that $a \vee c = a \vee b$. From $c < b$ we get $a \wedge c \leqslant a \wedge b$. Further, since $a \wedge b < c$ we get $a \wedge (a \wedge b) = a \wedge b \leqslant a \wedge c$. Thus $a \wedge c = a \wedge b$.

Hence $a, c, b, a \wedge b, a \vee b$ forms a sublattice isomorphic to N_5 (see Figure 4). Let $x = \langle \uparrow \underline{a}, \downarrow \overline{b} \rangle$, $y = \langle \uparrow \underline{b}, \downarrow \overline{c} \rangle$ and $z = \langle \uparrow \underline{c}, \downarrow \overline{a} \rangle$, be arbitrary maximal disjoint extensions of $\langle \uparrow a, \downarrow b \rangle$, $\langle \uparrow b, \downarrow c \rangle$ and $\langle \uparrow c, \downarrow a \rangle$, respectively. Then by Lemma 5.1, the induced subgraph $\{x, y, z\}$ of G_L is isomorphic to $G_0 = G_{N_5}$, G_1, or G_2. Using the assumption (FIS), $\{x, y, z\}$ must be isomorphic to G_2.

In particular, it follows that G_L does not have the arc yEz. Therefore $\underline{b} \leqslant \overline{a}$. Suppose $a = \overline{a}$. Then $\underline{b} \leqslant a$, so $\underline{b} \leqslant a \wedge b$. This gives $\underline{b} \leqslant c \leqslant \overline{c}$, which contradicts the fact that $y = \langle \uparrow \underline{b}, \downarrow \overline{c} \rangle$ is a disjoint filter-ideal pair. Hence $a < \overline{a}$. Now either $\overline{a} < a \vee b$ or $\overline{a} || a \vee b$, since if $\overline{a} \geqslant a \vee b > c \geqslant \underline{c}$ then $z = \langle \uparrow \underline{c}, \downarrow \overline{a} \rangle$ could not be a disjoint filter-ideal pair.

If $a < \overline{a} < a \vee b$, this contradicts $a \prec a \vee b$, so $\overline{a} || a \vee b$. If $\underline{b} > a$ then $b > \underline{b} > a$, which contradicts $a || b$. If $\underline{b} \leqslant a$, then $\underline{b} \leqslant a \wedge b \leqslant c \leqslant \overline{c}$, which contradicts that $y = \langle \uparrow \underline{b}, \downarrow \overline{c} \rangle$ is a disjoint filter-ideal pair. This proves that $\underline{b} || a$. Since $\underline{b} \leqslant b$, $a \vee \underline{b} \leqslant a \vee b$. If $a \vee \underline{b} = a \vee b$, then since $a < \overline{a}$ and $\underline{b} \leqslant \overline{a}$, we get $\overline{a} \geqslant a \vee \underline{b} = a \vee b$, which contradicts $\overline{a} || a \vee b$. This establishes that $a \vee \underline{b} < a \vee b$ and $a < a \vee \underline{b}$ (since $\underline{b} || a$), which contradicts $a \prec a \vee b$. Hence, our assumption that L does not satisfy (LSM) leads to a contradiction. □

Figure 4: The isomorphic copies of N_5 constructed in Proposition 5.2 (left) and Proposition 5.3 (right).

Below we give the result dual to Proposition 5.2. The proof is similar to the above argument, so we omit some of the details.

Proposition 5.3. *Let L be a finite lattice and assume that its dual digraph $G_L = (V, E)$ satisfies* (FIS). *Then L is upper semimodular.*

Proof. Suppose L does not satisfy (USM). Then there are elements $a, b \in L$ such that $a \wedge b \prec b$ but $a \not\prec a \vee b$, *i.e.* there is $d \in L$ such that $a < d < a \vee b$. Analogous to the proof of Proposition 5.2, it can be shown that the elements $b, a, d, a \wedge b, a \vee b$ form a sublattice isomorphic to N_5 (see Figure 4).

Then by Lemma 5.1, arbitrary maximal disjoint extensions of $\langle \uparrow b, \downarrow d \rangle$, $\langle \uparrow d, \downarrow a \rangle$ and $\langle \uparrow a, \downarrow b \rangle$, denoted by $x = \langle \uparrow \underline{b}, \downarrow \overline{d} \rangle$, $y = \langle \uparrow \underline{d}, \downarrow \overline{a} \rangle$ and $z = \langle \uparrow \underline{a}, \downarrow \overline{b} \rangle$, respectively, form an induced subgraph $\{x, y, z\}$ of G_L that is isomorphic either to $G_0 = G_{N_5}$, G_1, or G_2. Using (FIS), $\{x, y, z\}$ is isomorphic to G_2.

In particular, it follows that G_L does not have the arc xEy. Hence, $\underline{b} \leqslant \overline{a}$. We can then get $\underline{b} < b$ (as we got $a < \overline{a}$ in Proposition 5.2—see the left lattice in Figure 4). Now either $a \wedge b < \underline{b}$ or

$a \wedge b || \underline{b}$.

If $a \wedge b < \underline{b} < b$, this contradicts $a \wedge b \prec b$, so $\underline{b} || a \wedge b$. We can also show $b || \overline{a}$ (as we showed $\underline{b} || a$ in Proposition 5.2).

Since $a \leqslant \overline{a}$, we get $a \wedge b \leqslant \overline{a} \wedge b$. We can again establish that $a \wedge b < \overline{a} \wedge b$ and $\overline{a} \wedge b < b$ (since $b || \overline{a}$), which contradicts $a \wedge b \prec b$. Hence, our assumption that L does not satisfy (USM) leads to a contradiction. □

Now we can deduce that the condition (FIS) is a sufficient condition for modularity of a finite lattice.

Theorem 5.4. (**Sufficient condition for modularity**) *Let L be a finite lattice with dual TiRS digraph G_L. If G_L satisfies* (FIS) *then L is modular.*

Proof. If follows by Propositions 5.2 and 5.3 that L satisfies both (LSM) and (USM). Since L is finite, we have that L is modular [9, Corollary 376]. □

We notice that the dual digraph of the modular lattice M_3 has neither $G_0 = G_{N_5}$ nor G_1 as an induced subgraph (see Figure 5), hence it satisfies (FIS). The following example shows that the digraphs G_0 and G_1 cannot be dropped as forbidden induced subgraphs in the condition (FIS) for the dual digraph G_L, which guarantees the modularity of a finite lattice L.

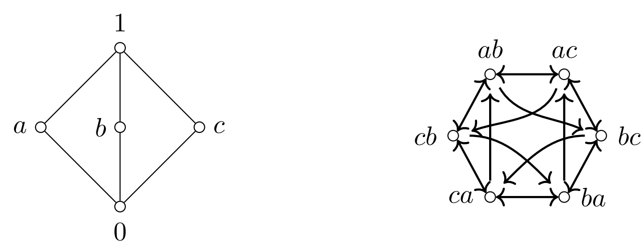

Figure 5: M_3 and its dual digraph.

Example 5.5. *The dual digraph of L_3^∂ in Figure 3 contains G_0 as an induced subgraph, but not G_1. Hence the lattice L_3^∂ (in addition to N_5) witnesses that the digraph G_0 cannot be dropped from the condition* (FIS).

The dual digraphs of the lattices L_4 and L_4^∂ in Figure 1 do not contain G_0 as an induced subgraph but they both contain G_1 as an induced subgraph. Hence these two examples witness that the digraph G_1 cannot be dropped from the condition (FIS).

Now we are going to show that the condition (FIS) is not necessary for modularity. Indeed, it is not the case that every lattice whose dual digraph has $G_0 = G_{N_5}$ as an induced subgraph is a non-modular lattice. The next example gives a modular lattice whose dual digraph has G_0 as an induced subgraph (but does not have G_1 as an induced subgraph).

Example 5.6. (Condition (FIS) not necessary for modularity) *Figure 6 shows a modular lattice K on the left, and its dual digraph on the right. The induced subgraph isomorphic to G_0 is shown with the dotted arrows ($dcEcb$ and $cbEed$).*

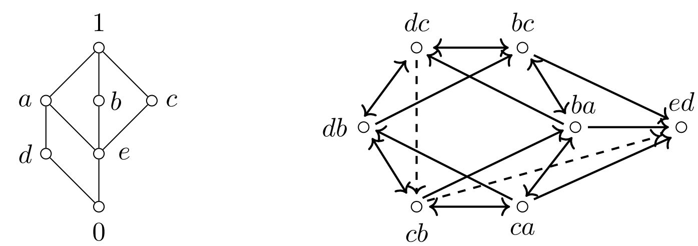

Figure 6: A finite modular lattice K whose dual digraph contains $G_0 = G_{N_5}$ as an induced subgraph.

The fact that the dual TiRS digraph $G_L = (V, E)$ of a finite modular lattice L does not contain $G_0 = G_{N_5}$ as an induced subgraph can be understood as some form of a "weak transitivity" condition for G_L. We cannot have the arcs xEy and yEz in G_L without having also the arc xEz or at least the arc zEx (provided there are no "opposite" arcs yEx and zEy in G_L):

(wT0) for all vertices $x, y, z \in V$, if xEy and yEz, but $(y, x) \notin E$ and $(z, y) \notin E$, then xEz or zEx.

Similarly, the fact that the dual TiRS digraph $G_L = (V, E)$ of a finite modular lattice L does not contain the digraph G_1 as an induced subgraph can be understood as some form of a "weak transitivity" condition for G_L:

(wT1) for all vertices $x, y, z \in V$, if xEy but $(y, x) \notin E$ and $(y, z) \notin E$ and $(z, y) \notin E$ then xEz or zEx.

Example 5.7. *It is easy to see that the dual digraph of the lattice M_3 (Figure 5) satisfies the weak transitivity conditions* (wT0) *and* (wT1). *The lattices L_4 and L_4^∂ in Figure 1, and L_3^∂ in Figure 3 are non-modular lattices. The weak transitivity condition* (wT0) *is not satisfied in the dual digraph of L_3^∂. In the dual digraphs of the lattices L_4 and L_4^∂ we see the failures of* (wT1).

We notice that the weak transitivity conditions (wT0) and (wT1) are essentially expressing on the digraph side that the digraph G_L does not contain respectively the graphs G_0 and G_1 as induced subgraphs.

Hence the sufficiency of the quasi-equations (wT0) and (wT1) on the dual TiRS digraphs G_L for the modularity of L comes as no surprise:

Corollary 5.8 (Sufficient condition for modularity by "weak transitivity")**.** *Let L be a finite lattice with dual TiRS digraph $G_L = (V, E)$. If G_L satisfies the weak transitivity conditions* (wT0) *and* (wT1)*, then L is modular.*

Proof. Let the weak transitivity conditions (wT0) and (wT1) be satisfied in G_L. Suppose for contradiction that the lattice L is not modular. Then by Theorem 5.4, for some $i \in \{0, 1\}$ the digraph G_L contains the digraph G_i as an induced subgraph on certain vertices $x, y, z \in V$. It follows that the weak transitivity condition (wTi) is not satisfied. □

6 Conclusions and future work

In Section 3 we defined two lattice conditions which generalise lower semimodularity and (upper) semimodularity respectively. We were motivated by Figure 2, taken from Ganter and Wille's book [8] (see also the PhD thesis of Reppe [13, Chapter 3.7]). There, weakenings of (LSM) and (USM) are given using complicated conditions on standard contexts. Our lattice-theoretic conditions on finite lattices that are weakenings of (LSM) and (USM), which we call (JM-LSM) and (JM-USM), seem to be simpler than the mentioned conditions in Figure 2 and they are easily seen to be generalisations of (LSM) and (USM). The top left and top right conditions in Figure 2 were shown to be equivalent to (JM-LSM) and (JM-USM) by Kadima [10, Theorem 4.9].

In Section 4 we used the results of Section 3 to obtain a new characterisation of meet-distributive lattices in Theorem 4.1. Combining this with previous results [5], we obtained a characterisation of the dual digraphs of finite meet-distributive lattices. Theorem 4.13 shows that we have identified a new class of structures that is in a one-to-one correspondence with finite convex geometries.

In Remark 4.3 we gave a condition, (W-JSD), which is a weakening of join-semidistributivity. The lattice M_3 satisfies (LSM) but not (W-JSD) and hence shows that (LSM) is not equivalent to (JM-LSM) and (W-JSD). This leads us to ask the following question.

Problem 6.1. *Is there another weakening of* (JSD) *such that when it is combined with* (JM-LSM)*, this will be equivalent to* (LSM)*?*

Theorem 4.9 gave three conditions ((dJSD), (R) and (LTi)) on reflexive digraphs, which characterise the dual digraphs of finite meet-distributive lattices. This leads to the posing of the following open problem.

Problem 6.2. *Can the conditions* (dJSD), (R) *and* (LTi) *be combined to give fewer, and possibly simpler, conditions?*

In Section 5 we introduced the condition (FIS) on dual digraphs and showed that it implies both lower and upper semimodularity of a finite lattice. Hence (FIS) was shown to be a sufficient condition for modularity of a finite lattice (Theorem 5.4). We also formulated a sufficient condition for modularity in different terms in Corollary 5.8. The condition (FIS) was shown not to be necessary for modularity of a finite lattice and hence we raise the following open question.

Problem 6.3. *Is it possible to find forbidden induced subgraphs that characterise the dual digraphs of finite modular lattices in an analogous way to how* N_5 *characterises modularity?*

The task of representing structures (in our case digraphs) dual to finite modular lattices has proved to be very challenging. We note that in the setting of formal contexts dual to finite lattices, a condition dual to semimodularity has been obtained (*cf.* item (4) of [8, Theorem 42]). We have attempted to translate this condition to TiRS digraphs and the result was a complicated and opaque condition. We do not believe that the translation of this condition and its dual will yield a useful characterisation of the TiRS digraphs dual to finite modular lattices.

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Extremal functions and best approximate formulas for the Hankel-type Fock space

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ABSTRACT

In this paper we recall some properties for the Hankel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. This space was introduced by Cholewinsky in 1984 and plays a background to our contribution. Especially, we examine the extremal functions for the difference operator D, and we deduce best approximate inversion formulas for the operator D on the the Hankel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$.

RESUMEN

En este artículo, resumimos algunas propiedades para el espacio de Fock the tipo Hankel $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. Este espacio fue introducido por Cholewinsky en 1984 y es un antecedente para nuestra contribución. Especialmente examinamos las funciones extremales para el operador de diferencia D y deducimos fórmulas de inversión del mejor aproximante para el operador D en el espacio de Fock de tipo Hankel $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$.

Keywords and Phrases: Analytic functions, Hankel-type Fock space, extremal functions.

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1 Introduction

The classical Fock space $\mathscr{F}(\mathbb{C}^d)$ is the Hilbert space of entire functions f on $\mathbb{C}^d$ such that

$$\|f\|^2_{\mathscr{F}(\mathbb{C}^d)} := \frac{1}{\pi^d}\int_{\mathbb{C}^d} |f(z)|^2 e^{-|z|^2}\mathrm{d}x\mathrm{d}y < \infty, \quad z = x + iy,$$

where $|z|^2 = \sum_{k=1}^d (x_k^2 + y_k^2)$ and $\mathrm{d}x\mathrm{d}y = \prod_{k=1}^d \mathrm{d}x_k\mathrm{d}y_k$.

This space was introduced by Bargmann [3], is called also Segal-Bargmann space [5] and it was the aim of many works [4, 6, 22, 28]. Recently the author of the paper studied the extremal functions for the difference and primitive operators on the Fock space $\mathscr{F}(\mathbb{C}^d)$ (see [20, 21]).

Cholewinsky [7] defined the Hankel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ associated with the poly-axially operator. The space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is the Hilbert space of entire functions f on $\mathbb{C}^d$, even with respect to the last variable, such that

$$\|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} := \left[\int_{\mathbb{C}^d} |f(z)|^2 \mathrm{d}m_\alpha(z)\right]^{1/2} < \infty,$$

where m_α is the measure defined for $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ by

$$\mathrm{d}m_\alpha(z) := \frac{1}{\pi^d}\prod_{k=1}^d \frac{|z_k|^{2\alpha_k+2}K_{\alpha_k}(|z_k|^2)}{2^{\alpha_k}\Gamma(\alpha_k+1)}\mathrm{d}z_k, \tag{1.1}$$

and K_{α_k}, $\alpha_k > -1/2$, is the Macdonald function [8].

The generalized Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is equipped with the inner product

$$\langle f, g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} := \int_{\mathbb{C}^d} f(w)\overline{g(w)}\mathrm{d}m_\alpha(w).$$

The Hankel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is also studied in [24], when the author proved an uncertainty principle of Heisenberg type for this space.

Let D be the difference operator defined for $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu\in\mathbb{N}^d} a_\nu z^{2\nu}$, by

$$Df(z) := \sum_{\nu\in\mathbb{N}^d} a_{\nu+\mathbf{1}} z^{2\nu}.$$

The main goal of the paper is to find the minimizer (denoted by $F^*_{\lambda,D}(h)$) for the extremal problem:

$$\inf_{f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \left\{\lambda\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} + \|Df - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\right\},$$

where $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ and $\lambda > 0$. We prove that the extremal function $F^*_{\lambda,D}(h)$ is given by

$$F^*_{\lambda,D}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

where $\Psi_z(w)$ is the kernel given later in Section 3.

