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SUMMARY

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A note on Buell's theorem on length four Büchi sequences

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ABSTRACT

Büchi sequences are sequences whose second difference of squares is the sequence $(2,\dots,2)$, like for instance $(6,23,32,39)$ — so they can be seen as a generalization of arithmetic progressions. No (non-trivial) length 5 Büchi sequence is known to exist. Length four Büchi sequences were parameterized by D. A. Buell in 1987. We revisit his theorem, fixing the statement (about 26% of the Büchi sequences from R. G. E. Pinch's 1993 table were missed), and giving a much simpler proof.

RESUMEN

Las secuencias de Büchi son secuencias para las cuales la segunda diferencia de sus cuadrados es la sucesión $(2,\dots,2)$, como por ejemplo $(6,23,32,39)$ — luego pueden ser vistas como una generalización de las progresiones aritméticas. No se sabe de la existencia de ninguna secuencia de Büchi (no-trivial) de largo 5. Las secuencias de Büchi de largo 4 fueron parametrizadas por D. A. Buell en 1987. Revisitamos este teorema, corrigiendo el enunciado (faltan alrededor del 26% de las secuencias de Büchi de la tabla de R. G. E. Pinch de 1993), y dando una demostración bastante más simple.

Keywords and Phrases: Representation of systems of quadratic forms, Büchi's n-squares problem, second difference of squares.

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1 Introduction and result

Recall that the first (forward) difference of a sequence $(y_n)_n$ is the sequence $(y_{n+1} - y_n)_n$, so the second difference is $((y_{n+2} - y_{n+1}) - (y_{n+1} - y_n))_n = (y_{n+2} - 2y_{n+1} + y_n)_n$. A Büchi sequence is a sequence $(x_1, \dots, x_M)$ whose second difference of its sequence of squares is the constant sequence $(\dots, 2, \dots)$, namely, it is a sequence which satisfy the system of Büchi equations $x_{n+2}^2 - 2x_{n+1}^2 + x_n^2 = 2$, for $n = 1, \dots, M-2$. We call *trivial Büchi sequence* any such sequence such that $x_{n+1}^2 = (x_n \pm 1)^2$ for every $n = 1, \dots, M-1$. Büchi's problem asks whether there exists an M such that every Büchi sequence of integers of length M is trivial. It is not known whether any such M exists, and actually no non-trivial length 5 Büchi sequence of integers is known to exist. However, Büchi's problem has a positive answer, namely, an M can be proved to exist, under some classical conjectures in Number Theory — see [11] and [6]. For a general survey on Büchi's problem and variations, see [5] and the references therein.

Length 3 Büchi sequences of integers were characterized by D. Hensley [2,3] through a parametrization in two variables coming from the line and circle method, and later by P. Sáez and the second author [8] using matrices. In [1], D. A. Buell builds on Hensley's parametrization to find a parametrized family, say by a pair (k, ℓ) of integers, of quadratic equations whose solutions correspond to length 4 Büchi sequences of integers (BS4 in the sequel) — see Equation (1.1) below. As J. Lipman pointed out in [4, page 4], it is not clear how to characterize the pairs (k, ℓ) for which the equation is solvable.

See [7], [10] and [9] for other approaches to the problem of understanding the BS4.

In this short note, we fix two mistakes in the statement of the original theorem — see the comments before the proof — and give a much simpler and more transparent proof.

Theorem 1.1 (D. A. Buell, 1987, *revisited*)**.** *A sequence $\sigma = (x_1, \dots, x_4)$ is a Büchi sequence of integers if and only if there exist coprime integers k and ℓ of opposite parity, an integer x, and a rational number y such that $3y \in \mathbb{Z}$, which satisfy*

$$\begin{cases} x_1 &= x(-2\ell + 3k) + y(-3\ell + 6k) \\ x_2 &= x(-\ell + 2k) + y(-2\ell + 3k) \\ x_3 &= xk + y\ell \\ x_4 &= x\ell + 3yk \end{cases}$$

and

$$(\ell - k)^2 x^2 + (2\ell^2 - 6k\ell + 6k^2)xy + (\ell - 3k)^2 y^2 = 1. \tag{1.1}$$

The proof below allows to find easily some of the possible parameters k and ℓ from a given BS4

— this was our original motivation, as this is not clear how to do it from [1]. This is also how we realized that the possibility of having a 3 in the denominator of the y cannot be dropped, as can be seen with the Büchi sequence $(16, 87, 122, 149)$, for which $yk = -\frac{40}{3}$. Indeed, about 26% of the sequences with some entry at most 1000 need a 3 in the denominator (see [7] for the list). This phenomenon was overlooked in Buell's statement, though one could detect it while going through his intricate proof: his quotient $\frac{a+t}{\ell-3k}$, line 4 before the Theorem, can actually have a 3 in the denominator. The other issue in Buell's original statement has to do with trivial sequences, which cannot be put aside in the statement, as our proof shows.

Proof. If direction. When computing the second difference of squares of the x_i, one obtains the left hand-side of Equation (1.1) multiplied by two. So σ is a Büchi sequence. If y is an integer, there is nothing else to prove. Otherwise, replacing y by $\frac{y'}{3}$ in Equation (1.1), then multiplying by 9 and taking modulo 3, we see that 3 divides ℓ, so the x_i are indeed integers.

Only if direction. Assume that $(x_1, \ldots, x_4)$ is a Büchi sequence of integers. The idea is to pretend that $\omega_1 := xk$ is a variable, as well as $\omega_2 := x\ell$, $\omega_3 := yk$ and $\omega_4 := y\ell$, so that the system of the statement can be seen as a linear system:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 6 & -3 \\ 2 & -1 & 3 & -2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} xk \\ x\ell \\ yk \\ y\ell \end{pmatrix}. \tag{1.2}$$

By inverting the system we get:

$$\begin{cases} 2\omega_1 & = -x_1 + 2x_2 + x_3 \\ 2\omega_2 & = -2x_1 + 3x_2 + x_4 \\ 6\omega_3 & = 2x_1 - 3x_2 + x_4 \\ 2\omega_4 & = x_1 - 2x_2 + x_3. \end{cases} \tag{1.3}$$

Observe that, since x_i and x_{i+1} have opposite parity for each i (which can be easily seen from the Büchi equations), ω_1, ω_2, $3\omega_3$ and ω_4 are integers.

If $\omega_1 = \omega_2 = 0$, then one can choose $x = 0$, and $y = 1$, $\ell = x_3$, $k = \frac{x_4}{3}$ if 3 divides x_4, and $y = \frac{1}{3}$, $\ell = 3x_3$ and $k = x_4$ if not. From (1.3), we get $x_2 + 2x_3 - x_4 = 0$, which, together with the Büchi equation $x_4^2 = 2x_3^2 - x_2^2 + 2$ gives $(x_2 + x_3)^2 = 1$, hence the sequence is trivial. Similarly, if $\omega_3 = \omega_4 = 0$, then one can choose $y = 0$, $x = 1$, $k = x_3$ and $\ell = x_4$, and again the sequence is trivial. Since in both cases the sequence is trivial, we have $x_4 = \pm x_3 \pm 1$, so in particular, k and ℓ are coprime and of opposite parity. One readily checks that (1.2) and (1.1) are satisfied in both cases.

We assume now that $(\omega_1, \omega_2) \neq (0, 0)$ and $(\omega_3, \omega_4) \neq (0, 0)$. A direct computation gives

$$12(\omega_1\omega_4 - \omega_2\omega_3) = x_1^2 - 3x_2^2 + 3x_3^2 - x_4^2 = (x_1^2 - 2x_2^2 + x_3^2) - (x_2^2 - 2x_3^2 + x_4^2) = 0,$$

so we have

$$\omega_1\omega_4 = \omega_2\omega_3. \tag{1.4}$$

Hence $\omega_1 = 0$ if and only if $\omega_3 = 0$, in which case we choose $k = 0$, $\ell = 1$, $x = \omega_2$ and $y = \omega_4$, so that $xk = \omega_1$, $x\ell = \omega_2$, $yk = \omega_3$ and $y\ell = \omega_4$. Similarly, $\omega_2 = 0$ if and only if $\omega_4 = 0$, in which case we choose $\ell = 0$, $k = 1$, $x = \omega_1$ and $y = \omega_3$, so that $xk = \omega_1$, $x\ell = \omega_2$, $yk = \omega_3$ and $y\ell = \omega_4$.

