ISSN 0719-0646
ONLINE VERSION

UNIVERSIDAD
DE LA FRONTERA

VOLUME 26 · ISSUE 3
2024

Cubo
A Mathematical Journal

Departamento de Matemática y Estadística
Facultad de Ingeniería y Ciencias
Temuco - Chile

www.cubo.ufro.cl

CUBO
A Mathematical Journal

CUBO, A Mathematical Journal, is a scientific journal founded in 1985, and published by the Department of Mathematics and Statistics of the Universidad de La Frontera, Temuco, Chile. CUBO appears in three issues per year and is indexed in the Web of Science, Scopus, MathSciNet, zbMATH Open, DOAJ, SciELO-Chile, Dialnet, REDIB, Latindex and MIAR. The journal publishes original results of research papers, preferably not more than 20 pages, which contain substantial results in all areas of pure and applied mathematics.

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CUBO
A MATHEMATICAL JOURNAL
Universidad de La Frontera
Volume 26/N$^{\underline{o}}$3 – DECEMBER 2024

SUMMARY

CUBO, A Mathematical Journal
Vol. 26, no. 3, pp. 367–386, December 2024
DOI: 10.56754/0719-0646.2603.367

Characterizations of kites as graceful graphs

Miroslav Haviar [1,2,3,✉]
Katarina Kotuľová [1]

[1] *Department of Mathematics, Faculty of Natural Sciences, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia.*
miroslav.haviar@umb.sk
kvasekova29@gmail.com

[2] *Mathematical Institute, Slovak Academy of Sciences, Ďumbierska 1, 974 11 Banská Bystrica, Slovakia.*

[3] *Department of Mathematics and Applied Mathematics, University of Johannesburg, PO Box 524, Auckland Park, 2006 South Africa.*

ABSTRACT

We introduce and study an infinite family of graceful graphs, which we call kites. The kites are graphs where a path is joined with a graph "forming" a kite. We study and characterize three classes of the kites: kites formed by cycles known to be graceful, fan kites and lantern kites. Beside showing in a transparent way that all these graphs are graceful, we provide characterizations of these graphs among all simple graphs via three tools: via Sheppard's labelling sequences introduced in the 1970s and via labelling relations and graph chessboards. The latter are relatively new tools for the study of graceful graphs introduced by Haviar and Ivaška in 2015. The labelling relations are closely related to Sheppard's labelling sequences while the graph chessboards provide a nice visualization of the graceful labellings.

RESUMEN

Introducimos y estudiamos una familia infinita de grafos agraciados que llamamos cometas. Las cometas son grafos en los cuales un camino está unido con un grafo "formando" una cometa. Estudiamos y caracterizamos tres clases de cometas: cometas formadas por ciclos conocidas por ser agraciadas, cometas abanicos y cometas linternas. Además de mostrar de manera transparente que todos estos grafos son agraciados, entregamos caracterizaciones de estos grafos entre todos los grafos simples a través de tres herramientas: a través de sucesiones de etiquetados de Sheppard introducidos en los 1970s y vía relaciones de etiquetados y tableros de ajedrez de grafos. Los últimos son herramientas relativamente nuevas en el estudio de grafos agraciados introducidos por Haviar e Ivaška en 2015. Las relaciones de etiquetados están estrechamente relacionadas con las sucesiones de etiquetados de Sheppard mientras que los tableros de ajedrez de grafos entregan una visualización agradable para los etiquetados agraciados.

Keywords and Phrases: Graph, graceful labelling, graph chessboard, labelling sequence, labelling relation.

2020 AMS Mathematics Subject Classification: 05C78.

Published: 01 October, 2024
Accepted: 15 August, 2024
Received: 14 July, 2023

1 Introduction

The Graceful Tree Conjecture stated by Rosa in the mid 1960s says that every tree can be gracefully labelled. The conjecture is one of the most attractive open problems in Graph Theory. It has led to a great interest in the study of gracefulness of simple graphs. Yet not much is known about the structure of graceful graphs after almost sixty years.

A *graceful labelling* of a graph of size m is a vertex labelling by numbers from the set $\{0, 1, \ldots, m\}$ such that no two vertices share the same label, each edge is assigned the label, which is the absolute value of the difference of the vertex labels, and the edge labels cover all values of the set $\{1, 2, \ldots, m\}$. If a graph is gracefully labelled, we say it is a *graceful graph*.

The Graceful Tree Conjecture was stated by Rosa in [7] and [8]. The best source of information on attacks of the conjecture and on the study of labellings of graphs is the electronic book *A Dynamic Survey of Graph Labeling* by Gallian [1].

In this paper we introduce and study an infinite family of graceful graphs, which we call kites. The kites $K_n(G)$ are graphs where a path P_n is joined with a graph G "forming" a kite. In our work the graph G can be a cycle C_m known to be graceful (*i.e.* $m = 0 \pmod 4$ or $m = 3 \pmod 4$), a fan graph F_m or a lantern L_m. These kites $K_n(G)$ have been studied in the second author's M.Sc. thesis [5].

Characterizations of the kites are presented using the tools of labelling sequences, labelling relations and graph chessboards. Labelling sequences were introduced in 1976 by Sheppard [9]. The labelling relations and graph chessboards as new tools for the study of graceful graphs were introduced and applied in 2015 by Haviar and Ivaška [3]. We also refer to recent papers [6] and [2], in which another classes of graceful graphs were studied by these tools.

The basic terms and facts needed in this paper are presented in Section 2. This includes the concepts of graph chessboards, labelling sequences and labelling relations. In Section 3 we describe graceful labellings of the kites formed by graceful cycles and we present their characterizations by the mentioned concepts. In Section 4 we introduce and similarly characterize other two classes of the kites: fan kites and lantern kites.

2 Preliminaries

In this section we recall necessary basic terms concerning the graph labellings as well as the concepts of labelling sequences, labelling relations and simple chessboards. These definitions are taken primarily from [8] and [3].

Throughout this paper we consider only finite *simple graphs*, that is, finite unoriented graphs without loops and multiple edges. The following concept was called *valuation* by Rosa in his seminal paper [8].

Definition 2.1 ([3,8])**.** *A vertex labelling f of a simple graph $G = (V, E)$ is a one-to-one mapping of its vertex set V into the set of non-negative integers assigning so-called vertex labels to the vertices of G.*

In this paper by a *labelling* we mean a *vertex labelling.* The number $|f(u) - f(v)|$, where $f(u), f(v)$ are the labels of the vertices u, v respectively, will be called the *induced label of the edge uv* in the labelling f.

Definition 2.2 ([3, Definition 1.2.3])**.** *Let $G = (V, E)$ be a graph of size m and let $f : V \to \mathbb{N}$ be its labelling. Then f is called a graceful labelling if*

(1) $f(V) \subseteq \{0, 1, \dots, m\}$*, and*

(2) $f(E) = \{1, 2, \dots, m\}$.

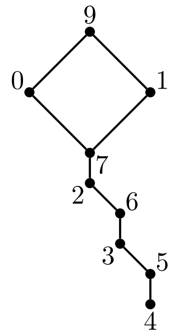
A *simple chessboard* is a square table with n rows and n columns, and dots which represent the edges of a graph are placed in the cells of the table. Every edge uv corresponds to the dots with coordinates $[u, v]$ and $[v, u]$ (the dot with coordinates $[i, j]$ means the dot in the i-th row and the j-th column of the table, where $i, j \in \{1, \dots, n\}$). Let the r-th diagonal be the set of all cells with the coordinates $[i, j]$ where $i - j = r$ and $i \geq j$. The 0-th diagonal (also called the *main* diagonal) has no dots, because we consider simple graphs, the other diagonals are called *associate.* Simple chessboard is called *graceful* if there is exactly one dot on each of its associate diagonals.

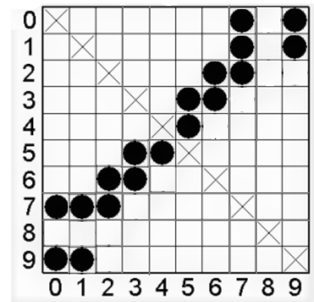
The simple chessboard is a useful visualization of a graph because via it one can easily see some properties of the graph such as its size, degrees of vertices, gracefulness, etc. In Figure 1 we see the simple chessboard of a graph of size 9 (the kite $K_6(C_4)$ formed by the cycle C_4).

To represent gracefully labelled graphs, we will use other two tools: *labelling sequences* and *labelling relations.* A concept of the labelling sequence was introduced by Sheppard in [9]. He proved that there is a unique correspondence between gracefully labelled graphs and labelling sequences. Later in [3] Haviar and Ivaška proved a correspondence between labelling sequences and labelling relations. Let us now define these concepts.

Definition 2.3 ([3,9])**.** *For a positive integer m, the sequence $(j_1, j_2, \dots, j_m)$ of integers, denoted (j_i), is a labelling sequence if $0 \leq j_i \leq m - i$ for all $i \in \{1, 2, \dots, m\}$.*

The correspondence between gracefully labelled graphs and labelling sequences is described in the following theorem.





1	2	3	4	5	6	7	8	9
4	3	3	2	2	1	0	1	0
5	5	6	6	7	7	7	9	9

Figure 1: Representations of the kite $K_6(C_4)$

Theorem 2.4 ([3, 9]). *There exists a one-to-one correspondence between graphs of size m having a graceful labelling f and between labelling sequences (j_i) of m terms. The correspondence is given by $j_i = \min\{f(u), f(v)\}$, $i \in \{1, 2, \dots, m\}$, where u, v are the end-vertices of the edge labelled i.*

Definition 2.5 ([3, Definition 3.5.1]). *Let $L = (j_1, j_2, \dots, j_m)$ be a labelling sequence. Then the relation $A(L) = \{[j_i, j_i + i] \mid i \in \{1, 2, \dots, m\}\}$ will be called the labelling relation assigned to the labelling sequence L.*

In [3] also a *labelling table* was assigned to a graceful graph of size m, which displays its labelling sequence and the labelling relation together. The labelling table consists of a header and two rows. The header just lists the numbers $1, 2, \dots, m$. The first row of the labelling table consists of the labelling sequence j_i as defined in Definition 2.3. The numbers in the second row are sums of the numbers from the header and the numbers of the first row (the members of the labelling sequence). The pairs from the first and second rows in each column are the elements of the labelling relation.

In Figure 1 we see the labelling table of the kite $K_6(C_4)$. In the first row of the table we see the labelling sequence $(4, 3, 3, 2, 2, 1, 0, 1, 0)$ of this graceful graph. The pairs from the first and second rows in each column of the labelling table form the labelling relation representing the edges of this graph. For example the pair $[4, 5]$ represents the last edge of the path.

3 Characterizations of kites formed by graceful cycles

It is well-known (see [8], [3] or [1]) that a cycle C_m is graceful if and only if $m = 0 \pmod 4$ or $m = 3 \pmod 4$. Therefore our first two studied classes of kites are those formed by graceful cycles C_m.

3.1 Kites formed by cycles C_m for $m = 0 \pmod 4$

In this subsection we present our characterization of the kites $K_n(C_m)$ formed by cycles C_m for $m = 0 \pmod 4$, where $m \geq 4$ and $n \geq 1$. The special case are the quadrangular kites.

By a *quadrangular kite* we mean a graph obtained by joining the cycle C_4 to the end-point of the path P_n with $n \geq 1$. We denote it $K_n(C_4)$. The size of this graph is $s = n + 3$, where $n - 1$ is the length of the path P_n.

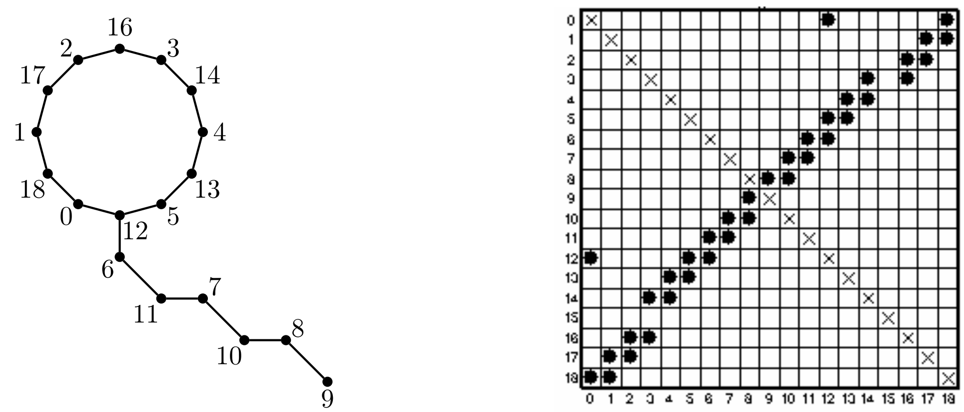
Example 3.1. *We again consider the quadrangular kite $K_6(C_4)$ presented in Figure 1. Its labelling sequence (LS, for short) $(4, 3, 3, 2, 2, 1, 0, 1, 0)$ consists of two groups: the first is $(4, 3, 3, 2, 2)$, which is the LS of the path P_6 and the second is $(1, 0, 1, 0)$, which is the LS of the cycle C_4. These two groups are clearly seen in the graph chessboard as the dots forming the "stairway" representing the path pattern and the dots forming the "square block" representing the cycle C_4.*

Definition 3.2. *Let $s = n+3$ for some $n \geq 1$. By a QK-graph chessboard (QK standing for "quadrangular kite") of size s we mean a simple chessboard such as in Figure 1 described in the previous example. Its dots start in the lower left corner with two dots in the column 0 and two dots in the column 1, which together create the "square block". The remaining dots form a "stairway" attached to the square block (the "stairway" starts with the dot with coordinates $[s - 2, 2]$).*

The characterization of the quadrangular kites via their chessboards, labelling sequences and labelling relations is a special case of Theorem 3.4, which will be presented with a full proof. It follows from it that a graph G of size $s = n + 3$ for some $n \geq 1$ is the quadrangular kite $K_n(C_4)$ if and only if G has a graceful labelling and a QK-graph chessboard of size s.

Consider now the kite $K_7(C_{12})$ and its representations in Figure 2. In the first row of the labelling table is the LS $(8, 8, 7, 7, 6, 6, 5, 5, 4, 4, 3, 0, 3, 2, 2, 1, 1, 0)$.

Definition 3.3. *Let $s = m + n - 1$ for $m = 0 \pmod 4$ and $m \geq 4$, $n \geq 1$. By a C0K-graph chessboard (C0K standing for "kite formed by cycle C_m for $m = 0 \pmod 4$") of size s we mean a simple chessboard such as in Figure 2 whose dots can be divided into three groups: the first group of dots form a "stairway" starting in the lower left corner (with the dot with coordinates $[s, 0]$), the second group consists of a single dot with coordinates $[\frac{m}{2} + (n - 1), 0]$ and the third group is again a "stairway". (It starts with the dot with coordinates $[\lfloor \frac{s-i}{2} \rfloor + i, \lfloor \frac{s-i}{2} \rfloor]$ for $i = \frac{m}{2} + (n - 2)$.)*



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
8	8	7	7	6	6	5	5	4	4	3	0	3	2	2	1	1	0
9	10	10	11	11	12	12	13	13	14	14	12	16	16	17	17	18	18

Figure 2: Representations of the kite $K_7(C_{12})$

The following theorem characterizes the kites $K_n(C_m)$ for $m = 0 \pmod 4$ via their graph chessboards, labelling sequences and labelling relations.

Theorem 3.4. *Let G be a graph of size $s = m + n - 1$ for $m = 0 \pmod 4$ and $m \geq 4$, $n \geq 1$. Then the following are equivalent:*

(1) G is the kite $K_n(C_m)$.

(2) G has a graceful labelling and a C0K-graph chessboard of size s.

(3) There exists a labelling sequence $L = (j_1, j_2, \dots, j_s)$ of G such that

$$j_i = \begin{cases} \left\lfloor \frac{s-i}{2} \right\rfloor, & \text{if } i < \frac{m}{2} + (n-1); \\ 0, & \text{if } i = \frac{m}{2} + (n-1); \\ \left\lfloor \frac{s-i+1}{2} \right\rfloor, & \text{if } i > \frac{m}{2} + (n-1). \end{cases} \tag{LSC0K}$$

(4) There exists a labelling sequence L of G with the labelling relation

$$A(L) = \left\{ \left[\left\lfloor \frac{s-i}{2} \right\rfloor, \left\lfloor \frac{s-i}{2} \right\rfloor + i \right] \middle| i < \frac{m}{2} + (n-1) \right\} \cup \left\{ \left[0, \frac{m}{2} + (n-1) \right] \right\} \cup \left\{ \left[\left\lfloor \frac{s-i+1}{2} \right\rfloor, \left\lfloor \frac{s-i+1}{2} \right\rfloor + i \right] \middle| i > \frac{m}{2} + (n-1) \right\}.$$

Proof. (1) ⇒ (2)**:** Let G be the kite $K_n(C_m)$ for $m = 0 \pmod 4$. Let us label its vertices as follows: we label the vertex joining the cycle C_m with the path P_n (let us call it the "joining vertex") by number $s - \frac{m}{2}$, and we label every second vertex from the joining vertex in the clockwise direction by numbers $s, s-1, s-2, \ldots,$ but we skip the number $\frac{3}{4}m + n - 1$. The remaining vertices of the cycle C_m will be labelled in the clockwise direction from the joining vertex by numbers $0, 1, 2, \ldots, \frac{m}{2} - 1$. Next we label the path P_n. We start from the joining vertex labelled by $s - \frac{m}{2}$ and we label every second vertex from it by numbers $s-(\frac{m}{2}+1), s-(\frac{m}{2}+2), \ldots$ The remaining vertices of the path P_n will be labelled by numbers $\frac{m}{2}, \frac{m}{2}+1, \ldots, \lfloor \frac{s}{2} \rfloor$. The labelling is illustrated in Figure 3.

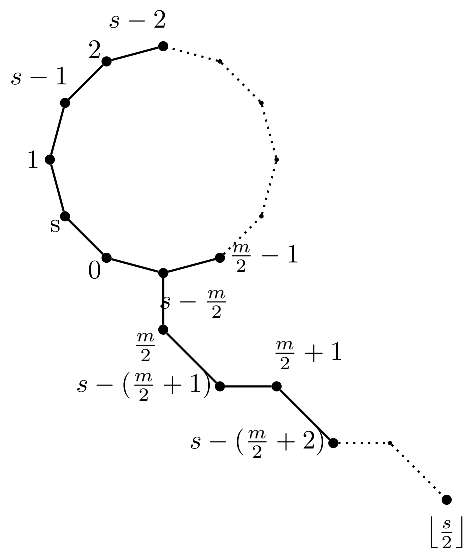


Figure 3: Vertex labelling of the kite $K_n(C_m)$ for $m = 0 \pmod 4$

Gracefulness of the described vertex labelling of G will be shown by using visualization via the corresponding chessboard of G. The dots in the $(\frac{m}{2}) \times (\frac{m}{2}+1)$ left lower rectangle of the chessboard represent the cycle C_m of the kite. (Specifically, the columns 0 to $\frac{m}{2} - 1$ and the rows s to $s - \frac{m}{2}$.) The remaining dots represent the path P_n and form a "stairway". So we obtain a C0K graph chessboard. This yields that the labelling is graceful because each diagonal of the chessboard has exactly one dot.

(2) $\Rightarrow$ (3)**:** Let G be a gracefully labelled graph with a C0K graph chessboard. We show that the corresponding labelling sequence (LS) satisfies (LSC0K). The "stairway" in the direction from the main diagonal corresponds in the LS to numbers $\lfloor \frac{s-i}{2} \rfloor$ for $i < \frac{m}{2} + (n-1)$. The "single" dot represents in the LS the number 0. The remaining "stairway" corresponds in the LS to numbers $\lfloor \frac{s-i+1}{2} \rfloor$ for $i > \frac{m}{2} + (n-1)$. So the formula (LSC0K) holds.

(3) $\Rightarrow$ (4)**:** Let (LSC0K) hold. We show that the LS L has the labelling relation $A(L)$ as described in (4). The numbers $\lfloor \frac{s-i}{2} \rfloor$ in the LS correspond in $A(L)$ to the pairs $\left[\lfloor \frac{s-i}{2} \rfloor, \lfloor \frac{s-i}{2} \rfloor + i \right]$ for $i < \frac{m}{2} + (n-1)$. The number 0 clearly corresponds to the pair $[0, \frac{m}{2} + (n-1)]$. The numbers $\lfloor \frac{s-i+1}{2} \rfloor$ in the LS correspond in $A(L)$ to the pairs $\left[\lfloor \frac{s-i+1}{2} \rfloor, \lfloor \frac{s-i+1}{2} \rfloor + i \right]$ for $i > \frac{m}{2} + (n-1)$.

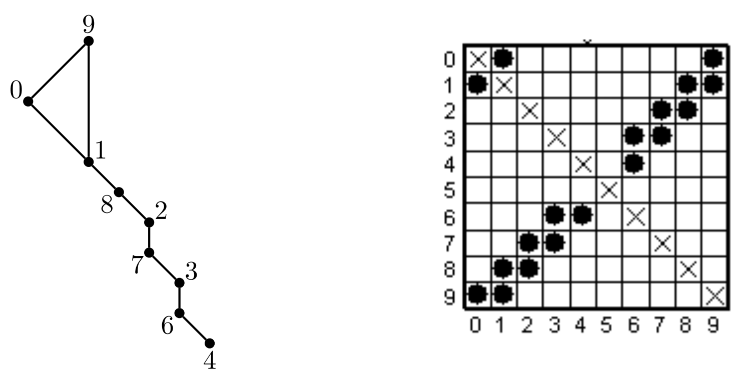
The first coordinates of these pairs are members of the LS and the second coordinates are obtained by adding numbers i to the members of the LS. So the formula $A(L)$ from (4) holds.

(4) $\Rightarrow$ (1): Let (4) hold, *i.e.* there exists a LS L of the graph G with the labelling relation $A(L)$ as in (4). We show that the pairs in $A(L)$ represent the edges of the kite $K_n(C_m)$. The pairs $\left[\left\lfloor \frac{s-i}{2} \right\rfloor, \left\lfloor \frac{s-i}{2} \right\rfloor + i\right]$ for $i < n$ clearly represent the path P_n. The pairs $\left[\left\lfloor \frac{s-i}{2} \right\rfloor, \left\lfloor \frac{s-i}{2} \right\rfloor + i\right]$ for $i = n, n+1, \ldots, \frac{m}{2}+n-2$ represent a part of the cycle C_m starting from the joining vertex and going in the anticlockwise direction. The pair $[0, \frac{m}{2} + (n-1)]$ represents the edge of the cycle C_m with the joining vertex $\frac{m}{2} + (n-1)$ and the vertex 0. The pairs $\left[\left\lfloor \frac{s-i+1}{2} \right\rfloor, \left\lfloor \frac{s-i+1}{2} \right\rfloor + i\right]$ for $i > \frac{m}{2} + (n-1)$ represent the remaining edges of the cycle C_m. Hence, G is the kite $K_n(C_m)$. □

3.2 Kites formed by cycles C_m for $m = 3 \pmod 4$

By the *triangular kite* $K_n(C_3)$ we mean a graph obtained by joining the cycle C_3 to end-point of the path P_n with $n \geq 1$. The size of the triangular kite $K_n(C_3)$ is $s = n + 2$.

In this subsection we present our characterization of the kites $K_n(C_m)$ formed by cycle C_m for $m = 3 \pmod 4$ and n sufficiently big, more precisely $n \geq \left\lfloor \frac{m}{2} \right\rfloor$. This will cover all the triangular kites $K_n(C_3)$. For general $m \geq 3$ with $m = 3 \pmod 4$ and $n \geq \left\lfloor \frac{m}{2} \right\rfloor$ we distinguish two subclasses of the kites $K_n(C_m)$ according to the order of their path P_n: n is even and n is odd. Both cases are similar, but they differ in details.



1	2	3	4	5	6	7	8	9
0	4	3	3	2	2	1	1	0
1	6	6	7	7	8	8	9	9

Figure 4: Representations of the triangular kite $K_7(C_3)$

Example 3.5. *In Figure 4 we see the triangular kite $K_7(C_3)$ obtained by joining the cycle C_3 to the path P_7. Its graceful labelling is depicted in the graph diagram. We also see the corresponding graph chessboard and the labelling relation. The labelling sequence (LS) is $(0,4,3,3,2,2,1,1,0)$.*

We can easily recognize the LS of the path P_7, which is $(4, 3, 3, 2, 2, 1)$, and the LS of the cycle C_3, which is $(0, 1, 0)$.

Definition 3.6. *Let $s = n + 2$ for some $n \geq 1$. By a TK-graph chessboard (TK standing for the "triangular kite") we mean a simple chessboard such as in Figure 4. It has the "single" dot with coordinates $[1, 0]$ and the remaining dots form a "stairway" in the chessboard starting in the lower left corner (with the dot with coordinates $[s, 0]$).*

The characterization of the triangular kites $K_n(C_3)$ by the simple chessboards, the labelling sequences and the labelling relations is a special case of the coming Theorem 3.9 (for even n) and Theorem 3.12 (for odd n), where $n \geq \lfloor \frac{m}{2} \rfloor$. It follows from these theorems that a graph G of size $s = n + 2$ for some $n \geq 1$ is the triangular kite $K_n(C_3)$ if and only if G has a graceful labelling and a TK-graph chessboard of size s.

Now we are going to describe the kites $K_n(C_m)$ formed by cycle C_m for $m = 3 \pmod 4$ and general parameter $m \geq 3$ where we assume that n is sufficiently big, more precisely $n \geq \lfloor \frac{m}{2} \rfloor$. We will start with the subcase where n is even.

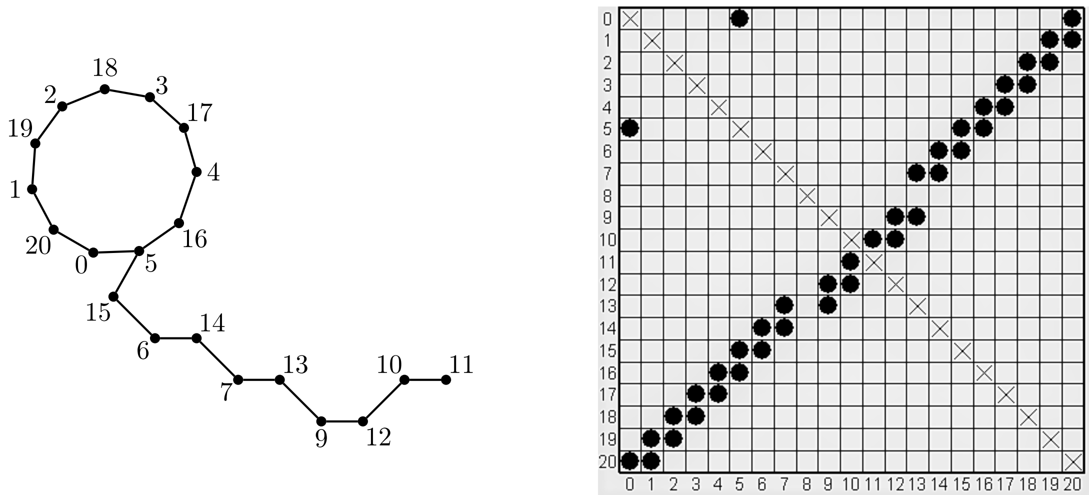
Example 3.7. *In Figure 5 we see a gracefully labelled graph diagram of the kite $K_{10}(C_{11})$ and its corresponding simple chessboard.*

The chessboard can be divided into three parts: the first part is a "stairway" (from column 0 to column 7), the second part is the "single dot" with coordinates $[5, 0]$ and the third part is a "stairway" starting with two vertical *dots.*

The labelling sequence (LS) is $(10, 10, 9, 9, 0, 7, 7, 6, 6, 5, 5, 4, 4, 3, 3, 2, 2, 1, 1, 0)$. We can divide it into four parts. The first part $(10, 10, 9, 9)$ is the LS of one part of the path, then $(7, 7, 6, 6, 5)$ represents the remaining part of the path. The third part is the number 0 in the place $\lfloor \frac{m}{2} \rfloor = 5$ and the last part $(5, 4, 4, 3, 3, 2, 2, 1, 1, 0)$ represents the cycle C_{11}.

Definition 3.8. *Let $s = m + n - 1$ for $m = 3 \pmod 4$ and $n \geq \lfloor \frac{m}{2} \rfloor$, where $m \geq 3$, $n \geq 1$ and n is even. By an even C3K-graph chessboard of size s we mean the simple chessboard such as in Figure 5, which has three parts: the first part is a "stairway" (from column 0 to column $\lceil \frac{1}{2}(s - \lfloor \frac{m}{2} \rfloor) \rceil - 1$), the second part is the "single" dot with coordinates $[\lfloor \frac{m}{2} \rfloor, 0]$ and the third part is again a "stairway" starting with two vertical dots (starting in the column $\lceil \frac{1}{2}(s - \lfloor \frac{m}{2} \rfloor) \rceil + 1$).*

Now we are ready for the characterization of the kites formed by cycle C_m for $m = 3 \pmod 4$ and $m \geq 3$ with $n \geq \lfloor \frac{m}{2} \rfloor$ in the case of even n.



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
10	10	9	9	0	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0
11	12	12	13	5	13	14	14	15	15	16	16	17	17	18	18	19	19	20	20

Figure 5: Representations of the kite $K_{10}(C_{11})$ with path P_n for even n

Theorem 3.9. *Let G be a graph of size $s = m + n - 1$ for $m = 3 \pmod 4$ and $n \geq \lfloor \frac{m}{2} \rfloor$, where $m \geq 3$, $n \geq 1$ and n is even. Then the following are equivalent:*

(1) G is the kite $K_n(C_m)$.

(2) G has a graceful labelling and an even C3K-graph chessboard of size s.

(3) There exists a labelling sequence $L = (j_1, j_2, \dots, j_s)$ of G such that

$$j_i = \begin{cases} \left\lceil \frac{s-i+1}{2} \right\rceil, & \text{if } i < \left\lfloor \frac{m}{2} \right\rfloor; \\ 0, & \text{if } i = \left\lfloor \frac{m}{2} \right\rfloor; \\ \left\lceil \frac{s-i}{2} \right\rceil, & \text{if } i > \left\lfloor \frac{m}{2} \right\rfloor. \end{cases} \qquad \text{(LSC3K-even)}$$

(4) There exists a labelling sequence L of G with the labelling relation

$$A(L) = \left\{ \left[\left\lceil \frac{s-i+1}{2} \right\rceil, \left\lceil \frac{s-i+1}{2} \right\rceil + i \right] \;\middle|\; i < \left\lfloor \frac{m}{2} \right\rfloor \right\} \cup \left\{ \left[0, \left\lfloor \frac{m}{2} \right\rfloor \right] \right\} \cup \left\{ \left[\left\lceil \frac{s-i}{2} \right\rceil, \left\lceil \frac{s-i}{2} \right\rceil + i \right] \;\middle|\; i > \left\lfloor \frac{m}{2} \right\rfloor \right\}.$$

Proof. (1) ⇒ (2)**:** Let G be the kite $K_n(C_m)$. We label it such that we start labelling the cycle C_m: we label the vertex joining the cycle with the path (the "joining vertex") by number $\lfloor \frac{m}{2} \rfloor$ and we follow in the clockwise direction by numbers $0, s, 1, s-1, \dots, s - \lfloor \frac{m}{2} \rfloor + 1$. Next we label the path P_n. We start from number $\lfloor \frac{m}{2} \rfloor$ and label every second vertex by numbers $\lfloor \frac{m}{2} \rfloor + 1, \lfloor \frac{m}{2} \rfloor + 2, \ldots$, but we skip the number $\lceil \frac{1}{2}(s - \lfloor \frac{m}{2} \rfloor) \rceil$. In the remaining part of the path we start with the vertex next to the joining vertex and we label every second vertex by numbers $s - \lfloor \frac{m}{2} \rfloor, s - (\lfloor \frac{m}{2} \rfloor + 1), \ldots$ To show that this labelling is graceful, we use the corresponding simple chessboard of G (see Figure 5). The cycle C_m of the kite is in the corresponding chessboard represented by the "stairway" in columns 0 to $\lfloor \frac{m}{2} \rfloor$ and by the "single" dot with coordinates $[\lfloor \frac{m}{2} \rfloor, 0]$. The path P_n is in the chessboard represented by part of the "stairway" starting with the upper dot in column $\lfloor \frac{m}{2} \rfloor$ and by another "stairway" starting with two vertical dots, but the column $\lceil \frac{1}{2}(s - \lfloor \frac{m}{2} \rfloor) \rceil$ is without any dots. So we obtain an even C3K-graph chessboard, which means that our labelling is graceful, because each diagonal of the simple chessboard has exactly one dot.

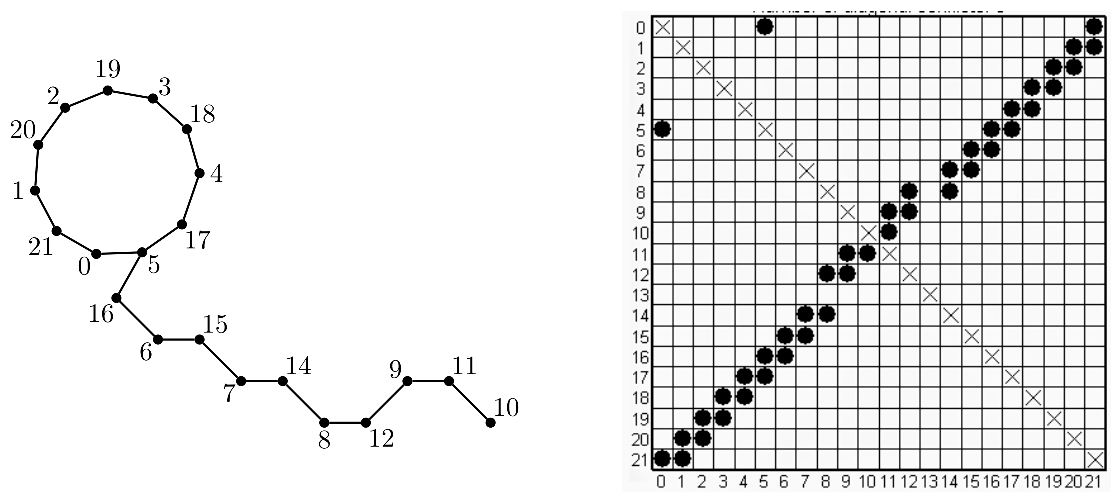
(2) $\Rightarrow$ (3)**:** Assume we have a graceful labelling of the graph G with an even C3K-graph chessboard. We show that the corresponding LS satisfies the formula (LSC3K-even). The "stairway" in the direction from the main diagonal represents in the corresponding LS numbers $\lceil \frac{s-i+1}{2} \rceil$ for $i < \lfloor \frac{m}{2} \rfloor$. The "single" dot represents the number 0. The remaining "stairway" represents numbers $\lceil \frac{s-i}{2} \rceil$ for $i > \lfloor \frac{m}{2} \rfloor$. So the formula (LSC3K-even) holds.

(3) $\Rightarrow$ (4)**:** Assume now that (3) holds, *i.e.* there exists a LS $L = (j_1, j_2, \ldots, j_s)$ that satisfies the formula (LSC3K-even). We show that this LS has the labelling relation $A(L)$ as described in (4). The numbers $\lceil \frac{s-i+1}{2} \rceil$ obviously correspond in $A(L)$ to the pairs $[\lceil \frac{s-i+1}{2} \rceil, \lceil \frac{s-i+1}{2} \rceil + i\ ]$ for $i < \lfloor \frac{m}{2} \rfloor$. The number 0 clearly corresponds to the pair $[0, \lfloor \frac{m}{2} \rfloor]$. The numbers $\lceil \frac{s-i}{2} \rceil$ correspond in $A(L)$ to the pairs $[\lceil \frac{s-i}{2} \rceil, \lceil \frac{s-i}{2} \rceil + i\ ]$ for $i > \lfloor \frac{m}{2} \rfloor$. The first coordinates of these pairs are members of the LS and the second coordinates arise by adding the number i to them.

(4) $\Rightarrow$ (1)**:** Let (4) hold, *i.e.* there exists a LS L with the labelling relation $A(L)$ as described in (4). We show that the pairs in $A(L)$ represent the edges of the kite $K_n(C_m)$. The pairs $[\lceil \frac{s-i+1}{2} \rceil, \lceil \frac{s-i+1}{2} \rceil + i\ ]$ for $i < \lfloor \frac{m}{2} \rfloor$ represent the ending part of a path. The pairs $[\lceil \frac{s-i}{2} \rceil, \lceil \frac{s-i}{2} \rceil + i\ ]$ for $i = \lfloor \frac{m}{2} \rfloor + 1, \ldots, s - m + 1$ represent the remaining edges of the path P_n. The pairs $[\lceil \frac{s-i}{2} \rceil, \lceil \frac{s-i}{2} \rceil + i\ ]$ for $i = s - m + 2, \ldots, s$ represent the edges of the cycle C_m starting from the joining vertex in the anticlockwise direction and skipping the last edge of the cycle ending in the joining vertex. This last edge of the cycle is represented by the pair $[0, \lfloor \frac{m}{2} \rfloor]$ in $A(L)$. Hence G is the kite $K_n(C_m)$. □

The description for the subcase with odd n is similar and it will follow now.

Example 3.10. *In Figure 6 we see the kite $K_{11}(C_{11})$, so it differs from Example 3.7 only by the length of the path, which is now an odd number. We see a gracefully labelled graph diagram of this kite and its corresponding simple chessboard. Again, the simple chessboard can be divided into*



1	2	3	4	5	6	7	8	9	10	11
10	9	9	8	0	8	7	7	6	6	5
11	11	12	12	5	14	14	15	15	16	16

12	13	14	15	16	17	18	19	20	21
5	4	4	3	3	2	2	1	1	0
17	17	18	18	19	19	20	20	21	21

Figure 6: Representations of the kite $K_{11}(C_{11})$

three parts: the first part is a "stairway", the second part is the "single dot" with coordinates $[5,0]$ *and the third part is again a "stairway" starting now with two* horizontal *dots.*

The labelling sequence (LS) is $(10,9,9,8,0,8,7,7,6,6,5,5,4,4,3,3,2,2,1,1,0)$. *We can divide it into four parts: the first part* $(10,9,9,8)$ *is the LS of a part of the path* P_{11}*, then* $(8,7,7,6,6,5)$ *is the LS of the remaining part of the path* P_{11}*. The number* 0 *and the last part of the LS, which is* $(5,4,4,3,3,2,2,1,1,0)$*, together represent the cycle* C_{11}*.*

We now define an odd C3K-graph chessboard.

Definition 3.11. *Let* $s=m+n-1$ *for* $m=3 \pmod 4$ *and* $n \geq \lfloor \frac{m}{2} \rfloor$*, where* $m \geq 3$*,* $n \geq 1$ *and* n *is odd. By an odd C3K-graph chessboard of size* s *we mean the simple graph chessboard such as in Figure 6, which has three parts: the first part is a "stairway" starting from the left lower corner, the second part is the "single" dot with coordinates* $\left[\lfloor \frac{m}{2} \rfloor, 0\right]$ *and the third part is a 'stairway" starting with two* horizontal *dots.*

The proof of the following characterization of the kites $K_n(C_m)$ formed by cycle C_m for $m=3$ (mod 4) in this subcase with odd n, which is sufficiently big, is analogous to the proof of Theorem 3.9 and we leave it for the reader.

Theorem 3.12. *Let G be a graph of size $s = m + n - 1$ for $m = 3 \pmod 4$ and $n \geq \lfloor \frac{m}{2} \rfloor$, where $m \geq 3$, $n \geq 1$ and n is odd. Then the following are equivalent:*

(1) G is the kite $K_n(C_m)$.

(2) G has a graceful labelling and an odd C3K-graph chessboard of size s.

(3) There exists a labelling sequence $L = (j_1, j_2, \dots, j_s)$ of G such that

$$j_i = \begin{cases} \lfloor \frac{s-i}{2} \rfloor, & \text{if } i < \lfloor \frac{m}{2} \rfloor; \\ 0, & \text{if } i = \lfloor \frac{m}{2} \rfloor; \\ \lfloor \frac{s-i+1}{2} \rfloor, & \text{if } i > \lfloor \frac{m}{2} \rfloor. \end{cases} \tag{LSC3K-odd}$$

(4) There exists a labelling sequence L of G with the labelling relation

$$A(L) = \left\{ \left[\left\lfloor \frac{s-i}{2} \right\rfloor, \left\lfloor \frac{s-i}{2} \right\rfloor + i \right] \;\middle|\; i < \left\lfloor \frac{m}{2} \right\rfloor \right\} \cup \left\{ \left[0, \left\lfloor \frac{m}{2} \right\rfloor \right] \right\} \cup$$
$$\left\{ \left[\left\lfloor \frac{s-i+1}{2} \right\rfloor, \left\lfloor \frac{s-i+1}{2} \right\rfloor + i \right] \;\middle|\; i > \left\lfloor \frac{m}{2} \right\rfloor \right\}.$$

4 Characterizations of fan kites and lantern kites

In this section we describe graceful labellings of other two classes of the kites: fan kites and lantern kites. We present their characterizations again by the graph chessboards, labelling sequences and labelling relations.

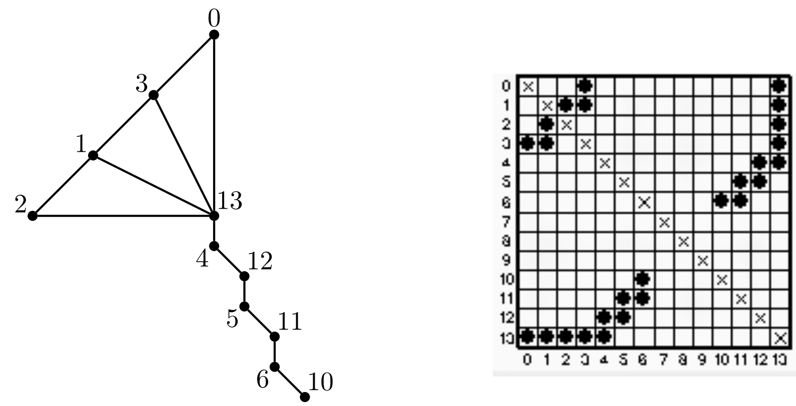
4.1 Fan kites

By a *fan kite* we mean a fan-graph kite, which is a graph obtained by joining the fan-graph F_m (see Definition 4.1) with end-point of the path P_n (see Figure 7). We will denote it by $K_n(F_m)$. The fan kite $K_n(F_m)$ is the graph of size $s = 2m + n - 2$, where $2m - 1$ is the size of the fan graph F_m and $n - 1$ is the length of the path P_n.

Definition 4.1 ([3, Section 4.4.7], [6, Section 4.1])**.** *Let $m \geq 2$. The fan graph F_m is a join of the path P_m and a single vertex K_1.*

Clearly, the fan graph F_m has order $m + 1$ and size $2m - 1$.

Example 4.2. *The fan kite $K_7(F_4)$ is obtained by joining the fan-graph F_4 with the path P_7. In Figure 7 we see its gracefully labelled graph diagram and its corresponding graph chessboard. In the chessboard we can recognize a "fan-graph pattern" in the columns from 0 to 3 and a "path pattern".*



1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0	6	6	5	5	4	4	3	2	1	0
2	3	3	10	11	11	12	12	13	13	13	13	13

Figure 7: Representations of the fan kite $K_7(F_4)$

The labelling sequence (LS) is $(1,1,0,6,6,5,5,4,4,3,2,1,0)$. *It consists of the LS* $(1,1,0,3,2,1,0)$ *of the fan-graph* F_4 *and the LS* $(6,6,5,5,4,4)$ *of the path* P_7. *The pairs from the second and third rows of the table form the labelling relation.*

Definition 4.3. *Let* $s = 2m + n - 2$ *for* $m \geq 2$, $n \geq 1$. *By an FK-graph chessboard of size* s *we mean a simple chessboard such as in Figure 7. It starts with* m *dots in the last row and continues with dots creating a "path pattern". In the upper left corner of the simple chessboard there are* $m-1$ *dots forming a "stairway".*

Now we present the result where we show gracefulness of the fan kites. We present their characterization by the graph chessboards, labelling sequences and labelling relations.

Theorem 4.4. *Let* G *be a graph of size* $s = 2m + n - 2$ *for* $m \geq 2$, $n \geq 1$. *Then the following are equivalent:*

(1) G *is the fan kite* $K_n(F_m)$.

(2) G *has a graceful labelling and an FK-graph chessboard of size* s.

(3) There exists a labelling sequence $L = (j_1, j_2, \ldots, j_s)$ *of* G *such that*

$$j_i = \begin{cases} \left\lceil \frac{m-1-i}{2} \right\rceil, & \text{if } i \leq m-1; \\ \left\lceil \frac{s-i+m-1}{2} \right\rceil, & \text{if } m \leq i < s-m+1; \\ s-i, & \text{if } i \geq s-m+1. \end{cases} \tag{LSFK}$$

(4) There exists a labelling sequence L of G with the labelling relation

$$A(L) = \left\{ \left[\left\lceil \frac{m-1-i}{2} \right\rceil, \left\lceil \frac{m-1-i}{2} \right\rceil + i \right] \;\middle|\; i \leq m-1 \right\} \cup \left\{ \left[\left\lceil \frac{s-i+m-1}{2} \right\rceil, \left\lceil \frac{s-i+m-1}{2} \right\rceil + i \right] \;\middle|\; m \leq i < s-m+1 \right\} \cup \{[\, s-i, s \,] \mid i \geq s-m+1\}.$$

Proof. (1) ⇒ (2)**:** Let G be the fan kite $K_n(F_m)$ of size s. It contains two paths: the path P_m as the path of the fan-graph F_m, and the main path P_n of the kite. We label the vertex connecting the fan-graph F_m with the path P_n (the "joining vertex") by number s. The joining vertex is adjacent to every vertex of the path P_m of the fan-graph F_m. We label P_m by numbers $0, m-1, 1, m-2, 2, \ldots \lceil \frac{m-1}{2} \rceil - 1, \lceil \frac{m-1}{2} \rceil$. Now we label the path P_n: we start from the joining vertex and we continue gradually with numbers $m, s-1, m+1, \ldots$ We show that the labelling of $K_n(F_m)$ is graceful by using its corresponding chessboard. The dots in the left upper corner represent the path P_m of the fan graph F_m, the dots in the last row represent edges connecting the joining vertex with the vertices of the path P_m. The remaining dots represent the path P_n. There is exactly one dot on each diagonal, so the labelling is graceful.

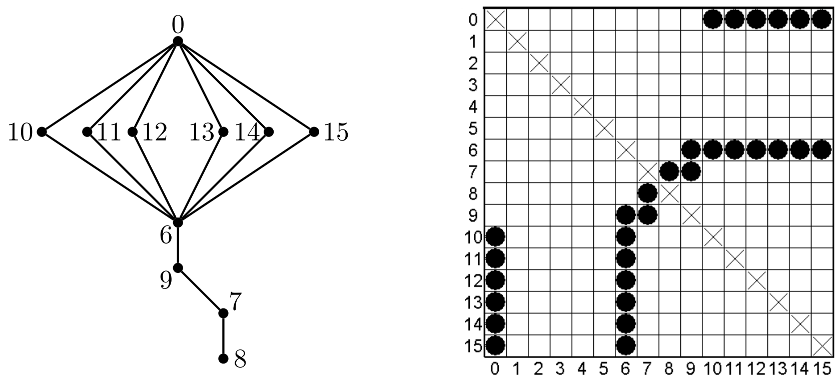
(2) ⇒ (3)**:** Let G have a graceful labelling with an FK-graph chessboard. The dots in the left upper corner of the graph chessboard correspond in the labelling sequence (LS) to numbers $\lceil \frac{m-1-i}{2} \rceil$ for $i \leq m-1$. The "path pattern" in the bottom right of the graph chessboard corresponds in the LS to numbers $\lceil \frac{s-i+m-1}{2} \rceil$ for $m \leq i < s-m+1$. The m dots in the last row correspond in the LS to numbers $s-i$ for $i \geq s-m+1$.

(3) ⇒ (4)**:** Let (3) hold, we show that this LS L has the labelling relation $A(L)$ as in (4). Every member of the LS L creates in the labelling relation the first coordinate. The second coordinate in each of the pairs in $A(L)$ is obtained by adding the number i to the first coordinate. So $A(L)$ satisfies (4).

(4) ⇒ (1)**:** Let (4) hold. We show that the pairs in $A(L)$ represent the edges of the graph $K_n(F_m)$. The pairs $\left[\lceil \frac{m-1-i}{2} \rceil, \lceil \frac{m-1-i}{2} \rceil + i\right]$ for $i \leq m-1$ correspond to the path P_m. The pairs $\left[\lceil \frac{s-i+m-1}{2} \rceil, \lceil \frac{s-i+m-1}{2} \rceil + i\right]$ for $m \leq i < s-m+1$ correspond to the path P_n. The pairs $[s-i, s]$ for $i \geq s-m+1$ correspond to the edges connecting the vertex labelled s with the path P_m. So G is the fan kite $K_n(F_m)$. □

4.2 Lantern kites

By a *lantern kite* we mean a graph obtained by joining a "lantern" to the end-point of a path. By a lantern we mean a complete bipartite graph $K_{2,m}$, but we will denote it simply by L_m. We denote the lantern kite obtained by joining the lantern L_m to the end-point of the path P_n by $K_n(L_m)$ and we assume that $m \geq 2$, $n \geq 1$. The size of the graph is $s = 2m + n - 1$ where $2m$ is the size of lantern L_m and $n-1$ is the length of the path P_n. We note that the lantern kite $K_n(L_2)$ is just the quadrangular kite $K_n(C_4)$.



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
7	7	6	6	6	6	6	6	6	0	0	0	0	0	0
8	9	9	10	11	12	13	14	15	10	11	12	13	14	15

Figure 8: Representations of the lantern kite $K_4(L_6)$

Example 4.5. *The lantern kite $K_4(L_6)$ is obtained by joining the vertex labelled by $m = 6$ of the lantern L_6 to the path P_4. In Figure 8 we see a gracefully labelled graph diagram of the kite $K_4(L_6)$ and its corresponding graph chessboard. The second row of the labelling table is the labelling sequence (LS) $(7,7,6,6,6,6,6,6,6,0,0,0,0,0,0)$. It consists of two parts: one is the LS $(7,7,6)$ of the path P_4, and the second is the LS $(6,6,6,6,6,6,0,0,0,0,0,0)$ of the lantern L_6.*

Definition 4.6. *Let $s = 2m + n - 1$ for $m \geq 2$, $n \geq 1$. By an LK-graph chessboard of size s we mean a simple chessboard as in Figure 8. It has m dots in the column 0 and m dots in the column m, and the pattern continues from the mth column with dots forming a "path pattern".*

Now we characterize via our tools the lantern kites $K_n(L_m)$:

Theorem 4.7. *Let G be a graph of size $s = 2m + n - 1$ for $m \geq 2$, $n \geq 1$. Then the following are equivalent:*

(1) G is the lantern kite $K_n(L_m)$.

(2) G has a graceful labelling and an LK-graph chessboard of size s.

(3) There exists a labelling sequence $L = (j_1, j_2, \dots, j_s)$ *of* G *such that*

$$j_i = \begin{cases} \left\lceil \frac{s-i}{2} \right\rceil, & \textit{if } i \leq s - 2m; \\ m, & \textit{if } s - 2m < i \leq s - m; \\ 0, & \textit{if } s - m < i \leq s. \end{cases} \tag{LSLK}$$

(4) There exists a labelling sequence L *of* G *with the labelling relation*

$$A(L) = \left\{ \left[\left\lceil \frac{s-i}{2} \right\rceil, \left\lceil \frac{s-i}{2} \right\rceil + i \right] \;\middle|\; i \leq s - 2m \right\} \cup \\ \{ [m, m+i] \mid s - 2m < i \leq s - m \} \cup \{ [0, i] \mid s - m < i \leq s \}.$$

Proof. (1) ⇒ (2)**:** Let G be the lantern kite $K_n(L_m)$. We label the vertex joining the path P_n with the lantern L_m (the "joining vertex") by m and the vertex on the top of the lantern by 0. We label the vertices in the middle of the lantern (adjacent to the vertices 0 and m) by $s - m + 1, s - m + 2, \dots, s$. We finally label the vertices of the path P_n from the joining vertex by numbers $s - m, m + 1, s - (m + 1), m + 2, \dots, \left\lceil \frac{s}{2} \right\rceil$. To show that this labelling is graceful we use the corresponding graph chessboard. The edges connecting the vertex 0 with the "middle" vertices of the lantern are represented in the column 0 of the graph chessboard. The edges connecting the joining vertex labelled m with the "middle" vertices of the lantern are represented in the column m of the graph chessboard. The path P_n is represented by a "path pattern". We obtain an LK-graph chessboard, where each diagonal has exactly one dot, so the labelling is graceful.

(2) ⇒ (3)**:** Let G be a gracefully labelled graph with an LK-graph chessboard. The dots in the column 0 of the graph chessboard correspond in the labelling sequence (LS) to the number 0. The dots in column m of the graph chessboard correspond in the LS to the number m. The dots forming the "path pattern" correspond in the LS to numbers $\left\lceil \frac{s-i}{2} \right\rceil$. So the formula (LSLK) holds.

(3) ⇒ (4)**:** Let (3) hold. We verify that the labelling relation $A(L)$ of the LS L satisfying (LSLK) consists of the pairs as described in (4). The numbers $\left\lceil \frac{s-i}{2} \right\rceil$ from the LS L correspond in $A(L)$ to the pairs $\left[\left\lceil \frac{s-i}{2} \right\rceil, \left\lceil \frac{s-i}{2} \right\rceil + i \right]$ for $i \leq s - 2m$. The numbers m from L corresponds in $A(L)$ to the pairs $[m, m + i]$ for $s - 2m < i \leq s - m$. Finally, the numbers 0 from L corresponds in $A(L)$ to the pairs $[0, i]$ for $s - m < i \leq s$. So $A(L)$ is exactly as in (4).

(4) ⇒ (1)**:** Let (4) hold. We show that the pairs in $A(L)$ represent the edges of $K_n(L_m)$. The pairs $\left[\left\lceil \frac{s-i}{2} \right\rceil, \left\lceil \frac{s-i}{2} \right\rceil + i \right]$ for $i \leq s - 2m$ represent the edges of the path P_n. The pairs $[m, m+i]$ for $s-2m < i \leq s-m$ represent the edges from the lantern L_m connecting the joining vertex with the "middle" vertices. Finally, the pairs $[0, i]$ for $s - m < i \leq s$ represent the edges from

the lantern L_m connecting the top vertex with the "middle" vertices. So G is the lantern kite $K_n(L_m)$. □

5 Conclusion and further research directions

We introduced and studied classes of graceful graphs, which we call kites. We described kites formed by cycles known to be graceful, fan kites and lantern kites. We showed in a transparent way that the studied graphs are graceful and we provided characterizations of these graphs among all simple graphs via Sheppard's labelling sequences, labelling relations and graph chessboards. The labelling relations are closely related to Sheppard's labelling sequences while the graph chessboards provide a nice visualization of the graceful labellings.

In particular, in Section 3 we firstly presented the characterization of the kites $K_n(C_m)$ formed by cycles C_m for $m = 0 \pmod 4$, where $m \geq 4$ and $n \geq 1$. It follows from it as a special case that a graph G of size $s = n + 3$ for some $n \geq 1$ is the quadrangular kite $K_n(C_4)$ if and only if G has a graceful labelling and a QK-graph chessboard of size s. Then in Section 3 we presented the characterization of the kites $K_n(C_m)$ formed by cycle C_m for $m = 3 \pmod 4$ and $n \geq \lfloor \frac{m}{2} \rfloor$. We distinguished two subclasses of the kites $K_n(C_m)$ with n even and n odd. Both cases are rather similar, yet they differ in details. Our theorems also cover all triangular kites $K_n(C_3)$. It follows from them as a special case that a graph G of size $s = n + 2$ for some $n \geq 1$ is the triangular kite $K_n(C_3)$ if and only if G has a graceful labelling and a TK-graph chessboard of size s.

In Section 4 we described graceful labellings of other two classes of the kites: fan kites and lantern kites. We showed that a graph G of size $s = 2m + n - 2$ for $m \geq 2$, $n \geq 1$ is the fan kite $K_n(F_m)$ if and only if G has a graceful labelling and an FK-graph chessboard of size s. We finally proved that a graph G of size $s = 2m + n - 1$ for $m \geq 2$, $n \geq 1$ is the lantern kite $K_n(L_m)$ if and only if G has a graceful labelling and an LK-graph chessboard of size s. For both fan kites and lantern kites we also gave characterizations of these graphs among all simple graphs via Sheppard's labelling sequences and the labelling relations.

Before we present possible further research directions, we notice that the gracefulness of certain similar graphs was studied in 1980 by Koh, Rogers, Teo, and Yap [4] and in 1984 by Truszczyński [10]. In [4] the authors call them *tadpoles*, but the journal with the paper has not been accessible to us, and so we do not know which tadpoles exactly were studied there. Yet we are sure they could not be described by the graph chessboards and labelling relations as the concepts invented much later.

M. Truszczyński in [10] refers to his graphs as *dragons* and denotes by $D_k(m)$ a dragon with the cycle C_k and the path P_{m+1}. He gives a proof that all dragons are graceful for $k \geq 3$ and $m \geq 1$. His proof uses a method that is laborious, technical, has lots of sub-cases and is hardly visualizable. We proved here in Section 3 two cases, $k = 0 \pmod 4$ and $k = 3 \pmod 4$, the latter with sufficiently big path, but we use visualization, from which gracefulness of the graph is clearly seen. In addition, we characterized these kites formed by graceful cycles by the simple chessboards and we gave formulas for Sheppard's labelling sequences and the labelling relations. The aim of our approach has been to study interesting kites, find their graceful labellings and characterize them by the simple chessboards, labelling sequences and labelling relations. Finally, the author of the paper [10] studied gracefulness of the so-called unicyclic graphs, *i.e.* those with one cycle and connected to anything possible (the path P_m was only one of the possibilities, he also connects them *e.g.* to stars). He has expressed his belief that all unicyclic graphs are graceful. So our and his approach overlap a bit, but both approaches have different intentions.

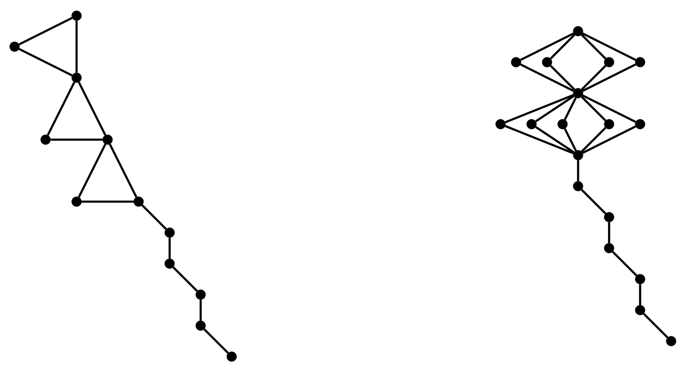


Figure 9: Further interesting kites

The first possible further research direction that we propose here is to characterize some tadpoles from the paper [4] and the remaining kites formed by the cycles C_m for $m = 1, 2 \pmod 4$ from the paper [10] via the simple chessboards, labelling sequences and labelling relations similarly as here. The second possible research direction is to take some further classes of gracefully labelled graphs (like fan graphs here) from Chapter 4 of [3] and describe them in the analogous manner like here. Another possible research direction is to consider "chain kites" in the way that the chain is a collection of C_3 graphs ("triangular chain kites", see the left graph in Figure 9) or a collection of any C_m graphs for a fixed $m \geq 4$. (We notice that the case $m = 4$, so-called "quadrangular chain kites", were studied and described in [5].) Or one could consider the chain kites in the way that the chain is a collection of C_m graphs of different sizes. Also interesting could be the chains as collections of the lanterns L_m of different sizes (see the right graph in Figure 9).

6 Acknowledgements

The first author acknowledges a support by Slovak VEGA grants 2/0078/20 and 1/0152/22, and a Visiting Professorship at University of Johannesburg. The authors would like to express their appreciation for the referees' recommendations.

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CUBO, A Mathematical Journal
Vol. 26, no. 3, pp. 387–406, December 2024
DOI: 10.56754/0719-0646.2603.387

New values of the Julia Robinson number

Carlos Muñoz Sandoval[1]

[1] *Departamento de Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Concepción, Chile.*
cmunoz2016@udec.cl

ABSTRACT

We extend results of Vidaux and Videla concerning the set of Julia Robinson numbers.

RESUMEN

Extendemos los resultados de Vidaux y Videla respecto del conjunto de números de Julia Robinson.

Keywords and Phrases: Decidability, definability, 2-towers, totally real towers, iterates of quadratic polynomials.

2020 AMS Mathematics Subject Classification: 03B25, 11U05, 11R80.

Published: 07 October, 2024
Accepted: 03 September, 2024
Received: 28 June, 2023

1 Introduction

Given a ring R of totally real algebraic integers and $t \in \mathbb{R} \cup \{\infty\}$, consider the set

$$R_t = \{x \in R \colon 0 \ll x \ll t\},$$

where $x \ll y$ means that all the conjugates of $y - x$ are positive, the interval (or singleton $\{\infty\}$)

$$\{t \in \mathbb{R} \cup \{\infty\} \colon \#R_t = \infty\}$$

and the so-called *Julia Robinson number*

$$\mathrm{JR}(R) = \inf\{t \in \mathbb{R} \cup \{\infty\} \colon \#R_t = \infty\}.$$

When the interval is closed or $\{\infty\}$, we say that R has the JR property. Notice that $\mathrm{JR}(R) \geq 4$ by a result of Kronecker (see [3]). Using the above definition, J. Robinson proved in [6] a result that can be formulated as

Theorem 1.1. *Let R be a ring of totally real algebraic integers. If R has the JR property, then it is possible to define $\mathbb{N}$ in R, and hence, R has undecidable first-order theory.*

Originally Robinson only considered R when it was the ring of integers of a totally real field, but it is not difficult to see that the proof of this theorem can be adapted to apply to any subring of the ring of integers of a totally real field (see [1, Theorem 1.2.2 and Lemma 1.2.3] for more details).

In the same work, J. Robinson proved that the ring of integers of the field $\mathbb{Q}^{\mathrm{tr}}$ of all totally real algebraic numbers (whose conjugates are all real numbers) has the JR property with JR number equal to 4, and the ring of integers of $K = \mathbb{Q}(\sqrt{p} \colon p \text{ prime})$ also has the JR property with JR number equal to ∞. In the case of the ring of integers of a totally real number field K has JR number equal to ∞ and hence, has undecidable theory. In [5] J. Robinson proved that every ring of integers of a number field (not necessarily totally real) has undecidable theory.

So, all known examples at that time had JR numbers equal to 4 or ∞ and the natural question, asked by J. Robinson in [6], was

Does the JR property hold for every ring of integers of any totally real algebraic field?

Motivated by the attempt to find rings that do not satisfy one or the other of these two properties of the JR number, Vidaux and Videla constructed in [7] infinitely many rings $\mathcal{O}$ depending on two parameters (ν, x_0) for which the JR number of $\mathcal{O}$ is a minimum but is not 4 or ∞, and also also show that for infinitely many values of (ν, x_0) the JR number is not a minimum, but satisfies another topological property called *isolation property* defined as:

R has the isolation property if and only if R does not have the JR property and there exists $M > 0$ such that for every $\varepsilon > 0$, if $\varepsilon < M$ then the set $R_{\mathrm{JR}(R)+M} \setminus R_{\mathrm{JR}(R)+\varepsilon}$ is finite.

In case that R has the *isolation property* then the natural numbers are definable in R, so in particular the theory of the ring R is undecidable (see [7] for details).