Moreover, we establish best approximate inversion formulas for the difference operator D on the weighted Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. A pointwise approximate inversion formula for the operator D are also discussed.

Recently, the analog results are also proved, for the Fock space $\mathscr{F}(\mathbb{C}^d)$ (see [20, 21]), and for the Bessel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C})$ (see [23, 25]).

The paper is organized as follows. In Section 2 we recall some properties for the Hankel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$. In Section 3 we examine the extremal functions for the difference operator D. Finally, in Section 4, we establish best approximate inversion formulas for the operator D on the Hankel-type Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$.

Throughout this paper we shall use on $\mathbb{C}^d$ the multi-index notations.

- For all $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, $z^\nu = \prod_{k=1}^d z_k^{\nu_k}$.
- For any $\nu \in \mathbb{N}^d$, the partial ordering $\geq$ on $\mathbb{N}^d$, which is defined by

$$\nu \geq \mathbf{1} \iff \nu_j \geq 1, \quad \forall j = 1, \ldots, d, \quad \text{with } \mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^d.$$

2 Hankel-type Fock space

In this section, we recall some properties for the Fock space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ associated with the poly-axially operator.

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$, we denote by Δ_α, the poly-axially operator [1, 9, 27] defined for $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ by

$$\Delta_\alpha := \sum_{k=1}^d \Delta_{\alpha_k, z_k}, \quad \Delta_{\alpha_k, z_k} := \frac{\partial^2}{\partial z_k^2} + \frac{2\alpha_k + 1}{z_k} \frac{\partial}{\partial z_k}.$$

This operator has important applications in both pure and applied mathematics and give rise to a generalization of multi-variable analytic structures like the Hankel transform, and the Hankel convolution [2, 15–18]. For any $w \in \mathbb{C}^d$, the system

$$\Delta_\alpha u(z) = |w|^2 u(z), \quad u(0) = 1, \quad \frac{\partial}{\partial z_k} u(z)\Big|_{z_k=0} = 0, \quad k = 1, \ldots, d,$$

admits a unique solution $I_\alpha(w,z)$, given by

$$I_\alpha(w,z) := \prod_{k=1}^{d} j_{\alpha_k}(iw_k z_k),$$

where j_{α_k} is the spherical Bessel function [26] given by

$$j_{\alpha_k}(x) := \Gamma(\alpha_k+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha_k+1)}\left(\frac{x}{2}\right)^{2n}.$$

The Bessel kernel I_α can be extended in a power series in the form

$$I_\alpha(w,z) = \sum_{\nu\in\mathbb{N}^d}\frac{w^{2\nu}z^{2\nu}}{c_\nu(\alpha)},$$

where

$$c_\nu(\alpha) = 2^{2\langle\nu\rangle}\nu!\prod_{k=1}^{d}\frac{\Gamma(\nu_k+\alpha_k+1)}{\Gamma(\alpha_k+1)} = \prod_{k=1}^{d} c_{\nu_k}(\alpha_k). \tag{2.1}$$

Here

$$c_{\nu_k}(\alpha_k) = 2^{2\nu_k}\nu_k!\frac{\Gamma(\nu_k+\alpha_k+1)}{\Gamma(\alpha_k+1)}$$

and

$$\langle\nu\rangle = \sum_{k=1}^{d}\nu_k, \quad \nu! = \prod_{k=1}^{d}\nu_k!, \quad \nu=(\nu_1,\dots,\nu_d)\in\mathbb{N}^d.$$

In the statement, and later in this work we use the following notations.

- $\mathscr{H}_*(\mathbb{C}^d)$, is the space of entire functions on $\mathbb{C}^d$ and even with respect to each variable.

- $L^2_\alpha(\mathbb{C}^d)$, is the Hilbert space of measurable functions f on $\mathbb{C}^d$, such that

$$\|f\|_{L^2_\alpha(\mathbb{C}^d)} := \left[\int_{\mathbb{C}^d}|f(z)|^2\mathrm{d}m_\alpha(z)\right]^{1/2} < \infty,$$

  where m_α being the measure on $\mathbb{C}^d$ given by (1.1).

Cholewinsky [7] defined the Hilbert space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ as

$$\mathscr{F}_{\alpha,*}(\mathbb{C}^d) := \mathscr{H}_*(\mathbb{C}^d)\cap L^2_\alpha(\mathbb{C}^d).$$

The space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is equipped with the inner product

$$\langle f,g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} := \int_{\mathbb{C}^d} f(z)\overline{g(z)}\mathrm{d}m_\alpha(z).$$

The space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ has the reproducing kernel

$$\mathscr{K}_\alpha(w,z) = I_\alpha(w,\overline{z}), \quad w,z\in\mathbb{C}^d.$$

If $f,g\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z)=\sum_{\nu\in\mathbb{N}^d} a_\nu z^{2\nu}$ and $g(z)=\sum_{\nu\in\mathbb{N}^d} b_\nu z^{2\nu}$, then

$$\langle f,g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} a_\nu \overline{b_\nu} c_\nu(\alpha), \tag{2.2}$$

where $c_\nu(\alpha)$ are the constants given by (2.1).

Then, the set $\left\{\frac{z^{2\nu}}{\sqrt{c_\nu(\alpha)}}\right\}_{\nu\in\mathbb{N}^d}$ forms a Hilbertian basis for the space $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$; and each $f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ can be written as

$$f(z)=\sum_{\nu\in\mathbb{N}^d}\frac{\langle f,z^{2\nu}\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}}{c_\nu(\alpha)}z^{2\nu},$$

and

$$\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d}\frac{\left|\langle f,z^{2\nu}\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\right|^2}{c_\nu(\alpha)}.$$

Bargmann [3] introduced the classical Fock space $\mathscr{F}(\mathbb{C}^d)$. Let $f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z)=\sum_{\nu\in\mathbb{N}^d} a_\nu z^{2\nu}$. From [3], we have

$$\|f\|^2_{\mathscr{F}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d}|a_\nu|^2\nu!.$$

Using the inequality $\nu!\le c_\nu(\alpha)$, we obtain

$$\|f\|^2_{\mathscr{F}(\mathbb{C}^d)} \le \sum_{\nu\in\mathbb{N}^d}|a_\nu|^2 c_\nu(\alpha) = \|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}.$$

Therefore

$$\mathscr{F}_{\alpha,*}(\mathbb{C}^d)\subset\mathscr{F}(\mathbb{C}^d).$$

3 Difference operator

In this section, building on the ideas of Saitoh [12–14] we examine the extremal function associated with the difference operator D. The results that are written here are a special case of [14].

Let D be the difference operator defined for $f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z)=\sum_{\nu\in\mathbb{N}^d} a_\nu z^{2\nu}$, by

$$Df(z) := \sum_{\nu\in\mathbb{N}^d} a_{\nu+\mathbf{1}} z^{2\nu}. \tag{3.1}$$

In particular, for $f \in \mathscr{F}_{\alpha,*}(\mathbb{C})$, the difference operator [23, 25] is given

$$Df(z) := \begin{cases} \dfrac{1}{z^2}(f(z) - f(0)), & z \neq 0, \\ \dfrac{1}{2} f''(0), & z = 0. \end{cases}$$

We also define, the operators E and H for $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu z^{2\nu}$, by

$$Ef(z) := \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} \frac{c_{\nu-\mathbf{1}}(\alpha)}{c_\nu(\alpha)} a_{\nu-\mathbf{1}} z^{2\nu}, \tag{3.2}$$

and

$$Hf(z) := \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} \frac{c_{\nu-\mathbf{1}}(\alpha)}{c_\nu(\alpha)} a_\nu z^{2\nu}, \tag{3.3}$$

where $c_\nu(\alpha)$ are the constants given by (2.1).

Lemma 3.1. (i) *The operator D maps continuously from $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ into $\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, and*

$$\|Df\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{2^d \sqrt{\prod_{k=1}^d (\alpha_k + 1)}} \|f\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}, \quad f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d).$$

(ii) *If $D^* : \mathscr{F}_{\alpha,*}(\mathbb{C}^d) \longrightarrow \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ is the adjoint operator of D, then*

$$E = D^* \quad \textit{and} \quad H = D^* D.$$

Proof. (i) Let $f \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu z^{2\nu}$. From (3.1), we have

$$\|Df\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu \in \mathbb{N}^d} |a_{\nu+\mathbf{1}}|^2 c_\nu(\alpha) = \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} |a_\nu|^2 c_{\nu-\mathbf{1}}(\alpha).$$

Using the fact that $c_\nu(\alpha) = \left[2^{2d} \prod_{k=1}^d \nu_k(\nu_k + \alpha_k)\right] c_{\nu-\mathbf{1}}(\alpha)$, we deduce that

$$\|Df\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{2^{2d} \prod_{k=1}^d (\alpha_k + 1)} \sum_{\nu \in \mathbb{N}^d} |a_\nu|^2 c_\nu(\alpha) = \frac{1}{2^{2d} \prod_{k=1}^d (\alpha_k + 1)} \|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}.$$

(ii) If $f, g \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu z^{2\nu}$ and $g(z) = \sum_{\nu \in \mathbb{N}^d} b_\nu z^{2\nu}$, then by (2.2) and (3.1) we obtain

$$\langle Df, g \rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu \in \mathbb{N}^d} a_{\nu+\mathbf{1}} \overline{b_\nu} c_\nu(\alpha) = \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} a_\nu \overline{b_{\nu-\mathbf{1}}} c_{\nu-\mathbf{1}}(\alpha).$$

On the other hand, from (2.2) and (3.2) we have

$$\langle f, Eg \rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu \in \mathbb{N}^d,\, \nu \geq \mathbf{1}} a_\nu \overline{b_{\nu-\mathbf{1}}} c_{\nu-\mathbf{1}}(\alpha).$$

Then $\langle Df, g\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \langle f, Eg\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$ and consequently $E = D^*$.

Finally, by relations (3.1), (3.2) and (3.3) we deduce that

$$D^*Df(z) = EDf(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{c_{\nu-\mathbf{1}}(\alpha)}{c_\nu(\alpha)} a_\nu z^{2\nu} = Hf(z).$$

The lemma is proved. □

Theorem 3.2. *For any $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ and for any $\lambda > 0$, the Tikhonov regularization problem*

$$\inf_{f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \left\{\lambda\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} + \|Df - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\right\}$$

has a unique extremal function denoted $F^*_{\lambda,D}(h)$ and is given by

$$F^*_{\lambda,D}(h)(z) = \langle h, \Psi_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

where

$$\Psi_z(w) = \sum_{\nu\in\mathbb{N}^d} \frac{(\overline{z})^{2(\nu+\mathbf{1})} w^{2\nu}}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)}, \quad w \in \mathbb{C}^d.$$

Proof. First, from [12, Theorem 2.5, Section 2], the Tikhonov regularization problem

$$\inf_{f\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \left\{\lambda\|f\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} + \|Df - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\right\}$$

has a unique extremal function denoted $F^*_{\lambda,D}(h)$ and is given by

$$F^*_{\lambda,D}(h)(z) = (\lambda I + D^*D)^{-1}D^*h(z), \quad z \in \mathbb{C}^d, \tag{3.4}$$

where I is the unit operator. We put $h(z) = \sum_{\nu\in\mathbb{N}^d} h_\nu z^{2\nu}$ and $F^*_{\lambda,D}(h)(z) = \sum_{\nu\in\mathbb{N}^d} d_\nu z^{2\nu}$. From Lemma 3.1 (ii) and (3.4) we have

$$(\lambda I + H)F^*_{\lambda,D}(h)(z) = Eh(z).$$

By relations (3.2) and (3.3) we deduce that

$$d_\nu = 0, \quad \text{if } \exists\ \nu_k = 0,$$

$$d_\nu = \frac{c_{\nu-\mathbf{1}}(\alpha)h_{\nu-\mathbf{1}}}{\lambda c_\nu(\alpha) + c_{\nu-\mathbf{1}}(\alpha)}, \quad \nu \geq \mathbf{1}.$$

Thus,

$$F^*_{\lambda,D}(h)(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} \frac{c_{\nu-\mathbf{1}}(\alpha)h_{\nu-\mathbf{1}}}{\lambda c_\nu(\alpha) + c_{\nu-\mathbf{1}}(\alpha)} z^{2\nu}. \tag{3.5}$$

Then by (2.2) and (3.5) we obtain

$$F^*_{\lambda,D}(h)(z) = \sum_{\nu\in\mathbb{N}^d} \frac{c_\nu(\alpha)h_\nu}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)} z^{2(\nu+\mathbf{1})} = \langle h, \Psi_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}, \tag{3.6}$$

where

$$\Psi_z(w) = \sum_{\nu\in\mathbb{N}^d} \frac{(\overline{z})^{2(\nu+\mathbf{1})} w^{2\nu}}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}, \quad w\in\mathbb{C}^d.$$

The theorem is proved. □

4 Approximate inversion formulas

In this section we establish the estimate properties of the extremal function $F^*_{\lambda,D}(h)(z)$, and we deduce approximate inversion formulas for the difference operator D. These formulas are the analogous of Calderón's reproducing formulas for the Fourier type transforms [10, 11, 19]. A pointwise approximate inversion formulas for the operator D are also discussed.

The extremal function $F^*_{\lambda,D}(h)$ given by (3.6) satisfies the following properties.

Lemma 4.1. *If $\lambda > 0$ and $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, then*

(i) $|F^*_{\lambda,D}(h)(z)| \leq \frac{1}{2\sqrt{\lambda}}(I_\alpha(z,\overline{z}))^{1/2}\|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$,

(ii) $|DF^*_{\lambda,D}(h)(z)| \leq \frac{1}{2^{d+1}\sqrt{\lambda\prod_{k=1}^d(\alpha_k+1)}}(I_\alpha(z,\overline{z}))^{1/2}\|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$,

(iii) $\|F^*_{\lambda,D}(h)\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{2\sqrt{\lambda}}\|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}$.

Proof. Let $\lambda > 0$ and $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $h(z) = \sum_{\nu\in\mathbb{N}^d} h_\nu z^{2\nu}$. From (3.6) we have

$$|F^*_{\lambda,D}(h)(z)| \leq \|\Psi_z\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}\|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}.$$

Using the fact that $(x+y)^2 \geq 4xy$ we obtain

$$\|\Psi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d}\left|\frac{(\overline{z})^{2(\nu+\mathbf{1})}}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}\right|^2 c_\nu(\alpha) \leq \frac{1}{4\lambda}\sum_{\nu\in\mathbb{N}^d}\frac{|(\overline{z})^{2\nu}|^2}{c_\nu(\alpha)} = \frac{1}{4\lambda}I_\alpha(z,\overline{z}).$$

This gives (i).

On the other hand, from (3.1) and (3.5) we have

$$DF^*_{\lambda,D}(h)(z) = \sum_{\nu\in\mathbb{N}^d} \frac{c_\nu(\alpha)h_\nu}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)} z^{2\nu} = \langle h, \Phi_z\rangle_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}, \tag{4.1}$$

where

$$\Phi_z(w) = \sum_{\nu\in\mathbb{N}^d} \frac{(\overline{z})^{2\nu} w^{2\nu}}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)}.$$

Then

$$|DF^*_{\lambda,D}(h)(z)| \leq \|\Phi_z\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \|h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

and

$$\|\Phi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} \left| \frac{(\overline{z})^{2\nu}}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)} \right|^2 c_\nu(\alpha) \leq \frac{1}{4\lambda} \sum_{\nu\in\mathbb{N}^d} \frac{|(\overline{z})^{2\nu}|^2}{c_{\nu+\mathbf{1}}(\alpha)}.$$

By using the fact that $c_{\nu+\mathbf{1}}(\alpha) = \left[2^{2d}\prod_{k=1}^d (\nu_k+1)(\nu_k+\alpha_k+1)\right] c_\nu(\alpha)$, we deduce that

$$\|\Phi_z\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{2^{2(d+1)}\lambda \prod_{k=1}^d (\alpha_k+1)} \sum_{\nu\in\mathbb{N}^d} \frac{|(\overline{z})^{2\nu}|^2}{c_\nu(\alpha)} = \frac{I_\alpha(z,\overline{z})}{2^{2(d+1)}\lambda \prod_{k=1}^d (\alpha_k+1)}.$$

This gives (ii).

Finally, from (3.5) we have

$$\|F^*_{\lambda,D}(h)\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} c_\nu(\alpha) \left[\frac{c_{\nu-\mathbf{1}}(\alpha)|h_{\nu-\mathbf{1}}|}{\lambda c_\nu(\alpha) + c_{\nu-\mathbf{1}}(\alpha)} \right]^2.$$

Then we obtain

$$\|F^*_{\lambda,D}(h)\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} \leq \frac{1}{4\lambda} \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} c_{\nu-\mathbf{1}}(\alpha)|h_{\nu-\mathbf{1}}|^2 = \frac{1}{4\lambda}\|h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)},$$

which gives (iii) and completes the proof of the lemma. □

We establish approximate inversion formulas for the difference operator D.