Assume that $\omega_1\omega_2\omega_3\omega_4 \neq 0$. Let ε be the sign of $\omega_1\omega_3$. Choose $x = \varepsilon \gcd(\omega_1, \omega_2)$, $k = \frac{\omega_1}{x}$, $\ell = \frac{\omega_2}{x}$ (so k and ℓ are coprime integers), and $y = \frac{y'}{3}$, where $y' = \gcd(3\omega_3, 3\omega_4)$. Note that if both ω_1 and ω_3 are positive, then we obtain

$$3\omega_3 \gcd(\omega_1, \omega_2) = \gcd(3\omega_1\omega_3, 3\omega_2\omega_3) = \gcd(3\omega_1\omega_3, 3\omega_1\omega_4) = \omega_1 \gcd(3\omega_3, 3\omega_4).$$

In general, we have $3\omega_3 \gcd(\omega_1, \omega_2) = \varepsilon\omega_1 \gcd(3\omega_3, 3\omega_4)$, hence

$$3\omega_3 = \frac{\varepsilon\omega_1}{\gcd(\omega_1, \omega_2)} \times \gcd(3\omega_3, 3\omega_4) = ky'$$

hence $\omega_3 = yk$. Since $\omega_1 \neq 0$, we have $\omega_4 = \frac{\omega_2\omega_3}{\omega_1} = \frac{x\ell \cdot yk}{xk} = y\ell$. By inverting the system (1.3), we see that the system (1.2) is satisfied.

Equation (1.1) comes from replacing the x_i in $x_4^2 - 2x_3^2 + x_2^2 = 2$ (for instance) by their expression in terms of x, y, k and ℓ. Equation (1.1) implies immediately that k and ℓ cannot have the same parity. □

While working on this note, we realized that the solutions of (1.1) with $k = \ell + 1$, described in Section 5 of [1], are precisely the BS4 that were found by the second author in [10] with a different method.

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Minkowski type inequalities for a generalized fractional integral

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ABSTRACT

In this paper we introduce a new generalized fractional integral unifying most of previous existing fractional integrals. Then, we prove some essential properties of this new operator under some classical assumptions. As application, we use this novel fractional integral to establish a several inequalities of Minkowski type. Our results recover a large number of a well known inequalities in the literature.

RESUMEN

En este artículo introducimos una nueva integral fraccionaria generalizada, que unifica la mayoría de las integrales fraccionarias existentes. Luego demostramos algunas propiedades esenciales de este nuevo operador bajo algunas suposiciones clásicas. Como aplicación, usamos esta nueva integral fraccionaria para establecer varias desigualdades de tipo Minkowski. Nuestros resultados recuperan un amplio número de desigualdades bien conocidas en la literatura.

Keywords and Phrases: Fractional calculus, fractional integral, Riemann-Liouville integral, reverse Minkowski inequality.

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1 Introduction

Fractional calculus has been the subject of a lot of works during the last years. Fractional models has been used to diverse problems in various domains of science, see [43]. In fact, it is mainly used in modeling different phenomena, as mechanics [6], economy [45], human body modeling [14], visco-elasticity [18, 30], biology [28], circuits [31], material sciences [41], porous-medium equations [36] and many other domains. In order to modeling such problems, different integral operators or differential operators were defined. Nevertheless, some of fractional operators defined with a special "kernel" are used only in some cases. In [27], the authors defined a fractional integral according to another function ψ as a general integral. Choosing a particular function ψ, we obtain a pre-existing non-integer integral. This allows us to select the most adapted integral for proving the result under examination.

In [47], Sousa-Oliveira defined a new fractional-derivative according to another function; the "ψ-Hilfer fractional derivative". They proved many interesting properties and they presented also a large number of integrals and derivatives as a special cases of the ψ-Hilfer derivative and the integral according to another function.

In [25], Katugampola defined the following new fractional integral

$$({}^{\rho}I^{\alpha,\beta}_{a+;\eta,k}f)(x) = \frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{a}^{x}\frac{u^{\rho(1+\eta)-1}}{(x^{\rho}-u^{\rho})^{1-\alpha}}f(u)du.$$

He proved that the above integral unifies six pre-existing fractional integrals.

With the numerous propositions of fractional derivatives and integrals, it was very important to propose a new definition of fractional integral that unifies most of the pre-existing definitions. The new generalized ψ-fractional integral proposed in this paper, will be the first step in order to obtain a single general model, which can be used to different problems and to prove different results only for this general model, rather than proving similar results each time in each different model. In the first part of this paper, our purpose is to define this new fractional integral. Then, we prove some important properties to justify the originality of this new generalization. Among other, we show that the new operator is well defined, bounded and satisfies the semigroup property.

As application of the numerous fractional integrals proposed in the last years, a large number of works are interested to several important inequalities for different definitions of fractional integrals. See for example [2, 8, 19, 48, 49] for the Ostrowski type inequalities, [7, 10, 16, 20, 40] for the Grüss type inequalities, [4, 11, 12, 17, 26, 35] for the Hermite-Hadamard type inequalities, [21, 23, 34, 42, 46] for the Čebyšev type inequalities, [5, 13, 15, 17, 32, 33, 37–39, 44] for the Minkowski type inequalities and many others, (see [3, 29]). Such types of inequalities are very important in different areas of science, (see [32, 38]).

Motivated by the above large literature and as application of the new generalized ψ-fractional integral defined in the first part, our second aim in this work is to generalize the Minkowski type inequalities using the new generalized ψ- fractional integral. Our results recover the Minkowski type inequalities proved in [5, 13, 15, 17, 33, 37–39] and [44]. Then, we prove different other inequalities related to the Minkowski's inequalities.

The remainder of the paper is organized as follows. In the next section we present the definition of the new generalized ψ-fractional integral and some examples. In section three we give some principal properties of this new operators. In section four, we prove the main results related to Minkowski inequality and in the last section, we prove other inequalities related to the new fractional integral.

2 Definition and examples

Definition 2.1 ([24])**.** *Let $f \in L_1(a,b)$ and ψ be a positive function such that its derivative is continuous and satisfying $\psi'(x) > 0, \forall x \in (a,b)$. For $1 \leq p < \infty$, we denote*

$$X^p_\psi(a,b) := \{f : (a,b) \to \mathbb{R},\ \text{Lebesgue-measurable s.t. } \|f\|_{X^p_\psi} < \infty\},$$

where

$$\|f\|^p_{X^p_\psi} = \int_a^b |f(s)|^p \psi'(s) ds.$$

For $p = \infty$,

$$\|f\|_{X^\infty_\psi} = \operatorname*{ess\,sup}_{s\in(a,b)} |\psi'(s) f(s)|.$$

When $\psi(s) = s$, the space $X^p_\psi(a,b)$, $(1 \leq p < \infty)$, is identical to $L_p(a,b)$.