In [2] P. Gillibert and G. Ranieri built infinite rings with JR number strictly between 4 and infinity, which are the ring of integers of their field of fractions, however, the JR number of each of these rings is a minimum, also leaving J. Robinson's question open.

The objective of this article is to obtain new Julia Robinson numbers, having either the JR property or the isolation property, and hence produce new examples of totally real undecidable rings — see [4] for recent results on this *spectrum* problem.

Given (non-zero) natural numbers ν, λ and x_0, put $x_n = \sqrt{\nu + \lambda x_{n-1}}$ for every $n \geq 1$, and consider the ring $\mathcal{O}$ equal to the union of all the $\mathbb{Z}[x_n]$. Vidaux and Videla [7], and Castillo [1], study the definability of $\mathbb{N}$ in $\mathcal{O}$ when $\lambda = 1$. Generalizing their results, we have the following:

In section 2 we will start studying properties of the sequence (x_n) and we will give necessary and sufficient conditions for the ring $\mathcal{O}$ to be totally real (which is necessary to be able to apply Julia Robinson's techniques).

Theorem 2.7. $\mathcal{O}$ is totally real if and only if either $\nu > x_0^2 - \lambda x_0$ and $\nu \geq 2\lambda^2$ or $\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_0 < \nu^2 - \lambda^2 \nu$.

Later, we will give sufficient conditions for the tower $(K_n)_{n\geq 0}$, of the fraction fields of $\mathcal{O}_n = \mathbb{Z}[x_n]$ is a 2-tower, that is, such that $[K_{n+1} \colon K_n] = 2$ for all $n \geq 0$ (the latter is necessary to apply the argument given by Vidaux and Videla in [7]). More precisely, we will show that the tower grows when $\nu + \lambda x_0$ is congruent to 2 or 3 modulo 4, and λ is congruent to 1 or 3 modulo 4 (Proposition 2.13).

In section 3 we will study the increasing case, giving rise to our main result (in the following theorem, the case $\lambda = 1$ is done in [7] and [1]):

Theorem 1.2. *Assume $\nu > x_0^2 - \lambda x_0$ and $\nu \geq 2\lambda^2$. Assume that for every $n \geq 0$ we have $[K_{n+1} \colon K_n] = 2$. If $\lambda = 1$ and $\nu \neq 3$, then $\mathcal{O}$ has JR number equal to $\lceil \alpha \rceil + \alpha$ and satisfies the JR property. If $\lambda \geq 2$, $\nu \geq 2\lambda^2 + 2$, and $x_0 \geq \frac{\lambda}{4}$, then $\mathcal{O}$ has JR number equal to $\lceil \alpha \rceil + \alpha$ and satisfies the JR property.*

This theorem gives us new values of JR numbers, *e.g.* for the parameters $\lambda = 3$, $\nu = 20$ and $x_0 = 2$, the JR number is equal to 13.217 approximately, but with $\lambda = 1$ this number is not obtained.

In section 4 we present two new theorems: the first of them is a direct adaptation of [7, Proposition 3.4, Proposition 3.5 and Proposition 3.6]:

Theorem 1.3. *Assume $\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_0 < \nu^2 - \lambda^2 \nu$. Assume that for every $n \geq 0$ we have $[K_{n+1} \colon K_n] = 2$. Assume that $\nu - \lambda x_1 \geq 1$ and $x_1 < \lfloor \alpha \rfloor + 1$. The JR number of $\mathcal{O}$ is $\lfloor \alpha \rfloor + \alpha + 1$ and satisfies the isolation property. Moreover, there are infinitely many rings $\mathcal{O}$ that satisfy these hypotheses.*

The following theorem solves the problem for infinitely many values of the parameters ν and x_0 when $\lambda = 3$, removing the hypothesis $\nu - \lambda x_1 \geq 1$. The proof of this theorem can be easily adapted to $\lambda = 2, 4, 5, \ldots,$ as long as λ is not too large, nevertheless, despite the fact that the number of cases to be considered seems to decrease as λ grows, we were not able to find a pattern that would allow as to write a proof for arbitrary λ.

Theorem 1.4. *Assume $\nu < x_0^2 - 3x_0$ and $27x_0 < \nu^2 - 9\nu$. Assume that for every $n \geq 0$ we have $[K_{n+1} \colon K_n] = 2$. If $x_1 < \lfloor \alpha \rfloor + 1$ and $\nu \neq 19$, then $\mathcal{O}$ has JR number equal to $\lfloor \alpha \rfloor + \alpha + 1$ and satisfies the isolation property. Moreover, there are infinitely many rings $\mathcal{O}$ that satisfy these hypotheses.*

This article is a contribution to two long term projects:

1) Does the ring of integers of any 2-tower over a number field have undecidable theory?

2) Study the topology of the set of JR numbers on the interval $[4, +\infty)$ — *e.g.* is it a dense set?

2 Basic properties of the tower

We define the sequence (x_n) whose general term is $x_n = \sqrt{\nu + \lambda x_{n-1}}$ and

- ν and x_0 are non-negative integers and not zero simultaneously,
- $\lambda > 0$ is a rational integer, and
- $\nu \neq x_0^2 - \lambda x_0$ (in order to avoid $x_1 = x_0$).

We define the following rings and their field of fractions:

$$
\begin{aligned}
\mathcal{O}_0 &= \mathbb{Z} & K_0 &= \mathbb{Q} \\
\mathcal{O}_n &= \mathcal{O}_{n-1}[x_n] & K_n &= K_{n-1}[x_n] \\
\mathcal{O} &= \bigcup_{n \geq 0} \mathcal{O}_n & K &= \bigcup_{n \geq 0} K_n
\end{aligned}
$$

Let us begin by stating the following lemma, whose proof is essentially the same as those given in [7, Lemma 2.2, 2.3 and 2.14].

Lemma 2.1. (1) *The sequence* (x_n) *is strictly increasing if and only if* $\nu > x_0^2 - \lambda x_0$ *or is strictly decreasing if and only if* $\nu < x_0^2 - \lambda x_0$.

(2) *The sequence* (x_n) *converges to the limit* $\alpha = \frac{\lambda + \sqrt{\lambda^2 + 4\nu}}{2}$.

(3) *If* $\mathcal{O}$ *is totally real, then the JR number of* $\mathcal{O}$ *is finite, in particular the extension of* K *over* $\mathbb{Q}$ *is infinite.*

Lemma 2.2. *There exists an integer* $n_0 \geq 0$ *such that for every* $n \geq 0$*, we have* $n \leq n_0$ *if and only if* x_n *is a rational integer.*

Proof. If $x_n \notin \mathbb{Z}$ for some $n \geq 0$, then $x_n \notin \mathbb{Q}$ since x_n is an algebraic integer. Hence, $\lambda x_n \notin \mathbb{Q}$ for every $\lambda \geq 1$. So, $x_{n+1} = \sqrt{\nu + \lambda x_n} \notin \mathbb{Z}$. Since (x_n) is bounded, the sequence takes finitely many integer values. We choose n_0 to be the largest index n such that x_n is a rational integer. □

2.1 The totally real condition

As was indicated in [7], Julia Robinson's criterion is only applicable for rings of totally real algebraic integers. In this section we will give a sufficient and necessary condition for the ring $\mathcal{O}$ to be totally real.

Lemma 2.3. *We have* $\nu \geq 2\lambda^2$ *if and only if* $\nu \geq \lambda\alpha$.

Proof. Observe that $\nu \geq \lambda\alpha$ if and only if

$$\nu \geq \lambda \left(\frac{\lambda + \sqrt{\lambda^2 + 4\nu}}{2} \right) \geq \frac{\lambda^2}{2},$$

which implies $2\nu \geq \lambda^2$. Therefore, we have

$$\begin{aligned} \nu \geq 2\lambda^2 &\iff 4\nu^2 \geq 8\lambda^2\nu \iff 4\nu^2 - 4\lambda^2\nu + \lambda^4 \geq \lambda^4 + 4\lambda^2\nu \\ &\iff (2\nu - \lambda^2)^2 \geq \lambda^2(\lambda^2 + 4\nu) \iff 2\nu - \lambda^2 \geq \lambda\sqrt{\lambda^2 + 4\nu} \\ &\iff \nu \geq \lambda\alpha. \end{aligned}$$

□

Lemma 2.4. *If* $\mathcal{O}$ *is totally real and* $\nu > x_0^2 - \lambda x_0$*, then* $\nu \geq 2\lambda^2$.

Proof. Since K_{n+1} has degree 2 over K_n for infinitely many n by Lemma 2.1, we have a subsequence of (x_n), namely (x_{n_k}), such that $\sqrt{\nu - \lambda x_{n_k}}$ is a conjugate of x_{n_k+1}. In particular, $\nu \geq \lambda x_{n_k}$ for every $k \geq 1$ since the ring $\mathcal{O}$ is totally real. From this, and the fact that x_n converges to α, we can deduce $\nu \geq \lambda\alpha$. We can conclude using Lemma 2.3. □

Lemma 2.5. *If $\mathcal{O}$ is totally real, then we have $\lambda^3 x_{n_0} < \nu^2 - \lambda^2\nu$, where n_0 comes from Lemma 2.2.*

Proof. We write $n_1 = n_0 + 1$. By the definition of n_0, we have $x_{n_1} \notin K_{n_0}$ and therefore K_{n_1} is a quadratic extension over K_{n_0}. Thus $\sqrt{\nu - \lambda x_{n_1}}$ is a conjugate of x_{n_1+1}. Since $\mathcal{O}$ is totally real, $\sqrt{\nu - \lambda x_{n_1}}$ will be a real number, which is not zero because λx_{n_1} is an irrational number and ν is a rational integer. So we have $\nu > \lambda x_{n_1} = \lambda\sqrt{\nu + \lambda x_{n_0}}$ if and only if $\lambda^3 x_{n_0} < \nu^2 - \lambda^2\nu$. □

Remark 2.6. *Let $x \in \mathcal{O}$. We use the notation $\overline{|x|}$ for the largest absolute value of conjugates of x over $\mathbb{Q}$.*

The following theorem gives us a characterization of when our ring $\mathcal{O}$ is totally real and therefore, will allow us to use Julia Robinson's methods.

Theorem 2.7. *The ring $\mathcal{O}$ is totally real if and only if*

(1) *either $\nu > x_0^2 - \lambda x_0$ and $\nu \geq 2\lambda^2$, or*

(2) *$\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_{n_0} < \nu^2 - \lambda^2\nu$.*

If $\mathcal{O}$ is totally real, then $\overline{|x_n|} = x_n$ for each $n \geq 0$.

Proof. Let us start proving that $\overline{|x_n|} = x_n$ for each $n \geq 0$ if $\mathcal{O}$ is totally real. We will show this by induction over n. The case $n = 0$ is trivial. Assume $\overline{|x_n|} = x_n$ for some n. We have

$$x_{n+1} = \sqrt{\nu + \lambda x_n} \geq \pm\sqrt{\nu + \lambda x_n^\sigma}$$

for every embedding σ and since the only possible conjugates of x_{n+1} are of the form $\pm\sqrt{\nu + \lambda x_n^\sigma}$ for some embedding σ, we are done. For the rest of the proof, the implication from left to right is an immediate consequence of Lemma 2.4 and Lemma 2.5. We show the other implication by induction on n. Let $n_1 = n_0 + 1$. If $n \leq n_0$, then $\mathcal{O}_n = \mathbb{Z}$ which is totally real and hence $\overline{|x_n|} = x_n$. For n_1 we have $x_{n_1} \notin \mathbb{Z}$ and hence its conjugates are of the form $\pm\sqrt{\nu + \lambda x_{n_0}}$. Therefore, $\mathcal{O}_{n_1} = \mathbb{Z}[x_{n_1}]$ is totally real and $\overline{|x_{n_1}|} = x_{n_1}$. Suppose that for some $n \geq n_1$, $\mathcal{O}_n$ is totally real and $\overline{|x_n|} = x_n$. Note that the conjugates of x_{n+1} are of the form $\pm\sqrt{\nu + \lambda x_n^\sigma}$ for some embedding σ. Since $\overline{|x_n|} = x_n$, we have $\overline{|x_{n+1}|} = x_{n+1}$ and it will be enough to prove that $\nu \geq \lambda x_n$ for each $n \geq n_1$. We can separate the proof into cases where the sequence (x_n) is increasing or decreasing:

- If $\nu > x_0^2 - \lambda x_0$ and $\nu \geq 2\lambda^2$, then (x_n) is strictly increasing by Lemma 2.1 and hence $\lambda x_n < \lambda\alpha \leq \nu$ by Lemma 2.3.
- If $\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_{n_0} < \nu^2 - \lambda^2\nu$, then (x_n) is strictly decreasing by Lemma 2.1 and $\lambda x_{n_1} < \nu$. Hence, $\lambda x_n \leq \lambda x_{n_1} < \nu$ for each $n \geq n_1$. □

We can assume, without loss of generality, that $n_0 = 0$, since if $n_0 > 0$, then we can define a new sequence $y_n = x_{n+n_0}$, and the rings $\mathcal{O}$ corresponding to (x_n) and (y_n) are the same.

Assumption 2.8. *The number x_1 is a non-rational integer*

Lemma 2.9. *In the decreasing case, we have $\nu \geq 3$ and $x_0 \geq 3$.*

Proof. This is an immediate consequence of the inequalities $\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_0 < \nu^2 - \lambda^2 \nu$, and the fact that λ is at least 1. □

Lemma 2.10. *Assume that (x_n) is increasing. If $\nu \geq 2\lambda^2 + 2$, then $x_n \geq 2$ for each $n \geq 1$.*

Proof. Since the sequence (x_n) is increasing, we have

$$x_n \geq x_1 = \sqrt{\nu + \lambda x_0} \geq \sqrt{2\lambda^2 + 2} \geq 2.$$

for each $n \geq 1$. □

Lemma 2.11. *We have $\alpha \geq 2$.*

Proof. If (x_n) is decreasing, then by Lemma 2.9 we have $\nu \geq 3$, and if (x_n) is increasing, then $\nu \geq 2\lambda^2 \geq 2$. In all cases, we have $\nu \geq 2$. Hence, we have

$$2\alpha = \lambda + \sqrt{\lambda^2 + 4\nu} \geq 4$$

because $\lambda \geq 1$ and $\nu \geq 2$. □

2.2 Conditions for the tower to increase at each step

For the induction arguments to work in the next sections, we will need the tower (K_n) to increase at each step. In this subsection, we will provide sufficient conditions for that.

Let $f(t) = \dfrac{t^2 - \nu}{\lambda}$ be a function of the real variable t. We define for each $n \geq 1$

$$P_n = \lambda^{2^n - 1} f^{\circ n}(t) - \lambda^{2^n - 1} x_0,$$

where $f^{\circ n}$ stands for the composition of f with itself n times.

Lemma 2.12. *The polynomial P_n is monic for each $n \geq 1$.*

Proof. We prove it by induction on n. If $n = 1$, then $P_1 = \lambda f(t) - \lambda x_0 = t^2 - \nu - \lambda x_0$ is monic. Suppose that for some $n \geq 2$ the polynomial P_n is monic. We have

$$\begin{aligned} P_{n+1}(t) &= \lambda^{2^{n+1}-1} f^{\circ(n+1)}(t) - \lambda^{2^{n+1}-1} x_0 = \lambda^{2^{n+1}-1}\left(\frac{(f^{\circ n}(t))^2 - \nu}{\lambda}\right) - \lambda^{2^{n+1}-1} x_0 \\ &= \lambda^{2^{n+1}-2}(f^{\circ n}(t))^2 - \lambda^{2^{n+1}-2}\nu - \lambda^{2^{n+1}-1}x_0 = \left(\lambda^{2^n-1} f^{\circ n}(t)\right)^2 - \lambda^{2^{n+1}-2}\nu - \lambda^{2^{n+1}-1}x_0 \\ &= \left(P_n(t) + \lambda^{2^n-1}x_0\right)^2 - \lambda^{2^{n+1}-2}\nu - \lambda^{2^{n+1}-1}x_0, \end{aligned}$$

and since P_n is monic by hypothesis, P_{n+1} is monic too. □

Proposition 2.13. *If $\nu + \lambda x_0$ is congruent to 2 or 3 modulo 4 and λ is congruent to 1 or 3 modulo 4, then for each $n \geq 1$, we have $[K_{n+1} \colon K_n] = 2$.*

Proof. From the definition of f we have $f^{\circ n}(x_n) = x_0$ for each $n \geq 1$. Therefore, x_n is a root of P_n. Also note that, by Lemma 2.12, P_n is monic for each $n \geq 1$. Given $a, b \in \mathbb{Z}$, we have

$$P_1(t+a) = (t+a)^2 - \nu - \lambda x_0 = t^2 + 2at + a^2 - (\nu + \lambda x_0), \tag{2.1}$$

and

$$\begin{aligned} P_2(t+b) &= \lambda^3 f^{\circ 2}(t+b) - \lambda^3 x_0 \\ &= t^4 + 4bt^3 + 2(3b^2 - \nu)t^2 + 4(b^3 - b\nu)t + (b^4 - 2b^2\nu + \nu^2 - \lambda^2(\nu + \lambda x_0)). \end{aligned} \tag{2.2}$$

Also, for each $n \geq 1$, we have

$$\begin{aligned} P_{n+2}(t) &= \lambda^{2^{n+2}-1}(f^{\circ(n+2)}(t) - x_0) = \lambda^{2^{n+2}-1}(f^{\circ 2}(f^{\circ n}(t)) - x_0) \\ &= \lambda^{2^{n+2}-1}\left(f^{\circ 2}\left(\frac{P_n(t)}{\lambda^{2^n-1}} + x_0\right) - x_0\right) \\ &= \lambda^{2^{n+2}-1}\left(\left(\frac{P_2\left(\frac{P_n(t)}{\lambda^{2^n-1}} + x_0\right)}{\lambda^3} + x_0\right) - x_0\right) = \lambda^{4(2^n-1)} P_2\left(\frac{P_n(t)}{\lambda^{2^n-1}} + x_0\right) \\ &= P_n^4(t) + 4\lambda^{2^n-1}x_0 P_n^3(t) + 2\lambda^{2(2^n-1)}(3x_0^2 - \nu)P_n^2(t) + 4\lambda^{3(2^n-1)}(x_0^3 - x_0\nu)P_n(t) \\ &\quad + \lambda^{4(2^n-1)}\left(x_0^4 - 2x_0^2\nu + \nu^2 - \lambda^2(\nu + \lambda x_0)\right). \end{aligned} \tag{2.3}$$

We prove by induction on n that the polynomial P_n is irreducible. If $n = 1$, then using Equation (2.1) we choose $a = 0$ if $\nu + \lambda x_0$ is congruent to 2 modulo 4, and $a = 1$ if $\nu + \lambda x_0$ is congruent to 3 modulo 4. In both cases $P_1(t+a)$ is an Eisenstein polynomial for 2. If $n = 2$, then using Equation (2.2), we have that $P_2(t+x_0)$ is an Eisenstein polynomial for 2, because $x_0^4 - 2x_0^2\nu + \nu^2 - \lambda^2(\nu + \lambda x_0)$

is congruent to 2 modulo 4 when $\nu + \lambda x_0$ is congruent to 2 or 3 modulo 4 and λ is congruent to 1 or 3 modulo 4 (we leave the verification to the reader). Note that λ^2 is congruent to 1 modulo 4 by hypothesis. Therefore, the constant term of $P_{n+2}(t)$, seen as a polynomial in $P_n(t)$, is congruent to 2 modulo 4. So, using Equation (2.3), if $P_n(t+c)$ is an Eisenstein polynomial for 2 for some $c \in \mathbb{Z}$, then $P_{n+2}(t+c)$ is an Eisenstein polynomial for 2 too. Thus, we can prove the irreducibility of P_n by induction on n, separating into two cases:

- If n is odd, then $P_n(t+a)$ is an Eisenstein polynomial for 2 (with the respective choice of a).
- If n is even, then $P_n(t+x_0)$ is an Eisenstein polynomial for 2. □

From now on, we assume

Assumption 2.14. *K_n is a quadratic extension of K_{n-1} and the ring $\mathcal{O}$ is a totally real.*

Lemma 2.15 ([7, Lemma 2.19])**.** *Let r be any real number and $a, b \in \mathcal{O}_{n-1}$ with $n \geq 1$. For $n = 1$, if $0 \ll a + bx_1 \ll 2r$, then $|b| < \frac{r}{x_1}$. For $n \geq 2$, if $0 \ll a + bx_n \ll 2r$, then $|b^\sigma| < \frac{r}{\sqrt{\nu - \lambda x_{n-1}}}$ for every embedding σ of $\mathcal{O}_n$.*

3 Increasing case

Assumption 3.1. *For this section, let us assume $\nu \geq 2\lambda^2 + 2$, $x_0 \geq \frac{\lambda}{4}$ and the sequence (x_n) is strictly increasing.*

Definition 3.2. *For each $n \geq 1$, let k_n be the only rational integer such that*

$$\lceil \alpha \rceil - (k_n + 1) < x_n < \lceil \alpha \rceil - k_n.$$

Remember that x_1 is not a rational integer by Assumption 2.14 and note that the sequence (k_n) is (non strictly) decreasing, hence the k_n take only finitely many values, and since the sequence (x_n) tends to α, eventually k_n is 0.

The main result we use to compute the JR number in the increasing case is the following lemma:

Lemma 3.3. *Assume $x \in \mathcal{O}$. We have $0 \ll x \ll 2\lceil \alpha \rceil$ if and only if $x \in X$.*

The set X is defined as follows:

$$\begin{aligned} X_0 &= \{1, 2, \ldots, 2\lceil \alpha \rceil - 1\}, \\ X_n &= X_0 \cup \{\lceil \alpha \rceil \pm j \pm x_s \colon 0 \leq j \leq k_s \text{ and } 1 \leq s \leq n\}, \\ X &= \bigcup_{n \geq 0} X_n. \end{aligned}$$

Lemma 3.4. *If $\lambda \geq 2$, then $x_1 + x_2 + \lceil x_1 \rceil > 2\lceil \alpha \rceil$.*

Proof. It is enough to prove that we have $x_2 + 2x_1 > 2(\alpha + 1)$. We have

$$\begin{aligned} 2\sqrt{\nu + \lambda x_0} + \sqrt{\nu + \lambda\sqrt{\nu + \lambda x_0}} &\geq \sqrt{4\nu + \lambda^2} + \sqrt{2\lambda^2 + 2 + \lambda\sqrt{2\lambda^2 + 2 + \frac{\lambda^2}{4}}} \\ &\geq \sqrt{4\nu + \lambda^2} + \sqrt{\lambda^2 + 4\lambda + 4} = 2(\alpha + 1), \end{aligned}$$

where the first inequality is by Assumption 3.1. □

Lemma 3.5. *Let $n \geq 1$. If $0 \ll a \pm bx_n \ll 2\lceil \alpha \rceil$, with $a, b \in \mathcal{O}_{n-1}$, then $|b| < 2$ (in the inequality, the plus-minus means that both inequalities hold).*

Proof. Since $\nu \geq 2\lambda^2 + 2$ and $\nu \in \mathbb{N}$, we can write $\nu = 2\lambda^2 + k$, for some $k \geq 2$. Since $0 < a \pm bx_n < 2\lceil \alpha \rceil$, combining both inequalities we obtain $|b| < \frac{\lceil \alpha \rceil}{x_n}$. So, we have

$$\begin{aligned} |b| < \frac{\lceil \alpha \rceil}{x_n} \leq \frac{\alpha + 1}{\sqrt{2\lambda^2 + k + \lambda x_{n-1}}} &= \frac{\lambda + \sqrt{\lambda^2 + 4(2\lambda^2 + k)} + 2}{2\sqrt{2\lambda^2 + k + \lambda x_{n-1}}} \\ \leq \frac{\lambda + 2 + \sqrt{\lambda^2} + \sqrt{8\lambda^2 + 4k}}{\sqrt{8\lambda^2 + 4k}} &\leq 1 + \frac{2\lambda + 2}{\sqrt{8\lambda^2 + 4k}} \leq 1 + \frac{2\lambda + 2}{\sqrt{8\lambda^2 + 8}} \leq 2, \end{aligned}$$

where the last inequality is true because $2\lambda + 2 \leq \sqrt{8\lambda^2 + 8}$ for every $\lambda \geq 1$. □

Lemma 3.6. *We have $\nu - \lambda\alpha > 1$.*

Proof. Since $\nu \geq 2\lambda^2 + 2$ and $\nu \in \mathbb{N}$, we can write $\nu = 2\lambda^2 + k$, for some $k \geq 2$. Hence, we have

$$\begin{aligned} &(2\lambda^2 + k) - \lambda\left(\frac{\lambda + \sqrt{\lambda^2 + 4(2\lambda^2 + k)}}{2}\right) > 1 \iff 3\lambda^2 + 2k - 2 > \lambda\sqrt{9\lambda^2 + 4k} \\ &\iff 4k^2 + 12k\lambda^2 - 8k + 9\lambda^4 - 12\lambda^2 + 4 > 9\lambda^4 + 4k\lambda^2 \iff 4k^2 + (8\lambda^2 - 8)k + 4 - 12\lambda^2 > 0, \end{aligned}$$

and since $k \geq 0$, the latter is true for

$$k > \frac{8 - 8\lambda^2 + \sqrt{64\lambda^4 + 64\lambda^2}}{8} = 1 - \lambda^2 + \sqrt{\lambda^4 + \lambda^2}.$$

We consider the continuous function $x \mapsto 1 - x^2 + \sqrt{x^4 + x^2}$. The line $y = \frac{3}{2}$ is an horizontal asymptote for this function, hence we have

$$1 - \lambda^2 + \sqrt{\lambda^4 + \lambda^2} < \frac{3}{2},$$

for every $\lambda \geq 1$. □

Lemma 3.7. *Let $x = a + bx_1 \in \mathcal{O}_1$, with $a, b \in \mathbb{Z}$. If $0 < a \pm bx_1 < 2\lceil\alpha\rceil$, then $x \in X_1$.*

Proof. By Lemma 3.5, we have $b = \pm 1$ or $b = 0$.

- If $a \leq \lceil\alpha\rceil - (k_1 + 1)$, then $b = 0$. Indeed, if $|b| = 1$, by choosing σ such that $x^\sigma = a - |b|x_1$, we obtain:
$$a - |b|x_1 \leq \lceil\alpha\rceil - (k_1 + 1) - x_1 \leq 0,$$
by the definition of k_1, contradicting our hypothesis.

- If $a \geq \lceil\alpha\rceil + (k_1 + 1)$, then $b = 0$. If $|b| = 1$, by choosing σ such that $x^\sigma = a + |b|x_1$, we obtain:
$$a + |b|x_1 \geq \lceil\alpha\rceil + (k_1 + 1) + x_1 \geq 2\lceil\alpha\rceil,$$
again contradicting our hypothesis.

Therefore, we have either $|a - \lceil\alpha\rceil| \geq k_1 + 1$ and $b = 0$, or $|a - \lceil\alpha\rceil| < k_1 + 1$ and $|b| \leq 1$. In both cases, we have $x \in X_1$. □

Lemma 3.8. *Assume $n > m \geq 1$ and $\lambda \geq 2$.*

(1) *We have $\lceil\alpha\rceil \pm j + x_m + x_n \geq 2\lceil\alpha\rceil$ for every $0 \leq j \leq k_m$.*

(2) *We have $\lceil\alpha\rceil \pm j - x_m - x_n \leq 0$ for every $0 \leq j \leq k_m$.*

Proof.

(1) Note that $\lceil x_1\rceil = \lceil\alpha\rceil - k_1$. By Lemma 3.4, and using the fact that (x_n) is increasing, we have
$$x_m + x_n + \lceil\alpha\rceil - k_1 \geq 2\lceil\alpha\rceil,$$
for each $n > m \geq 1$. Since $k_1 \geq k_m$ for each $m \geq 1$, we have
$$x_m + x_n + \lceil\alpha\rceil \pm j \geq 2\lceil\alpha\rceil,$$
for every $0 \leq j \leq k_m$.

(2) For every $0 \leq j \leq k_m$, we have $\lceil\alpha\rceil \pm j - x_m - x_n \leq 0$ if and only if $x_m + x_n + \lceil\alpha\rceil \pm j \geq 2\lceil\alpha\rceil$. So we can conclude by item (1). □

Lemma 3.9. *Assume $\lambda \geq 2$. We have $\lceil x_n \rceil + x_n \geq \lceil \alpha \rceil + 2$ for each $n \geq 1$. In particular, we have $x_n \geq k_n + 2$ for each $n \geq 1$.*

Proof. Since (x_n) is increasing, it is enough to prove that we have $x_1 + \lceil x_1 \rceil > \alpha + 3$. If $\lambda = 2$, then we have (recalling that we have $x_0 \geq 1$ and $\nu \geq 10$ by Assumption 3.1)

$$x_1 + \lceil x_1 \rceil \geq \sqrt{\nu + 2} + \lceil \sqrt{\nu + 2} \rceil > \sqrt{\nu + 1} + \lceil \sqrt{12} \rceil = 3 + \alpha.$$

For $\lambda \geq 3$, we have

$$2x_1 + 2\lceil x_1 \rceil \geq \sqrt{4\nu + \lambda^2} + \sqrt{9\lambda^2 + 8} > \sqrt{4\nu + \lambda^2} + \lambda + 6 = 2(\alpha + 3),$$

where the first inequality is by Assumption 3.1, and the second inequality is because $\lambda \geq 3$. In particular, using $\lceil x_n \rceil = \lceil \alpha \rceil - k_n$ for each $n \geq 1$, we have $\lceil x_n \rceil + x_n \geq \lceil \alpha \rceil + 2$ if and only if $x_n \geq k_n + 2$. □

Lemma 3.10. *Let $x \in \mathcal{O}$. If $0 \ll x \ll 2\lceil \alpha \rceil$, then $x \in X$.*

Proof. For $\lambda = 1$, this is [7, Lemma 4.9]. For $\lambda \geq 2$, which we now assume, we start as in [7, Lemma 4.9]. We prove by induction on n that if $x \in \mathcal{O}_n$ is such that $0 \ll x \ll 2\lceil \alpha \rceil$, then $x \in X_n$. This is clear for $n = 0$. For $n = 1$, we have $x \in X_1$ by Lemma 3.7. Assume $n \geq 2$. Let us fix $x = a + bx_n \in \mathcal{O}_n$ with $a, b \in \mathcal{O}_{n-1}$. By Lemma 2.15, we have $0 \ll a \ll 2\lceil \alpha \rceil$, so $a \in X_{n-1}$ by induction hypothesis. Also, by Lemma 2.15, we have

$$\overline{|b|} < \frac{\lceil \alpha \rceil}{\sqrt{\nu - \lambda x_{n-1}}} < \frac{\lceil \alpha \rceil}{\sqrt{\nu - \lambda \alpha}} \leq \lceil \alpha \rceil,$$

since $\sqrt{\nu - \lambda\alpha} \geq 1$ by Lemma 3.6. Hence, we have $0 \ll \lceil \alpha \rceil + b \ll 2\lceil \alpha \rceil$, and by induction hypothesis we have $\lceil \alpha \rceil + b \in X_{n-1}$. From the definition of X_{n-1}, we have either $b \in \mathbb{Z}$, or $|b| = |j \pm x_s|$ for some $1 \leq s \leq n-1$ and $0 \leq j \leq k_s$. In the first case, we have either $b = 0$ or $b = \pm 1$ by Lemma 3.5. In the second case, we have, also by Lemma 3.5, either $|j + x_s| < 2$ or $|x_s - j| < 2$. If $|j + x_s| < 2$, then $x_s < 2 - j \leq 2$ and we have a contradiction by Lemma 3.9. If $|x_s - j| < 2$, then $x_s < j + 2 \leq k_s + 2$, which is a contradiction, again by Lemma 3.9. Therefore, we have $b \in \{-1, 0, 1\}$. For $b = 0$, there is nothing to prove, as we already know that $x = a$ lies in X_{n-1}. Assume $|b| = 1$. We can write $x = a \pm x_n$, and since $a \in X_{n-1}$, we have either $a \in \{1, \ldots, 2\lceil \alpha \rceil - 1\}$, or $a = \lceil \alpha \rceil \pm j \pm x_s$ for some $1 \leq s \leq n-1$ and $0 \leq j \leq k_s$.

- If $a \in \{1, \ldots, \lceil \alpha \rceil - (k_n + 1)\}$, then we can choose an embedding σ such that:

$$x^\sigma = a - x_n \leq \lceil \alpha \rceil - (k_n + 1) - x_n < 0,$$

by definition of k_n, which contradicts our hypothesis.

- If $a \in \{\lceil\alpha\rceil + (k_n + 1), \ldots, 2\lceil\alpha\rceil - 1\}$, then again we can choose σ such that

$$x^\sigma = a + x_n \geq \lceil\alpha\rceil + (k_n + 1) + x_n > 2\lceil\alpha\rceil,$$

which again contradicts our hypothesis on x.

- If $a = \lceil\alpha\rceil \pm j + x_s$, with $0 \leq j \leq k_s$, then

$$a + x_n = \lceil\alpha\rceil \pm j + x_s + x_n \geq 2\lceil\alpha\rceil,$$

by Lemma 3.8, a contradiction.

- If $a = \lceil\alpha\rceil \pm j - x_s$, with $0 \leq j \leq k_s$, then

$$a - x_n = \lceil\alpha\rceil \pm j - x_s - x_n \leq 0,$$

also by Lemma 3.8, again a contradiction.

So, we have $a \in \{\lceil\alpha\rceil - k_n, \ldots, \lceil\alpha\rceil + k_n\}$. Therefore, if $|b| = 1$, then x is of the form $\lceil\alpha\rceil \pm j \pm x_n$ where $0 \leq j \leq k_n$. In all the cases we obtain $x \in X$. □

Proof Lemma 3.3. Thanks to Lemma 3.10, we need only to prove the lemma from right to left. Assume $x \in X$. For $x \in X_0$, there is nothing to prove. Assume $x \in X_n$ for some $n \geq 1$, so that $x = \lceil\alpha\rceil \pm j \pm x_s$ for some s and j such that $1 \leq s \leq n$ and $0 \leq j \leq k_s$. By definition of k_s, we have $x_s + k_s < \lceil\alpha\rceil$. Hence, we have

$$\lceil\alpha\rceil \pm j + x_s \leq \lceil\alpha\rceil + k_s + x_s < 2\lceil\alpha\rceil,$$

and

$$\lceil\alpha\rceil \pm j - x_s \geq \lceil\alpha\rceil - k_s - x_s > 0.$$

Thus, we have $0 < x^\sigma < 2\lceil\alpha\rceil$ for every embedding σ of $\mathcal{O}_s$ since $\overline{|x_s|} = x_s$ by Lemma 2.7. □

Proposition 3.11. *The ring $\mathcal{O}$ has the JR property and $JR(\mathcal{O}) = \lceil\alpha\rceil + \alpha$.*

Proof. For each n we have $x_n + \lceil\alpha\rceil < \alpha + \lceil\alpha\rceil$. By Theorem 2.7, we have $\overline{|x_n|} = x_n$, and hence, there are infinitely many $x \in \mathcal{O}$ such that $0 \ll x \ll \lceil\alpha\rceil + \alpha$. Since the sequence (x_n) is increasing and converges to α, for each $\varepsilon > 0$, there are only finitely many n such that $x_n + \lceil\alpha\rceil < \alpha + \lceil\alpha\rceil - \varepsilon$. Moreover, almost all n we have $k_n = 0$. Hence, there are only finitely many elements of the form $x_n + \lceil\alpha\rceil + j$ where $0 \leq j \leq k_n$ and $k_n \geq 1$. In particular, only finitely many of them satisfy

$0 \ll x_n + \lceil \alpha \rceil + j \ll \lceil \alpha \rceil + \alpha$. Therefore, by Lemma 3.3, for each $\varepsilon > 0$, there are only finitely many $x \in \mathcal{O}$ such that $0 \ll x \ll \lceil \alpha \rceil + \alpha - \varepsilon$. □

4 Decreasing case

Assumption 4.1. *For this section, let us assume that the sequence (x_n) is strictly decreasing.*

We define the following sets:

$$X_0 = \{1, 2, \ldots, 2\lfloor \alpha \rfloor + 1\}$$
$$X_n = X_0 \cup \{\lfloor \alpha \rfloor + 1 \pm x_k : \ 1 \leq k \leq n\}$$
$$X = \bigcup_{n \geq 0} X_n.$$

The following lemma and theorem are exactly as [7, Lemma 3.2, Proposition 3.4 and Proposition 3.5], changing their hypothesis $\nu - x_1 \geq 1$ by $\nu - \lambda x_1 \geq 1$. For this reason, we will omit the proof.

Lemma 4.2 ([7, Lemma 3.2])**.** *Assume $\nu - \lambda x_1 \geq 1$ and $x_1 < \lfloor \alpha \rfloor + 1$. For each $n \geq 0$, if $x \in \mathcal{O}_n$ and $0 \ll x \ll 2\lfloor \alpha \rfloor + 2$, then $x \in X_n$.*

Theorem 4.3 ([7, Propositions 3.4 and 3.5])**.** *Assume $\nu - \lambda x_1 \geq 1$ and $x_1 < \lfloor \alpha \rfloor + 1$. The JR number of $\mathcal{O}$ is $\lfloor \alpha \rfloor + \alpha + 1$ and $\mathcal{O}$ satisfies the isolation property.*

The following proposition proves that there are infinitely many pairs (ν, x_0) for which Theorem 4.3 holds.

Proposition 4.4. *For any λ congruent to 1 or 3 modulo 4, there are infinitely many distinct values of α corresponding to pairs (ν, x_0) of rational integers such that*

(1) $\nu < x_0^2 - \lambda x_0$,

(2) $\sqrt{\nu + \lambda x_0}$ *is not a rational integer,*

(3) *For every $n \geq 1$, we have $[K_n \colon K_{n-1}] = 2$,*

(4) $\lambda^3 x_0 < \nu^2 - \lambda^2 \nu$,

(5) $\nu - \lambda x_1 \geq 1$,

(6) $\sqrt{\nu + \lambda x_0} < \lfloor \alpha \rfloor + 1$.

Proof. For any $\lambda \geq 1$ which is congruent to 1 or 3 modulo 4, we choose $\nu = 4\lambda^4$ and $x_0 = 2\lambda^2 + \lambda$. The first two conditions are immediate. The condition 3 holds by Proposition 2.13. The condition

4 holds because $\nu^2 > \lambda^2\nu + \lambda^3 x_0$ iff $16\lambda^4 > 4\lambda^2 + 2\lambda + 1$ which is true for all $\lambda \geq 1$. For the condition 5 we have

$$\nu - \lambda x_1 > \nu - \lambda x_0 = 4\lambda^4 - 2\lambda^3 - \lambda^2 \geq 1.$$

for each $\lambda \geq 1$. Finally, we have

$$\alpha = \frac{\lambda + \sqrt{\lambda^2 + 4\nu}}{2} = \frac{\lambda + \sqrt{16\lambda^4 + \lambda^2}}{2} = \frac{\lambda + 4\lambda^2 + \varepsilon}{2} = \frac{\lambda - 1}{2} + 2\lambda^2 + \frac{1}{2} + \frac{\varepsilon}{2}$$

for some $0 < \varepsilon < 1$. Since λ is congruent to 1 or 3 modulo 4, we have $\lfloor \alpha \rfloor = 2\lambda^2 + \frac{\lambda - 1}{2}$. Therefore, we have

$$(\lfloor \alpha \rfloor + 1)^2 = 4\lambda^4 + 2\lambda^3 + 2\lambda^2 + \left(\frac{\lambda + 1}{2}\right)^2 > 4\lambda^4 + 2\lambda^3 + \lambda^2 = \nu + \lambda x_0,$$

so the last condition is satisfied. □

For $\lambda = 1$, M. Castillo [1, Theorem 1] was able to remove the hypothesis $\nu - x_1 \geq 1$ and $x_1 < \lfloor \alpha \rfloor + 1$, and obtain the following theorem:

Theorem 4.5. *Assuming $\lambda = 1$ and $\nu > 3$, $\mathcal{O}$ has JR number $\lfloor \alpha \rfloor + \alpha + 1$ and it satisfies the isolation property.*

Now we will present some new results for $\lambda = 3$. The same proof can be easily adapted to the case $\lambda = 2, 4, 5 \ldots$ We could not find the general pattern that would let us write a general proof since for each value of λ there are cases that must be studied independently.

We will prove the following theorem at the end of this section.

Theorem 4.6. *Assume $\lambda = 3$. If $x_1 < \lfloor \alpha \rfloor + 1$ and $\nu \neq 19$, then $\mathcal{O}$ has JR number $\lfloor \alpha \rfloor + \alpha + 1$ and it satisfies the isolation property.*

Assumption 4.7. *For the following lemmas we assume that $\lambda = 3$.*

Lemma 4.8. *If $x_1 < \lfloor \alpha \rfloor + 1$ and $\nu \neq 19$, then $\nu - 3x_2 \geq 1$.*

Proof. Since $x_1 < \lfloor \alpha \rfloor + 1$, we have

$$\nu - 3x_2 > \nu - 3(\lfloor \alpha \rfloor + 1) \geq \nu - 3\alpha - 3.$$

Therefore, it suffices to prove $\nu - 3\alpha - 3 \geq 1$. This is satisfied if and only if $2\nu - 17 \geq 3\sqrt{9 + 4\nu}$, which is true for every $\nu \geq 24$. By Lemma 2.9, we have $\nu \geq 3$, so we must analyze the cases when $\nu \in \{3, \ldots, 23\}$. A simple calculation shows that for $\nu \in \{3, \ldots, 18\}$, there is no x_0 that satisfies the inequalities given in Theorem 2.7. Hence, $\nu \in \{19, \ldots, 23\}$, and again solving the inequalities given in Theorem 2.7, we obtain the following cases:

ν	x_0	x_1	x_2	$\nu - 3x_1$	$\nu - 3x_2$
19	7	$\sqrt{40}$	$\sqrt{19+3\sqrt{40}}$	0.03	0.51
20	7	$\sqrt{41}$	$\sqrt{20+3\sqrt{41}}$	0.79	1.21
	8	$\sqrt{44}$	$\sqrt{20+3\sqrt{44}}$	0.10	1.05
21	7	$\sqrt{42}$	$\sqrt{21+3\sqrt{42}}$	1.56	1.92
	8	$\sqrt{45}$	$\sqrt{21+3\sqrt{45}}$	0.88	1.76
	9	$\sqrt{48}$	$\sqrt{21+3\sqrt{48}}$	0.22	1.61
22	7	$\sqrt{43}$	$\sqrt{22+3\sqrt{43}}$	2.33	2.63
	8	$\sqrt{46}$	$\sqrt{22+3\sqrt{46}}$	1.65	2.48
	9	7	$\sqrt{22+3\sqrt{49}}$	1	2.33
	10	$\sqrt{52}$	$\sqrt{22+3\sqrt{52}}$	0.37	2.18
23	7	$\sqrt{44}$	$\sqrt{23+3\sqrt{44}}$	3.10	3.35
	8	$\sqrt{47}$	$\sqrt{23+3\sqrt{47}}$	2.43	3.20
	9	$\sqrt{50}$	$\sqrt{23+3\sqrt{50}}$	1.79	3.05
	10	$\sqrt{53}$	$\sqrt{23+3\sqrt{53}}$	1.16	2.91
	11	$\sqrt{56}$	$\sqrt{23+3\sqrt{56}}$	0.55	2.78

Table 1: Approximate values of $\nu - 3x_1$ and $\nu - 3x_2$ for $\nu \in \{19, \ldots, 23\}$. □

Lemma 4.9. *Let* $x \in \mathcal{O}_1$ *be such that* $0 \ll x \ll 2\lfloor\alpha\rfloor+2$. *If* $(\nu, x_0) \in \{(20,7),(20,8),(21,8),(21,9)\}$, *then* $x \in X_1$.

Proof. Let $x = a + bx_1 \in \mathcal{O}_1$, with $a, b \in \mathbb{Z}$, be such that $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$. Note that in all cases we have $\lfloor\alpha\rfloor + 1 = 7$ and $x_1 \geq \sqrt{41}$. Since $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$, by Lemma 2.15, we have $a \in \{1, \ldots, 13\}$ and

$$|b| < \frac{\lfloor\alpha\rfloor + 1}{x_1} \leq \frac{7}{\sqrt{41}},$$

so we have $b \in \{-1, 0, 1\}$. Finally, using a computer program (we used SageMath 9.2, see below) we can analyze all the cases to see that x is indeed in X_1. □

```
def cases_X1(x_0, nu, l):
    x_1=sqrt(nu+l*x_0)
    floor_alpha=math.floor((l+sqrt(l**2+4*nu))/2)
    for a in srange(1,2*floor_alpha+2,1):
        for b in [-1,0,1]:
            if 0<a+b*x_1<2*floor_alpha+2 and
            0<a-b*x_1<2*floor_alpha+2:
                print(a+b*x_1)
cases_X1(7,20,3)
cases_X1(8,20,3)
cases_X1(8,21,3)
cases_X1(9,21,3)
```

Lemma 4.10. *Let* $x \in \mathcal{O}_2$ *be such that* $0 \ll x \ll 2\lfloor\alpha\rfloor+2$. *If* $(\nu, x_0) \in \{(20,7),(20,8),(21,8),(21,9)\}$, *then* $x \in X_2$.

Proof. Let $x = a + bx_2 \in \mathcal{O}_2$, with $a, b \in \mathcal{O}_1$. Note that in all cases we have $x_1 \geq \sqrt{41}$, $x_2 \geq \sqrt{20+3\sqrt{41}}$ and $\lfloor\alpha\rfloor + 1 = 7$. Since $0 \ll a + bx_2 \ll 2\lfloor\alpha\rfloor + 2$, by Lemma 2.15 we have $0 \ll a \ll 2\lfloor\alpha\rfloor + 2$. Hence, $a \in \{1, \ldots, 13\} \cup \{7 \pm x_1\}$ by Lemma 4.9. We will prove that we have $\overline{|b|} < 1.2$. Assume, for the sake of contradiction, that this is not the case. We will see that for whatever choice of a, there is an embedding σ such that x^σ is either negative or larger than 14, contradicting our hypothesis.

- Assume first $a \in \{1,2,3,4,5,6\}$: We choose σ such that $x^\sigma = a - |b|x_2$, so that we have

$$x^\sigma = a - |b|x_2 \leq 6 - x_2 < 0.$$

- Assume $a \in \{8,9,10,11,12,13\}$: We choose σ such that $x^\sigma = a + |b|x_2$, so that we have

$$(a + bx_2)^\sigma = a + |b|x_2 \geq 8 + x_2 > 14.$$

- Assume $a = 7 + x_1$: We choose σ such that $x^\sigma = a + |b|x_2$, so that we have

$$a + \overline{|b|}x_2 \geq 7 + x_1 + x_2 > 14.$$

- Assume $a = 7 - x_1$: We choose σ such that $x^\sigma = a - |b|x_2$, so that we have

$$a - \overline{|b|}x_2 \leq 7 - x_1 - x_2 < 0.$$

- Assume $a = 7$. We choose σ such that $x^\sigma = a - \overline{|b|}x_2$, so that we have

$$a + |b|x_2 \geq 7 + 1.2x_2 \geq 7 + 1.2\sqrt{20+3\sqrt{41}} > 14.$$

We conclude $\overline{|b|} < 1.2$.

Write $b = b_1 + b_2x_1$, with $b_1, b_2 \in \mathbb{Z}$, so that

$$\overline{|b_1 + b_2x_1|} < 1.2.$$

Hence in particular, we have $|b_1| < 1.2$ and $|b_2| < \frac{1.2}{\sqrt{41}}$. The only choices for b_1 and b_2 are $(b_1, b_2) \in \{(-1,0),(0,0),(1,0)\}$. Therefore, if $a \in \{1,\ldots,6\} \cup \{8,\ldots,13\} \cup \{7 \pm x_1\}$, then $b = 0$ by the first four cases above. Otherwise, if $a = 7$, then we can have either $x = 7 - x_2$, or $x = 7 + x_2$, or $x = 7$. In all cases we obtain $x \in X_2$. ☐

Lemma 4.11. *Assume* $x_1 < \lfloor\alpha\rfloor+1$ *and* $\nu \neq 19$. *For each* $n \geq 0$, *if* $x \in \mathcal{O}_n$ *and* $0 \ll x \ll 2\lfloor\alpha\rfloor+2$, *then* $x \in X_n$.

Proof. If $\nu - 3x_1 \geq 1$, then we are done by Lemma 4.2. Assume $\nu - 3x_1 < 1$. By Lemma 4.8, the only cases where $\nu - 3x_1 < 1$ are when

$$(\nu, x_0) \in \{(20,7),(20,8),(21,8),(21,9),(22,10),(23,11)\}$$

(see Table 1). However, when $(\nu, x_0) \in \{(22,10),(23,11)\}$, a simple calculation shows that $x_1 > \lfloor\alpha\rfloor + 1$, so we may assume $(\nu, x_0) \in \{(20,7),(20,8),(21,8),(21,9)\}$. We will prove by induction on n that if $x \in \mathcal{O}_n$ is such that $0 \ll x \ll 2\lceil\alpha\rceil + 2$, then $x \in X_n$. It is clear for $n = 0$. For $n = 1$ and $n = 2$ we are done by Lemmas 4.9 and 4.10 respectively. Assume $n \geq 3$. By Lemmas 2.15 and 4.8 we have

$$|b^\sigma| < \frac{\lfloor\alpha\rfloor + 1}{\sqrt{\nu - 3x_{n-1}}} \leq \frac{\lfloor\alpha\rfloor + 1}{\sqrt{\nu - 3x_2}} \leq \lfloor\alpha\rfloor + 1$$

for every $n \geq 3$. The rest of the proof goes exactly as the proof of [7, Lemma 3.2]. □

Lemma 4.12. *Assume* $x_1 < \lfloor\alpha\rfloor + 1$ *and* $\nu \neq 19$. *Let* $x \in \mathcal{O}$. *We have* $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$ *if and only if* $x \in X$.

Proof. By Lemma 4.11, we need only to prove the lemma from right to left. Let $x \in X$. If $x \in X_0$, then there is nothing to prove. Assume $x \in X_n$ for some $n \geq 1$, so that $x = \lfloor\alpha\rfloor + 1 \pm x_k$ for some $1 \leq k \leq n$. Since the sequence (x_n) is decreasing, we have

$$\lfloor\alpha\rfloor + 1 + x_k < 2\lfloor\alpha\rfloor + 2,$$

and

$$\lfloor\alpha\rfloor + 1 - x_k > 0$$

for every $1 \leq k \leq n$. Therefore, we have $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$ since $\overline{|x_k|} = x_k$ by Theorem 2.7. □

Proof Theorem 4.6. We will prove that $\lfloor\alpha\rfloor + \alpha + 1$ is the JR number of $\mathcal{O}$ and that it satisfies the isolation property. Since (x_n) is a decreasing sequence and converges to α, for every $\varepsilon > 0$ there exist infinitely many n such that

$$x_n + \lfloor\alpha\rfloor + 1 < \lfloor\alpha\rfloor + \alpha + 1 + \varepsilon.$$

So, by Lemma 4.12 and Theorem 2.7, for every $\varepsilon > 0$, there exist infinitely many $x \in \mathcal{O}$ such that $0 \ll x \ll \lfloor\alpha\rfloor + \alpha + 1 + \varepsilon$. Also, for each $n \geq 1$, we have $\lfloor\alpha\rfloor + 1 + x_n > \lfloor\alpha\rfloor + 1 + \alpha$. Hence, if $x \in \mathcal{O}$ is such that $0 \ll x \ll \lfloor\alpha\rfloor + \alpha + 1$, by Lemma 4.12, then we have $x \in \{1, \ldots, 2\lfloor\alpha\rfloor + 1\}$.

Therefore, $\lfloor \alpha \rfloor + \alpha + 1$ is the JR number of $\mathcal{O}$, and it is not a minimum. We now show that it satisfies the isolation property. Let $M = \lfloor \alpha \rfloor + 1 - \alpha$ and $x \in \mathcal{O}$ be such that

$$0 \ll x \ll \mathrm{JR}(\mathcal{O}) + M = 2\lfloor \alpha \rfloor + 2.$$

By Lemma 4.12, we have

$$x \in \{1, 2, \ldots, 2\lfloor \alpha \rfloor + 1\} \cup \{\lfloor \alpha \rfloor + 1 \pm x_n : n \geq 1\}$$

and since (x_n) is decreasing with limit α, we have

$$\lfloor \alpha \rfloor + 1 + x_n \geq \lfloor \alpha \rfloor + 1 + \alpha + \varepsilon$$

for only finitely many n. □

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Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 26, no. 3, pp. 407–430, December 2024
DOI: 10.56754/0719-0646.2603.407

Fractional Sobolev space: Study of Kirchhoff-Schrödinger systems with singular nonlinearity

Elhoussain Arhrrabi[1,2]
Hamza El-Houari[1,✉]

[1] *Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Faculty of Sciences and Technology, Béni Mellal, Morocco.*
`arhrrabi.elhoussain@gmail.com`

[2] *ENSI - School of New Sciences and Engineering, Tangier, Morocco.*
`h.elhouari94@gmail.com`✉

ABSTRACT

This study extensively investigates a specific category of Kirchhoff-Schrödinger systems in fractional Sobolev space with Dirichlet boundary conditions. The main focus is on exploring the existence and multiplicity of non-negative solutions. The non-linearity of the problem generally exhibits singularity. By employing minimization arguments involving the Nehari manifold and a variational approach, we establish the existence and multiplicity of positive solutions for our problem with respect to the parameters η and ζ in suitable fractional Sobolev spaces. Our key findings are novel and contribute significantly to the literature on coupled systems of Kirchhoff-Schrödinger system with Dirichlet boundary conditions.

RESUMEN

Este estudio investiga en detalle una categoría específica de sistemas de Kirchhoff-Schrödinger en espacios de Sobolev fraccionarios con condiciones de borde de Dirichlet. El objetivo principal es explorar la existencia y multiplicidad de soluciones no-negativas. La no-linealidad del problema generalmente exhibe singularidades. Empleando argumentos de minimización que involucran la variedad de Nehari y un enfoque variacional, establecemos la existencia y multiplicidad de soluciones positivas para nuestro problema con respecto a los parámetros η y ζ en espacios de Sobolev fraccionarios apropiados. Nuestros hallazgos principales son novedosos y contribuyen significativamente a la literatura de sistemas de Kirchhoff-Schrödinger acoplados con condiciones de borde de Dirichlet.

Keywords and Phrases: Fractional p-Laplacian operator, Kirchhoff-Schrödinger system, Nehari manifold, fibering map approach.

2020 AMS Mathematics Subject Classification: 35R11, 35A15, 47G20.

Published: 08 October, 2024
Accepted: 04 September, 2024
Received: 18 January, 2024

1 Introduction

The Kirchhoff-Schrödinger problem is a class of partial differential equations that combines aspects of Kirchhoff and Schrödinger equations. It takes the form:

$$M\left(\int_Q |\nabla u|^2 dx\right)(-\Delta u) + V(x)u = f(x,u),$$

where M is a function representing the Kirchhoff-type nonlinearity, $V(x)$ is a potential function, and $f(x,u)$ denotes the nonlinearity in the system. This type of system generalizes the classical Schrödinger equation by incorporating a nonlinear term dependent on the integral of the gradient, reflecting the influence of the entire domain on the local behavior of the solution.

The Kirchhoff-Schrödinger system arises in various physical contexts, such as the study of quantum mechanical systems, nonlinear optics, and the dynamics of elastic strings and membranes. These systems are particularly challenging due to their nonlocal nature and the potential presence of singularities in the nonlinearity $f(x,u)$, which can complicate both theoretical analysis and numerical simulations.

In this study, we consider the following fractional Kirchhoff-Schrödinger equations with singular nonlinearity,

$$\begin{cases} \mathbf{K}\left(\int_Q V(\kappa)|w|^p d\kappa + \int_{Q\times Q}\frac{|w(\kappa)-w(y)|^p}{|\kappa-y|^{d+sp}}d\kappa\, dy\right)\left[(-\Delta)_p^s w + V(\kappa)|w|^{p-2}w\right] \\ \qquad = \eta\alpha(\kappa)|w|^{q-2}u + \dfrac{1-\varrho}{2-\varrho-\tau}\xi(\kappa)|w|^{-\varrho}|v|^{1-\tau}, \qquad \text{in } Q, \\ \mathbf{K}\left(\int_Q V(\kappa)|v|^p d\kappa + \int_{Q\times Q}\frac{|v(\kappa)-v(y)|^p}{|\kappa-y|^{d+sp}}d\kappa\, dy\right)\left[(-\Delta)_p^s v + V(\kappa)|v|^{p-2}v\right] \\ \qquad = \zeta\beta(\kappa)|v|^{q-2}v + \dfrac{1-\tau}{2-\varrho-\tau}\xi(\kappa)|w|^{1-\varrho}|v|^{-\tau}, \qquad \text{on } Q, \\ w = v = 0, \qquad \text{on } \mathbb{R}^d\setminus Q, \end{cases} \tag{1.1}$$

where $Q\subset\mathbb{R}^d$ $(d\geq 3)$ is a bounded domain with smooth boundary, $s\in(0,1)$, $0<\tau<1$, $0<\varrho<1$, $d>ps$, $2-\varrho-\tau<p\leq p\sigma<q<p_s^*=\frac{dp}{d-sp}$, $\alpha,\beta,\xi\in C(\overline{Q})$ are non-negative weight functions, η,ζ are two parameters, $(-\Delta)_p^s$ is the fractional p-Laplacian operator defined as (see [10])

$$(-\Delta)_p^s w(\kappa) = 2\lim_{\epsilon\searrow 0}\int_{Q\setminus B_\epsilon}\frac{|w(\kappa)-w(y)|^{p-2}(w(\kappa)-w(y))}{|\kappa-y|^{d+sp}}\,\mathrm{d}y, \quad \kappa\in\mathbb{R}^d,$$

and $\mathbf{K}:(0,+\infty)\longrightarrow(0,+\infty)$ is the continuous Kirchhoff function defined by

$$\mathbf{K}(t) = k + lt^{\sigma-1} \quad \text{with } k>0,\ l,\ \sigma\geq 1. \tag{1.2}$$

Recently, there has been a lot of interest in examining non local problems of this kind. For an interesting one, we refer to learn more about Kirchhoff problems, specifically those dealing with the Laplace operator and a singular term, in references like [19–22]. Additionally, the study of the fractional Kirchhoff problem, which involves a singular term like $u^{-\gamma}$, can be found in [14]. This research combines a variational approach with a specific truncation argument. For more details on the fractional system, you can check out [23, 36].

These problems involve studying how things spread unevenly in complicated environments. This happens because of random movements, like jumps, where entities can move to nearby places or make longer trips using a specific kind of flight pattern called Lévy flights. These issues are also used to model things like turbulence, chaotic movements, plasma physics, and financial dynamics. Check [1, 7] and references therein for more information.

The system expressed in (1.1) without a Kirchhoff function and potential function has been thoroughly explored in recent years. For the case involving the fractional p-Laplacian, the existence results have been investigated using Morse theory, as discussed in [18]. Perera-Squassina-Yang [25] introducing a novel abstract result based on a pseudo-index associated with the $\mathbb{Z}_2$-cohomological index. These constraints are employed to establish the existence within a certain range of the Palais-Smale condition. It is worth noting that, in this study, bifurcation and multiplicity results are obtained with specific limitations on the parameter η. Additionally, the investigation into the multiplicity of solutions is conducted through the Nehari manifold and fibering maps in works like [6, 15, 29, 31].

In a distinct context, the investigation of the problem was undertaken in [6].

$$\begin{cases} (-\Delta)_p^s u = \eta|w|^{q-2}u + \dfrac{2\varrho}{\varrho+\tau}|w|^{\varrho-2}u|v|^{\tau}, \quad \text{in } Q, \\ \\ (-\Delta)_p^s v = \zeta|v|^{q-2}v + \dfrac{2\tau}{\varrho+\tau}|w|^{\varrho}|v|^{\tau-2}v, \quad \text{in } Q, \\ \\ u = v = 0, \quad \text{on } \mathbb{R}^d \setminus Q, \end{cases}$$

where Q is a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂Q, $d > sp$, $s \in (0,1)$, $p < \varrho+\tau < p_s^*$, η, ζ are two parameters. The scholars investigated the Nehari manifold associated with the problem, employing fibering maps, and established the existence of solutions under certain conditions for the parameter pair (η, ζ).

The problem expressed in (1.1) without a Kirchhoff coefficient has been thoroughly explored in recent years. For the case involving the fractional p-Laplacian, the existence results have been investigated using Morse theory, as discussed in [24]

$$\begin{cases} (-\Delta)_p^s u + |w|^{q-2}u = \dfrac{H_w(\kappa, u, v)}{|\kappa|^\gamma}, & \text{in } \mathbb{R}^d, \\ \\ (-\Delta)_p^s v + |w|^{q-2}u = \dfrac{H_w(\kappa, u, v)}{|\kappa|^\gamma} & \text{in } \mathbb{R}^d, \end{cases}$$

where $d \geq 1$, $0 < s < 1$, $d = ps$, $\gamma \in (0, d)$ and H has exponential growth. By using a version of the mountain pass theorem without (PS) condition, they established the existence of nontrivial solution to the above system. In [35] the authors studied the existence of solutions to the following quasi linear Schrödinger system

$$\begin{cases} (-\Delta)_p^s u + \alpha(\kappa)|w|^{q-2}u = H_w(\kappa, u, v) & \text{in } \mathbb{R}^d, \\ \\ (-\Delta)_p^s v + \beta(\kappa)|w|^{q-2}u = H_w(\kappa, u, v) & \text{in } \mathbb{R}^d, \end{cases}$$

where $1 < q \leq p$, $sp < d$, they used the critical approach, to obtain the existence of nontrivial and non negative solutions for the above system.

Following this, the issue has been explored by various authors in the context of Laplacian, p-Laplacian, and fractional N-Laplacian operators, employing either the technique employed in this paper or employing critical point methods. Noteworthy references encompass [2, 4, 5, 9, 12, 16, 28, 30, 34].

Motivated by the results above, by using minimization arguments and implicit function theorem together with variational approach, we prove the existence and multiplicity of nontrivial, non-negative solutions for the singular fractional Kirchhoff-Schrödinger system described in (1.1) within suitable fractional Sobolev spaces.

This paper is organized as follow: In the second section, we discuss familiar properties and results related to fractional Sobolev spaces. In the third section, we show the existence theorem and its proof, which uses the Nehari manifold and fibering map approach. In the fourth section, we demonstrate the existence of multiple nontrivial positive solutions for our problem (1.1).

2 Preliminaries

In this paper, $Q \subset \mathbb{R}^d$ represents a bounded domain with a smooth boundary, and $\langle \cdot, \cdot \rangle$ denotes the standard duality between X and its dual space X^*.

Let $u : Q \times Q \longrightarrow \mathbb{R}$ be a measurable function,

$$[w]_{s,p} = \left(\int_{Q \times Q} \frac{|w(\kappa) - w(y)|^p}{|\kappa - y|^{d+ps}} d\kappa \, dy \right)^{1/p},$$

is the Gagliardo seminorm. We denote by $\mathbb{W}^{s,p}(Q)$ the fractional Sobolev space given by

$$\mathbb{W}^{s,p}(Q) := \{u \in \mathbb{L}^p(Q) : \; [w]_{s,p} < \infty\},$$

with the norm

$$\|w\|_{s,p} := \left(\|w\|^p_{\mathbb{L}^p(Q)} + [w]^p_{s,p} \right)^{1/p},$$

where

$$\|w\|_{\mathbb{L}^p(Q)} = \left(\int_Q |w|^p d\kappa \right)^{1/p}.$$

For our analysis, we assume the following assumption:

(V) $V \in \mathbb{L}^\infty_{loc}(Q) \backslash \{0\}$, ess $\inf_{\kappa \in Q} V(\kappa) > 0$ and meas$(\{x \in q : V(x) \leq L\}) < \infty$, for all $L > 0$,

where meas$(\cdot)$ denotes the Lebesgue measure in Q.