Theorem 4.2. *If $\lambda > 0$ and $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, then*

(i) $\lim_{\lambda\to 0^+} \|DF^*_{\lambda,D}(h) - h\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = 0,$

(ii) $\lim_{\lambda\to 0^+} \|F^*_{\lambda,D}(Dh) - h_0\|_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = 0,$ *where $h_0(z) = \sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}} h_\nu z^{2\nu}$ if $h(z) = \sum_{\nu\in\mathbb{N}^d} h_\nu z^{2\nu}$.*

Proof. Let $\lambda > 0$ and $h \in \mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $h(z) = \sum_{\nu\in\mathbb{N}^d} h_\nu z^{2\nu}$. From (4.1) we have

$$DF^*_{\lambda,D}(h)(z) - h(z) = \sum_{\nu\in\mathbb{N}^d} \frac{-\lambda c_{\nu+\mathbf{1}}(\alpha) h_\nu}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)} z^{2\nu}. \tag{4.2}$$

Therefore

$$\|DF^*_{\lambda,D}(h) - h\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)} = \sum_{\nu\in\mathbb{N}^d} c_\nu(\alpha) \left[\frac{\lambda c_{\nu+\mathbf{1}}(\alpha)|h_\nu|}{\lambda c_{\nu+\mathbf{1}}(\alpha) + c_\nu(\alpha)} \right]^2.$$

Again, by dominated convergence theorem and the fact that

$$c_\nu(\alpha)\left[\frac{\lambda c_{\nu+\mathbf{1}}(\alpha)|h_\nu|}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}\right]^2 \leq c_\nu(\alpha)|h_\nu|^2,$$

we deduce (i).

Finally, from (3.1) and (3.5) we have

$$F^*_{\lambda,D}(Dh)(z)-h_0(z)=\sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}}\frac{-\lambda c_\nu(\alpha)h_\nu}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}z^{2\nu}. \tag{4.3}$$

So, one has

$$\|F^*_{\lambda,D}(Dh)-h_0\|^2_{\mathscr{F}_{\alpha,*}(\mathbb{C}^d)}=\sum_{\nu\in\mathbb{N}^d,\ \nu\geq\mathbf{1}}c_\nu(\alpha)\left[\frac{\lambda c_\nu(\alpha)|h_\nu|}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}\right]^2.$$

Using the dominated convergence theorem and the fact that

$$c_\nu(\alpha)\left[\frac{\lambda c_\nu(\alpha)|h_\nu|}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}\right]^2 \leq c_\nu(\alpha)|h_\nu|^2,$$

we deduce (ii). □

We deduce also pointwise approximate inversion formulas for the operator D.

Theorem 4.3. *If $\lambda>0$ and $h\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$, then*

(i) $\lim_{\lambda\to0^+} DF^*_{\lambda,D}(h)(z)=h(z)$,

(ii) $\lim_{\lambda\to0^+} F^*_{\lambda,D}(Dh)(z)=h_0(z)$.

Proof. Let $h\in\mathscr{F}_{\alpha,*}(\mathbb{C}^d)$ with $h(z)=\sum_{\nu\in\mathbb{N}^d}h_\nu z^{2\nu}$. From (4.2) and (4.3), by using the dominated convergence theorem and the fact that

$$\frac{\lambda c_{\nu+\mathbf{1}}(\alpha)|h_\nu|}{\lambda c_{\nu+\mathbf{1}}(\alpha)+c_\nu(\alpha)}|z^{2\nu}|,\ \frac{\lambda c_\nu(\alpha)|h_\nu|}{\lambda c_\nu(\alpha)+c_{\nu-\mathbf{1}}(\alpha)}|z^{2\nu}|\leq|h_\nu||z^{2\nu}|,$$

we obtain (i) and (ii). □

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Approximation and inequalities for the factorial function related to the Burnside's formula

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ABSTRACT

In this paper, we present a continued fraction approximation and some inequalities of the factorial function based on the Burnside's formula. This approximation is fast in comparison with the recently discovered asymptotic series. Finally, some numerical computations are provided for demonstrating the superiority of our approximation over the Burnside's formula and the classical Stirling's series.

RESUMEN

En este artículo, presentamos una aproximación con una fracción continua y algunas desigualdades para la función factorial basada en la fórmula de Burnside. Esta aproximación es rápida en comparación con las series asintóticas descubiertas recientemente. Finalmente, se entregan algunos cálculos numéricos para demostrar la superioridad de nuestra aproximación por sobre la fórmula de Burnside y la serie de Stirling clásica.

Keywords and Phrases: Factorial function, Stirling's formula, Burnside's formula, approximation, continued fraction.

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1 Introduction and main results

It is well known that we often need to deal with the big factorials in many situations in pure mathematics and other branches of science. To the best of our knowledge, the Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \quad n \to \infty \tag{1.1}$$

is one of the most known formulas for approximation of the factorial function. Up to now, many researchers made great efforts in the area of establishing more precise inequalities and more accurate approximation for the factorial function and its extension, called gamma function, and had a lot of inspiring results. For example, the Stirling series [1]

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \dots\right), \quad n \to \infty \tag{1.2}$$

is an extension of (1.1). Furthermore, there is a variety of approaches to Stirling's formula, ranging from elementary to advanced methods. We mention the estimations given by Schuster in [14], or the formula

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} = \sigma_n, \quad n \to \infty, \tag{1.3}$$

with $n! < \sigma_n$, due to Burnside, whose superiority over Stirling's formula was proved in [3]. There are also some approximations which are better than (1.3), Gosper's formula [7]

$$n! \sim \sqrt{2\pi\left(n + \frac{1}{6}\right)}\left(\frac{n}{e}\right)^n, \quad n \to \infty, \tag{1.4}$$

and Ramanujan's formula [13]

$$n! \sim \sqrt{2\pi}\left(\frac{n}{e}\right)^n \left(n^3 + \frac{1}{2}n^2 + \frac{1}{8}n + \frac{1}{240}\right)^{1/6}, \quad n \to \infty, \tag{1.5}$$

and Nemes's formula [12]

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n^2 - 1/10}\right)^n, \quad n \to \infty. \tag{1.6}$$

In [2], Batir obtained an asymptotic formula as follows:

$$n! \sim \sqrt{2\pi}\frac{n^{n+1}e^{-n}}{\sqrt{n - 1/6}}, \quad n \to \infty. \tag{1.7}$$

The following more accurate approximation for $n!$

$$n! \sim \sqrt{2\pi}\left(\frac{n^2+n+1/6}{e^2}\right)^{n/2+1/4}, \quad n \to \infty. \tag{1.8}$$

can be found in the literature [9].

Recently, Mortici [8] proved that for every $x \geq 0$,

$$\sqrt{2\pi e}\cdot e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+\frac{1}{2}} < \Gamma(x+1) \leq \alpha\sqrt{2\pi e}\cdot e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+\frac{1}{2}}, \tag{1.9}$$

where $\omega = \dfrac{3-\sqrt{3}}{6}$, $\alpha = 1.072042464\ldots$, and

$$\beta\sqrt{2\pi e}\cdot e^{-\zeta}\left(\frac{x+\zeta}{e}\right)^{x+\frac{1}{2}} \leq \Gamma(x+1) < \sqrt{2\pi e}\cdot e^{-\zeta}\left(\frac{x+\zeta}{e}\right)^{x+\frac{1}{2}}, \tag{1.10}$$

where $\zeta = \dfrac{3+\sqrt{3}}{6}$, $\beta = 0.988503589\ldots$

Estimates and approximations for the factorial function (and the gamma function) are a popular subject, with many papers appearing on this topic over the years. More results involving the asymptotic formulas or bounds for $n!$ or gamma function can be found in the references cited therein.

A natural question arises. It is true that the behavior of the Burnside's formula for n approaches infinity is of the form

$$n! \sim \sqrt{2\pi e}\cdot e^{-p}\left(\frac{n+p}{e}\right)^{n+q}, \tag{1.11}$$

where p, q are some constants? We propose the following sharp approximation formula as $n \to \infty$:

$$n! \sim \sqrt{2\pi e}\cdot e^{-\frac{3\pm\sqrt{3}}{6}}\left(\frac{n+\frac{3\pm\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}}. \tag{1.12}$$

These constants p, q in (1.11) given by (1.12), namely

$$p = \frac{3\pm\sqrt{3}}{6}, \quad q = \frac{1}{2}$$

are justified by the result in Theorem 1.1. Then we prove the following stronger approximation formula using continued fraction for the factorial function by the *multiple-correction method* [4–6].

Theorem 1.1. *For the factorial function, we have*

$$n! \sim \sqrt{2\pi e} \cdot e^{-p} \left(\frac{n+p}{e} \right)^{n+q} \exp \left(\cfrac{u_1}{n^2 + v_1 n + v_0 + \cfrac{s_1}{n + t_1 + \cfrac{s_2}{n + t_2 + \ddots}}} \right), \quad n \to \infty, \tag{1.13}$$

where

$$p = \frac{3 \pm \sqrt{3}}{6}, \quad q = \frac{1}{2}; \quad u_1 = \mp \frac{1}{72\sqrt{3}}, \quad v_1 = \frac{10 \pm 3\sqrt{3}}{10}, \quad v_0 = \frac{47 \pm 15\sqrt{3}}{10};$$
$$s_1 = \pm \frac{163}{21000\sqrt{3}}, \quad t_1 = \frac{815 \pm 11596\sqrt{3}}{1630};$$
$$s_2 = \frac{15531525}{106276}, \quad t_2 = \frac{19139187627 \mp 259913623163\sqrt{3}}{38278375254}; \ldots.$$

Using Theorem 1.1, we provide some inequalities for the factorial function.

Theorem 1.2. *For every $n \in \mathbb{N}$, it holds:*

$$\sqrt{2\pi e} \cdot e^{-\frac{3-\sqrt{3}}{6}} \left(\frac{n + \frac{3-\sqrt{3}}{6}}{e} \right)^{n+\frac{1}{2}} < n! < \sqrt{2\pi e} \cdot e^{-\frac{3+\sqrt{3}}{6}} \left(\frac{n + \frac{3+\sqrt{3}}{6}}{e} \right)^{n+\frac{1}{2}}. \tag{1.14}$$

To obtain Theorem 1.1, we need the following lemma which was used in [8,10,11] and is very useful for constructing asymptotic expansions.

Lemma 1.3. *If the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to zero and there exists the limit*

$$\lim_{n\to+\infty} n^s (x_n - x_{n+1}) = l \in [-\infty, +\infty] \tag{1.15}$$

with $s > 1$, then

$$\lim_{n\to+\infty} n^{s-1} x_n = \frac{l}{s-1}. \tag{1.16}$$

Lemma 1.3 was proved by Mortici in [8]. From Lemma 1.3, we can see that the speed of convergence of the sequences $(x_n)_{n\in\mathbb{N}}$ increases together with the values s satisfying (1.15).

2 Proof of Theorem 1.1

Step 0: The initial-correction.

Based on the Burnside's formula $n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}$, $n \to \infty$, we need to find the values p, q which produces the most accurate approximation of the form

$$n! \sim \sqrt{2\pi e} \cdot e^{-p}\left(\frac{n+p}{e}\right)^{n+q}, \quad n \to \infty.$$

To measure the accuracy of this approximation, a method is to define a sequence $(u_0(n))_{n\geq 1}$ by the relations

$$n! = \sqrt{2\pi e} \cdot e^{-p}\left(\frac{n+p}{e}\right)^{n+q} \exp u_0(n), \tag{2.1}$$

and to say that the approximation $n! \sim \sqrt{2\pi e} \cdot e^{-p}\left(\frac{n+p}{e}\right)^{n+q}$, $n \to \infty$ is better if $u_0(n)$ converges to zero faster.

From (2.1), we have

$$u_0(n) = \ln n! - \frac{1}{2}\ln(2\pi e) + p - (n+q)\ln(n+p) + (n+q). \tag{2.2}$$

Thus,

$$u_0(n) - u_0(n+1) = -1 - \ln(n+1) - (n+q)\ln(n+p) + (n+1+q)\ln(n+1+p). \tag{2.3}$$

Developing (2.3) into power series expansion in $1/n$, we have

$$\begin{aligned} u_0(n) - u_0(n+1) = {} & \frac{-1+2q}{2}\frac{1}{n} + \frac{2+3p^2-3q-6pq}{6}\frac{1}{n^2} \\ & + \frac{-3-6p^2-8p^3+4q+12pq+12p^2q}{12}\frac{1}{n^3} + O\left(\frac{1}{n^4}\right). \end{aligned} \tag{2.4}$$

The fastest possible sequence $(u_0(n))_{n\geq 1}$ is obtained as the first two items on the right of (2.4) vanishes, we get $p = \frac{3 \pm \sqrt{3}}{6}$, $q = \frac{1}{2}$. Thus, using Lemma 1.3, from (2.4) we have

$$u_0(n) - u_0(n+1) = \mp\frac{1}{36\sqrt{3}}\frac{1}{n^3} + O\left(\frac{1}{n^4}\right),$$

and the rate of convergence of the sequence $(u_0(n))_{n\geq 1}$ is at least n^{-2}.

Step 1: The first-correction.

Next, we define the sequence $(u_1(n))_{n\geq 1}$ by the relation

$$n! = \sqrt{2\pi e}\cdot e^{-\frac{3\pm\sqrt{3}}{6}}\left(\frac{n+\frac{3\pm\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}}\exp\left(\frac{u_1}{n^2+v_1 n+v_0}\right)\exp u_1(n). \tag{2.5}$$

From (2.5), we have

$$\begin{aligned} u_1(n)-u_1(n+1) &= -1-\ln(n+1)-\left(n+\frac{1}{2}\right)\ln\left(n+\frac{3\pm\sqrt{3}}{6}\right) \\ &+\left(n+\frac{3}{2}\right)\ln\left(n+1+\frac{3\pm\sqrt{3}}{6}\right)-\frac{u_1}{n^2+v_1 n+v_0}+\frac{u_1}{(n+1)^2+v_1(n+1)+v_0}. \end{aligned} \tag{2.6}$$

Developing (2.6) into power series expansion in $1/n$, we have

$$\begin{aligned} u_1(n)-u_1(n+1) &= \left(\mp\frac{1}{36\sqrt{3}}-2u_1\right)\frac{1}{n^3}+\left(\frac{1}{80}\pm\frac{1}{12\sqrt{3}}+3u_1+3u_1v_1\right)\frac{1}{n^4} \\ &+\left(-\frac{1}{20}\mp\frac{11}{60\sqrt{3}}-4u_1+4u_1v_0-6u_1v_1-4u_1v_1^2\right)\frac{1}{n^5} \\ &+\left(\frac{599}{4536}\pm\frac{13}{36\sqrt{3}}+5u_1-10u_1v_0+10u_1v_1-10u_1v_0v_1+10u_1v_1^2+5u_1v_1^3\right)\frac{1}{n^6}+O\left(\frac{1}{n^7}\right). \end{aligned} \tag{2.7}$$

By Lemma 1.3, the fastest possible sequence $(u_1(n))_{n\geq 1}$ is obtained as the first three items on the right of (2.7) vanishes. So we can obtain

$$u_1 = \mp\frac{1}{72\sqrt{3}},\quad v_1=\frac{10\pm 3\sqrt{3}}{10},\quad v_0=\frac{47\pm 15\sqrt{3}}{100},$$

and from (2.7) we have

$$u_1(n)-u_1(n+1)=\frac{163}{907200}\frac{1}{n^6}+O\left(\frac{1}{n^7}\right),$$

and the rate of convergence of the sequence $(u_1(n))_{n\geq 1}$ is at least n^{-5}.

Step 2: The second-correction.

Furthermore, we define the sequence $(u_2(n))_{n\geq 1}$ by the relation

$$n! = \sqrt{2\pi e}\cdot e^{-\frac{3\pm\sqrt{3}}{6}}\left(\frac{n+\frac{3\pm\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}}\exp\left(\frac{\mp\frac{1}{72\sqrt{3}}}{n^2+\frac{10\pm 3\sqrt{3}}{10}n+\frac{47\pm 15\sqrt{3}}{100}+\frac{s_1}{n+t_1}}\right)\exp u_2(n) \tag{2.8}$$

Using the same method as above, we obtain that the sequence $(u_2(n))_{n\geq 1}$ converges fastest only if $s_1 = \pm\dfrac{163}{21000\sqrt{3}}$, $t_1 = \dfrac{815 \pm 11596\sqrt{3}}{1630}$, and the rate of convergence of the sequence $(u_2(n))_{n\geq 1}$ is at least n^{-7}. We can get

$$u_2(n) - u_2(n+1) = -\frac{69029}{1877760}\frac{1}{n^8} + O\left(\frac{1}{n^9}\right).$$

Step 3: The third-correction.

Similarly, define the sequence $(u_3(n))_{n\geq 1}$ by the relation

$$n! = \sqrt{2\pi e} \cdot e^{-\frac{3\pm\sqrt{3}}{6}} \left(\frac{n + \frac{3\pm\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}} \tag{2.9}$$

$$\exp\left(\frac{\mp\frac{1}{72\sqrt{3}}}{n^2 + \frac{10\pm3\sqrt{3}}{10}n + \frac{47\pm15\sqrt{3}}{100} + \frac{\pm\frac{163}{21000\sqrt{3}}}{n+\frac{815\pm11596\sqrt{3}}{1630} + \frac{s_2}{n+t_2}}}\right)\exp u_3(n).$$

Using the same method as above, we obtain that the sequence $(u_3(n))_{n\geq 1}$ converges fastest only if $s_2 = \dfrac{15531525}{106276}$, $t_2 = \dfrac{19139187627 \mp 259913623163\sqrt{3}}{38278375254}$.
The new asymptotic (1.13) is obtained.