Definition 2.2. *For $1 \leq p \leq \infty$, let $f \in X^p_\psi(a,b)$ and ψ as defined in the previous Definition 2.1. For $\alpha > 0$, $\beta, \gamma, \eta, k, \rho, \in \mathbb{R}$, we define the following new generalized ψ-fractional integral, (left side and right side), by*

$$I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x) = \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta} \int_a^x [\psi(u)]^\eta \psi'(u) \exp\big(-\gamma\psi(u)\big)\big(\psi(x)-\psi(u)\big)^{\alpha-1} f(u) du \quad (2.1)$$

$$I^{\alpha,\beta;\psi}_{b-;\eta,k,\gamma,\rho} f(x) = \frac{[\psi(x)]^\eta \exp(-\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta} \int_x^b [\psi(u)]^k \psi'(u) \exp\big(\gamma\psi(u)\big)\big(\psi(u)-\psi(x)\big)^{\alpha-1} f(u) du \quad (2.2)$$

Remark 2.3. *Most of the pre-existing fractional integrals are a particular cases of integrals* (2.1) *and* (2.2)*. For example, if $\psi(x) = x$, $\alpha > 0$, $\gamma = 0$, $k = 0$, $\eta = 0$, $\rho > 0$, $\beta = 0$, then we obtain the integral of Riemann Liouville (left sided). For $a = -\infty$, we obtain the integral of Liouville ${}^L I^\alpha_+ f(x)$. If $a = 0$ then* (2.1) *is the analogue of the integral of Riemann ${}^R I^\alpha_+ f(x)$. For a general case of function ψ,* (2.1) *is reduced to the integral of Riemann Liouville according to a function ψ,*

$$^{RL}I_{a+}^{\alpha;\psi}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\alpha-1}f(s)ds.$$

If $\psi(x) = \ln x$, $\alpha > 0$, $\gamma = 0$, $k = 0$, $\eta = 0$, $\rho > 0$, $\beta = 0$ then (2.1) is reduced to the integral of Hadamard $^{H}I_{a+}^{\alpha}f(x)$ and for $\gamma \in \mathbb{R}$ and $a = 0$, we get the integral of Hadamard type (called also Butzer et al. integral),

$$^{H}I_{a+;\gamma}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \left(\frac{u}{x}\right)^{-\gamma}\left(\ln\left(\frac{x}{u}\right)\right)^{\alpha-1}f(u)\frac{du}{u}.$$

If $\psi(x) = x^{\rho}$, $\alpha > 0$, $\gamma = 0$, $k = -\alpha-\eta$, $\rho > 0$, $\beta = 0$, then we get the fractional-integral of "Erdélyi-Kober",

$$^{EK}I_{a+;\eta,\rho}^{\alpha}f(x) = \frac{\rho x^{-\rho(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x f(s)s^{\rho(1+\eta)-1}\big(x^{\rho}-s^{\rho}\big)^{\alpha-1}ds.$$

For $a = 0$, we get the fractional-integral of "Erdélyi" $^{E}I_{a+;\eta,\rho}^{\alpha}f(x)$ and for $\rho = 1$, $a = 0$, we get the fractional-integral of "Kober", $^{K}I_{a+;\eta,\rho}^{\alpha}f(x)$.

If $\psi(x) = x^{\rho}$, $\alpha > 0, \rho \in \mathbb{R}$, $\gamma = 0$, $\beta = \alpha$, $\eta = k = 0$, then (2.1) is reduced to "Katugampola" integral, and for $\eta \in \mathbb{R}$, $\beta \in \mathbb{R}$, $k = s/\rho$, we get the "generalized Katugampola" fractional integral

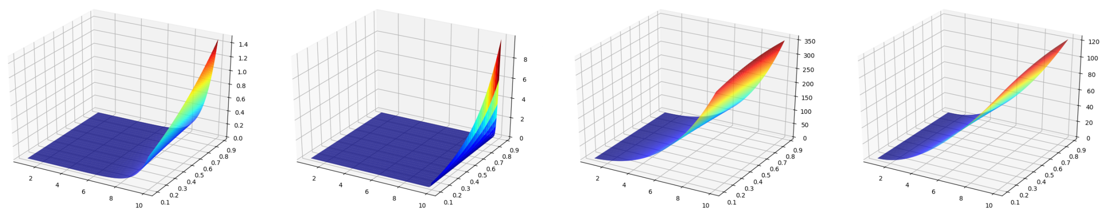
$$^{\rho}I_{a+;\eta,s}^{\alpha,\beta}f(x) = \frac{x^{s}}{\Gamma(\alpha)\rho^{\beta-1}}\int_a^x u^{\rho(1+\eta)-1}\big(x^{\rho}-u^{\rho}\big)^{\alpha-1}f(u)du.$$

If $\psi(x) = x$, $\gamma = \frac{\rho-1}{\rho}$, $\alpha > 0$, $k = \eta = 0$, $\rho \in (0,1]$, $\beta = \alpha$, we obtain the fractional (left sided) generalized proportional integral $I_{a+;\rho}^{\alpha}f(x)$ (Jarad-Abdeljawad-Alzabut integral) and for a general case of ψ, we obtain the fractional (left sided) proportional integral in the general form according to a function ψ,

$$I_{a+;\rho}^{\alpha;\psi}f(x) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^x \psi'(u)\exp\left[\frac{\rho-1}{\rho}\big(\psi(x)-\psi(s)\big)\right]\big(\psi(x)-\psi(s)\big)^{\alpha-1}f(s)ds.$$

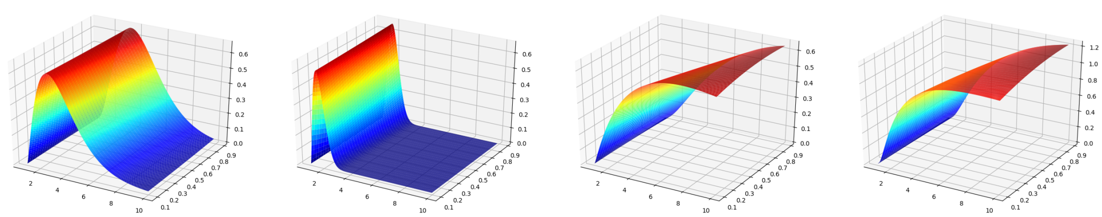
If $\psi(x) = \frac{x^{\rho+r}}{\rho+r}$, $\gamma = 0$, $\alpha > 0$, $k = \eta = \beta = 0$, $\rho \in (0,1]$, and $r \in \mathbb{R}$, we obtain the generalized conformable fractional (left sided) integrals, $^{r}K_{a+;\rho}^{\alpha}f(x)$. If $\psi(x) = \ln x$, $\gamma = \frac{\rho-1}{\rho}$, $\alpha > 0$, $k = \eta = 0$, $\rho \in (0,1]$, $\beta = \alpha$, we obtain the generalized proportional integral of "Hadamard" (left sided), $I_{a+;\rho}^{\alpha;\psi}f(x)$.

In the following, we plot some examples of the new ψ-fractional integral of the function $f(x)$ from Theorem 3.2, in the case $\rho = 0.5$, $\beta = 1$, $k = 1$ for a different example of ψ and different values of γ. The first two figures plot the expression of $I_{a+;\eta,k,\gamma,\rho}^{\alpha,\beta;\psi}f(x)$ against the variables x and α. The third and fourth figures plot the expression of $I_{a+;\eta,k,\gamma,\rho}^{\alpha,\beta;\psi}f(x)$ against the variables x and λ. Since the fractional integral of the above mentioned function $f(x) = [\psi(x)]^{-\eta}\exp\big(\gamma\psi(x)\big)\big(\psi(x)-\psi(a)\big)^{\lambda-1}$ is the solution of many well known fractional differential equations (see [43]), each figure is the solution of a specific differential equation. This fact will be the subject of a forthcoming work.



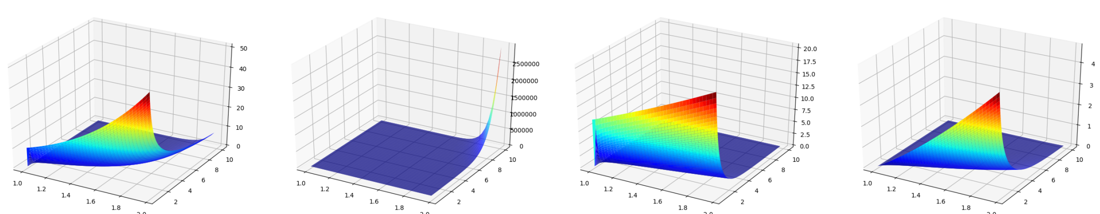
$\psi(x) = x$ $\psi(x) = x^2$ $\psi(x) = \sqrt{x}$ $\psi(x) = \log(x)$

Figure 1: ψ-fractional integral $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)$ where $f(x) = [\psi(x)]^{-\eta} \exp\big(\gamma\psi(x)\big)\big(\psi(x) - \psi(a)\big)^{\lambda-1}$ with $\gamma = 1$, $\lambda = 2$, $1 \leq x \leq 10$ and $0.1 \leq \alpha \leq 0.9$.