When V satisfies **(V)**, the basic space

$$\mathbb{W}_s(Q) := \left\{ w \in \mathbb{W}^{s,p}(Q) : \; V|w|^p \in \mathbb{L}^1(Q); \quad u = 0 \text{ in } \mathbb{R}^d \backslash Q \right\}$$

denotes the completion of $C_0^\infty(Q)$ with respect to the norm

$$\|w\|_{\mathbb{W}_s} := \left(\|w\|^p_{\mathbb{L}^p(V,Q)} + [w]^p_{s,p} \right)^{1/p},$$

where

$$\|w\|_{\mathbb{L}^p(V,Q)} = \left(\int_Q V(\kappa)|w|^p d\kappa \right)^{1/p}.$$

In $\mathbb{W}_s$ we have the following embedding

Lemma 2.1 ([33]). *Let $0 < s < 1 < p < +\infty$ with $ps < d$ and suppose that the assumption **(V)** holds. Then,*

$$\mathbb{W}_s(Q) \hookrightarrow \mathbb{L}^q(Q) \quad \textit{for all } q \in [p, p_s^*). \tag{2.1}$$

When $r + r' \in (p, p^*)$, then, for any $u \in \mathbb{W}_s$, we obtain

$$\|w\|_{\mathbb{L}^{r+r'}(Q)} \leq S\|w\|_{\mathbb{W}_s}. \tag{2.2}$$

Let us define the functional $\Psi_{s,p} : \mathbb{W}_s \longrightarrow \mathbb{R}$ by

$$\Psi_{s,p}(w) = \int_{Q\times Q} \frac{|w(\kappa) - w(y)|^p}{|\kappa - y|^{d+ps}} d\kappa\, dy + \int_Q V(\kappa)|w|^p d\kappa.$$

At this point, we introduce our working space $\mathbb{W} = \mathbb{W}_s \times \mathbb{W}_s$, which is a reflexive Banach space endowed with the norm

$$\|(w, v)\|_{\mathbb{W}} = \Big(\Psi_{s,p}(w) + \Psi_{s,p}(v)\Big)^{1/p}. \tag{2.3}$$

We say that $(w, v) \in \mathbb{W}$ is a weak solution to system (1.1) if $u, v > 0$ in Q, one has

$$\begin{aligned}
&\mathbf{K}\big(\|w\|_{\mathbb{W}_s}\big)\left(\int_Q V(\kappa)|w|^{p-2}u\phi d\kappa + \int_Q \frac{|w(\kappa) - w(y)|^{p-2}(w(\kappa) - w(y))(\phi(\kappa) - \phi(y))}{|\kappa - y|^{d+sp}}\, d\kappa\, dy\right)\\
&+ \mathbf{K}\big(\|v\|_{\mathbb{W}_s}\big)\left(\int_Q V(\kappa)|v|^{p-2}v\phi d\kappa + \int_Q \frac{|v(\kappa) - v(y)|^{p-2}(v(\kappa) - v(y))(\psi(\kappa) - \psi(y))}{|\kappa - y|^{d+sp}}\, d\kappa\, dy\right)\\
&= \int_Q \big(\eta\alpha(\kappa)|w|^{q-2}u\phi + \zeta\beta(\kappa)|v|^{q-2}v\phi\big)d\kappa + \frac{1-\varrho}{2-\varrho-\tau}\int_Q \xi(\kappa)u^{-\varrho}v^{1-\tau}\psi\, d\kappa\\
&\quad + \frac{1-\tau}{2-\varrho-\tau}\int_Q \xi(\kappa)u^{1-\varrho}v^{-\tau}\psi\, d\kappa,
\end{aligned}$$

for all $(\phi, \psi) \in \mathbb{W}$.

Now, with the essential tools in place, we are ready to state our main results, which take the following form:

Theorem 2.2. *There exists*

$$\Lambda_0 = \left(\frac{q+\varrho+\tau-2}{\|\zeta\|_\infty k(q-p)}\right)^{\frac{p}{p+\varrho+\tau-2}} \left(\frac{2-\varrho-\tau-q}{k(2-\varrho-\tau-p)}|Q|^{\frac{p_s^*-q}{p_s^*}}\right)^{-\frac{p}{p-q}} S^{\frac{2-\varrho-\tau}{p+\varrho+\tau-2}},$$

such that if

$$0 < \big(\eta\|\alpha\|_\infty\big)^{\frac{p}{p-q}} + \big(\zeta\|\beta\|_\infty\big)^{\frac{p}{p-q}} < \Lambda_0,$$

then system (1.1) *has at least two nontrivial positive solutions.*

3 Nehari manifold & fibering map analysis

In this part, we gather basic information about a Nehari manifold and discuss fibering maps.

Obviously, the energy functional $\mathfrak{J}_{\eta,\zeta} : \mathbb{W}_s \longrightarrow \mathbb{R}$ associated with problem (1.1) is given by

$$\mathfrak{J}_{\eta,\zeta}(w,v) = \frac{1}{p}\Big(\hat{\mathbf{K}}(||w||^p_{\mathbb{W}_s}) + \hat{\mathbf{K}}(||v||^p_{\mathbb{W}_s})\Big) - \frac{1}{q}\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)d\kappa \\ - \frac{1}{2-\varrho-\tau}\int_Q \xi(\kappa)(w^+)^{1-\varrho}(v^+)^{1-\tau}\, d\kappa,$$

where $\hat{\mathbf{K}}(t) = \displaystyle\int_0^t \mathbf{K}(\varrho)d\varrho$. This together with (1.2) gets to

$$\begin{aligned}\mathfrak{J}_{\eta,\zeta}(w,v) &= \frac{k}{p}\Big(\Psi_{s,p}(w) + \Psi_{s,p}(v)\Big)^p + \frac{l}{p\sigma}\Big(\Psi_{s,p}(w) + \Psi_{s,p}(v)\Big)^{p\sigma} \\ &\quad - \frac{1}{q}\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)d\kappa - \frac{1}{2-\varrho-\tau}\int_Q \xi(\kappa)(w^+)^{1-\varrho}(v^+)^{1-\tau}\, d\kappa \\ &= \frac{k}{p}\|(w,v)\|^p_{\mathbb{W}} + \frac{l}{p\sigma}\|(w,v)\|^{p\sigma}_{\mathbb{W}} - \frac{1}{q}\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)d\kappa \\ &\quad - \frac{1}{2-\varrho-\tau}\int_Q \xi(\kappa)(w^+)^{1-\varrho}(v^+)^{1-\tau}\, d\kappa, \end{aligned} \tag{3.1}$$

where $r^+ = \max\{r, 0\}$ and $r^- = \max\{-r, 0\}$ for $r \in \mathbb{R}$.

Keep in mind that $\mathfrak{J}_{\eta,\zeta}$ does not behave smoothly in $\mathbb{W}$. So, standard variational methods will not work here. If (w,v) is a weak solution for the problem (1.1), it means that both w and v are positive in Q and satisfy the equation

$$\mathbf{K}\big(\|w\|_{\mathbb{W}_s}\big)\Psi_{s,p}(w) + \mathbf{K}\big(\|v\|_{\mathbb{W}_s}\big)\Psi_{s,p}(\mathbf{V}) - \eta\int_Q \alpha(\kappa)|w|^q\, d\kappa \\ - \zeta\int_Q \beta(\kappa)|v|^q\, d\kappa - \int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\, d\kappa = 0,$$

which implies by using (1.2) that

$$k\|(w,v)\|^p_{\mathbb{W}} + l\|(w,v)\|^{p\sigma}_{\mathbb{W}} - \eta\int_Q \alpha(\kappa)|w|^q\, d\kappa - \zeta\int_Q \beta(\kappa)|v|^q\, d\kappa - \int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\, d\kappa = 0. \tag{3.2}$$

It is simple to confirm that the energy functional $\mathfrak{J}_{\eta,\zeta}(w,v)$ is not bounded below in the space $\mathbb{W}$. However, we will demonstrate that on the Nehari manifold, defined below, $\mathfrak{J}_{\eta,\zeta}(w,v)$ is bounded below. We will establish a solution by minimizing this functional over specific subsets. The Nehari manifold is defined as follows:

$$\mathbf{N}_{\eta,\zeta}=\left\{(w,v)\in\mathbb{W}\setminus\{(0,0)\};\ \frac{k}{p}\|(w,v)\|_{\mathbb{W}}^{p}+\frac{l}{k}\|(w,v)\|_{\mathbb{W}}^{p\sigma}-\eta\int_Q\alpha(\kappa)|w|^q\,d\kappa\right.$$
$$\left.-\zeta\int_Q\beta(\kappa)|v|^q\,d\kappa-\int_Q\xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa=0\right\}.$$

Now, understanding that the Nehari manifold is intricately connected to a fibering maps which is introduced by Drábek and Pohozaev in [11]. The form of the fibrering maps is as follows, $\Upsilon_{w,v}:t\mapsto\mathfrak{J}_{\eta,\zeta}(tw,tv)$ for $t>0$ defined by

$$\Upsilon_{w,v}(t)=\frac{1}{p}\left(\hat{\mathbf{K}}\big(t^P||w||_{\mathbb{W}_s}^p\big)+\hat{\mathbf{K}}\big(t^p||v||_{\mathbb{W}_s}^p\big)\right)-\frac{t^q}{q}\int_Q\left(\eta\alpha(\kappa)|w|^q+\zeta\beta(\kappa)|v|^q\right)d\kappa$$
$$-\frac{t^{2-\varrho-\tau}}{2-\varrho-\tau}\int_Q\xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa.$$

The first and second derivative of Υ respectively, is given by

$$\begin{aligned}\Upsilon'_{w,v}(t)&=kt^{p-1}\|(w,v)\|_{\mathbb{W}}^{p}+lt^{p\sigma-1}\|(w,v)\|_{\mathbb{W}}^{p\sigma}-t^{q-1}\int_Q\left(\eta\alpha(\kappa)|w|^q+\zeta\beta(\kappa)|v|^q\right)d\kappa\\&\quad-t^{1-\varrho-\tau}\int_Q\xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}\Upsilon''_{w,v}(t)&=(p-1)kt^{p-2}\|(w,v)\|_{\mathbb{W}}^{p}+l(p\sigma-1)t^{p\sigma-2}\|(w,v)\|_{\mathbb{W}}^{p\sigma}\\&\quad-(q-1)t^{q-2}\int_Q\left(\eta\alpha(\kappa)|w|^q+\zeta\beta(\kappa)|v|^q\right)d\kappa\\&\quad-(1-\varrho-\tau)t^{-\varrho-\tau}\int_Q\xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa.\end{aligned}\tag{3.4}$$

Now, we prove some useful inequality. Using Hölder's and Sobolev inequalities, one has

$$\begin{aligned}\int_Q\left(\eta\alpha(\kappa)|w|^q+\zeta\beta(\kappa)|v|^q\right)d\kappa&\le|Q|^{\frac{p_s^*-q}{p_s^*}}\left(\eta\|\alpha\|_\infty\|w\|_{p_s^*}^q+\zeta\|\beta\|_\infty\|v\|_{p_s^*}^q\right)\\&\le|Q|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\left(\eta\|\alpha\|_\infty\|w\|^q+\zeta\|\beta\|_\infty\|v\|^q\right)\\&\le|Q|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}}+\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}(\|w\|^q+\|v\|^q)\\&\le C|Q|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}}+(\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\|(w,v)\|_{\mathbb{W}}^q\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}\int_Q\xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa&\le\|\zeta\|_\infty\left(\frac{1-\varrho}{2-\varrho-\tau}\int_Q|w|^{2-\varrho-\tau}\,d\kappa+\frac{1-\tau}{2-\varrho-\tau}\int_Q|v|^{2-\varrho-\tau}\,d\kappa\right)\\&\le\|\zeta\|_\infty S^{-\frac{2-\varrho-\tau}{p}}\|(w,v)\|_{\mathbb{W}}^{2-\varrho-\tau}.\end{aligned}\tag{3.6}$$

Lemma 3.1. *Let $(w,v) \in \mathbb{W} \setminus \{(0,0)\}$. Then $(tw,tv) \in \mathbf{N}_{\eta,\zeta}$ if and only if $\Upsilon'_{w,v}(t) = 0$.*

Proof. The conclusion is derived from the observation that

$$\begin{aligned}\Upsilon'_{w,v}(t) &= \langle \mathfrak{J}'_{\eta,\zeta}(w,v),(w,v)\rangle \\ &= kt^{p-1}\|(w,v)\|^p_{\mathbb{W}} + lt^{p\sigma-1}\|(w,v)\|^{p\sigma}_{\mathbb{W}} - t^{q-1}\left(\int_Q \eta\alpha(\kappa)|w|^q\,d\kappa - \int_Q \zeta\beta(\kappa)|v|^q\,d\kappa\right) \\ &\quad - t^{1-\varrho-\tau}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa = 0\end{aligned}$$

if and only if $(tw,tv) \in \mathbf{N}_{\eta,\zeta}$. □

Due to Lemma 3.1, we have $(w,v) \in \mathbf{N}_{\eta,\zeta}$ are associated with stationary points of $\Upsilon_{w,v}(tw,tv)$ and in particular, $(w,v) \in \mathbf{N}_{\eta,\zeta}$ if and only if $\Upsilon'_{w,v}(1) = 0$. Hence, we split $\mathbf{N}_{\eta,\zeta}$ into three parts:

$$\begin{aligned}\mathbf{N}^+_{\eta,\zeta} &= \Big\{(w,v) \in \mathbf{N}_{\eta,\zeta} : \Upsilon''_{w,v}(1) > 0\Big\} = \Big\{(tw,tv) \in \mathbb{W} \setminus \{0,0\} : \Upsilon'_{w,v}(t) = 0, \Upsilon''_{w,v}(t) > 0\Big\}, \\ \mathbf{N}^-_{\eta,\zeta} &= \Big\{(w,v) \in \mathbf{N}_{\eta,\zeta} : \Upsilon''_{w,v}(1) < 0\Big\} = \Big\{(tw,tv) \in \mathbb{W} \setminus \{0,0\} : \Upsilon'_{w,v}(t) = 0, \Upsilon''_{w,v}(t) < 0\Big\}, \\ \mathbf{N}^0_{\eta,\zeta} &= \Big\{(w,v) \in \mathbf{N}_{\eta,\zeta} : \Upsilon''_{w,v}(1) = 0\Big\} = \Big\{(tw,tv) \in \mathbb{W} \setminus \{0,0\} : \Upsilon'_{w,v}(t) = 0, \Upsilon''_{w,v}(t) = 0\Big\}.\end{aligned}$$

For the proof of the following lemma we refer to [32].

Lemma 3.2. *If (w,v) is a minimizer of $\mathfrak{J}_{\eta,\zeta}$ on $\mathbf{N}_{\eta,\zeta}$ such that $(w,v) \notin \mathbf{N}^0_{\eta,\zeta}$. Then, (w,v) is a critical point for $\mathfrak{J}_{\eta,\zeta}$.*

Our initial result is as follows:

Lemma 3.3. *$\mathfrak{J}_{\eta,\zeta}$ is bounded below on $\mathbf{N}_{\eta,\zeta}$ and coercive.*

Proof. As $(w,v) \in \mathbf{N}_{\eta,\zeta}$, then using (3.2) and the embedding of $\mathbb{W}_s$ in $\mathbb{L}^{2-\varrho-\tau}(Q)$, we obtain

$$\mathfrak{J}_{\eta,\zeta}(w,v) = k\left(\frac{1}{p} - \frac{1}{q}\right)\|(w,v)\|^p_{\mathbb{W}} + l\Big(\frac{1}{p\sigma} - \frac{1}{q}\Big)\|(w,v)\|^{p\sigma}_{\mathbb{W}} - \left(\frac{1}{2-\varrho-\tau} - \frac{1}{q}\right)\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa.$$

Then by (3.6), we obtain

$$\begin{aligned}\mathfrak{J}_{\eta,\zeta}(w,v) \geq{}& k\left(\frac{1}{p} - \frac{1}{q}\right)\|(w,v)\|^p_{\mathbb{W}} + l\left(\frac{1}{p\sigma} - \frac{1}{q}\right)\|(w,v)\|^{p\sigma}_{\mathbb{W}} \\ & - \left(\frac{1}{2-\varrho-\tau} - \frac{1}{q}\right)\|\zeta\|_\infty S^{-\frac{2-\varrho-\tau}{2}}\|(w,v)\|^{2-\varrho-\tau}_{\mathbb{W}}.\end{aligned}$$

Since $2-\varrho-\tau < p \leq p\sigma$, it follows that $\mathfrak{J}_{\eta,\zeta}$ is coercive and bounded below on $\mathbf{N}_{\eta,\zeta}$. □

Lemma 3.4. *For every* $(w,v)\in \mathbf{N}^{-}_{\eta,\zeta}$ *(respectively* $\mathbf{N}^{+}_{\eta,\zeta}$*) with* $u,v\geq 0$*, and all* $(\phi,\psi)\in \mathbf{N}_{\eta,\zeta}$ *with* $(\phi,\psi)\geq 0$*, there exist* $\varepsilon>0$ *and a continuous function* $h=h(r)>0$ *such that for all* $r\in\mathbb{R}$ *with* $|r|<\varepsilon$ *we have* $h(0)=1$ *and* $h(r)(w+r\phi,v+r\psi)\in \mathbf{N}^{-}_{\eta,\zeta}$ *(respectively* $\mathbf{N}^{+}_{\eta,\zeta}$*).*

Proof. First, let us introduce the function $f:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ defined by

$$\begin{aligned}
f(t,r)&=kt^{p+\varrho+\tau-2}\|(w+r\phi,v+r\psi)\|^{p}+lt^{p\sigma+\varrho+\tau-2}\|(w+r\phi,v+r\psi)\|^{p\sigma}\\
&\quad-(q+\varrho+\tau-2)t^{q+\varrho+\tau-3}\int_Q\Big(\eta\alpha(\kappa)(w+r\phi)^q+\zeta\beta(\kappa)(v+r\psi)^q\Big)d\kappa\\
&\quad-\int_Q\xi(\kappa)(w+r\phi)^{1-\varrho}(v+r\psi)^{1-\tau}d\kappa.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{df}{dt}(t,r)&=k\big(p+\varrho+\tau-\big)t^{p+\varrho+\tau-3}\|(w+r\phi,v+r\psi)\|^{p}\\
&\quad+l\big(p\sigma+\varrho+\tau-2\big)t^{p\sigma+\varrho+\tau-3}\|(w+r\phi,v+r\psi)\|^{p\sigma}\\
&\quad-t^{q+\varrho+\tau-2}\int_Q\Big(\eta\alpha(\kappa)(w+r\phi)^q+\zeta\beta(\kappa)(v+r\psi)^q\Big)d\kappa.
\end{aligned}$$

Hence, $\frac{df}{dt}$ is continuous. Recall that $(w,v)\in\mathbf{N}^{-}_{\eta,\zeta}\subset\mathbf{N}_{\eta,\zeta}$, we have $f(1,0)=0$, and

$$\begin{aligned}
\frac{df}{dt}(1,0)&=k\big(p+\varrho+\tau-2\big)\|(w,v)\|^{p}_{\mathbb{W}}+l\big(p\sigma+\varrho+\tau-2\big)\|(w,v)\|^{p\sigma}_{\mathbb{W}}\\
&\quad-\big(q+\varrho+\tau-2\big)\int_Q\Big(\eta\alpha(\kappa)w^q+\zeta\beta(\kappa)v^q\Big)d\kappa<0.
\end{aligned}$$

Thus, by applying the implicit function theorem to the function f at the point $(1,0)$, we deduce the existence of $\delta>0$ and a positive continuous function $h=h(r)>0$, defined for $r\in\mathbb{R}$ with $|r|<\delta$, satisfying:

$$h(0)=1\quad\text{and}\quad h(r)(w+r\phi,v+r\psi)\in\mathbf{N}_{\eta,\zeta},\quad\text{for all } r\in\mathbb{R},\quad |r|<\delta.$$

Hence, for a small possible $\varepsilon>0$ $(\varepsilon<\delta)$, we obtain

$$h(r)(w+r\phi,v+r\psi)\in\mathbf{N}^{-}_{\eta,\zeta},\quad\forall r\in\mathbb{R},\ |r|<\varepsilon.$$

Similarly, we prove the other case. □

Lemma 3.5. *There exists*

$$\Lambda_0 = \left(\frac{q+\varrho+\tau-2}{\|\zeta\|_\infty k(q-p)}\right)^{\frac{p}{p+\varrho+\tau-2}} \left(\frac{2-\varrho-\tau-q}{k(2-\varrho-\tau-p)}|Q|^{\frac{2_s^*-q}{2_s^*}}\right)^{-\frac{p}{p-q}} S^{\frac{2-\varrho-\tau}{p+\varrho+\tau-2}},$$

such that for $0 < \left(\eta\|\alpha\|_\infty\right)^{\frac{p}{p-q}} + \left(\zeta\|\beta\|_\infty\right)^{\frac{p}{p-q}} < \Lambda_0$ *we have:*

(i) If $\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\,d\kappa > 0$, *then, there exist a unique* $T_l > 0$ *and* $t_0 < T_l < t_1$ *such that*

$$\Upsilon_{w,v}(t_0) = \Upsilon_{w,v}(t_1),$$
$$\Upsilon'_{w,v}(t_0) < 0 < \Upsilon'_{w,v}(t_1);$$

that is, $(t_0w, t_0v) \in \mathbf{N}^+_{\eta,\zeta}$, $(t_1w, t_1v) \in \mathbf{N}^-_{\eta,\zeta}$ *and*

$$\mathfrak{J}_{\eta,\zeta}(t_0w, t_0v) = \min_{0\le t\le t_1} \mathfrak{J}_{\eta,\zeta}(tw, tv),$$
$$\mathfrak{J}_{\eta,\zeta}(t_1w, t_1v) = \max_{t\ge T_l} \mathfrak{J}_{\eta,\zeta}(tw, tv).$$

(ii) If $\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\,d\kappa < 0$, *then there exists a unique* $T_l > 0$ *such that* $(T_lw, T_lv) \in \mathbf{N}^-_{\eta,\zeta}$ *and* $\mathfrak{J}_{\eta,\zeta}(T_lw, T_lv) = \max_{t\ge 0} \mathfrak{J}_{\eta,\zeta}(tw, tv)$.

Proof. (i) Suppose that $\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\,d\kappa > 0$. Define the function $\psi_{w,v} : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$\psi_{w,v}(t) = kt^{p-q}\|(w,v)\|^p_{\mathbb{W}} + lt^{p\sigma-q}\|(w,v)\|^{p\sigma}_{\mathbb{W}} - t^{2-\varrho-\tau-q}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa.$$

Note that $(tw, tv) \in \mathbf{N}_{\eta,\zeta}$ if and only if

$$\psi_{w,v}(t) = \int_Q \left(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\right)d\kappa.$$

Now, the first derivative of the function ψ is

$$\begin{aligned}
\psi'_{w,v}(t) &= k(p-q)t^{p-q-1}\|(w,v)\|^p_{\mathbb{W}} + (p\sigma-q)lt^{p\sigma-q-1}\|(w,v)\|^{p\sigma}_{\mathbb{W}} \\
&\quad - (2-\varrho-\tau-q)t^{1-\varrho-\tau-q}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa \\
&= t^{-q-1}\Big(k(p-q)t^p\|(w,v)\|^p_{\mathbb{W}} + (p\sigma-q)lt^{p\sigma}\|(w,v)\|^{p\sigma}_{\mathbb{W}} \\
&\quad - (2-\varrho-\tau-q)t^{-\varrho-\tau+2}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa\Big).
\end{aligned} \tag{3.7}$$

It is clear that $\psi_{w,v}(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$. Moreover, using (3.7), it is simple to see that $\lim_{t\longrightarrow 0^+} \psi'_{w,v}(t) > 0$ and $\lim_{t\longrightarrow \infty} \psi'_{w,v}(t) < 0$. Thus, there exists $T_l > 0$ such that $\psi_{w,v}(t)$ is decreasing on (T_l, ∞), increasing on $(0, T_l)$, and $\psi'_{w,v}(T_l) = 0$. Thus,

$$\psi_{w,v}(T_l) = kT_l^{p-q}\|(w,v)\|_{\mathbb{W}}^p + lT_l^{p\sigma-q}\|(w,v)\|_{\mathbb{W}}^{p\sigma} - T_l^{2-\varrho-\tau-q}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa,$$

where T_l is the solution of

$$\begin{aligned} &k(p-q)t^p\|(w,v)\|_{\mathbb{W}}^p + (p\sigma - q)lt^{p\sigma}\|(w,v)\|_{\mathbb{W}}^{p\sigma} \\ &\qquad - (2-\varrho-\tau-q)t^{-\varrho-\tau+2}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa = 0. \end{aligned} \tag{3.8}$$

Then, using (3.8), we obtain

$$T_0 := \left(\frac{(2-\varrho-\tau-q)\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa}{k(p-q)\|(w,v)\|_{\mathbb{W}}^p}\right)^{\frac{1}{p+\tau+\varrho-2}} \leq T_l. \tag{3.9}$$

From inequality (3.9), we can find a constant $C = C(p,q,\varrho,\tau) > 0$ such that

$$\begin{aligned} \psi_{w,v}(T_l) &\geq \psi_{w,v}(T_0) \\ &\geq kT_0^{p-q}\|(w,v)\|_{\mathbb{W}}^p - T_0^{2-\varrho-\tau-q}\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa \\ &\geq k\Big(\frac{\varrho+\tau}{q+\varrho+\tau-2}\Big)\Big(\frac{q+\varrho+\tau-2}{k(q-2)}\Big)^{\frac{2-q}{\tau+\varrho}} \frac{\|(w,v)\|_{\mathbb{W}}^{2^{\frac{q+\varrho+\tau-2}{\tau+\varrho}}}}{\big(\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa\big)^{\frac{q-2}{\tau+\varrho}}} \\ &\quad - |Q|^{\frac{2_s^*-q}{2_s^*}} S^{-\frac{q}{2}}\left((\eta\|\alpha\|_\infty)^{\frac{2}{2-q}} + (\zeta\|\beta\|_\infty)^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\|(w,v)\|_{\mathbb{W}}^q > 0, \end{aligned}$$

if and only if

$$\begin{aligned} &(\eta\|\alpha\|_\infty)^{\frac{2}{2-q}} + (\zeta\|\beta\|_\infty)^{\frac{2}{2-q}} \\ &\quad < \left(\frac{k(q-2)}{\|\zeta\|_\infty(q+\varrho+\tau-2)}\right)^{-\frac{2}{\varrho+\tau}} \left(\frac{q+\varrho+\tau-2}{k(\varrho+\tau)}|Q|^{\frac{2_s^*-q}{2_s^*}}\right)^{-\frac{2}{2-q}} S^{\frac{\varrho+\tau-2}{\varrho+\tau}+\frac{q}{2-q}} = \Lambda_0. \end{aligned}$$

Then, there exist exactly two points $t_0 < T_l$ and $t_1 > T_l$ with

$$\psi'_{w,v}(t_0) = \int_Q \big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\big)\,d\kappa = \psi'_{w,v}(t_1).$$

Also, $\psi'_{w,v}(t_0) > 0$ and $\psi'_{w,v}(t_1) < 0$. That is, $(t_0u, t_0v) \in \mathbf{N}^+_{\eta,\zeta}$ and $(t_1u, t_1v) \in \mathbf{N}^-_{\eta,\zeta}$. Since

$$\Upsilon'_{w,v}(t) = t^q\left(\psi_{w,v}(t) - \int_Q \big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\big)\,d\kappa\right).$$

Thus, $\Upsilon'_{w,v}(t) < 0$ for all $t \in [0, t_0)$ and $\Upsilon'_{w,v}(t) > 0$ for all $t \in (t_0, t_1)$. Hence $\mathfrak{J}_{\eta,\zeta}(t_0 w, t_0 v) = \min_{0 \leq t \leq t_1} \mathfrak{J}_{\eta,\zeta}(tw, tv)$. In the same way, $\Upsilon'_{w,v}(t) > 0$ for all $t \in (t_0, t_1)$, $\Upsilon'_{w,v}(t) = 0$ and $\Upsilon'_{w,v}(t) < 0$ for all $t \in (t_1, \infty)$ that is $\mathfrak{J}_{\eta,\zeta}(t_1 w, t_1 v) = \max_{t \geq T_l} \mathfrak{J}_{\eta,\zeta}(tw, tv)$.

(ii) Suppose that $\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\, d\kappa < 0$. So $\psi_{w,v}(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$. Therefore, for all (η, ζ) there exists $T_l > 0$ such that $(T_l w, T_l v) \in \mathbf{N}^-_{\eta,\zeta}$ and $\mathfrak{J}_{\eta,\zeta}(T_l w, T_l v) = \max_{t \geq 0} \mathfrak{J}_{\eta,\zeta}(tw, tv)$. □

The consequence of Lemma 3.5 is summarized in the following Lemma.

Lemma 3.6. *There exists*

$$\Lambda_0 = \left(\frac{q + \varrho + \tau - 2}{\|\zeta\|_\infty k(q-p)}\right)^{\frac{p}{p+\varrho+\tau-2}} \left(\frac{2 - \varrho - \tau - q}{k(2 - \varrho - \tau - p)} |Q|^{\frac{p_s^* - q}{p_s^*}}\right)^{-\frac{p}{p-q}} S^{\frac{2-\varrho-\tau}{p+\varrho+\tau-2}},$$

such that for $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, *we have* $\mathbf{N}^{\pm}_{\eta,\zeta} \neq \emptyset$ *and* $\mathbf{N}^0_{\eta,\zeta} = \emptyset$.

Proof. From Lemma 3.4, we infer that $\mathbf{N}^{\pm}_{\eta,\zeta}$ are non-empty for all (η, ζ) with $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$. Next, we employ a proof by contradiction to show that $\mathbf{N}^0_{\eta,\zeta} = \emptyset$ for all (η, ζ), with $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$. Let $(w, v) \in \mathbf{N}^0_{\eta,\zeta}$. Then, we have two cases:

Case 1: $(w, v) \in \mathbf{N}^+_{\eta,\zeta}$ and $\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\, d\kappa = 0$. Using (3.3) and (3.4) with $t = 1$, it follows that

$$\begin{aligned}(p-1)k\|(w,v)\|^p_{\mathbb{W}} + l(p\sigma - 1)\|(w,v)\|^{p\sigma}_{\mathbb{W}} - (1 - \varrho - \tau)\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\, d\kappa \\ = (p + \varrho + \tau - 2)k\|(w,v)\|^p_{\mathbb{W}} + l(p\sigma + \varrho + \tau - 2)\|(w,v)\|^{p\sigma}_{\mathbb{W}} > 0,\end{aligned}$$

which is a contradiction.

Case 2: Let $(w, v) \in \mathbf{N}^-_{\eta,\zeta}$ and $\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\, d\kappa = 0$. Using (3.3) and (3.4) with $t = 1$, it follows that

$$(p-q)k\|(w,v)\|^p_{\mathbb{W}} + l(p\sigma - q)\|(w,v)\|^{p\sigma}_{\mathbb{W}} = -(q + \varrho + \tau)\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\, d\kappa, \tag{3.10}$$

$$\begin{aligned}(2 - \varrho - \tau - p)k\|(w,v)\|^p_{\mathbb{W}} + l(2 - \varrho - \tau - p\sigma)\|(w,v)\|^{p\sigma}_{\mathbb{W}} \\ = (2 - \varrho - \tau - q)\int_Q \Big(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\Big)\, d\kappa.\end{aligned} \tag{3.11}$$

Now, define $\mathfrak{E}_{\eta,\zeta} : \mathbf{N}_{\eta,\zeta} \longrightarrow \mathbb{R}$ as follows

$$\mathfrak{E}_{\eta,\zeta}(w,v) = \frac{2-\varrho-\tau-p}{2-\varrho-\tau-q} k\|(w,v)\|_{\mathbb{W}}^{p} + \frac{2-\varrho-\tau-p\sigma}{2-\varrho-\tau-q} l\|(w,v)\|_{\mathbb{W}}^{p\sigma} - \int_Q \left(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\right) d\kappa.$$

Therefore, from (3.11), $\mathfrak{E}_{\eta,\zeta}(w,v) = 0$ for all $(w,v) \in \mathbf{N}^0_{\eta,\zeta}$. Furthermore,

$$\begin{aligned}
\mathfrak{E}_{\eta,\zeta}(w,v) &\geq \frac{2-\varrho-\tau-p}{2-\varrho-\tau-q} k\|(w,v)\|_{\mathbb{W}}^{p} - \int_Q \left(\eta\alpha(\kappa)|w|^q + \zeta\beta(\kappa)|v|^q\right) d\kappa \\
&\geq \frac{2-\varrho-\tau-p}{2-\varrho-\tau-q} k\|(w,v)\|_{\mathbb{W}}^{p} \\
&\quad - C|Q|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}} \left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} \|(w,v)\|_{\mathbb{W}}^{q} \\
&\geq \|(w,v)\|_{\mathbb{W}}^{q} \left(\frac{2-\varrho-\tau-p}{2-\varrho-\tau-q} k\|(w,v)\|_{\mathbb{W}}^{p-q} \right.\\
&\quad \left. - C|Q|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}} \left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\right).
\end{aligned}$$

Then, utilizing (3.6) and (3.10), we get

$$\|(u,v)\| \geq \frac{1}{\|\zeta\|_\infty} S^{-\frac{2-\varrho-\tau}{p(p+\varrho+\tau-2)}} \left(\frac{k(p-q)}{2-\varrho-\tau-q}\right)^{-\frac{1}{p+\varrho+\tau-2}}. \tag{3.12}$$

From (3.12) we get

$$\begin{aligned}
\mathfrak{E}_{\eta,\zeta}(w,v) \geq \|(w,v)\|_{\mathbb{W}}^{q} &\left(\frac{2-\varrho-\tau-p}{2-\varrho-\tau-q} k \left(k(p-q)\|\zeta\|_\infty S^{\frac{2-\varrho-\tau}{p(p+\varrho+\tau-2)}}\right) \left(\frac{k(p-q)}{2-\varrho-\tau-q}\right)^{\frac{q-p}{p+\varrho+\tau-2}} \right.\\
&\left. - C|Q|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}} \left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\right).
\end{aligned}$$

This implies that for $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, we have $\mathfrak{E}_{\eta,\zeta}(w,v) > 0$, for all $(w,v) \in \mathbf{N}^0_{\eta,\zeta}$. The proof is complete. □

Due to Lemmas 3.3 and 3.4, for $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, we can write $\mathbf{N}_{\eta,\zeta} = \mathbf{N}^+_{\eta,\zeta} \cup \mathbf{N}^-_{\eta,\zeta}$ and define

$$c^+_{\eta,\zeta} = \inf_{(w,v)\in\mathbf{N}^+_{\eta,\zeta}} \mathfrak{J}_{\eta,\zeta}(w,v), \quad c^-_{\eta,\zeta} = \inf_{(w,v)\in\mathbf{N}^-_{\eta,\zeta}} \mathfrak{J}_{\eta,\zeta}(w,v).$$

3.1 Existence of a minimizer on $\mathbf{N}^+_{\eta,\zeta}$.

In this subsection, we establish that the minimum of $\mathfrak{J}_{\eta,\zeta}$ is found within $\mathbf{N}^+_{\eta,\zeta}$. Furthermore, we demonstrate that this minimizer also serves as a solution to problem (1.1).

Lemma 3.7. *If $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, then for all $(w, v) \in \mathbf{N}^+_{\eta,\zeta}$, we have $c^+_{\eta,\zeta} < 0$.*

Proof. Let $(w_0^+, v_0^+) \in \mathbf{N}^+_{\eta,\zeta}$, then $\Upsilon''_{(w_0^+,v_0^+)}(1) > 0$ which from (3.2) gives

$$\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa < \frac{k(p-q)}{2-\varrho-\tau-q}\|(w,v)\|^p_{\mathbb{W}} + \frac{l(p\sigma-q)}{2-\varrho-\tau-q}\|(w,v)\|^{p\sigma}_{\mathbb{W}}. \tag{3.13}$$

Thus, according to (3.2) with (3.13), we obtain

$$\begin{aligned}
\mathfrak{J}_{\eta,\zeta}(w,v) &\le k\Big(\frac{1}{p}-\frac{1}{q}\Big)\|(w,v)\|^p_{\mathbb{W}} + l\left(\frac{1}{p\sigma}-\frac{1}{q}\right)\|(w,v)\|^{p\sigma}_{\mathbb{W}} \\
&\quad - \left(\frac{1}{2-\varrho-\tau}-\frac{1}{q}\right)\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\,d\kappa \\
&\le \left[k\left(\frac{1}{p}-\frac{1}{q}\right) - \left(\frac{1}{2-\varrho-\tau}-\frac{1}{q}\right)\frac{k(p-q)}{2-\varrho-\tau-q}\right]\|(w,v)\|^p_{\mathbb{W}} \\
&\quad + \left[l\left(\frac{1}{p\sigma}-\frac{1}{q}\right) - \left(\frac{1}{2-\varrho-\tau}-\frac{1}{q}\right)\frac{l(p\sigma-q)}{2-\varrho-\tau-q}\right]\|(w,v)\|^{p\sigma}_{\mathbb{W}}.
\end{aligned} \tag{3.14}$$

Hence, using (3.14), we get

$$\mathfrak{J}_{\eta,\zeta}(w,v) < -\left(\frac{k(q-p)(p+\varrho+\tau-2)}{pq(2-\varrho-\tau)}\|(w,v)\|^p + \frac{l(q-p)(p+\varrho+\tau-2)}{pq(2-\varrho-\tau)}\|(w,v)\|^{p\sigma}\right) < 0.$$

Therefore, the definition of $c^+_{\eta,\zeta}$ owing to $c^+_{\eta,\zeta} < 0$. □

Theorem 3.8. *If $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, then there exists (w_0^+, v_0^+) in $\mathbf{N}^+_{\eta,\zeta}$ satisfying $\mathfrak{J}_{\eta,\zeta}(w_0^+, v_0^+) = \inf\limits_{(w,v)\in\mathbf{N}^+_{\eta,\zeta}} \mathfrak{J}_{\eta,\zeta}(w, v)$.*

Proof. From the fact that $\mathfrak{J}_{\eta,\zeta}$ is bounded below on $\mathbf{N}_{\eta,\zeta}$, then it bounded on $\mathbf{N}^+_{\eta,\zeta}$. Thus, there exists $\{(w_n^+, v_n^+)\} \subset \mathbf{N}^+_{\eta,\zeta}$ a sequence such that

$$\mathfrak{J}_{\eta,\zeta}(w_n^+, v_n^+) \longrightarrow \inf_{(w,v)\in\mathbf{N}^+_{\eta,\zeta}} \mathfrak{J}_{\eta,\zeta}(w,v) \quad \text{as } n \longrightarrow \infty.$$

Since $\mathfrak{J}_{\eta,\zeta}$ is coercive, $\{w_n, v_n\}$ is bounded in $\mathbb{W}$. Then, there exists a sub-sequence, still denoted by (w_n^+, v_n^+) and $(w_0^+, v_0^+) \in \mathbb{W}$ such that, as $n \longrightarrow \infty$,

$$w_n^+ \rightharpoonup w_0^+,\ v_n^+ \rightharpoonup v_0^+ \quad \text{weakly in } \mathbb{W}_s(Q),$$
$$w_n^+ \longrightarrow w_0^+,\ v_n^+ \longrightarrow v_0^+ \quad \text{strongly in } \mathbb{L}^r(Q) \text{ for } 1 \leq r < p_s^*,$$
$$w_n^+ \longrightarrow w_0^+,\ v_n^+ \longrightarrow v_0^+ \quad \text{a.e. in } Q.$$

Claim:

$$\lim_{n\longrightarrow\infty} \int_Q \alpha(\kappa)|w_n^+|^{1-\varrho} d\kappa = \int_Q \alpha(\kappa)|w_0^+|^{1-\varrho} d\kappa. \tag{3.15}$$

Indeed, due to Vitali's theorem (see [26, pp. 133]), we only need to prove that

$$\left\{\int_Q \alpha(\kappa)|w_n^+|^{1-\varrho} d\kappa, n \in N\right\} \quad \text{is equi-absolutely-continuous.}$$

Since $\{w_n\}$ is bounded, by the Sobolev embedding theorem, there exists a constant $C > 0$ such that $|w_n|_{p_s^*} \leq C < \infty$. Moreover, by the Hölder inequality we have

$$\int_Q \alpha(\kappa) w^{1-\varrho} d\kappa \leq \|\alpha\|_\infty \int_Q |w|^{1-\varrho} d\kappa \leq \|\alpha\|_\infty |Q|^{\frac{p_s^*}{p_s^*+\varrho-1}} |w|_{p_s^*}^{1-\varrho}. \tag{3.16}$$

From (3.16), for every $\varepsilon > 0$, setting

$$\delta = \left(\frac{\varepsilon}{\|\alpha\|_\infty C^{1-\varrho}}\right)^{\frac{p_s^*}{p_s^*+\gamma-1}},$$

when $A \subset Q$ with $\text{meas}(A) < \delta$, we have

$$\int_A \alpha(\kappa)|w_n^+|^{1-\varrho} d\kappa \leq \|\alpha\|_\infty \|w\|_{p_s^*}^{1-\varrho} \big(\text{meas}(A)\big)^{\frac{p_s^*+\varrho-1}{p_s^*}} \leq \|\alpha\|_\infty C^{1-\varrho} \delta^{\frac{p_s^*+\varrho-1}{p_s^*}} < \varepsilon.$$

Thus, our claim is true. Similarly, we claim that

$$\lim_{n\longrightarrow\infty} \int_Q \beta(\kappa)|v_n^+|^{1-\tau} d\kappa = \int_Q \beta(\kappa)|v_0^+|^{1-\tau} d\kappa. \tag{3.17}$$

On the other hand, by [3] there exists $l \in \mathbb{L}^r(\mathbb{R}^d)$ such that

$$|w_n^+(\kappa)| \leq l(\kappa), \quad |v_n^+(\kappa)| \leq l(\kappa), \quad \text{as } k \longrightarrow \infty$$

for $1 \leq r < p_s^*$. Therefore by the dominated convergence theorem,

$$\int_Q \Big(\eta|w_n^+|^q + \zeta|v_n^+|^q\Big) d\kappa \longrightarrow \int_Q \Big(\eta|w_0^+|^q + \zeta|v_0^+|^q\Big) d\kappa.$$

Furthermore, from Lemma 3.5, there exists t_0 such that $(t_0 w_0^+, t_0 v_0^+) \in \mathbf{N}_{\eta,\zeta}^+$. Now, we shall prove that $w_n^+ \longrightarrow w_0^+$ strongly in $\mathbb{W}_s$, $v_n^+ \longrightarrow v_0^+$ strongly in $\mathbb{W}_s$. Suppose otherwise, then

$$\|(w_0^+, v_0^+)\|_{\mathbb{W}} \leq \liminf_{n\longrightarrow\infty} \|(w_n^+, v_n^+)\|_{\mathbb{W}}.$$

On the other hand, since $(w_n^+, v_n^+) \in \mathbf{N}_{\eta,\zeta}^+$, one has

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \Upsilon'_{w_n^+,v_n^+}(t_0) &= \lim_{n\longrightarrow\infty} \Big(kt_0^{p-1}\|(w_n^+, v_n^+)\|^p + lt_0^{p\sigma-1}\|(w_n^+, v_n^+)\|^{p\sigma} \\
&\quad - t_0^{q-1}\int_Q \big(\eta\alpha(\kappa)|w_n^+|^q + \zeta\beta(\kappa)|v_n^+)|^q\big)\, d\kappa - t_0^{1-\varrho-\tau}\int_Q \xi(\kappa)|w_n^+|^{1-\varrho}|v_n^+|^{1-\tau}\, d\kappa\Big) \\
&> kt_0^{p-1}\|(w_0^+, v_0^+)\|^p + lt_0^{p\sigma-1}\|(w_0^+, v_0^+)\|^{p\sigma} \\
&\quad - t_0^{q-1}\int_Q \big(\eta\alpha(\kappa)|w_0^+|^q + \zeta\beta(\kappa)|v_0^+)|^q\big)\, d\kappa - t_0^{1-\varrho-\tau}\int_Q \xi(\kappa)|w_0^+|^{1-\varrho}|v_0^+|^{1-\tau}\, d\kappa \\
&= \Upsilon'_{w_0^+,v_0^+}(t_0) = 0.
\end{aligned}$$

Therefore, $\Upsilon'_{w_n^+,v_n^+}(t_0) > 0$ for n large enough. Furthermore, $(w_n^+, v_n^+) \in \mathbf{N}_{\eta,\zeta}^+$, and we can see for all n that $\Upsilon'_{w_n^+,v_n^+}(t) < 0$ for $t \in (0,t)$ and $\Upsilon'_{w_n^+,v_n^+}(1) = 0$. Thus we must have $t_0 > 1$. Moreover $\Upsilon_{w_n^+,v_n^+}(1)$ is decreasing for $t \in (0, t_0)$ and that is

$$\mathfrak{J}_{\eta,\zeta}(t_0 w_0^+, t_0 v_0^+) < \mathfrak{J}_{\eta,\zeta}(w_0^+, v_0^+) = \lim_{n\longrightarrow\infty} \mathfrak{J}_{\eta,\zeta}(w_n^+, v_n^+) = \inf_{(w,v)\in\mathbf{N}_{\eta,\zeta}^+} \mathfrak{J}_{\eta,\zeta}(w, v)$$

which gives a contradiction. Thus, $w_n^+ \longrightarrow w_0^+$ strongly in $\mathbb{W}_s$, $v_n^+ \longrightarrow v_0^+$ strongly in $\mathbb{W}_s$ and $\mathfrak{J}_{\eta,\zeta}(w_0^+, v_0^+) = \inf_{(w,v)\in\mathbf{N}_{\eta,\zeta}^+} \mathfrak{J}_{\eta,\zeta}(w, v)$. The proof of Theorem 3.8 is complete. □

3.2 Existence of a minimizer on $\mathbf{N}_{\eta,\zeta}^-$.

In this subsection, we aim to establish the existence of a solution to problem (1.1) by demonstrating the existence of a minimizer for $\mathfrak{J}_{\eta,\zeta}$ within the set $\mathbf{N}_{\eta,\zeta}^-$.

Lemma 3.9. *If $0 < (\eta\|\alpha\|_\infty)^{\frac{p}{p-q}} + (\zeta\|\beta\|_\infty)^{\frac{p}{p-q}} < \Lambda_0$, then for all $(w, v) \in \mathbf{N}_{\eta,\zeta}^+$, one has $c_{\eta,\zeta}^- > d_0$ for some $d_0 = d_0\big(\varrho, \tau, p, q, \alpha, \beta, \eta, \zeta, |Q|\big) > 0$.*

Proof. Let $(w_0^-, v_0^-) \in \mathbf{N}_{\eta,\zeta}^-$, then we have $\Upsilon''_{w_0^-,v_0^-}(1) < 0$ which from (3.2) gives

$$\int_Q \xi(\kappa)|w|^{1-\varrho}|v|^{1-\tau}\, d\kappa > \frac{k(p-q)}{2-\varrho-\tau-q}\|(w, v)\|_{\mathbb{W}}^p + \frac{l(p\sigma-q)}{2-\varrho-\tau-q}\|(w, v)\|_{\mathbb{W}}^{p\sigma}. \tag{3.18}$$

Hence, using (3.6), we get

$$\|(w, v)\|_{\mathbb{W}} > \frac{1}{\|\zeta\|_\infty} S^{-\frac{2-\varrho-\tau}{p(p+\varrho+\tau-2)}} \left(\frac{k(p-q)}{2-\varrho-\tau-q}\right)^{-\frac{1}{p+\varrho+\tau-2}}. \tag{3.19}$$

Therefore, by (3.5) and (3.19), we obtain

$$\begin{aligned}
\mathfrak{J}_{\eta,\zeta}(w,v) &\geq k\left(\frac{1}{p}-\frac{1}{2-\varrho-\tau}\right)\|(w,v)\|_{\mathbb{W}}^{p}-\left(\frac{1}{q}-\frac{1}{2-\varrho-\tau}\right)|Q|^{\frac{p_s^*-q}{p_s^*}} \\
&\quad \times S^{-\frac{q}{p}}\left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}}+(\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\|(w,v)\|_{\mathbb{W}}^{q} \\
&= \|(w,v)\|_{\mathbb{W}}^{q}\Bigg[k\left(\frac{1}{p}-\frac{1}{2-\varrho-\tau}\right)\|(w,v)\|_{\mathbb{W}}^{p-q}-\left(\frac{1}{q}-\frac{1}{2-\varrho-\tau}\right)|Q|^{\frac{p_s^*-q}{p_s^*}} \\
&\quad \times S^{-\frac{q}{p}}\left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}}+(\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\Bigg] \\
&> \|(w,v)\|_{\mathbb{W}}^{q}\Bigg[k\left(\frac{1}{p}-\frac{1}{2-\varrho-\tau}\right)S^{\frac{(p-q)}{p}}\left(\frac{p-q}{2-\varrho-\tau-q}\right)^{\frac{q-p}{p+\varrho+\tau-2}} \\
&\quad -\left(\frac{1}{q}-\frac{1}{2-\varrho-\tau}\right)|Q|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\left((\eta\|\alpha\|_\infty)^{\frac{p}{p-q}}+(\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\Bigg].
\end{aligned}$$

Hence, if $0<(\eta\|\alpha\|_\infty)^{\frac{p}{p-q}}+(\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}<\Lambda_0$, then $\mathfrak{J}_{\eta,\zeta}(w,v)>d_0$ for all $(w,v)\in\mathbf{N}_{\eta,\zeta}^{-}$ for some $d_0=d_0\big(\varrho,\tau,p,q,\alpha,\beta,\eta,\zeta,|Q|\big)>0$. Therefore $c_{\eta,\zeta}^{-}>d_0$ follows from the definition $c_{\eta,\zeta}^{-}$. □

Theorem 3.10. *If $0<(\eta\|\alpha\|_\infty)^{\frac{p}{p-q}}+(\zeta\|\beta\|_\infty)^{\frac{p}{p-q}}<\Lambda_0$, then there exists (w_0^-,v_0^-) in $\mathbf{N}_{\eta,\zeta}^{-}$ satisfying $\mathfrak{J}_{\eta,\zeta}(w_0^-,v_0^-)=\inf\limits_{(w,v)\in\mathbf{N}_{\eta,\zeta}^{-}}\mathfrak{J}_{\eta,\zeta}(w,v)$.*

Proof. As $\mathfrak{J}_{\eta,\zeta}$ is bounded below on $\mathbf{N}_{\eta,\zeta}$ and then on $\mathbf{N}_{\eta,\zeta}^{-}$. Thus, there exists $\{(w_n^-,v_n^-)\}\subset\mathbf{N}_{\eta,\zeta}^{-}$, a sequence such that

$$\mathfrak{J}_{\eta,\zeta}(w_n^-,v_n^-)\longrightarrow\inf_{(w,v)\in\mathbf{N}_{\eta,\zeta}^{-}}\mathfrak{J}_{\eta,\zeta}(w,v)\quad\text{as } n\longrightarrow\infty.$$

Since $\mathfrak{J}_{\eta,\zeta}$ is coercive, $\{(w_n,v_n)\}$ is bounded in $\mathbb{W}$. Then there exists a sub-sequence, still denoted by (w_n^-,v_n^-) and $(w_0^-,v_0^-)\in\mathbb{W}$ such that, as $n\longrightarrow\infty$,

$$\begin{gathered}
w_n^+\rightharpoonup w_0^-,\ v_n^-\rightharpoonup v_0^-\quad\text{weakly in } \mathbb{W}_s(Q),\\
w_n^-\longrightarrow w_0^-,\ v_n^-\longrightarrow v_0^-\quad\text{strongly in } \mathbb{L}^r(Q) \text{ for } 1\leq r<p_s^*,\\
w_n^-\longrightarrow w_0^-,\ v_n^-\longrightarrow v_0^-\quad\text{a.e. in } Q.
\end{gathered}$$

Furthermore, similar to the proof in Lemma 3.8, we have

$$\begin{gathered}
\lim_{n\longrightarrow\infty}\int_Q|w_n^-|^{1-\varrho}d\kappa=\int_Q|w_0^-|^{1-\varrho}d\kappa,\\
\lim_{n\longrightarrow\infty}\int_Q|v_n^-|^{1-\tau}d\kappa=\int_Q|v_0^-|^{1-\tau}d\kappa,\\
\int_Q\Big(\eta\alpha(\kappa)|w_n^+|^q+\zeta\beta(\kappa)|v_n^+|^q\Big)d\kappa\longrightarrow\int_Q\Big(\eta\alpha(\kappa)|w_0^+|^q+\zeta\beta(\kappa)|v_0^+|^q\Big)d\kappa.
\end{gathered}$$

Moreover, by Lemma 3.5, there exists t_1 such that $(t_1 w_0^-, t_1 v_0^-) \in \mathbf{N}_{\eta,\zeta}^-$. Now, we prove that $w_n^- \longrightarrow w_0^-$ strongly in $\mathbb{W}_s$, $v_n^- \longrightarrow v_0^-$ strongly in $\mathbb{W}_s$. Suppose otherwise, then

$$\|(w_0^-, v_0^-)\|_{\mathbb{W}} \leq \liminf_{n \longrightarrow \infty} \|(w_n^-, v_n^-)\|_{\mathbb{W}}.$$

Thus, since $(w_n^-, v_n^-) \in \mathbf{N}_{\eta,\zeta}^-$ and $\mathfrak{J}_{\eta,\zeta}(tw_0^-, tv_0^-) \leq \mathfrak{J}_{\eta,\zeta}(w_0^-, v_0^-)$, for all $t \geq 0$ we have

$$\mathfrak{J}_{\eta,\zeta}(t_1 w_0^-, t_1 v_0^-) < \lim_{n \longrightarrow \infty} \mathfrak{J}_{\eta,\zeta}(t_1 u_n^-, t_1 v_n^-) \leq \lim_{n \longrightarrow \infty} \mathfrak{J}_{\eta,\zeta}(w_n^-, v_n^-) = c_{\eta,\zeta}^-,$$

which gives a contradiction. Hence, $w_n^- \longrightarrow w_0^-$ strongly in $\mathbb{W}_s(Q)$, $v_n^- \longrightarrow v_0^-$ strongly in $\mathbb{W}_s(Q)$ and $\mathfrak{J}_{\eta,\zeta}(w_0^-, v_0^-) = \inf_{(w,v) \in \mathbf{N}_{\eta,\zeta}^-} \mathfrak{J}_{\eta,\zeta}(w, v)$. Which complete the proof. □

4 Multiple solutions

In this section, we shall prove Theorem (2.2), which gives the multiplicity of solutions for problem (1.1).

Proof of Theorem 2.2. To begin, let us establish the existence of non-negative solutions. Initially, according to Theorems 3.8 and 3.10, there exist$(w_0^+, v_0^+) \in \mathbf{N}_{\eta,\zeta}^+$, $(w_0^-, v_0^-) \in \mathbf{N}_{\eta,\zeta}^-$ satisfying

$$\mathfrak{J}_{\eta,\zeta}(w_0^+, v_0^+) = \inf_{(w,v) \in \mathbf{N}_{\eta,\zeta}^+} \mathfrak{J}_{\eta,\zeta}(w, v),$$

$$\mathfrak{J}_{\eta,\zeta}(w_0^-, v_0^-) = \inf_{(w,v) \in \mathbf{N}_{\eta,\zeta}^-} \mathfrak{J}_{\eta,\zeta}(w, v).$$

Also, from the fact that $\mathfrak{J}_{\eta,\zeta}(w_0^+, v_0^+) = \mathfrak{J}_{\eta,\zeta}(|w_0^+|, |v_0^+|)$ and $(|w_0^+|, |v_0^+|) \in \mathbf{N}_{\eta,\zeta}^+$. Similarly we have $\mathfrak{J}_{\eta}(w_0^-, v_0^-) = \mathfrak{J}_{\eta,\zeta}(|w_0^-|, |v_0^-|)$ and $(|w_0^-|, |v_0^-|) \in \mathbf{N}_{\eta,\zeta}^-$, thus we can assume $(w_0^\pm, v_0^\pm) \geq 0$. Due to Lemma 3.2, $(w_0^\pm, v_0^\pm)$ are the nontrivial non-negative solutions of problem (1.1). Finally, we need to establish that the solutions obtained in Theorems 3.8 and 3.10 are distinct. Given that $\mathbf{N}_{\eta,\zeta}^- \cap \mathbf{N}_{\eta,\zeta}^+ = \emptyset$, it follows that $(w_0^\pm, v_0^\pm)$ are indeed distinct. This completes the proof. □

Authors contributions

All authors of this manuscript contributed equally to this work.

Funding

Not applicable.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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CUBO, A Mathematical Journal
Vol. 26, no. 3, pp. 431–442, December 2024
DOI: 10.56754/0719-0646.2603.431

Solutions of two open problems on inequalities involving trigonometric and hyperbolic functions

Rupali Shinde[1]
Christophe Chesneau[2,✉]
Nitin Darkunde[1]

[1] *School of Mathematical Sciences, SRTM University Nanded, India.*
rupalishinde260@gmail.com
darkundenitin@gmail.com

[2] *Department of Mathematics, LMNO, University of Caen, 14032 Caen, France.*
chesneau.christophe@gmail.com✉

ABSTRACT

In 2019, Bagul *et al.* posed two open problems dealing with inequalities involving trigonometric and hyperbolic functions and an adjustable parameter. This article is an attempt to solve these open problems. The results are supported by three-dimensional graphics, taking into account the variation of the parameter involved.

RESUMEN

En 2019, Bagul *et al.* propusieron dos problemas relacionados con desigualdades que involucran funciones trigonométricas e hiperbólicas y un parámetro ajustable. Este artículo es un intento de resolver estos problemas abiertos. Los resultados están apoyados con gráficas tridimensionales, tomando en consideración la variación del parámetro involucrado.

Keywords and Phrases: Trigonometric inequalities, series expansion, open problems.

2020 AMS Mathematics Subject Classification: 26D05, 26D07.

Published: 15 October, 2024
Accepted: 06 September, 2024
Received: 08 March, 2024

1 Introduction

When studying mathematical inequalities, it is often useful to find generalizations of known results. These generalizations can provide deep insights into the structure of inequalities and their applications in various areas of mathematics. They can be established using integer series expansions of well-known elementary functions. For a more rigorous treatment of this topic, see [2, 5, 7, 9, 11–15].

A brief discussion of recent progress in the inequalities of some trigonometric and hyperbolic functions is given below. In 2021, Bagul *et al.* [2] studied the following inequalities: For $r > 0$ and $x \in (0, r)$, we have

$$\left(1+\frac{x^2}{\pi^2}\right)e^{ax^2} < \frac{\sinh x}{x} < \left(1+\frac{x^2}{\pi^2}\right)e^{bx^2},$$

where $a = \ln\left[\pi^2(\sinh r)/r(\pi^2+r^2)\right]/r^2$ and $b = 1/6 - 1/\pi^2$ are the best possible constants in the exponential term. To prove these inequalities, the author used the concept of series expansion. For the details, see [2]. Later, in 2023, Li *et al.* [8] presented the proof of the following inequalities involving sine and hyperbolic sine functions using power series expansion: For $|x| < \pi/2$, we have

$$\frac{4}{15}\left(\cos x+\frac{11}{4}\right)^2-\frac{3}{4} \leq \frac{\sin(2x)}{2x}+2\frac{\sin x}{x} \leq \frac{4}{15}\left(\cos x+\frac{11}{4}\right)^2-\frac{3}{4}+\frac{1}{1260}x^6$$

and, for $x \in \mathbb{R}$ and an integer $n \geq 2$, we have

$$1+2\cosh x+\sum_{k=2}^{n} b_k x^{2k} \leq \frac{\sinh(2x)}{2x}+2\frac{\sinh x}{x} \leq \frac{4}{15}\left(\cosh x+\frac{11}{4}\right)^2-\frac{3}{4},$$

where $b_k = (2^{2k}-4k)/(2k+1)!$ for $k = 2, 3, \ldots, n$.

In 2018, Malešević, *et al.* [10] gave the following generalized inequalities: For $x \in (0, \pi/2)$ and an integer $n \geq 1$, we have

$$\frac{2+\cos x}{3}+\sum_{k=2}^{2n}(-1)^{k+1}B(k)x^{2k} < \frac{\sin x}{x} < \frac{2+\cos x}{3}+\sum_{k=2}^{2n+1}(-1)^{k+1}B(k)x^{2k},$$

where $B(k) = 2(k-1)/[2(2k+1)!]$.

The following result gives us sharper bounds on the above inequalities established by Bagul *et al.* [4]: For an integer $n \geq 1$, $m = 2n-1$, and $x \in (0, \pi)$, we have

$$F(x) < \frac{\sin x}{x} < G(x),$$

where

$$F(x)=\frac{2m+\cos x}{2m+1}+\frac{2}{2m+1}\sum_{k=1}^{m+1}\frac{k-m}{(2k+1)!}(-1)^{k+1}x^{2k}$$

and

$$G(x)=\frac{(2m+1)+\cos x}{2m+3}+\frac{2}{2m+3}\sum_{k=1}^{m+2}\frac{k-m-1}{(2k+1)!}(-1)^{k+1}x^{2k}.$$

In parallel to these remarkable modern results, there are some open problems on similar functions. For example, in 2019, Bagul *et al.* [3] posed the following open problems on some trigonometric and hyperbolic functions:

(1) For $x\in(0,\pi/2)$ and $p\geq 2$, we have

$$p+(\cos x)^p>\frac{\sin(px)}{px}+p\left(\frac{\sin x}{x}\right).$$

(2) For $x\in(0,\pi/2)$ and $p\in(0,2]$, we have

$$\frac{\sin(px)}{px}+p\left(\frac{\sin x}{x}\right)>1+p\cos x.$$

(3) For $x\in\mathbb{R}-\{0\}$ and $p\in(0,2]$, we have

$$p+(\cosh x)^p>\frac{\sinh(px)}{px}+p\left(\frac{\sinh x}{x}\right).$$

(4) For $x\in\mathbb{R}-\{0\}$ and $p\geq 2$, we have

$$\frac{\sinh(px)}{px}+p\left(\frac{\sinh x}{x}\right)>1+p\cosh x.$$

This article is an attempt to prove two open problems, namely those presented in Items 2 and 4, which are further listed in the main results. Our focus is to show that these inequalities hold for a wide range of p. It is important to note that while the inequalities we prove are useful for a wide range of p and x, we do not claim that these inequalities are optimal in the sense of sharpness. There is scope in the future to find sharper bounds on these inequalities for particular values of p. In particular, the first inequality we prove holds for $p\geq 2$, and the second for $p\in(0,2)$, which may also be useful to researchers for further development.

The plan is as follows: First, Section 2 gives some preliminary remarks that will be useful for the gradual development of this article. Section 3 deals with our main results, supported by graphics, and Section 4 is the concluding part.

2 Preliminaries

Well-known power series expansions derived from $\sinh x$ and $\cosh x$ are the following formulas:

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \tag{2.1}$$

$$\sinh(px) = \sum_{n=0}^{\infty} \frac{p^{2n+1}x^{2n+1}}{(2n+1)!}, \tag{2.2}$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

and, an immediate consequence of the previous formula,

$$x\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n)!}. \tag{2.3}$$

We may refer to [1] and [6].

3 Main results

In this section, using power series expansion and some trigonometric identities, we present the proof of two inequalities.