3 Proof of Theorem 1.2

The double-side inequality (1.14) may be written as follows:

$$f(n) = \ln\Gamma(n+1) - \frac{1}{2}\ln(2\pi e) + \frac{3+\sqrt{3}}{6} - \left(n+\frac{1}{2}\right)\left(\ln\left(n + \frac{3+\sqrt{3}}{6}\right) - 1\right) < 0$$

and

$$g(n) = \ln\Gamma(n+1) - \frac{1}{2}\ln(2\pi e) + \frac{3-\sqrt{3}}{6} - \left(n+\frac{1}{2}\right)\left(\ln\left(n + \frac{3-\sqrt{3}}{6}\right) - 1\right) > 0.$$

Suppose $F(n) = f(n+1) - f(n)$ and $G(n) = g(n+1) - g(n)$. For every $x > 1$, we can get

$$F''(x) = \frac{36(-1+4\sqrt{3}+4\sqrt{3}n)}{(1+n)^2(3+\sqrt{3}+6n)^2(9+\sqrt{3}+6n)^2} > 0 \tag{3.1}$$

and

$$G''(x) = -\frac{36(1+4\sqrt{3}+4\sqrt{3}n)}{(1+n)^2(3-\sqrt{3}+6n)^2(9-\sqrt{3}+6n)^2} < 0. \tag{3.2}$$

It shows that $F(x)$ is strictly convex and $G(x)$ is strictly concave on $(0,\infty)$. According to Theorem 1.1, when $n \to \infty$, it holds that $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = 0$; thus $\lim_{n\to\infty} F(n) = \lim_{n\to\infty} G(n) = 0$. As a result, we can make sure that $F(x) > 0$ and $G(x) < 0$ on $(0,\infty)$. Consequently, the sequence $f(n)$ is strictly increasing and $g(n)$ is strictly decreasing while they both converge to 0. As a result, we conclude that $f(n) < 0$, and $g(n) > 0$ for every integer $n > 1$.

The proof of Theorem 1.2 is completed.

4 Numerical computations

In this section, we give Table 1 to demonstrate the superiority of our new series respectively. From what has been discussed above, we found out some new approximations as follows:

$$n! \approx \sqrt{2\pi e} \cdot e^{-\frac{3+\sqrt{3}}{6}} \left(\frac{n+\frac{3+\sqrt{3}}{6}}{e} \right)^{n+\frac{1}{2}} = \beta_1(n), \tag{4.1}$$

or

$$n! \approx \sqrt{2\pi e} \cdot e^{-\frac{3-\sqrt{3}}{6}} \left(\frac{n+\frac{3-\sqrt{3}}{6}}{e} \right)^{n+\frac{1}{2}} = \beta_2(n) \tag{4.2}$$

or

$$n! \approx \sqrt{2\pi e} \cdot e^{-\frac{3+\sqrt{3}}{6}} \left(\frac{n+\frac{3+\sqrt{3}}{6}}{e} \right)^{n+\frac{1}{2}} \exp\left(\frac{-\frac{1}{72\sqrt{3}}}{n^2+\frac{10+3\sqrt{3}}{10}n+\frac{47+15\sqrt{3}}{100}} \right) = \beta_3(n) \tag{4.3}$$

For simplicity, we only give three approximations $\beta_1(n)$, $\beta_2(n)$, $\beta_3(n)$, more formulas can be directly obtained from Theorem 1.1.

Burnside [3] gave the formula:

$$n! \approx \sqrt{2\pi} \left(\frac{n+\frac{1}{2}}{e} \right)^{n+\frac{1}{2}} = \beta(n). \tag{4.4}$$

The great advantage of our continued fraction approximation $\beta_3(n)$ consists in its simple form and its accuracy. From Table 1, we can see that the relative error of $\beta_3(n)$ is -1.1137×10^{-18} when calculating 500! and the relative error of $\beta(n)$ is 8.2540×10^{-4} when calculating 50!. Our formula

Table 1: Simulations for $\beta(n)$ and $\beta_i(n)$, $i = 1, 2, 3$.

n	$\frac{\beta(n)-n!}{n!}$	$\frac{\beta_1(n)-n!}{n!}$	$\frac{\beta_2(n)-n!}{n!}$	$\frac{\beta_3(n)-n!}{n!}$
50	8.2540×10^{-4}	-3.1767×10^{-6}	3.1120×10^{-6}	-8.1273×10^{-14}
500	8.3254×10^{-5}	-3.2044×10^{-8}	3.1978×10^{-8}	-1.1137×10^{-18}
1000	4.1647×10^{-5}	-8.0149×10^{-9}	8.0066×10^{-9}	-3.5367×10^{-20}
2000	2.0828×10^{-5}	-2.0042×10^{-9}	2.0032×10^{-9}	-1.1141×10^{-21}

$\beta_3(n)$ converges faster than the approximation of the Burnside's formula $\beta(n)$.

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Some norm inequalities for accretive Hilbert space operators

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ABSTRACT

New norm inequalities for accretive operators on Hilbert space are given. Among other inequalities, we prove that if $A, B \in \mathbb{B}(\mathbb{H})$ and B is self-adjoint and also $C_{m,M}(iAB)$ is accretive, then

$$\frac{4\sqrt{Mm}}{M+m}\|AB\| \leq \omega(AB - BA^*),$$

where M and m are positive real numbers with $M > m$ and $C_{m,M}(A) = (A^* - mI)(MI - A)$. Also, we show that if $C_{m,M}(A)$ is accretive and $(M - m) \leq k\|A\|$, then

$$\omega(AB) \leq (2 + k)\omega(A)\omega(B).$$

RESUMEN

Entregamos nuevas desigualdades para normas de operadores acretivos en espacios de Hilbert. Entre otras desigualdades, probamos que si $A, B \in \mathbb{B}(\mathbb{H})$ y B es auto-adjunto y también $C_{m,M}(iAB)$ es acretivo, entonces

$$\frac{4\sqrt{Mm}}{M+m}\|AB\| \leq \omega(AB - BA^*),$$

donde M y m son números reales positivos con $M > m$ y $C_{m,M}(A) = (A^* - mI)(MI - A)$. También mostramos que si $C_{m,M}(A)$ es acretivo y $(M - m) \leq k\|A\|$, entonces

$$\omega(AB) \leq (2 + k)\omega(A)\omega(B).$$

Keywords and Phrases: Bounded linear operator, Hilbert space, norm inequality, numerical radius.

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1 Introduction and preliminaries

Let $\mathbb{B}(\mathbb{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathbb{H}$ with inner product $\langle\cdot,\cdot\rangle$. The numerical radius of $A \in \mathbb{B}(\mathbb{H})$ is defined by

$$\omega(A) = \sup\{ \ |\langle Ax, x\rangle| : \|x\| = 1 \ \}.$$

In [14], Yamazaki proved that for any $A \in \mathbb{B}(\mathbb{H})$

$$\omega(A) = \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}A)\|. \tag{1.1}$$

It is well known that $\omega(\cdot)$ is a norm on $\mathbb{B}(\mathbb{H})$ which is equivalent to the usual operator norm $\|.\|$. In fact, for all $A \in \mathbb{B}(\mathbb{H})$,

$$\frac{\|A\|}{2} \le \omega(A) \le \|A\|. \tag{1.2}$$

The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A is normal. Several numerical radius inequalities improving the inequalities in (1.2) have been recently given in [1–3, 7, 9, 11, 12, 15, 16] and [17]. Holbrook in [6] showed that, for any $A, B \in \mathbb{B}(\mathbb{H})$,

$$\omega(AB) \le 4\omega(A)\omega(B). \tag{1.3}$$

In the case $AB = BA$, then

$$\omega(AB) \le 2\omega(A)\omega(B).$$

The question about the best constant k such that the inequality

$$w\,(AB) \le k\|A\|\omega(B) \tag{1.4}$$

holds for all operators $A, B \in \mathbb{B}(\mathbb{H})$ is still open. It is shown in [4] that, for any $A, B \in \mathbb{B}(\mathbb{H})$,

$$\omega(AB \pm BA^*) \le 2\|A\|\omega(B). \tag{1.5}$$

Let $D_A = \inf\limits_{\lambda\in\mathbb{C}} \|A - \lambda I\|$ and let R_A denote the radius of the smallest disk in the complex plane containing $\sigma(A)$ (the spectrum of A). Stampfli in [13] proved that if $A \in \mathbb{B}(\mathbb{H})$ and A is normal, then

$$D_A = R_A. \tag{1.6}$$

The following result from [10] may be stated as well: if $A, B \in \mathbb{B}(\mathbb{H})$,then

$$w\,(AB) \leq \omega(A)\omega(B) + D_A D_B. \tag{1.7}$$

Also, the authors in [8] proved that if $A, B \in \mathbb{B}(\mathbb{H})$ and A is self-adjointable, then

$$\omega(BA) \leq D_B \|A\|. \tag{1.8}$$

We consider the nonlinear functional $V_s : \mathbb{B}(\mathbb{H}) \longrightarrow R$, given by

$$V_s(A) = \sup_{\|x\|=1} Re\langle Ax, x\rangle.$$

Recall that, for all $A \in \mathbb{B}(\mathbb{H})$,

$$V_s(A) \leq \omega(A) \leq \|A\|. \tag{1.9}$$

We say that an operator $A : \mathbb{H} \longrightarrow \mathbb{H}$ is accretive, if $Re\langle Ax, x\rangle \geq 0$ for any $x \in \mathbb{H}$. In [3], Dragomir has shown that if M and m are positive real numbers with $M > m$ and the operator $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive, then

$$\|A\| \leq \frac{M+m}{2\sqrt{Mm}} V_s(A) \tag{1.10}$$

and

$$\|A\| \leq \frac{M+m}{2\sqrt{Mm}} \omega(A). \tag{1.11}$$

A sufficient simple condition for $C_{m,M}(A)$ to be accretive is that A is a self-adjoint operator on $\mathbb{H}$ such that $mI \leq A \leq MI$ in the usual operator order of $\mathbb{B}(\mathbb{H})$. Moreover, if $0 < m < M$, a sufficient condition for $C_{m,M}(A)$ to be accretive is that

$$\left\| A - \frac{M+m}{2} I \right\| < \frac{(M-m)}{2}.$$

The following result from [5] may be stated as well: if M and m are positive real numbers with $M > m$ and $A, B \in \mathbb{B}(\mathbb{H})$ and also $C_{m,M}(A)$ is accretive, then

$$\omega(AB - BA^*) \leq (M-m)\omega(B). \tag{1.12}$$

And also

$$\|A\| \leq \frac{M+m}{2\sqrt{Mm}} \|Re(A)\|, \tag{1.13}$$

which is a refinement of inequality (1.11).

In Section 2, we introduce some inequalities between the operator norm and the numerical radius of accretive operators on Hilbert spaces. More precisely, we establish the generalization of the inequalities (1.11) and (1.13). Also, we find a lower bound for $\omega(AB - BA^*)$.

2 Main results

We need the following lemma, to achieve our goal.

Lemma 2.1. *If $A \in \mathbb{B}(\mathbb{H})$, then*

$$V_s(A) \leq \|Re(A)\|.$$

Proof. Suppose that $x \in \mathbb{H}$ with $\|x\| = 1$. We have

$$Re\langle Ax, x\rangle = \frac{\langle (A + A^*)x, x\rangle}{2} \leq \frac{\|A + A^*\|}{2} \leq \|Re(A)\|.$$

Hence

$$Re\langle Ax, x\rangle \leq \|Re(A)\|.$$

Taking the supremum over $x \in \mathbb{H}$ with $\|x\| = 1$ gives

$$V_s(A) \leq \|Re(A)\|,$$

which is exactly the desired result. □

Remark 2.2. *Let M and m be positive real numbers with $M > m$ and $A \in \mathbb{B}(\mathbb{H})$ and also $C_{m,M}(A)$ is accretive. By* (1.10) *and Lemma 2.1 we deduce that*

$$\frac{M + m}{2\sqrt{Mm}} V_s(A) \leq \frac{M + m}{2\sqrt{Mm}} \|Re(A)\|.$$

Therefore, the inequality (1.10) *strengthens* (1.11) *and* (1.13)*. Then, we continue this section and introduce some norm inequalities for products of two Hilbert space operators with inequality* (1.10).

The following result may be as well.

Theorem 2.3. *If $A, B \in \mathbb{B}(\mathbb{H})$, then*

$$V_s(AB) \leq \frac{\|B+B^*\|\omega(A)}{2} + \frac{D_A D_{B+B^*}}{2} + \frac{1}{2}\omega(AB - BA^*).$$

Proof. Clearly, $\|Re(AB)\| = \omega(Re(AB))$. Then

$$\begin{aligned}
\|Re(AB)\| &= \omega\left(\frac{AB + B^*A^*}{2}\right) \\
&= \omega\left(\frac{AB + AB^* - AB^* + B^*A^*}{2}\right) \\
&\leq \omega\left(\frac{AB + AB^*}{2}\right) + \omega\left(\frac{-AB^* + B^*A^*}{2}\right) \\
&= \frac{1}{2}\omega(A(B+B^*)) + \frac{1}{2}\omega(AB - BA^*) \\
&\leq \frac{\|B+B^*\|\omega(A)}{2} + \frac{1}{2}D_A D_{B+B^*} + \frac{1}{2}\omega(AB - BA^*). \qquad \text{(by (1.7))}
\end{aligned}$$

Hence

$$\|Re(AB)\| \leq \frac{\|B+B^*\|\omega(A)}{2} + \frac{1}{2}D_A D_{B+B^*} + \frac{1}{2}\omega(AB - BA^*) \tag{2.1}$$

and the result follows from Lemma 2.1. □

Corollary 2.4. *If $A, B \in \mathbb{B}(\mathbb{H})$, then*

$$V_s(AB) \leq \omega(B)\left(\omega(A) + D_A\right) + \frac{1}{2}\omega(AB - BA^*).$$

Proof. By Theorem 2.3,

$$V_s(AB) \leq \frac{\|B+B^*\|\omega(A)}{2} + \frac{D_A D_{B+B^*}}{2} + \frac{1}{2}\omega(AB - BA^*).$$

Since $D_{B+B^*} \leq \|B+B^*\|$, then

$$\begin{aligned}
V_s(AB) &\leq \|Re(B)\|(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*) \\
&\leq \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}B)\|(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*) \\
&\leq \omega(B)\left(\omega(A) + D_A\right) + \frac{1}{2}\omega(AB - BA^*). \qquad \text{(by (1.1))}
\end{aligned}$$

Therefore,

$$V_s(AB) \leq \omega(B)\left(\omega(A) + D_A\right) + \frac{1}{2}\omega(AB - BA^*).$$

This completes the proof. □

Corollary 2.5. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If there exist $\theta_0 \in \mathbb{R}$ such that $C_{m,M}(e^{i\theta_0}AB)$ is accretive , then*

$$\|AB\| \leq \frac{M+m}{2\sqrt{Mm}} \left(\omega(B)(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*) \right). \tag{2.2}$$

Proof. By (2.1),

$$\|Re(AB)\| \leq \frac{\|B + B^*\|\omega(A)}{2} + \frac{1}{2}D_A D_{B+B^*} + \frac{1}{2}\omega(AB - BA^*).$$

Since $D_{B+B^*} \leq \|B + B^*\|$, gives

$$\|Re(AB)\| \leq \|Re(B)\|(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*). \tag{2.3}$$

Suppose that $\theta_0 \in \mathbb{R}$. Replacing B by $e^{i\theta_0}B$ in the inequality (2.3) gives

$$\begin{aligned}
\|Re(e^{i\theta_0}AB)\| &\leq \|Re(e^{i\theta_0}B)\|(\omega(A) + D_A) + \frac{1}{2}\omega(e^{i\theta_0}(AB - BA^*)) \\
&= \|Re(e^{i\theta_0}B)\|(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*) \\
&\leq \sup_{\theta_0 \in \mathbb{R}} \|Re(e^{i\theta_0}B)\|(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*) \\
&= \omega(B)(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*). \qquad \text{(by (1.1))}
\end{aligned}$$

Hence,

$$\|Re(e^{i\theta_0}AB)\| \leq \omega(B)(\omega(A) + D_A) + \frac{1}{2}\omega(AB - BA^*). \tag{2.4}$$

Since $C_{m,M}(e^{i\theta_0}AB)$ is accretive, from the inequality (1.13) we have

$$\frac{2\sqrt{Mm}}{M+m}\|AB\| \leq \|Re(e^{i\theta_0}AB)\|$$

and the result follows from (2.4). □

Remark 2.6. *The result stated in Corollary 2.5 is stronger than inequality* (1.11)*. To explain that, suppose that $C_{m,M}(B)$ is accretive. Replacing A by I in inequality* (2.2)*. Since $D_I = 0$ and $\omega(I) = \|I\| = 1$, then we have $\|B\| \leq \dfrac{M+m}{2\sqrt{Mm}}\omega(B)$.*

The following result may be as well.