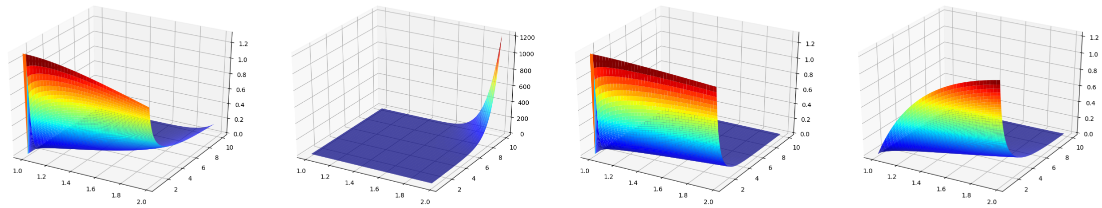
$\psi(x) = x$ $\psi(x) = x^2$ $\psi(x) = \sqrt{x}$ $\psi(x) = \log(x)$

Figure 2: ψ-fractional integral $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)$ where $f(x) = [\psi(x)]^{-\eta} \exp\big(\gamma\psi(x)\big)\big(\psi(x) - \psi(a)\big)^{\lambda-1}$ with $\gamma = -1$, $\lambda = 2$, $0.1 \leq \alpha \leq 0.9$ and $1 \leq x \leq 10$.



$\psi(x) = x$ $\psi(x) = x^2$ $\psi(x) = \sqrt{x}$ $\psi(x) = \log(x)$

Figure 3: ψ-fractional integral $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)$ where $f(x) = [\psi(x)]^{-\eta} \exp\big(\gamma\psi(x)\big)\big(\psi(x) - \psi(a)\big)^{\lambda-1}$ with $\gamma = 1$, $\alpha = 0.5$, $1 \leq x \leq 2$ and $0.5 \leq \lambda \leq 10$.



$\psi(x) = x$ $\quad$ $\psi(x) = x^2$ $\quad$ $\psi(x) = \sqrt{x}$ $\quad$ $\psi(x) = \log(x)$

Figure 4: ψ-fractional integral $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)$ where $f(x) = [\psi(x)]^{-\eta} \exp\big(\gamma\psi(x)\big)\big(\psi(x) - \psi(a)\big)^{\lambda-1}$ with $\gamma = -1$, $\alpha = 0.5$, $1 \leq x \leq 2$ and $0.5 \leq \lambda \leq 10$.

3 Main properties of the new generalized ψ-fractional integral

We present in this section some essential properties of the new generalized ψ-fractional integral. First, we give some elementary properties having obvious proofs.

Theorem 3.1. *Let $\alpha > 0$, $\beta, \gamma, k, \eta, \rho, \in \mathbb{R}$ and ψ as defined in Definition 2.1. For $1 \leq p \leq \infty$ and $f \in X^p_\psi(a,b)$, we have the following properties:*

- $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} \psi(x)^\lambda f(x) = I^{\alpha,\beta;\psi}_{a+;\eta+\lambda,k,\gamma,\rho} f(x)$,
- $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} \exp(\lambda\psi(x))\; f(x) = \exp(\lambda\psi(x))\; I^{\alpha,\beta;\psi}_{a+;\eta+\lambda,k,\gamma-\lambda,\rho} f(x)$,
- $I^{\alpha,\beta;\psi}_{b-;\eta,k,\gamma,\rho} \psi(x)^\lambda f(x) = I^{\alpha,\beta;\psi}_{b-;\eta,k+\lambda,\gamma,\rho} f(x)$,
- $I^{\alpha,\beta;\psi}_{b-;\eta,k,\gamma,\rho} \exp(\lambda\psi(x))\; f(x) = \exp(\lambda\psi(x))\; I^{\alpha,\beta;\psi}_{b-;\eta+\lambda,k,\gamma+\lambda,\rho} f(x)$.

Theorem 3.2. *Let $\alpha > 0$, $\beta, \gamma, k, \eta, \rho, \in \mathbb{R}$ and ψ as defined in Definition 2.1. We have*

- $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} [\psi(t)]^{-\eta} \exp\big(\gamma\psi(t)\big)\Big(\psi(t) - \psi(a)\Big)^{\lambda-1} = \dfrac{\Gamma(\lambda)[\psi(t)]^k \exp\big(\gamma\psi(t)\big)}{\Gamma(\lambda+\alpha)\rho^\beta}\Big(\psi(t) - \psi(a)\Big)^{\lambda+\alpha-1}$,
- $I^{\alpha,\beta;\psi}_{b-;\eta,k,\gamma,\rho} [\psi(t)]^{-k} \exp\big(-\gamma\psi(t)\big)\Big(\psi(b) - \psi(t)\Big)^{\lambda-1} = \dfrac{\Gamma(\lambda)[\psi(t)]^\eta \exp\big(-\gamma\psi(t)\big)}{\Gamma(\lambda+\alpha)\rho^\beta}\Big(\psi(b) - \psi(t)\Big)^{\lambda+\alpha-1}$.

Next we prove that for a positive increasing function ψ on (a, b), the new ψ-fractional operator $I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}$ is well-defined and bounded on the space $X^p_\psi(a,b)$.

Theorem 3.3. *Let $\gamma > 0$, $\alpha > 0$, $\beta, k, \eta, \rho \in \mathbb{R}$ and ψ as defined in Definition 2.1. For $1 \leq p \leq \infty$ and $f \in X^p_\psi(a,b)$, we have*

$$\left\| I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f \right\|_{X^p_\psi} \leq K \|f\|_{X^p_\psi},$$

where

$$K = \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta}[\psi(b)]^{\alpha+\eta+k} \int_{\psi^{-1}(1)}^{\psi^{-1}\left(\frac{\psi(b)}{\psi(a)}\right)} [\psi(u)]^{-\alpha-\eta}\big[\psi(u) - 1\big]^{\alpha-1} \psi'(u)du.$$

Proof. Let $1 \leq p < \infty$. Using (2.1) and Definition 2.1, we have

$$\begin{aligned}
\left\| I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f \right\|_{X^p_\psi} &= \left(\int_a^b \left| \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta} \int_a^x \psi'(u)[\psi(u)]^\eta \exp(-\gamma\psi(u)) \right.\right. \\
&\qquad \left.\left. \times \big(\psi(x)-\psi(u)\big)^{\alpha-1} f(u)du \right|^p \psi'(x)dx \right)^{\frac{1}{p}} \\
&= \frac{1}{\Gamma(\alpha)\rho^\beta} \left(\int_a^b \left| \int_a^x [\psi(x)]^k \exp\big(\gamma(\psi(x)-\psi(u))\big)[\psi(u)]^\eta \psi'(u)\big(\psi(x)-\psi(u)\big)^{\alpha-1} f(u)du \right|^p \psi'(x)dx \right)^{\frac{1}{p}} \\
&\leq \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta} \left(\int_a^b \left| \int_a^x [\psi(x)]^k \psi'(u)[\psi(u)]^{\eta+\alpha-1} \left(\frac{\psi(x)}{\psi(u)} - 1\right)^{\alpha-1} f(u)du \right|^p \psi'(x)dx \right)^{\frac{1}{p}}.
\end{aligned}$$

If we suppose $\frac{\psi(x)}{\psi(u)} = \psi(s)$, we get

$$\begin{aligned}
\left\| I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f \right\|_{X^p_\psi} \leq \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta} &\left(\int_a^b \left| \int_{\psi^{-1}(1)}^{\psi^{-1}\left(\frac{\psi(x)}{\psi(a)}\right)} [\psi(x)]^{k+\eta+\alpha} \psi'(s)[\psi(s)]^{-1-\eta-\alpha} \big[\psi(s)-1\big]^{\alpha-1} \right.\right. \\
&\left.\left. \times f\left(\psi^{-1}\left(\frac{\psi(x)}{\psi(s)}\right)\right) ds \right|^p \psi'(x)dx \right)^{\frac{1}{p}}.
\end{aligned}$$