Theorem 3.1. *For $x > 0$ and $p \geq 2$, we have*

$$\frac{\sinh(px)}{px} + p\left(\frac{\sinh x}{x}\right) > 1 + p\cosh x.$$

Proof. To prove this result, let us consider the following function:

$$f(x) = \sinh(px) + p^2 \sinh x - px - p^2 x\cosh x.$$

A differentiation work gives

$$\begin{aligned} f'(x) &= p\cosh(px) + p^2\cosh x - p - p^2\{\cosh x + x\sinh x\} \\ &= p\cosh(px) + p^2\cosh x - p - p^2\cosh x - p^2 x\sinh x \\ &= p\cosh(px) - p - p^2 x\sinh x \end{aligned}$$

and

$$f''(x) = p^2\sinh(px) - p^2\{\sinh x + x\cosh x\}.$$

From Equations (2.1), (2.2) and (2.3), we can decompose $f''(x)$ as

$$
\begin{aligned}
f''(x) &= p^2 \left[\sum_{n=0}^{\infty} \frac{p^{2n+1}x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n)!} \right] \\
&= p^2 \left[\sum_{n=0}^{\infty} \frac{p^{2n+1}x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n+1}}{(2n+1)!} \right] \\
&= p^2 \left[\sum_{n=0}^{\infty} \frac{p^{2n+1}x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(2+2n)x^{2n+1}}{(2n+1)!} \right] = p^2 \sum_{n=0}^{\infty} \frac{\left[p^{2n+1} - (2+2n)\right] x^{2n+1}}{(2n+1)!}.
\end{aligned}
$$

For $p \geq 2$, the Bernoulli inequality gives

$$
p^{2n+1} - (2+2n) \geq (1+1)^{2n+1} - (2+2n) \geq 1 + (2n+1) - (2+2n) = 0.
$$

Therefore, for any $x > 0$, we have $f''(x) > 0$. Hence, we conclude that, for $x > 0$, $f'(x)$ is strictly increasing. As a result, we have $f'(x) > f'(0)$ with $f'(0) = p - p = 0$. This implies that $f(x)$ is strictly increasing, so $f(x) > f(0)$ with $f(0) = 0$. By taking into account the definition of $f(x)$, we find

$$
\frac{\sinh(px)}{px} + p\left(\frac{\sinh x}{x}\right) > 1 + p\cosh x.
$$

The proof ends. □

Thus, through Theorem 3.1, we provide the solution to one of the open problems in Bagul *et al.* [3]. Figures 1 and 2 illustrate the validity of Theorem 3.1 by considering the following bivariate function with respect to x and p:

$$
f_\star(x,p) = \frac{\sinh(px)}{px} + p\left(\frac{\sinh x}{x}\right) - 1 - p\cosh x,
$$

with $x > 0$ and $p \geq 2$.

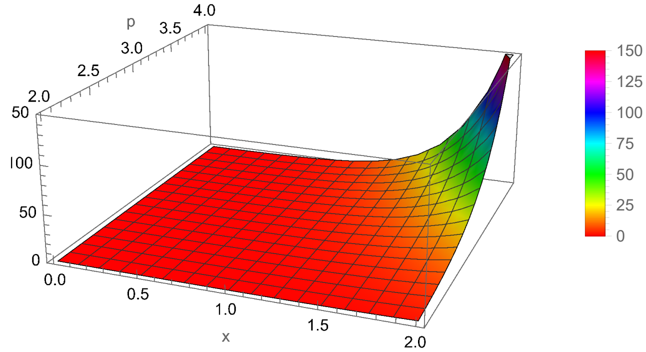

Figure 1: Three-dimensional shape plots of the function $f_\star(x,p)$ for $x \in (0,2)$ and $p \in [2,4)$.

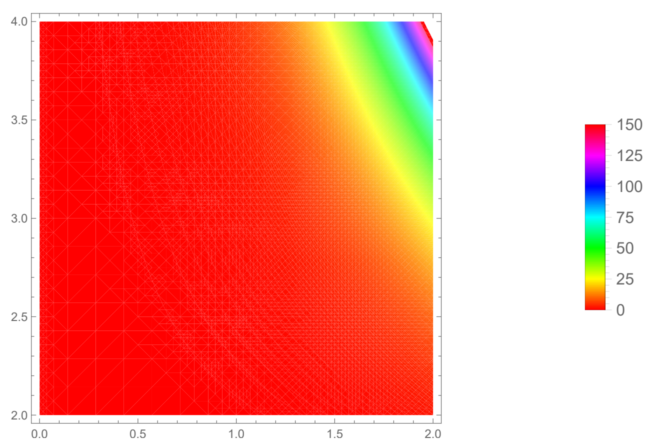

Figure 2: Three-dimensional intensity plots of the function $f_\star(x,p)$ for $x \in (0,2)$ and $p \in [2,4)$.

It is clear that the zone corresponding to the negative values is never reached, implying that $f_\star(x,p) > 0$ for the considered configuration, which is consistent with Theorem 3.1 as expected.

The next result concerns another open problem in Bagul *et al.* [3].

Theorem 3.2. *For $x \in (0,\pi/2)$ and $p \in (0,2]$, we have*

$$\frac{\sin(px)}{px} + p\left(\frac{\sin x}{x}\right) > 1 + p\cos x.$$

Proof. To prove this theorem, let us consider the following function:

$$g(x) = \sin(px) + p^2 \sin x - px - p^2 x \cos x.$$

A differentiation work gives

$$g'(x) = p\cos(px) + p^2 \cos x - p - p^2(\cos x - x \sin x) = p\cos(px) - p + p^2 x \sin x$$

and

$$g''(x) = -p^2 \sin(px) + p^2(\sin x + x\cos x) = p^2[\sin x + x\cos x - \sin(px)].$$

Owing to basic trigonometric identities, we obtain

$$\sin(px) = \sin[(p-1)x + x] = \cos[(p-1)x]\sin x + \sin[(p-1)x]\cos x.$$

Hence, we can rewrite $g''(x)$ as

$$\begin{aligned} g''(x) &= p^2\{\sin x + x\cos x - \cos[(p-1)x]\sin x - \sin[(p-1)x]\cos x\} \\ &= p^2[\{1 - \cos[(p-1)x]\}\sin x + \{x - \sin[(p-1)x]\}\cos x]. \end{aligned}$$

We know that, for $x \in (0, \pi/2)$, we have $\sin x > 0$ and $\cos x > 0$. Also, for any $p \in (0, 2]$ we have $\cos[(p-1)x] \leq 1$, implying that $1 - \cos[(p-1)x] \geq 0$.

Now, let us discuss the sign of the term $x - \sin[(p-1)x]$ by distinguishing the cases $p \in (0, 1]$ and $p \in (1, 2]$.

For $p \in (0, 1]$ and $x \in (0, \pi/2)$, it is immediate that

$$-\sin[(p-1)x] = \sin[(1-p)x] \geq 0.$$

Hence, we can conclude that $x - \sin[(p-1)x] > 0$.

Now for $p \in (1, 2]$ and $x \in (0, \pi/2)$, thanks to the classical sine inequality: $\sin y < y$ for $y > 0$, we have

$$\sin[(p-1)x] < (p-1)x \leq x.$$

Thus, we have $x - \sin[(p-1)x] > 0$.

As a result, we can conclude that $g''(x) > 0$. Thus, for $x \in (0, \pi/2)$, $g'(x)$ is strictly increasing. As a result, we have $g'(x) > g'(0)$ with $g'(0) = p - p = 0$. This implies that $g(x)$ is strictly increasing, so $g(x) > g(0)$ with $g(0) = 0$. Thanks to the definition of $g(x)$, we establish that

$$\frac{\sin(px)}{px} + p\left(\frac{\sin x}{x}\right) > 1 + p\cos x.$$

This achieves the proof. □

Hence, through Theorem 3.2, we offer a solution to one of the open problems in Bagul *et al.* [3]. Figures 3 and 4 illustrate the validity of Theorem 3.2 by considering the following bivariate function with respect to x and p:

$$g_\star(x,p) = \frac{\sin(px)}{px} + p\left(\frac{\sin x}{x}\right) - 1 - p\cos x,$$

with $x \in (0, \pi/2)$ and $p \in (0, 2]$.

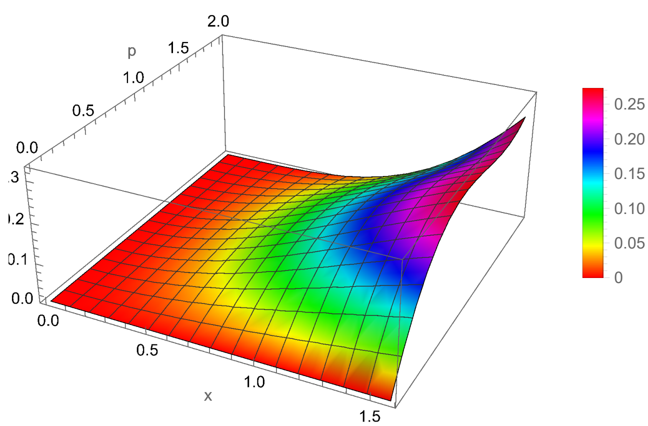

Figure 3: Three-dimensional shape plots of the function $g_\star(x,p)$ for $x \in (0, \pi/2)$ and $p \in (0, 2]$.

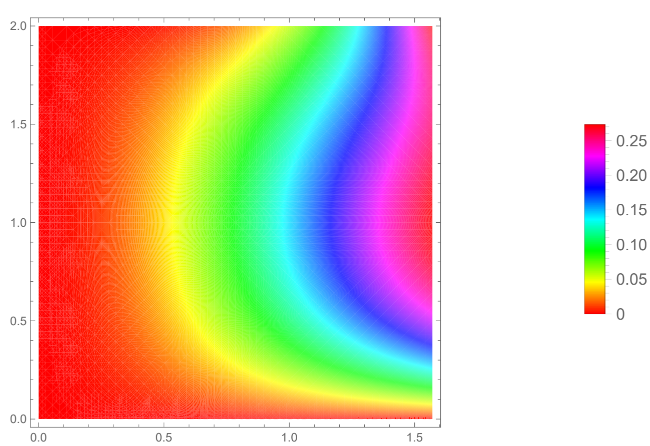

Figure 4: Three-dimensional intensity plots of the function $g_\star(x,p)$ for $x \in (0, \pi/2)$ and $p \in (0, 2]$.

We note that the zone associated with the negative values is never reached, suggesting that $g_\star(x,p) > 0$ for the configuration under consideration, which is in expected agreement with Theorem 3.2.

During our graphical investigation, we found that Theorem 3.2 can be conjectured to be valid for $x \in (0,\pi)$ instead of just $x \in (0,\pi/2)$, as shown in Figure 5 with the absence of a negative value zone. The rigorous proof, however, remains a new challenge to be investigated in the future.

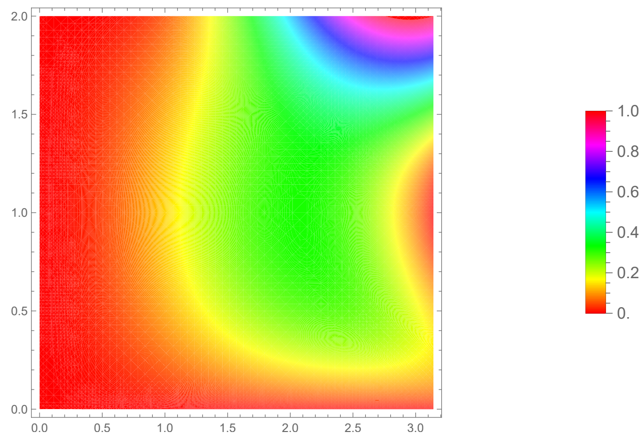

Figure 5: Three-dimensional intensity plots of the function $g_\star(x,p)$ for $x \in (0,\pi)$ and $p \in (0,2]$.

4 Conclusion

In this article, we have given simple and elegant proofs for two open problems posed by Bagul *et al.* in 2019 [3], which concern inequalities related to trigonometric and hyperbolic functions for a large range of p and x. The presented inequalities generalize existing results for large values of p and provide researchers with valuable insights and tools for further developments in this area.

Availability of data and material

Not applicable.

Conflict of interests

The authors declare no conflict of interest.

Ethics approval

The authors declare that the article was not submitted or published anywhere.

Acknowledgements

First author would like to acknowledge CSMNRF-2022 (SARATHI) for the financial assistance which helped to carry out this research work. First and third author would like to acknowledge the DST-FIST (India) for providing research and infrastructural facilities at School of Mathematical Sciences, SRTM University, Nanded.

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CUBO, A Mathematical Journal
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DOI: 10.56754/0719-0646.2603.443

Congruences of finite semidistributive lattices

J. B. Nation[1,✉]

[1] *Department of Mathematics, University of Hawaii, Honolulu, HI 96822, USA.*
jb@math.hawaii.edu

ABSTRACT

We show that there are finite distributive lattices that are not the congruence lattice of any finite semidistributive lattice. For $0 \leq k \leq 2$, the distributive lattice $(\mathbf{B}_k)_{++} = \mathbf{2} + \mathbf{B}_k$, where $\mathbf{B}_k$ denotes the boolean lattice with k atoms, is not the congruence lattice of any finite semidistributive lattice. Neither can these lattices be a filter in the congruence lattice of a finite semidistributive lattice. However, each $(\mathbf{B}_k)_{++}$ with $k \geq 3$ is the congruence lattice of a finite semidistributive lattice, say $\mathbf{L}_k$. These lattices $\mathbf{L}_k$ cannot be bounded (in the sense of McKenzie), as no $(\mathbf{B}_k)_{++}$ $(k \geq 0)$ is the congruence lattice of a finite bounded lattice. A companion paper shows that every $(\mathbf{B}_k)_{++}$ $(k \geq 0)$ can be represented as the congruence lattice of an infinite semidistributive lattice. We also find sufficient conditions for a finite distributive lattice to be representable as the congruence lattice of a finite bounded (and hence semidistributive) lattice.

Published: 18 October, 2024
Accepted: 12 September, 2024
Received: 13 November, 2023

RESUMEN

Mostramos que existen reticulados distributivos finitos que no son el reticulado de congruencia de cualquier reticulado semidistributivo finito. Para $0 \leq k \leq 2$, el reticulado distributivo $(\mathbf{B}_k)_{++} = \mathbf{2} + \mathbf{B}_k$, donde $\mathbf{B}_k$ denota el reticulado booleano con k átomos, no es el reticulado de congruencia de cualquier reticulado semidistributivo finito. Estos reticulados tampoco pueden ser un filtro en el reticulado de congruencia de un reticulado semidistributivo finito. De todas formas, cada $(\mathbf{B}_k)_{++}$ con $k \geq 3$ es el reticulado de congruencia de un reticulado semidistributivo finito, digamos $\mathbf{L}_k$. Estos reticulados $\mathbf{L}_k$ no pueden ser acotados (en el sentido de McKenzie), puesto que ningún $(\mathbf{B}_k)_{++}$ $(k \geq 0)$ es el reticulado de congruencia de un reticulado finito acotado. Un artículo acompañante muestra que todo $(\mathbf{B}_k)_{++}$ $(k \geq 0)$ puede ser representado como el reticulado de congruencia de un reticulado infinito semidistributivo. También encontramos condiciones suficientes para que un reticulado finito distributivo sea representable como el reticulado de congruencia de un reticulado finito acotado (y por lo tanto semidistributivo).

Keywords and Phrases: Distributive lattice, semidistributive lattice, congruence lattice.

2020 AMS Mathematics Subject Classification: 06B10, 06B15.

1 Introduction

R. P. Dilworth proved in the 1940's that every finite distributive lattice is the congruence lattice of a finite lattice. Not every finite distributive lattice is isomorphic to the congruence lattice of a finite join semidistributive (or meet semidistributive) lattice, but it turns out that there is only one restriction; see Theorem 1.1 below, from [1]. This note shows that there is at least one additional restriction on a finite distributive lattice $\mathbf{D}$ in order for $\mathbf{D}$ to be the congruence lattice of a finite (meet and join) semidistributive lattice; see Theorem 3.1.

In the paper K. Adaricheva *et al.* [1] it was shown that a finite distributive lattice $\mathbf{D} \cong \mathcal{O}(\mathbf{P})$ is the congruence lattice of a finite *join* semidistributive lattice if and only if every non-maximal element of $\mathbf{P}$ is below at least two maximal elements. In fact, the equivalence of five conditions is proved in that paper.

Theorem 1.1. *The following are equivalent for a finite distributive lattice* $\mathbf{D}$*. Let* $\mathbf{D} \cong \mathcal{O}(\mathbf{P})$ *for an ordered set* $\mathbf{P}$ *(isomorphic to* $\mathrm{J}(\mathbf{D})$*).*

(1) $\mathbf{D} \cong \operatorname{Con} \mathbf{L}$ *for a finite join semidistributive lattice* $\mathbf{L}$*.*

(2) $\mathbf{D} \cong \operatorname{Con} \mathbf{S}$ *for a finite lower bounded lattice* $\mathbf{S}$*.*

(3) $\mathbf{D} \cong \operatorname{Con} \mathbf{G}$ *for a finite convex geometry* $\mathbf{G}$*.*

(4) $\mathbf{D} \cong \operatorname{Con} \mathbf{A}$ *for a finite, lower bounded, atomistic convex geometry* $\mathbf{A}$*.*

(5) *Every non-maximal element of* $\mathbf{P}$ *is below at least two maximal elements.*

(6) *The three-element chain is not a filter in* $\mathbf{D}$*.*

We will show that there is at least one additional restriction for the congruence lattice $\operatorname{Con} \mathbf{K}$ when $\mathbf{K}$ is a finite lattice that is both join and meet semidistributive. The restrictions are perhaps best expressed in terms of the lattices $(\mathbf{B}_k)_{++}$ obtained by adjoining a new zero *twice* to a boolean lattice with k atoms. Theorem 3.1 is that neither $(\mathbf{B}_0)_{++}$ nor $(\mathbf{B}_2)_{++}$ can be a filter in the congruence lattice of a finite semidistributive lattice. (Since $(\mathbf{B}_0)_{++}$ is a three-element chain and $(\mathbf{B}_1)_{++}$ is a four-element chain, excluding the latter is redundant.) We can show that every $(\mathbf{B}_k)_{++}$ with $k \geq 3$ is the congruence lattice of a finite semidistributive lattice (Theorem 4.8). However, a lattice $\mathbf{K}$ with $\operatorname{Con} \mathbf{K} \cong (\mathbf{B}_k)_{++}$ for $\mathbf{K}$ finite, semidistributive and $k \geq 3$ cannot be bounded in the sense of McKenzie (Theorem 4.2). To complicate matters, it turns out that every lattice $(\mathbf{B}_k)_{++}$ with $k \geq 0$ is isomorphic to the congruence lattice of an *infinite* semidistributive lattice, as shown by the author and G. Grätzer [11]. It remains open whether every finite distributive lattice is the congruence lattice of an infinite semidistributive lattice.

2 Preliminaries on congruence lattices and semidistributivity

A lattice is *join semidistributive* if it satisfies the condition

$$x \vee y = x \vee z \text{ implies } x \vee y = x \vee (y \wedge z) .$$

The dual is called *meet semidistributive*, and a lattice is *semidistributive* if it is both join and meet semidistributive. This notion was introduced in Jónsson [12] as a basic property of free lattices; summaries of semidistributive lattices can be found in [3, 7].

A lattice homomorphism $h : \mathbf{K} \to \mathbf{L}$ is *lower bounded* if for every $a \in L$, $h^{-1}(\uparrow a)$ is either empty or has a least element. Dually, h is an *upper bounded* homomorphism if $h^{-1}(\downarrow a)$ has a greatest element whenever it is nonempty. A homomorphism that is both lower and upper bounded is called *bounded*.

A finitely generated lattice is said to be *bounded* if it is a bounded homomorphic image of a free lattice. The basic historical sources are R. McKenzie [15] and A. Day [5]; again more recent summaries can be found in [3, 7]. Bounded lattices inherit semidistributivity from free lattices.

For $k \geq 0$, $\mathbf{B}_k$ denotes the boolean lattice with k atoms; in particular, $\mathbf{B}_0$ is a one-element lattice. Given a lattice $\mathbf{K}$, let $\mathbf{K}_+$ denote the lattice obtained by adjoining a new zero element. The lattices $(\mathbf{B}_k)_{++}$ will play an important role in this paper.

For finite subsets X, Y of a lattice $\mathbf{L}$, we say that X refines Y, written $X \ll Y$, if for each $x \in X$ there exists $y \in Y$ such that $x \leq y$. An inclusion $p \leq \bigvee Q$, where $p \in L$ and $Q \subseteq L$ is a finite nonempty subset, is a *minimal nontrivial join cover* if $p \nleq q$ for all $q \in Q$ and Q cannot be properly refined, *i.e.*, if $p \leq \bigvee R$ and $R \ll Q$, then $Q \subseteq R$. When $p \leq \bigvee Q$ is a minimal nontrivial join cover, then Q is an antichain of join irreducible elements. We say that a minimal nontrivial join cover $p \leq \bigvee Q$ is *doubly minimal* if there is no minimal nontrivial join cover S with $p \leq \bigvee S < \bigvee Q$.

A join irreducible element p in a finite lattice has a unique lower cover, denoted p_*. A finite lattice $\mathbf{L}$ is meet semidistributive if and only if for each join irreducible element $p \in \mathrm{J}(\mathbf{L})$, there is a unique element $\kappa(p)$ that is maximal with respect to the property of being above p_* and not above p; see *e.g.* Theorem 2.56 of [7]. Thus $x \leq \kappa(p)$ if and only if $p_* \vee x \ngeq p$. Indeed, $\kappa(p)$ will be meet irreducible with the unique upper cover $\kappa(p)^* = p \vee \kappa(p)$. Note that if $p \leq \bigvee Q$ is a minimal nontrivial join cover and $q \in Q$, then $\bigvee(Q \setminus \{q\}) \leq \kappa(q)$; else q could be replaced by q_* for a proper refinement.

Let us review congruence lattices of finite lattices and the special properties of bounded ones.

Define five relations on the set of join irreducible elements $\mathrm{J}(\mathbf{L})$, the first three requiring meet semidistributivity.

- $p\,A\,q$ if $q < p < q \vee \kappa(q)$,
- $p\,B\,q$ if $p \neq q$, $p \leq p_* \vee q$, $p \nleq p_* \vee q_*$, or equivalently, $p \neq q$, $q \nleq \kappa(p)$, $q_* \leq \kappa(p)$,
- $C = A \cup B$,
- $p\,D\,q$ if $q \in Q$ for some minimal nontrivial join cover Q of p,
- $p\,E\,q$ if $q \in R$ for some doubly minimal nontrivial join cover R of p.

Now in a finite semidistributive lattice $E \subseteq C \subseteq D$, and the containments can be proper (Theorem 2.59 of [7]). Form the reflexive, transitive closures of the last three: $\overline{C}$, $\overline{D}$, $\overline{E}$. These are quasi-orders.

An *order ideal* of a quasi-ordered set $(\mathbf{Q}, \leq)$ is a subset $I \subseteq Q$ such that $s \leq t \in I$ implies $s \in I$. The order ideals of $\mathbf{Q}$ form a distributive lattice $\mathcal{O}(\mathbf{Q}, \leq)$. A standard result is that for any finite lattice, Con $\mathbf{L} \cong (\,\mathrm{J}(\mathbf{L}), \overline{D})$, see Chapter 10 of [17] or Section II.3 of [7]. But for bounded finite lattices, we also have Con $\mathbf{L} \cong \mathcal{O}(\mathrm{J}(\mathbf{L}), \overline{E})$, see Section 6.6 of [3] or Section 9 of [4]. (This does not mean that D and E are the same, but their reflexive, transitive closures $\overline{D}$ and $\overline{E}$ are.)

We assume a familiarity with the following basic facts of lattice theory.

- Every finite distributive lattice is isomorphic to the lattice of order ideals of its join irreducible elements, $\mathbf{D} \cong \mathcal{O}(\mathbf{P})$ where $\mathbf{P} = (\mathrm{J}(\mathbf{D}), \leq)$. By convention, $\mathcal{O}(\mathbf{P})$ includes the empty ideal.
- Equivalently, $\mathbf{D}$ is isomorphic to the lattice of order filters of meet irreducible elements, $\mathbf{D} \cong \mathcal{F}(\mathbf{Q})$, where $\mathbf{Q} = (\mathrm{M}(\mathbf{D}), \leq)$ and filters are ordered by *reverse* set inclusion.
- For disjoint unions of ordered sets, $\mathcal{O}(\mathbf{P} \,\dot{\cup}\, \mathbf{Q}) \cong \mathcal{O}(\mathbf{P}) \times \mathcal{O}(\mathbf{Q})$, while lattices satisfy Con$(\mathbf{K} \times \mathbf{L}) \cong (\text{Con } \mathbf{K}) \times (\text{Con } \mathbf{L})$. Hence we may restrict our attention to connected finite ordered sets.
- For any finite lattice $\mathbf{L}$, the congruence lattice Con $\mathbf{L}$ is isomorphic to the ideal lattice of the quasi-ordered set $\mathbf{Q} = (\mathrm{J}(\mathbf{L}), \overline{D})$, *i.e.*, Con $\mathbf{L} \cong \mathcal{O}(\mathrm{J}(\mathbf{L}), \overline{D})$.
- In particular, maximal members of $\mathbf{Q}$ correspond to simple homomorphic images of $\mathbf{L}$.
- The two-element lattice $\mathbf{2}$ is the only finite simple semidistributive lattice. (Infinite simple semidistributive lattices exist; see [8].)
- Thus for a finite semidistributive lattice, coatoms of Con $\mathbf{L}$ correspond to maximal members of $\mathbf{Q} = (\mathrm{J}(\mathbf{L}), \overline{D})$, which in turn correspond to join prime elements of $\mathbf{L}$. That is, the maximal proper congruences on a finite semidistributive lattice are exactly those with two classes, $\uparrow p$ and $\downarrow \kappa(p)$, where p is a join prime element.

- Every element in a finite join semidistributive lattice has a canonical join representation [13]. This canonical representation is the unique non-refinable join representation of the element, and refines every other join representation. Thus if $a = \bigvee B$ canonically and also $a = \bigvee C$, then $B \ll C$.
- In a finite join semidistributive lattice, the canonical joinands of $1_{\mathbf{L}}$ are join prime.
- The atoms of a finite meet semidistributive lattice are join prime.

While the whole theory of Day doubling of intervals is relevant to bounded lattices, for this paper we need only double points, which is easily described; see [5, 6, 10]. If $\mathbf{L}$ is a lattice and $a \in L$, let $\mathbf{L}[a]$ be the lattice on the set $L \setminus \{a\} \cup \{(a,0),(a,1)\}$ with the order $\leq'$ such that, for x, $y \in L \setminus \{a\}$ and $i \in \{0,1\}$,

- $x \leq' y$ iff $x \leq y$,
- $(a,0) \leq' (a,1)$,
- $x \leq' (a,i)$ iff $x \leq a$,
- $(a,i) \leq' y$ iff $a \leq y$.

Note that $(a,1)$ is join irreducible in $\mathbf{L}[a]$. Doubling intervals, and in particular points, preserves both meet and join semidistributivity, and both lower and upper boundedness [5].

3 Congruence lattices of finite semidistributive lattices

Consider the two ordered sets in Figure 1.

Theorem 3.1. *The distributive lattices $\mathcal{O}(\mathbf{2})$ and $\mathcal{O}(\mathbf{Y})$ are not the congruence lattice of a finite semidistributive lattice.*

Recall that the homomorphic images of a finite semidistributive lattice $\mathbf{L}$ are semidistributive. (More generally, bounded homomorphisms preserve semidistributivity; see the proof of Theorem 2.20 in [7].) It follows that neither $\mathbf{2}$ nor $\mathbf{Y}$ can be a filter in $(\mathrm{J}(\mathbf{L}), \overline{D})$ when $\mathbf{L}$ is a finite semidistributive lattice. Note that $\mathcal{O}(\mathbf{2}) = \mathbf{3}$ is the three-element chain $(\mathbf{B}_0)_{++}$, while $\mathcal{O}(\mathbf{Y}) = (\mathbf{B}_2)_{++}$. The four-element chain $\mathbf{4} = (\mathbf{B}_1)_{++}$ has $\mathbf{3}$ as a filter, so neither is it the congruence lattice of a finite semidistributive lattice. See also Lemma 3.3 below.

Proof. Elements of $(\mathrm{J}(\mathbf{L}), \overline{D})$ may be equivalence classes induced by the quasi-order $\overline{D}$, but maximal elements of $(\mathrm{J}(\mathbf{L}), \overline{D})$ correspond to singleton classes with one join prime element. This is because

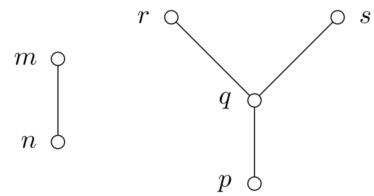

Figure 1: Ordered sets **2** and **Y**

every finite nontrivial semidistributive lattice contains join prime elements, and a join prime element p has no nontrivial join cover, making $p\,D\,q$ impossible when p is maximal in $(\mathrm{J}(\mathbf{L}), \overline{D})$. Now $\mathbf{2}$ is the only finite semidistributive lattice with only one join prime element, and its congruence lattice is $\mathbf{2} = \mathcal{O}(\mathbf{1})$, not $\mathbf{3} = \mathcal{O}(\mathbf{2})$. We conclude that Con $\mathbf{L} \cong \mathbf{3}$ cannot occur. (This argument applies with join semidistributivity only; see Theorem 1.1.)

So suppose $\mathbf{L}$ is a finite semidistributive lattice with Con $\mathbf{L} \cong \mathcal{O}(\mathbf{Y})$. Then $\mathbf{L}$ has two join prime elements, which includes its atoms and the canonical joinands of 1. The trivial case with one atom and $1_\mathbf{L}$ join prime would give $\mathbf{L} \cong \mathbf{3}$, while Con $\mathbf{3} \cong \mathbf{2} \times \mathbf{2}$, so that does not occur. Thus $\mathbf{L}$ has exactly two atoms, say r and s, and $1_\mathbf{L} = r \vee s$. Since r is an atom, $\kappa(r)$ is the largest element not above r, and similarly for $\kappa(s)$. So the coatoms of $\mathbf{L}$ are $\kappa(r)$ and $\kappa(s)$, and they satisfy $\kappa(r) \wedge \kappa(s) = 0_\mathbf{L}$. Thus $L = \{0, 1\} \,\dot\cup\, [r, \kappa(s)] \,\dot\cup\, [s, \kappa(r)]$, as in Figure 2.

Put $\mathbf{U} = [r, \kappa(s)]$ and $\mathbf{V} = [s, \kappa(r)]$. Note $u \vee v = 1$ and $u \wedge v = 0$ for any $u \in U$ and $v \in V$. Hence congruences behave independently on the sublattices $\mathbf{U}$ and $\mathbf{V}$. It follows that Con $\mathbf{L}$ is isomorphic to Con $\mathbf{U}$ $\times$ Con $\mathbf{V}$ with three additional elements on top, as illustrated in Figure 2. If Con $\mathbf{L} \cong \mathcal{O}(\mathbf{Y})$, then Con $\mathbf{U}$ $\times$ Con $\mathbf{V} \cong \mathbf{3} = \mathcal{O}(\mathbf{2})$, which is impossible by the first part. Therefore there is no finite semidistributive lattice with Con $\mathbf{L} \cong \mathcal{O}(\mathbf{Y})$. □

In the preceding argument, $\mathbf{U}$ and $\mathbf{V}$ are intervals of $\mathbf{L}$, and hence finite semidistributive lattices. On the other hand, one or both of these could have only one element. Hence, from the proof we conclude:

Corollary 3.2. *The following are equivalent for a finite distributive lattice* $\mathbf{D}$ *with two coatoms.*

(1) $\mathbf{D} \cong$ Con $\mathbf{L}$ *for some finite semidistributive lattice* $\mathbf{L}$.

(2) $\mathbf{D}$ *is a glued sum* $\mathbf{D} = \mathbf{E} \oplus (\mathbf{2} \times \mathbf{2})$ *where* $\mathbf{E} \cong$ Con $\mathbf{K}$ *for some finite semidistributive lattice* $\mathbf{K}$.

While the corollary applies only to distributive lattices with two coatoms, it allows us to construct a multitude of examples of distributive lattices, both representable and non-representable as congruence lattices of finite semidistributive lattices. The construction in the positive direction mimics Figure 2 with say $\mathbf{U} = \mathbf{K}$ and $|\mathbf{V}| = 1$.

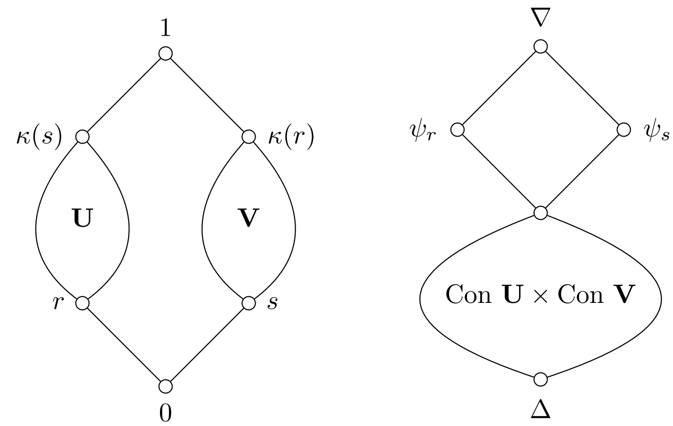

Figure 2: **L** and Con **L** for a finite semidistributive lattice with exactly two join prime elements, r and s. The congruence ψ_r collapses the intervals $[r, 1]$ and $[0, \kappa(r)]$, the congruence ψ_s collapses the interval $[s, 1]$ and $[0, \kappa(s)]$, so that $\mathbf{L}/(\psi_r \wedge \psi_s) \cong \mathbf{2} \times \mathbf{2}$.

It is currently unknown whether additional restrictions apply to congruence lattices of finite semidistributive lattices. By analogy with situation for finite join semidistributive lattices (Theorem 1.1) we conjecture that the restrictions of Theorem 3.1 are the only ones.

Conjecture: *A finite distributive lattice* $\mathbf{D}$ *is the congruence lattice of a finite semidistributive lattice if and only if neither the* 3*-element chain nor* $\mathcal{O}(\mathbf{Y}) = (\mathbf{B}_2)_{++}$ *is a filter of* $\mathbf{D}$.

With respect to such characterizations, we remind the reader of an elementary fact.

Lemma 3.3. *Let* $\mathbf{S}$ *and* $\mathbf{P}$ *be finite ordered sets. Then* $\mathcal{O}(\mathbf{S})$ *is isomorphic to a filter of* $\mathcal{O}(\mathbf{P})$ *if and only if* $\mathbf{S}$ *is a filter of* $\mathbf{P}$.

Proof. If $\mathbf{S}$ is a filter of $\mathbf{P}$, let $L = P \setminus S$. Clearly $\uparrow L$ is a filter of $\mathcal{O}(\mathbf{P})$ isomorphic to $\mathcal{O}(\mathbf{S})$.

Conversely, assume that $\mathcal{O}(\mathbf{S})$ is isomorphic to a filter of $\mathcal{O}(\mathcal{P})$, say $\mathcal{O}(\mathbf{S}) \cong \uparrow K$. Set $T = P \setminus K$ and $\mathbf{T} = (T, \leq)$ with the order inherited from $\mathbf{P}$. As the complement of an ideal, T is a filter in $\mathbf{P}$. Now $\mathbf{S} \cong \mathrm{J}(\mathcal{O}(\mathbf{S}))$. We want to establish an isomorphism $\nu : \mathbf{T} \cong \mathrm{J}(\uparrow K)$ between $\mathbf{T}$ and the ideals that are join irreducible in the filter $\uparrow K$ (which need not be join irreducible in $\mathcal{O}(\mathbf{P})$).

For $t \in T$, define $\nu(t) = K \cup \downarrow t$. Note that $\nu(t)$ is join irreducible in $\uparrow K$. In fact, for an ideal $L \geq K$, $t \in L$ iff $L \geq \nu(t)$.

On the other hand, if L is join irreducible in $\uparrow K$, then there is a unique ideal $L_\dagger$ with $L \succ L_\dagger \geq K$. There is only one element in $L \setminus L_\dagger$, and it must be in T. Denote this element by $\tau(L)$, so that $\tau(L) \in T$ and $L = L_\dagger \,\dot\cup\, \{\tau(L)\}$.

Now $\tau\nu(t) = t$ because $\nu(t) \succ \nu(t) \setminus \{t\} \geq K$. Let us show that $\nu\tau(L) = K \cup \downarrow \tau(L) = L$ when L is join irreducible in $\uparrow K$. Clearly $K \cup \downarrow \tau(L) \subseteq L$. Suppose the reverse inclusion fails. That means there exists an element $\ell_0 \in L \cap T$ with $\ell_0 \not\leq \tau(L)$. Let $\ell_1 \geq \ell_0$ be maximal in L, so also

$\ell_1 \in T$ and $\ell_1 \ngeq \tau(L)$. Then $L \succ L \setminus \{\ell_1\} \geq K$, yielding another lower cover of L in $\uparrow K$ besides $L_\dagger$, contrary to the assumption that L is join irreducible in the interval. Therefore $\nu(\tau(L)) = L$.

It remains to show that for I, L join irreducible in $\uparrow K$, $\tau(I) \leq \tau(L)$ iff $I \leq L$. But $I = \nu\tau(I) = K \cup \downarrow\tau(I)$ and $L = \nu\tau(L) = K \cup \downarrow\tau(L)$ with $\tau(I)$ and $\tau(L)$ not in K, from which the claim follows immediately. □

4 Congruence lattices of finite bounded lattices

Now we turn to finite lattices that are bounded homomorphic images of a free lattice. These inherit semidistributivity from the free lattice. Finite bounded lattices have many special properties, which we summarize here from [3], Sections 3-2.6 and 3-2.7, or [7], Sections II-4 and II-5, both of which have references to the original sources.

Theorem 4.1. *The following are equivalent for a finite semidistributive lattice* $\mathbf{L}$.

(1) $\mathbf{L}$ *is bounded.*

(2) $\mathbf{L}$ *is lower bounded.*

(3) $\mathbf{L}$ *is upper bounded.*

(4) $\mathrm{J}(\mathbf{L})$ *contains no* D*-cycle*

$$p_0 \, D \, p_1 \, D \, p_2 \, D \, \ldots \, D \, p_{m-1} \, D \, p_0.$$

(5) $\mathrm{J}(\mathbf{L})$ *contains no* E*-cycle*

$$p_0 \, E \, p_1 \, E \, p_2 \, E \, \ldots \, E \, p_{n-1} \, E \, p_0.$$

(6) $|\,\mathrm{J}(\mathrm{Con}\ \mathbf{L})| = |\,\mathrm{J}(\mathbf{L})|$.

Condition (6), from Pudlák and Tůma [19], is particularly important for us: if $\mathbf{L}$ is a finite bounded lattice with $\mathrm{Con}\ \mathbf{L} \cong \mathcal{O}(\mathbf{P})$, then there is a bijection between $\mathrm{J}(\mathbf{L})$ and $\mathbf{P}$. This is not true for unbounded semidistributive lattices in general, because of the presence of D-cycles as in (4).

Moreover, bounded finite lattices have $\mathrm{Con}\ \mathbf{L} \cong \mathcal{O}(\mathrm{J}(\mathbf{L}), \overline{E})$, where E is the relation on $\mathrm{J}(\mathbf{L})$ determined by doubly minimal join covers. This need not be true for unbounded lattices.

Recall that $\mathbf{B}_k$ denotes the boolean lattice with k atoms, and $\mathbf{L}_+$ denotes the lattice obtained by adding a new least element 0 to $\mathbf{L}$.

Theorem 4.2. *For* $k \geq 0$*, the distributive lattice* $(\mathbf{B}_k)_{++}$ *is not the congruence lattice of a finite bounded lattice.*

The proof uses a technical lemma, which is Theorem 2.60 in [7].

Lemma 4.3. *Let* $\mathbf{K}$ *be a finite semidistributive lattice, and let* $q \in \mathrm{J}(\mathbf{K})$*. Assume* $q \leq \bigvee R$ *is a doubly minimal nontrivial join cover. Then there is a unique* $r_0 \in R$ *such that* $r_0 \nleq q$*, and* $q\,B\,r_0$ *holds. The remaining* $r \in R \setminus \{r_0\}$ *satisfy* $r < q$ *and* $q\,A\,r$*.*

Proof. Assume that $q \leq \bigvee R$ is doubly minimal, and consider any $r \in R$. Let $S = R \setminus \{r\}$, noting that $\bigvee S \leq \kappa(r)$ else $r \vee \bigvee S = r_* \vee \bigvee S$, contradicting minimality. If $r < q$, then $r < q < r \vee \bigvee S \leq r \vee \kappa(r) = \kappa(r)^*$ so that $q\,A\,r$ holds.

Meanwhile, $q \leq \bigvee R$ implies $\bigvee R \nleq \kappa(q)$. Hence $q \leq q_* \vee r_0$ for at least one $r_0 \in R$. This refines to a minimal nontrivial join cover $q \leq \bigvee T$ with $T \ll \{q_*, r_0\}$. Clearly $\bigvee T \leq q \vee r_0 \leq \bigvee R$; by the double minimality, $\bigvee T = q \vee r_0 = \bigvee R$.

We have $q \leq q_* \vee r_0 = \bigvee R$. Suppose $q \leq q_* \vee r_{0*}$. Then by the double minimality of $\bigvee R$ we get $\bigvee R = q_* \vee r_{0*}$. Put $S = R \setminus \{r_0\}$, noting $\bigvee S \leq q_*$ by the preceding paragraph. Recall that in a join semidistributive lattice, $u = \bigvee a_i = \bigvee b_j$ implies $u = \bigvee_{i,j}(a_i \wedge b_j)$. (This is Theorem 1.21 in [7], from Jónsson and Kiefer [13].) Thus we calculate

$$\bigvee R = \bigvee S \vee r_0 = q_* \vee r_{0*} = \bigvee S \vee (r_0 \wedge q_*) \vee r_{0*} = \bigvee S \vee r_{0*}$$

which contradicts $q \leq \bigvee R$ being a minimal (nonrefinable) join cover. So $q \nleq q_* \vee r_{0*}$, whence $q\,B\,r_0$ holds.

By $(\mathrm{SD}_\vee)$, $R = T$ consists of the canonical joinands of $q \vee r_0$, all except one of which, namely r_0, lie below q_*. □

Corollary 4.4. *If* $q\,E\,s$ *and* $s \nleq q$ *in a finite semidistributive lattice, then* $q\,B\,s$*.*

Now we can prove Theorem 4.2.

Proof. We may assume $k \geq 3$, as the cases $0 \leq k \leq 2$ are covered by Theorem 3.1.

Suppose that $\mathbf{L}$ is a bounded lattice and that $\mathrm{Con}\,\mathbf{L} \cong (\mathbf{B}_k)_{++}$. Then $(\mathrm{J}(\mathbf{L}), \overline{E})$ is isomorphic to the ordered set drawn in Figure 3. Note that because $\mathbf{L}$ is bounded, the relation $\overline{E}$ is antisymmetric (as there are no E-cycles), making $\overline{E}$-classes singletons. So each point in Figure 3 represents an element of $\mathrm{J}(\mathbf{L})$.

Moreover, the elements labeled $r_1, \ldots, r_k$ in the top row are join prime in $\mathbf{L}$. Let $R_1 = \{r_1, \ldots, r_\ell\}$ be the join prime elements with $r_i < q$, and let $R_2 = \{r_{\ell+1}, \ldots, r_k\}$ be those with $r_j \nleq q$. As the diagram indicates, we have $p\,E\,q$ and $q\,E\,r_i$ for all i. Since $p\,E\,q$, in $\mathbf{L}$ there is at least one doubly minimal nontrivial join cover $p \leq q \vee \bigvee S$ with $S \subseteq R_1 \cup R_2$.

Clearly $S \cap R_1 = \varnothing$, *i.e.*, we cannot have $s < q$ with both in the same minimal join cover. So $S \subseteq R_2$. But if $s_0 \in R_2$, then $q\,B\,s_0$ by Corollary 4.4, so $q \leq q_* \vee s_0$. Thus $p \leq q_* \vee \bigvee S$, contradicting the minimality of $\{q\} \cup S$. □

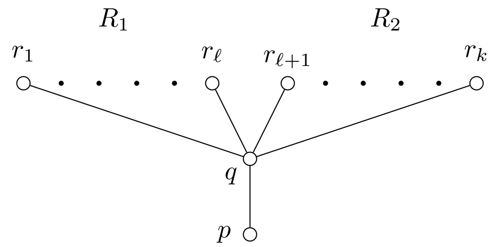

Figure 3: $(\mathrm{J}(\mathbf{L}), \overline{E})$ for $(\mathbf{B}_k)_{++}$

But now we encounter an unexpected surprise. The lattice $\mathbf{U}$ in Figure 4 is obtained by doubling the point p_0 in a lattice $\mathbf{U}_0$ from [14]. Now $\mathbf{U}$ is not bounded, because it has the D-cycle $p_0\,A\,p_1\,A\,p_2\,B\,p_3\,B\,p_0$. However, its congruence lattice Con $\mathbf{U}$ is $(\mathbf{B}_3)_{++}$.

Theorem 4.5. *The lattice $(\mathbf{B}_3)_{++}$ is the congruence lattice of a finite semidistributive lattice.*

A couple of lemmas are required to prove this.

When a point a is doubled in a finite lattice $\mathbf{L}$, then the principal congruence $\alpha = \mathrm{Cg}((a,0),(a,1))$ has only one nontrivial congruence class, so that α is an atom of Con $\mathbf{L}[a]$ with $\mathbf{L}[a]/\alpha \cong \mathbf{L}$. But we need a little more information as to which congruences lie above α. The calculation is based on the following straightforward lemmas.

Lemma 4.6. *Let $\mathbf{L}$ be a finite lattice. Double a join irreducible element $a \in \mathrm{J}(\mathbf{L})$, replacing a by $(a,0)$ and $(a,1)$. Note that both $(a,0)$ and $(a,1)$ are join irreducible in $\mathbf{L}[a]$.*

(1) *If $a\,D\,b$ in $\mathbf{L}$, then $(a,0)\,D\,b$ and $(a,1)\,D\,b$ in $\mathbf{L}[a]$.*

(2) *If $c\,D\,a$ in $\mathbf{L}$, then $c\,D\,(a,0)$ in $\mathbf{L}[a]$, but $c\;\not{D}\,(a,1)$.*

(3) *If $\mathbf{L}$ is meet semidistributive and $\kappa(a) \neq a_*$, then $(a,1)\,D\,(a,0)$.*

If p and q are join irreducible elements with $p\,D\,q$, then we have the congruence inclusion $\mathrm{Cg}(p,p_*) \leq \mathrm{Cg}(q,q_*)$. Thus whenever there is a D-cycle $p_0\,D\,p_1\,D\,\ldots\,D\,p_{n-1}\,D\,p_0$, then $\mathrm{Cg}(p_i,p_{i*}) = \mathrm{Cg}(p_j,p_{j*})$ for all i, j. We refer to this congruence as the *congruence generated by the cycle.*

Lemma 4.7. *Let $\mathbf{L}$ be a finite, subdirectly irreducible, semidistributive lattice with* Con $\mathbf{L} \cong \mathbf{D}$. *Suppose the monolith μ of $\mathbf{L}$ is generated by a proper D-cycle, and let a be a join irreducible element in that cycle. Then* Con $\mathbf{L}[a] \cong \mathbf{D}_+$.

The crucial observation is that if $p_0\,D\,p_1\,D\,\ldots\,D\,p_{n-1}\,D\,p_0$ is a D-cycle in $\mathbf{L}$, then by Lemma 4.6(1) and (2),

$$p_0\,D\,p_1\,D\,\ldots\,D\,(p_j,0)\,D\,\ldots\,D\,p_{n-1}\,D\,p_0$$

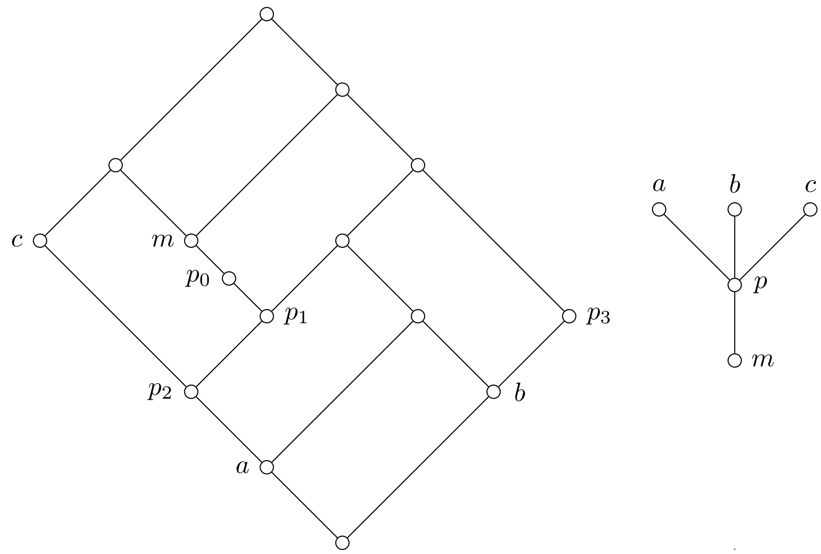

Figure 4: A finite semidistributive lattice with Con $\mathbf{U} \cong (\mathbf{B}_3)_{++}$. On the left $\mathbf{U}$, on the right $(\mathrm{J}(\mathbf{U}), \overline{D})$.

is a D-cycle in $\mathbf{L}[p_j]$.

In our situation, for Theorem 4.5 we have the original lattice from [14] with congruence lattice isomorphic to $(\mathbf{B}_3)_+$. Doubling p_0 to get the element labeled m in $\mathbf{U}$, as in the figure, yields Con $\mathbf{U} \cong (\mathbf{B}_3)_{++}$ and thus Theorem 4.5.

With the lattice $\mathbf{U}$ as a pattern, we can find more examples. The lattice $\mathbf{U}_0$ from [14] has a D-cycle of the form $AABB$ and 3 join prime elements. We would like to find finite, subdirectly irreducible, semidistributive lattices $\mathbf{L}_0$ whose join irreducibles consist of a D-cycle and k join prime elements, so that Con $\mathbf{L}_0 \cong (\mathbf{B}_k)_+$. Then double an element p in the D-cycle to obtain $(\mathbf{B}_k)_{++}$ as the congruence lattice of $\mathbf{L}_0[p]$.

In [18] there is a finite, semidistributive, unbounded lattice $\mathbf{V}_6$ based on a D-cycle of the form $(AB)^3$ that has Con $\mathbf{V}_6 \cong (\mathbf{B}_6)_+$. Doubling a join irreducible in the cycle yields another finite semidistributive lattice $\mathbf{W}_6$ with Con $\mathbf{W}_6 \cong (\mathbf{B}_6)_{++}$. A straightforward generalization of the construction in [18], using a cycle of the form $(AB)^m$ for $m \geq 3$, gives a finite semidistributive lattice $\mathbf{W}_{2m}$ with Con $\mathbf{W}_{2m} \cong (\mathbf{B}_{2m})_{++}$. The general construction to represent all $(\mathbf{B}_n)_{++}$ with $n \geq 4$ is somewhat more complicated.

Theorem 4.8. *For all $k \geq 3$, the lattice $(\mathbf{B}_k)_{++}$ is the congruence lattice of a finite semidistributive lattice.*

As noted earlier, all the lattices $(\mathbf{B}_k)_{++}$ $(k \geq 0)$ can be represented as the congruence lattice of an *infinite* semidistributive lattice [11].

Proof. The case $k = 3$ is Theorem 4.5, so let us consider $n \geq 4$. We will construct a finite semidistributive lattice $\mathbf{X}_n$ whose join irreducible elements have the following properties:

- there is a D-cycle of the form $B^2 A^{n-2}$,

$$p_0 \, B \, p_1 \, B \, p_2 \, A \, p_3 \, A \, \ldots \, A \, p_{n-1} \, A \, p_0 \ ;$$

- there are n join prime elements p_{0*}, p_{1*}, $x_3, \ldots, x_n$;
- for each join prime element q there is a p_j such that $p_j \, D \, q$;
- there are no more join irreducible elements in $\mathbf{X}_n$.

Thus Con $\mathbf{X}_n \cong (\mathbf{B}_n)_+$. Applying Lemma 4.7 to double an element in the cycle yields a lattice $\mathbf{Y}_n$ with Con $\mathbf{Y}_n \cong (\mathbf{B}_n)_{++}$.

A standard duality for finite lattices is to regard $\mathbf{L}$ as a closure system on the ordered set of its join irreducibles $\mathbf{J} = (\mathrm{J}(\mathbf{L}), \leq)$. Given $\mathbf{L}$, the map $a \mapsto \downarrow a \cap \mathbf{J}$ represents the lattice as an intersection-closed collection of subsets of $\mathbf{J}$. The corresponding closure operator γ on $\mathbf{J}$ is given by

$$x \in \gamma(\{y\}) \text{ if } x \leq y,$$
$$x \in \gamma(Y) \text{ if } x \leq \bigvee Y$$

for x, $y \in J$ and $Y \subseteq J$. Then $\mathbf{L}$ is isomorphic to the lattice of γ-closed subsets of $\mathbf{J}$ (which are automatically order ideals by the first rule, including the empty ideal $\varnothing$).

To construct a lattice using the duality, we must specify the ordered set $\mathbf{J}$ and a basis for the desired join operation. Following custom, we write the closure rules as $x \leq y$ and $x \leq \bigvee Y$, respectively. Part of the verification will include checking that $\leq$ is a partial order, and that $\gamma(x) \setminus \{x\} = \downarrow x \setminus \{x\}$ is closed for $x \in J$, so that x is join irreducible.

To construct $\mathbf{X}_n$ with the properties described above, for $n \geq 4$ we take

$$\mathbf{J}_n = \{p_{0*}, p_0, p_{1_*}, p_1, p_2, \ldots, p_{n-1}, x_3, \ldots, x_n\}.$$

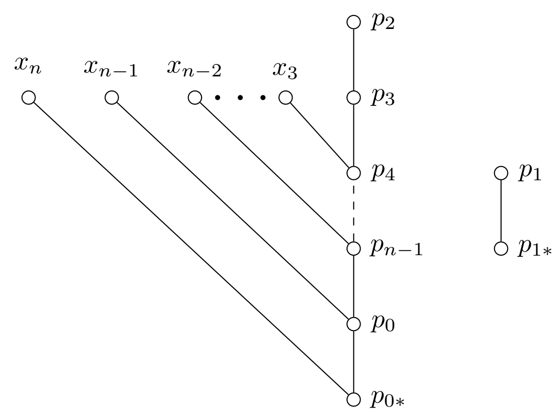

Figure 5: The order on the join irreducibles of $\mathbf{X}_n$

The order on $\mathbf{J}_n$ is given by

$$
\begin{aligned}
&p_2 > p_3 > \cdots > p_{n-1} > p_0 > p_{0*} \\
&x_j > p_{j+1} \text{ for } 3 \leq j < n-1 \\
&x_{n-1} > p_0 \\
&x_n > p_{0*} \\
&p_1 > p_{1*}
\end{aligned}
$$

as illustrated in Figure 5. The defining join covers are

$$
(\ddagger) \qquad
\begin{aligned}
&p_0 \leq p_{0*} \vee p_1 \\
&p_1 \leq p_{1*} \vee p_2 \\
&p_2 \leq p_3 \vee x_3 \\
&\cdots \\
&p_{n-2} \leq p_{n-1} \vee x_{n-1} \\
&p_{n-1} \leq p_0 \vee x_n \\
&p_1 \leq p_{1*} \vee x_3
\end{aligned}
$$

The last is a bit of a mystery, but is required for meet semidistributivity, and does the job.

Set $\mathbf{X}_n$ to be the lattice of closed ideals of $\mathbf{J}_n$. Routine checks, with multiple cases, show that the elements of $\mathbf{J}_n$ are join irreducible, with the lower covers u_* as indicated in Figure 5, and that the join covers given in the basis (‡) are minimal (nonrefinable). Thus for each inclusion $u \leq y \vee z$ in (‡) we have $u \, D \, y$ and $u \, D \, z$. These facts give the desired properties from the first paragraph of the proof. It remains to prove that $\mathbf{X}_n$ is semidistributive.

To see that $\mathbf{X}_n$ is meet semidistributive, we must show that every join irreducible element q has a unique element $\kappa(q) \in \mathbf{X}_n$ that is maximal w.r.t. being above q_* and not above q. The elements $p_{0*}, p_{1_*}, x_3, \ldots, x_n$ are join prime, so for them $\kappa(q) = \bigvee\{u \in J : u \ngeq q\}$. For the rest, we calculate as follows.

$$\begin{aligned}
&\kappa(p_0) = p_{1*} \vee x_n \\
&\kappa(p_1) = p_{1*} \vee p_3 \vee \bigvee_{4 \le j \le n} x_j \\
&\kappa(p_2) = p_1 \vee p_3 \vee \bigvee_{4 \le j \le n} x_j \\
&\kappa(p_3) = p_1 \vee x_3 \vee \bigvee_{5 \le j \le n} x_j \\
&\kappa(p_4) = p_1 \vee x_4 \vee \bigvee_{6 \le j \le n} x_j \\
&\ldots \\
&\kappa(p_{n-2}) = p_1 \vee x_{n-2} \vee x_n \\
&\kappa(p_{n-1}) = p_1 \vee x_{n-1}
\end{aligned}$$

Now we appeal to two lemmas from [18], the second one slightly enhanced.

Lemma 4.9. *Let* $\mathbf{L}$ *be a finite lattice. Then* $\mathbf{L}$ *satisfies* $(\mathrm{SD}_\wedge)$ *if and only if* $\kappa(a)$ *exists for each* $a \in \mathrm{J}(\mathbf{L})$.

Lemma 4.10. *Let* $\mathbf{L}$ *be a finite lattice that satisfies* $(\mathrm{SD}_\wedge)$. *The following are equivalent.*

(1) $\mathbf{L}$ *satisfies* $(\mathrm{SD}_\vee)$.

(2) *There do not exist* a, $b \in \mathrm{J}(\mathbf{L})$ *such that* $a\,B\,b\,B\,a$.

(3) *There do not exist* a, $b \in \mathrm{J}(\mathbf{L})$ *such that* $a \neq b$ *and* $\kappa(a) = \kappa(b)$.

Proof. The equivalence of (1) and (2) is Theorem 8 of [18].

The definition of $a\,B\,b$ is equivalent to $a \neq b$, $b_* \le \kappa(a)$, $b \nleq \kappa(a)$. Thus $a\,B\,b$ implies $\kappa(a) \le \kappa(b)$ in a meet semidistributive lattice (though not conversely). If $a\,B\,b\,B\,a$, then $a \neq b$ and $\kappa(a) = \kappa(b)$.

Finally, assume $a \neq b$ and $\kappa(a) = \kappa(b) = m$, say. Then $a \vee m = m^* = b \vee m > (a \wedge b) \vee m$, since $a \wedge b \le a_*$ or $a \wedge b \le b_*$. This is a failure of $(\mathrm{SD}_\vee)$. □

We have just checked that $\kappa(a)$ exists for each $a \in \mathrm{J}(\mathbf{X}_n)$, with the values given above. By Lemma 4.9, $\mathbf{X}_n$ satisfies $(\mathrm{SD}_\wedge)$. Moreover, it is straightforward to check that the values of $\kappa(a)$ are all distinct, so $\mathbf{X}_m$ satisfies $(\mathrm{SD}_\vee)$ by Lemma 4.10. Thus $\mathbf{X}_n$ is semidistributive.

This completes the proof of Theorem 4.8. □

Some comments on the differences between representing congruence lattices of bounded versus unbounded lattices are in order. The problems are twofold.

First, while Con $\mathbf{L} \cong \mathcal{O}(\mathrm{J}(\mathbf{L}), \overline{D})$ holds for all finite lattices, we would like to use the order induced by the E-relation. However, Con $\mathbf{L} \cong \mathcal{O}(\mathrm{J}(\mathbf{L}), \overline{E})$ holds for all bounded finite lattices, does not hold for all finite join semidistributive lattices, and it is unknown whether the E-relation suffices for finite semidistributive lattices. See the discussion in Section 6.6 of [2].

The second difficulty is that unbounded finite semidistributive lattices contain D-cycles, making the order on $\mathrm{J}(\mathbf{L})$ a proper quasi-order rather than a partial order. In that case it is necessary to work with $\overline{D}$-equivalence classes of join irreducibles. Little is known about the structure of unbounded finite semidistributive lattices, except that they fail Whitman's condition (W) [16]. The examples used above, from [14] and [18], may be the only examples in the literature. W. Geyer constructed others using formal concept analysis in connection with [9], but they may not have been published. Our general construction was modeled on [18].

5 A sufficient condition

If behooves us then to find *sufficient* conditions for a finite distributive lattice to be the congruence lattice of a finite semidistributive lattice.

Theorem 5.1. *Let* $\mathbf{P}$ *be a finite ordered set satisfying*

($\diamondsuit$) $\mathbf{P}$ *is a tree, i.e., no element has more than one lower cover,*

($\clubsuit$) *every non-maximal element in* $\mathbf{P}$ *has at least two upper covers.*

Then $\mathcal{O}(\mathbf{P})$ *is isomorphic to the congruence lattice of a finite bounded (and in particular semidistributive) lattice.*

In fact, the condition ($\diamondsuit$) that $\mathbf{P}$ be a tree is much stronger than needed for the construction to work, and is just the simplest way to guarantee that the technical condition of Theorem 5.11 holds.

Here is a sketch of our itinerary. We are given the ordered set $\mathbf{P} = (P, \leq)$. Define a new ordered set $\overline{\mathbf{P}} = (P, \sqsubseteq)$ with the same base set but a different order, described below. In fact, it will have the property that $x \sqsubseteq y$ implies $x \geq y$. The lattice $\mathbf{M}$ that we construct with Con $\mathbf{M} \cong \mathcal{O}(\mathbf{P})$ will be the lattice of closed ideals of a closure operator on $\overline{\mathbf{P}}$. The join irreducible elements of $\mathbf{M}$ will be the principal ideals $\downarrow_{\sqsubseteq} u$ with $u \in P$.

When there is any chance of confusion, we write either $(\mathbf{P}, \leq)$ or $(\overline{\mathbf{P}}, \sqsubseteq)$. The base set of both is P, and $\mathbf{P}$ by itself means $(\mathbf{P}, \leq)$.

The order $\sqsubseteq$ uses a function $\dagger : P \to P$ so that if x is not maximal in $(\mathbf{P}, \leq)$, then $x_\dagger$ is the unique lower cover of x in $(\overline{\mathbf{P}}, \sqsubseteq)$. This will imply that if $\downarrow_\sqsubseteq x$ is not an atom of $\mathbf{M}$, then $\downarrow_\sqsubseteq x_\dagger$ is its unique lower cover in $\mathbf{M}$, whence $\downarrow_\sqsubseteq x$ is join irreducible. The basic idea is to define the join on $\mathbf{M}$ so that $x \sqsubseteq x_\dagger \vee y$ whenever $x_\dagger \neq y \succ x$ in $(\mathbf{P}, \leq)$. There is a slight complication: if $x \sqsubseteq x_\dagger \vee y$ is a minimal nontrivial join cover and $y \sqsubseteq y_\dagger \vee z$, then meet semidistributivity implies $x \sqsubseteq x_\dagger \vee z$. The recursive definitions in the construction are a way of addressing this difficulty. With that guide, let us proceed.