Theorem 2.7. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}\left(\begin{bmatrix} 0 & AB \\ 0 & 0 \end{bmatrix}\right)$ is accretive, then*

$$\frac{2\sqrt{Mm}}{M+m}\|AB\| \leq \frac{\|B\|}{2}(\omega(A) + D_A) + \frac{\|AB - BA^*\|}{4}.$$

Proof. Let $A_1 = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $B_1 = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$. Since $C_{m,M}(A_1B_1) = C_{m,M}\left(\begin{bmatrix} 0 & AB \\ 0 & 0 \end{bmatrix}\right)$ is accretive, from the inequality (1.10) and Theorem 2.3 we have

$$\begin{aligned}
\frac{2\sqrt{Mm}}{M+m}\|AB\| &= \frac{2\sqrt{Mm}}{M+m}\|A_1B_1\| \\
&\leq \frac{\|B_1 + {B_1}^*\|\omega(A_1)}{2} + \frac{D_{A_1}D_{B_1+{B_1}^*}}{2} + \frac{1}{2}\omega(A_1B_1 - B_1{A_1}^*) \\
&= \frac{\|B\|\omega(A)}{2} + \frac{D_A D_{B_1+{B_1}^*}}{2} + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & AB - BA^* \\ 0 & 0 \end{bmatrix}\right) \\
&\leq \frac{\|B\|\omega(A)}{2} + \frac{D_A\|B_1\|}{2} + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & AB - BA^* \\ 0 & 0 \end{bmatrix}\right) \\
&= \frac{\|B\|\omega(A)}{2} + \frac{D_A\|B\|}{2} + \frac{\|AB - BA^*\|}{4}.
\end{aligned}$$

Consequently,

$$\frac{2\sqrt{Mm}}{M+m}\|AB\| \leq \frac{\|B\|}{2}(\omega(A) + D_A) + \frac{\|AB - BA^*\|}{4},$$

which is exactly the desired result. □

As a direct consequence of Theorem 2.7, we have:

Corollary 2.8. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}\left(\begin{bmatrix} 0 & AB \\ 0 & 0 \end{bmatrix}\right)$ is accretive and $AB = BA^*$, then*

$$\|AB\| \leq \frac{M+m}{4\sqrt{Mm}}\|B\|(\omega(A) + D_A).$$

We need the following lemma to give some applications of Theorem 2.3.

Lemma 2.9. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}(A)$ and $C_{m,M}(B)$ are accretive, then $C_{m,M}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)$ is accretive.*

Proof. Put $X = \begin{bmatrix} x \\ y \end{bmatrix}$, where $x, y \in \mathbb{H}$. First we show that if A and B are accretive, then $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is accretive. We have

$$\begin{aligned} Re(\langle TX, X\rangle) &= Re\left(\left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right\rangle\right) = Re\left(\left\langle \begin{bmatrix} Ax \\ By \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right\rangle\right) \\ &= Re(\langle Ax, x\rangle) + Re(\langle By, y\rangle). \end{aligned}$$

Since $Re(\langle Ax, x\rangle) \geq 0$ and $Re(\langle By, y\rangle) \geq 0$, then

$$Re(\langle TX, X\rangle) \geq 0 \tag{2.5}$$

and so T is accretive. On the other hand,

$$\begin{aligned} C_{m,M}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) &= \left(\begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} - \begin{bmatrix} mI & 0 \\ 0 & mI \end{bmatrix}\right)\left(\begin{bmatrix} MI & 0 \\ 0 & MI \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} A^* - mI & 0 \\ 0 & B^* - mI \end{bmatrix}\right)\left(\begin{bmatrix} MI - A & 0 \\ 0 & MI - B \end{bmatrix}\right) \\ &= \begin{bmatrix} (A^* - mI)(MI - A) & 0 \\ 0 & (B^* - mI)(MI - A) \end{bmatrix} \\ &= \begin{bmatrix} C_{m,M}(A) & 0 \\ 0 & C_{m,M}(B) \end{bmatrix}. \end{aligned}$$

Consequently,

$$C_{m,M}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \begin{bmatrix} C_{m,M}(A) & 0 \\ 0 & C_{m,M}(B) \end{bmatrix}. \tag{2.6}$$

Since $C_{m,M}(A)$ and $C_{m,M}(B)$ are accretive, the result follows from (2.5) and (2.6). □

In the following, we provide a lower bound of the $\omega(AB - BA^*)$ in terms of $\|AB\|$ for some case.

Theorem 2.10. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If B is self-adjoint and $C_{m,M}(iAB)$ is accretive, then*

$$\frac{4\sqrt{Mm}}{M+m}\|AB\| \leq \omega(AB - BA^*). \tag{2.7}$$

Proof. By Theorem 2.3,

$$V_s(AB) \leq \frac{\|B+B^*\|\omega(A)}{2} + \frac{D_A D_{B+B^*}}{2} + \frac{1}{2}\omega(AB-BA^*).$$

Replacing B by iB in the last inequality gives

$$V_s(iAB) \leq \frac{1}{2}\omega(AB-BA^*). \tag{2.8}$$

Since $C_{m,M}(iAB)$ is accretive, from the inequality (1.10) and (2.8) we have

$$\frac{2\sqrt{Mm}}{M+m}\|AB\| \leq V_s(iAB) \leq \frac{1}{2}\omega(AB-BA^*).$$

Therefore,

$$\frac{2\sqrt{Mm}}{M+m}\|AB\| \leq \frac{1}{2}\omega(AB-BA^*).$$

This completes the proof. □

Recently, some inequalities have been presented by mathematicians to find the upper bound of $\omega(AB-BA^*)$, for example inequalities (1.5) and (1.12). On the other hand, we have to use the first inequality (1.2) to find a lower bound of $\omega(AB-BA^*)$. Now, in the following we give an example to show how Theorem 2.10 improves the first inequality (1.2).

Example 2.11. *Let* $B = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$, $A = \begin{bmatrix} -1.5i & 0.2i \\ 0 & -3.2i \end{bmatrix}$, $M = 3$, *and* $m = 1$. *Clearly* B *is self-adjoint and with a simple calculation, we have*

$$\left\|iAB - \frac{M+m}{2}I\right\| = \left\|\begin{bmatrix} 0.5 & 0.1 \\ 0 & 0.4 \end{bmatrix}\right\| \simeq 0.52 \leq 1 = \frac{M-m}{2}.$$

Therefore, $C_{m,M}(iAB)$ *is accretive. On the other hand,*

$$\|AB\| = \left\|\begin{bmatrix} -1.5i & 0.1i \\ 0 & -1.6i \end{bmatrix}\right\| \simeq 1.62$$

and

$$\|AB-BA^*\| = \left\|\begin{bmatrix} -3i & 0.1i \\ 0.1i & -3.2i \end{bmatrix}\right\| \simeq 3.24.$$

In this case

$$\frac{\|AB-BA^*\|}{2} \simeq 1.62$$

while

$$\frac{4\sqrt{Mm}}{M+m}\|AB\| \simeq 2.80.$$

Remark 2.12. *Let M and m are positive real numbers with $M > m$ and $A \in \mathbb{B}(\mathbb{H})$ and also $C_{m,M}(A)$ is accretive. Replacing B by I and A by $-iA$ in Theorem 2.10 gives*

$$\frac{2\sqrt{Mm}}{M+m}\|A\| \leq \frac{1}{2}\omega(A+A^*) = \|Re(A)\|.$$

Therefore, the inequality (2.7) *strengthens* (1.13)*.*

Corollary 2.13. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If B is self-adjoint and $C_{m,M}(A)$ and also $C_{m,M}(iAB)$ is accretive, then*

$$\|AB\| \leq \frac{(M^2-m^2)}{4\sqrt{Mm}}\|B\|.$$

Proof. By Theorem 2.10,

$$\frac{4\sqrt{Mm}}{M+m}\|AB\| \leq \omega(AB-BA^*).$$

From the hypothesis $C_{m,M}(A)$ is accretive and (1.12),

$$\frac{4\sqrt{Mm}}{M+m}\|AB\| \leq (M-m)\omega(B),$$

which is exactly the desired result. □

At the end of this section, we introduce some numerical radius inequalities for products of two operators.

Theorem 2.14. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}(A)$ is accretive, then*

$$\omega(AB) \leq \left(D_A + \frac{M-m}{2}\right)\omega(B).$$

Proof. Clearly, $\|Re(AB)\| = \omega(Re(AB))$. Then

$$\begin{aligned}
\|Re(AB)\| &= \omega\left(\frac{AB+B^*A^*}{2}\right) \\
&= \omega\left(\frac{AB+AB^*-AB^*+B^*A^*}{2}\right) \\
&\leq \omega\left(\frac{AB+AB^*}{2}\right) + \omega\left(\frac{-AB^*+B^*A^*}{2}\right) \\
&= \frac{1}{2}\omega(A(B+B^*)) + \frac{1}{2}\omega(AB-BA^*) \\
&\leq \frac{1}{2}D_A\|B+B^*\| + \frac{1}{2}\omega(AB-BA^*) \qquad \text{(by (1.8))}
\end{aligned}$$

$$\begin{aligned}
&= D_A\|Re(B)\| + \frac{1}{2}\omega(AB - BA^*) \\
&\leq D_A \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}B)\| + \frac{1}{2}\omega(AB - BA^*) \\
&= D_A\omega(B) + \frac{1}{2}\omega(AB - BA^*) \\
&\leq D_A\omega(B) + \frac{M-m}{2}\omega(B). \qquad \text{(by (1.12))} \\
&= \left(D_A + \frac{M-m}{2}\right)\omega(B).
\end{aligned}$$

Hence,

$$\|Re(AB)\| \leq \left(D_A + \frac{M-m}{2}\right)\omega(B). \tag{2.9}$$

Suppose that $\theta_0 \in \mathbb{R}$. Replacing B by $e^{i\theta_0}B$ in the inequality (2.9) gives

$$\|Re(e^{i\theta_0}AB)\| \leq \left(D_A + \frac{M-m}{2}\right)\omega(B).$$

Taking the supremum over $\theta_0 \in \mathbb{R}$ gives

$$\omega(AB) \leq \left(D_A + \frac{M-m}{2}\right)\omega(B),$$

which is exactly the desired result. □

Corollary 2.15. *Let M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}(A)$ is accretive, then*

$$\omega(AB) \leq \left(\|A\| + \frac{M-m}{2}\right)\omega(B).$$

Proof. Since $D_A \leq \|A\|$, the result follows from Theorem 2.14. □

Concerning the inequality (1.4), the following result is interesting.

Theorem 2.16. *Let k, M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}(A)$ is accretive and $(M-m) \leq k\|A\|$, then*

$$\omega(AB) \leq \left(1 + \frac{k}{2}\right)\|A\|\omega(B).$$

Proof. By Corollary 2.15,

$$\omega(AB) \leq \left(\|A\| + \frac{M-m}{2}\right)\omega(B). \tag{2.10}$$

From the hypothesis $(M-m) \leq k\|A\|$ and inequality (2.10),

$$\omega(AB) \leq \left(\|A\| + \frac{k\|A\|}{2}\right)\omega(B),$$

which is exactly the desired result. □

Corollary 2.17. *Let k, M and m (with $M > m$) are positive real numbers and $A, B \in \mathbb{B}(\mathbb{H})$. If $C_{m,M}(A)$ is accretive and $(M - m) \leq k\|A\|$, then*

$$\omega(AB) \leq (2 + k)\omega(A)\omega(B).$$

Proof. Since $\|A\| \leq 2\omega(A)$, the result follows from Theorem 2.16. □

Remark 2.18. *If $k < 2$, Corollary 2.16 refines the inequality* (1.3).

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Declarations

Competing interests The authors declare that they do not have any competing interests.

Conflicts of interest All authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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A simple construction of a fundamental solution for the extended Weinstein equation

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ABSTRACT

In this article, we study the extended Weinstein equation

$$Lu = \Delta u + \frac{k}{x_n}\frac{\partial u}{\partial x_n} + \frac{\ell}{x_n^2}u,$$

where u is a sufficiently smooth function defined in $\mathbb{R}^n$ with $x_n > 0$ and $n \geq 3$. We find a detailed construction for a fundamental solution for the operator L. The fundamental solution is represented by the associated Legendre functions Q_ν^μ.

RESUMEN

En este artículo estudiamos la ecuación de Weinstein extendida

$$Lu = \Delta u + \frac{k}{x_n}\frac{\partial u}{\partial x_n} + \frac{\ell}{x_n^2}u,$$

donde u es una función suficientemente suave definida en $\mathbb{R}^n$ con $x_n > 0$ y $n \geq 3$. Encontramos una construcción detallada para una solución fundamental del operador L. La solución fundamental está representada por las funciones de Legendre asociadas Q_ν^μ.

Keywords and Phrases: Extended Weinstein equation, fundamental solution, associated Legendre function.

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1 Introduction

In this paper, we will study the *extended Weinstein* or the *Leutwiler-Weinstein equation*

$$Lu := \Delta u + \frac{k}{x_n}\frac{\partial u}{\partial x_n} + \frac{\ell}{x_n^2}u = 0, \tag{1.1}$$

where $k, \ell \in \mathbb{R}$. The *Weinstein operator* L plays an interesting special role in the theory of partial differential equations, hyperbolic geometry and in other areas of mathematics (*cf.* Section 5). With the trivial choice of parameters $k = \ell = 0$, the Weinstein operator is the usual Euclidean *Laplacian*

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

acting on functions defined on $\mathbb{R}^n$. The solutions are called harmonic functions and the theory is well elaborated, see *e.g.* [3, 13, 14]. The next natural step is to just require the condition $\ell = 0$ to be fulfilled, in which case we are in the case presented by Alexander Weinstein, see [21] and also [4, 11]. In this case, equation (1.1) is a classic Weinstein equation and the operator L is singular on the surface $x_n = 0$. In this case, we usually look at functions that are defined in the upper half-space

$$\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0, \infty).$$

For more recent research on the Weinstein equation, see *e.g.* [5, 8]. The extended Weinstein equation (1.1) with arbitrary parameters $k, \ell \in \mathbb{R}$ was initially studied by Heinz Leutwiler in [12]. The equation has continued to be studied quite actively until these days, see *e.g.* [2].

The purpose of this article is to present the simplest possible construction (from the point of view of the authors) for the fundamental solution for the Weinstein operator L represented in (1.1). We try to present the theory in such a way that basic knowledge of partial differential equations and vector analysis are sufficient to follow the presentation, *i.e.* the so-called graduate student level. The structure of the article is as follows:

- In Section 2, we outline the required preliminaries, *i.e.* the Weinstein operator with its reduced version, and some useful notions from the theory of distributions.
- In Section 3, find the special type of “radial” solutions for (1.1).
- In Section 4, we use the “radial” solutions to define the fundamental solution and compute its proper coefficient.

2 Preliminaries

2.1 Weinstein operator

Let us look at some basic properties of the Weinstein operator L. Note that the variable x_n plays a special role in the operator. We denote elements $x = (x', x_n) \in \mathbb{R}^n_+$, where $x_n > 0$. We observe that keeping x_n fixed, the operator L admits with respect to the variable x' the same invariance properties as the Laplacian in $\mathbb{R}^{n-1}$, *i.e.* invariance under the Euclidean rigid motions (*cf.* [3]). Particularly important in what follows is the invariance with respect to translations

$$x' \mapsto x' + a' \tag{2.1}$$

for any $a' \in \mathbb{R}^{n-1}$. In the previous section, we did not discuss the fourth possible canonical special case for the Weinstein equation, namely the situation $k = 0$. In fact, this situation is significantly related to solving the extended Weinstein equation as follows. As a direct computation gives

$$L(x_n^{-\frac{k}{2}} u) = x_n^{-\frac{k}{2}} \widetilde{L} u, \tag{2.2}$$

where

$$\widetilde{L} u = \Delta u + \frac{k(2-k) + 4\ell}{4} \frac{u}{x_n^2},$$

we call the operator $\widetilde{L}$ the *reduced operator*. Subsequently, we will base our calculations largely on the reduced operator, as it is relatively close to the Laplace operator in its properties.

The reduced operator is especially useful from the point of view of the integration theory. Let U be a bounded subset of $\mathbb{R}^n_+$ with a sufficiently smooth boundary ∂U and let u and v twice differentiable real valued functions defined in an open set containing U. Hence, the usual *Green formula* for the Laplace operator is

$$\int_U (u\Delta v - v\Delta u)\, dx = \int_{\partial U} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS,$$

where the derivative with respect to the outer unit normal n is defined by

$$\frac{\partial u}{\partial n} = n \cdot \nabla u.$$

The Green formula for the reduced operator is obtained by adding and subtracting the term $\frac{k(2-k) + 4\ell}{4} \frac{uv}{x_n^2}$ in the volume integral, *i.e.*

$$\int_U \left(u\widetilde{L} v - v\widetilde{L} u \right) dx = \int_{\partial U} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS. \tag{2.3}$$

2.2 Generalized functions

Generalized functions or distributions are a standard tool in modern partial differential equation theory. Their history begins in 1936, when Sergei Sobolev introduced his "*l'espace fonctionnel*" and applied them to solve a Cauchy problem of second-order partial differential equations in [17]. After this, the theory was further developed, see *e.g.*, the first larger representation of Laurent Schwartz [16]. A key work in the theory of partial differential equations is the classic book [9] by Gelfand and Shilov. In this book, distributions are examined from the point of view of solving partial differential equations, and the key tool is the connection between distributions and complex analytical functions. All the following information can be found in more detail in the literature mentioned above.