Using the generalized Minkowski-inequality ([1]), we obtain

$$\begin{aligned}
\left\| I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f \right\|_{X^p_\psi} &\leq \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta} \int_{\psi^{-1}(1)}^{\psi^{-1}\left(\frac{\psi(b)}{\psi(a)}\right)} (\psi(s))^{-\eta-\alpha} \big(\psi(s)-1\big)^{\alpha-1} \psi'(s)[\psi(b)]^{\eta+k+\alpha} \\
&\qquad \times \left(\int_{\psi^{-1}(\psi(a)\psi(t))}^b \frac{\psi'(x)}{\psi(s)} \left| f\left(\psi^{-1}\left(\frac{\psi(x)}{\psi(s)}\right)\right) \right|^p dx \right)^{\frac{1}{p}} ds \\
&\leq \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta} \int_{\psi^{-1}(1)}^{\psi^{-1}\left(\frac{\psi(b)}{\psi(a)}\right)} (\psi(s))^{-\eta-\alpha} \big(\psi(s)-1\big)^{\alpha-1} \psi'(s)[\psi(b)]^{k+\eta+\alpha} \left(\int_a^{\psi^{-1}\left(\frac{\psi(b)}{\psi(s)}\right)} \Big| f(t) \Big|^p \psi'(t)dt \right)^{\frac{1}{p}} ds \\
&\leq K \|f\|_{X^p_\psi},
\end{aligned}$$

where

$$K = \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta} [\psi(b)]^{k+\eta+\alpha} \int_{\psi^{-1}(1)}^{\psi^{-1}\left(\frac{\psi(b)}{\psi(a)}\right)} (\psi(u))^{-\eta-\alpha} \big(\psi(u)-1\big)^{\alpha-1} \psi'(u)du.$$

Thus, the result is proved for $1 \leq p < \infty$. For $p = \infty$, we have

$$\begin{aligned}
\left\| I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f \right\|_{X^\infty_\psi} &= \operatorname*{ess\,sup}_{t\in(a,b)} \left| \psi'(t)\, I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(t) \right| \\
&\leq \frac{[\psi(b)]^k \exp(\gamma\psi(b))}{\Gamma(\alpha)\rho^\beta} \int_a^x [\psi(u)]^\eta \psi'(x) \big(\psi(x)-\psi(u)\big)^{\alpha-1} \Big| \psi'(u) f(u) \Big| du \\
&\leq \frac{\exp\big(\gamma\psi(b)\big)}{\Gamma(\alpha)\rho^\beta} [\psi(b)]^{k+\eta+\alpha} \int_{\psi^{-1}(1)}^{\psi^{-1}\left(\frac{\psi(b)}{\psi(a)}\right)} [\psi(s)]^{-\eta-\alpha} \big(\psi(s)-1\big)^{\alpha-1} \psi'(s)ds \, \|f\|_{X^\infty_\psi}.
\end{aligned}$$

Theorem 3.3 is thereby proved. □

In the next result, we prove that the new generalized ψ-fractional integral satisfies the property of semigroup.

Theorem 3.4. *Let $\alpha > 0$, $\beta, k, \eta, \rho, \gamma \in \mathbb{R}$ and ψ as defined in Definition 2.1. For $1 \le p \le \infty$ and $f \in X^p_\psi(a,b)$, we have:*

$$I^{\alpha_1,\beta_1;\psi}_{a+;\eta_1,k_1,\gamma,\rho}\, I^{\alpha_2,\beta_2;\psi}_{a+;\eta_2,-\eta_1,\gamma,\rho} f(x) = I^{\alpha_1+\alpha_2,\beta_1+\beta_2;\psi}_{a+;\eta_2,k_1,\gamma,\rho} f(x),$$
$$I^{\alpha_1,\beta_1;\psi}_{b-;\eta_1,-\eta_2,\gamma,\rho}\, I^{\alpha_2,\beta_2;\psi}_{b-;\eta_2,k_2,\gamma,\rho} f(x) = I^{\alpha_1+\alpha_2,\beta_1+\beta_2;\psi}_{b-;\eta_1,k_2,\gamma,\rho} f(x).$$

Proof. Using Definition 2.1, we have

$$\begin{aligned}
I^{\alpha_1,\beta_1;\psi}_{a+;\eta_1,k_1,\gamma,\rho} I^{\alpha_2,\beta_2;\psi}_{a+;\eta_2,k_2,\gamma,\rho} f(x) &= \frac{[\psi(x)]^{k_1}\exp(\gamma\psi(x))}{\Gamma(\alpha_1)\rho^{\beta_1}} \int_a^x [\psi(t)]^{\eta_1} \exp(-\gamma\psi(t))\big(\psi(x)-\psi(t)\big)^{\alpha_1-1} \\
&\quad \times \frac{[\psi(t)]^{k_2}\exp(\gamma\psi(t))}{\Gamma(\alpha_2)\rho^{\beta_2}} \psi'(t) \int_a^t \psi'(s)[\psi(s)]^{\eta_2}\exp(-\gamma\psi(s))\big[\psi(t)-\psi(s)\big]^{\alpha_2-1} f(s)ds\, dt \\
&= \frac{[\psi(x)]^{k_1}\exp(\gamma\psi(x))}{\Gamma(\alpha_1)\Gamma(\alpha_2)\rho^{\beta_1+\beta_2}} \int_a^x \psi'(s)[\psi(s)]^{\eta_2}\exp(-\gamma\psi(s)) f(s) \int_s^x \psi'(t)[\psi(t)]^{\eta_1+k_2} \\
&\quad \times \big(\psi(x)-\psi(t)\big)^{\alpha_1-1}\big(\psi(t)-\psi(s)\big)^{\alpha_2-1} dt\, ds.
\end{aligned}$$

For $k_2 = -\eta_1$ and supposing that $u := \dfrac{\psi(t)-\psi(s)}{\psi(x)-\psi(s)}$, we derive that

$$\begin{aligned}
&\int_s^x (\psi(t))^{\eta_1+k_2}\psi'(t)\big(\psi(x)-\psi(t)\big)^{\alpha_1-1}\big(\psi(t)-\psi(s)\big)^{\alpha_2-1} dt \\
&\qquad = \big(\psi(x)-\psi(s)\big)^{\alpha_1+\alpha_2-1}\int_0^1 (1-u)^{\alpha_1-1}u^{\alpha_2-1}du \\
&\qquad = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\big(\psi(x)-\psi(s)\big)^{\alpha_1+\alpha_2-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&I^{\alpha_1,\beta_1;\psi}_{a+;\eta_1,k_1,\gamma,\rho}\, I^{\alpha_2,\beta_2;\psi}_{a+;\eta_2,-\eta_1,\gamma,\rho} f(x) \\
&\qquad = \frac{[\psi(x)]^{k_1}\exp(\gamma\psi(x))}{\Gamma(\alpha_1+\alpha_2)\rho^{\beta_1+\beta_2}} \int_a^x \psi'(s)[\psi(s)]^{\eta_2}\exp(-\gamma\psi(s))\big(\psi(x)-\psi(s)\big)^{\alpha_1+\alpha_2-1} f(s)ds \\
&\qquad = I^{\alpha_1+\alpha_2,\beta_1+\beta_2;\psi}_{a+;\eta_2,k_1,\gamma,\rho} f(x).
\end{aligned}$$

The first identity in Theorem 3.4 is thereby proved. The second one follows using the same arguments. □

Theorem 3.5. *Let $\alpha > 0$, $\beta, k, \eta, \rho, \gamma \in \mathbb{R}$ and ψ as defined in Definition 2.1. For $1 \le p \le \infty$, we have*

$$\int_a^b f(u)\Big[I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g\Big](u)\psi'(u)du = \int_a^b g(u)\Big[I^{\alpha,\beta;\psi}_{b-;\eta,k,\gamma,\rho} f\Big](u)\psi'(u)du.$$