Assume that $\mathbf{P}$ satisfies (♣). For each non-maximal $p \in P$, choose an element $p_\dagger \succ p$ in $(\mathbf{P}, \leq)$. If p is maximal, then $p_\dagger$ is undefined. Let $p_\dagger \sqsubset p$, and take the reflexive, transitive closure of $\sqsubset$ as the order $\sqsubseteq$ on P. (There will in general be many options for the $\dagger$-function, but choose one.)

Let us consider the order $\sqsubseteq$ on the elements of P. The ideal of $(\overline{\mathbf{P}}, \sqsubseteq)$ generated by an element $u \in P$ is $\downarrow_\sqsubseteq u = \{u, u_\dagger, u_{\dagger\dagger}, \dots\}$. Use $u_{(k)}$ to denote $u_{\dagger\cdots\dagger}$ with k daggers.

Lemma 5.2. *The order $\sqsubseteq$ on $\overline{\mathbf{P}}$ satisfies the following.*

(1) $u \sqsubseteq v$ *if and only if* $u = v_{(k)}$ *for some* $k \geq 0$.

(2) $u \sqsubseteq v$ *implies* $u \geq v$.

(3) $\downarrow_\sqsubseteq v$ *is a chain.*

(4) $\uparrow_\sqsubseteq u$ *is a tree.*

The proofs are straightforward. Note that (3) and (4) are equivalent in any ordered set.

Next, for each $x \in P$, we partition P into subsets $K(x)$ and $L(x) = P \setminus K(x)$. This is done recursively on the depth of x in $(\mathbf{P}, \leq)$. If x is a maximal element, then

$$K(x) = \{z \in P : x \not\sqsubseteq z\}$$
$$L(x) = \{z \in P : x \sqsubseteq z\} = \uparrow_\sqsubseteq x\,.$$

If x is not maximal in $(\mathbf{P}, \leq)$ and $K(u)$, $L(u)$ are defined for all $u > x$, set

$$K(x) = \bigcap_{x_\dagger \neq y \succ x} K(y) \;\cap\; \{z \in P : x \not\sqsubseteq z\}$$
$$L(x) = \bigcup_{x_\dagger \neq y \succ x} L(y) \;\cup\; \{z \in P : x \sqsubseteq z\}\,.$$

By induction on the depth of x in $(\mathbf{P}, \leq)$, and using DeMorgan's laws, one can show that $P = K(x) \,\dot\cup\, L(x)$ for all $x \in P$.

Lemma 5.3. *For $x \in P$,*

(1) $x \in L(x)$, *whence* $x \notin K(x)$,

(2) *if* $x_\dagger \neq y \succ x$, *then* $y \in L(x)$,

(3) *if* $z \sqsubseteq u \in K(x)$, *then* $z \in K(x)$,

(4) *if* $t \sqsupseteq v \in L(x)$, *then* $t \in L(x)$.

The last pair says that $K(x)$ is an order ideal in $(\overline{\mathbf{P}}, \sqsubseteq)$ and $L(x)$ is an order filter. Item (3) requires an easy induction, and (4) follows by complementation.

Let us describe $L(x)$ and $K(x)$ more completely. Recursively define subsets $L^*(x) \subseteq P$ for $x \in P$ by

$$L^*(x) = \begin{cases} \{x\} & \text{if } x \text{ is maximal in } (\mathbf{P}, \leq), \\ \{x\} \cup \bigcup_{x_\dagger \neq y \succ x} L^*(y) & \text{otherwise.} \end{cases}$$

Lemma 5.4. *For all $x \in P$,*

(1) $L^*(x)$ *is contained in* $\uparrow_\leq x$,

(2) $u \in L(x)$ *if and only if* $u \sqsupseteq v$ *for some* $v \in L^*(x)$.

The proofs are straightforward induction using the definitions of $L(x)$ and $L^*(x)$.

Now to describe $K(x)$.

Lemma 5.5. *For each $x \in P$,*

$$K(x) = P \setminus \bigcup_{u \in L^*(x)} \uparrow_\sqsubseteq u$$

Proof. If x is maximal, $K(x) = P \setminus \uparrow_\sqsubseteq x$. So assume the statement holds for all $u > x$. Then

$$\begin{aligned} K(x) &= \bigcap_{x_\dagger \neq y \succ x} K(y) \ \cap \ \{z \in P : x \not\sqsubseteq z\} \\ &= \bigcap_{x_\dagger \neq y \succ x} \left(P \setminus \bigcup_{u \in L^*(y)} \uparrow_\sqsubseteq u \right) \ \cap \ (P \setminus \uparrow_\sqsubseteq x) \\ &= \bigcap_{u \in L^*(x)} (P \setminus \uparrow_\sqsubseteq u) \\ &= P \setminus \bigcup_{u \in L^*(x)} \uparrow_\sqsubseteq u \end{aligned}$$

by DeMorgan's laws. □

Now we make additional assumptions about $(\mathbf{P}, \leq)$ and $\dagger$:

($\heartsuit$) if $x_\dagger \neq y \succ x$ in $(\mathbf{P}, \leq)$, then

(a) $x_\dagger \in K(y)$,

(b) $y \in K(x_\dagger)$,

(c) $y_\dagger \in K(x)$.

The condition looks mysterious, so some discussion is in order.

Long aside on ($\heartsuit$).

The first observation is straight from the definitions, using (a), but important.

Lemma 5.6. *If* ($\heartsuit$) *holds and* $x \in P$ *is not maximal in* $(\mathbf{P}, \leq)$*, then* $x_\dagger \in K(x)$*.*

Consequently, condition (c) is equivalent to

(c′) if both y and y' satisfy $x_\dagger \neq u \succ x$, then $y_\dagger \in K(y')$.

Corollary 5.7. *Assume* ($\heartsuit$) *holds and* $x \in P$ *is not maximal in* $(\mathbf{P}, \leq)$*. If* $w \in P$ *satisfies* $w \sqsupseteq x_\dagger$ *and* $w \not\sqsupseteq x$*, then* $w \in K(x)$*.*

Proof. If $w \notin K(x)$ then $w \in L(x)$, which means that $w \sqsupseteq t$ for some $t \in L^*(x)$. Remember that $\downarrow_\sqsubseteq w$ is a chain, so $x_\dagger = w_{(i)}$ and $t = w_{(j)}$ for some pair i, j. But $w \not\sqsupseteq x$, so $t \in L^*(y)$ for some y with $x_\dagger \neq y \succ x$. This implies $t > x$. Hence $i < j$, making $x_\dagger \sqsupset t$, which is a contradiction since $x_\dagger \in K(x)$ and $t \in L(x)$. □

How could ($\heartsuit$) fail? Consider $x_\dagger \neq y \succ x$ in $(\mathbf{P}, \leq)$, and for (iii) also $x_\dagger \neq y' \succ x$. Here are some failures of (a), (b), and (c′) respectively.

(i) If $y_\dagger \neq x_{(k)} \succ y$ for some $k > 0$, then $x_{(k)} \in L(y)$, whence $x_\dagger \in L(y)$ since $x_\dagger \sqsupseteq x_{(k)}$.

(ii) If $x_{\dagger\dagger} \neq y_{(\ell)} \succ x_\dagger$ for some $\ell > 0$, then $y_{(\ell)} \in L(x_\dagger)$, whence $y \in L(x_\dagger)$ since $y \sqsupseteq y_{(\ell)}$.

(iii) If $y'_\dagger \neq y_{(m)} \succ y'$ for some $m > 0$, then $y_{(m)} \in L(y')$, whence $y_\dagger \in L(y')$ since $y_\dagger \sqsupseteq y_{(m)}$, contra (c′).

Figures 6 and 7 illustrate these situations. Figure 6 shows the conditions (i)–(iii) prohibited by ($\heartsuit$), while Figure 7 indicates the exceptions allowed. Solid black lines are covers, solid red lines are covers of the form $u_\dagger \succ u$, and dashed red lines indicate sequences of covers from u to $u_{(k)}$ with $k \geq 1$.

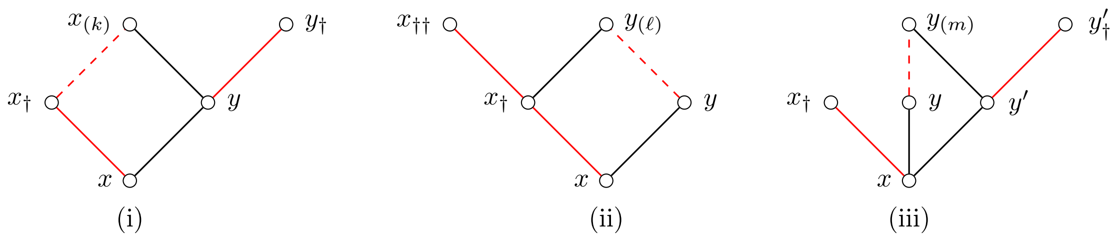

Figure 6: Configurations prohibited in $(\mathbf{P}, \leq)$ by $(\heartsuit)$

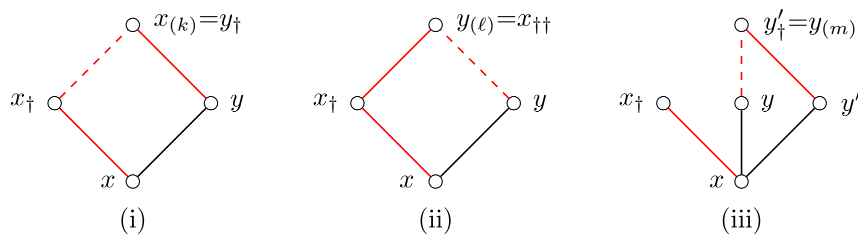

Figure 7: Exceptions allowed by $(\heartsuit)$

The failures of $(\heartsuit)$ in (i)–(iii) were *direct*, in that they used $x_\dagger$ and y. The next type of failures are *once removed*, using a cover z of one of those elements. Again let $x_\dagger \neq y \succ x$.

(iv) If $x_{(k)} \succ z \succ y$ for some $k > 0$, with $x_{(k)} \neq z_\dagger$ and $z \neq y_\dagger$, then $x_{(k)} \in L(y)$, whence $x_\dagger \in L(y)$.

(v) If $y_{(\ell)} \succ z \succ x_\dagger$ for some $\ell > 0$, with $y_{(\ell)} \neq z_\dagger$ and $z \neq x_{\dagger\dagger}$, then $y_{(\ell)} \in L(x_\dagger)$, whence $y \in L(x_\dagger)$.

(vi) If $y_{(m)} \succ z \succ y'$ for some $m > 0$, with $y_{(m)} \neq z_\dagger$ and $z \neq y'_\dagger$, then $y_{(m)} \in L(y')$, whence $y \in L(y')$, contra (c$'$).

Cases (iv)–(vi) are illustrated in Figure 8.

Continuing in this manner, we arrive at the following characterization.

Theorem 5.8. *An ordered set* $\mathbf{P}$ *with a* $\dagger$*-function satisfies* $(\heartsuit)$ *if and only if there do not exist* k, $\ell \geq 2$ *and elements* u, x *and covering chains*

$$u = c_0 \succ c_1 \succ \cdots \succ c_{k-1} \succ x$$

$$u = d_0 \succ d_1 \succ \cdots \succ d_{\ell-1} \succ x$$

with $c_{i-1} = c_{i\dagger}$ *for* $1 \leq i \leq k-1$ *and* $d_{j-1} \neq d_{j\dagger}$ *for* $1 \leq j \leq \ell - 1$.

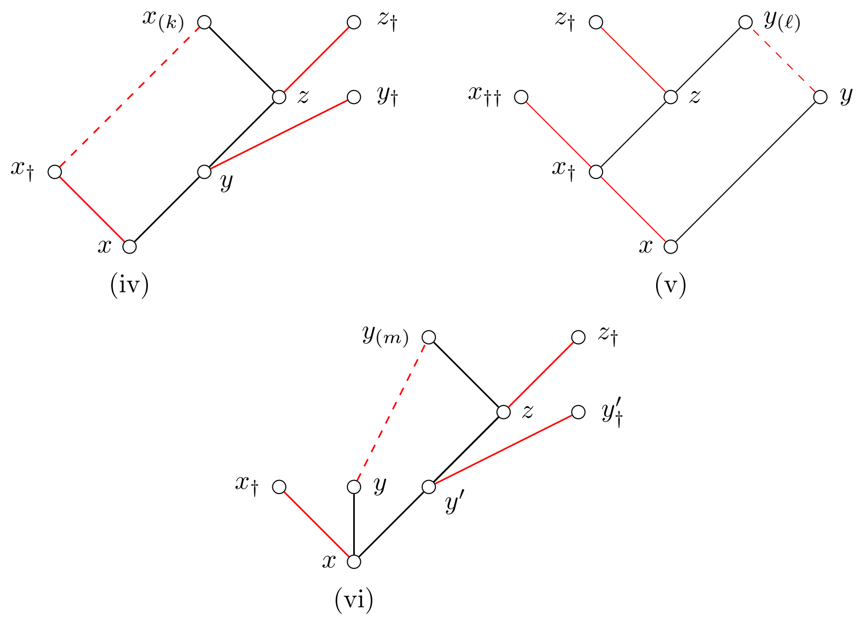

Figure 8: More configurations prohibited in $(\mathbf{P}, \leq)$ by $(\heartsuit)$, cases (iv)–(vi)

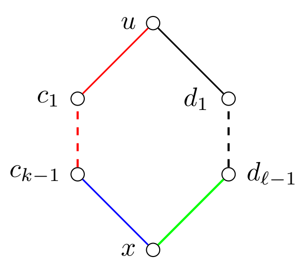

Figure 9: Prohibited configuration from Theorem 5.8. The blue and green edges can be red or black, but not both red.

The forbidden configuration is illustrated in Figure 9 where again red edges indicate $c_{i-1} = c_{i\dagger}$ and black edges indicate $d_{j-1} \neq d_{j\dagger}$. The blue and green edges can be either, except they cannot both be red, *i.e.*, we can have $x_\dagger = c_{k-1}$ or $x_\dagger = d_{\ell-1}$ or neither, but not both.

This must be balanced with the requirement that $x_\dagger$ be defined for every non-maximal $x \in P$. As an immediate consequence of Theorem 5.8, we see that there is a †-function satisfying $(\heartsuit)$ whenever

- $\mathbf{P}$ is a tree (the condition $(\diamondsuit)$ of Theorem 5.1), or
- the height of $\mathbf{P}$ is at most 2, *i.e.*, $\mathbf{P}$ contains no 3-element chain.

To find a more general sufficient condition for $\mathbf{P}$, satisfying $(\clubsuit)$, to admit a †-function satisfying $(\heartsuit)$, we imagine that † is given, and color the edges (covers) of the form $(c, c_\dagger)$ of $\mathbf{P}$ red, the

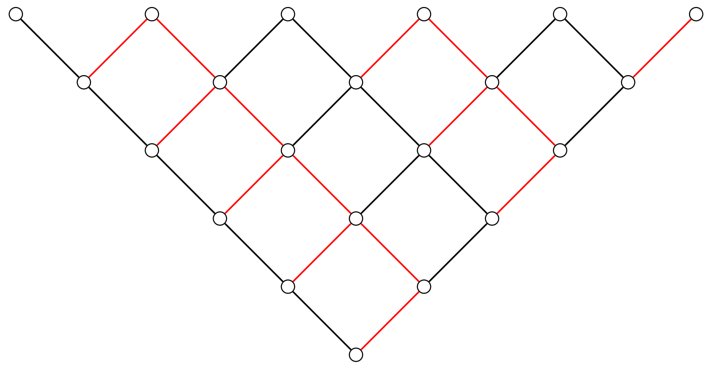

Figure 10: An ordered set with an edge-coloring that satisfies the conditions of Theorem 5.9, and hence $\mathcal{O}(\mathbf{P})$ is representable by Theorem 5.11.

remaining edges black. Classify the non-minimal vertices of $\mathbf{P}$ thusly.

- An element of P is a Λ-*node* if it has ≥ 2 lower covers.
- An element with 1 lower cover is an S-*node*.
- s is a *red* Λ-*node* if all its lower covering edges are red.
- t is a *black* Λ-*node* if all its lower covering edges are black.
- u is a *red* S-*node* if its unique lower covering edge is red.
- v is a *black* S-*node* if its unique lower covering edge is black.
- w is a *mixed node* if it is a Λ-node with both red and black lower covers.

Theorem 5.9. *Let* $\mathbf{P}$ *be a finite ordered set that satisfies* ($\clubsuit$)*. Suppose there is a coloring of the edges of* $\mathbf{P}$ *such that*

(i) $\mathbf{P}$ *has no mixed nodes,*

(ii) every non-maximal node has exactly 1 red upper cover and ≥ 1 *black upper covers.*

For non-maximal elements $x \in P$*, define* $x_\dagger$ *to be the red upper cover of* x*. Then* $\mathbf{P}$ *with the function* $\dagger$ *satisfies* ($\heartsuit$)*.*

For the configuration of Theorem 5.8 cannot occur, as every Λ-node is either red or black. Item (ii) guarantees that there is a unique choice for $x_\dagger$. Examples are given in Figures 10 and 12.

Conjecture: *If* $\mathbf{P}$ *is planar, then it has a coloring satisfying the conditions of* Theorem 5.9*.*

The ordered sets $\mathbf{P}$ for which $\mathcal{O}(\mathbf{P})$ is known not to be representable as the congruence lattice of a finite semidistributive or bounded lattice are all excluded by the condition ($\clubsuit$). It takes a

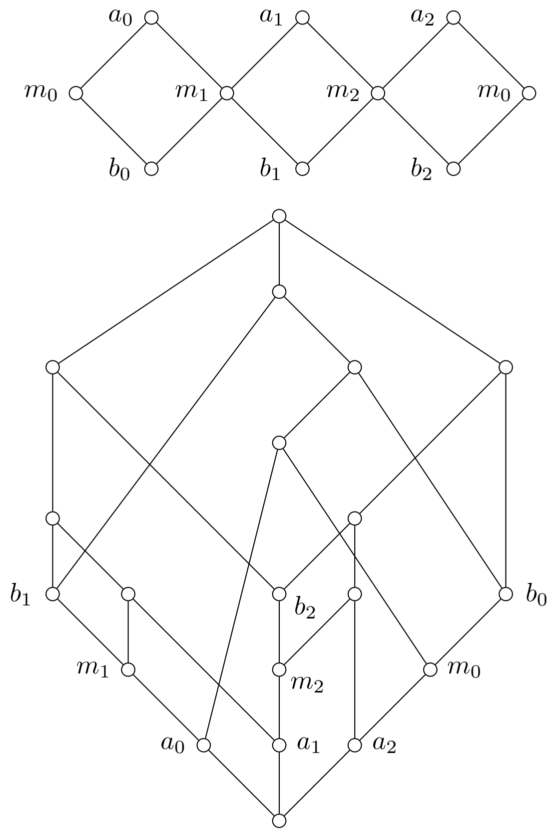

Figure 11: The ordered set $\mathbf{Q}$ at the top that satisfies (♣) but has no †-function satisfying (♡). Note that $\mathbf{Q}$ is a torus: m_0 is depicted twice. Nonetheless, $\mathcal{O}(\mathbf{Q}) \cong \text{Con}\ \mathbf{K}$ for the bounded lattice at the bottom.

little effort to find an ordered set $\mathbf{Q}$ that satisfies (♣) but fails (♡). Nonetheless, they exist, and the ordered set $\mathbf{Q}$ at the top of Figure 11 gives one such. By circular symmetry we may assume $m_{0\dagger} = a_0$. To avoid the configuration of Figure 9, that implies $m_{1\dagger} = a_0$. Hence $m_{1\dagger} \neq a_1$, whence $m_{2\dagger} \neq a_1$. That in turn leads to $m_{2\dagger} = a_2$ and $m_{0\dagger} = a_2$, a contradiction.

Even though Theorem 5.11 does not apply, $\mathcal{O}(\mathbf{Q})$ is the congruence lattice of a finite bounded lattice. The lattice $\mathbf{K}$ at the bottom of Figure 11 was obtained from $\mathbf{B}_3$ by two sets of doubling three intervals, so it is bounded. The minimal nontrivial join covers in $\mathbf{K}$ are

$$m_i \leq a_i \vee a_{i+2}$$
$$b_i \leq m_i \vee a_{i+1}$$
$$b_i \leq m_{i+1} \vee a_{i+2}$$

where the subscripts are taken modulo 3. Thus $(\mathrm{J}(\mathbf{K}), \overline{D}) \cong \mathbf{Q}$.

The argument against the ordered set in Figure 11 satisfying (♡) depended on having an odd

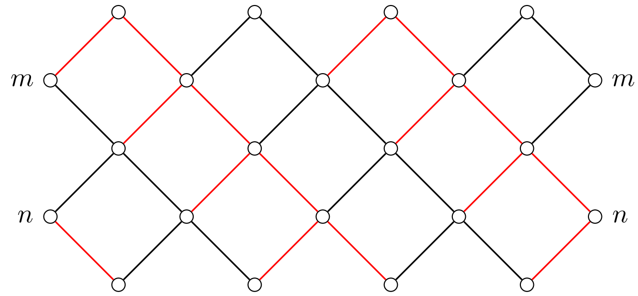

Figure 12: An ordered set that is a torus and has an edge-coloring satisfying the conditions of Theorem 5.9.

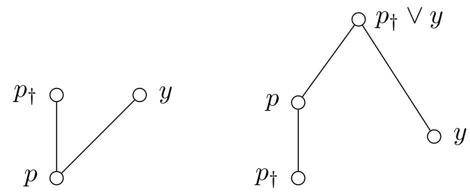

Figure 13: Illustrating a basic closure rule: on left $p_\dagger$, $y \succ p$ in $\mathbf{P}$, on right $p \sqsubseteq p_\dagger \vee y$ in $\mathbf{M}$.

number of squares across the top row. With an even number, there is no problem satisfying the conditions of Theorem 5.9, and the pattern can be extended downward as well, as in Figure 12.

Finally we are in position to construct the lattice $\mathbf{M}$. Assume that $(\mathbf{P}, \leq)$ satisfies (♣) and that the †-function has been chosen to satisfy (♡). Form the ordered set $\overline{\mathbf{P}} = (P, \sqsubseteq)$ with $u \sqsubseteq v$ iff $u = v_{(k)}$ for some $k \geq 0$. Then define closure rules on P by setting $p \in \gamma(\{y\})$ if $p \sqsubseteq y$, and

$$p \in \gamma(\{p_\dagger, u\})$$

for each non-$\sqsubseteq$-minimal $p \in P$ and every $u \in L(p)$. With a slight abuse of notation, it is convenient to think of γ as a join operation and write the closure rule as

$$p \sqsubseteq p_\dagger \vee u$$

for each $u \in L(p)$. The condition (♣) makes this not vacuous. Let $\mathbf{M}$ be the lattice of γ-closed order ideals of $(\overline{\mathbf{P}}, \sqsubseteq)$, *i.e.*, subsets closed under joins and downward containment $\sqsupseteq$.

The closure rule $p \sqsubseteq p_\dagger \vee y$ when $p_\dagger \neq y \succ p$ in $(\mathbf{P}, \leq)$ is illustrated in Figure 13. In general there will be other closure rules: if $p_\dagger \neq y \succ p$ and $y_\dagger \neq z \succ y$, then we also have $p \sqsubseteq p_\dagger \vee z$, etc.

Lemma 5.10. *Let $\mathbf{M}$ be the lattice constructed above. Then the order $\sqsubseteq$ on P and the set of closure rules $x \sqsubseteq x_\dagger \vee u$ with $u \in L^*(x)$ are a basis for $\mathbf{M}$. Moreover, the join irreducible elements of $\mathbf{M}$ are exactly the ideals $\downarrow_\sqsubseteq u$ for $u \in P$.*

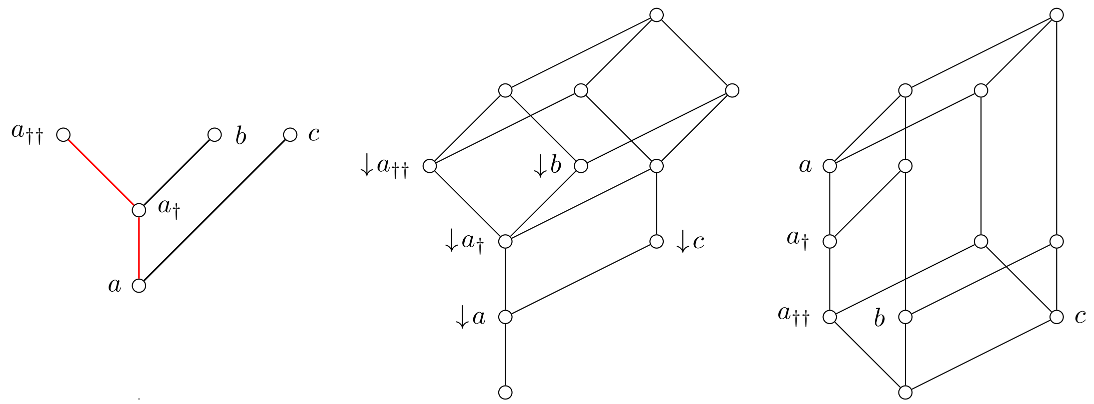

Figure 14: Representing a small distributive lattice as Con $\mathbf{M}$ with $\mathbf{M}$ semidistributive: $\mathbf{P}$, $\mathcal{O}(\mathbf{P})$, $\mathbf{M}$.

Proof. The first part follows from Lemma 5.4.

Clearly every ideal in $\mathbf{M}$ is the join of the principal ideals $\downarrow_{\sqsubseteq} u$ that it contains. Note that for any $u \in P$, $\downarrow_{\sqsubseteq} u$ is closed with respect to the join operation in $\mathbf{M}$, since $\downarrow_{\sqsubseteq} u$ is a chain. Thus you can identify u with the ideal $\downarrow_{\sqsubseteq} u$, as usual, and observe that $u_\dagger$ is the unique lower cover of u in $\mathbf{M}$. In particular, each $u \in P$ is join irreducible in $\mathbf{M}$. (This is slightly more subtle than it appears. If we had $p_\dagger \sqsubseteq u$ and $p \not\sqsubseteq u$, then $u \in K(p)$ by Corollary 5.7. That implies $\downarrow_{\sqsubseteq} u \subseteq K(p)$, so no closure rule can apply in $\downarrow_{\sqsubseteq} u$.) □

Now we can state the stronger version of Theorem 5.1.

Theorem 5.11. *Let* $\mathbf{P}$ *be a finite ordered set with a* $\dagger$*-function satisfying* ($\heartsuit$) *and* ($\clubsuit$). *Then* $\mathcal{O}(\mathbf{P})$ *is isomorphic to the congruence lattice of a finite bounded (and hence semidistributive) lattice.*

Figure 14 provides an example of the construction, giving $\mathbf{P}$, $\mathcal{O}(\mathbf{P})$, and $\mathbf{M}$. The defining relations for $\mathbf{M}$ are $a \sqsupset a_\dagger \sqsupset a_{\dagger\dagger}$, $a \sqsubseteq a_\dagger \vee c$, and $a_\dagger \sqsubseteq a_{\dagger\dagger} \vee b$. It is straightforward to verify that Con $\mathbf{M} \cong \mathcal{O}(\mathbf{P})$.

Figure 15 provides another example of the construction. The defining relations for $\mathbf{M}$ are $a \sqsupset a_\dagger$, $b \sqsupset b_\dagger$, $a \sqsubseteq a_\dagger \vee b_\dagger$, and $b \sqsubseteq a_\dagger \vee b_\dagger$.

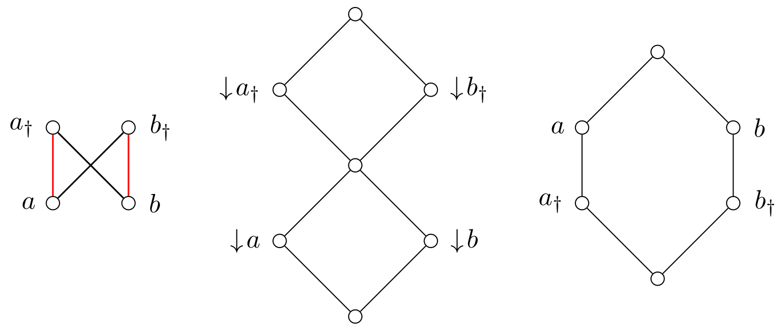

Figure 15: Representing a small distributive lattice as Con $\mathbf{M}$ with $\mathbf{M}$ semidistributive: $\mathbf{P}$, $\mathcal{O}(\mathbf{P})$, $\mathbf{M}$.

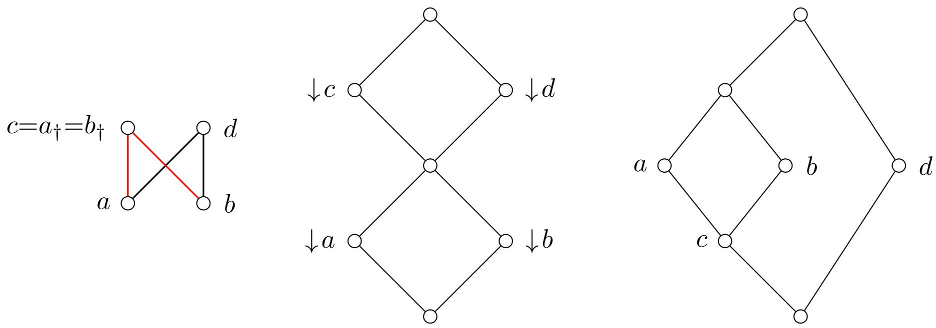

Figure 16: The same $\mathbf{P}$ as Figure 15 with a different †-function: $\mathbf{P}$, $\mathcal{O}(\mathbf{P})$, $\mathbf{M}'$.

Figure 16 has the same ordered set $\mathbf{P}$ as Figure 15, with a different †-function. Thus $\mathcal{O}(\mathbf{P})$ remains the same, but the closure rules for $\mathbf{M}'$ are $c \sqsubseteq a$, b and a, $b \sqsubseteq c \vee d$.

Now let us return to the business of proving that the construction works, *i.e.*, produces a bounded lattice $\mathbf{M}$ with Con $\mathbf{M} \cong \mathcal{O}(\mathbf{P})$ when the two conditions (♡) and (♣) are satisfied.

Lemma 5.12. *Let $Q \subseteq P$. The join $\bigvee Q$ in $\mathbf{M}$ is obtained by:*

(1) *for each $q \in Q$, add $\downarrow_{\sqsubseteq} q$ to obtain Q_1;*

(2) *recursively, if $x_\dagger$, $u \in Q_j$ with $u \in L^*(x)$, let $Q_{j+1} = Q_j \cup \{x\}$.*

If Q_m denotes the end result of applying (2) as long as possible, then Q_m is a closed ideal of $(\overline{\mathbf{P}}, \sqsubseteq)$, and hence $\bigvee Q = Q_m$. In particular, one need not go back to (1).

The crucial observation here is that when one adds x to Q_j in step (2), we already have $\downarrow_{\sqsubseteq} x_\dagger \subseteq Q_j$.

Lemma 5.13. *Let $x \in P$. Then every join cover $x \sqsubseteq \bigvee Q$ refines to a join cover $x \sqsubseteq \bigvee R$ with $R \ll Q$ and $R \subseteq \uparrow_{\leq} x$. Thus every minimal nontrivial join cover of x is contained in $\uparrow_{\leq} x$.*

Proof. Suppose $x \in Q_{m'}$ with $m' \leq m$ from Lemma 5.12. If $m' = 1$ then $x \sqsubseteq q$ for some $q \in Q$; note that implies $x \geq q$ in $(\mathbf{P}, \leq)$, *i.e.*, the trivial cover $\{x\}$ refines Q. So assume $m' > 1$. Then there exists $u \in L^*(x)$ such that $x_\dagger$, $u \in Q_{m'-1}$. Note $x \leq x_\dagger$, $x \leq u$, and both $x_\dagger$, $u \leq \bigvee Q$. By induction, there exist $R_1 \subseteq \uparrow_\leq x_\dagger$ with $R_1 \ll Q$ and $x_\dagger \sqsubseteq \bigvee R_1$, and $R_2 \subseteq \uparrow_\leq u$ with $R_2 \ll Q$ and $u \sqsubseteq \bigvee R_2$. Then $R_1 \cup R_2 \ll Q$, $R_1 \cup R_2 \subseteq (\uparrow_\leq x_\dagger) \cup (\uparrow_\leq u) \subseteq \uparrow_\leq x$, and

$$x \sqsubseteq x_\dagger \vee u \sqsubseteq \bigvee R_1 \vee \bigvee R_2$$

as desired. □

Lemma 5.14. *For each $x \in P$, $K(x)$ is a closed ideal of $(\overline{\mathbf{P}}, \sqsubseteq)$.*

Proof. Since maximal elements of $(\mathbf{P}, \leq)$ are join prime in $\mathbf{M}$, this certainly holds for them. So assume x is not maximal and that $K(y)$ is a closed ideal for every $y > x$. Recall that

$$K(x) = \bigcap_{x_\dagger \neq y \succ x} K(y) \ \cap \ S_x$$

where $S_x = \{z \in P : x \not\sqsubseteq z\}$.

Suppose $K(x)$ is not a closed ideal. Now $K(x)$ is an ideal with respect to $\sqsubseteq$ by Lemma 5.3(3). Assume it is not join-closed. Let $u \sqsubseteq u_\dagger \vee v$ be the first instance where a basic closure rule applies, *i.e.*, $u \notin K(x)$ but $u_\dagger$, $v \in K(x)$ and $v \in L^*(u)$. (We can do this because $K(x)$ is $\sqsupseteq$-closed.) Then, since each $K(y)$ is closed, we must have $u \notin S_x$ and $u_\dagger$, $v \in S_x$. Now $u_\dagger \in S_x$ means $u_{\dagger(k)} \neq x$ for all $k \geq 0$. But that implies $u_{(k+1)} \neq x$ for all $k \geq 0$. Meanwhile $u \notin S_x$ says $u_{(\ell)} = x$ for some $\ell \geq 0$. This only makes sense if $\ell = 0$, *i.e.*, $u = x$. But then $v \in L^*(x) \subseteq L(x)$, whence $v \notin K(x)$, a contradiction. □

Lemma 5.15. *If $x_\dagger \neq y \succ x$ in $(\mathbf{P}, \leq)$, then $x \sqsubseteq x_\dagger \vee y$ is a minimal nontrivial join cover in $\mathbf{M}$. Hence $x \, D \, x_\dagger$ and $x \, D \, y$.*

Proof. Let $x \in P$, so $\downarrow_\sqsubseteq x \in \mathbf{M}$. We have $x_{\dagger\dagger} \in K(x_\dagger)$ by Lemma 5.6, while $y \in K(x_\dagger)$ by (♡)(b). Thus $x_{\dagger\dagger} \vee y \sqsubseteq \bigvee K(x_\dagger) = K(x_\dagger)$ using Lemma 5.14, while $x \notin K(x_\dagger)$ since $x \sqsupset x_\dagger$. Therefore $x \not\sqsubseteq x_{\dagger\dagger} \vee y$.

Similarly, $x_\dagger \in K(x)$ by Lemma 5.6, while $y_\dagger \in K(x)$ by (♡)(c). Thus $x_\dagger \vee y_\dagger \sqsubseteq \bigvee K(x) = K(x)$, while $x \notin K(x)$. Hence $x \not\sqsubseteq x_\dagger \vee y_\dagger$. □

Lemma 5.13 does not tell us exactly which join covers are minimal. It is often the case in semidistributive lattices that compounding the defining join covers produces more minimal nontrivial join covers (though not *doubly* minimal join covers!). However, we know the following.

(i) The join irreducible elements of $\mathbf{M}$ are exactly the ideals $\downarrow_{\sqsubseteq} x$ for $x \in P$ (Lemma 5.10).

(ii) If $z \succ p$ in $(\mathbf{P}, \leq)$ then $x\, D\, z$ (Lemma 5.15).

(iii) If $p \sqsubseteq \bigvee Q$ is a minimal nontrivial join cover in $\mathbf{M}$, then $p < q$ in $(\mathbf{P}, \leq)$ for each $q \in Q$ (Lemma 5.13).

Consequently, the dependency relation D on $\mathbf{M}$ satisfies $\prec_{\mathbf{P}} \subseteq D \subseteq \leq_{\mathbf{P}}$ and we get $\operatorname{Con} \mathbf{M} \cong \mathcal{O}(\mathbf{P})$. Moreover, in view of (iii), there can be no D-cycles. Thus $\mathbf{M}$ is lower bounded, and hence join semidistributive

(It is interesting to note how the construction fails on the ordered set $\mathbf{Y}$, which fails (♣). On the other hand, in nature the defining closure operators need not use only covers.)

Now let us prove that $\mathbf{M}$ is meet semidistributive by showing that $\kappa(x)$ exists for each $x \in P$. It is useful to have a slightly enhanced technical version of Lemma 4.9.

Lemma 5.16. *In a finite lattice* $\mathbf{L}$*, the following conditions are equivalent (to* $(\mathrm{SD}_\wedge)$*).*

(1) *For all* $x \in \mathrm{J}(\mathbf{L})$ *there exists* $\kappa(x)$ *such that* $x \nleq \kappa(x)$ *and for all* $u \in L$*,* $x \nleq x_* \vee u$ *implies* $u \leq \kappa(x)$*.*

(2) *For all* $x \in \mathrm{J}(\mathbf{L})$ *there exists* $\kappa(x)$ *such that* $x \nleq \kappa(x)$ *and for all join irreducible elements* $w \in \mathrm{J}(\mathbf{L})$*,* $x \nleq x_* \vee w$ *implies* $w \leq \kappa(x)$*.*

Condition (1) is a traditional equivalent to meet semidistributivity, and (2) allows us to check it at join irreducibles only.

Proof. Clearly (1) implies (2). Conversely, assume that $\mathbf{L}$ satisfies (2) and that $x \nleq x_* \vee u$ for some $u \in L$. Let $u = \bigvee u_i$ with each $u_i \in \mathrm{J}(\mathbf{L})$. Since $u_i \leq u$ we have $x \nleq x_* \vee u_i$ for all i, whence $u_i \leq \kappa(x)$ by (2). Thus $u = \bigvee u_i \leq \kappa(x)$ as well. □

For each $x \in P$ we claim that $K(x) \subseteq P$ has these properties.

(a) $K(x)$ is a closed ideal of $(P, \sqsubseteq)$, *i.e.*, $\bigvee K(x) = K(x)$,

(b) $x_\dagger \in K(x)$,

(c) $x \notin K(x)$,

(d) for all $u \in P$ we have $x \not\sqsubseteq x_\dagger \vee u$ if and only if $u \in K(x)$.

Indeed, (a) is Lemma 5.13, (b) is Lemma 5.6, and (c) is Lemma 5.3(1). For (d), if $u \in L(x)$ then $x \sqsubseteq x_\dagger \vee u$ by the definition of join in $\mathbf{M}$. If $u \in K(x)$, though, then $x_\dagger \vee u \in K(x)$ by (a) and (b), while $x \notin K(x)$. Thus we cannot have $x \sqsubseteq x_\dagger \vee u$ if $u \in K(x)$.

We conclude by Lemma 5.16 that $\mathbf{M}$ is meet semidistributive. Moreover, since $\mathbf{M}$ is lower bounded and semidistributive, it is also upper bounded by Theorem 4.1.

Thus $\mathbf{M}$ as constructed is a finite bounded lattice with Con $\mathbf{M} \cong \mathcal{O}(\mathbf{P}, \leq)$, completing the proof of Theorem 5.1.

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Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 26, no. 3, pp. 475–488, December 2024
DOI: 10.56754/0719-0646.2603.475

The metric dimension of cyclic hexagonal chain honeycomb triangular mesh and pencil graphs

R. Nithya Raj[1,✉]
R. Sundara Rajan[1]
Ismail Naci Cangul[2]

[1] *Department of Mathematics, Hindustan Institute of Technology and Science, Chennai 603103, India.*
nithyarajmaths@gmail.com
vprsundar@gmail.com

[2] *Department of Mathematics, Faculty of Arts and Science, Bursa Uludag University, Gorukle 16059, Turkey.*
cangul@uludag.edu.tr

ABSTRACT

The metric dimension of a graph serves a fundamental role in organizing structures of varying dimensions and establishing their foundations through diverse perspectives. Studying symmetric network characteristics like connectedness, diameter, vertex centrality, and complexity depends heavily on the distance parameter. In this article, we explore the exact value for different hexagonal networks' metric dimensions, such as cyclic hexagonal chains, triangular honeycomb mesh, and pencil graphs.

RESUMEN

La dimensión métrica de un grafo cumple un rol fundamental para organizar estructuras de dimensiones variables y establecer sus fundamentos a través de perspectivas diversas. Estudiar características de redes simétricas como la conexidad, diámetro, centralidad de vértices y complejidad depende fuertemente del parámetro de distancia. En este artículo exploramos el valor exacto de la dimensión métrica de diferentes redes hexagonales, tales como cadenas hexagonales cíclicas, la malla triangular panal y grafos lápices.

Keywords and Phrases: Metric basis, metric dimension, cyclic hexagonal chain, triangular honeycomb mesh, pencil graph.

2020 AMS Mathematics Subject Classification: 05C12, 05C90, 05C15, 05C62.

Published: 27 November, 2024
Accepted: 18 October, 2024
Received: 10 November, 2023

1 Introduction

In the field of robotics, the metric dimension problem is important. A robot is an automated machine designed to move through space while avoiding obstacles. It does not understand either direction or visibility. However, it is presumable that it can detect the separation of a collection of landmarks. Evidently, the robot can establish its precise location in space if it is aware of the distances to a significant number of landmarks. In order to perform this, the idea of "landmarks in a graph" was created [12], and later it was expanded to the metric dimension in which networks are taken into consideration within the framework of the graph structure.

Finding a metric basis for the graph is the goal of the metric dimension problem in graph theory; the landmarks that make up a metric basis are known as landmarks, and the cardinality of a metric basis is referred to as the metric dimension of the graph. Harary and Melter [10] did the first investigation into the metric dimension problem. They provided a description of the trees' metric dimensions. Melter and Tomescu investigated the grid graphs' metric dimension problem [18]. For each arbitrary graph, the metric dimension problem is NP-complete [9]. Since then, a great deal of study has been conducted on this problem. In many fields of science and technology, the metric dimension has several uses. For grid graphs and trees, the metric dimension problem has been studied [12], hexagonal and honeycomb networks [15], silicate networks [16], torus networks [14], and enhanced hypercubes [17]. Metric dimension is used to address issues with robot navigation and pattern recognition [12], network discovery and validation [5], and issues with coin weighing and graph joins [20, 22].

In this paper, in Section 2, preliminaries and basis definitions are discussed. Section 3, deals with the metric dimension of the cyclic hexagonal chain, honeycomb triangular mesh, and pencil graph. Finally, the Significance and Contributions of the Results, concluding remarks and, open problem are given in Section 4 and Section 5 respectively.

2 Basis concepts

A finite simple connected graph $G = (V, E)$ is used in this paper, where V and E are the set of vertices and edges respectively. The distance between two vertices a and b in a graph G, denoted as $d(a, b)$, is defined as the minimum number of edges in any path from a to b. It is normal to have questions about the characterizations of graphs based on their metric dimension. Researchers are continuously interested in determining whether the metric dimension of a network family is constant, bounded, or unbounded. Consequently, there has been significant research focused on finding the metric dimension of networks, resulting in numerous findings. Examples of such findings include: Muhammad *et al.* [19] investigated the metric dimension of some chemical structures. Akhter and Farooq [3] investigated the metric dimension of the Indu-Bala product of graphs. The metric dimension of the subdivided honeycomb network and Aztec diamond network was determined by Xiujun *et al.* [23]. Ahmad *et al.* [1] found the metric dimension for benzenoid hammer graph. A bicyclic network's metric dimension was examined by Khan *et al.* [11]. Bokhary *et al.* [11] studied the metric dimension of the subdivision graph of a circulant network. Koam *et al.* [13] investigated the metric dimension and exchange property of nanotubes. Resolving sets have been discussed across the literature [2, 4, 8, 10, 18].

In this study, we obtain the metric dimension of specific planar architectures. To prove the main results we need the following.

Definition 2.1. *The diameter of a graph is the greatest distance between any pair of vertices, where the distance is defined as the length of the shortest path connecting them.*

Definition 2.2. *The metric basis or resolving set for a graph $G = (V, E)$, a resolving set of G is a subset of vertices $S \subseteq V$ such that every vertex $v \in V$ is uniquely determined by its distance vector to the vertices in S. For each vertex $v \in V$, its distance vector with respect to S is defined as $(d(v, s_1), d(v, s_2), \ldots, d(v, s_k))$, where $s_1, s_2, \ldots, s_k \in S$, and $d(v, s_i)$ is the shortest distance between v and s_i in the graph.*

The subset S is a metric basis if, for any two distinct vertices $u, v \in V$, their distance vectors relative to S are distinct, i.e.,

$$d(u, s_i) \neq d(v, s_i) \quad \text{for at least one } s_i \in S.$$

The cardinality of the metric basis or resolving set S is called the metric dimension of the graph and is denoted as $\dim(G)$.

Theorem 2.3 ([7])**.** *A simple connected graph G has a metric dimension 1 if and only if it is precisely identical in structure to the path graph P_n.*

Theorem 2.4 ([12])**.** *Suppose G is a graph with a minimum metric dimension of 2, and let $\{a, b\}$ be a subset of the vertices set V that forms a metric basis in B. In this context, the subsequent statements hold true:*

(a) Only one shortest route is possible between a and b.

(b) Each a and b has a maximum degree of three.

3 Main results

In this section, we determine the metric dimension of the cyclic hexagonal chain, honeycomb triangular mesh, and pencil graph.

3.1 Cyclic hexagonal chain

A catacondensed hexagonal structure known as a hexagonal chain has each hexagon being next to no more than two other hexagons. The graph representation of linear polyacene is a linear hexagonal chain, which is a hexagonal chain. A cyclic hexagonal chain is created when the ends of a linear hexagonal chain are bent to touch. The symbol H_n will be used to represent a cyclic hexagonal chain of dimension n respectively. We split the vertices of H_n as I and J, where I and J are the set of all vertices in the inner and outer cycle respectively. The cyclic hexagonal chain is symmetric in rotation and has $4n$ vertices in which $2n$ vertices are in each of the inner and outer cycles labeled as $I = \{i_1, i_2, i_3, \ldots, i_{2n}\}$ and $J = \{j_1, j_2, j_3, \ldots, j_{2n}\}$ in the clockwise direction respectively. For example, the labeling of a cyclic hexagonal chain of dimension n is given in Figure 1.

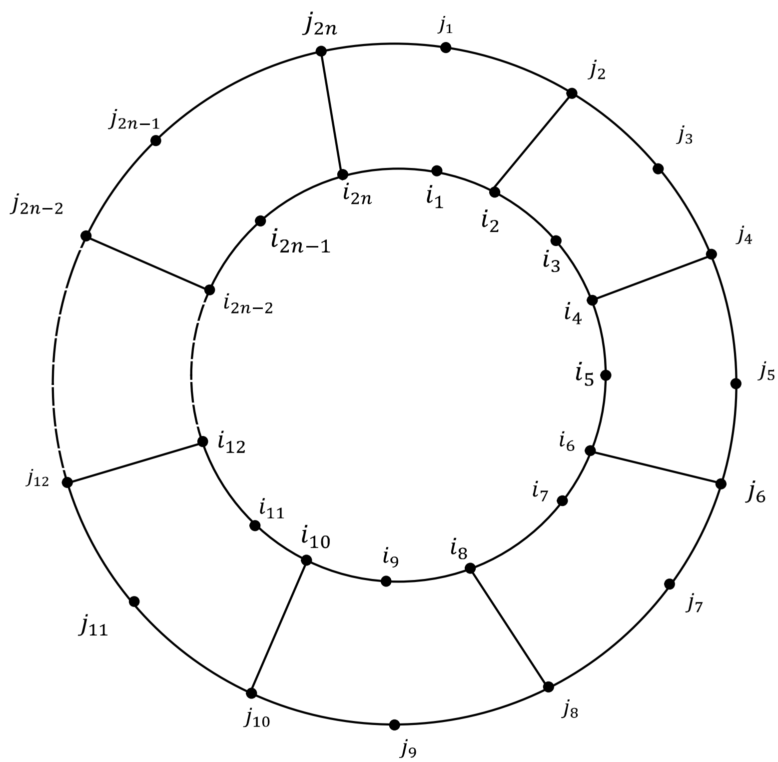

Figure 1: Labeling of cyclic hexagonal chain H_n

Theorem 3.1. *The metric dimension of the graph of the cyclic hexagonal chain H_n is more than 2 for $n \geq 2$.*

Proof. Based on Theorem 2.4, suppose that there exists a resolving set T with size 2. There are two cases for T.

Case 1. Suppose that $T = \{j_k, j_l\}$ for some k and l where $1 \leq k \leq n+1$ (by the symmetry of H_n, it is enough to consider the first half of the cycle). Then we have $r(i_{l+1}|T) = r(j_{l+2}|T) = (2, l-k+2)$.

Case 2. Suppose that $T = \{i_k, j_l\}$ for some $k, l = 1, 2, \ldots, n+1$. If $k = 1$, then we have $r(i_{k-1}|T) = r(i_{k+1}|T) = (1, 2)$. If $k < l$ (without loss of generality), then we have two subcases: if l is odd, then $r(i_l|T) = r(j_{l-1}|T) = (1, l-k)$, and if l is even, then $r(i_{k+2}|T) = r(j_{k+1}|T) = (2, l-k-1)$.

From these two cases, we find two vertices having the same representations. Therefore, T is not a resolving set of H_n, a contradiction. □

Theorem 3.2. *The metric dimension of the graph of the cyclic hexagonal chain H_n is 3 for $n \geq 2$.*

Proof. Let $T = \{j_1, j_2, j_{n+1}\}$ be a resolving set of H_n. To prove that T is a resolving set. It is enough to prove that all the vertices j_l, i_l $1 \leq l \leq 2n$ of H_n have unique representations with respect to T.

For $1 \leq l \leq 2n$, the representation j_l of H_n with respect to T is given as follows:

$$r(j_l|T) = \begin{cases} (l-1, 1, n), & \text{if } l = 1 \\ (l-1, l-2, n-l+1) & \text{if } 2 \leq l \leq n \\ (l-1, l-2, 0) & \text{if } l = n+1 \\ (2n-l+1, 2n-l+2, l-n-1) & \text{if } n+2 \leq l \leq 2n. \end{cases}$$

For $1 \leq l \leq 2n$, the representation of i_l of H_n with respect to T is given as follows:

$$r(i_l|T) = \begin{cases} (3, 2, n+1) & \text{if } l = 1 \\ (l, l-1, n+2-l) & \text{if } 2 \leq l \leq n \\ (n+1, n, 1) & \text{if } l = n+1 \\ (2n-l+2, 2n-l+3, l-n) & \text{if } n+2 \leq l \leq 2n. \end{cases}$$

We can see that each vertex of H_n has a distinct representation and satisfies the notion of a resolving set with regard to T. Hence $\dim(H_n) = 3$. □

3.2 Honeycomb triangular mesh

In this section, we show that the construction and the metric dimension of the honeycomb triangular mesh are discussed. Honeycomb triangular mesh is built recursively using hexagonal tessellations with three pendant edges. The honeycomb triangular mesh HTM_1 is a single vertex. The honeycomb triangular mesh HTM_2 is obtained by adding 3 pendant edges to HTM_1. In a similar manner, the n-dimensional honeycomb triangular mesh HTM_n is adding $(n-2)$ hexagons to the boundary of HTM_{n-1} with three pendent edges in the triangular form. The number of vertices, edges, faces, and diameter of HTM_n are n^2, $\frac{3(n^2-n)}{2}$, $\frac{n^2-3n+4}{2}$, and $(2n-2)$ respectively. A honeycomb triangular mesh HTM_1, HTM_2, HTM_3, and HTM_4 are shown in Figure 2.

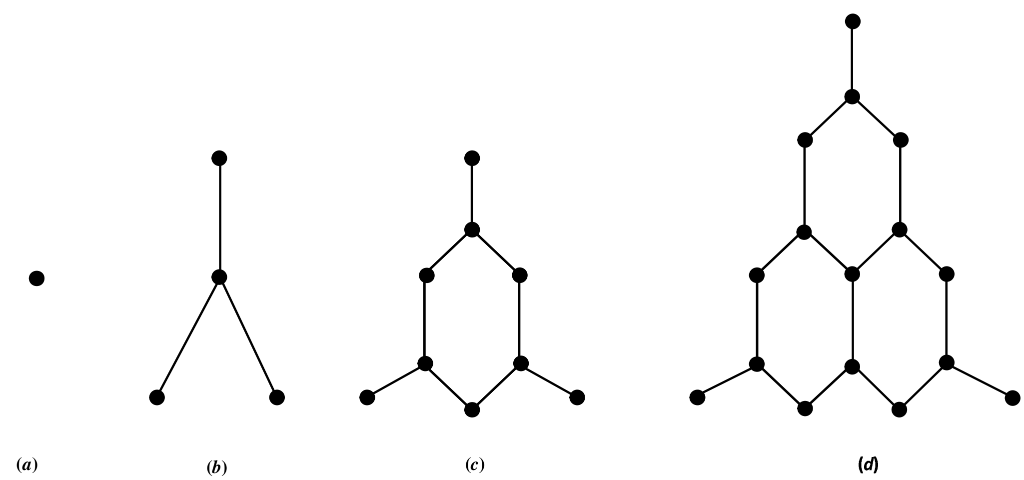

Figure 2: Honeycomb triangular mesh (a) HTM_1, (b) HTM_2, (c) HTM_3, and (d) HTM_4

The strip between two successive lines is marked in Honeycomb Triangular mesh is called the segments and it is denoted by S_L. The representation of any two points $p(l_1, m_1)$ and $q(l_2, m_2)$ in the honeycomb triangular mesh is defined by if $l_1 = l_2$, then p and q lies in the same segment, and if $l_1 \neq l_2$, then p and q are lies in the different segments. The distance between any two vertices $p(l_1, m_1)$ and $q(l_2, m_2)$ is non zero, when p and q lie in the same and different segments. We partition the vertices of HTM_n into n segments, namely S_1, S_2, S_3, $\ldots$, S_n, and the segment representation of Honeycomb triangular mesh is shown in Figure 3.

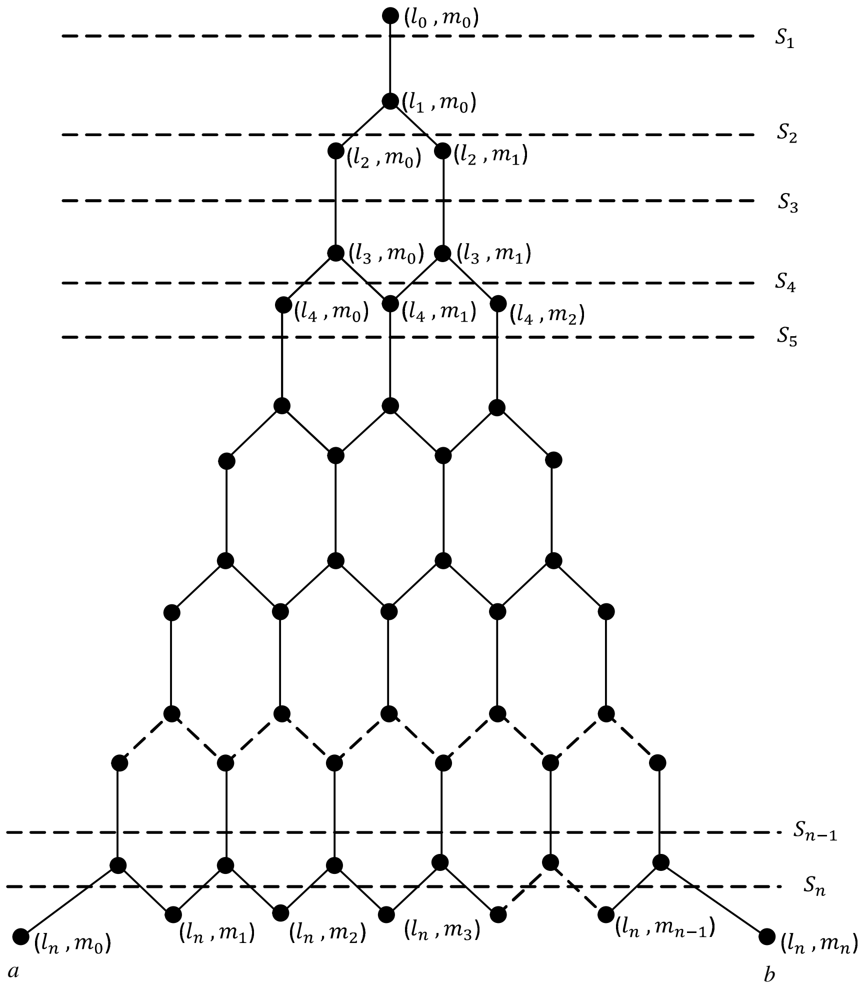

Figure 3: Honeycomb triangular mesh of dimension HTM_n with segments $S_1, S_2, \ldots, S_n$

Theorem 3.3. *The metric dimension of the graph of the honeycomb triangular mesh HTM_n is 2 for $n \geq 2$.*

Proof. Based on Theorem 2.3, we have $\dim(HTM_n) \geq 2$. Next, we will show that $\dim(HTM_n) \leq 2$.

Let $A = \{x : deg(x) = 1\}$ and $T = \{a, b\}$ where $a, b \in A$. We will show that T is a resolving set or not.

Now, we have the following cases.

Let $p = (l_1, m_1)$ and $q = (l_2, m_2)$ be any two distinct vertices in HTM_n.

Case 1: If $l_1 = l_2$ and $m_1 \neq m_2$, then p and q are resolved by either a or b. Suppose that, if $d(p, a) = d(q, b)$, then p and q are resolved by either a or b, *i.e.*, $d(p, a) \neq d(q, a)$ or $d(p, b) \neq d(q, b)$.

Case 2: If $l_1 \neq l_2$ and $m_1 = m_2$, then both p and q are resolved by a and b.

Case 3: If $l_1 \neq l_2$ and $m_1 \neq m_2$, then p and q are resolved by either a or b. Suppose that, if p and q are at equal distance to a, then p and q must be resolved by b, *i.e.*, if $d(p, a) = d(q, a)$, then $d(p, b) \neq d(q, b)$.

From the above cases if we take any two vertices in a honeycomb triangular mesh are resolved by a and b. Therefore $\dim(HTM_n) \geq 2$. Hence, $\dim(HTM_n) = 2$ □

3.3 Pencil graph

In this section, we determine the pencil graph's metric dimension. In 2015, Simamora and Salman [23] introduced and studied vertex rainbow connection numbers for a new cubic graph called pencil graph. Pencil graph are a specific type of graph in graph theory that consist of a central hub vertex connected to a set of outer vertices called spokes. Pencil graphs have applications in various areas, including network topology, and algorithm design.

Definition 3.4. *Suppose that n is a positive integer with $n \geq 2$. The graph PC_n is a pencil graph with $2n+2$ vertices and the vertex and edge sets are as follows: $V(PC_n) = \{a\} \cup \{b\} \cup \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$ and $E(PC_n) = \{(ax_1), (ay_1), (ab), (bx_n), (by_n)\} \cup \{(x_i x_{i+1}), (y_i y_{i+1}) : 1 \leq i \leq n-1\} \cup \{(x_i y_i) : 1 \leq i \leq n\}$*

For $n \geq 2$, the pencil graph PC_n is a 3-regular graph with diameter $\lceil n/2 \rceil + 1$ and $3(n+1)$.

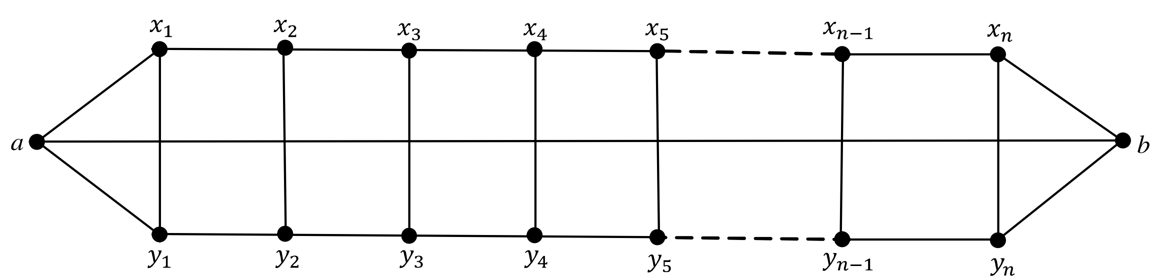

Figure 4: Labeling of pencil graph of dimension n

Theorem 3.5. *The metric dimension of the graph of the pencil graph PC_n, for $n \geq 1$ is*

$$\dim(G) = \begin{cases} 2 & \textit{if } n \textit{ is even} \\ 3 & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. **Case 1 (For n even):**

Let $T = \{a, x_{\frac{n}{2}}\}$ be a resolving set of PC_n. To prove that T is a resolving set, it is enough to prove all the vertices a, b, $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, y_3, \ldots, y_n$ of PC_n have distinct representations with respect to T.

The representation of a and b in PC_n with respect to T as $r(a|T) = (0, \frac{n}{2})$ and $r(b|T) = (1, \frac{n}{2} + 1)$.

For $1 \leq i \leq n$, the representation of x_i in PC_n with respect to T is given as follows:

$$r(x_i|T) = \begin{cases} (i, \frac{n-i}{2}) & \text{if } 1 \leq i \leq \frac{n}{2} \\ (n-i+2, \frac{2i-n}{2}) & \text{if } \frac{n}{2}+1 \leq i \leq n \end{cases}$$

For $1 \leq i \leq n$, the representation of y_i in PC_n with respect to T is given as follows:

$$r(y_i|T) = \begin{cases} (i, \frac{n-2i+2}{2}) & \text{if } 1 \leq i \leq \frac{n}{2} \\ (n-i+2, \frac{2i-n+2}{2}) & \text{if } \frac{n}{2}+1 \leq i \leq n \end{cases}$$

Since all vertices have distinct representations we obtain $\dim(PC_n) = 2$ in this case.

Case 2 (For n odd): Let $T = \{a, x_{\frac{n+1}{2}}, y_{\frac{n-1}{2}}\}$ be a resolving set of PC_n. To prove that T is a resolving set, it is enough to prove all the vertices a, b, $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, y_3, \ldots, y_n$ of PC_n have distinct representations with respect to T.

The representation of a and b in PC_n with respect to T as $r(a|T) = (0, \frac{n+1}{2}, \frac{n-1}{2})$ and $r(b|T) = (1, \frac{n+1}{2}, \frac{n+1}{2})$.

For $1 \leq i \leq n$, the representation of x_i in PC_n with respect to T as follows

$$r(x_i|T) = \begin{cases} (i, \frac{n+1-2i}{2}, \frac{n+1-2i}{2}) & \text{if } 1 \leq i \leq \frac{n-1}{2} \\ (i, 0, 2) & \text{if } i = \frac{n+1}{2} \\ (n-i+2, \frac{2i-n-1}{2}, \frac{2i-n+3}{2}) & \text{if } \frac{n+3}{2} \leq i \leq n \end{cases}$$

For $1 \leq i \leq n$, the representation of y_i in PC_n with respect to T as follows

$$r(y_i|T) = \begin{cases} (i, \frac{n+3-2i}{2}, \frac{n-1-2i}{2}) & \text{if } 1 \leq i \leq \frac{n-1}{2} \\ (i, 1, 1) & \text{if } i = \frac{n+1}{2} \\ (n-i+2, \frac{2i-n+1}{2}, \frac{2i-n+1}{2}) & \text{if } \frac{n+3}{2} \leq i \leq n \end{cases}$$

It is clear that every vertex of PC_n has a unique representation with repect to T. Therefore $\dim(PC_n) \leq 3$.