Let Ω be an open subset of $\mathbb{R}^n$ (or $\mathbb{R}^n_+$). We denote $\mathcal{D}(\Omega)$ as the space of compactly supported functions

$$C_0^\infty(\Omega) := \{\varphi \in C^\infty(\Omega) : \text{supp}(\varphi) \text{ is compact and } \text{supp}(\varphi) \subset \Omega\}$$

equipped with the topology of uniform convergence in compact subsets $K \subset \Omega$. Indeed, $\varphi_j \to \varphi$ in $\mathcal{D}(\Omega)$, if there exists a compact subset $K \subset \Omega$ such that $\text{supp}(\varphi_j) \subset K$ for any j and all derivatives $\partial^\alpha \varphi_j \to \partial^\alpha \varphi$ uniformly, *i.e.* the convergence in the Fréchet space $C^\infty(K)$. Above, multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

is used. The preceding $\mathcal{D}(\Omega)$ is called the *test function space*. We denote by $\mathcal{D}'(\Omega)$ the space of continuous linear functionals over $\mathcal{D}(\Omega)$, and we call its elements *distributions* or *generalized functions*. If $T \in \mathcal{D}'(\Omega)$, we denote

$$T(\varphi) =: \langle T, \varphi \rangle$$

for all $\varphi \in \mathcal{D}(\Omega)$. The continuity of a functional T means, that $T(\varphi) \to 0$ for all $\varphi \to 0$ in $\mathcal{D}(\Omega)$. The convergence in $\mathcal{D}'(\Omega)$ is defined in the weak form, *i.e.* a sequence $\{T_j\}$ of distributions converges to a distribution T if and only if

$$\langle T_j, \varphi \rangle \to \langle T, \varphi \rangle, \quad \text{for} \quad j \to \infty, \tag{2.4}$$

for any $\varphi \in \mathcal{D}(\Omega)$. Important basic properties of distributions are that they have all derivatives defined by setting

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and multiplying by a smooth function $f \in C^\infty(\Omega)$ produces a distribution, *i.e.*,

$$\langle fT, \varphi \rangle = \langle T, f\varphi \rangle.$$

The above properties allow differential operators to be defined in distributional sense, *e.g.*,

$$\langle \widetilde{L}T, \varphi \rangle = \langle T, \widetilde{L}\varphi \rangle, \tag{2.5}$$

for any $T \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ when $\Omega \subset \mathbb{R}^n_+$. We also denote

$$\langle T, \varphi \rangle = \langle T(x), \varphi(x) \rangle,$$

where the x is a dummy variable (*cf.* the use of variables in integrals). Any *locally integrable* function $g \in L^1_{\text{loc}}(\Omega)$ defines a distibution via the L^2-inner product by

$$\langle g, \varphi \rangle := \langle g, \varphi \rangle_{L^2(\Omega)} = \int_\Omega g(x)\varphi(x)\, dx. \tag{2.6}$$

Remark 2.1. *The starting point for the theory of distributions can be also in measure theory. Let us elaborate on the equivalence of perspectives. If $\Omega \subset \mathbb{R}^n$ is an open set and μ a complex Borel measure on it with $\mu(K) < \infty$ for any compact $K \subset \Omega$, then*

$$T(\varphi) = \int_\Omega \varphi\, d\mu,$$

defines a distribution, where $\varphi \in \mathcal{D}(\Omega)$. If $f \in L^1_{loc}(\Omega)$, then the measure

$$\mu(E) = \int_E f(x)\, dx$$

for any Borel set $E \subset \Omega$ is a complex Borel measure with $\mu(K) < \infty$. Then the Radon-Nikodym derivative $\frac{d\mu}{dx} = f$. Hence, we can intuitively identify distributions with functions f or equivalently with measures μ.

The most important example of distributions is the Dirac delta distribution, which is defined by setting

$$\langle \delta(x-y), \varphi(x) \rangle := \varphi(y),$$

for $y \in \mathbb{R}^n$. The Dirac delta is not a distribution generated by a locally integrable function. In the distributional sense, one can see that $\delta(x-y) = 0$ for any $x \neq y$. Moreover, the Dirac delta has the obvious property

$$f(x)\delta(x-y) = f(y)\delta(x-y), \tag{2.7}$$

for $f \in C^\infty(\Omega)$, which plays a central role in this paper. If

$$Pu(x) = \sum_{k=0}^{m} \sum_{|\alpha|=k} a_\alpha(x)\partial^\alpha u(x)$$

is a linear differential operator P acting on a suitable function u, where $a_\alpha \in C^\infty(\Omega)$, we call a distribution $G(\cdot, y) \in \mathcal{D}'(\Omega)$ a *fundamental solution* at $y \in \Omega$, if it satisfies the equation

$$PG(x,y) = \delta(x-y).$$

The main motivation to find a fundamental solution is to study solutions of the equation $Pu = f$. One can see, that the solution of the problem is given by $u = G * f$, where $*$ is the convolution of a distribution and a function. See details, *e.g.* in [9].

3 Classical "radial" solutions

In this paper, our aim is to find a fundamental solution G for the Weinstein operator (1.1), *i.e.*

$$LG(x,y) = \delta(x-y)$$

where $y \in \mathbb{R}^n_+$. Our first observation is that due to the formulas (2.2) and (2.7), we obtain the following formula.

Proposition 3.1. *If* $\widetilde{L}v = \delta(x-y)$, *then* $L\left(\left(\frac{y_n}{x_n}\right)^{\frac{k}{2}} v\right) = \delta(x-y)$.

Hence, it is enough to find a fundamental solution for the reduced operator $\widetilde{L}$. A usual problem with any non-constant coefficient differential operator is that the symmetry of the operator does not match with the symmetry of the Dirac delta. We know that the Dirac delta is rotationally invariant (see [9]), *i.e.*

$$\delta(Ax) = \delta(x)$$

for any $A \in SO(n)$, but as we mentioned above, $\widetilde{L}$ is rotation invariant only around the x_n-axis, or more precisely, it is invariant under the subgroup $SO(n-1)$ in $SO(n)$ defined as the stabiliser of the x_n-axis. Hence, the x_n-direction will play a special role. Since the operator is translation invariant with respect to x', we can try to find first a fundamental solution only at the point $y = (0', y_n)$. Thus,

$$\delta(x-y) = \delta(x')\delta(x_n - y_n).$$

Consequently, the fundamental solution must be a "radial function", *i.e.*, it depends on $|x-y|$, with the special role of x_n. Hyperbolic geometry gives us an idea how to proceed. In [15], one can find the expression

$$|x-y|^2 = 2x_n y_n(\lambda(x,y) - 1), \tag{3.1}$$

where $\lambda : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [1, \infty)$ is defined by

$$\lambda(x, y) = 1 + \frac{|x-y|^2}{2x_n y_n}.$$

The reader should note that

$$\lambda \geq 1. \tag{3.2}$$

The function λ is an invariant with respect to the invariance group of the hyperbolic upper-half space, *cf.* [15]. Based on this, one can try to find a solution for the extended Weinstein equation in the form

$$u(x) = x_n^{\alpha} v(\lambda), \tag{3.3}$$

for a fixed $y \in \mathbb{R}^n_+$. We want to substitute this into equation (1.1). First, we compute the following technical lemma.

Lemma 3.2. *If u is of the form* (3.3) *and $y' = 0$, we have*

$$x_n^{2-\alpha} \widetilde{L} u = \left(\lambda^2 - 1\right) v''(\lambda) + \left((n - 2 + 2\alpha)\frac{x_n}{y_n} + 2(1-\alpha)\lambda\right) v'(\lambda) + \left(\alpha(\alpha - 1) + \frac{k(2-k) + 4\ell}{4}\right) v(\lambda).$$

Proof. Since $v = v(\lambda(x, y))$, we compute

$$\begin{aligned}
\frac{\partial v}{\partial x_j} &= \frac{x_j}{x_n y_n} v'(\lambda), \\
\frac{\partial v}{\partial x_n} &= \frac{x_n - y_n\lambda}{x_n y_n} v'(\lambda), \\
\frac{\partial^2 v}{\partial x_j^2} &= \frac{x_j^2 v''(\lambda) + x_n y_n v'(\lambda)}{x_n^2 y_n^2}, \\
\frac{\partial^2 v}{\partial x_n^2} &= \frac{(x_n - y_n\lambda)^2 v''(\lambda) + (2y_n^2\lambda - x_n y_n) v'(\lambda)}{x_n^2 y_n^2},
\end{aligned}$$

for $j = 1, \ldots, n-1$. Then we compute

$$\begin{aligned}
\frac{\partial u}{\partial x_j} &= x_n^{\alpha-1} \frac{x_j}{y_n} v'(\lambda), \\
\frac{\partial u}{\partial x_n} &= x_n^{\alpha-1} \left(\alpha v(\lambda) + \left(\frac{x_n}{y_n} - \lambda\right) v'(\lambda)\right), \\
\frac{\partial^2 u}{\partial x_j^2} &= \frac{x_n^{\alpha-2}}{y_n^2} \Big(x_j^2 v''(\lambda) + y_n x_n v'(\lambda)\Big), \\
\frac{\partial^2 u}{\partial x_n^2} &= \frac{x_n^{\alpha-2}}{y_n^2} \Big((x_n - y_n\lambda)^2 v''(\lambda) + \big((2\alpha - 1)x_n y_n + (y_n^2 - \alpha y_n^2)2\lambda\big) v'(\lambda) + \alpha(\alpha-1) y_n^2 v(\lambda)\Big),
\end{aligned}$$

for $j = 1, \ldots, n-1$. Then we observe

$$\sum_{j=1}^{n-1} \frac{\partial^2 u}{\partial x_j} = \frac{x_n^{\alpha-2}}{y_n^2} \Big(|x'|^2 v''(\lambda) + (n-1) y_n x_n v'(\lambda)\Big).$$

Since $y' = 0$, we have $|x'|^2 = 2x_n y_n \lambda - x_n^2 - y_n^2$ and we obtain

$$\Delta u = x_n^{\alpha-2}\left((\lambda^2-1)v''(\lambda) + \left((n-2+2\alpha)\frac{x_n}{y_n} + 2(1-\alpha)\lambda\right)v'(\lambda) + \alpha(\alpha-1)v(\lambda)\right),$$

completing the proof. □

We observe that we obtain the ordinary differential equation with respect to λ if we choose $\alpha = \frac{2-n}{2}$. Since $\alpha(\alpha-1) = \frac{1}{4}(n-2)n$, we obtain the following result.

Proposition 3.3. *The function $u(x) = x_n^{\frac{2-n}{2}} v(\lambda)$ is a solution of $\widetilde{L}u = 0$ if and only if*

$$(\lambda^2-1)v''(\lambda) + n\lambda v'(\lambda) + \frac{1}{4}(k(2-k) + (n-2)n + 4\ell)v(\lambda) = 0.$$

We denote $\beta := \frac{1}{4}(k(2-k) + (n-2)n + 4\ell)$. To solve the equation

$$(\lambda^2-1)v''(\lambda) + n\lambda v'(\lambda) + \beta v(\lambda) = 0, \tag{3.4}$$

we first observe that it is not far from the associated Legendre equation

$$(z^2-1)w''(z) + 2zw'(z) - \left(\nu(\nu+1) + \frac{\mu^2}{z^2-1}\right)w(z) = 0, \tag{3.5}$$

with parameters $\mu, \nu \in \mathbb{C}$. The associated Legendre equation has two solutions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ defined outside of singularities $z = \pm 1$, see *e.g.* [1,10]. The solutions are called *associated Legendre functions*. The solutions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are linearly independent if and only if $\mu \pm \nu \notin -\mathbb{N}$. We need to exclude this case in the future.

Assume $x \neq y$, that is, from (3.2) we obtain $\lambda = \lambda(x,y) > 1$. We look for a solution for (3.4) in the form

$$v(\lambda) = (\lambda^2-1)^\delta w(\lambda).$$

Substituting this into the equation (3.4), the equation becomes

$$(\lambda^2-1)w'' + (4\delta+n)\lambda w' + \left(2\delta + \beta + \frac{(4\delta(\delta-1) + 2n\delta)\lambda^2}{\lambda^2-1}\right)w = 0. \tag{3.6}$$

To obtain the associated Legendre equation, it is required that

$$4\delta + n = 2 \iff \delta = \frac{2-n}{4}.$$

We obtain the following result.

Proposition 3.4. *The function* $v(\lambda) = (\lambda^2-1)^{\frac{2-n}{4}} w(\lambda)$ *satisfies equation* (3.4) *if and only if* w *is a solution of the associated Legendre equation*

$$(\lambda^2-1)w'' + 2\lambda w' - \left(-\frac{n(2-n)+4\beta}{4} + \frac{\frac{1}{4}(n-2)^2}{\lambda^2-1}\right) w = 0.$$

Proof. Using (3.6), we have

$$(\lambda^2-1)w'' + 2\lambda w' + \left(\frac{2-n}{2} + \beta - \frac{1}{4}(n-2)^2\frac{\lambda^2}{\lambda^2-1}\right) w = 0.$$

On the other hand,

$$\frac{\lambda^2}{\lambda^2-1} = 1 + \frac{1}{\lambda^2-1},$$

and we obtain the result. □

Hence we obtain the solutions as follows.

Theorem 3.5. *Equation* (3.4) *has two linearly independent solutions*

$$(\lambda^2-1)^{\frac{2-n}{4}} P^{\pm\frac{n-2}{2}}_{-\frac{1}{2}\pm\frac{\sqrt{n(n-2)+1-4\beta}}{2}}(\lambda) \quad \text{and} \quad (\lambda^2-1)^{\frac{2-n}{4}} Q^{\pm\frac{n-2}{2}}_{-\frac{1}{2}\pm\frac{\sqrt{n(n-2)+1-4\beta}}{2}}(\lambda)$$

where we can choose any $\pm$ *combination for any indices (4 possible combinations).*

Proof. To solve the reduced equation, we need to find the right parameters in the Legendre equation (3.5), that is

$$\nu(\nu+1) = -\frac{n(2-n)+4\beta}{4} \iff \nu = -\frac{1}{2} \pm \frac{\sqrt{n(n-2)+1-4\beta}}{2}$$

and

$$\mu^2 = \frac{1}{4}(n-2)^2 \iff \mu = \pm\frac{n-2}{2}.$$

Equation (3.4) admits two linearly independent solutions

$$(\lambda^2-1)^{\frac{2-n}{4}} P^{\pm\frac{n-2}{2}}_{-\frac{1}{2}\pm\frac{\sqrt{n(n-2)+1-4\beta}}{2}}(\lambda) \text{ and } (\lambda^2-1)^{\frac{2-n}{4}} Q^{\pm\frac{n-2}{2}}_{-\frac{1}{2}\pm\frac{\sqrt{n(n-2)+1-4\beta}}{2}}(\lambda).$$

Then, using the formulas 8.2.1 and 8.2.2 from [1], both functions $P^{\mu}_{-\kappa-\frac{1}{2}}$ and $Q^{\mu}_{-\kappa-\frac{1}{2}}$ can be represented by the functions $P^{\mu}_{\kappa-\frac{1}{2}}$ and $Q^{\mu}_{\kappa-\frac{1}{2}}$. Similarly, using the formulas 8.2.5 and 8.2.6, we can represent $P^{-\mu}_{\nu}$ and $Q^{-\mu}_{\nu}$ by the functions P^{μ}_{ν} and Q^{μ}_{ν}. Hence, the any $\pm$ combination gives us two linear independent solutions. □

Corollary 3.6. *Using* (2.2), *we obtain the solutions*

$$x_n^{-\frac{k}{2}}(\lambda^2-1)^{\frac{2-n}{4}}P^{\pm\frac{n-2}{2}}_{-\frac{1}{2}\pm\frac{\sqrt{n(n-2)+1-4\beta}}{2}}(\lambda) \quad \textit{and} \quad x_n^{-\frac{k}{2}}(\lambda^2-1)^{\frac{2-n}{4}}Q^{\pm\frac{n-2}{2}}_{-\frac{1}{2}\pm\frac{\sqrt{n(n-2)+1-4\beta}}{2}}(\lambda)$$

for the Weinstein equation (1.1).