Proof. We have

$$\int_a^b f(u)\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}g\Big)(u)\psi'(u)du = \int_a^b f(u)\psi'(u)\frac{[\psi(u)]^k\exp(\gamma\psi(u))}{\Gamma(\alpha)\rho^\beta}$$
$$\times\int_a^u [\psi(t)]^\eta \exp\big(-\gamma\psi(t)\big)\big(\psi(u)-\psi(t)\big)^{\alpha-1}g(t)\psi'(t)dt\,du$$
$$=\int_a^b g(t)\frac{[\psi(t)]^\eta\exp(-\gamma\psi(t))}{\Gamma(\alpha)\rho^\beta}\psi'(t)\int_t^b [\psi(u)]^k\psi'(u)\exp\big(\gamma\psi(u)\big)\big(\psi(u)-\psi(t)\big)^{\alpha-1}f(u)du\,dt$$
$$=\int_a^b \psi'(u)g(u)\Big(I^{\alpha,\beta;\psi}_{b-;\eta,k,\gamma,\rho}f\Big)(u)du. \qquad\square$$

Theorem 3.6. *Let $\alpha>0$, $\beta,k,\eta,\rho,\gamma\in\mathbb{R}$ and ψ as defined in Definition 2.1. For $f\in X^\infty_\psi(a,b)$ and $x,y\in(a,b)$, we have*

$$\Big\|I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f(x)-I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f(y)\Big\|\le\frac{2\Big\|[\psi(u)]^\eta\exp(-\gamma\psi(u))f(u)\Big\|_{X^\infty_\psi}}{\Gamma(\alpha+1)\rho^\beta}[\psi(y)]^k\exp(\gamma\psi(y))\Big[\psi(y)-\psi(x)\Big]^\alpha.$$

Proof. We have

$$\Big\|I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f(x)-I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}f(y)\Big\| = \Big\|\frac{\exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x[\psi(u)]^\eta\psi'(u)\exp\big(-\gamma\psi(u)\big).\big[\psi(x)-\psi(u)\big]^{\alpha-1}f(u)du$$
$$-\frac{\exp(\gamma\psi(y))}{\Gamma(\alpha)\rho^\beta}\int_a^y[\psi(u)]^\eta\psi'(u)\exp\big(-\gamma\psi(u)\big)\big(\psi(y)-\psi(u)\big)^{\alpha-1}f(u)du\Big\|$$
$$=\Big\|\frac{1}{\Gamma(\alpha)\rho^\beta}\int_a^x f(u)(\psi(u))^\eta\psi'(u)\exp\big(-\gamma(\psi(u))\big)$$
$$\times\Big(\exp(\gamma(\psi(x)))[\psi(x)]^k\big(\psi(x)-\psi(u)\big)^{\alpha-1}-\exp(\gamma(\psi(y)))[\psi(y)]^k\big[\psi(y)-\psi(u)\big]^{\alpha-1}\Big)dt$$
$$-\frac{\exp(\gamma\psi(y))}{\Gamma(\alpha)\rho^\beta}\int_x^y[\psi(u)]^\eta\psi'(u)\exp\big(-\gamma\psi(u)\big)\big(\psi(y)-\psi(u)\big)^{\alpha-1}f(u)du\Big\|$$
$$\le\frac{\|[\psi(u)]^\eta\exp(-\gamma\psi(u))f(u)\|_{X^\infty_\psi}}{\Gamma(\alpha)\rho^\beta}\int_a^x\psi'(u)\Big([\psi(x)]^k\exp(\gamma(\psi(x)))\big(\psi(x)-\psi(u)\big)^{\alpha-1}$$
$$-[\psi(y)]^k\exp(\gamma(\psi(y)))\big(\psi(y)-\psi(u)\big)^{\alpha-1}\Big)du$$
$$+\frac{\|[\psi(u)]^\eta\exp\big(-\gamma\psi(u)\big)f(u)\|_{X^\infty_\psi}}{\Gamma(\alpha)\rho^\beta}\int_x^y\psi'(u)[\psi(y)]^k\exp(\gamma\psi(y))\big(\psi(y)-\psi(u)\big)^{\alpha-1}du$$
$$\le\frac{\|[\psi(u)]^\eta\exp(-\gamma\psi(u))f(u)\|_{X^\infty_\psi}}{\Gamma(\alpha+1)\rho^\beta}\Big([\psi(x)]^k\exp(\gamma(\psi(x)))\Big(\psi(x)-\psi(a)\Big)^\alpha$$
$$-\exp(\gamma(\psi(y)))[\psi(y)]^k\Big(\big[\psi(y)-\psi(a)\big]^\alpha-\Big[\psi(y)-\psi(x)\Big]^\alpha\Big)\Big)$$
$$+\frac{\|[\psi(u)]^\eta\exp(-\gamma\psi(u))f(u)\|_{X^\infty_\psi}}{\Gamma(\alpha+1)\rho^\beta}[\psi(y)]^k\exp(\gamma\psi(y))\Big(\psi(y)-\psi(x)\Big)^\alpha$$
$$\le\frac{2\|[\psi(u)]^\eta\exp(-\gamma\psi(u))f(u)\|_{X^\infty_\psi}}{\Gamma(\alpha+1)\rho^\beta}[\psi(y)]^k\exp(\gamma\psi(y))\Big[\psi(y)-\psi(x)\Big]^\alpha. \qquad\square$$

Theorem 3.7. *Let $n-1<\alpha<n$, $\beta,\eta,k,\rho,\gamma\in\mathbb{R}$ and ψ as defined in Definition 2.1. For $(f_n)_{n\geq 1}$ a sequence uniformly convergent in $X_\psi^\infty(a,b)$, we have*

$$I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f_n(x).$$

Proof. Let $f(x)=\lim\limits_{n\to\infty} f_n(x)$. We have

$$\begin{aligned}&\Big|I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f_n(x)-I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)\Big|\\ &\leq \frac{([\psi(x)]^k\exp\big(\gamma(\psi(x))\big)}{\Gamma(\alpha)\rho^\beta}\int_a^x[\psi(u)]^\eta\psi'(u)\exp\big(-\gamma(\psi(u))\big)\big(\psi(x)-\psi(u)\big)^{\alpha-1}\big|f_n(u)-f(u)\big|du\\ &\leq \Big\|\exp(-\gamma(\psi(u)))[\psi(u)]^\eta\big(f_n(u)-f(u)\big)\Big\|_{X_\psi^\infty}\frac{([\psi(x)]^k\exp(\gamma(\psi(x)))}{\Gamma(\alpha+1)\rho^\beta}\big(\psi(x)-\psi(a)\big)^\alpha.\end{aligned}$$

Since the sequence $(f_n)_{n\geq1}$ is uniformly convergence, the result follows. ☐

Theorem 3.8. *Let f be a uniformly continuous function on $[0,b]$. For $\beta,\eta,k,\rho,\gamma\in\mathbb{R}$ and ψ as defined in Definition 2.1, if there exists $\alpha\in(0,1]$ satisfying*

$$\lim_{x\to\infty} I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}|f(x)|=0,$$

then

$$\lim_{x\to\infty}|f(x)|=0.$$

Proof. Arguing by contradiction, we assume that there exists an unbounded sequence $(x_i)_{i\in\mathbb{N}}$ and $\varepsilon>0$ such that

$$|f(x_i)|\geq\varepsilon,\quad \forall x_i\in[0,b].$$

Using the fact that f is uniformly continuous, we deduce that for each x_i, $\exists\,\mu>0$ such that

$$\big|f(x_i)-f(x)\big|<\frac{\varepsilon}{2},\quad \forall x\in[x_i-\mu,x_i+\mu]$$

Thus, for all $x\in[x_i-\mu,x_i+\mu]$ we have:

$$|f(x)|\geq\Big||f(x_i)|-|f(x_i)-f(x)|\Big|\geq\frac{\varepsilon}{2}. \tag{3.1}$$