Next we show that $\dim(PC_n) \geq 3$. We suppose on contrary that $\dim(PC_n) = 2$. Now we have the following cases.

Subcase 2.1: For $1 \leq i \leq n$, let $T = \{a, b\}$ be a resolving set, then $r(x_i|T) = r(y_i|T)$, which is a contradiction to our assumption.

Subcase 2.2: Let $T = \{a, x_1\}$ be a resolving set, then $r(x_n|T) = r(y_n|T)$, which leads to a contradiction.

Subcase 2.3: For $2 \leq i \leq \frac{n+1}{2}$, let $T = \{a, x_i\}$ be a resolving set, then $r(x_{\frac{n+3}{2}}|T) = r(y_{\frac{n+1}{2}}|T)$, which is a contradiction to our assumption.

Subcase 2.4: For $\frac{n+3}{2} \leq i \leq n$, let $T = \{a, x_i\}$ be a resolving set, then $r(x_{\frac{n+1}{2}}|T) = r(y_{\frac{n+3}{2}}|T)$, which is a contradiction to our assumption.

Subcase 2.5: For $2 \leq i \leq \frac{n+1}{2}$, let $T = \{x_1, x_i\}$ be a resolving set, then $r(x_{i+1}|T) = r(y_i|T)$, which is a contradiction to our assumption.

Subcase 2.6: For $\frac{n+3}{2} \leq i \leq n$, let $T = \{x_1, x_i\}$ be a resolving set, then $r(x_{i+1}|T) = r(y_{i-1}|T)$, which is a contradiction to our assumption.

Subcase 2.7: Let $T = \{x_1, y_1\}$ be a resolving set, then $r(x_n|T) = r(y_n|T)$, which leads to a contradiction.

Subcase 2.8: For $2 \leq i \leq \frac{n+3}{2}$, let $T = \{x_1, y_i\}$ be a resolving set, then $r(x_{\frac{n+3}{2}}|T) = r(y_{\frac{n+5}{2}}|T)$, which is a contradiction to our assumption.

Subcase 2.9: For $\frac{n+5}{2} \leq i \leq n$, let $T = \{x_1, y_i\}$ be a resolving set, then $r(x_{\frac{n+3}{2}}|T) = r(y_{\frac{n+1}{2}}|T)$, which is a contradiction to our assumption.

By the symmetrical nature of the pencil graph the remaining possibility of resolving sets for $1 \leq i \leq n$, $T = \{\{a, b\}, \{a, y_i\}, \{b, x_i\}, \{b, y_i\}, \{y_i, x_i\}\}$ is ruled out. From all the above cases it is clear that $\dim(PC_n) \geq 3$. Hence $\dim(PC_n) = 3$. $\square$

4 Significance and contributions of the results

This research offers valuable insights into the metric dimension of cyclic hexagonal chains, honeycomb triangular meshes, and pencil graphs, with direct applications in modern network design, particularly in the field of robot navigation for smart home environments. The primary significance and contributions are as follows:

Novel Metric Dimension Analysis: This study presents a detailed investigation of the metric dimension of three distinct graph structures: cyclic hexagonal chains, honeycomb triangular meshes, and pencil graphs. The results contribute to expanding the mathematical foundation of graph theory, particularly in relation to chemical, geometric, and computational networks.

Applications in Robot Navigation: By determining the metric dimension of these structures, the research provides optimized strategies for robot navigation. The results are critical for localization and pathfinding within networks like smart homes, where robots or autonomous agents need precise positioning with minimal sensors.

Insights for Chemical Graph Theory: Cyclic hexagonal chains represent fundamental structures in chemical graph theory, modeling molecular systems. Understanding their metric dimension helps chemists analyze molecular distances and design efficient chemical compounds or materials with predictable properties.

Optimizing Network Design: The honeycomb triangular mesh and pencil graphs offer useful models for wireless networks and sensor systems. Analyzing their metric dimension improves the efficiency of node placement and minimizes redundancy, supporting the development of more reliable and cost-effective communication networks.

Bridging Theory and Practical Applications: This work bridges theoretical graph metrics with real-world applications, especially in robot-assisted smart homes. The findings enable better design of indoor networks, where efficient navigation plays a critical role in tasks such as surveillance, cleaning, and elderly assistance.

Framework for Future Studies: The approach and results of this research provide a basis for future investigations into other graph families with similar structures. Researchers working on emerging technologies, such as smart cities or the Internet of Things (IoT), can build on the analytical methods presented here.

In summary, this study significantly advances the understanding of the metric dimension in three important graph classes, contributing to both theory and practice. It offers practical solutions for smart environments while enriching the field of graph theory with new perspectives and methods.

5 Concluding remarks

In this paper, we investigated the metric dimension of three significant graph structures: cyclic hexagonal chains, honeycomb triangular meshes, and pencil graphs. Metric dimensions of honeycomb networks and hexagonal-type derived networks have constant metric dimensions, according to research by Manuel *et al.* [15]. In this article, a different kind of honeycomb network known as a triangular honeycomb mesh was created and it was demonstrated that its metric dimension is 2. This research also looked at pencil graphs and the metric dimension of cyclic hexagonal chains. Further obtaining the metric dimensions for symmetric types of honeycomb and hexagonal networks is under investigation. Moreover, computing the metric dimension of the triangular honeycomb network is still an open problem.

Acknowledgements

We would like to thank the referees for their comments and suggestions which were very helpful for improving the quality of this paper. We are especially grateful to one of the referees for his explicit suggestions, including the calculation of finding the representation in Theorem 3.3 and Theorem 3.5.

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CUBO, A Mathematical Journal
Vol. 26, no. 3, pp. 489–505, December 2024
DOI: 10.56754/0719-0646.2603.489

On the approximation of the δ-shell interaction for the 3-D Dirac operator.

Mahdi Zreik [1,2,✉]

[1] *Institut de Mathématiques de Bordeaux, UMR 5251, Université de Bordeaux 33405 Talence Cedex, France.*
mzreik@bcamath.org
mahdi.zreik@math.u-bordeaux.fr

[2] *Departamento de Matemáticas, Universidad del País Vasco, Barrio Sarriena s/n 48940 Leioa, Spain, and Basque Center for Applied Mathematics (BCAM), Spain.*

ABSTRACT

We consider the three-dimensional Dirac operator coupled with a combination of electrostatic and Lorentz scalar δ-shell interactions. We approximate this operator with general local interactions V. Without any hypotheses of smallness on the potential V, we investigate convergence in the strong resolvent sense to the Dirac Hamiltonian coupled with a δ-shell potential supported on Σ, a bounded smooth surface. However, the coupling constant depends nonlinearly on the potential V.

RESUMEN

Consideramos el operador de Dirac tridimensional acoplado con una combinación de interacciones electrostáticas y δ-cáscara escalar de Lorentz. Aproximamos este operador con interacciones locales generales V. Sin ninguna hipótesis en la pequeñez del potencial V, investigamos la convergencia en el sentido resolvente fuerte del Hamiltoniano de Dirac acoplado con un potencial δ-cáscara soportado en Σ, una superficie suave acotada. Sin embargo, la constante de acoplamiento depende no-linealmente del potencial V.

Keywords and Phrases: Dirac operators, self-adjoint operators, shell interactions, non critical and non-confining interaction strengths, approximations.

2020 AMS Mathematics Subject Classification: 81Q10, 81V05, 35P15, 58C40.

Published: 11 December, 2024
Accepted: 08 November, 2024
Received: 22 December, 2023

1 Introduction

Dirac Hamiltonians of the type $D_m + V$, where D_m is the free Dirac operator and V represents a suitable perturbation, are used in many problems where the implications of special relativity play an important role. This is the case, for example, in the description of elementary particles such as quarks, or in the analysis of graphene, which is used in research for batteries, water filters, or photovoltaic cells. For these problems, mathematical investigations are still in their infancy. The current study focuses on analyzing the three-dimensional Dirac operator with a singular interaction on a closed surface Σ.

Mathematically, the Hamiltonian of interest is formally represented as

$$D_{\eta,\tau} = D_m + B_{\eta,\tau}\delta_\Sigma = D_m + \big(\eta\,\mathbb{I}_4 + \tau\beta\big)\delta_\Sigma, \tag{1.1}$$

where $B_{\eta,\tau} := \big(\eta\,\mathbb{I}_4 + \tau\beta\big)$ is a combination of *electrostatic* and *Lorentz scalar* potentials of strengths η and τ, respectively. Physically, the Hamiltonian $D_{\eta,\tau}$ is used as an idealized model for Dirac operators with strongly localized electric and massive potential near an interface Σ (*e.g.*, an annulus), *i.e.*, it replaces a Hamiltonian of the form

$$\mathbb{H}_{\tilde{\eta},\tilde{\tau}} = D_m + \big(\tilde{\eta}\,\mathbb{I}_4 + \tilde{\tau}\beta\big)\mathfrak{B}_\Sigma, \tag{1.2}$$

where $\mathfrak{B}_\Sigma$ is a regular potential localized in a thin layer containing the interface Σ.

The operators $D_{\eta,\tau}$ have been studied in detail recently. The initial direct study on the spectral analysis of the Hamiltonian $D_{\eta,\tau}$ dates back to Ref. [9], in which the authors treated all self-adjoint realizations for spherical surfaces. Besides, they also noted that a shell can confine a particle under the coupling constants assumption: $\eta^2 - \tau^2 = -4$, a phenomenon known in physics as the *confinement case*, which indicates the stability of a particle (for example, an electron) within its initial region during time evolution. In other words, if the particle is confined within a region $\Omega \subset \mathbb{R}^3$ at time $t = 0$, it cannot cross the boundary $\partial\Omega$ and enter the region $\mathbb{R}^3\setminus\overline{\Omega}$ for all subsequent times $t > 0$. Mathematically, this implies that the Dirac operator under consideration can be decomposed into a direct sum of two Dirac operators acting on Ω and $\mathbb{R}^3\setminus\overline{\Omega}$, respectively, each with appropriate boundary conditions. Subsequently, spectral analyses involving Schrödinger operators coupled to δ-shell interactions have developed considerably, while research into the spectral aspects of δ-shell interactions associated with Dirac operators were comparatively inactive. However, in 2014, a resurgence in the spectral study of δ-shell interactions of Dirac operators occurred in [1], where the authors developed a new technique to characterize the self-adjointness of the free Dirac operator coupled to a δ-shell potential. In a special case, they treated pure electrostatic δ-shell interactions (*i.e.*, $\tau = 0$) supported on the boundary of a bounded regular domain and proved that the perturbed operator is self-adjoint. The same authors continued their investigation into the spectral analysis of the electrostatic case, exploring the existence of a point spectrum and associated issues in works such as [2] and [3].

The approximation of the Dirac operator $D_{\eta,\tau}$ by Dirac operators with regular potentials with shrinking support (*i.e.*, of the form (1.2)) provides a justification of the considered idealized model. In the one-dimensional framework, the analysis is carried out in [17], where Šeba showed that convergence is true in the norm resolvent sense. Subsequently, Hughes and Tušek established strong resolvent convergence and norm resolvent convergence for Dirac operators with general point interactions in [11, 12] and [20], respectively. In the two-dimensional case, [8, Section 8] addressed the approximation of Dirac operators with electrostatic, Lorentz scalar, and magnetic δ-shell potentials on closed and bounded curves. A related problem was also considered in [7] for a straight line scenario. More precisely, taking parameters $(\tilde{\eta}, \tilde{\tau}) \in \mathbb{R}^2$ in (1.2) and a potential $\mathfrak{V}^{\varepsilon}_{\Sigma}$ that converges to δ_Σ when ε tends to 0 (in the sense of distributions), then $D_m + \big(\tilde{\eta}\, \mathbb{I}_4 + \tilde{\tau}\beta\big)\mathfrak{V}^{\varepsilon}_{\Sigma}$ converges to the Dirac operator $D_{\eta,\tau}$ with different coupling constants $(\eta, \tau) \in \mathbb{R}^2$ that depend nonlinearly on the potential $\mathfrak{V}^{\varepsilon}_{\Sigma}$. This dependence has been observed in the one-dimensional case, for example [17, 20], and in higher dimensional cases, see [8, 15].

In the three-dimensional case, the situation seems to be even more complex, as recently shown in [15]. There, too, the authors were able to show convergence in the strong resolvent sense in the non-confining case, however, a smallness assumption on the potential $\mathfrak{V}^{\varepsilon}_{\Sigma}$ was required to achieve such a result. On the other hand, this assumption unfortunately prevents us from obtaining an approximation of the operator $D_{\eta,\tau}$ with the physically or mathematically more relevant parameters η and τ. Recognizing this limitation, the authors of the recent paper [4] delved into and verified the approximation problem for two- and three-dimensional Dirac operators with δ-shell potential in the norm resolvent sense. Without the smallness assumption on the potential $\mathfrak{V}^{\varepsilon}_{\Sigma}$ no results could be obtained here either. Finally, in [14], the authors of [15] treated the approximation of the operator (1.2) in the case of the sphere without assuming any hypothesis of smallness on the potential.

The primary aim of our work is to extend the approximation result explored in [8, Section 8] to the three-dimensional case. We seek to verify whether the methodologies employed in the two-dimensional context allow us to establish a comparable approximation in terms of strong resolvent. Specifically, we aim to achieve this in the non-critical and non-confinement cases (*i.e.*, when $\eta^2 - \tau^2 \neq \pm 4$) without relying on the smallness assumption as stipulated in [15].

Organization of the paper. The present paper is structured as follows. We start with Section 2, where we define the free Dirac operators D_m and the model to be studied in our paper by introducing the family $\{\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\}_\varepsilon$, which is the approximate Dirac operators family of operator $D_{\eta,\tau}$. We also discuss our main results by establishing Theorem 2.2. Moreover, in this section we give some geometric aspects characterizing the surface Σ, as well as some spectral properties of the

Dirac operator coupled with the δ-shell interaction presented in Lemma 2.6. Section 3 is devoted to the proof of Theorem 2.2, which approximates the Dirac operator with δ-shell interaction by sequences of Dirac operators with regular potentials at the appropriate scale in the strong resolvent sense.

2 Model and main results

First, let me define the free Dirac operator and describe some of its properties. Given $m > 0$, the free Dirac operator D_m on $\mathbb{R}^3$ is defined by

$$D_m := -i\alpha \cdot \nabla + m\beta,$$

where

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \text{for } k = 1,2,3, \quad \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad \mathbb{I}_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

is the family of Dirac and Pauli matrices satisfying the anticommutation relations:

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk}\mathbb{I}_4, \quad \{\alpha_j, \beta\} = 0, \quad \text{and} \quad \beta^2 = \mathbb{I}_4, \quad j,k \in \{1,2,3\}, \tag{2.1}$$

where $\{\cdot,\cdot\}$ is the anticommutator bracket. We use the notation $\alpha \cdot x = \sum_{j=1}^{3} \alpha_j x_j$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We recall that $(D_m, \mathrm{dom}(D_m))$ is self-adjoint (see, *e.g.*, [18, Subsection 1.4]), and that

$$\mathrm{Sp}(D_m) = \mathrm{Sp}_{\mathrm{ess}}(D_m) = (-\infty, -m] \cup [m, +\infty).$$

Throughout this paper, for $\Omega \subset \mathbb{R}^3$ a C^∞-smooth bounded domain with boundary $\Sigma := \partial\Omega$, we refer to $H^1(\Omega, \mathbb{C}^4) := H^1(\Omega)^4$ as the first order Sobolev space

$$H^1(\Omega)^4 = \{\varphi \in L^2(\Omega, \mathbb{C}^4) : \text{ there exists } \tilde{\varphi} \in H^1(\mathbb{R}^3)^4 \text{ such that } \tilde{\varphi}|_\Omega = \varphi\}.$$

We denote by $H^{1/2}(\Sigma, \mathbb{C}^4) := H^{1/2}(\Sigma)^4$ the Sobolev space of order $1/2$ along the boundary Σ, and by $t_\Sigma : H^1(\Omega)^4 \to H^{1/2}(\Sigma)^4$ the classical trace operator. The surface Σ divides the Euclidean space into the disjoint union $\mathbb{R}^3 = \Omega_+ \cup \Sigma \cup \Omega_-$, where $\Omega_+ := \Omega$ is a bounded domain and $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega_+}$. We denote by ν and dS the unit outward pointing normal to Ω and the surface measure on Σ, respectively. We also denote by $f_\pm := f \upharpoonright \Omega_\pm$ the restriction of f in $\Omega_\pm$, for all $\mathbb{C}^4$–valued functions f defined on $\mathbb{R}^3$. Then, we define the distribution $\delta_\Sigma f$ by

$$\langle \delta_\Sigma f,\, g\rangle := \frac{1}{2}\int_\Sigma (t_\Sigma f_+ + t_\Sigma f_-)\, g\, \mathrm{dS}, \qquad \text{for any test function } g \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4).$$

Finally, we define the Dirac operator coupled with a combination of electrostatic and Lorentz scalar δ-shell interactions of strengths η and τ, respectively, which we will denote $D_{\eta,\tau}$ in what follows.

Definition 2.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\Sigma = \partial\Omega$. Let $(\eta,\tau)\in\mathbb{R}^2$. Then, $D_{\eta,\tau} = D_m + B_{\eta,\tau}\delta_\Sigma := D_m + (\eta\mathbb{I}_4 + \tau\beta)\delta_\Sigma$ acting in $L^2(\mathbb{R}^3)^4$ is defined as follows:*

$$\begin{aligned} D_{\eta,\tau} f &= D_m f_+ \oplus D_m f_-,\\ \forall f \in \mathrm{dom}(D_{\eta,\tau}) &:= \{ f = f_+ \oplus f_- \in H^1(\Omega)^4 \oplus H^1(\mathbb{R}^3\setminus\overline{\Omega})^4 :\\ &\qquad \textit{the transmission condition (T.C) below holds in } H^{1/2}(\Sigma)^4\}. \end{aligned}$$

Transmission condition:

$$i\alpha\cdot\nu(t_\Sigma f_+ - t_\Sigma f_-) + \frac{1}{2}(\eta\,\mathbb{I}_4 + \tau\beta)(t_\Sigma f_+ + t_\Sigma f_-) = 0, \tag{2.2}$$

where ν is the outward pointing normal to Ω.

Recall that for $\eta^2 - \tau^2 \neq 4$, the Dirac operator $(D_{\eta,\tau}, \mathrm{dom}(D_{\eta,\tau}))$ is self-adjoint and verifies the following assertions (see, *e.g.*, [6, Theorem 3.4, 4.1])

(i) $\mathrm{Sp}_{\mathrm{ess}}(D_{\eta,\tau}) = (-\infty, m] \cup [m, +\infty)$.

(ii) $\mathrm{Sp}_{\mathrm{dis}}(D_{\eta,\tau}) \cap (-m, m)$ is finite.

Now, we explicitly construct regular symmetric potentials $V_{\tilde\eta,\tilde\tau,\varepsilon} \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times4})$ supported on a tubular ε-neighbourhood of Σ and such that

$$V_{\tilde\eta,\tilde\tau,\varepsilon} \xrightarrow[\varepsilon\to0]{} (\tilde\eta\,\mathbb{I}_4 + \tilde\tau\beta)\delta_\Sigma \quad \text{in the sense of distributions.}$$

To explicitly describe the approximate potentials $V_{\tilde\eta,\tilde\tau,\varepsilon}$, we will introduce some additional notations. For $\gamma > 0$, we define $\Sigma_\gamma := \{x \in \mathbb{R}^3,\, \mathrm{dist}(x,\Sigma) < \gamma\}$ a tubular neighborhood of Σ with width γ. For $\gamma > 0$ small enough, Σ_γ is parametrized in a similar way as in [5, 15], given by

$$\Sigma_\gamma = \{x_\Sigma + p\nu(x_\Sigma),\, x_\Sigma \in \Sigma \quad \text{and} \quad p \in (-\gamma,\gamma)\}. \tag{2.3}$$

For $0 < \varepsilon < \gamma$, let $h_\varepsilon(p) := \frac{1}{\varepsilon} h\left(\frac{p}{\varepsilon}\right)$, for all $p \in \mathbb{R}$, with the function h verifies the following

$$h \in L^\infty(\mathbb{R},\mathbb{R}), \quad \mathrm{supp}\, h \subset (-1,1) \text{ and } \int_{-1}^{1} h(t)\,\mathrm{d}t = 1.$$

Thus, we have:

$$\operatorname{supp} h_\varepsilon \subset (-\varepsilon, \varepsilon), \int_{-\varepsilon}^{\varepsilon} h_\varepsilon(t)\,\mathrm{d}t = 1, \text{ and } \lim_{\varepsilon \to 0} h_\varepsilon = \delta_0 \text{ in the sense of distributions,} \tag{2.4}$$

where δ_0 is the Dirac δ-function supported at the origin. Finally, for any $\varepsilon \in (0, \gamma)$, we define the symmetric approximate potentials $V_{\tilde{\eta},\tilde{\tau},\varepsilon} \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4})$, as follows:

$$V_{\tilde{\eta},\tilde{\tau},\varepsilon}(x) := \begin{cases} B_{\tilde{\eta},\tilde{\tau}} h_\varepsilon(p), & \text{if } \; x = x_\Sigma + p\nu(x_\Sigma) \in \Sigma_\varepsilon, \\ 0, & \text{if } \; x \in \mathbb{R}^3 \setminus \Sigma_\varepsilon. \end{cases} \tag{2.5}$$

It is easy to see that $\lim_{\varepsilon \to 0} V_{\tilde{\eta},\tilde{\tau},\varepsilon} = B_{\tilde{\eta},\tilde{\tau}} \delta_\Sigma$, in $\mathcal{D}'(\mathbb{R}^3, \mathbb{C}^{4\times 4})$. For $0 < \varepsilon < \gamma$, we define the family of Dirac operators $\{\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\}_\varepsilon$ as follows:

$$\begin{aligned} &\operatorname{dom}(\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}) := \operatorname{dom}(D_m) = H^1(\mathbb{R}^3)^4, \\ &\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon} \psi = D_m \psi + V_{\tilde{\eta},\tilde{\tau},\varepsilon} \psi, \quad \text{for all } \psi \in \operatorname{dom}(\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}). \end{aligned} \tag{2.6}$$

The main purpose of the present manuscript is to study the strong resolvent limit of $\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}$ at $\varepsilon \to 0$. The following theorem is the main result of this paper.

Theorem 2.2. *Let $(\tilde{\eta}, \tilde{\tau}) \in \mathbb{R}^2$ such that $\tilde{d} := \tilde{\eta}^2 - \tilde{\tau}^2$. Let $(\eta, \tau) \in \mathbb{R}^2$ be defined as follows:*

- *if $\tilde{d} < 0$, then $(\eta, \tau) = \dfrac{\tanh\left(\sqrt{-\tilde{d}}/2\right)}{\sqrt{-\tilde{d}}/2}(\tilde{\eta}, \tilde{\tau})$,*
- *if $\tilde{d} = 0$, then $(\eta, \tau) = (\tilde{\eta}, \tilde{\tau})$,*
- *if $\tilde{d} > 0$ such that $d \neq (2k+1)^2\pi^2$, $k \in \mathbb{N} \cup \{0\}$, then $(\eta, \tau) = \dfrac{\tan\left(\sqrt{\tilde{d}}/2\right)}{\sqrt{\tilde{d}}/2}(\tilde{\eta}, \tilde{\tau})$.*

Now, let $\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}$ be defined as in (2.6) and $D_{\eta,\tau}$ as in Definition 2.1. Then,

$$\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon} \xrightarrow[\varepsilon \to 0]{} D_{\eta,\tau} \quad \textit{in the strong resolvent sense.} \tag{2.7}$$

Remark 2.3. *We mention that in this work we find approximations by regular potentials in the sense of strong resolvent for the Dirac operator with δ-shell potentials $\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}$ in the non-critical case (i.e., when $d \neq 4$) and the non-confining case, (i.e., when $d \neq -4$) everywhere on Σ. This is what we will show in the proof of Theorem 2.2.*

Now, we will introduce some notations and geometrical aspects which we will use in the rest of the paper.

2.1 Notations and geometric aspects

Let Σ be parametrized by the family $\{\phi_j, U_j, V_j,\}_{j\in J}$ with J a finite set, $U_j \subset \mathbb{R}^2$, $V_j \subset \mathbb{R}^3$, $\Sigma \subset \bigcup_{j\in J} V_j$ and $\phi_j(U_j) = V_j \cap \Sigma \subset \Sigma \subset \mathbb{R}^3$ for all $j \in J$. We set $s = \phi_j^{-1}(x_\Sigma)$ for any $x_\Sigma \in \Sigma$.

Definition 2.4 (Weingarten map)**.** *For $x_\Sigma = \phi_j(s) \in \Sigma \cap V_j$ with $s \in U_j$, the Weingarten map (arising from the second fundamental form) is defined as the following linear operator*

$$\begin{aligned} W_{x_\Sigma} := W(x_\Sigma) : T_{x_\Sigma} &\to T_{x_\Sigma} \\ \partial_i \phi_j(s) &\mapsto W(x_\Sigma)[\partial_i \phi_j](s) := -\partial_i \nu(\phi_j(s)), \end{aligned}$$

where T_{x_Σ} denotes the tangent space of Σ on x_Σ and $\{\partial_i \phi_j(s)\}_{i=1,2}$ are the basis vectors of T_{x_Σ}.

Proposition 2.5 ([19, Chapter 9 (Theorem 2), 12 (Theorem 2)])**.** *Let Σ be an $n-$surface in $\mathbb{R}^{n+1}$, oriented by the unit normal vector field ν, and let $x \in \Sigma$. The principal curvatures of Σ at x (i.e., the eigenvalues $k_1(x), \dots, k_n(x)$ of the Weingarten map W_x) are uniformly bounded on Σ.*

2.1.1 Tubular neighborhood of Σ

Recall that for $\Omega \subset \mathbb{R}^3$ a bounded domain with smooth boundary Σ parametrized by $\phi \in \{\phi_j\}_{j\in J}$. Let $\{\phi, U_\phi, V_\phi\}$ belong to $\{\phi_j, U_j, V_j\}_{j\in J}$ and set $\nu_\phi = \nu \circ \phi : U_\phi \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, with ν the outward pointing unit normal to Ω.

For $\gamma > 0$, Σ_γ (2.3) is a tubular neighborhood of Σ with width γ. We define the diffeomorphism Φ_ϕ as follows:

$$\begin{aligned} \Phi_\phi : U_\phi \times (-\gamma, \gamma) &\longrightarrow \mathbb{R}^3 \\ (s, p) &\longmapsto \Phi_\phi(s, p) = \phi(s) + p\nu(\phi(s)). \end{aligned}$$

For sufficiently small γ, Φ_ϕ is a smooth parametrization of Σ_γ. Moreover, the matrix of the differential $\mathrm{d}\Phi_\phi$ of Φ_ϕ in the canonical basis of $\mathbb{R}^3$ is given by

$$\mathrm{d}\Phi_\phi(s, p) = \Big(\partial_1\phi(s) + p\,\mathrm{d}\nu(\partial_1\phi)(s) \quad \partial_2\phi(s) + p\,\mathrm{d}\nu(\partial_2\phi)(s) \quad \nu_\phi(s)\Big). \tag{2.8}$$

Thus, the differential on U_ϕ and the differential on $(-\gamma, \gamma)$ of Φ_ϕ are respectively given by

$$\begin{aligned} &\mathrm{d}_s\Phi_\phi(s, p) = \partial_i\phi_j(s) - pW(x_\Sigma)\partial_i\phi_j(s) \quad \text{ for } \ i = 1, 2 \ \text{ and } \ x_\Sigma = \phi(s) \in \Sigma, \\ &\mathrm{d}_p\Phi_\phi(s, p) = \nu_\phi(s), \end{aligned} \tag{2.9}$$

where $\partial_i\phi$, ν_ϕ should be understood as column vectors, and $W(x_\Sigma)$ is the Weingarten map defined in Definition 2.4. Next, we define

$$\begin{aligned} \mathscr{P}_\phi &:= \left(\Phi_\phi^{-1}\right)_1 : \Sigma_\gamma \longrightarrow U_\phi \subset \mathbb{R}^2; \quad \mathscr{P}_\phi\big(\phi(s) + p\nu(\phi(s))\big) = s \in \mathbb{R}^2, \\ \mathscr{P}_\perp &:= \left(\Phi_\phi^{-1}\right)_2 : \Sigma_\gamma \longrightarrow (-\gamma, \gamma); \quad \mathscr{P}_\perp\big(\phi(s) + p\nu(\phi(s))\big) = p. \end{aligned} \tag{2.10}$$

Using the inverse function theorem and equation (2.8), for $x = \phi(s) + p\nu(\phi(s)) \in \Sigma_\gamma$, we obtain the following differential

$$\nabla \mathscr{P}_\phi(x) = \left(J_{\Phi_\phi^{-1}}\right)_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} J_{\Phi_\phi^{-1}} \quad \text{and} \quad \nabla \mathscr{P}_\perp(x) = \nu_\phi(s), \tag{2.11}$$

with $J_{\Phi_\phi^{-1}}$ the Jacobian matrix of the diffeomorphism Φ_ϕ^{-1} given by the following formula:

$$J_{\Phi_\phi^{-1}} = \frac{1}{\det(J_{\Phi_\phi})} \times \operatorname{Adj}(J_{\Phi_\phi}).$$

Here $\operatorname{Adj}(J_{\Phi_\phi})$ is expressed in terms of the partial derivatives of ϕ, J_{Φ_ϕ} is the Jacobian matrix of the diffeomorphism Φ_ϕ and $\det(J_{\Phi_\phi}) = 1 + p\kappa_1 + p^2\kappa_2$ (see, for example [13, Lemma 2.3 (1)]), where κ_1 and κ_2 depend on the principal curvatures $k_1, \dots, k_n$ of Σ.

2.2 Preparations for the proof

Before presenting the tools for the proof of Theorem 2.2, we state several properties satisfied by the operator $D_{\eta,\tau}$, which appeared in almost the same form in several papers, for example, [8, Section 5] and [6, Section 3].

Lemma 2.6. *Let $(\eta, \tau) \in \mathbb{R}^2$, and let $D_{\eta,\tau}$ be defined as in Definition 2.1. Then, the following hold:*

(i) If $\eta^2 - \tau^2 \neq -4$, there exists an invertible matrix $R_{\eta,\tau}$ such that a function $f = f_+ \oplus f_- \in H^1(\Omega_+)^4 \oplus H^1(\Omega_-)^4$ belongs to $\operatorname{dom}(D_{\eta,\tau})$ *if and only if $t_\Sigma f_+ = R_{\eta,\tau} t_\Sigma f_-$, with $R_{\eta,\tau}$ given by*

$$R_{\eta,\tau} := \left(\mathbb{I}_4 - \frac{i\alpha\cdot\nu}{2}(\eta\,\mathbb{I}_4 + \tau\beta)\right)^{-1}\left(\mathbb{I}_4 + \frac{i\alpha\cdot\nu}{2}(\eta\,\mathbb{I}_4 + \tau\beta)\right). \tag{2.12}$$

(ii) If $\eta^2 - \tau^2 = -4$, then a function $f = f_+ \oplus f_- \in H^1(\Omega_+)^4 \oplus H^1(\Omega_-)^4$ belongs to $\operatorname{dom}(D_{\eta,\tau})$ *if and only if*

$$\left(\mathbb{I}_4 - \frac{i\alpha\cdot\nu}{2}(\eta\,\mathbb{I}_4 + \beta\tau)\right) t_\Sigma f_+ = 0 \quad \textit{and} \quad \left(\mathbb{I}_4 + \frac{i\alpha\cdot\nu}{2}(\eta\,\mathbb{I}_4 + \beta\tau)\right) t_\Sigma f_- = 0.$$

Proof. Let us show (i). Using the transmission condition in equation (2.2), we find that for all $f = f_+ \oplus f_- \in \mathrm{dom}(D_{\eta,\tau})$,

$$\left(i\alpha \cdot \nu + \frac{1}{2}(\eta\mathbb{I}_4 + \tau\beta)\right) t_\Sigma f_+ = \left(i\alpha \cdot \nu - \frac{1}{2}(\eta\mathbb{I}_4 + \tau\beta)\right) t_\Sigma f_-.$$

Thanks to properties in (2.1) and the fact that $(i\alpha \cdot \nu)^{-1} = -i\alpha \cdot \nu$, we have

$$\big(\mathbb{I}_4 - M\big) t_\Sigma f_+ = \big(\mathbb{I}_4 + M\big) t_\Sigma f_-, \tag{2.13}$$

with M a 4×4 matrix having the following form

$$M = \frac{i\alpha \cdot \nu}{2}(\eta\,\mathbb{I}_4 + \beta\tau),$$

thus (2.12) is established.

Now, using the anticommutation relations from (2.1), we have:

$$M^2 = -\frac{d}{4}\mathbb{I}_4 \quad \text{and} \quad (\mathbb{I}_4 - M)(\mathbb{I}_4 + M) = \frac{4+d}{4}\mathbb{I}_4,$$

where $d := \eta^2 - \tau^2$. When $d \neq -4$, then $\mathbb{I}_4 - M$ is invertible with $(\mathbb{I}_4 - M)^{-1} = \dfrac{4}{4+d}(\mathbb{I}_4 + M)$. Consequently, using (2.13) we obtain that $t_\Sigma f_+ = R_{\eta,\tau} t_\Sigma f_-$, where $R_{\eta,\tau}$ has the explicit form

$$R_{\eta,\tau} = \frac{4}{4+d}\left(\frac{4-d}{4}\mathbb{I}_4 + i\alpha \cdot \nu(\eta\mathbb{I}_4 + \tau\beta)\right). \tag{2.14}$$

For assertion (ii), we multiply (2.13) by $(\mathbb{I}_4 \pm M)$, giving

$$(\mathbb{I}_4 + M)^2 t_\Sigma f_- = 0 \quad \text{and} \quad (\mathbb{I}_4 - M)^2 t_\Sigma f_+ = 0.$$

Moreover, we mention that in the case $d = -4$, we have $(\mathbb{I}_4 \pm M)^2 = 2(\mathbb{I}_4 \pm M)$. This completes the proof of Lemma 2.6. □

3 Proof of Theorem 2.2

Proof. Following the ideas in [8, Section 8], the key step in proving Theorem 2.2 is to establish the convergence (2.7) in the strong graph limit sense. Let $\{\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\}_{\varepsilon\in(0,\gamma)}$ and $D_{\eta,\tau}$ be as defined in (2.6) and Definition 2.1, respectively. Since the singular interactions $V_{\tilde{\eta},\tilde{\tau},\varepsilon}$ are bounded and symmetric, the Kato-Rellich theorem implies that the operators $\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}$ are self-adjoint in $L^2(\mathbb{R}^3, \mathbb{C}^4)$. Moreover, we know that $D_{\eta,\tau}$ is self-adjoint, with $\mathrm{dom}(D_{\eta,\tau}) \subset H^1(\mathbb{R}^3 \setminus \Sigma)^4$. Thus, the convergence of $\{\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\}_{\varepsilon\in(0,\gamma)}$ to $D_{\eta,\tau}$ in the strong resolvent sense as $\varepsilon \to 0$ holds if and only if it converges in the strong graph limit sense, as shown in [16, Theorem VIII.26]. This means we must show the following:

For $\psi \in \mathrm{dom}(D_{\eta,\tau})$, there is a family of vectors $\{\psi_\varepsilon\}_{\varepsilon\in(0,\gamma)} \subset H^1(\mathbb{R}^3)^4$ such that

$$\text{(a)} \lim_{\varepsilon\to 0}\psi_\varepsilon = \psi \quad \text{and} \quad \text{(b)} \lim_{\varepsilon\to 0}\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\psi_\varepsilon = D_{\eta,\tau}\psi \quad \text{in } L^2(\mathbb{R}^3)^4, \tag{3.1}$$

with $H^1(\mathbb{R}^3)^4 = \mathrm{dom}(\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon})$ for all $\varepsilon \in (0,\gamma)$.

Let $\psi \equiv \psi_+ \oplus \psi_- \in \mathrm{dom}(D_{\eta,\tau})$. From Theorem 2.2, we have that

$$\begin{aligned} d = \eta^2 - \tau^2 &= -4\tanh^2\left(\sqrt{-\tilde{d}}/2\right), \quad \text{if} \quad \tilde{d} < 0,\\ d = \eta^2 - \tau^2 &= 4\tan^2\left(\sqrt{\tilde{d}}/2\right), \quad \text{if} \quad \tilde{d} > 0,\\ d = \eta^2 - \tau^2 &= 0, \quad \text{if} \quad \tilde{d} = 0. \end{aligned} \tag{3.2}$$

In all cases, we have that $d > -4$ (in particular $d \neq -4$). Then, by Lemma 2.6 (i),

$$t_\Sigma\psi_+ = R_{\eta,\tau}t_\Sigma\psi_-,$$

where $R_{\eta,\tau}$ is given in (2.14). Moreover, using Definition 2.1, we obtain that $t_\Sigma\psi_\pm \in H^{1/2}(\Sigma)^4$.

Show that

$$e^{i\alpha\cdot\nu B_{\tilde{\eta},\tilde{\tau}}} = R_{\eta,\tau}. \tag{3.3}$$

Recall the definition of the family $\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}$ and the potential $V_{\tilde{\eta},\tilde{\tau},\varepsilon}$ defined in (2.6) and (2.5), respectively. We have that

$$(i\alpha\cdot\nu B_{\tilde{\eta},\tilde{\tau}})^2 = (i\alpha\cdot\nu(\tilde{\eta}\mathbb{I}_4 + \tilde{\tau}\beta))^2 = -(\tilde{\eta}^2 - \tilde{\tau}^2) =: \tilde{D}^2, \quad \text{with } \tilde{D} = \sqrt{-(\tilde{\eta}^2 - \tilde{\tau}^2)} = \sqrt{-\tilde{d}}.$$

Using this equality, we can write: $e^{i\alpha\cdot\nu B_{\tilde{\eta},\tilde{\tau}}} = e^{-\tilde{D}}\Pi_- + e^{\tilde{D}}\Pi_+$, with $\pm\tilde{D}$ the eigenvalues of $i\alpha\cdot\nu B_{\tilde{\eta},\tilde{\tau}}$, and $\Pi_\pm$ the eigenprojections are given by:

$$\Pi_\pm := \frac{1}{2}\left(\mathbb{I}_4 \pm \frac{i\alpha\cdot\nu B_{\tilde{\eta},\tilde{\tau}}}{\tilde{D}}\right).$$

Therefore,

$$\begin{aligned} e^{(i\alpha\cdot\nu B_{\tilde{\eta},\tilde{\tau}})} &= \left(\frac{e^{\tilde{D}} + e^{-\tilde{D}}}{2}\right)\mathbb{I}_4 + \frac{i\alpha\cdot\nu B_{\tilde{\eta},\tilde{\tau}}}{\tilde{D}}\left(\frac{e^{\tilde{D}} - e^{-\tilde{D}}}{2}\right)\\ &= \cosh(\tilde{D})\mathbb{I}_4 + \frac{\sinh(\tilde{D})}{\tilde{D}}(i\alpha\cdot\nu(\tilde{\eta}\mathbb{I}_4 + \tilde{\tau}\beta)). \end{aligned}$$

Now, the idea is to show (3.3), *i.e.*, that it remains to show

$$\frac{4}{4+d}\left(\frac{4-d}{4}\mathbb{I}_4 + i\alpha\cdot\nu(\eta\mathbb{I}_4+\tau\beta)\right) = \cosh(\tilde{D})\mathbb{I}_4 + \frac{\sinh(\tilde{D})}{\tilde{D}}\left(i\alpha\cdot\nu(\tilde{\eta}\mathbb{I}_4+\tilde{\tau}\beta)\right). \tag{3.4}$$

To this end, set $\mathfrak{U} = \dfrac{4-d}{4+d} - \cosh(\tilde{D})$ and $\mathfrak{V} = \dfrac{4}{4+d} - \dfrac{\sinh(\tilde{D})}{\tilde{D}}$. If we apply (3.4) to the unit vector $e_1 = (1\ 0\ 0\ 0)^t$, and, since the matrices $\mathbb{I}_4$ and $\alpha\cdot\nu(\eta\mathbb{I}_2+\tau\beta)$ are linearly independent for $(\eta,\tau)\neq(0,0)$, then we find that $\mathfrak{U}=\mathfrak{V}=0$. Hence, (3.4) makes sense if and only if

$$\cosh(\tilde{D}) = \frac{4-d}{4+d} \quad \text{and} \quad \frac{\sinh(\tilde{D})}{\tilde{D}}(\tilde{\eta},\tilde{\tau}) = \frac{4}{4+d}(\eta,\tau).$$

Consequently, we have $R_{\eta,\tau} = e^{i\alpha\cdot\nu B_{\tilde{\eta},\tilde{\tau}}}$.

Dividing $\dfrac{\sinh(\tilde{D})}{\tilde{D}}$ by $(1+\cosh(\tilde{D}))$ we obtain $(\eta,\tau) = \dfrac{\sinh(\tilde{D})}{1+\cosh(\tilde{D})}\dfrac{1}{\tilde{D}/2}(\tilde{\eta},\tilde{\tau})$.

Now, applying the elementary identity $\tanh\left(\dfrac{\theta}{2}\right) = \dfrac{\sinh(\theta)}{1+\cosh(\theta)}$, for all $\theta\in\mathbb{C}\setminus\{i(2k+1)\pi,$ $k\in\mathbb{Z}\}$. We conclude that

$$\frac{\tanh(\sqrt{-\tilde{d}}/2)}{\sqrt{-\tilde{d}}/2}(\tilde{\eta},\tilde{\tau}) = (\eta,\tau), \quad \text{if } \tilde{d}<0,$$

and so, for $\tilde{d}>0$ we apply the elementary identity $-i\tanh(i\theta) = \tan(\theta)$ for all $\theta\in\mathbb{C}\setminus\left\{\pi\left(k+\dfrac{1}{2}\right),\, k\in\mathbb{Z}\right\}$, and then we get that

$$\frac{\tanh\left(\sqrt{-\tilde{d}}/2\right)}{\sqrt{-\tilde{d}}/2} = \frac{\tan\left(\sqrt{\tilde{d}}/2\right)}{\sqrt{\tilde{d}}/2}.$$

Hence, for $\tilde{d}>0$ such that $\tilde{d}\neq(2k+1)^2\pi^2$, we obtain $(\eta,\tau) = \dfrac{\tan(\sqrt{\tilde{d}}/2)}{\sqrt{\tilde{d}}/2}(\tilde{\eta},\tilde{\tau})$. Consequently, the equality $e^{i\alpha\cdot\nu B_{\tilde{\eta},\tilde{\tau}}} = R_{\eta,\tau}$ is shown, with the following parameters satisfying:

- $\dfrac{\tanh(\sqrt{-\tilde{d}}/2)}{\sqrt{-\tilde{d}}/2}(\tilde{\eta},\tilde{\tau}) = (\eta,\tau)$, if $\tilde{d}<0$,
- $\dfrac{\tan(\sqrt{\tilde{d}}/2)}{\sqrt{\tilde{d}}/2}(\tilde{\eta},\tilde{\tau}) = (\eta,\tau)$, if $\tilde{d}>0$,
- $(\tilde{\eta},\tilde{\tau}) = (\eta,\tau)$, if $\tilde{d}=0$.

Moreover, the fact that $\int_{-\varepsilon}^{\varepsilon} h_\varepsilon(t)dt = 1$ (see, (2.4)) with the statement (3.3) make it possible to write

$$\exp\left[\left(-i\int_{-\varepsilon}^{0} h_\varepsilon(t)\,\mathrm{d}t\right)(\alpha\cdot\nu B_{\eta,\tau})\right]t_\Sigma\psi_+ = \exp\left[\left(i\int_{0}^{\varepsilon} h_\varepsilon(t)\,\mathrm{d}t\right)(\alpha\cdot\nu B_{\eta,\tau})\right]t_\Sigma\psi_-. \tag{3.5}$$

Remark 3.1. *We mention that, in the case where $\tilde{D} = 0$, the phenomenon of renormalization of the coupling constants does not arise. This was already observed in the one-dimensional setting in [20]. Indeed, using* (3.2) *and equation* (3.4)*, we find that* $(\tilde{\eta}, \tilde{\tau}) = (\eta, \tau)$*, where* $\dfrac{\sinh(\tilde{D})}{\tilde{D}}$ *is taken to be equal to* 1 *when* $\tilde{D} = 0$*.*

Construction of the family $\{\psi_\varepsilon\}_{\varepsilon\in(0,\gamma)}$. Proceeding as in the construction of [8, Section 8], one can construct the following family. The reader should look at that paper for the details. For all $0 < \varepsilon < \gamma$, we define the function $H_\varepsilon : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ as follows:

$$H_\varepsilon(p) := \begin{cases} \displaystyle\int_p^\varepsilon h_\varepsilon(t)\,\mathrm{d}t, & \text{if} \quad 0 < p < \varepsilon, \\ \displaystyle -\int_{-\varepsilon}^p h_\varepsilon(t)\,\mathrm{d}t, & \text{if} \quad -\varepsilon < p < 0, \\ 0, & \text{if} \quad |p| \geq \varepsilon. \end{cases}$$

Clearly, $H_\varepsilon \in L^\infty(\mathbb{R})$ and is supported in $(-\varepsilon, \varepsilon)$. Since $||H_\varepsilon||_{L^\infty} \leq ||h||_{L^1}$, we get that $\{H_\varepsilon\}_\varepsilon$ is bounded uniformly in ε. For all $\varepsilon \in (0, \gamma)$, the restrictions of H_ε to $\mathbb{R}_\pm$ are uniformly continuous, with finite limits at $p = 0$ exist, and are differentiable a.e., with a bounded derivative, since $h_\varepsilon \in L^\infty(\mathbb{R}, \mathbb{R})$. Using these functions, we set the matrix functions $\mathbb{U}_\varepsilon : \mathbb{R}^3 \setminus \Sigma \to \mathbb{C}^{4\times4}$ such that

$$\mathbb{U}_\varepsilon(x) := \begin{cases} e^{(i\alpha\cdot\nu)B_{\tilde{\eta},\tilde{\tau}} H_\varepsilon(\mathscr{P}_\perp(x))}, & \text{if} \quad x \in \Sigma_\varepsilon \setminus \Sigma, \\ \mathbb{I}_4, & \text{if} \quad x \in \mathbb{R}^3 \setminus \Sigma_\varepsilon, \end{cases} \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times4}), \tag{3.6}$$

where the mapping $\mathscr{P}_\perp$ is defined as in (2.10). The functions $\mathbb{U}_\varepsilon$ are bounded, uniformly in ε, and uniformly continuous in $\Omega_\pm$, with a jump discontinuity across Σ. Then, ψ_ε can be constructed by

$$\psi_\varepsilon = \psi_{\varepsilon,+} \oplus \psi_{\varepsilon,-} := \mathbb{U}_\varepsilon \psi \in L^2(\mathbb{R}^3, \mathbb{C}^4), \quad \text{where} \quad \forall x_\Sigma \in \Sigma,\, y_\pm \in \Omega_\pm :$$

$$\begin{aligned} \mathbb{U}_\varepsilon(x_\Sigma^-) &:= \lim_{y_-\to x_\Sigma} \mathbb{U}_\varepsilon(y_-) = \exp\left[i\left(\int_0^\varepsilon h_\varepsilon(t)\,\mathrm{d}t\right)(\alpha\cdot\nu(x_\Sigma))B_{\tilde{\eta},\tilde{\tau}}\right], \\ \mathbb{U}_\varepsilon(x_\Sigma^+) &:= \lim_{y_+\to x_\Sigma} \mathbb{U}_\varepsilon(y_+) = \exp\left[-i\left(\int_{-\varepsilon}^0 h_\varepsilon(t)\,\mathrm{d}t\right)(\alpha\cdot\nu(x_\Sigma))B_{\tilde{\eta},\tilde{\tau}}\right]. \end{aligned} \tag{3.7}$$

Since $\mathbb{U}_\varepsilon$ are bounded, uniformly in ε, using the construction of ψ_ε we get that $\psi_\varepsilon - \psi := (\mathbb{U}_\varepsilon - \mathbb{I}_4)\psi$. Then, by the dominated convergence theorem and the fact that $\operatorname{supp}(\mathbb{U}_\varepsilon - \mathbb{I}_4) \subset |\Sigma_\varepsilon|$ with $|\Sigma_\varepsilon| \to 0$ as $\varepsilon \to 0$, it is easy to show that

$$\psi_\varepsilon \xrightarrow[\varepsilon\to0]{} \psi \quad \text{in } L^2(\mathbb{R}^3, \mathbb{C}^4). \tag{3.8}$$

This proves assertion (a).

Show that $\psi_\varepsilon \in \mathrm{dom}(\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}) = H^1(\mathbb{R}^3)^4$. This means that we must show, for all $0 < \varepsilon < \gamma$,

$$\text{(i)}\, \psi_{\varepsilon,\pm} \in H^1(\Omega_\pm)^4 \quad \text{and} \quad \text{(ii)}\, t_\Sigma \psi_{\varepsilon,+} = t_\Sigma \psi_{\varepsilon,-} \in H^{1/2}(\Sigma)^4.$$

Let us show point (i). By the construction of ψ_ε, we have $\psi_\varepsilon \in L^2(\mathbb{R}^3, \mathbb{C}^4)$. It remains to show that $\partial_j \mathbb{U}_\varepsilon \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4})$, for $j = 1, 2, 3$. To do so, recall the parametrization ϕ of Σ defined at the beginning of Subsection 2.1 and let $A \in C^\infty(\mathbb{R}^2, \mathbb{C}^{4\times 4})$ such that $A(s) := i\alpha \cdot \nu(\phi(s)) B_{\tilde{\eta},\tilde{\tau}}$, for $s = (s_1, s_2) \in U \subset \mathbb{R}^2$. Thus, the matrix functions $\mathbb{U}_\varepsilon$ in (3.6) can be written

$$\mathbb{U}_\varepsilon(x) = \begin{cases} e^{A(\mathscr{P}_\phi(x)) H_\varepsilon(\mathscr{P}_\perp(x))}, & \text{if} \quad x \in \Sigma_\varepsilon \setminus \Sigma, \\ \mathbb{I}_4, & \text{if} \quad x \in \mathbb{R}^3 \setminus \Sigma_\varepsilon, \end{cases} \quad \in L^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4}),$$

where $\mathscr{P}_\phi$ is defined as in (2.10).

For $j = 1, 2, 3$, we have $\operatorname{supp} \partial_j \mathbb{U}_\varepsilon \subset \Sigma_\varepsilon$. By the Wilcox formula as used in [8, Eq. 8.12], we obtain that

$$\partial_j \mathbb{U}_\varepsilon(x) = \int_0^1 \Big[\exp\Big(z A(\mathscr{P}_\phi(x)) H_\varepsilon(\mathscr{P}_\perp(x))\Big) \partial_j \Big(A(\mathscr{P}_\phi(x)) H_\varepsilon(\mathscr{P}_\perp(x))\Big) \times \\ \exp\Big((1-z) A(\mathscr{P}_\phi(x)) H_\varepsilon(\mathscr{P}_\perp(x))\Big)\Big] \mathrm{d}z.$$

Based on the quantities (2.11), for $x = \phi(s) + p\nu(\phi(s)) \in \Sigma_\gamma$, and for $s = \mathscr{P}_\phi(x)$, $p = \mathscr{P}_\perp(x)$, with $\mathscr{P}_\phi(x)$ and $\mathscr{P}_\perp(x)$ the mappings introduced in (2.10), together with

$$\partial_j \Big(A(\mathscr{P}_\phi(x)) H_\varepsilon(\mathscr{P}_\perp(x))\Big) = \partial_j \Big(A(\mathscr{P}_\phi(x)\Big) H_\varepsilon(p) - A(s) h_\varepsilon(p) (\nu_\phi(s))_j,$$

yields that $\partial_j \mathbb{U}_\varepsilon$ has the following form

$$\partial_j \mathbb{U}_\varepsilon(x) = -A(s) h_\varepsilon(p) (\nu_\phi(s))_j \mathbb{U}_\varepsilon(x) + \int_0^1 e^{zA(s)H_\varepsilon(p)} \Big[\partial_j \Big(A(\mathscr{P}_\phi(x)\Big) H_\varepsilon(p)\Big] e^{(1-z)A(s)H_\varepsilon(p)} \,\mathrm{d}z, \tag{3.9}$$

with

$$\partial_j \Big(A(\mathscr{P}_\phi(x)\Big) = \sum_{k=1}^{2} \frac{\partial A(s)}{\partial s_k} (J_{\Phi_\phi^{-1}})_{kj},$$

where $(J_{\Phi_\phi^{-1}})_{kj}$ is the coefficient of the k-th row and j-th column of the matrix $\Big(J_{\Phi_\phi^{-1}}\Big)$ given in (2.11).

We denote by $\mathbb{E}_{\varepsilon,j}$ the second term of the right-hand side of the equality (3.9), *i.e.*,

$$\mathbb{E}_{\varepsilon,j} = \int_0^1 e^{zA(s)H_\varepsilon(p)} \Big[\partial_s A(s) \times \sum_{k=1}^{2} \frac{\partial A(s)}{\partial s_k} (J_{\Phi_\phi^{-1}})_{kj} \times H_\varepsilon(p)\Big] e^{(1-z)A(s)H_\varepsilon(p)} \,\mathrm{d}z. \tag{3.10}$$

Thanks to Proposition 2.5, the matrix-valued functions $\mathbb{E}_{\varepsilon,j}$ are bounded, uniformly for $0<\varepsilon<\gamma$, and $\operatorname{supp}\mathbb{E}_{\varepsilon,j}\subset\Sigma_\varepsilon$. Moreover, we have $\mathbb{U}_\varepsilon$ and $\partial_j\mathbb{U}_\varepsilon\in L^\infty(\Omega_\pm,\mathbb{C}^{4\times4})$, and we deduce that for all $\psi_\pm\in H^1(\Omega_\pm)^4$ we have that $\psi_{\varepsilon,\pm}=\mathbb{U}_\varepsilon\psi_\pm\in H^1(\Omega_\pm)^4$ and statement (i) is verified.

Let us now check point (ii). Since $\psi_{\varepsilon,\pm}\in H^1(\Omega_\pm)^4$, we get that $t_\Sigma\psi_{\varepsilon,\pm}\in H^{1/2}(\Sigma)^4$. On the other hand, as $\mathbb{U}_\varepsilon$ is continuous in $\overline{\Omega_\pm}$, we get

$$t_\Sigma\psi_{\varepsilon,\pm}(x_\Sigma)=\mathbb{U}_\varepsilon(x_\Sigma^\pm)t_\Sigma\psi_\pm(x_\Sigma)\quad\text{ for a.e. } x_\Sigma\in\Sigma;$$

see [10, Chapter 4 (p.133)] and [8, Section 8] for a similar argument.

Consequently, (3.5) with (3.7) give us that $t_\Sigma\psi_{\varepsilon,+}=t_\Sigma\psi_{\varepsilon,-}\in H^{1/2}(\Sigma)^4$. With this, (ii) is valid and $\psi_\varepsilon\in\operatorname{dom}(\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon})$.

To complete the proof of Theorem 2.2, it remains to show the property (b), mentioned in (3.1). Since $(\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon}\psi_\varepsilon-D_{\eta,\tau}\psi)$ belongs to $L^2(\mathbb{R}^3,\mathbb{C}^4)$, it suffices to prove the following:

$$\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}-D_{\eta,\tau}\psi_\pm\xrightarrow[\varepsilon\to0]{}0\quad\text{in } L^2(\Omega_\pm,\mathbb{C}^4).\tag{3.11}$$

To do this, let $\psi\equiv\psi_+\oplus\psi_-\in\operatorname{dom}(D_{\eta,\tau})$ and $\psi_\varepsilon\equiv\psi_{\varepsilon,+}\oplus\psi_{\varepsilon,-}\in\operatorname{dom}(\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon})$. We have

$$\begin{aligned}\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}-D_{\eta,\tau}\psi_\pm&=-i\alpha\cdot\nabla\psi_{\varepsilon,\pm}+m\beta(\psi_{\varepsilon,\pm}-\psi_\pm)+V_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}+i\alpha\cdot\nabla\psi_\pm\\&=-i\alpha\cdot\nabla(\mathbb{U}_\varepsilon\psi_\pm)+i\alpha\cdot\nabla\psi_\pm+m\beta(\mathbb{U}_\varepsilon-\mathbb{I}_4)\psi_\pm+V_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}\\&=-i\sum_{j=1}^3\alpha_j\big[(\partial_j\mathbb{U}_\varepsilon)\psi_\pm+(\mathbb{U}_\varepsilon-\mathbb{I}_4)\partial_j\psi_\pm\big]+m\beta(\mathbb{U}_\varepsilon-\mathbb{I}_4)\psi_\pm+V_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}.\end{aligned}\tag{3.12}$$

Using the form of $\partial_j\mathbb{U}_\varepsilon$ given in (3.9), the quantity $-i\sum_{j=1}^3\alpha_j(\partial_j\mathbb{U}_\varepsilon)\psi_\pm$ yields

$$\begin{aligned}-i\sum_{j=1}^3\alpha_j(\partial_j\mathbb{U}_\varepsilon)\psi_\pm&=-i\sum_{j=1}^3\alpha_j\big[-i\alpha\cdot\nu V_{\tilde\eta,\tilde\tau,\varepsilon}\nu_j\mathbb{U}_\varepsilon\psi_\pm+\mathbb{E}_{\varepsilon,j}\psi_\pm\big]\\&=-(\alpha\cdot\nu)^2V_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}-i\sum_{j=1}^3\alpha_j\mathbb{E}_{\varepsilon,j}\psi_\pm=-V_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}+\mathbb{R}_\varepsilon\psi_\pm,\end{aligned}$$

where $\mathbb{E}_{\varepsilon,j}$ is given in (3.10) and $\mathbb{R}_\varepsilon=-i\sum_{j=1}^3\alpha_j\mathbb{E}_{\varepsilon,j}$, a matrix-valued function in $L^\infty(\mathbb{R}^3,\mathbb{C}^{4\times4})$, verifies the same property of $\mathbb{E}_{\varepsilon,j}$ for all $\varepsilon\in(0,\gamma)$. Thus, (3.12) becomes

$$\mathscr{E}_{\tilde\eta,\tilde\tau,\varepsilon}\psi_{\varepsilon,\pm}-D_{\eta,\tau}\psi_\pm=-i\sum_{j=1}^3\alpha_j\big[(\mathbb{U}_\varepsilon-\mathbb{I}_4)\partial_j\psi_\pm\big]+m\beta(\mathbb{U}_\varepsilon-\mathbb{I}_4)\psi_\pm+\mathbb{R}_\varepsilon\psi.$$

Since $\psi_\pm\in H^1(\Omega_\pm)^4$, $(\mathbb{U}_\varepsilon-\mathbb{I}_4)$ and $\mathbb{R}_\varepsilon$ are bounded, uniformly in $\varepsilon\in(0,\gamma)$ and supported in Σ_ε, and $|\Sigma_\varepsilon|$ tends to 0 as $\varepsilon\to0$. By the dominated convergence theorem, we conclude that

$$\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\psi_{\varepsilon,\pm} - D_{\eta,\tau}\psi_{\pm} \xrightarrow[\varepsilon\to 0]{} 0, \quad \text{holds in } L^2(\Omega_\pm, \mathbb{C}^4), \tag{3.13}$$

and this achieves the assertion (3.11).

Thus, both conditions mentioned in (3.1) $\big($*i.e.*, (a) and (b)$\big)$ of the convergence in the strong graph limit sense are proved (see, (3.8) and (3.13)). Hence, the family $\{\mathscr{E}_{\tilde{\eta},\tilde{\tau},\varepsilon}\}_{\varepsilon\in(0,\gamma)}$ converges in the strong resolvent sense to $D_{\eta,\tau}$ as $\varepsilon \to 0$. The proof of the Theorem 2.2 is complete. □

4 Acknowledgment

This work was supported by LTC Transmath, BERC.2022-2025 program and BCAM Severo Ochoa research project. I wish to express my gratitude to my thesis advisor Luis VEGA for suggesting the problem and for many stimulating conversations, for his patient advice, and enthusiastic encouragement. I would also like to thank my supervisor Vincent BRUNEAU for his reading, comments, and helpful criticism.

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CUBO, A Mathematical Journal
Vol. 26, no. 3, pp. 507–523, December 2024
DOI: 10.56754/0719-0646.2603.507

Perturbed weighted trapezoid inequalities for convex functions with applications

Sever Silvestru Dragomir[1,2]
Eder Kikianty[3,✉]

[1] *Applied Mathematics Research Group, ISILC, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.*
sever.dragomir@vu.edu.au

[2] *School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, Private Bag 3, Wits 2050, South Africa.*

[3] *Department of Mathematics and Applied Mathematics, University of Pretoria, Private Bag X20, Hatfield 0028, South Africa.*
eder.kikianty@up.ac.za

ABSTRACT

We consider trapezoid type inequalities for twice differentiable convex functions, perturbed by a non-negative weight. Applications on a normed space $(X, \|\cdot\|)$ are considered, by establishing bounds for the term

$$\frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\|x\|^p + \|y\|^p}{2}\right] - \int_0^1 \|(1-t)x + ty\|^p \, dt, \quad x, y \in X,$$

which can be seen as a combination of both the midpoint and the trapezoid p-norm (with $2 \le p < \infty$) inequalities.

RESUMEN

Consideramos desigualdades de tipo trapezoidal para funciones convexas dos veces diferenciables, perturbadas por un peso no-negativo. Se consideran aplicaciones en un espacio normado $(X, \|\cdot\|)$, estableciendo cotas para el término

$$\frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\|x\|^p + \|y\|^p}{2}\right] - \int_0^1 \|(1-t)x + ty\|^p \, dt, \quad x, y \in X,$$

que se puede ver como una combinación de las desigualdades de punto medio y trapezoidal para las p-normas (con $2 \le p < \infty$).

Keywords and Phrases: Trapezoid inequality, midpoint inequaliy, Ostrowski's inequality, Čebyšev's inequality, norm inequality, semi-inner product.

2020 AMS Mathematics Subject Classification: 26D15, 46C50.

Published: 12 December, 2024
Accepted: 14 November, 2024
Received: 25 July, 2024

1 Introduction

The following inequality, which is known in the literature as the Hermite-Hadamard inequality, holds for any convex function f defined on $\mathbb{R}$ and all $a, b \in \mathbb{R}$:

$$f\left(\frac{a+b}{2}\right)(b-a) \leq \int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}(b-a). \tag{1.1}$$

Let $a, b \in \mathbb{R}$ with $a < b$, $f\colon [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (a,b)$. Then the following inequality, known as the Ostrowski inequality:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M, \tag{1.2}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant. Note that when f is convex and $x = (a+b)/2$, the Ostrowski inequality (1.2) provides a sharp bound for the midpoint difference

$$\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a), \tag{1.3}$$

in view of the middle and the left-hand terms of (1.1). The following result provides some sharp bounds for the midpoint difference (*cf.* [5, Corollary 2.3]). We note the use of the notation $f'_{\pm}$ to denote the right-hand and left-hand derivatives of f, which exist for any convex function f.

Proposition 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then we have the inequality*

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a)^2 \leq \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a) \\ &\leq \frac{1}{8}\left[f'_-(b) - f'_+(a)\right](b-a)^2. \end{aligned} \tag{1.4}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

In what follows, a similar result provides some sharp bounds for the trapezoid difference (*cf.* [6, Corollary 2.3]):

$$\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt, \tag{1.5}$$

in view of the middle and the right-hand terms of (1.1).

Proposition 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then we have the inequality*

$$0\leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a)^{2}\leq \frac{f(a)+f(b)}{2}(b-a)-\int_{a}^{b}f(t)\,dt \qquad (1.6)$$
$$\leq \frac{1}{8}\left[f'_{-}(b)-f'_{+}(a)\right](b-a)^{2}.$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

There are many results in the literature which provide bounds for both midpoint and trapezoids differences. We refer the readers to the survey paper [9].