Remark 3.7. *We observe that if* $n(n-2)+1-4\beta<0$, *we obtain solutions with the functions*

$$P^{\mu}_{-\frac{1}{2}+i\theta}(\lambda) \quad \textit{and} \quad Q^{\mu}_{-\frac{1}{2}+i\theta}(\lambda),$$

with some $\theta\in\mathbb{R}$. *These functions are called the Mehler functions or conical functions, see e.g. Section 8.12. in [1] or Section 8.84 in [10]. The first of the functions is real-valued, while the second is complex-valued in general. To find completely real-valued solutions, see e.g. [6].*

Remark 3.8. *The special case* $n(n-2)+1-4\beta=0$, *i.e.* $k(2-k)+4\ell=1$, *corresponds to the equation*

$$\widetilde{L}u=\Delta u+\frac{1}{4}\frac{u}{x_n^2}=0 \quad \textit{or} \quad Lu=\Delta u+\frac{k}{x_n}\frac{\partial u}{\partial x_n}+\frac{\frac{1}{4}(k-1)^2}{x_n^2}u=0,$$

and the solutions are given by

$$P^{\mu}_{-\frac{1}{2}}(\lambda) \quad \textit{and} \quad Q^{\mu}_{-\frac{1}{2}}(\lambda).$$

4 Finding fundamental solutions

The solutions given in Theorem 3.5 can be used as candidates for a fundamental solution. From (3.1), we infer that $x\to y$ if and only if $\lambda\to 1+$. Next, let us examine the asymptotic behavior of functions in general. In the following, we assume that the argument z of the functions is real.

Proposition 4.1. *If* $Re(\mu)>0$, *then*

$$\lim_{z\to 1}\left((z^2-1)^{\frac{\mu}{2}}P_\nu^{-\mu}(z)\right)=0.$$

Proof. For $|1-z|<2$ the associated Legendre function P_ν^μ admits the representation (see 8.1.2 in [1])

$$P_\nu^{-\mu}(z)=\frac{1}{\Gamma(1+\mu)}\left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}}{}_2F_1\left(-\nu,\nu+1;1+\mu;\frac{1-z}{2}\right),$$

where ${}_2F_1$ represents the usual hypergometric functions (see [1, 10]). Hence

$$(z^2-1)^{\frac{\mu}{2}}P_\nu^{-\mu}(z)=\frac{1}{\Gamma(1+\mu)}(z-1)^{\mu}{}_2F_1\left(-\nu,\nu+1;1+\mu;\frac{1-z}{2}\right),$$

completing the proof. □

Proposition 4.2. *If* $Re(\mu) > 0$ *and* $\nu + \frac{1}{2} \notin -\mathbb{N}$, *then*

$$\lim_{z \to 1+} \left((z^2-1)^{\frac{\mu}{2}} Q_\nu^\mu(z) \right) = e^{i\pi\mu} 2^{\mu-1} \Gamma(\mu).$$

Proof. Using the representation 8.703 in [10], we obtain the representation

$$(z^2-1)^{-\frac{\mu}{2}} Q_\nu^\mu(z) = e^{i\pi\mu} \frac{\Gamma(\nu+\mu+1)\sqrt{\pi}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2}) z^{\nu+\mu+1}} {}_2F_1\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2} \right).$$

Using the transformation formula 9.131.1 in [10], we have

$${}_2F_1\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2} \right) = \frac{(z^2-1)^{-\mu}}{z^{-2\mu}} {}_2F_1\left(\frac{\nu-\mu+1}{2}, \frac{\nu-\mu+2}{2}; \nu+\frac{3}{2}, \frac{1}{z^2} \right).$$

Hence

$$(z^2-1)^{\frac{\mu}{2}} Q_\nu^\mu(z) = e^{i\pi\mu} \frac{\Gamma(\nu+\mu+1)\sqrt{\pi}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2}) z^{\nu-\mu+1}} {}_2F_1\left(\frac{\nu-\mu+2}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2} \right).$$

If $\mathrm{Re}(c-a-b) > 0$ and $c \notin -\mathbb{N}_0$, the identity 15.1.20 in [1] says

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

If $a = \frac{\nu-\mu+1}{2}$, $b = \frac{\nu-\mu+2}{2}$ and $c = \nu+\frac{3}{2}$, we have $\mathrm{Re}(c-a-b) = \mathrm{Re}(\mu) > 0$ and $\nu+\frac{3}{2} \neq 0, -1, -2, \ldots$, *i.e.*

$${}_2F_1\left(\frac{\nu-\mu+1}{2}, \frac{\nu-\mu+2}{2}; \nu+\frac{3}{2}; 1 \right) = \frac{\Gamma(\nu+\frac{3}{2})\Gamma(\mu)}{\Gamma(\frac{\nu+\mu+2}{2})\Gamma(\frac{\nu+\mu+1}{2})}.$$

Using the doubling formula for the gamma function 8.335.1 in [10], we obtain

$${}_2F_1\left(\frac{\nu-\mu+1}{2}, \frac{\nu-\mu+2}{2}; \nu+\frac{3}{2}; 1 \right) = \frac{2^{\nu+\mu}\Gamma(\nu+\frac{3}{2})\Gamma(\mu)}{\sqrt{\pi}\Gamma(\nu+\mu+1)}.$$

Thus

$$\lim_{z \to 1+} \left((z^2-1)^{\frac{\mu}{2}} Q_\nu^\mu(z) \right) = e^{i\pi\mu} \frac{\Gamma(\nu+\mu+1)\sqrt{\pi}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} \frac{2^{\nu+\mu}\Gamma(\nu+\frac{3}{2})\Gamma(\mu)}{\sqrt{\pi}\Gamma(\nu+\mu+1)} = e^{i\pi\mu} 2^{\mu-1}\Gamma(\mu). \qquad \square$$

We know that the homogenity of the Dirac delta distribution is $-n$ and the reduced operator is a homogeneous differential operator of degree 2. Hence, the fundamental solutions should have the homogenity $-n+2$. From (3.1), we obtain, that $(\lambda-1)^{\frac{n-2}{2}}$ has the needed homogenity. Hence the solution $(\lambda^2-1)^{-\frac{\mu}{2}} Q_\nu^\mu(\lambda)$ has the suitable homogenity. We see it by writing it in the form

$$(\lambda^2-1)^{-\frac{\mu}{2}} Q_\nu^\mu(\lambda) = \frac{(\lambda^2-1)^{\frac{\mu}{2}} Q_\nu^\mu(\lambda)}{(\lambda+1)^\mu(\lambda-1)^\mu}$$

for $\mu = \frac{n-2}{2}$. Let us define the following function with the canonical asymptotic behavior, which we can use as a candidate for a fundamental solution. This means that we fixing the constant, and proving directly that it indeed satisfies the correct equation.

Proposition 4.3. *Let* $\mu = \frac{n-2}{2}$ *and* $\mu = -\frac{1}{2} \pm \frac{\sqrt{n(n-2)+1-4\beta}}{2}$. *The function*

$$F(x,y) = \frac{f(\lambda)}{(\lambda-1)^{\mu}},$$

where

$$f(\lambda) = \frac{2e^{-\pi\mu i}}{\Gamma(\nu)} \frac{(\lambda^2-1)^{\frac{\mu}{2}} Q_{\nu}^{\mu}(\lambda)}{(\lambda+1)^{\mu}}$$

is a null solution of the reduced operator for $x \neq y$ *and*

$$\lim_{\lambda\to 1} f(\lambda) = 1.$$

Proof. Using the preceding proposition, we obtain

$$\lim_{\lambda\to 1} \left(\frac{(\lambda^2-1)^{\frac{\mu}{2}} Q_{\nu}^{\mu}(\lambda)}{(\lambda+1)^{\mu}} \right) = \frac{e^{\pi\mu i}}{2}\Gamma(\mu). \qquad \square$$

Next we need to evaluate $\widetilde{L}F(\cdot,y)$ in the distribution sense. We proceed as follows. We take a test function $\varphi \in \mathcal{D}(\mathbb{R}^n_+)$ and choose a bounded open set $U \subset \mathbb{R}^n_+$ with a sufficiently smooth boundary satisfying $\mathrm{supp}(\varphi) \subset U$, and we define $U_r(y) := U \backslash B_r(y)$ for $0 < r < R$, where $R = \inf\{|x-y| : x \in \partial U\}$. If $\chi_{U_r(y)}$ is the characteristic function of $U_r(y)$, then we define the sequence of locally integrable functions $\{F_r\}$ by $F_r := \chi_{U_r(y)} F(\cdot,y)$. Obviously, the sequence converges to the $F(\cdot,y)$ in the distributional sense (2.4). Then, using this convergence and (2.5), we obtain

$$\langle \widetilde{L}F(\cdot,y), \varphi\rangle = \langle F(\cdot,y), \widetilde{L}\varphi\rangle = \lim_{r\to 0}\langle F_r, \widetilde{L}\varphi\rangle. \tag{4.1}$$

Since F_r is locally integrable, we have using (2.6) and the Green formula (2.3),

$$\langle F_r, \widetilde{L}\varphi\rangle = \int_{U_r(y)} F(x,y)\widetilde{L}\varphi(x)\,dx = \int_{U_r(y)} \widetilde{L}F(x,y)\varphi(x)\,dx + \int_{\partial U_r(y)} \left(F\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial F}{\partial n}\right) dS.$$

We observe that $\widetilde{L}F(x,y) = 0$ for $x \neq y$ and split $\partial U_r(y) = \partial U \cup (-\partial B_r(y))$, where the minus sign denotes the opposite (*i.e.* inward) orientation. Since $\mathrm{supp}(\frac{\partial\varphi}{\partial n}) \subset \mathrm{supp}(\varphi)$, we observe that the surface integral over ∂U vanishes. Hence

$$\langle F_r, \widetilde{L}\varphi\rangle = \int_{\partial B_r(y)} \left(\varphi\frac{\partial F}{\partial n} - F\frac{\partial \varphi}{\partial n}\right) dS. \tag{4.2}$$

To compute the surface integral in (4.2), we need the following technical lemma.

Lemma 4.4. *If $x \in \partial B_r(y)$ with $y = (0', y_n)$, then the normal derivative of $\lambda(x,y)$ satisfies*

$$\frac{\partial \lambda}{\partial n} = r\frac{x_n + y_n}{2x_n^2 y_n}.$$

Proof. We compute

$$\frac{\partial \lambda}{\partial x_j} = \frac{x_j}{x_n y_n}, \quad \text{for} \quad j = 1, \ldots, n-1,$$
$$\frac{\partial \lambda}{\partial x_n} = \frac{-|x'|^2 + x_n^2 - y_n^2}{2x_n^2 y_n},$$

i.e.

$$\nabla \lambda = \frac{\left(x', \frac{-|x'|^2 + x_n^2 - y_n^2}{2x_n}\right)}{x_n y_n}.$$

At $x \in \partial B_r(y)$, the outward pointing unit normal is

$$n(x) = \frac{(x', x_n - y_n)}{r}.$$

Since $|x'|^2 = r^2 - (x_n - y_n)^2$, we compute

$$\frac{\partial \lambda}{\partial n} = n \cdot \nabla \lambda = r\frac{x_n + y_n}{2x_n^2 y_n}.$$ □

We also need the following asymptotics.

Remark 4.5 (Integrals over spheres). *If $f : U \to \mathbb{R}$ is a continuous function, $y \in U$ and $R > 0$ a radius such that $B_r(y) \subset U$ for all $0 < r < R$. Then there is the classical asymptotic formula of the surface integrals, depending on the singularity of the integrand. A direct consequence of the continuity of the function f is*

$$\lim_{r\to 0} \int_{\partial B_r(y)} \frac{f(x)}{r^\alpha} dS(x) = \begin{cases} 0, & \textit{for} \quad 0 < \alpha < n-1, \\ \omega_{n-1} f(y), & \textit{for} \quad \alpha = n-1, \\ \pm\infty, & \textit{for} \quad \alpha > n-1, \end{cases} \tag{4.3}$$

where ω_{n-1} is the surface area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. These are a special case of the so-called potential type integrals, see e.g. [19].

Then we are ready to prove:

Theorem 4.6. *If $y=(0',y_n)$, then*

$$\widetilde{L}F(\cdot,y)=-\frac{n-2}{2}y_n^{n-2}\omega_{n-1}\delta(x')\delta(x_n-y_n),$$

where ω_{n-1} is the surface area of the unit sphere $S^{n-1}\subset\mathbb{R}^n$ and $n\geq 3$.

Proof. Since $\lambda-1=\frac{r^2}{2x_ny_n}$, we have

$$F(x,y)=\frac{(2x_ny_n)^\mu f(\lambda)}{r^{n-2}},$$

where $\mu=\frac{n-2}{2}$. Hence using (4.3), we obtain

$$\lim_{r\to 0}\int_{\partial B_r(y)}F\frac{\partial\varphi}{\partial n}dS=\lim_{r\to 0}\int_{\partial B_r(y)}\frac{(2x_ny_n)^\mu f(\lambda)}{r^{n-2}}\frac{\partial\varphi}{\partial n}dS=0.$$

Then we compute using Lemma 4.4

$$\begin{aligned}\frac{\partial F}{\partial n}&=\frac{d}{d\lambda}\left(\frac{f(\lambda)}{(\lambda-1)^\mu}\right)\frac{\partial\lambda}{\partial n}\\&=\left(\frac{f'(\lambda)}{(\lambda-1)^\mu}-\frac{\mu f(\lambda)}{(\lambda-1)^{\mu+1}}\right)\frac{\partial\lambda}{\partial n}\\&=\frac{1}{2}\left(\frac{(2x_ny_n)^\mu f'(\lambda)}{r^{n-3}}-\frac{\mu(2x_ny_n)^{\frac{n}{2}}f(\lambda)}{r^{n-1}}\right)\frac{x_n+y_n}{x_n^2y_n}.\end{aligned}$$

Hence, we can compute

$$\begin{aligned}\lim_{r\to 0}\int_{\partial B_r(y)}\varphi\frac{\partial F}{\partial n}dS=&\frac{1}{2}\underbrace{\lim_{r\to 0}\int_{\partial B_r(y)}\frac{(2x_ny_n)^\mu f'(\lambda)}{r^{n-3}}\frac{x_n+y_n}{x_n^2y_n}\varphi dS}_{=0,\text{ using (4.3)}}\\&-\frac{1}{2}\lim_{r\to 0}\int_{\partial B_r(y)}\frac{\mu(2x_ny_n)^{\frac{n}{2}}f(\lambda)}{r^{n-1}}\frac{x_n+y_n}{x_n^2y_n}\varphi dS\\=&-\mu y_n^{n-2}\omega_{n-1}\varphi(y)\end{aligned}$$

again using (4.3). Hence, using (4.1) and (4.2), we conclude

$$\langle\widetilde{L}F(\cdot,y),\varphi\rangle=-\mu y_n^{n-2}\omega_{n-1}\varphi(y).$$

Using the definition of the Dirac delta distribution, we obtain the result. □

Since $\widetilde{L}$ is invariant under (2.1), we obtain a fundamental solution by the simple substitution.

Theorem 4.7. *Let $n \geq 3$. The distribution*

$$H(x, y) = \frac{h(\lambda(x, y))}{(\lambda(x, y) - 1)^{\frac{n-2}{2}}}$$

where

$$h(\lambda) = -\frac{4e^{-\pi\mu i}}{(n-2)y_n^{n-2}\omega_{n-1}\Gamma(\nu)} \frac{(\lambda^2 - 1)^{\frac{\mu}{2}} Q_\nu^\mu(\lambda)}{(\lambda + 1)^\mu}$$

is a fundamental solution for $\widetilde{L}$, i.e.

$$\widetilde{L}H(\cdot, y) = \delta(x - y),$$

for any $y \in \mathbb{R}_+^n$.

Proposition 3.1 gives the following theorem.

Theorem 4.8. *Let $n \geq 3$. The distribution*

$$G(x, y) = \frac{g(\lambda(x, y))}{(\lambda(x, y) - 1)^{\frac{n-2}{2}}}$$

where

$$g(\lambda) = -\frac{1}{x_n^{\frac{k}{2}} y_n^{n-2-\frac{k}{2}}} \frac{4e^{-\pi\mu i}}{(n-2)\omega_{n-1}\Gamma(\nu)} \frac{(\lambda^2 - 1)^{\frac{\mu}{2}} Q_\nu^\mu(\lambda)}{(\lambda + 1)^\mu}$$

is a fundamental solution for L, i.e.

$$LG(\cdot, y) = \delta(x - y),$$

for any $y \in \mathbb{R}_+^n$.

Above, the special case $n = 2$ is not considered and is left as a future research topic. The question is a natural deformation for the hyperbolic Laplace operator on the complex upper half-plane.

5 Conclusions

In this paper, we derive the fundamental solution for the operator L in detail. The reader can see that to find the fundamental solution for an operator with a non-constant coefficient is much more challenging than in the case of constant coefficients. The reader should also bear in mind how the only constant multiplication special case $k = \ell = 0$ makes calculations significantly easier. By doing the calculations presented in the paper in this case, we recover a classical derivation, based on differential equations, for the fundamental solution of the Laplace operator.

Finally, the authors would like to point out that the results of the paper may be interesting in addition to analysis in other areas of mathematics, such as analytical number theory, because the extended Weinstein equation also encompasses the famous Maaß wave equation, including the famous Maaß forms as special solutions, see *e.g.* [7, 18].