From another part, we have

$$\begin{aligned}I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}|f(x_i)|=\frac{([\psi(x_i)]^k\exp(\gamma(\psi(x_i)))}{\Gamma(\alpha)\rho^\beta}\Bigg(&\int_{x_0}^{x_i-1}[\psi(u)]^\eta\psi'(u)\exp\big(-\gamma(\psi(u))\big)\big(\psi(x_i)-\psi(u)\big)^{\alpha-1}|f(u)|du\\ &+\int_{x_i-1}^{x_i-\mu}[\psi(u)]^\eta\psi'(u)\exp\big(-\gamma(\psi(u))\big)\big(\psi(x_i)-\psi(u)\big)^{\alpha-1}|f(u)|du\end{aligned}$$

$$+\int_{x_i-\mu}^{x_i}[\psi(u)]^{\eta}\psi'(u)\exp\big(-\gamma(\psi(u))\big)\big(\psi(x_i)-\psi(u)\big)^{\alpha-1}|f(u)|du\bigg).$$

If we suppose that $[\psi(u)]^{\eta}\psi'(u)\exp\big(-\gamma(\psi(u))\big)\big(\psi(x_i)-\psi(u)\big)^{\alpha-1}\geq 1,\ \forall t\in[x_i-1,x_i]$, then

$$\begin{aligned}I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}|f(x_i)| \geq \frac{([\psi(x_i)]^k\exp(\gamma(\psi(x_i)))}{\Gamma(\alpha)\rho^{\beta}}\bigg(\int_{x_0}^{x_i-1}[\psi(u)]^{\eta}\psi'(u)\exp\big(-\gamma(\psi(t))\big) \\ \times\big(\psi(x_i)-\psi(u)\big)^{\alpha-1}\big|f(u)\big|du+\int_{x_i-1}^{x_i-\mu}\big|f(u)\big|du+\int_{x_i-\mu}^{x_i}\big|f(u)\big|du\bigg).\end{aligned} \tag{3.2}$$

Using (3.1) and (3.2) and denoting $c=\big([\psi(0)]^k\exp(\gamma(\psi(0)))\big)$, we obtain

$$I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}|f(x_i)|\geq\frac{c\varepsilon}{2\Gamma(\alpha)\rho^{\beta}},$$

which contradicts the hypothesis of the Theorem. □

4 On a Minkowski type inequality

First, we recall the celebrated Minkowski inequality as follows, (see [1, 22]).

Theorem 4.1. *If $p\geq 1$ and f,g two positives functions in $L^p([a,b])$, then*

$$\left(\int_a^b|f(t)+g(t)|^pdt\right)^{1/p}\leq\left(\int_a^b|f(t)|^pdt\right)^{1/p}+\left(\int_a^b|g(t)|^pdt\right)^{1/p}.$$

As a reverse of Minkowski's inequality, Bougoffa [9] proved the following result.

Theorem 4.2. *If $p\geq 1$, f and g two positives functions satisfying $0<m\leq\frac{f(t)}{g(t)}\leq M$, $\forall t\in[a,b]$, then*

$$\left(\int_a^b|f(t)|^pdt\right)^{1/p}+\left(\int_a^b|g(t)|^pdt\right)^{1/p}\leq c\left(\int_a^b|f(t)+g(t)|^pdt\right)^{1/p},$$

where $c=\frac{M}{M+1}+\frac{1}{m+1}$.

The above result was generalized by Dahmani [17] using Riemann-Liouville fractional integral, by Chinchane-Pachpatte [13] and Taf-Brahim [44] using the Hadamard fractional integral, by Sousa-Oliveira [15] using Katugampola generalized fractional integral, by Aljaaidi-Pachpatte [5] using the ψ Riemann Liouville integral, by Rahman *et al.* [37] using generalized proportional fractional integral, by Rachid-Jarad-Chu [39] using generalized proportional integral according to another function, by Rachid *et al.* [38] using generalized conformable integral, by Nale-Panchal-Chinchane [33] using generalized proportional Hadamard fractional integral.

In the following, we prove the reverse of the Minkowski inequality using the new generalized ψ-Hilfer integral, recovering the results of the above cited papers.

Theorem 4.3. *Let $\beta, k, \eta, \rho, \gamma \in \mathbb{R}, \alpha > 0$, $p \geq 1$ and ψ as defined in Definition 2.1. Let also f, g be two positive functions in $X^p_\psi(a,b)$. If $0 < m \leq \dfrac{f(t)}{g(t)} \leq M$, $\forall t \in [a,b]$ for m and M two strictly positive constants, then:*

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(t)\right)^{\frac{1}{p}} + \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(t)\right)^{\frac{1}{p}} \leq c \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} (f+g)^p(t)\right)^{\frac{1}{p}},$$

where $c = \dfrac{1}{m+1} + \dfrac{M}{M+1}$.

Proof. Since $\dfrac{f(s)}{g(s)} \leq M$, $\forall s \in [a,b]$, then

$$f(s) + Mf(s) \leq M\big(g(s) + f(s)\big), \quad \forall s \in [a,b],$$

thus

$$(M+1)^p f^p(s) \leq M^p \big(f(s) + g(s)\big)^p, \quad \forall s \in [a,b].$$

Multiplying both sides by $\dfrac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta} \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s)) \big[\psi(x) - \psi(s)\big]^{\alpha-1}$ and integrating with respect to s, we obtain

$$\begin{aligned}(M+1)^p \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta} &\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s)) \big[\psi(x) - \psi(s)\big]^{\alpha-1} f^p(s) ds \\ &\leq M^p \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta} \int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s)) \big[\psi(x) - \psi(s)\big]^{\alpha-1} (f+g)^p(s) ds.\end{aligned}$$

Which implies that

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}} \leq \frac{M}{M+1} \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} (g+f)^p(x)\right)^{\frac{1}{p}}. \tag{4.1}$$

From another part, since $0 < m \leq \dfrac{f(s)}{g(s)}$, for all $s \in [a,b]$, then

$$g(s) \leq \frac{f(s)}{m}, \quad \forall s \in [a,b].$$

Thus

$$g(s)\left(1 + \frac{1}{m}\right) \leq \frac{f(s)}{m} + \frac{g(s)}{m}, \quad \forall s \in [a,b],$$

and consequently

$$g^p(s)\left(1 + \frac{1}{m}\right)^p \leq \left(\frac{1}{m}\right)^p \Big[g(s) + f(s)\Big]^p.$$

Multiplying both sides by $\dfrac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1}$ and integrating with respect to s, we derive that

$$\begin{aligned}&\left(1+\frac{1}{m}\right)^p \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1} g^p(s)ds\\ &\leq \left(\frac{1}{m}\right)^p \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1}\Big[g(s)+f(s)\Big]^p ds.\end{aligned}$$

Thus,

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq \frac{1}{m+1}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}(g+f)^p(x)\right)^{\frac{1}{p}}. \tag{4.2}$$

Using (4.1) and (4.2), the result follows. □

Theorem 4.4. *Let $\beta,\rho,\gamma,k,\eta\in\mathbb{R}$, $\alpha>0$, $p\geq 1$ and ψ as defined in Definition 2.1. Let also f,g be two positive functions in $X^p_\psi(a,b)$. If $0<m\leq \dfrac{f(t)}{g(t)}\leq M$, $\forall t\in[a,b]$ for $m>0, M>0$, then:*

$$\left[I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right]^{\frac{2}{p}} + \left[I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right]^{\frac{2}{p}} \geq \hat{c}\left[I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right]^{\frac{1}{p}}\left[I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right]^{\frac{1}{p}},$$

where $\hat{c}=\dfrac{(M+1)(m+1)}{M}-2.$

Proof. From (4.1) and (4.2), we have:

$$\frac{(M+1)(m+1)}{M}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}(g+f)^p(x)\right)^{\frac{2}{p}}. \tag{4.3}$$

Using Minkowski's inequality, we obtain

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}(g+f)^p(x)\right)^{\frac{1}{p}} \leq \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}} + \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \tag{4.4}$$

Using (4.3) and (4.4), we deduce that

$$\frac{(M+1)(m+1)}{M}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq \left(\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}} + \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}}\right)^2.$$

Thus,

$$\left(\frac{(M+1)(m+1)}{M}-2\right)\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{2}{p}} + \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{2}{p}}.$$ □

Remark 4.5. *Using Remark 2.3, it is easy to see that Theorems 4.3 and 4.4 recover Theorems 2.1 and 2.3 of [17], Theorems 3.1 and 3.2 of [13], Theorems 2.9 and 2.10 of [44], Theorems 7 and 8 of [15], Theorems 3.1 and 3.2 of [5], Theorems 3.1 and 3.2 of [37], Theorems 5 and 6 of [39], Theorems 3.1 and 3.2 of [38] and Theorems 3.1 and 3.2 of [33].*

5 Other inequalities related to the Minkowski type inequality

In this section, we state other inequalities related to the Minkowski type inequality, using generalized ψ-fractional integral.