Let X be a real linear space, $x,\ y\in X$, $x\neq y$ and let $[x,y]:=\{(1-\lambda)x+\lambda y,\ \lambda\in[0,1]\}$ be the *segment* generated by x and y. We consider the function $f:[x,y]\to\mathbb{R}$ and the associated function $g(x,y):[0,1]\to\mathbb{R}$,

$$g(x,y)(t):=f[(1-t)x+ty],\quad t\in[0,1].$$

It is well known that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy the following properties:

(i) $g'_{\pm}(x,y)(s)=(\nabla_{\pm}f[(1-s)x+sy])(y-x)$, $s\in[0,1)$;

(ii) $g'_{+}(x,y)(0)=(\nabla_{+}f(x))(y-x)$;

(iii) $g'_{-}(x,y)(1)=(\nabla_{-}f(y))(y-x)$;

where $(\nabla_{\pm}f(x))(y)$ are the *Gâteaux lateral derivatives, i.e.*

$$(\nabla_{\pm}f(x))(y):=\lim_{h\to 0^{\pm}}\left[\frac{f(x+hy)-f(x)}{h}\right],$$

for $x,y\in X$.

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x\in X$, is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s:=(\nabla_{+}f_0(y))(x)=\lim_{t\to 0^{+}}\left[\frac{\|y+tx\|^2-\|y\|^2}{2t}\right]$;

(v) $\langle x,y\rangle_i:=(\nabla_{-}f_0(y))(x)=\lim_{t\to 0^{-}}\left[\frac{\|y+tx\|^2-\|y\|^2}{2t}\right]$;

for any $x,y\in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

In [14] Kikianty *et al.* obtained, among others, the following *midpoint p-norm inequalities*:

$$0 \le \int_0^1 \|(1-t)x+ty\|^p dt - \left\|\frac{x+y}{2}\right\|^p \tag{1.7}$$
$$\le p\|y-x\| \begin{cases} \frac{1}{4}\max\{\|x\|^{p-1}, \|y\|^{p-1}\}, \\ \frac{1}{2(q'+1)^{\frac{1}{q'}}}\left(\frac{\|x\|^{q(p-1)}+\|y\|^{q(p-1)}}{2}\right)^{\frac{1}{q}}, \quad q>1,\ \frac{1}{q}+\frac{1}{q'}=1; \\ \frac{1}{4}(\|x\|^{p-1}+\|y\|^{p-1}), \end{cases}$$

that hold for any $x, y \in X$. The constants in the first and second cases of (1.7) are sharp. Furthermore, in [13], the following *trapezoid p-norm inequalities* are obtained:

$$\begin{aligned} 0 &\le \frac{1}{8}p\left\|\frac{x+y}{2}\right\|^{p-2}\left[\left\langle y-x, \frac{x+y}{2}\right\rangle_s - \left\langle y-x, \frac{x+y}{2}\right\rangle_i\right] \\ &\le \frac{\|y\|^p+\|x\|^p}{2} - \int_0^1 \|(1-t)\,x+ty\|^p\, dt \\ &\le \frac{1}{8}p\left[\|y\|^{p-2}\langle y-x, y\rangle_i - \|x\|^{p-2}\langle y-x, x\rangle_s\right] \end{aligned} \tag{1.8}$$

that hold for any $x, y \in X$ whenever $p \ge 2$; otherwise, they hold for linearly independent $x, y \in X$. The constant $\frac{1}{8}$ is best in (1.8).

In this paper, we provide bounds for the term

$$\frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\|x\|^p+\|y\|^p}{2}\right] - \int_0^1 \|(1-t)x+ty\|^p\, dt$$

which can be seen as a combination of both the midpoint and the trapezoid p-norm inequalities. This is done via a series of results on twice differentiable convex functions and we take integrals with respect to a weight function as outlined in Section 2.

2 Main results

Let φ be a twice differentiable convex function on $[0,1]$, w integrable and non-negative on $[0,1]$, and $\lambda \in (0,1)$. In this section, we establish bounds for the following

$$\begin{aligned} &\left(\int_\lambda^1 w(s)\,ds\right)\varphi(1) + \left(\int_0^\lambda w(s)\,ds\right)\varphi(0) - \int_0^1 w(t)\,\varphi(t)\,dt \\ &- \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (1-t)\,w(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda tw(t)\,dt, \end{aligned}$$

using Ostrowski's and Čebyšev's inequalities (*cf.* Propositions 2.1 and 2.4 below).

Recall the following inequality by Ostrowski [16], which was proven in 1970.

Proposition 2.1 (Ostrowski)**.** *Let h be integrable and $n \leq h \leq N$ for some constants n, N on $[a,b]$, while g is absolutely continuous and its derivative is essentially bounded. Then,*

$$\left| \frac{1}{b-a} \int_a^b g(t) h(t) \, dt - \frac{1}{b-a} \int_a^b g(t) \, dt \frac{1}{b-a} \int_a^b h(t) \, dt \right| \leq \frac{1}{8} (b-a)(N-n) \|g'\|_\infty . \tag{2.1}$$

The constant $\frac{1}{8}$ is best possible in the general case.

We derive the first set of inequalities.

Theorem 2.2. *Let φ be a twice differentiable convex function on $[0,1]$, w integrable and non-negative on $[0,1]$ and $\lambda \in (0,1)$. Then*

$$\begin{aligned} 0 &\leq \left(\int_\lambda^1 w(s) \, ds \right) \varphi(1) + \left(\int_0^\lambda w(s) \, ds \right) \varphi(0) - \int_0^1 w(t) \varphi(t) \, dt \\ &\quad - \frac{\varphi(1) - \varphi(\lambda)}{1-\lambda} \int_\lambda^1 (1-t) w(t) \, dt + \frac{\varphi(\lambda) - \varphi(0)}{\lambda} \int_0^\lambda t w(t) \, dt \\ &\leq \frac{1}{8} \left[(1-\lambda)^2 \left(\int_\lambda^1 w(s) \, ds \right) + \lambda^2 \left(\int_0^\lambda w(s) \, ds \right) \right] \|\varphi''\|_{\infty,[0,1]} . \end{aligned} \tag{2.2}$$

Proof. Let $\lambda \in [0,1]$. By using integration by parts, we have

$$\begin{aligned} \int_0^\lambda \left(\int_t^\lambda w(s) \, ds \right) \varphi'(t) \, dt &= \left(\int_t^\lambda w(s) \, ds \right) \varphi(t) \Big|_0^\lambda + \int_0^\lambda w(t) \varphi(t) \, dt \\ &= \int_0^\lambda w(t) \varphi(t) \, dt - \left(\int_0^\lambda w(s) \, ds \right) \varphi(0) \end{aligned}$$

and

$$\begin{aligned} \int_\lambda^1 \left(\int_\lambda^t w(s) \, ds \right) \varphi'(t) \, dt &= \left(\int_\lambda^t w(s) \, ds \right) \varphi(t) \Big|_\lambda^1 - \int_\lambda^1 w(t) \varphi(t) \, dt \\ &= \left(\int_\lambda^1 w(s) \, ds \right) \varphi(1) - \int_\lambda^1 w(t) \varphi(t) \, dt. \end{aligned}$$

Then we have the following identity of interest

$$\begin{aligned} \int_0^1 \left(\int_\lambda^t w(s) \, ds \right) \varphi'(t) \, dt &= \int_\lambda^1 \left(\int_\lambda^t w(s) \, ds \right) \varphi'(t) \, dt - \int_0^\lambda \left(\int_t^\lambda w(s) \, ds \right) \varphi'(t) \, dt \\ &= \left(\int_\lambda^1 w(s) \, ds \right) \varphi(1) + \left(\int_0^\lambda w(s) \, ds \right) \varphi(0) - \int_0^1 w(t) \varphi(t) \, dt \end{aligned} \tag{2.3}$$

for $\lambda \in [0,1]$. If we use (2.1) for $h(t) = \int_\lambda^t w(s) \, ds$ and $g(t) = \varphi'(t)$ on the interval $[\lambda, 1]$, then we get

$$\begin{aligned}0 &\leq \int_\lambda^1 \left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1\left(\int_\lambda^t w(s)\,ds\right)dt\\ &= \int_\lambda^1 \left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (1-t)w(t)dt\\ &\leq \frac{1}{8}(1-\lambda)^2\left(\int_\lambda^1 w(s)\,ds\right)\|\varphi''\|_{\infty,[\lambda,1]}.\end{aligned}$$

We also have, again using (2.1) for $h(t) = -\int_t^\lambda w(s)\,ds$ and $g(t) = \varphi'(t)$ on the interval $[0,\lambda]$, that

$$\begin{aligned}0 &\leq -\int_0^\lambda \left(\int_t^\lambda w(s)\,ds\right)\varphi'(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda\left(\int_t^\lambda w(s)ds\right)dt\\ &= -\int_0^\lambda \left(\int_t^\lambda w(s)\,ds\right)\varphi'(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda tw(t)\,dt \leq \frac{1}{8}\lambda^2\left(\int_0^\lambda w(s)\,ds\right)\|\varphi''\|_{\infty,[0,\lambda]}.\end{aligned}$$

If we add these inequalities, then we get

$$\begin{aligned}0 &\leq \int_\lambda^1 \left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt - \int_0^\lambda \left(\int_t^\lambda w(s)\,ds\right)\varphi'(t)\,dt\\ &\quad - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (1-t)\,w(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda tw(t)\,dt\\ &\leq \frac{1}{8}(1-\lambda)^2\left(\int_\lambda^1 w(s)\,ds\right)\|\varphi''\|_{\infty,[\lambda,1]} + \frac{1}{8}\lambda^2\left(\int_0^\lambda w(s)\,ds\right)\|\varphi''\|_{\infty,[0,\lambda]}\\ &\leq \frac{1}{8}\left[(1-\lambda)^2\left(\int_\lambda^1 w(s)\,ds\right) + \lambda^2\left(\int_0^\lambda w(s)\,ds\right)\right]\|\varphi''\|_{\infty,[0,1]}\end{aligned}$$

and by (2.3) we obtain (2.2). □

When $\lambda = 1/2$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *With the assumptions of Theorem 2.2, we have*

$$\begin{aligned}0 &\leq \left(\int_{\frac{1}{2}}^1 w(s)\,ds\right)\varphi(1) + \left(\int_0^{\frac{1}{2}} w(s)\,ds\right)\varphi(0) - \int_0^1 w(t)\,\varphi(t)\,dt\\ &\quad - 2\left[\left[\varphi(1) - \varphi\left(\frac{1}{2}\right)\right]\int_{\frac{1}{2}}^1 (1-t)\,w(t)\,dt - \left[\varphi\left(\frac{1}{2}\right) - \varphi(0)\right]\int_0^{\frac{1}{2}} tw(t)\,dt\right] \qquad (2.4)\\ &\leq \frac{1}{32}\left(\int_0^1 w(s)\,ds\right)\|\varphi''\|_{\infty,[0,1]}.\end{aligned}$$

The following result obtained by Čebyšev in 1882, [2]. For a function φ with a bounded derivative, we use the following notation

$$\|\varphi'\|_\infty = \sup_{t\in[a,b]} |\varphi'(t)|.$$

Proposition 2.4. *Let g, h be differentiable functions such that g', h' exist and are continuous on $[a,b]$. Then,*

$$\left|\frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt - \frac{1}{b-a}\int_a^b g(t)\,dt \frac{1}{b-a}\int_a^b h(t)\,dt\right| \le \frac{1}{12}(b-a)^2 \|h'\|_\infty \|g'\|_\infty. \tag{2.5}$$

The constant $\frac{1}{12}$ cannot be improved in the general case.

We now derive the second set of inequalities.

Theorem 2.5. *Let φ be a twice differentiable convex function on $[0,1]$, w bounded and non-negative on $[0,1]$ and $\lambda \in (0,1)$. Then*

$$\begin{aligned}
0 &\le \left(\int_\lambda^1 w(s)\,ds\right)\varphi(1) + \left(\int_0^\lambda w(s)\,ds\right)\varphi(0) - \int_0^1 w(t)\,\varphi(t)\,dt \\
&\quad - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (1-t)\,w(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda tw(t)\,dt \\
&\le \frac{1}{12}\left[(1-\lambda)^3 \|w\|_{\infty,[\lambda,1]} + \lambda^3 \|w\|_{\infty,[0,\lambda]}\right]\|\varphi''\|_{\infty,[0,1]} \\
&\le \frac{1}{12}\left[\frac{1}{4} + 3\left(\lambda - \frac{1}{2}\right)^2\right]\|w\|_{\infty,[0,1]}\|\varphi''\|_{\infty,[0,1]}.
\end{aligned} \tag{2.6}$$

Proof. If we use (2.5) for $h(t) = \int_\lambda^t w(s)\,ds$ and $g(t) = \varphi'(t)$ on the interval $[\lambda,1]$, then we get

$$\begin{aligned}
0 &\le \int_\lambda^1 \left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1\left(\int_\lambda^t w(s)\,ds\right)dt \\
&= \int_\lambda^1 \left(\int_\lambda^t w(s)\,ds\right)\varphi'(t)\,dt - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (1-t)w(t)\,dt \\
&\le \frac{1}{12}(1-\lambda)^3 \sup_{s\in[\lambda,1]} w(s)\,\|\varphi''\|_{\infty,[\lambda,1]}.
\end{aligned}$$

Again, by (2.5) for $h(t) = -\int_t^\lambda w(s)\,ds$ and $g(t) = \varphi'(t)$ on the interval $[0,\lambda]$, we get

$$\begin{aligned}
0 &\le -\int_0^\lambda \left(\int_t^\lambda w(s)\,ds\right)\varphi'(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda\left(\int_t^\lambda w(s)\,ds\right)dt \\
&= -\int_0^\lambda \left(\int_t^\lambda w(s)\,ds\right)\varphi'(t)\,dt + \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda tw(t)\,dt \\
&\le \frac{1}{12}\lambda^3 \sup_{s\in[0,\lambda]} w(s)\,\|\varphi''\|_{\infty,[0,\lambda]}.
\end{aligned}$$

If we add these inequalities, then we get

$$\begin{aligned}
0 &\le \int_{\lambda}^{1}\left(\int_{\lambda}^{t} w(s)\,ds\right)\varphi'(t)\,dt-\int_{0}^{\lambda}\left(\int_{t}^{\lambda} w(s)\,ds\right)\varphi'(t)\,dt\\
&-\frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_{\lambda}^{1}(1-t)\,w(t)\,dt+\frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_{0}^{\lambda} tw(t)\,dt\\
&\le \frac{1}{12}\left[(1-\lambda)^{3}\sup_{s\in[\lambda,1]} w(s)\,\|\varphi''\|_{\infty,[\lambda,1]}+\lambda^{3}\sup_{s\in[0,\lambda]} w(s)\,\|\varphi''\|_{\infty,[0,\lambda]}\right],
\end{aligned}$$

which proves (2.6). □

When $\lambda = 1/2$ in Theorem 2.5, we have the following corollary.

Corollary 2.6. *With the assumptions of Theorem 2.5, we have*

$$\begin{aligned}
0 &\le \left(\int_{\frac{1}{2}}^{1} w(s)\,ds\right)\varphi(1)+\left(\int_{0}^{\frac{1}{2}} w(s)\,ds\right)\varphi(0)-\int_{0}^{1} w(t)\,\varphi(t)\,dt\\
&-2\left[\left[\varphi(1)-\varphi\left(\frac{1}{2}\right)\right]\int_{\frac{1}{2}}^{1}(1-t)\,w(t)\,dt-\left[\varphi\left(\frac{1}{2}\right)-\varphi(0)\right]\int_{0}^{\frac{1}{2}} tw(t)\,dt\right] \qquad (2.7)\\
&\le \frac{1}{96}\left[\|w\|_{\infty,[\frac{1}{2},1]}+\|w\|_{\infty,[0,\frac{1}{2}]}\right]\|\varphi''\|_{\infty,[0,1]}\le \frac{1}{48}\|w\|_{\infty,[0,1]}\|\varphi''\|_{\infty,[0,1]}.
\end{aligned}$$

3 Symmetrical weight functions

Simpler forms of the inequalities in Theorems 2.2 and 2.5 (also, Corollaries 2.3 and 2.6) are obtained when we consider the case that the weight w is symmetrical on $[0,1]$. Assume that w is symmetrical on $[0,1]$. Then,

$$\int_{\frac{1}{2}}^{1}(1-t)\,w(t)\,dt=\int_{0}^{\frac{1}{2}} tw(t)\,dt.$$

By assuming the symmetry of w on $[0,1]$ in Corollary 2.3, we have

$$\begin{aligned}
0 &\le \frac{\varphi(1)+\varphi(0)}{2}\left(\int_{0}^{1} w(s)\,ds\right)-\int_{0}^{1} w(t)\,\varphi(t)\,dt \qquad (3.1)\\
&-4\left[\frac{\varphi(1)+\varphi(0)}{2}-\varphi\left(\frac{1}{2}\right)\right]\int_{0}^{\frac{1}{2}} tw(t)\,dt\le \frac{1}{32}\left(\int_{0}^{1} w(s)\,ds\right)\|\varphi''\|_{\infty,[0,1]},
\end{aligned}$$

and similarly in Corollary 2.6, we get

$$\begin{aligned}
0 &\le \frac{\varphi(1)+\varphi(0)}{2}\left(\int_{0}^{1} w(s)\,ds\right)-\int_{0}^{1} w(t)\,\varphi(t)\,dt \qquad (3.2)\\
&-4\left[\frac{\varphi(1)+\varphi(0)}{2}-\varphi\left(\frac{1}{2}\right)\right]\int_{0}^{\frac{1}{2}} tw(t)\,dt\le \frac{1}{48}\|w\|_{\infty,[0,1/2]}\|\varphi''\|_{\infty,[0,1]}.
\end{aligned}$$

We give now some examples for simple symmetrical weights.

Example 3.1. *First, consider the weight* $w(t) = \left|t - \frac{1}{2}\right|, \ t \in [0,1]$. *Observe that*

$$\int_{\frac{1}{2}}^{1} (1-t)\left(t - \frac{1}{2}\right) dt = \frac{1}{48}, \quad \int_{0}^{\frac{1}{2}} t\left(\frac{1}{2} - t\right) dt = \frac{1}{48}$$

and

$$\int_{\frac{1}{2}}^{1} \left(t - \frac{1}{2}\right) dt = \int_{0}^{\frac{1}{2}} \left(\frac{1}{2} - t\right) dt = \frac{1}{8}.$$

From (3.1) *we obtain*

$$0 \le \frac{1}{12}\left[\varphi(1) + \varphi(0) + \varphi\left(\frac{1}{2}\right)\right] - \int_{0}^{1} \left|t - \frac{1}{2}\right| \varphi(t)\, dt \le \frac{1}{128} \|\varphi''\|_{\infty,[0,1]}, \tag{3.3}$$

while from (3.2)

$$0 \le \frac{1}{12}\left[\varphi(1) + \varphi(0) + \varphi\left(\frac{1}{2}\right)\right] - \int_{0}^{1} \left|t - \frac{1}{2}\right| \varphi(t)\, dt \le \frac{1}{96} \|\varphi''\|_{\infty,[0,1]} \tag{3.4}$$

which is not as good as (3.3).

In the above example, we choose a weight function w for which the bound obtained from Corollary 2.3 (and thus Theorem 2.2) is better than that obtained from Corollary 2.6 (and thus Theorem 2.5). Is this always the case? In what follows, we choose a weight function for which we obtain identical bounds.

Example 3.2. *Consider the weight* $w(t) = t(1-t), \ t \in [0,1]$. *Observe that*

$$\int_{\frac{1}{2}}^{1} (1-t)^2 t\, dt = \frac{5}{192}, \quad \int_{0}^{\frac{1}{2}} t^2 (1-t)\, dt = \frac{5}{192}$$

and

$$\int_{\frac{1}{2}}^{1} t(1-t)\, dt = \int_{0}^{\frac{1}{2}} t(1-t)\, dt = \frac{1}{12}.$$

From (3.1) *we get*

$$0 \le \frac{1}{96}\left[3\left[\varphi(1) + \varphi(0)\right] + 10\varphi\left(\frac{1}{2}\right)\right] - \int_{0}^{1} t(1-t)\varphi(t)\, dt \le \frac{1}{192} \|\varphi''\|_{\infty,[0,1]}, \tag{3.5}$$

while from (3.2)

$$0 \le \frac{1}{96}\left[3\left[\varphi(1) + \varphi(0)\right] + 10\varphi\left(\frac{1}{2}\right)\right] - \int_{0}^{1} t(1-t)\varphi(t)\, dt \le \frac{1}{192} \|\varphi''\|_{\infty,[0,1]}, \tag{3.6}$$

which is the same as (3.5).

In other cases, the bound obtained from Theorem 2.2 is better than that of Theorem 2.5, as outlined in the next two examples.

Example 3.3. *If $w \equiv 1$ in Theorem 2.2, then*

$$0 \leq \frac{1}{2}\left[\varphi(\lambda) + (1-\lambda)\varphi(1) + \lambda\varphi(0)\right] - \int_0^1 \varphi(t)\,dt \leq \frac{1}{8}\left[(1-\lambda)^3 + \lambda^3\right]\|\varphi''\|_{\infty,[0,1]} \tag{3.7}$$

for all $\lambda \in (0,1)$. Since

$$\lambda^3 + (1-\lambda)^3 = \frac{1}{4} + 3\left(\lambda - \frac{1}{2}\right)^2,$$

then (3.7) can be written as

$$0 \leq \frac{1}{2}\left[\varphi(\lambda) + (1-\lambda)\varphi(1) + \lambda\varphi(0)\right] - \int_0^1 \varphi(t)\,dt \leq \frac{1}{8}\left[\frac{1}{4} + 3\left(\lambda - \frac{1}{2}\right)^2\right]\|\varphi''\|_{\infty,[0,1]}. \tag{3.8}$$

In particular, we derive the inequality

$$0 \leq \frac{1}{2}\left[\frac{\varphi(1)+\varphi(0)}{2} + \varphi\left(\frac{1}{2}\right)\right] - \int_0^1 \varphi(t)\,dt \leq \frac{1}{32}\|\varphi''\|_{\infty,[0,1]}. \tag{3.9}$$

Example 3.4. *If $w \equiv 1$ in Theorem 2.5, then*

$$0 \leq \frac{1}{2}\left[\varphi(\lambda) + (1-\lambda)\varphi(1) + \lambda\varphi(0)\right] - \int_0^1 \varphi(t)\,dt \leq \frac{1}{12}\left[\frac{1}{4} + 3\left(\lambda - \frac{1}{2}\right)^2\right]\|\varphi''\|_{\infty,[0,1]} \tag{3.10}$$

for all $\lambda \in (0,1)$. In particular, we have

$$0 \leq \frac{1}{2}\left[\frac{\varphi(1)+\varphi(0)}{2} + \varphi\left(\frac{1}{2}\right)\right] - \int_0^1 \varphi(t)\,dt \leq \frac{1}{48}\|\varphi''\|_{\infty,[0,1]}. \tag{3.11}$$

These inequalities are better than the ones in Example 3.3.

4 Applications for norms

We assume that $(X, \|\cdot\|)$ is a real normed space throughout the sequel.

4.1 Smoothness of the norms and semi-inner products

The terminologies, definitions, and results in this subsection follow those of [7]. Let $x, y \in X$ with $x \neq 0$, then the following limits exist

$$\lim_{t\to 0^{\pm}} \frac{\|x+ty\|^2 - \|x\|^2}{2t}.$$

The mapping $[\cdot,\cdot] : X \times X \to \mathbb{R}$ given by

$$[y,x] := \lim_{t\to 0^+} \frac{\|x+ty\|^2 - \|x\|^2}{2t},$$

is called the T-semi-inner-product.

Definition 4.1. *The T-semi-inner-product $[\cdot,\cdot]$ is said to be continuous on X if*

$$\lim_{t\to 0} [y, x+ty] = [y,x], \quad \text{for all } x,y \in X.$$

Proposition 4.2. *The normed space X is smooth if and only if*

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t} = \lim_{t\to 0^-} \frac{\|x+ty\| - \|x\|}{t}$$

for all $x, y \in X$ with $x \neq 0$.

Proposition 4.3. *The normed space X is smooth if and only if the T-semi-inner-product is continuous.*

Definition 4.4. *A smooth normed space $(X, \|\cdot\|)$ is of (D)-type if the following limit*

$$\lim_{t\to 0} \frac{[y, x+ty] - [y,x]}{t}$$

exists for all $x, y \in X$, in which case the above limit is denoted as $[y,x]'$.

Every inner product space is a smooth normed space of (D)-type. Every ℓ^p space is a smooth normed space of (D)-type when $p \geq 2$.

Proposition 4.5. *Let $(X, \|\cdot\|)$ be a smooth normed space of (D)-type and $x, y \in X$. Then, the mapping $\varphi_{x,y} \colon \mathbb{R} \to \mathbb{R}$ given by*

$$\varphi_{x,y}(t) := \|x+ty\|^2$$

is twice differentiable on $\mathbb{R}$,

$$\varphi'_{x,y}(t) = 2[y, x+ty], \quad \varphi''_{x,y}(t) = 2[y, x+ty]', \quad \text{for all } t \in \mathbb{R},$$

and $\varphi''_{x,y}$ is non-negative on $\mathbb{R}$.

Definition 4.6. *A smooth normed space of (D)-type is of (BD)-type if there exists a real number $k \geq 1$ such that*

$$[y,x]' \leq k^2 \|y\|^2, \quad \text{for all } x,y \in X. \tag{4.1}$$

The least number k such that (4.1) holds will be called the boundedness modulus of $[\cdot,\cdot]'$ and we denote such a number by k_0.

Example 4.7. *Every inner product space is of (BD)-type. In fact, X is an inner product space if and only if its boundedness modulus k_0 is exactly 1. For all $x, y \in X$, we have $[y,x]' = \|y\|^2$.*

Example 4.8. *Every ℓ^p space is a smooth normed space of (BD)-type when $p \geq 2$. In particular, for all $x, y \in \ell^p$, $x \neq 0$, we have*

$$[y,x]' \leq (4k+1)\|y\|^2$$

with $k = (p-2)/2$.

4.2 Convex functions on normed spaces

Let $(X, \|\cdot\|)$ be a smooth normed space of (D)-type and $x, y \in X$. Let $f_{x,y} \colon \mathbb{R} \to \mathbb{R}$ be given by

$$f_{x,y}(t) := \|(1-t)x + ty\|^2 = \|x + t(y-x)\|^2 .$$

By Proposition 4.5, f is convex and twice differentiable on $\mathbb{R}$, and

$$f'_{x,y}(t) = 2[y-x, (1-t)x+ty], \quad \text{and} \quad f''_{x,y}(t) := 2[y-x,(1-t)x+ty]',$$

for all $t \in \mathbb{R}$.

Let $(X, \|\cdot\|)$ be a smooth normed space of (D)-type, $x, y \in X$, and $1 \leq p < \infty$. Let $g_{x,y,p} \colon \mathbb{R} \to \mathbb{R}$ be given by

$$g_{x,y,p}(t) := \|(1-t)x+ty\|^p = \left(\|(1-t)x+ty\|^2\right)^{\frac{p}{2}} .$$

Then, for all $t \in \mathbb{R}$, we have

$$g'_{x,y,p}(t) = \frac{p}{2}\left(\|(1-t)x+ty\|^2\right)^{\frac{p}{2}-1}\frac{d}{dt}\|(1-t)x+ty\|^2 = p\|(1-t)x+ty\|^{p-2}[y-x,(1-t)x+ty],$$

and

$$\begin{aligned}
g''_{x,y,p}(t) &= p\left[[y-x,(1-t)x+ty]\frac{d}{dt}\|(1-t)x+ty\|^{p-2} + (\|(1-t)x+ty\|)^{p-2}\frac{d}{dt}[y-x,(1-t)x+ty]\right] \\
&= p\left[(p-2)\|(1-t)x+ty\|^{p-4}[y-x,(1-t)x+ty]^2 + (\|(1-t)x+ty\|)^{p-2}[y-x,(1-t)x+ty]'\right] \\
&= p\|(1-t)x+ty\|^{p-4}\left[(p-2)[y-x,(1-t)x+ty]^2 + (\|(1-t)x+ty\|)^2[y-x,(1-t)x+ty]'\right].
\end{aligned}$$

Note that since $[y-x,(1-\cdot)x+\cdot y]'$ is non-negative, then $g''_{x,y,p}$ is also non-negative and thus $g_{x,y,p}$ is convex. If we assume further that X is (BD)-smooth with constant $k \geq 1$, then, for all $t \in \mathbb{R}$, we have

$$\begin{aligned}
g''_{x,y,p}(t) &= p\|(1-t)x+ty\|^{p-4}\left[(p-2)[y-x,(1-t)x+ty]^2 + (\|(1-t)x+ty\|)^2[y-x,(1-t)x+ty]'\right] \\
&\leq p\|(1-t)x+ty\|^{p-4}\left[(p-2)\|y-x\|^2\|(1-t)x+ty\|^2 + k^2(\|(1-t)x+ty\|)^2\|y-x\|^2\right]
\end{aligned}$$

$$= p(p-2+k^2)\left\|(1-t)x+ty\right\|^{p-2}\left\|y-x\right\|^2.$$

Consequently,

$$\left\|g''_{x,y,p}\right\|_{\infty,[0,1]} \le p(p-2+k^2)\left\|y-x\right\|^2 \max\left\{\left\|x\right\|^{p-2}, \left\|y\right\|^{p-2}\right\}. \tag{4.2}$$

Remark 4.9. *Recall that when $X = \ell^p$ with $p \ge 2$, we have that*

$$[y,x]'_p \le (4k+1)\left\|y\right\|_p^2,$$

with $k = (p-2)/2$, that is

$$[y,x]'_p \le (2p-3)\left\|y\right\|_p^2,$$

for all $x, y \in \ell^p$ with $x \neq 0$. We use the subscripts p in the notation for the norms and semi-inner products here to highlight the fact that we consider the special case of ℓ^p spaces. Therefore (4.2) becomes

$$\begin{aligned}\left\|g''_{x,y,p}\right\|_{\infty,[0,1]} &\le p(p-2+(2p-3)^2)\left\|y-x\right\|_p^2 \max\left\{\left\|x\right\|_p^{p-2}, \left\|y\right\|_p^{p-2}\right\} \\ &= (4p^3-11p^2+7p)\left\|y-x\right\|_p^2 \max\left\{\left\|x\right\|_p^{p-2}, \left\|y\right\|_p^{p-2}\right\}.\end{aligned} \tag{4.3}$$

4.3 Application of Theorem 2.2

Let w be a non-negative, bounded, integrable weight on $[0,1]$ and $\lambda \in (0,1)$. Then, applying Theorem 2.2 to the function $g_{x,y,p}$, we have

$$\begin{aligned} 0 &\le \left(\int_\lambda^1 w(s)\,ds\right)\left\|y\right\|^p + \left(\int_0^\lambda w(s)\,ds\right)\left\|x\right\|^p - \int_0^1 w(t)\left\|(1-t)x+ty\right\|^p dt \\ &- \frac{\left\|y\right\|^p - \left\|(1-\lambda)x+\lambda y\right\|^p}{1-\lambda}\int_\lambda^1 (1-t)\,w(t)\,dt + \frac{\left\|(1-\lambda)x+\lambda y\right\|^p - \left\|x\right\|^p}{\lambda}\int_0^\lambda t w(t)\,dt \\ &\le \frac{1}{8}\left[(1-\lambda)^2\left(\int_\lambda^1 w(s)\,ds\right) + \lambda^2\left(\int_0^\lambda w(s)\,ds\right)\right]\left\|g''_{x,y,p}\right\|_{\infty,[0,1]}. \end{aligned}$$

When the weight w is symmetrical on $[0,1]$ and $\lambda = 1/2$, we have

$$\begin{aligned} 0 &\le \frac{\left\|x\right\|^p + \left\|y\right\|^p}{2}\left(\int_0^1 w(s)\,ds\right) - \int_0^1 w(t)\left\|(1-t)x+ty\right\|^p dt \\ &- 4\left[\frac{\left\|x\right\|^p + \left\|y\right\|^p}{2} - \left\|\frac{x+y}{2}\right\|^p\right]\int_0^{\frac{1}{2}} t w(t)\,dt \le \frac{1}{32}\left(\int_0^1 w(s)\,ds\right)\left\|g''_{x,y,p}\right\|_{\infty,[0,1]}. \end{aligned}$$

We obtain a simple inequality when $w \equiv 1$ and we assume further that X is (BD)-smooth

$$0 \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\left\|x\right\|^p + \left\|y\right\|^p}{2}\right] - \int_0^1 \left\|(1-t)x+ty\right\|^p dt \le \frac{1}{32}\left\|g''_{x,y,p}\right\|_{\infty,[0,1]}$$

$$\leq \frac{1}{32}p(p-2+k^2)\left\|y-x\right\|^2 \max\left\{\left\|x\right\|^{p-2}, \left\|y\right\|^{p-2}\right\}.$$

4.4 Application of Theorem 2.5

Let w be a non-negative, bounded, integrable weight on $[0,1]$ and $\lambda \in (0,1)$. Then, applying Theorem 2.5 to the function $g_{x,y,p}$, we have

$$\begin{aligned}
0 &\leq \left(\int_\lambda^1 w(s)\,ds\right)\left\|y\right\|^p + \left(\int_0^\lambda w(s)\,ds\right)\left\|x\right\|^p - \int_0^1 w(t)\left\|(1-t)x+ty\right\|^p dt \\
&\quad - \frac{\left\|y\right\|^p - \left\|(1-\lambda)x+\lambda y\right\|^p}{1-\lambda}\int_\lambda^1 (1-t)\,w(t)\,dt + \frac{\left\|(1-\lambda)x+\lambda y\right\|^p - \left\|x\right\|^p}{\lambda}\int_0^\lambda tw(t)\,dt \\
&\leq \frac{1}{12}\left[(1-\lambda)^3\left\|w\right\|_{\infty,[\lambda,1]} + \lambda^3\left\|w\right\|_{\infty,[0,\lambda]}\right]\left\|g''_{x,y,p}\right\|_{\infty,[0,1]} \\
&\leq \frac{1}{12}\left[\frac{1}{4} + 3\left(\lambda - \frac{1}{2}\right)^2\right]\left\|w\right\|_{\infty,[0,1]}\left\|g''_{x,y,p}\right\|_{\infty,[0,1]}.
\end{aligned}$$

When the weight w is symmetrical on $[0,1]$ and $\lambda = 1/2$, we have

$$\begin{aligned}
0 &\leq \frac{\left\|x\right\|^p + \left\|y\right\|^p}{2}\left(\int_0^1 w(s)\,ds\right) - \int_0^1 w(t)\left\|(1-t)x+ty\right\|^p dt \\
&\quad - 4\left[\frac{\left\|x\right\|^p + \left\|y\right\|^p}{2} - \left\|\frac{x+y}{2}\right\|^p\right]\int_0^{\frac{1}{2}} tw(t)\,dt \leq \frac{1}{48}\left\|w\right\|_{\infty,[0,1/2]}\left\|g''_{x,y,p}\right\|_{\infty,[0,1]}.
\end{aligned}$$

We obtain a simple inequality when $w \equiv 1$ and we assume further that X is (BD)-smooth

$$\begin{aligned}
0 &\leq \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\left\|x\right\|^p + \left\|y\right\|^p}{2}\right] - \int_0^1 \left\|(1-t)x+ty\right\|^p dt \leq \frac{1}{48}\left\|g''_{x,y,p}\right\|_{\infty,[0,1]} \\
&\leq \frac{1}{48}p(p-2+k^2)\left\|y-x\right\|^2 \max\left\{\left\|x\right\|^{p-2}, \left\|y\right\|^{p-2}\right\}.
\end{aligned}$$

4.5 Case of inner product spaces

In the case that $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, we have

$$\begin{aligned}
g''_{x,y,p}(t) &= p\left\|(1-t)x+ty\right\|^{p-4}\left[(p-2)\left\langle y-x,(1-t)x+ty\right\rangle^2 + (\left\|(1-t)x+ty\right\|)^2\left\langle y-x,(1-t)x+ty\right\rangle'\right] \\
&\leq p\left\|(1-t)x+ty\right\|^{p-4}\left[(p-2)\left\|y-x\right\|^2\left\|(1-t)x+ty\right\|^2 + \left\|(1-t)x+ty\right\|^2\left\|y-x\right\|^2\right] \\
&= p(p-1)\left\|(1-t)x+ty\right\|^{p-2}\left\|y-x\right\|^2,
\end{aligned}$$

and specifically when $p = 2$,

$$g''_{x,y,2}(t) \leq 2\left\|y-x\right\|^2,$$

We obtain simple inequalities when $w \equiv 1$, from (3.9) and (3.11),

$$0 \leq \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\|x\|^p + \|y\|^p}{2}\right] - \int_0^1 \|(1-t)x+ty\|^p \, dt \leq \frac{1}{32}p(p-1)\|y-x\|^2 \max\left\{\|x\|^{p-2}, \|y\|^{p-2}\right\} \tag{4.4}$$

and

$$0 \leq \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\|x\|^p + \|y\|^p}{2}\right] - \int_0^1 \|(1-t)x+ty\|^p \, dt \leq \frac{1}{48}p(p-1)\|y-x\|^2 \max\left\{\|x\|^{p-2}, \|y\|^{p-2}\right\}. \tag{4.5}$$

In particular, when $p = 2$

$$0 \leq \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^2 + \frac{\|x\|^2 + \|y\|^2}{2}\right] - \int_0^1 \|(1-t)x+ty\|^2 \, dt \leq \frac{1}{16}\|y-x\|^2 \tag{4.6}$$

and

$$0 \leq \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^2 + \frac{\|x\|^2 + \|y\|^2}{2}\right] - \int_0^1 \|(1-t)x+ty\|^2 \, dt \leq \frac{1}{24}\|y-x\|^2. \tag{4.7}$$

This last inequality is, in fact, an equality, since

$$\begin{aligned}\frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^2 + \frac{\|x\|^2 + \|y\|^2}{2}\right] - \int_0^1 \|(1-t)x+ty\|^2 \, dt &= \frac{1}{2}\left[\frac{1}{4}(\|x\|^2 + \|y\|^2 + 2\langle x,y\rangle) + \frac{\|x\|^2 + \|y\|^2}{2}\right] \\ &\quad - \int_0^1 [(1-t)^2\|x\|^2 + 2t(1-t)\langle x,y\rangle + t^2\|y\|^2]dt \\ &= \frac{1}{8}(3\|x\|^2 + 3\|y\|^2 + 2\langle x,y\rangle) - \frac{1}{3}(\|x\|^2 + \|y\|^2 + \langle x,y\rangle) \\ &= \frac{1}{24}(\|x\|^2 + \|y\|^2 - 2\langle x,y\rangle) = \frac{1}{24}\|y-x\|^2.\end{aligned}$$

The above shows that (4.5) is sharp, and consequently (2.6), (2.7), (3.2), (3.10), and (3.11), are also sharp.

We again consider $p = 2$, and we further consider the weight $w(t) = \left|t - \frac{1}{2}\right|$, $t \in [0,1]$, as in Example 3.1, then (3.3) becomes

$$0 \leq \frac{1}{12}\left[\|x\|^2 + \|y\|^2 + \left\|\frac{x+y}{2}\right\|^2\right] - \int_0^1 \left|t - \frac{1}{2}\right| \|(1-t)x+ty\|^2 \, dt \leq \frac{1}{64}\|y-x\|^2.$$

We conjecture that the above bound is not sharp.

We again consider $p = 2$ with the weight $w(t) = t(1-t)$, $t \in [0,1]$, as in Example 3.2, then (3.5) becomes

$$0 \leq \frac{1}{96}\left[3(\|x\|^2 + \|y\|^2) + 10\left\|\frac{x+y}{2}\right\|^2\right] - \int_0^1 t(1-t)\|(1-t)x+ty\|^2 \, dt \leq \frac{1}{96}\|y-x\|^2.$$

We conjecture that the above bound is not sharp.

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CUBO, A Mathematical Journal
Vol. 26, no. 3, pp. 525–540, December 2024
DOI: 10.56754/0719-0646.2603.525

Steffensen-like method in Riemannian manifolds

Chandresh Prasad[1]
P. K. Parida[1,✉]

[1] *Department of Mathematics, Central University of Jharkhand, Ranchi-835222, India.*
prasadchandresh20592@gmail.com
pkparida@cuj.ac.in[✉]

ABSTRACT

In this paper, we present semilocal convergence of Steffensen-like method for approximating zeros of a vector field in Riemannian manifolds. We establish the convergence of Steffensen-like method under Lipschitz continuity condition on first order covariant derivative of a vector field. Finally, two examples are given to show the application of our theorem.

RESUMEN

En este artículo, presentamos la convergencia semilocal del método de tipo Steffensen para aproximar los ceros de un campo de vectores en una variedad Riemanniana. Establecemos la convergencia del método de tipo Steffensen bajo la condición de continuidad Lipschitz de la derivada covariante de primer orden de un campo de vectores. Finalmente, damos dos ejemplos para mostrar la aplicabilidad de nuestro teorema.

Keywords and Phrases: Vector fields, Riemannian manifolds, Lipschitz condition, Steffensen-like method.

2020 AMS Mathematics Subject Classification: 65H10, 65D99.

Published: 13 December, 2024
Accepted: 19 November, 2024
Received: 13 March, 2024

1 Introduction

There are many problems in applied sciences and other including engineering, optimization, dynamic economic system, physics, biological problems which is formulated in an equation by using mathematical modeling to find the zeros of equations (see for example [7, 9, 10, 17, 19] and the references therein). To solve the nonlinear equations many types of iterative methods have been studied in Banach spaces. The most famous second order iterative method to solve a non-linear equation in Banach space is Newton's method. Recently, attention has been paid in studying iterative methods in Riemannian manifolds. There are many types of numerical methods that have been studied in manifolds which arise in many contexts. Some problems including eigenvalue problem, minimization problems with orthogonality constraints, optimization problems with equality constraints, invariant subspace computations (see for example [1–3, 6–8, 12–15, 21] and the references therein). To solve this problem, we have to find the zeros of a vector field in Riemannian manifolds. Generally convergence of iterative methods are usually centered on two types: semilocal and local convergence analysis. The convergence analysis which provides information around a solution and calculates the radius of convergence, it is local and when the convergence analysis provides information around an initial point, it is semilocal. The Steffensen-like method [5] which is second order method in Banach space is defined as:

$$\left.\begin{aligned} x_0 &\in \Omega,\\ y_{n-1} &= x_{n-1} - a\mathfrak{M}(x_{n-1}),\ a\in\mathbb{R}^+,\ n\in\mathbb{N},\\ z_{n-1} &= x_{n-1} + b\mathfrak{M}(x_{n-1}),\ b\in\mathbb{R}^+,\\ x_n &= x_{n-1} - [y_{n-1}, z_{n-1}; \mathfrak{M}]^{-1}\mathfrak{M}(x_{n-1}), \end{aligned}\right\} \tag{1.1}$$

where $\mathfrak{M}$ is a nonlinear operator defined in an open convex subset Ω of a Banach space B into itself and $\mathfrak{M}$ is first Fréchet differentiable in Ω. The computational efficiency of Steffensen-like method is the same as Newton's method, when it is applied to find the solution of finite dimensional system of nonlinear equations. The convergence of this second order method in Banach space has been studied in [5]. As motivation, the numerical solution of the vector field

$$G(u_1, u_2, u_3) = (-u_2,\ u_1 - u_1u_3^2,\ u_1u_2u_3)$$

using Newton's and Euler-Chebyshev's method on $\mathbb{R}^3$ is difficult to find as the Jacobian is a non-invertible matrix at the point $(0, 0, -1)^T$, but using the algorithm given in [11] such singularity is found on the two-dimensional sphere $\mathbf{S}^2$. In this paper, we extend the method (1.1) to the case of equations in Riemannian manifolds to find the singular point of a vector field.

The paper is organized as follows: Section 2, contains all the necessary background on fundamental properties and notation of Riemannian manifolds. In Section 3, we present the semilocal convergence of Steffensen-like method under Lipschitz continuity condition on the first order covariant derivative of vector field. In Section 4, two examples are given to show the application of our theorem. Finally, in Section 5, some brief conclusions are given.

2 Preliminaries

In this section, we introduce some basic definitions and properties of Riemannian manifolds (for more details see [16, 18, 20]).

Let Q be a real n-dimensional Riemannian manifold, $\mathfrak{X}(Q)$ be a set of all vector fields of class C^∞ on Q, T_uQ be a tangent space of Q at u, and TQ be a tangent bundle defined as $TQ = \bigcup_{u\in Q} T_uQ$. Suppose Q is equipped with a Riemannian metric $\langle .,.\rangle$ with corresponding norm $\|\cdot\|$. The arc length of piecewise smooth curve $\psi : [0,1] \to Q$ joining u to v is defined by $l(\psi) = \int_0^1 \|\psi'(z)\|dz$ and the Riemannian distance joining u to v is defined by $d(u,v) = \inf_\psi l(\psi)$. Let $D(Q)$ be the ring of real-valued functions of class C^∞ defined on Q. An affine connection ∇ on Q is a map

$$\begin{array}{rcrcl} \nabla & : & \mathfrak{X}(Q)\times\mathfrak{X}(Q) & \longrightarrow & \mathfrak{X}(Q) \\ & & (X,G) & \longmapsto & \nabla_X G \end{array}$$

which satisfies the properties

(i) $\nabla_{fX+gG}\mathfrak{V} = f\nabla_X\mathfrak{V} + g\nabla_G\mathfrak{V}$.

(ii) $\nabla_X(G+\mathfrak{V}) = \nabla_X G + \nabla_X\mathfrak{V}$.

(iii) $\nabla_X(fG) = f\nabla_X G + X(f)G$,

where $X, G, \mathfrak{V} \in \mathfrak{X}(Q)$ and $f, g \in D(Q)$. The covariant derivative of G determined by the connection ∇ defines at each point $u \in Q$ a linear application as

$$\begin{array}{rcrcl} DG(u) & : & T_uQ & \longrightarrow & T_uQ \\ & & v & \longmapsto & DG(u)(v) = \nabla_X G(u), \end{array}$$

where $G \in \mathfrak{X}(Q)$ of class C^1 on Q and X is a vector field that satisfies $X(u) = v$. We define the open and closed geodesic ball with centre u and radius v respectively, as

$$V(u,v) = \{t \in Q : d(u,t) < v\} \quad \text{and} \quad V[u,v] = \{t \in Q : d(u,t) \le v\}.$$

A parametrized curve $\psi : I \to Q$ is said to be a geodesic at $t_0 \in I$ if $\nabla_{\psi'(t)}\psi'(t) = 0$ in the point t_0. If ψ is a geodesic at t, for all $t \in I$, we say that ψ is a geodesic. If $[p,q] \subseteq I$, the restriction of ψ to $[a,b]$ is called a geodesic segment joining $\psi(p)$ to $\psi(q)$. By the Hopf-Rinow theorem, if Q is complete metric space then for any $u, t \in Q$ there exists a geodesic ψ, called minimizing geodesic joining u to t with

$$l(\psi) = d(u,t).$$

Also, if $v \in T_u Q$, then there exists a unique minimizing geodesic ψ such that $\psi(0) = u$ and $\psi'(0) = v$. The point $\psi(1)$ is called the image of v by the exponential map at u, *i.e.*

$$\exp_u : T_u Q \longrightarrow Q$$

such that $\exp_u(v) = \psi(1)$ and for all $p \in [0,1]$, $\psi(p) = \exp_u(pv)$. Let ψ be a piecewise smooth curve, then for any $x, y \in \mathbb{R}$, the parallel transport along ψ is a mapping from $T_{\psi(x)}Q$ to $T_{\psi(y)}Q$. It is denoted by $\mathbb{M}_{\psi,\cdot,\cdot}$ and given by

$$\begin{array}{rcccl} \mathbb{M}_{\psi,x,y} & : & T_{\psi(x)}Q & \longrightarrow & T_{\psi(y)}Q \\ & & v & \longmapsto & V(\psi(y)), \end{array}$$

where V is the unique vector field along ψ which satisfies $\nabla_{\psi'(t)}V = 0$ and $V(\psi(x)) = v$. It is easy to show that $\mathbb{M}_{\psi,x,y}$ is linear and one-to-one. Therefore $\mathbb{M}_{\psi,x,y} : T_{\psi(x)}Q \to T_{\psi(y)}Q$ is an isomorphism and inverse of parallel transport is denoted by $\mathbb{M}_{\psi,y,x}$. Thus $\mathbb{M}_{\psi,x,y}$ is an isometry between $T_{\psi(x)}Q$ and $T_{\psi(y)}Q$. For $i \in \mathbb{N}$, we define $\mathbb{M}^i_\psi$ as

$$\mathbb{M}^i_{\psi,x,y} : (T_{\psi(x)}Q)^i \longrightarrow (T_{\psi(y)}Q)^i,$$

where

$$\mathbb{M}^i_{\psi,x,y}(u_1, u_2, \ldots, u_i) = (\mathbb{M}_{\psi,x,y}(u_1), \mathbb{M}_{\psi,x,y}(u_2), \ldots, \mathbb{M}_{\psi,x,y}(u_i)).$$

It has the important properties:

$$\mathbb{M}^{-1}_{\psi,y,x} = \mathbb{M}_{\psi,x,y}, \quad \mathbb{M}_{\psi,x,y} \circ \mathbb{M}_{\psi,y,z} = \mathbb{M}_{\psi,x,z}.$$

Definition 2.1. *Let $G \in \mathfrak{X}(Q)$ of class C^k on Q and $j \in \mathbb{N}$. The covariant derivative of order j of G is denoted by D^jG and defined as:*

$$D^jG : \underbrace{C^k(TQ) \times C^k(TQ) \times \cdots \times C^k(TQ)}_{j\text{-times}} \longrightarrow C^{k-j}(TQ),$$

where

$$\begin{aligned} D^jG(A_1, A_2, \ldots, A_{j-1}, A) &= \nabla_A D^{j-1}G(A_1, A_2, \ldots, A_{j-1}) \\ &\quad - \sum_{i=1}^{j-1} D^{j-1}G(A_1, A_2, \ldots, \nabla_A A_i, \ldots, A_{j-1}) \end{aligned} \tag{2.1}$$

for all $A_1, A_2, \ldots, A_{j-1} \in C^k(TQ)$.

Definition 2.2. *Let $\mho \subseteq Q$ be an open convex set and $G \in \mathfrak{X}(Q)$. The covariant derivative $DG = \nabla_{(.)}G$ is Lipschitz with constant $\mathfrak{E} > 0$, if for any geodesic ψ and $x, y \in \mathbb{R}$ such that $\psi[x, y] \subseteq \mho$, and it holds the inequality*

$$\|\mathbb{M}_{\psi,y,x}DG(\psi(y))\mathbb{M}_{\psi,x,y} - DG(\psi(x))\| \leq \mathfrak{E}\int_x^y \|\psi'(t)\|dt,$$

and we write $DG \in Lip_{\mathfrak{E}}(\mho)$. If Q is finite dimensional Euclidean space, then it coincides with Lipschitz condition for $DG : Q \to Q$.

Definition 2.3. *Let $\mho \subseteq Q$, be an open convex set. Suppose ψ is a curve in Q, $[t, t+\delta e] \subset Dom(\psi)$ and $G \in \mathfrak{X}(Q)$ of class C^0 on Q. The divided difference of first order for G on the points $\psi(t)$ and $\psi(t + \delta e)$ in the direction $\psi'(t)$, is defined by*

$$[\psi(t + \delta x), \psi(t); G]\psi'(t) = \frac{1}{\delta e}(\mathbb{M}_{\psi,t+\delta e,t}G(\psi(t + \delta e)) - G(\psi(t))). \tag{2.2}$$

When Q is a Banach space, if ψ is the geodesic joining u_1 and u_2, such that

$$\psi(t) = u_1 + t(u_2 - u_1), \quad t \in \mathbb{R},$$

then from (2.2)*, we obtain*

$$[u_2, u_1; G](u_2 - u_1) = G(u_2) - G(u_1).$$

Also if $DG(u)$ exists, then $DG(u) = [u, u; G]$.

Proposition 2.4. *The covariant derivative of G in the direction of $\psi'(t)$ is defined as:*

$$\begin{aligned} DG(\psi(t))\psi'(t) &= \nabla_{\psi'(t)}G_{\psi(t)} \\ &= \lim_{\delta e \to 0} \frac{1}{\delta e}(\mathbb{M}_{\psi,t+\delta e,t}G(\psi(t + \delta e)) - G(\psi(t))), \end{aligned}$$

where ψ is a curve on Q and $G \in \mathfrak{X}(Q)$ of class C^1 on Q. If Q is finite dimensional Euclidean space, then it coincides with the directional derivative in finite dimensional Euclidean space.

Next, we will show Taylor-type expansions in Riemannian manifolds which will be used in the proof of the convergence of our iterative method.

Theorem 2.5. *Let ψ be a geodesic in Q and $G \in \mathfrak{X}(Q)$ of class C^1 on Q. Then*

$$\mathbb{M}_{\psi,t,0}G(\psi(t)) = G(\psi(0)) + \int_0^t \mathbb{M}_{\psi,e,0}DG(\psi(e))\psi'(e)de. \tag{2.3}$$

Proof. See [4]. □

3 Steffensen-like method in Riemannian manifolds

In this section, we will prove convergence and uniqueness of Steffensen-like method in Riemannian manifolds. The method (1.1) in Riemannian manifolds has the form

$$\left.\begin{aligned} u_0 &\in \mho, \\ L_{n-1} &= -aG(u_{n-1}), \quad a \in \mathbb{R}^+,\ n \in \mathbb{N}, \\ v_{n-1} &= \exp_{u_{n-1}}(L_{n-1}), \\ \psi_{n-1}(t) &= \exp_{u_{n-1}}(tL_{n-1}), \\ M_{n-1} &= bG(u_{n-1}), \quad b \in \mathbb{R}^+, \\ w_{n-1} &= \exp_{u_{n-1}}(M_{n-1}), \\ N_n &= -\mathbb{M}_{\psi_{n-1},1,0}[v_{n-1}, w_{n-1}; G]^{-1}\mathbb{M}_{\psi_{n-1},0,1}G(u_{n-1}), \\ u_n &= \exp_{u_{n-1}}(N_n). \end{aligned}\right\} \tag{3.1}$$

Assume that $G(u)$ satisfies the following conditions:

(1) $\|G(u_0)\| \leq \xi$,

(2) $\|DG(u_0)^{-1}\| \leq \zeta_0$,

(3) $\|\mathbb{M}_{\phi,j,i}DG(\phi(j))\mathbb{M}_{\phi,i,j} - DG(\phi(i))\| \leq K\int_i^j \|\phi'(x)\|dx$, where ϕ is a geodesic such that $\phi[i,j] \subseteq \mho$.

Firstly, we shall show that a operator $[v_0, w_0; G]^{-1}$ is bounded. Let $I_{u_0} : T_{u_0}Q \to T_{u_0}Q$ be a identity operator, ψ_n and α_n be a family of minimizing geodesics such that $\psi_n(0) = u_n$, $\psi_n(1) = v_n$, $\alpha_n(0) = w_n$, $\alpha_n(1) = v_n$ for each $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned} \|DG(u_0)^{-1}\mathbb{M}_{\psi_0,1,0}[v_0, w_0; G]\mathbb{M}_{\psi_0,0,1} - I_{u_0}\| &\leq \|DG(u_0)^{-1}\mathbb{M}_{\psi_0,1,0}([v_0, w_0; G] - DG(v_0))\mathbb{M}_{\psi_0,0,1}\| \\ &\quad + \|DG(u_0)^{-1}(\mathbb{M}_{\psi_0,1,0}DG(v_0)\mathbb{M}_{\psi_0,0,1} - DG(u_0))\| \\ &\leq \|DG(u_0)^{-1}\| \int_0^1 \|\mathbb{M}_{\alpha_0,1,0}DG(\alpha_0(t))\mathbb{M}_{\alpha_0,0,1} - DG(v_0)\|dt \\ &\quad + \|DG(u_0)^{-1}\|\|\mathbb{M}_{\psi_0,1,0}DG(v_0)\mathbb{M}_{\psi_0,0,1} - DG(u_0)\| \\ &\leq \zeta_0\Big(Kd(v_0, u_0) + \frac{K}{2}d(v_0, w_0)\Big) \leq \frac{(3a+b)}{2}K\zeta_0\xi, \end{aligned}$$

if $(3a+b)K\xi\zeta_0 < 2$, then the operator $\mathbb{M}_{\psi_0,1,0}[v_0, w_0; G]\mathbb{M}_{\psi_0,0,1}$ is invertible and

$$\|[v_0, w_0; G]^{-1}\| \leq \frac{2\zeta_0}{2 - (3a+b)K\xi\zeta_0} = c.$$

Now, we define the polynomial

$$z(f)=\frac{L}{2}f^2-\frac{f}{c}+\xi,\quad L=K\left(1+\frac{3a+b}{c}\right),\quad f\in[0,f']. \tag{3.2}$$

Let $f^*=\dfrac{1-\sqrt{1-2L\xi c^2}}{Lc}$ and $f^{**}=\dfrac{1+\sqrt{1-2L\xi c^2}}{Lc}$ be two positive roots of $z(f)$ such that $0<f^*\le f^{**}<f'$ if $L\xi c^2\le\frac{1}{2}$. Also for all $n\ge 0$, define the sequences

$$\begin{aligned} f_{n+1}&=f_n-\frac{z(f_n)}{z'(f_n)},\quad f_0=0,\\ \zeta_{n+1}&=\frac{\zeta_0}{1-\zeta_0 K d(u_{n+1},u_0)}. \end{aligned} \tag{3.3}$$

Before proving the convergence of iterative method firstly we will prove some lemmas which will be used to prove the theorem.

Lemma 3.1. *Let $G\in\mathfrak{X}(Q)$ of class C^1 on Q, then for any $n\in\mathbb{N}$, we have*

$$\begin{aligned}\mathbb{M}_{\phi,1,0}G(u_n)=\Big(\int_0^1\big(\mathbb{M}_{\phi,1,0}DG(\phi(t))\mathbb{M}_{\phi,0,1}-DG(u_{n-1})\big)dt\\ +\big(DG(u_{n-1})-\mathbb{M}_{\psi_{n-1},1,0}[v_{n-1},w_{n-1};G]\mathbb{M}_{\psi_{n-1},0,1}\big)\Big)N_n,\end{aligned}$$

where ϕ is a family of minimizing geodesics such that $\phi(0)=u_{n-1}$, $\phi(1)=u_n$.

Proof. We know that

$$[\phi(s+h),\phi(s);G]\phi'(s)=\frac{1}{h}\Big(\mathbb{M}_{\phi,s+h,s}G(\phi(s+h))-G(\phi(s))\Big),$$

put $s=0$ and $h=1$ in above equality, we get

$$[u_n,u_{n-1};G]\phi'(0)=\mathbb{M}_{\phi,1,0}G(u_n)-G(u_{n-1}).$$

Since $\phi(t)=\exp_{u_{n-1}}(tN_n)$, we have $\phi'(0)=N_n$.

We obtain that

$$[u_n,u_{n-1};G]N_n=\mathbb{M}_{\phi,1,0}G(u_n)-G(u_{n-1}). \tag{3.4}$$

By (3.1), we have

$$N_n=-\mathbb{M}_{\psi_{n-1},1,0}[v_{n-1},w_{n-1};G]^{-1}\mathbb{M}_{\psi_{n-1},0,1}G(u_{n-1})$$

or

$$G(u_{n-1})=-\mathbb{M}_{\psi_{n-1},1,0}[v_{n-1},w_{n-1};G]\mathbb{M}_{\psi_{n-1},0,1}N_n. \tag{3.5}$$

By (3.4) and (3.5), we obtain

$$\begin{aligned}\mathbb{M}_{\phi,1,0}G(u_n) &= \Big([u_n,u_{n-1};G] - \mathbb{M}_{\psi_{n-1},1,0}[v_{n-1},w_{n-1};G]\mathbb{M}_{\psi_{n-1},0,1}\Big)N_n \\ &= \Bigg(\int_0^1 \big(\mathbb{M}_{\phi,1,0}DG(\phi(t))\mathbb{M}_{\phi,0,1} - DG(u_{n-1})\big)dt \\ &\quad + \big(DG(u_{n-1}) - \mathbb{M}_{\psi_{n-1},1,0}[v_{n-1},w_{n-1};G]\mathbb{M}_{\psi_{n-1},0,1}\big)\Bigg)N_n.\end{aligned}$$

□

Lemma 3.2. *Suppose the sequence $\{f_n\}$ is generated by (3.3). If $L\xi c^2 \leq \frac{1}{2}$ and $f \in [0, f^*]$, then the sequence $\{f_n\}$ is increasing and bounded above. Hence converges to f^*.*

Proof. We define the function h by

$$h(f) = f - \frac{z(f)}{z'(f)}.$$

Then differentiating both sides, we get

$$h'(f) = \frac{z(f)z''(f)}{\big(z'(f)\big)^2},$$

as $z(f) \geq 0,\ z''(f) > 0,\ z'(f) < 0$ in $[0, f^*]$. We have

$$h'(f) = \frac{z(f)z''(f)}{\big(z'(f)\big)^2} \geq 0, \quad \forall f \in [0, f^*].$$

It shows that the function h is increasing on $[0, f^*]$. So, if $f_k \in [0, f^*]$ for some $k \in \mathbb{N}$, then

$$f_k \leq f_k - \frac{z(f_k)}{z'(f_k)} = f_{k+1}$$

and

$$f_{k+1} = f_k - \frac{z(f_k)}{z'(f_k)} \leq f^* - \frac{z(f^*)}{z'(f^*)} = f^*.$$

Thus, it completes the proof of Lemma 3.2. □

Now we can demonstrate the convergence of our method.

Theorem 3.3. *Let Q be a complete Riemannian manifold, $\mho \subseteq Q$ be an open convex set and $G \in \mathfrak{X}(Q)$ satisfies the conditions $(1) - (3)$ with:*

$$(3a+b)\xi K\zeta_0 < 2, \quad L\xi c^2 \leq \frac{1}{2}, \quad \zeta_0 K f^* < 1, \quad K\zeta_0(3f^* + \xi + f^{**}) < 2, \quad V(u_0, f^*) \subseteq \mho.$$

Then, the method given by (3.1) converges to a singular point u^* of the vector field G in $V[u_0, f^*]$ and the solution u^* is unique in $V[u_0, f^{**} + \xi]$.

Proof. To prove the theorem, at first we shall prove some conditions for all $i = 0, 1, 2, \ldots$

$(C1)$ $u_i \in V[u_0, f^*]$,

$(C2)$ $v_i \in V[u_0, f^*]$,

$(C3)$ $w_i \in V[u_0, f^*]$,

$(C4)$ $\|DG(u_i)^{-1}\| \leq \zeta_i$.