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L_p-boundedness of the Laplace transform

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ABSTRACT

In this paper, we discuss about the boundedness of the Laplace transform $\mathcal{L} : L_p([0,\infty)) \to L_p(A)$ ($p \geq 1$) for the cases $A = [0,\infty)$, $A = [1,\infty)$ and $A = [0,1]$. We also provide examples for the cases where $\mathcal{L}$ is unbounded.

RESUMEN

En este artículo, discutimos sobre la acotación de la transformada de Laplace $\mathcal{L} : L_p([0,\infty)) \to L_p(A)$ ($p \geq 1$) para los casos $A = [0,\infty)$, $A = [1,\infty)$ y $A = [0,1]$. También entregamos ejemplos para los casos donde $\mathcal{L}$ es no acotada.

Keywords and Phrases: Laplace transform, Integral transform, L_p-strong boundedness.

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1 Introduction

The Laplace transform $\mathcal{L}$ is a well-known classical linear integral operator defined for every appropriate function f on $[0, \infty)$ by

$$\mathcal{L}f(t) = \int_0^\infty e^{-st} f(s)\, ds, \quad t \in (0, \infty).$$

Laplace transform is widely used for solving ordinary and partial differential equations. Hence it is a useful tool not only for mathematicians but also for physicists and engineers. It is also useful in Probability Theory (see [1], [8] and [10]).

Searching among the literature, we found that the study of the boundedness of the Laplace transform for some unknown reason has been neglected. In this regard, we could only find the references [3] and [6, 7], in which the authors stated some results about the boundedness of the Laplace transform. In [3], the optimal rearrangement-invariant space on either side of $\mathcal{L} : X \to Y$ is characterized when the other space is given. In [6], the authors studied both the Laplace transform and a more general class of operators (also in weighted L_p spaces), and in [7], they provided for them a spectral representation in L_2. For more on the Laplace transform and its optimal domain of definition, the interested reader is invited to check [2, 9, 11] and the references therein.

In such a sense, in a self contained presentation, we study the boundedness of the Laplace transform on Lebesgue L_p-spaces. Our main goal is to show that:

(1) $\mathcal{L} : L_p([0, \infty)) \to L_p([0, \infty))$ is bounded only if $p = 2$.

(2) $\mathcal{L} : L_p([0, \infty)) \to L_p([1, \infty))$ is bounded only if $p > 2$.

(3) $\mathcal{L} : L_p([0, \infty)) \to L_p([0, 1])$ is bounded only if $1 < p < 2$.

2 Main results

We would like to discuss now about the boundedness of the Laplace transform $\mathcal{L}$. For example, for $f \in L_1([0, \infty))$, it holds that

$$|\mathcal{L}f(t)| \leq \int_0^\infty |f(s)||e^{-st}|\, ds \leq \int_0^\infty |f(s)|\, ds = \|f\|_{L_1([0,\infty))} < \infty.$$

This means that $\mathcal{L}(f)$ exists and it is bounded for all $t \geq 0$. By taking the supremum over $t \in [0, \infty)$, we obtain

$$\|\mathcal{L}(f)\|_{L_\infty([0,\infty))} \leq \|f\|_{L_1([0,\infty))},$$

which means that

$$\mathcal{L} : L_1([0,\infty)) \longrightarrow L_\infty([0,\infty)),$$

is a bounded operator.

For our next result we will use the so called *Minkowski integral inequality*, stated below. Details and proof of this inequality may be found in [4].

Theorem 2.1 (Minkowski integral inequality). *Let $(X,\mathscr{A},\mu)$ and $(Y,\mathscr{B},\nu)$ be σ-finite measure spaces. Suppose that f is $\mathscr{A}\times\mathscr{B}$-measurable function and $f(\cdot,y)\in L_p(\mu)$ for all $y\in Y$. Then for $1\le p\le\infty$ we have*

$$\left(\int_X \left|\int_Y f(x,y)\,d\nu\right|^p d\mu\right)^{\frac{1}{p}} \le \int_Y \left(\int_X |f(x,y)|^p\,d\mu\right)^{\frac{1}{p}} d\nu. \tag{2.1}$$

The next result is an exercise in the 1958 book of Dunford and Schwartz [5]. It states that

$$\mathcal{L} : L_2([0,\infty)) \to L_2([0,\infty)),$$

is a bounded operator. For the sake of completeness, we provide its proof.

Theorem 2.2. *Let $f\in L_2([0,\infty))$. Then*

$$\|\mathcal{L}f\|_{L_2([0,\infty))} \le \sqrt{\pi}\|f\|_{L_2([0,\infty))}.$$

Proof. Let $f\in L_2([0,\infty))$ and

$$\mathcal{L}f(t) = \int_0^\infty f(s)e^{-st}\,ds. \tag{2.2}$$

Now, making the change of variables $u=st$, (2.2) becomes

$$\mathcal{L}f(t) = \int_0^\infty e^{-u} f\left(\frac{u}{t}\right)\frac{dt}{t}.$$

By means of the Minkowski integral inequality (Theorem 2.1), one has

$$\begin{aligned}
\|\mathcal{L}\|_{L_2([0,\infty))} &= \left(\int_0^\infty |\mathcal{L}f(t)|^2\,dt\right)^{\frac{1}{2}} = \left(\int_0^\infty \left|\int_0^\infty e^{-u} f\left(\frac{u}{t}\right)\frac{du}{t}\right|^2 dt\right)^{\frac{1}{2}} \\
&\le \int_0^\infty \left(\int_0^\infty \left|e^{-u} f\left(\frac{u}{t}\right)t^{-1}\right|^2 dt\right)^{\frac{1}{2}} du = \int_0^\infty e^{-u}\left(\int_0^\infty \left|f\left(\frac{u}{t}\right)\right|^2 \frac{dt}{t^2}\right)^{\frac{1}{2}} du \\
&= \int_0^\infty u^{-\frac{1}{2}}e^{-u}\left(\int_0^\infty |f(\omega)|^2\,d\omega\right)^{\frac{1}{2}} du = \left(\int_0^\infty u^{\frac{1}{2}-1}e^{-u}\,du\right)\|f\|_{L_2([0,\infty))} \\
&= \Gamma\left(\frac{1}{2}\right)\|f\|_{L_2([0,\infty))}.
\end{aligned}$$

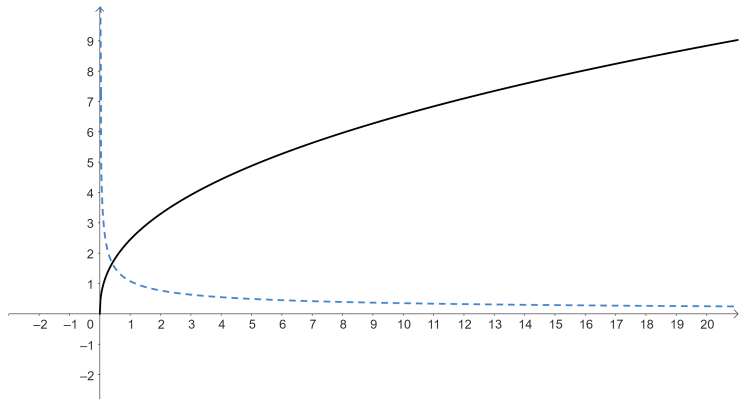

Figure 1: The graph of $F(a) = \frac{\|\mathcal{L}(f_a)\|_{L_p([0,\infty))}}{\|f_a\|_{L_p([0,\infty))}}$ for $p = 1.5$ (solid) and $p = 5$ (dashed).

It is a well known fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, so we finally arrive to

$$\|\mathcal{L}f\|_{L_2([0,\infty))} \leq \sqrt{\pi}\|f\|_{L_2([0,\infty))}. \qquad \square$$

Remark 2.3. *A routine calculation shows that, for $p > 1$, if $f_a(t) = e^{-at}$ where $a > 0$, we have*

$$\|f_a\|_{L_p([0,\infty))} = \left(\frac{1}{ap}\right)^{1/p}, \quad \|\mathcal{L}(f_a)\|_{L_p([0,\infty))} = \left(\frac{a^{1-p}}{p-1}\right)^{1/p}.$$

Hence

$$\frac{\|\mathcal{L}(f_a)\|_{L_p([0,\infty))}}{\|f_a\|_{L_p([0,\infty))}} = \left(\frac{p}{p-1}\right)^{1/p} a^{\frac{2}{p}-1} \to \infty,$$

as $a \to \infty$ for $1 < p < 2$, and as $a \to 0^+$ for $p > 2$ (see e.g. Figure 1 below). This shows that

$$\mathcal{L} : L_p([0,\infty)) \to L_p([0,\infty))$$

is not a bounded operator for $p \neq 2$.[1]

Our next result states that

$$\mathcal{L} : L_p([0,\infty)) \to L_p([1,\infty)),$$

is a bounded operator for $p > 2$.

[1]All plots in the present article were made using the software DESMOS.

Theorem 2.4. *Let $f \in L_p([0,\infty))$ with $2 < p < \infty$, then*

$$\|\mathcal{L}(f)\|_{L_p([1,\infty))} \le C_p \|f\|_{L_p([0,\infty))}.$$

Proof. Let $f \in L_p$ with $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder's inequality one has

$$\begin{aligned}
\|\mathcal{L}(f)\|^p_{L_p([1,\infty))} &= \int_1^\infty |\mathcal{L}f(x)|^p \, dx = \int_1^\infty \left(\int_0^\infty e^{-xy} f(y)\, dy \right)^p dx \\
&\le \int_1^\infty \left(\int_0^\infty |f(y)|^p \, dy \right) \left(\int_0^\infty e^{-qxy} \, dy \right)^{p/q} dy = \int_1^\infty \left(- \frac{e^{-qxy}}{qx} \Big|_0^\infty \right)^{p/q} \|f\|^p_{L_p([0,\infty))} \, dx \\
&= \left(\frac{1}{q} \right)^{p/q} \left(\int_1^\infty x^{-p/q} \, dx \right) \|f\|^p_{L_p([0,\infty))} = \left(\frac{1}{q} \right)^{p/q} \frac{1}{(2-p)x^{p-2}} \Big|_1^\infty \|f\|^p_{L_p([0,\infty))} \\
&= \left(\frac{1}{q} \right)^{p/q} \frac{1}{p-2} \|f\|^p_{L_p([0,\infty))}.
\end{aligned}$$

Finally,

$$\|\mathcal{L}(f)\|_{L_p(1,\infty)} \le \left(\frac{1}{q} \right)^{1/q} \left(\frac{1}{p-2} \right)^{1/p} \|f\|_{L_p(0,\infty)},$$

hence

$$\|\mathcal{L}(f)\|_{L_p(1,\infty)} \le \left(\frac{p-1}{p} \right)^{\frac{p-1}{p}} \left(\frac{1}{p-2} \right)^{1/p} \|f\|_{L_p(0,\infty)}.$$

□

Remark 2.5. *Theorem 2.4 does not hold for $1 < p < 2$. Let us check this. As in the previous remark, for $f_a(t) = e^{-at}$ with $a > 0$, we have $\|f_a\|_{L_p([0,\infty))} = \left(\frac{1}{ap} \right)^{1/p}$, and also*

$$\|\mathcal{L}(f_a)\|_{L_p(1,\infty)} = \left(\frac{1}{p-1} \right)^{1/p} \left((1+a)^{1-p} \right)^{1/p} = \left(\frac{1}{p-1} \right)^{1/p} (1+a)^{1/p-1}.$$

Hence

$$\frac{\|\mathcal{L}(f_a)\|_{L_p(1,\infty)}}{\|f_a\|_{L_p(0,\infty)}} = \frac{\left(\frac{1}{p-1} \right)^{1/p} (1+a)^{1/p-1}}{\left(\frac{1}{ap} \right)^{1/p}} = \left(\frac{p}{p-1} \right)^{1/p} \cdot \frac{(a+a^2)^{1/p}}{1+a} \to \infty$$

as $a \to \infty$ and $1 < p < 2$ (see, for example, Figure 2 below). So,

$$\mathcal{L} : L_p([0,\infty)) \to L_p([1,\infty)),$$

is not a bounded operator for $1 < p < 2$.

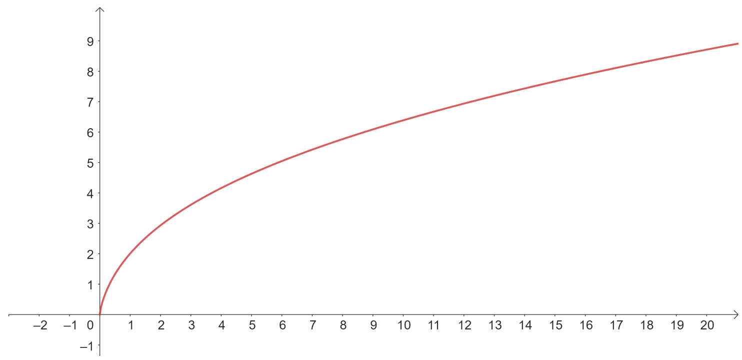

Figure 2: The graph of $G(a) = \frac{\|\mathcal{L}(f_a)\|_{L_p(1,\infty)}}{\|f_a\|_{L_p(0,\infty)}}$ for $p = 1.4$.

In our last result, we will show that

$$\mathcal{L} : L_p([0,\infty)) \to L_p([0,1]),$$

is a bounded operator for $1 < p < 2$.

Theorem 2.6. *Let $f \in L_p([0,\infty))$ with $1 < p < 2$. Then*

$$\|\mathcal{L}(f)\|_{L_p([0,1])} \leq C_p \|f\|_{L_p([0,\infty))}.$$

Proof. Let q denote the conjugate exponent of p, *i.e.* $1/p + 1/q = 1$. Assuming $1 < p < 2$, then $q > 2$ and also $1 - p/q > 0$. Now,

$$\begin{aligned}
\|\mathcal{L}f\|^p_{L_p([0,1])} &= \int_0^1 |\mathcal{L}f(t)|^p \, dt = \int_0^1 \left(\int_0^\infty e^{-st} f(s) \, ds \right)^p dt \\
&\leq \int_0^1 \left(\int_0^\infty |f(s)|^p \, ds \right) \left(\int_0^\infty e^{-sqt} \, ds \right)^{p/q} dt = \int_0^1 \left(- \left. \frac{e^{-sqt}}{qt} \right|_0^\infty \right)^{p/q} dt \cdot \|f\|^p_{L_p([0,\infty))} \\
&= \int_0^1 \left(\frac{1}{qt} \right)^{p/q} dt \cdot \|f\|^p_{L_p([0,\infty))} = \left(\frac{1}{q} \right)^{p/q} \int_0^1 t^{-p/q} \, dt \cdot \|f\|^p_{L_p([0,\infty))} \\
&= \left(\frac{1}{q} \right)^{p/q} \frac{1}{1 - p/q} \cdot \|f\|^p_{L_p([0,\infty))} = \left(\frac{p-1}{p} \right)^{p-1} \frac{1}{2-p} \cdot \|f\|^p_{L_p([0,\infty))},
\end{aligned}$$

where we used Hölder's inequality in the third line. Finally, we conclude that

$$\|\mathcal{L}f\|_{L_p([0,1])} \leq C_p \|f\|_{L_p([0,\infty))},$$

where $C_p = \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \left(\frac{1}{2-p}\right)^{\frac{1}{p}}$. □

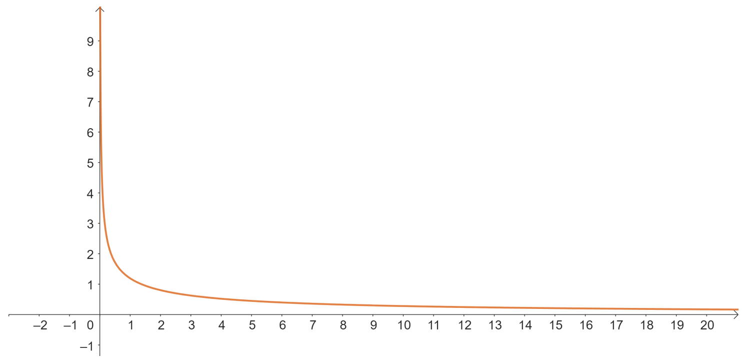

Figure 3: The graph of $H(a) = \frac{\|\mathcal{L}(f_a)\|_{L_p[0,1]}}{\|f_a\|_{L_p(0,\infty)}}$ for $p = 3.9$.

Remark 2.7. *Theorem 2.6 does not hold for $p > 2$. Again, for $f_a(t) = e^{-at}$ with $a > 0$, we have $\|f_a\|_{L_p([0,\infty))} = \left(\frac{1}{ap}\right)^{1/p}$, and also*

$$\|\mathcal{L}(f_a)\|_{L_p([0,1])} = \left(\frac{a^{1-p} - (1+a)^{1-p}}{p-1}\right)^{1/p}.$$

Hence

$$\frac{\|\mathcal{L}(f_a)\|_{L_p([0,1])}}{\|f_a\|_{L_p([0,\infty))}} = \left(\frac{p}{p-1}\right)^{1/p} \left(a^{2-p} - a(1+a)^{1-p}\right)^{1/p} \to \infty,$$

as $a \to 0^+$ and $p > 2$ (see e.g. Figure 3). So,

$$\mathcal{L} : L_p([0,\infty)) \to L_p([0,1]),$$

is not a bounded operator for $p > 2$.

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