For $i = 0$, $(C1)$ and $(C4)$ are trivial and since

$$d(v_0, u_0) = a\xi \leq f^*, \quad d(w_0, u_0) = b\xi \leq f^*,$$

therefore $(C1) - (C4)$ are true for $i = 0$. Now we will prove for $i \in \mathbb{N}$. We have

$$d(u_1, u_0) \leq \|[v_0, w_0; G]^{-1}\| \|G(u_0)\| \leq c\xi = f_1 - f_0 \leq f^*,$$

therefore $u_1 \in V[u_0, f^*]$. By Lemma 3.1, we have

$$\mathbb{M}_{\phi,1,0} G(u_n) = \Bigg(\int_0^1 \big(\mathbb{M}_{\phi,1,0} DG(\phi(t)) \mathbb{M}_{\phi,0,1} - DG(u_{n-1})\big) dt + \big(DG(u_{n-1}) - \mathbb{M}_{\psi_{n-1},1,0}[v_{n-1}, w_{n-1}; G] \mathbb{M}_{\psi_{n-1},0,1}\big) \Bigg) N_n.$$

For $n = 1$, we obtain that

$$\begin{aligned}
\|G(u_1)\| = \|\mathbb{M}_{\phi,1,0} G(u_1)\| &= \Bigg\| \Bigg(\int_0^1 \big(\mathbb{M}_{\phi,1,0} DG(\phi(t)) \mathbb{M}_{\phi,0,1} - DG(u_0)\big) dt \\
&\quad + \big(DG(u_0) - \mathbb{M}_{\psi_0,1,0}[v_0, w_0; G] \mathbb{M}_{\psi_0,0,1}\big) \Bigg) N_1 \Bigg\| \\
&\leq \int_0^1 \|\mathbb{M}_{\phi,1,0} DG(\phi(t)) \mathbb{M}_{\phi,0,1} - DG(u_0)\| \|N_1\| dt \\
&\quad + \|DG(u_0) - \mathbb{M}_{\psi_0,1,0}[v_0, w_0; G] \mathbb{M}_{\psi_0,0,1} \\
&\quad + \mathbb{M}_{\psi_0,1,0} DG(v_0) \mathbb{M}_{\psi_0,0,1} - \mathbb{M}_{\psi_0,1,0} DG(v_0) \mathbb{M}_{\psi_0,0,1}\| \|N_1\| \\
&\leq \frac{K}{2} d(u_1, u_0)^2 + \|\mathbb{M}_{\psi_0,1,0} DG(v_0) \mathbb{M}_{\psi_0,0,1} - DG(u_0)\| \|N_1\| \\
&\quad + \|[v_0, w_0; G] - DG(v_0)\| \|N_1\| \\
&= \frac{K}{2} d(u_1, u_0)^2 + \|\mathbb{M}_{\psi_0,1,0} DG(v_0) \mathbb{M}_{\psi_0,0,1} - DG(u_0)\| \|N_1\| \\
&\quad + \int_0^1 \|\big(\mathbb{M}_{\alpha_0,1,0} DG(\alpha_0(t)) \mathbb{M}_{\alpha_0,0,1} - DG(v_0)\big)\| \|N_1\| dt \\
&\leq \frac{K}{2} d(u_1, u_0)^2 + \frac{K(3a+b)}{2} \|G(u_0)\| d(u_1, u_0) \\
&\leq \frac{K}{2} (f_1 - f_0)^2 + \frac{K(3a+b)}{2} z(f_0)(f_1 - f_0) \leq \frac{L}{2} (f_1 - f_0)^2 = z(f_1).
\end{aligned}$$

As the sequence (3.3) is increasing and the polynomial (3.2) is decreasing in $[0, f^*]$, we have

$$\begin{aligned} d(v_1, u_0) &\le d(u_1, u_0) + d(v_1, u_1) = d(u_1, u_0) + \|L_1\| = d(u_1, u_0) + a\|G(u_1)\| \le f^*, \\ d(w_1, u_0) &\le d(u_1, u_0) + d(w_1, u_1) = d(u_1, u_0) + \|M_1\| = d(u_1, u_0) + b\|G(u_1)\| \le f^*, \end{aligned}$$

so that $v_1,\ w_1 \in V[u_0, f^*]$. We suppose that $u_i, v_{i-1}, w_{i-1} \in V[u_0, f^*]$, for $i = 2, 3, 4, \ldots, n$. Then we will prove for $i = n+1$. Since

$$\begin{aligned} z(f_n) &= \int_0^1 \Big(z'\big(f_{n-1} + x(f_n - f_{n-1})\big) - z'(f_{n-1}) \Big) dx (f_n - f_{n-1}) \\ &= L \int_0^1 x (f_n - f_{n-1})^2 dx = \frac{L}{2} (f_n - f_{n-1})^2, \end{aligned}$$

we have $\|G(u_n)\| \le z(f_n)$, for all $n \in \mathbb{N}$, as

$$\begin{aligned} \|G(u_n)\| = \|\mathbb{M}_{\phi,1,0} G(u_n)\| &= \Bigg\| \Bigg(\int_0^1 \big(\mathbb{M}_{\phi,1,0} DG(\phi(t)) \mathbb{M}_{\phi,0,1} - DG(u_{n-1})\big) dt \\ &\quad + \big(DG(u_{n-1}) - \mathbb{M}_{\psi_{n-1},1,0}[v_{n-1}, w_{n-1}; G] \mathbb{M}_{\psi_{n-1},0,1}\big) \Bigg) N_n \Bigg\| \\ &\le \int_0^1 \|\mathbb{M}_{\phi,1,0} DG(\phi(t)) \mathbb{M}_{\phi,0,1} - DG(u_{n-1})\| \|N_n\| dt \\ &\quad + \|DG(u_{n-1}) - \mathbb{M}_{\psi_{n-1},1,0}[v_{n-1}, w_{n-1}; G] \mathbb{M}_{\psi_{n-1},0,1} \\ &\quad + \mathbb{M}_{\psi_{n-1},1,0} DG(v_{n-1}) \mathbb{M}_{\psi_{n-1},0,1} - \mathbb{M}_{\psi_{n-1},1,0} DG(v_{n-1}) \mathbb{M}_{\psi_{n-1},0,1}\| \|N_n\| \\ &\le \frac{K}{2} d(u_n, u_{n-1})^2 + \|\mathbb{M}_{\psi_{n-1},1,0} DG(v_{n-1}) \mathbb{M}_{\psi_{n-1},0,1} - DG(u_{n-1})\| \|N_n\| \\ &\quad + \|[v_{n-1}, w_{n-1}; G] - DG(v_{n-1})\| \|N_n\| \\ &= \frac{K}{2} d(u_n, u_{n-1})^2 + \|\mathbb{M}_{\psi_{n-1},1,0} DG(v_{n-1}) \mathbb{M}_{\psi_{n-1},0,1} - DG(u_{n-1})\| \|N_n\| \\ &\quad + \int_0^1 \|\big(\mathbb{M}_{\alpha_{n-1},1,0} DG(\alpha_{n-1}(t)) \mathbb{M}_{\alpha_{n-1},0,1} - DG(v_{n-1})\big)\| \|N_n\| dt \\ &\le \frac{K}{2} d(u_n, u_{n-1})^2 + \frac{K(3a+b)}{2} \|G(u_{n-1})\| d(u_n, u_{n-1}) \\ &\le \frac{K}{2} (f_n - f_{n-1})^2 + \frac{K(3a+b)}{2} z(f_{n-1})(f_n - f_{n-1}) \le \frac{L}{2} (f_n - f_{n-1})^2 = z(f_n). \end{aligned}$$

We have

$$\begin{aligned} d(v_n, u_0) &\le d(u_n, u_0) + d(v_n, u_n) = d(u_n, u_0) + a\|L_n\| = d(u_n, u_0) + a\|G(u_n)\| \le f^*, \\ d(w_n, u_0) &\le d(u_n, u_0) + d(w_n, u_n) = d(u_n, u_0) + b\|M_n\| = d(u_n, u_0) + b\|G(u_n)\| \le f^*, \end{aligned}$$

so that $v_n, w_n \in V[u_0, f^*]$. Now we will show that the operator $[v_n, w_n; G]^{-1}$ is bounded. Let π

be a minimizing geodesic such that $\pi(0) = u_0$, $\pi(1) = v_n$. We have

$$\begin{aligned}\|DG(u_0)^{-1}\mathbb{M}_{\pi,1,0}[v_n, w_n; G]\mathbb{M}_{\pi,0,1} - I_{u_0}\| &\leq \|DG(u_0)^{-1}\mathbb{M}_{\pi,1,0}([v_n, w_n; G] - DG(v_n))\mathbb{M}_{\pi,0,1}\| \\ &+ \|DG(u_0)^{-1}(\mathbb{M}_{\pi,1,0}DG(v_n)\mathbb{M}_{\pi,0,1} - DG(u_0))\| \\ &\leq \|DG(u_0)^{-1}\| \int_0^1 \|\mathbb{M}_{\alpha_n,1,0}DG(\alpha_n(t))\mathbb{M}_{\alpha_n,0,1} - DG(v_n)\|dt \\ &+ \|DG(u_0)^{-1}\|\|\mathbb{M}_{\pi,1,0}DG(v_n)\mathbb{M}_{\pi,0,1} - DG(u_0)\| \\ &\leq \frac{K\zeta_0}{2}\big(2(f_n - f_0) + (3a+b)z(f_n)\big) < 1,\end{aligned}$$

therefore $\mathbb{M}_{\pi,1,0}[v_n, w_n; G]\mathbb{M}_{\pi,0,1}$ is invertible and

$$\begin{aligned}\|[v_n, w_n; G]^{-1}\| &= \|\mathbb{M}_{\pi,1,0}[v_n, w_n; G]^{-1}\mathbb{M}_{\pi,0,1}\| \\ &\leq \frac{\|DG(u_0)^{-1}\|}{1 - \|DG(u_0)^{-1}\|\|\mathbb{M}_{\pi,1,0}[v_n, w_n; G]^{-1}\mathbb{M}_{\pi,0,1} - DG(u_0)\|} \leq \frac{-1}{z'(f_n)}.\end{aligned}$$

We have

$$d(u_{n+1}, u_n) \leq \|[v_n, w_n; G]^{-1}\|\|G(u_n)\| \leq \frac{-z(f_n)}{z'(f_n)} = f_{n+1} - f_n \tag{3.6}$$

and

$$d(u_{n+1}, u_0) \leq d(u_{n+1}, u_n) + d(u_n, u_0) \leq f_{n+1} - f_n + f_n - f_0 = f_{n+1} - f_0 \leq f^*.$$

So that $u_{n+1} \in V[u_0, f^*]$. Suppose $(C4)$ holds for $i = 1, 2, \ldots, n$ and then we will prove for $i = n+1$. Let δ be a minimizing geodesic δ from $[0,1]$ to Q such that $\delta(0) = u_0$, $\delta(1) = u_{n+1}$, and $\|\delta'(0)\| = d(u_{n+1}, u_0)$.

We obtain that

$$\|\mathbb{M}_{\delta,1,0}DG(u_{n+1})\mathbb{M}_{\delta,0,1} - DG(u_0)\| \leq K\int_0^1 \|\delta'(0))\|ds = Kd(u_{n+1}, u_0) \leq Kf^*$$

and

$$\|DG(u_0)^{-1}\|\|\mathbb{M}_{\delta,1,0}DG(u_{n+1})\mathbb{M}_{\delta,0,1} - DG(u_0)\| \leq \zeta_0 K f^* < 1,$$

as $\zeta_0 K f^* < 1$. Therefore $\mathbb{M}_{\delta,1,0}DG(u_{n+1})\mathbb{M}_{\delta,0,1}$ is invertible by Banach's lemma and

$$\begin{aligned}\|DG(u_{n+1})^{-1}\| = \|\mathbb{M}_{\delta,1,0}DG(u_{n+1})^{-1}\mathbb{M}_{\delta,0,1}\| &\leq \frac{\|DG(u_0)^{-1}\|}{1 - \|DG(u_0)^{-1}\|\|\mathbb{M}_{\delta,1,0}DG(u_{n+1})\mathbb{M}_{\delta,0,1} - DG(u_0)\|} \\ &\leq \frac{\zeta_0}{1 - \zeta_0 K d(u_{n+1}, u_0)} = \zeta_{n+1},\end{aligned}$$

therefore it holds for $i = n+1$. Thus $(C1) - (C4)$ hold for all $i \in \mathbb{N}$.

Now we will prove the Theorem. Since $\{f_n\}$ is a convergent sequence and hence it is a Cauchy sequence therefore from (3.6) the sequence $\{u_n\}$ is also a convergent sequence and let the sequence

$\{u_n\}$ converges to $u^* \in V[u_0, f^*]$. Now we will show that u^* is a singularity of G. As for all $n \in \mathbb{N}$,

$$\|G(u_n)\| \leq z(f_n),$$

taking $n \to \infty$ both sides, we get

$$\|G(u^*)\| \leq z(f^*) = 0.$$

Then, we have $G(u^*) = 0$. Finally, we will show that the singularity is unique in $V[u_0, f^{**} + \xi]$. Let v^* be another singularity of G in $V[u_0, f^{**} + \xi]$. Let ρ be a minimizing geodesic from $[0, 1]$ to Q such that $\rho(0) = u^*$, $\rho(1) = v^*$, and $\|\rho'(0)\| = d(u^*, v^*)$.

We obtain

$$\|\mathbb{M}_{\rho,t,0} DG(\rho(t))\mathbb{M}_{\rho,0,t} - DG(u^*)\| \leq K \int_0^t \|\rho'(0)\| ds = Ktd(u^*, v^*) \leq Kt\left(d(u_0, u^*) + d(u_0, v^*)\right)$$

and

$$\begin{aligned}\|DG(u^*)^{-1}\| \int_0^1 \|\mathbb{M}_{\rho,t,0} DG(\rho(t))\mathbb{M}_{\rho,0,t} - DG(u^*)\| dt &\leq \left(\frac{1}{\zeta_0} - Kf^*\right)^{-1} \int_0^1 Kt\left(d(u_0, u^*) + d(u_0, v^*)\right) dt \\ &\leq \left(\frac{1}{\zeta_0} - Kf^*\right)^{-1} \frac{K}{2}(f^* + f^{**} + \xi) < 1.\end{aligned}$$

It shows that the operator

$$T = \int_0^1 \mathbb{M}_{\rho,t,0} DG(\rho(t))\mathbb{M}_{\rho,0,t} dt$$

is invertible by Banach's lemma and we have

$$0 = \mathbb{M}_{\rho,1,0} G(v^*) - G(u^*) = \int_0^1 \mathbb{M}_{\rho,t,0} DG(\rho(t))\mathbb{M}_{\rho,0,t}(\rho'(0)) dt.$$

So that $\rho'(0) = 0$. We have $0 = \|\rho'(0)\| = d(u^*, v^*)$, implies that $u^* = v^*$. Thus it completes the proof. □

Theorem 3.4. *Suppose that u^* is a singular point of G in $V[u_0, f^*]$, if $V(u_0, f^{**}) \subseteq \mho$, then the only singular point of G in $V[u_0, r]$ is u^*, where $f^* < r \leq f^{**}$.*

Proof. Let v^* be a singular point of G in $V[u_0, r]$. Let Λ be a minimizing geodesic such that $\Lambda(0) = u_0, \quad \Lambda(1) = v^*$. Then by (2.3), we have

$$\begin{aligned}\mathbb{M}_{\Lambda,1,0} G(v^*) &= \mathbb{M}_{\Lambda,1,0} G(v^*) - G(u_0) + G(u_0) + DG(u_0)\Lambda'(0) - DG(u_0)\Lambda'(0) \\ &= \int_0^1 \mathbb{M}_{\Lambda,t,0} DG(\Lambda(t))\mathbb{M}_{\Lambda,0,t}\Lambda'(0) dt - DG(u_0)\Lambda'(0) + G(u_0) + DG(u_0)\Lambda'(0) \\ &= \int_0^1 \big(\mathbb{M}_{\Lambda,t,0} DG(\Lambda(t))\mathbb{M}_{\Lambda,0,t} - DG(u_0)\big)\Lambda'(0) dt + G(u_0) + DG(u_0)\Lambda'(0).\end{aligned}$$

Thus, we have

$$\frac{Ld(u_0,v^*)^2}{2} \geq \frac{Kd(u_0,v^*)^2}{2} \geq \|G(u_0)+DG(u_0)\Lambda'(0)\| \geq \frac{1}{\|DG(u_0)^{-1}\|}\|DG(u_0)^{-1}G(u_0)+\Lambda'(0)\|$$
$$\geq \frac{1}{\zeta_0}\Big(\|\Lambda'(0)\| - \|DG(u_0)^{-1}G(u_0)\|\Big) \geq \left(\frac{d(u_0,v^*)}{\zeta_0} - \xi\right) \geq \left(\frac{d(u_0,v^*)}{c} - \xi\right).$$

Therefore

$$z(d(u_0,v^*)) = \frac{Ld(u_0,v^*)^2}{2} - \frac{d(u_0,v^*)}{c} + \xi \geq 0.$$

Since $d(u_0,v^*) \leq r \leq f^{**}$, we have $d(u_0,v^*) \leq f^*$, hence by Theorem 3.3, $u^* = v^*$. □

4 Numerical examples

In this section, two examples are given to show the application of our theorem.

Example 4.1. *Let us consider the vector field G from $\mho = (-1,1)^3 \subseteq Q = \mathbb{R}^3$ to $\mho = (-1,1)^3$ given by*

$$G\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = \begin{pmatrix}e^{u_1}-1\\u_2^2+u_2\\u_3\end{pmatrix}$$

with the max norm $\|\cdot\|_\infty$. For the point $\boldsymbol{u} = (u_1,u_2,u_3)^T$, the first and second Fréchet derivatives of G are:

$$DG(\boldsymbol{u}) = \begin{bmatrix}e^{u_1} & 0 & 0\\0 & 2u_2+1 & 0\\0 & 0 & 1\end{bmatrix}, \quad D^2G(\boldsymbol{u}) = \left[\begin{array}{ccc|ccc|ccc}e^{u_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{array}\right].$$

Initially for $\boldsymbol{u}_0 = (-0.005,-0.005,-0.005)^T$, we obtain

$$\|G(\boldsymbol{u}_0)\| = \max(|-0.005|,|-0.005|,|-0.005|) = 0.005 = \xi,$$
$$\|DG(\boldsymbol{u}_0)^{-1}\| = 1.0101 = \zeta_0, \quad \|D^2G(\boldsymbol{u})\| = \max(0.995,2,0) = 2 = K.$$

Now, for $a=1$, $b=1$, all the assumptions of the convergence theorem are satisfied and the Steffensen-like method can be applied to get the desired singular point.

Example 4.2. *Let us consider the vector field G from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by*

$$G\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}\frac{\cos u_1+4u_1}{4}\\u_2\end{pmatrix}$$

with the max norm $\|\cdot\|_\infty$. *For the point* $u = (u_1, u_2)^T$, *the first and second Fréchet derivatives of* G *are:*

$$DG(u) = \begin{bmatrix} \frac{-\sin u_1 + 4}{4} & 0 \\ 0 & 1 \end{bmatrix}, \quad D^2G(u) = \left[\begin{array}{cc|cc} \frac{-\cos u_1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Initially for $u_0 = (0,0)^T$, *we obtain*

$$\|G(u_0)\| = \frac{1}{4} = \xi, \quad \|DG(u_0)^{-1}\| = 1 = \zeta_0, \quad \|D^2G(u)\| \leq \frac{1}{4} = K.$$

Now, for $a = 1,\ b = 1,$ *all the assumptions for convergence are satisfied and the Steffensen-like method can be applied to get the desired singular point.*

5 Conclusion

In this paper, we have studied the semilocal convergence of Steffensen-like method for approximating the zeros of a vector field in Riemannian manifolds and established convergence theorem under Lipschitz continuity condition on the first order covariant derivative of a vector field. Finally, two examples are given to show the application of our theorem.

6 Acknowledgments

The first author wishes to thank the UGC, India, for providing financial support (No: NFO-2018-19-OBC-JHA-68560). Moreover, the authors express their gratitude to the anonymous referees for their valuable suggestions, which enhanced the quality of the paper.

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Cubo
A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 26, no. 3, pp. 541–558, December 2024
DOI: 10.56754/0719-0646.2603.541

Investigating the existence and multiplicity of solutions to $\varphi(x)$-Kirchhoff problem

Abolfazl Sadeghi[1]
Ghasem Alizadeh Afrouzi[1,✉]
Maryam Mirzapour[2]

[1] *Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.*
sadeghi31587@mail.com
afrouzi@umz.ac.ir✉

[2] *Department of Mathematics Education, Farhangian University, P.O. Box 14665-889, Tehran, Iran.*
m.mirzapour@cfu.ac.ir

ABSTRACT

In this article, we want to discuss variational methods such as the Mountain pass theorem and the Symmetric Mountain pass theorem, without the Ambrosetti-Rabinowitz condition. We prove the existence and multiplicity of nontrivial weak solutions for the problem of the following form

$$\begin{cases} -\left(\alpha - \beta \displaystyle\int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)} dx\right) \Delta_{\varphi(x)} \upsilon + |\upsilon|^{\psi(x)-2}\upsilon \\ \qquad = \lambda \eta(x,\upsilon), \\ \qquad x \in \Omega, \\ \left(\alpha - \beta \displaystyle\int_{\partial\Omega} \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx\right) |\nabla \upsilon|^{\varphi(x)-2} \dfrac{\partial \upsilon}{\partial \nu} = 0 \\ \qquad x \in \partial\Omega, \end{cases}$$

where $\alpha \geq \beta > 0$, $\Delta_{\varphi(x)}\upsilon$ is the $\varphi(x)$-Laplacian operator, Ω is a smooth bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and ν is the outer unit normal to $\partial\Omega$, $\varphi(x), \psi(x) \in C(\bar{\Omega})$ with $1 < \varphi(x) < N$, $\varphi(x) < \psi(x) < \varphi^*(x) := \dfrac{N\varphi(x)}{N - \varphi(x)}$, $\lambda > 0$ is a real parameter and $\eta(x,t) \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$.

Published: 13 December, 2024
Accepted: 29 November, 2024
Received: 14 April, 2024

RESUMEN

En este artículo discutimos métodos variacionales, como el teorema del paso de la montaña y el teorema simétrico del paso de la montaña, sin la condición de Ambrosetti-Rabinowitz. Demostramos la existencia y multiplicidad de soluciones débiles no triviales para el problema de la siguiente forma

$$\begin{cases} -\left(\alpha - \beta \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)} dx\right) \Delta_{\varphi(x)} \upsilon + |\upsilon|^{\psi(x)-2}\upsilon \\ \qquad = \lambda \eta(x, \upsilon), \\ \qquad x \in \Omega, \\ \left(\alpha - \beta \int_{\partial\Omega} \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx\right) |\nabla \upsilon|^{\varphi(x)-2} \frac{\partial \upsilon}{\partial \nu} = 0 \\ \qquad x \in \partial\Omega, \end{cases}$$

donde $\alpha \geq \beta > 0$, $\Delta_{\varphi(x)}\upsilon$ es el $\varphi(x)$ operador Laplaciano, Ω es un dominio acotado y suave en $\mathbb{R}^N$ con borde suave $\partial\Omega$ y ν es la normal unitaria exterior a $\partial\Omega$, $\varphi(x), \psi(x) \in C(\bar{\Omega})$ con $1 < \varphi(x) < N,\ \varphi(x) < \psi(x) < \varphi^*(x) := \frac{N\varphi(x)}{N - \varphi(x)}$, $\lambda > 0$ es un parámetro real y $\eta(x, t) \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$.

Keywords and Phrases: Generalized Lebesgue-Sobolev spaces, weak solutions, mountain pass theorem, symmetric mountain pass theorem.

2020 AMS Mathematics Subject Classification: 35J60, 35J20.

1 Introduction

In this article, we consider the following problem

$$\begin{cases} -\left(\alpha - \beta \int_\Omega \frac{1}{\varphi(x)}|\nabla \upsilon|^{\varphi(x)}\,dx\right)\Delta_{\varphi(x)}\upsilon + |\upsilon|^{\psi(x)-2}\upsilon = \lambda\,\eta(x,\upsilon), & x \in \Omega, \\ \left(\alpha - \beta \int_{\partial\Omega} \frac{1}{\varphi(x)}|\nabla \upsilon|^{\varphi(x)}\,dx\right)|\nabla \upsilon|^{\varphi(x)-2}\frac{\partial \upsilon}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\alpha \geq \beta > 0$, $\Delta_{\varphi(x)}\upsilon$ is the $\varphi(x)$-Laplacian operator, defined as $\Delta_{\varphi(x)}\upsilon := \operatorname{div}(|\nabla\upsilon|^{\varphi(x)-2}\nabla\upsilon) = \sum_{i=1}^N \left(|\nabla\upsilon|^{\varphi(x)-2}\frac{\partial\upsilon}{\partial x_i}\right)$, Ω is a smooth bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and ν is the outer unit normal to $\partial\Omega$ and $\varphi(x), \psi(x) \in C(\bar{\Omega})$ with $1 < \varphi(x) < N$, $\varphi(x) < \psi(x) < \varphi^*(x) := \frac{N\varphi(x)}{N - \varphi(x)}$, $\lambda > 0$ is a real parameter. We define φ_l and φ_s for convenience as follows: $\varphi_l := \inf_\Omega \varphi(x)$ and $\varphi_s := \sup_\Omega \varphi(x)$, for all $\varphi(x) \in C(\bar{\Omega})$. The function $\eta(x,t) \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies:

(η_1) $|\eta(x,t)| \leq c(1 + |t|^{r(x)-1})$, $\forall (x,t) \in \Omega \times \mathbb{R}$, where $c > 0$ and $\varphi(x) < r(x) < \varphi^*(x)$,

(η_2) $\lim\limits_{t\to 0} \frac{\eta(x,t)}{|t|^{\varphi(x)-2}t} = 0$, uniformly a.e. $x \in \Omega$,

(η_3) $\lim\limits_{|t|\to\infty} \frac{\eta(x,t)}{|t|^{\varphi_s}} = +\infty$, uniformly a.e. $x \in \Omega$,

(η_4) there exists a constant $c_0 > 0$ such that $\hat{H}(x,t) \leq \hat{H}(x,s) + c_0$ for each $x \in \Omega$, $0 < |t| < s$, where $\hat{H}(x,t) := t\,\eta(x,t) - \varphi_s H(x,t)$ and $H(x,t) := \int_0^t \eta(x,s)ds$,

(η_5) $\eta(x,-t) = -\eta(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$.

In addition to the conditions given for η, the functions $\varphi(x), \psi(x), r(x)$ must satisfy the following condition, which we call the $(\varphi\psi r)$-condition:

$$1 < \varphi_l < \varphi(x) < \varphi_s < \psi_l < \psi(x) < \psi_s < 2\varphi_l < r_l < r(x) < r_s < \varphi^*(x).$$

Sobolev spaces are essential in contemporary analysis, especially in the study of partial differential equations (PDEs) and functional analysis. These spaces generalize the classical concepts of differentiability and integrability, offering a more adaptable structure for analyzing functions whose derivatives might not be classically well-defined. By incorporating weak derivatives, Sobolev spaces allow for the examination of broader issues in areas such as mathematical physics, fluid dynamics, and engineering applications, see [1, 4, 5, 7–9, 12, 20, 21, 26, 27, 32, 34, 38].

The necessity of Sobolev spaces arises from their ability to handle irregularities and discontinuities in functions that appear naturally in real-world problems. For instance, solutions to PDEs often lack classical differentiability but possess weak derivatives that allow their analysis within Sobolev spaces. This makes them indispensable in addressing variational problems and boundary value problems.

Kirchhoff's problems, named after the German physicist Gustav Kirchhoff [28], are fundamental in the study of mechanics and mathematical physics, particularly in understanding wave propagation and elasticity theory. Kirchhoff's equations describe the motion of elastic surfaces and play a key role in modeling vibrating systems, such as strings, membranes, and plates. Recent research in this field has focused on nonlinear versions of Kirchhoff's equations, particularly in higher dimensions, where the complexity of the problem increases, see [2, 6, 10, 11, 14, 17–19, 24, 25, 31, 34, 37].

Variational methods have a relatively long history. Many scientists have studied in this field and have achieved many successes. Due to the applicability of this method in experimental sciences, it has always been of interest [**?**, 3, 8, 13, 15, 16, 22, 23, 26, 29, 33, 35, 36]. In these methods, especially those used to solve boundary value problems, the Palais-Smale condition ((PS)-condition in short) plays a crucial role in ensuring the existence of critical points, which correspond to solutions of the problem. This condition provides a framework for the analysis of functionals in infinite-dimensional spaces, such as Sobolev spaces. On the other hand, the Cerami condition ((C)-condition in short) is a variation of the (PS)-condition that is particularly useful in dealing with problems where the (PS)-condition might not hold. This modified condition is often more applicable in certain classes of problems, particularly those involving non-compact domains or complex geometries.

Now we state our main results.

Theorem 1.1. *Suppose* $(\eta_1) - (\eta_4)$ *and the* $(\varphi\psi r)$*-condition hold. Then problem* (1.1) *has at least a nontrivial weak solution for all* $\lambda < \lambda_0$ *(*λ_0 *which has been given in Section 3).*

Theorem 1.2. *Suppose* $(\eta_1), (\eta_2), (\eta_4), (\eta_5)$ *and the* $(\varphi\psi r)$*-condition hold. Then problem* (1.1) *has infinitely many weak solutions for all* $\lambda < \lambda_0$ *(*λ_0 *which has been given in Section 3).*

To prove our results, we will use inequalities and applied theorems such as Hölder and Poincaré inequalities and the embedding, Mountain pass and Symmetric Mountain pass theorems.

2 Preliminary results

In this section, we recall some important definitions and essential characteristics of the generalized Lebesgue-Sobolev spaces $L^{\varphi(x)}(\Omega)$ and $W^{1,\varphi(x)}(\Omega)$ where $\Omega \subset \mathbb{R}^N$ is an open set. In this regard, we refer readers to the book of Musielak [32] and the papers [20, 21]. Set

$$C_+(\bar{\Omega}) := \{h : h \in C(\bar{\Omega}), h(x) > 1 \quad \text{for all} \quad x \in \bar{\Omega}\},$$

and for each $\varphi(x) \in C_+(\bar{\Omega})$

$$L^{\varphi(x)}(\Omega) = \left\{ \upsilon : \text{a measurable real-valued function such that } \int_\Omega |\upsilon(x)|^{\varphi(x)}\,dx < \infty \right\},$$

is the definition of variable exponent Lebesgue space, that by mentioned the following norm (so-called Luxemburg norm) is reflexive and separable Banach space

$$\|\upsilon\|_{\varphi(x)} := \inf\left\{ \mu > 0; \int_\Omega \left| \frac{\upsilon(x)}{\mu} \right|^{\varphi(x)} dx \le 1 \right\}.$$

These spaces are similar to classical Lebesgue spaces in many aspects [35]:

a) If $0 < |\Omega| < \infty$ and $\varphi_1(x), \varphi_2(x)$ are variable exponents so that $\varphi_1(x) \le \varphi_2(x)$ a.e. $x \in \Omega$, then there is a continuous embedding

$$L^{\varphi_2(x)}(\Omega) \hookrightarrow L^{\varphi_1(x)}(\Omega).$$

b) The Hölder inequality holds, *i.e.*, if $L^{\varphi'(x)}(\Omega)$ is a conjugate of $L^{\varphi(x)}(\Omega)$, where $\dfrac{1}{\varphi(x)} + \dfrac{1}{\varphi'(x)} = 1$, we have

$$\left| \int_\Omega u\upsilon\,dx \right| \le \left(\frac{1}{\varphi_l} + \frac{1}{\varphi'_l} \right) \|u\|_{\varphi(x)} \|\upsilon\|_{\varphi'(x)}, \quad \forall u \in L^{\varphi(x)}(\Omega), \quad \forall \upsilon \in L^{\varphi'(x)}(\Omega).$$

The modular plays an essential role in manipulating the $L^{\varphi(x)}$ spaces and is defined by the following relation, $\rho_{\varphi(x)} : L^{\varphi(x)} \to \mathbb{R}$;

$$\rho_{\varphi(x)}(\upsilon) = \int_\Omega |\upsilon|^{\varphi(x)} dx.$$

Proposition 2.1 ([20])**.** *If $\upsilon, \upsilon_n \in L^{\varphi(x)}(\Omega)$ and $\varphi_s < +\infty$, then the following relations hold*

(1) $\|\upsilon\|_{\varphi(x)} > 1 \implies \|\upsilon\|_{\varphi(x)}^{\varphi_l} \le \rho_{\varphi(x)}(\upsilon) \le \|\upsilon\|_{\varphi(x)}^{\varphi_s}$;

(2) $\|\upsilon\|_{\varphi(x)} < 1 \implies \|\upsilon\|_{\varphi(x)}^{\varphi_s} \le \rho_{\varphi(x)}(\upsilon) \le \|\upsilon\|_{\varphi(x)}^{\varphi_l}$;

(3) $\|\upsilon\|_{\varphi(x)} < 1$ *(respectively,* $= 1; > 1$*)* $\iff \rho_{\varphi(x)}(\upsilon) < 1$ *(respectively,* $= 1; > 1$*);*

(4) $\|\upsilon_n\|_{\varphi(x)} \to 0$ *(respectively,* $\to +\infty$*)* $\iff \rho_{\varphi(x)}(\upsilon) = 0$ *(respectively,* $\to +\infty$*);*

(5) $\lim_{n\to\infty} \|\upsilon_n - \upsilon\|_{\varphi(x)} = 0 \iff \lim_{n\to\infty} \rho_{\varphi(x)}(\upsilon_n - \upsilon) = 0$*;*

(6) *For* $\upsilon \ne 0$, $\|\upsilon\|_{\varphi(x)} = \lambda \iff \rho\left(\frac{\upsilon}{\lambda}\right) = 1$.

Definition 2.2 ([21])**.** *If $\Omega \subset \mathbb{R}^N$, the Sobolev space with variable exponent $W^{1,\varphi(x)}(\Omega)$ is defined as*

$$W^{1,\varphi(x)}(\Omega) := \{\upsilon : \Omega \to \mathbb{R} : \upsilon \in L^{\varphi(x)}(\Omega),\ |\nabla\upsilon| \in L^{\varphi(x)}(\Omega)\},$$

endowed with the following norm

$$\|\upsilon\|_{W^{1,\varphi(x)}} := |||\upsilon||| = \|\upsilon\|_{\varphi(x)} + \|\nabla\upsilon\|_{\varphi(x)},$$

or equivalently

$$|||\upsilon||| = \inf\left\{\mu > 0, \int_\Omega \frac{\|\nabla\upsilon(x)\|_{\varphi(x)}^{\varphi(x)} + \|\upsilon\|_{\varphi(x)}^{\varphi(x)}}{\mu^{\varphi(x)}} dx \leq 1\right\}.$$

Proposition 2.3 ([20]). *The Poincaré inequality in $W^{1,\varphi(x)}(\Omega)$ holds, that is, there exists a positive constant c so that*

$$\|\upsilon\|_{\varphi(x)} \leq c\|\nabla\upsilon\|_{\varphi(x)}, \quad \forall\upsilon \in W^{1,\varphi(x)}(\Omega). \tag{2.1}$$

Proposition 2.4 (Sobolev embedding [21]). *If $\varphi(x), \psi(x) \in C_+(\bar{\Omega})$ and $1 \leq \psi(x) \leq \varphi^*(x)$ for each $x \in \bar{\Omega}$, then there exists a continuous embedding*

$$W^{1,\varphi(x)}(\Omega) \hookrightarrow L^{\psi(x)}(\Omega). \tag{2.2}$$

If $1 < \psi(x) < \varphi^*(x)$, the continuous embedding is compact.

In the sequel, the constant c_{emb} represents the Sobolev embedding quantity, and we denote by $X := W^{1,\varphi(x)}(\Omega)$; $X^* = (W^{1,\varphi(x)}(\Omega))^*$, the dual space and $\langle\cdot,\cdot\rangle$, the dual pair.

Lemma 2.5 ([21]). *Suppose*

$$J(\upsilon) = \int_\Omega \frac{1}{\varphi(x)}|\nabla\upsilon|^{\varphi(x)}\, dx, \quad \forall\upsilon \in X,$$

then $J(\upsilon) \in C^1(X, \mathbb{R})$ and the derivative operator J' of J is

$$\langle J'(\upsilon), \vartheta\rangle = \int_\Omega |\nabla\upsilon|^{\varphi(x)-2}\nabla\upsilon\nabla\vartheta\, dx, \quad \forall\upsilon, \vartheta \in X$$

and the following relations hold:

(1) *J is a convex functional,*

(2) *$J' : X \to X^*$ is a strictly monotone operator and bounded homeomorphism,*

(3) *J' is a mapping of type (S_+), it means, $\upsilon_n \rightharpoonup \upsilon$ (weakly) and $\lim_{n\to+\infty}\sup\langle J'(\upsilon), \upsilon_n - \upsilon\rangle \leq 0$, imply $\upsilon_n \to \upsilon$ (strongly) in $W_0^{1,\varphi(x)}(\Omega)$.*

Definition 2.6. *$\upsilon \in X$ is a weak solution of problem (1.1), if*

$$\left(\alpha - \beta \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx\right) \int_\Omega |\nabla \upsilon|^{\varphi(x)-2} \nabla \upsilon \nabla \nu \, dx + \int_\Omega |\upsilon|^{\psi(x)-2} \upsilon \nu \, dx = \lambda \int_\Omega \eta(x,\upsilon) \nu \, dx, \quad \forall \nu \in X.$$

The energy functional related to our problem, $J_\lambda : X \to \mathbb{R}$ such that

$$J_\lambda(\upsilon) = \alpha \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx - \frac{\beta}{2} \left(\int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx\right)^2 + \int_\Omega \frac{1}{\psi(x)} |\upsilon|^{\psi(x)}\, dx - \lambda \int_\Omega H(x,\upsilon)\, dx, \quad \forall \upsilon \in X, \tag{2.3}$$

which is also well defined and of class C^1 in $(X, \mathbb{R})$.

Now we define J'_λ as the derivative operator of J_λ in the weak sense, by the following formula,

$$\langle J'_\lambda(\upsilon), \nu\rangle = \left(\alpha - \beta \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx\right) \int_\Omega |\nabla \upsilon|^{\varphi(x)-2} \nabla \upsilon \nabla \nu \, dx + \int_\Omega |\upsilon|^{\psi(x)-2} \upsilon \nu \, dx - \lambda \int_\Omega \eta(x,\upsilon) \nu \, dx, \quad \forall \upsilon, \nu \in X. \tag{2.4}$$

A critical point of J_λ is clearly a weak solution of problem (1.1).

Definition 2.7. *If $(X, \|\cdot\|)$ is a real Banach space and $J \in C^1(X, \mathbb{R})$, then we can say that J ensures Cerami-condition in level c ($(C)_c$-condition in short), if for all sequence $\{\upsilon_n\} \subset X$ satisfying*

$$J(\upsilon_n) \to c \quad \text{and} \quad \|J'(\upsilon_n)\|_{X^*}(1 + \|\upsilon_n\|_X) \to 0, \tag{2.5}$$

then, $\{\upsilon_n\}$ contains a convergent subsequence.

If this condition holds for each $c \in \mathbb{R}$, it can be called (C)-condition.

3 Proof of Theorem 1.1

To prove Theorem 1.1, we will use the following Mountain pass theorem.

Theorem 3.1 (Mountain pass theorem [8])**.** *Let X be a real Banach space, let $J_\lambda : X \to \mathbb{R}$ as $J_\lambda \in C^1(X, \mathbb{R})$ that ensures the $(C)_c$-condition and $J_\lambda(0) = 0$, such that*

(a) there exists $R > 0$ and $\alpha > 0$, so that $J_\lambda(\upsilon) \geq \alpha$ for each $\upsilon \in X$ with $|||\upsilon||| = R$,

(b) there is a function $e \in X$ such that $|||e||| > R$ and $J_\lambda(e) \leq 0$.

So, J_λ has a critical value $c \geq \alpha$, that is $\upsilon \in X$, such that $J_\lambda(\upsilon) = c$ and $J'_\lambda(\upsilon) = 0$ in X^*.

First, we prove that J_λ has the geometry of the above Mountain pass theorem.

Lemma 3.2. *(a) Under the condition (η_3) the functional J_λ is unbounded from below.*

(b) Under the conditions (η_1) and (η_2), $\upsilon = 0$ is a strict local minimum for J_λ.

Proof. (a) By (η_3), we have

$$\forall M > 0,\ \exists c_M > 0; \quad \eta(x,t) \geq M|t|^{\varphi_s} - c_M, \quad \forall x \in \Omega, \quad t \in \mathbb{R}. \tag{3.1}$$

If $\upsilon \in X$ for $\upsilon > 0$, and (3.1), we have

$$\begin{aligned} J_\lambda(t\upsilon) &= \alpha \int_\Omega \frac{t^{\varphi(x)}}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx - \frac{\beta}{2} \left(\int_\Omega \frac{t^{\varphi(x)}}{\varphi(x)}\ \nabla \upsilon|^{\varphi(x)}\, dx \right)^2 + \int_\Omega \frac{t^{\psi(x)}}{\psi(x)} |\upsilon|^{\psi(x)}\, dx \\ &- \lambda \int_\Omega H(x, t\upsilon)\, dx \\ &\leq \alpha t^{\varphi_s} \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx - \frac{\beta}{2} t^{2\varphi_l} \left(\int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx \right)^2 + t^{\psi_s} \int_\Omega \frac{1}{\psi(x)} |\upsilon|^{\psi(x)}\, dx \\ &- M\lambda t^{\varphi_s} \int_\Omega |\upsilon|^{\varphi(x)}\, dx + \lambda c_M |\Omega| \to -\infty, \quad \text{as} \quad t \to +\infty, \end{aligned}$$

since $\varphi_s < \psi_s < 2\varphi_l$, thus, J_λ is unbounded from below.

(b) According to the conditions (η_1) and (η_2), we have

$$\forall \varepsilon > 0,\ \exists c_\varepsilon > 0; \quad H(x,t) \leq \varepsilon |t|^{\varphi(x)} + c_\varepsilon |t|^{r(x)}, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Therefore, if $\upsilon \in X$ with $|||\upsilon||| \leq 1$, by Poincaré inequality and Sobolev embedding (2.2), we have

$$\begin{aligned} J_\lambda(\upsilon) &= \alpha \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon|^{\varphi(x)}\, dx - \frac{\beta}{2} \left(\int_\Omega |\nabla \upsilon|^{\varphi(x)}\, dx \right)^2 + \int_\Omega \frac{1}{\psi(x)} |\upsilon|^{\psi(x)}\, dx - \lambda \int_\Omega H(x,\upsilon)\, dx, \\ &\geq \frac{\alpha}{\varphi_s} \int_\Omega |\nabla \upsilon|^{\varphi(x)} dx - \frac{\beta}{2\varphi_l^2} \left(\int_\Omega |\nabla \upsilon|^{\varphi(x)}\, dx \right)^2 - \varepsilon\lambda \int_\Omega |\upsilon|^{\varphi(x)}\, dx - c_\varepsilon \lambda \int_\Omega |\upsilon|^{r(x)}\, dx \\ &\geq \left(\frac{\alpha}{\varphi_s} - c_2 \lambda \varepsilon \right) \int_\Omega |\nabla \upsilon|^{\varphi(x)}\, dx - \frac{\beta}{2\varphi_l} \left(\int_\Omega |\nabla \upsilon|^{\varphi(x)}\, dx \right)^2 - c_\varepsilon \lambda \left(\|\upsilon\|_{r(x)}^{r_l} + \|\upsilon\|_{r(x)}^{r_s} \right) \\ &\geq \left(\frac{\alpha}{\varphi_s} - c_2 \varepsilon \lambda \right) |||\upsilon|||^{\varphi_s} - \frac{\beta}{2\varphi_l^2} |||\upsilon|||^{2\varphi_l} - c_\varepsilon \lambda \left(c_{emb}^{r_l} |||\upsilon|||^{r_l} + c_{emb}^{r_s} |||\upsilon|||^{r_s} \right) \\ &\geq \left(\frac{\alpha}{\varphi_s} - c_2 \varepsilon \lambda \right) |||\upsilon|||^{\varphi_s} - \frac{\beta}{2\varphi_l^2} |||\upsilon|||^{2\varphi_l} - c_\varepsilon \lambda \left(c_{emb}^{r_l} + c_{emb}^{r_s} \right) |||\upsilon|||^{r_l}, \end{aligned}$$

where embedding constant $c_{emb} > 0$. By selecting $\varepsilon \leq \dfrac{\alpha}{2c_2 \varphi_s \lambda}$, we have

$$J_\lambda(\upsilon) \geq \frac{\alpha}{2\varphi_s} |||\upsilon|||^{\varphi_s} - \frac{\beta}{2\varphi_l^2} |||\upsilon|||^{2\varphi_l} - c_\varepsilon \lambda \left(c_{emb}^{r_l} + c_{emb}^{r_s} \right) |||\upsilon|||^{r_l}.$$

By dividing the previous inequality sides on the positive value $|||\upsilon|||^{\varphi_s}$ and since, we know that $\varphi_s < 2\varphi_l < r_l$, we have

$$J_\lambda(\upsilon) \geq |||\upsilon|||^{\varphi_s}\left[\frac{\alpha}{2\varphi_s} - \frac{\beta}{2\varphi_l^2}|||\upsilon|||^{2\varphi_l-\varphi_s} - c_\varepsilon\lambda\left(c_{emb}^{r_l}+c_{emb}^{r_s}\right)|||\upsilon|||^{r_l-\varphi_s}\right],$$

now, we can choose $|||\upsilon||| = R > 0$, such that

$$\frac{\alpha}{2\varphi_s} - \frac{\beta}{2\varphi_l^2}R^{2\varphi_l-\varphi_s} - c_\varepsilon\lambda\left(c_{emb}^{r_l}+c_{emb}^{r_s}\right)R^{r_l-\varphi_s} > 0. \tag{3.2}$$

We can infer that

$$c_\varepsilon\lambda\left(c_{emb}^{r_l}+c_{emb}^{r_s}\right)R^{r_l-\varphi_s} < \frac{\alpha}{2\varphi_s} - \frac{\beta}{2\varphi_l^2}R^{2\varphi_l-\varphi_s} = \frac{\alpha\varphi_l^2 - \beta\varphi_s R^{2\varphi_l-\varphi_s}}{2\varphi_s\varphi_l^2},$$

since c_ε and $c_{emb} > 0$, we can infer that

$$\lambda < \frac{\alpha\varphi_l^2 - \beta\varphi_s R^{2\varphi_l-\varphi_s}}{2c_\varepsilon\left(c_{emb}^{r_l}+c_{emb}^{r_s}\right)\varphi_s\varphi_l^2 R^{r_l-\varphi_s}} := \lambda_0, \tag{3.3}$$

therefore, by (3.2) and (3.3) we have

$$\frac{\alpha}{2\varphi_s} - \frac{\beta}{2\varphi_l^2}R^{2\varphi_l-\varphi_s} - c_\varepsilon\lambda\left(c_{emb}^{r_l}+c_{emb}^{r_s}\right)R^{r_l-\varphi_s} > 0, \quad \forall\lambda\in(0,\lambda_0).$$

So, there exists $\delta > 0$ so that $J_\lambda(\upsilon) \geq \delta > 0$ for all $\upsilon \in X$ with $|||\upsilon||| = R$. Thus, the proof is complete. □

Now, we prove that J_λ ensures the $(C)_c$-condition.

Lemma 3.3. *If $(\eta_1) - (\eta_4)$ hold, then for all $\lambda \geq 0$, J_λ ensures the $(C)_c$-condition at any level $c \in \left(-\infty, \frac{\alpha^2}{2\beta}\right)$.*

Proof. At the beginning, we consider the boundary condition for $\{\upsilon_n\}$, let $\{\upsilon_n\} \subset X$ be a $(C)_c$ sequence related to the J_λ, such that

$$J_\lambda(\upsilon_n) \to c \quad \text{and} \quad \|J_\lambda'(\upsilon_n)\|_{X^*}(1+|||\upsilon_n|||) \to 0. \tag{3.4}$$

Using (η_3) and (3.4), we can write

$$\begin{aligned}\varphi_s c + O_n(1) &\geq \varphi_s J_\lambda(\upsilon_n) - \langle J_\lambda'(\upsilon_n), \upsilon_n\rangle \\ &= \alpha\int_\Omega\left(\frac{\varphi_s}{\varphi(x)} - 1\right)|\nabla\upsilon_n|^{\varphi(x)}\,dx + \int_\Omega\left(\frac{\psi_s}{\psi(x)} - 1\right)|\upsilon_n|^{\psi(x)}\,dx \\ &+ \lambda\int_\Omega \hat{H}(x,\upsilon_n)\,dx - \beta\left(\int_\Omega\frac{1}{\varphi(x)}|\nabla\upsilon_n|^{\varphi(x)}\,dx\right)\left(\int_\Omega\left[\frac{\varphi_s}{2\varphi(x)} - 1\right]|\nabla\upsilon_n|^{\varphi(x)}\,dx\right).\end{aligned}$$

Since $\alpha \geq \beta$ and $2\varphi_l > \varphi_s$ we have

$$\begin{aligned}\varphi_s c + O_n(1) &\geq \beta\left(\frac{1}{\varphi_s} - \frac{1}{2\varphi_l}\right)\left(\int_\Omega |\nabla \upsilon_n|^{\varphi(x)}\,dx\right)^2 + \int_\Omega \left(\frac{\psi_s}{\psi(x)} - 1\right)|\upsilon_n|^{\psi(x)}\,dx \\ &+ \lambda \int_\Omega (\hat{H}(x,0) - c_0)\,dx \\ &\geq \beta\left(\frac{1}{\varphi_s} - \frac{1}{2\varphi_l}\right)|||\upsilon_n|||^{2\varphi_l} + \int_\Omega \left(\frac{\psi_s}{\psi(x)} - 1\right)|\upsilon_n|^{\psi(x)}\,dxa + \lambda\int_\Omega (\hat{H}(x,0) - c_0)\,dx,\end{aligned}$$

therefore

$$\varphi_s c + O_n(1) \geq \beta\left(\frac{1}{\varphi_s} - \frac{1}{2\varphi_l}\right)|||\upsilon_n|||^{2\varphi_l} + \int_\Omega \left(\frac{\psi_s}{\psi(x)} - 1\right)|\upsilon_n|^{\psi(x)}\,dx + \lambda\int_\Omega (\hat{H}(x,0) - c_0)\,dx.$$

Since $\lambda \geq 0$, we have

$$\varphi_s c + O_n(1) \geq \beta\left(\frac{1}{\varphi_s} - \frac{1}{2\varphi_l}\right)|||\upsilon_n|||^{2\varphi_l} - \lambda c_0|\Omega|,$$

thus

$$\beta\left(\frac{1}{\varphi_s} - \frac{1}{2\varphi_l}\right)|||\upsilon_n|||^{2\varphi_l} \leq \varphi_s c + O_n(1) + \lambda c_0|\Omega|.$$

Since $\varphi_s < 2\varphi_l$, $\beta > 0$ and $\lambda \geq 0$, it is clear that $\{\upsilon_n\}$ is bounded in X. Then

$$\upsilon_n \rightharpoonup \upsilon \quad \text{weakly in} \quad X. \tag{3.5}$$

By Sobolev embedding (2.2), we have the following compact embedding

$$X \hookrightarrow L^{s(x)}(\Omega) \quad \text{for} \quad 1 \leq s(x) < \varphi^*(x). \tag{3.6}$$

From (3.5) and (3.6), we can infer that

$$\upsilon_n \rightharpoonup \upsilon \quad \text{in} \quad X, \quad \upsilon_n \to \upsilon \quad \text{in} \quad L^{s(x)}(\Omega), \quad \upsilon_n(x) \to \upsilon(x), \quad \text{a.e. in} \quad \Omega. \tag{3.7}$$

Using Hölder inequality and (3.7), we have

$$\begin{aligned}\left|\int_\Omega |\upsilon_n|^{\psi(x)-2}\upsilon_n(\upsilon_n - \upsilon)\,dx\right| &\leq \int_\Omega |\upsilon_n|^{\psi(x)-1}|\upsilon_n - \upsilon|\,dx \\ &\leq \||\upsilon_n|^{\psi(x)-1}\|_{\frac{\psi(x)}{\psi(x)-1}}\|\upsilon_n - \upsilon\|_{\psi(x)} \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

thus

$$\int_\Omega |\upsilon_n|^{\psi(x)-2}\upsilon_n(\upsilon_n - \upsilon)\,dx \to 0, \quad \text{as} \quad n \to \infty. \tag{3.8}$$

By (η_1) and (η_2), we have that for each $\varepsilon \in (0,1)$, there is $c_\varepsilon > 0$ so that

$$|\eta(x, \upsilon_n)| \leq \varepsilon |\upsilon_n|^{\varphi(x)-1} + c_\varepsilon |\upsilon_n|^{r(x)-1}. \tag{3.9}$$

By Sobolev embedding (2.2), Hölder inequality and (3.9), we have

$$\begin{aligned}\left|\int_\Omega \eta(x,\upsilon_n)(\upsilon_n-\upsilon)\,dx\right| &\leq \int_\Omega \left(\varepsilon|\upsilon_n|^{\varphi(x)-1}|\upsilon_n-\upsilon| + c_\varepsilon|\upsilon_n|^{r(x)-1}|\upsilon_n-\upsilon|\right)dx \\ &\leq \varepsilon \||\upsilon_n|^{\varphi(x)-1}\|_{\frac{\varphi(x)}{\varphi(x)-1}}\|\upsilon_n-\upsilon\|_{\varphi(x)} + c_\varepsilon\varepsilon\||\upsilon_n|^{r(x)-1}\|_{\frac{r(x)}{r(x)-1}}\|\upsilon_n-\upsilon\|_{r(x)} \to 0,\end{aligned}$$

as $n \to \infty$. Therefore

$$\int_\Omega \eta(x,\upsilon_n)(\upsilon_n-\upsilon)\,dx \to 0, \quad \text{as} \quad n \to \infty. \tag{3.10}$$

From (3.4), we have $\langle J'_\lambda(\upsilon_n), \upsilon_n - \upsilon\rangle \to 0$, as $n \to \infty$, so, we can infer that

$$\begin{aligned}\left(\alpha - \beta\int_\Omega \frac{1}{\varphi(x)}|\nabla\upsilon_n|^{\varphi(x)}\,dx\right)\int_\Omega |\nabla\upsilon_n|^{\varphi(x)-2}\nabla\upsilon_n(\nabla\upsilon_n-\nabla\upsilon)\,dx \\ + \int_\Omega |\upsilon_n|^{\psi(x)-2}\upsilon_n(\upsilon_n-\upsilon)\,dx - \lambda\int_\Omega \eta(x,\upsilon_n)(\upsilon_n-\upsilon)\,dx \to 0.\end{aligned} \tag{3.11}$$

From (3.8), (3.10), (3.11), we can write

$$\left(\alpha - \beta\int_\Omega \frac{1}{\varphi(x)}|\nabla\upsilon_n|^{\varphi(x)}\,dx\right)\int_\Omega |\nabla\upsilon_n|^{\varphi(x)-2}\nabla\upsilon_n(\nabla\upsilon_n-\nabla\upsilon)\,dx \to 0, \quad \text{as} \quad n\to\infty. \tag{3.12}$$

Since $\{\upsilon_n\}$ is bounded in X, therefore, it is necessary for the following positive sequence to converge to a non-negative value such as υ_p, which means,

$$\int_\Omega \frac{1}{\varphi(x)}|\nabla\upsilon_n|^{\varphi(x)}\,dx \to \upsilon_p \geq 0, \quad \text{as} \quad n\to\infty.$$

Similar to the proof of Lemma 3.1 in [23], we have the sequence $\left\{\alpha - \beta\int_\Omega \frac{1}{\varphi(x)}|\nabla\upsilon_n|^{\varphi(x)}\,dx\right\}$ is bounded, when n is large enough. So, it follows from (3.12) that

$$\int_\Omega |\nabla\upsilon_n|^{\varphi(x)-2}\nabla\upsilon_n(\nabla\upsilon_n-\nabla\upsilon)\,dx \to 0,$$

as $n\to\infty$. So, by the (S_+) property (see Lemma 2.5), we get $|||\upsilon_n||| \to |||\upsilon|||$ (strongly) in X, that means J_λ ensures the $(C)_c$-condition. Moreover, considering the proof of Lemma 3.1, Lemma 3.2 and Remark 3.1 in [23], we deduce that the $(C)_c$-condition is satisfied for $c < \dfrac{\alpha^2}{2\beta}$. □

3.1 Proof of Theorem 1.1

Proof. It is clear that $J_\lambda(0) = 0$, by Lemma 3.3, J_λ ensures the $(C)_c$-condition where $c \in \left(-\infty, \frac{\alpha^2}{2\beta}\right)$. Considering Lemma 3.2, we prove that J_λ has the geometry of the Mountain pass theorem, thus, all the assumptions of Mountain pass theorem are satisfied, therefore, for each $\lambda < \lambda_0$, our problem has at least a nontrivial weak solution in X. □

4 Proof of Theorem 1.2

In this section, we will prove that problem (1.1) has many pairs of solutions by using the following Symmetric Mountain pass theorem.

Theorem 4.1 ([8])**.** *Let X be a real Banach space, and $J_\lambda \in C^1(X, \mathbb{R})$ that ensures the $(C)_c$-condition and $J_\lambda(0) = 0$ and J_λ be an even functional, such as*

(A) *there exist two constants $a, R > 0$, so that $J_\lambda(\upsilon) \geq a$ for each $u \in X$ with $|||\upsilon||| = R$,*

(B) *for each finite dimensional subspace $E \subset X$, there exists $R_E > 0$ so that $J_\lambda(\upsilon) \leq 0$ on $E \backslash B_R$.*

Then J_λ has a sequence of critical points $\{\upsilon_n\}$ such that $J_\lambda(\upsilon_n) \to +\infty$.

It is clear that for the even functional J_λ, we have $J_\lambda(0) = 0$ and by Lemma 3.3, J_λ ensures the $(C)_c$-condition where $c \in \left(-\infty, \frac{\alpha^2}{2\beta}\right)$. Therefore, it suffices to prove that the two conditions (A) and (B) of the Theorem 4.1 are true for the functional J_λ. On the other hand by the proof of Lemma 3.2 (a), where

$$a_0 = \frac{\alpha\varphi_l^2 - \beta\varphi_s R^{2\varphi_l - \varphi_s}}{2c_\varepsilon \left(c_{emb}^{r_l} + c_{emb}^{r_s}\right)\varphi_s\varphi_l^2 R^{r_l - \varphi_s}}$$

and $a = a_0 R^{\varphi_s}$ for each $\lambda \in (0, a_0)$, there is $a > 0$ so that for each $\upsilon \in X$ with $|||\upsilon||| = R$, we have $J_\lambda(\upsilon) \geq a > 0$. Thus, it suffices to consider only the condition (B).

We use the indirect proof method, thus assume that $\{\upsilon_n\} \subset E$ such that if $|||\upsilon_n||| \to +\infty$ as $n \to +\infty$, then there is $M \in \mathbb{R}$ so that it is a fixed constant, then

$$J_\lambda(\upsilon_n) \geq M, \quad \forall n \in N. \tag{4.1}$$

Now, for any $\upsilon_n \in E \subseteq X$, put $V_n := \frac{\upsilon_n}{|||\upsilon_n|||}$. It is clear that $|||V_n||| = 1$. On the other hand, since $\dim E < +\infty$, we have

$$\exists V \in E \backslash \{0\}; \quad |||V_n - V||| \to 0.$$

We can infer that

$$V_n(x) \to V(x) \quad \text{a.e.} \quad x \in \Omega, \quad \text{as} \quad n \to \infty,$$

since $V(x) \neq 0 \to |\upsilon_n(x)| \to +\infty$, as $n \to +\infty$, (by (4.1)).

By (η_3), we can infer that

$$\lim_{n\to+\infty} \frac{H(x,\upsilon_n(x))}{|||\upsilon_n|||^{\varphi_s}} = \lim_{n\to+\infty} \frac{H(x,\upsilon_n(x))}{|\upsilon_n(x)|^{\varphi_s}} |V_n(x)|^{\varphi_s} = +\infty,$$

for all $x \in \Omega_0 := \{x \in \Omega : V(x) \neq 0\}$ and by (η_4), there is s_0, such that

$$\frac{H(x,s)}{|s|^{\varphi_s}} > 1, \quad \forall x \in \Omega \quad \text{and} \quad |s| > s_0. \tag{4.2}$$

Now by (η_1), we can write

$$\exists C_2 > 0; \quad |H(x,s)| \leq C_2, \quad \forall (x,s) \in \Omega \times [-s_0, s_0]. \tag{4.3}$$

Using (4.2) and (4.3), we conclude that

$$\exists C_4 \in \mathbb{R}, \quad H(x,s) \geq C_4, \quad \forall (x,s) \in \Omega \times \mathbb{R}. \tag{4.4}$$

Thus

$$\frac{H(x,\upsilon_n) - C_4}{|||\upsilon_n|||^{\varphi_s}} \geq 0, \quad \forall x \in \Omega, \quad \forall n \in \mathbb{N}.$$

Then, we have

$$\frac{H(x,\upsilon_n)}{|\upsilon_n(x)|^{\varphi_s}} |V_n(x)|^{\varphi_s} - \frac{C_4}{|||\upsilon_n|||^{\varphi_s}} \geq 0, \quad \forall x \in \Omega, \quad \forall n \in \mathbb{N}. \tag{4.5}$$

Thus, by Poincaré inequality, (4.1) and (4.5), we can infer that

$$\begin{aligned} 0 &\leq \lim_{n\to+\infty} \frac{J_\lambda(\upsilon_n)}{|||\upsilon_n|||^{\varphi_s}} \\ &\leq \lim_{n\to+\infty} \left[\frac{\alpha \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon_n|^{\varphi(x)}\, dx + \int_\Omega \frac{1}{\psi(x)} |\upsilon_n|^{\psi(x)}\, dx}{|||\upsilon_n|||^{\varphi_s}} - \lambda \int_\Omega \frac{H(x,\upsilon_n)}{|||\upsilon_n|||^{\varphi_s}}\, dx \right]. \end{aligned}$$

Since $\psi_s > \varphi_s$, and $\lambda > 0$, we have

$$\begin{aligned} 0 &\leq \lim_{n\to+\infty} \left[\frac{\alpha \int_\Omega \frac{1}{\varphi(x)} |\nabla \upsilon_n|^{\varphi(x)}\, dx}{|||\upsilon_n|||^{\varphi_s}} + \frac{\int_\Omega \frac{1}{\psi(x)} |\upsilon_n|^{\psi(x)}\, dx}{|||\upsilon_n|||^{\psi_s}} - \lambda \int_\Omega \frac{H(x,\upsilon_n)}{|||\upsilon_n|||^{\varphi_s}}\, dx \right] \\ &\leq \frac{\alpha}{\varphi_l} + \frac{C_5}{\psi_s} - \lambda \lim_{n\to+\infty} \int_\Omega \frac{H(x,\upsilon_n) - C_4}{|||\upsilon_n|||^{\varphi_s}}\, dx \\ &\leq \frac{\alpha}{\varphi_l} + \frac{C_5}{\psi_s} - \lambda \liminf_{n\to+\infty} \int_{\Omega_0} \frac{H(x,\upsilon_n) - C_4}{|||\upsilon_n|||^{\varphi_s}}\, dx \\ &\leq \frac{\alpha}{\varphi_l} + \frac{C_5}{\psi_s} - \lambda \liminf_{n\to+\infty} \int_{\Omega_0} \frac{H(x,\upsilon_n)}{|\upsilon_n(x)|^{\varphi_s}} |V_n(x)|^{\varphi_s}\, dx \to -\infty, \end{aligned}$$

which is a contradiction. Then, the proof of (B) in the Theorem 4.1 is complete.

4.1 Proof of Theorem 1.2

Proof. Now, by Theorem 4.1, we can deduce that J_λ has a sequence of critical points $\{\upsilon_n\}$ such that $J_\lambda(\upsilon_n) \to +\infty$, thus, we prove that our problem has infinitely many weak solutions and the Theorem 1.2 is proven. □

Acknowledgement

The authors gratefully ackowledge the anonymous referees for their useful comments that lead to an improved version of the paper.

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VOLUME 26 · ISSUE 3 · DECEMBER 2024

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