Theorem 5.1. *Let $\beta, \eta, \rho, \gamma, k \in \mathbb{R}$, $\alpha > 0$, $p \geq 1$ and ψ as defined in Definition 2.1. For f, g two positive functions in $X_\psi^p(a,b)$, if $0 < m \leq f(s) \leq M$ and $0 < n \leq g(s) \leq N$ for all $s \in [a,b]$, then*

$$\left(I_{a+;\eta,k,\gamma,\rho}^{\alpha,\beta;\psi} f^p(x)\right)^{\frac{1}{p}} + \left(I_{a+;\eta,k,\gamma,\rho}^{\alpha,\beta;\psi} g^p(x)\right)^{\frac{1}{p}} \leq \tilde{c}\left(I_{a+;\eta,k,\gamma,\rho}^{\alpha,\beta;\psi} (g+f)^p(x)\right)^{\frac{1}{p}}.$$

Here $\tilde{c} = \dfrac{M}{M+n} + \dfrac{N}{N+m}$.

Proof. Since $0 < n \leq g(s) \leq N$ for all $s \in [a,b]$, then

$$\frac{1}{N} \leq \frac{1}{g(s)} \leq \frac{1}{n}, \quad \forall s \in [a,b].$$

Thus,

$$\frac{m}{N} \leq \frac{f(s)}{g(s)} \leq \frac{M}{n}. \tag{5.1}$$

From (5.1), we deduce that

$$g(s)\left(\frac{m}{N}+1\right) \leq g(s)+f(s), \tag{5.2}$$

$$\left(\frac{n}{M}+1\right) f(s) \leq g(s)+f(s). \tag{5.3}$$

Thus,

$$g^p(s) \leq \left(\frac{N}{m+N}\right)^p (g(s)+f(s))^p, \tag{5.4}$$

$$f^p(s) \leq \left(\frac{M}{n+M}\right)^p (g(s)+f(s))^p. \tag{5.5}$$

Multiplying both sides of (5.4) by $\dfrac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1}$ and integrating with respect to s, we obtain

$$\begin{aligned}
&\frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1} g^p(s)ds \\
&\leq \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1}\left(\frac{N}{m+N}\right)^p (f+g)^p(s)ds,
\end{aligned}$$

which implies that

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq \frac{N}{m+N}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}(g+f)^p(x)\right)^{\frac{1}{p}}. \tag{5.6}$$

From another part, using the same argument to equation (5.5), we derive that

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}} \leq \frac{M}{n+M}\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}(g+f)^p(x)\right)^{\frac{1}{p}}. \tag{5.7}$$

Adding (5.6) and (5.7), the result follows. □

Theorem 5.2. *Let $\beta,\rho,k,\eta,\gamma,\in\mathbb{R}$, $\alpha>0$, $p\geq 1$ and ψ as defined in Definition 2.1. Let f,g two positive functions in $X^p_\psi(a,b)$. If $0<m\leq \frac{f(s)}{g(s)}\leq M, \forall s\in[a,b]$ for $m,M\in\mathbb{R}^*_+$, then*

$$\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\right)^{\frac{1}{p}} + \left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\right)^{\frac{1}{p}} \leq 2\left(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} h^p(g(x)+f(x))\right)^{\frac{1}{p}},$$

where $h(g(x)+f(x)) = \max\left\{\left(\frac{M}{m}+1\right)f(x)-Mg(x),\ \frac{(M+m)g(x)-f(x)}{m}\right\}$.

Proof. Since $0<m\leq\frac{f(s)}{g(s)}\leq M$, $\forall s\in[a,b]$, then

$$0<m\leq M-\frac{f(s)}{g(s)}+m.$$

Thus

$$g(s)\leq\frac{(M+m)g(s)-f(s)}{m},$$

which implies that

$$g(s)\leq h(f(s),g(s)). \tag{5.8}$$

From another part, since $0<\frac{1}{M}\leq\frac{g(s)}{f(s)}\leq\frac{1}{m}$, then

$$\frac{1}{M}\leq\frac{1}{M}+\frac{1}{m}-\frac{g(s)}{f(s)}.$$

Thus,

$$\frac{1}{M}\leq\frac{\left(\frac{1}{M}+\frac{1}{m}\right)f(s)-g(s)}{f(s)},$$

which implies that

$$f(s)\leq M\left(\frac{1}{M}+\frac{1}{m}\right)f(s)-Mg(s)\leq\left(\frac{M}{m}+1\right)f(s)-Mg(s)\leq h(f(s),g(s)). \tag{5.9}$$

From (5.8) and (5.9), we get

$$f^p(s) \leq h^p(f(s), g(s)), \tag{5.10}$$

$$g^p(s) \leq h^p(f(s), g(s)). \tag{5.11}$$

Multiplying both sides of (5.10) by $\frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x) - \psi(s)\big]^{\alpha-1}$ and integrating with respect to s, we derive that

$$\begin{aligned}\frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}&\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x) - \psi(s)\big]^{\alpha-1} f^p(s)ds\\ &\leq \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x) - \psi(s)\big]^{\alpha-1} h^p(f(s), g(s))ds.\end{aligned}$$

Which implies that

$$\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f^p(x)\Big)^{\frac{1}{p}} \leq \Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} h^p(g(x), f(x))\Big)^{\frac{1}{p}}. \tag{5.12}$$

Using the same argument to equation (5.11), we obtain

$$\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} g^p(x)\Big)^{\frac{1}{p}} \leq \Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} h^p(g(x), f(x))\Big)^{\frac{1}{p}} \tag{5.13}$$

and the result follows. □

Theorem 5.3. *Under the hypothesis of Theorem 5.2, we have*

$$\frac{1}{M}\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)g(x)\Big) \leq \frac{1}{(M+1)(m+1)}\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}(f+g)^2(x)\Big) \leq \frac{1}{m}\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)(g)(x)\Big).$$

Proof. Since $0 < m \leq \frac{f(s)}{g(s)} \leq M$ for all $s \in [a, b]$, then

$$g(s)(1+m) \leq f(s) + g(s) \leq g(s)(1+M). \tag{5.14}$$

Additionally, using the fact that $0 < \frac{1}{M} \leq \frac{g(s)}{f(s)} \leq \frac{1}{m}, \forall s \in [a, b]$, we obtain

$$f(s)\left(\frac{1}{M} + 1\right) \leq f(s) + g(s) \leq f(s)\left(1 + \frac{1}{m}\right). \tag{5.15}$$

From (5.14) and (5.15), we deduce that

$$\frac{g(s)f(s)}{M} \leq \frac{\big(g(s) + f(s)\big)^2}{(1+m)(1+M)} \leq \frac{f(s)g(s)}{m}. \tag{5.16}$$

Multiplying both sides of equation (5.16) by $\dfrac{[\psi(x)]^k \exp(\gamma\psi(x))}{\Gamma(\alpha)\rho^\beta}\psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x) - \psi(s)\big]^{\alpha-1}$ and integrating with respect to s, we obtain

$$\begin{aligned}
&\frac{[\psi(x)]^k \exp(\gamma\psi(x))}{M\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1} f(s)g(s)ds \\
&\quad\leq \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{(m+1)(M+1)\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1}(f+g)^2(s)ds \\
&\quad\leq \frac{[\psi(x)]^k \exp(\gamma\psi(x))}{m\Gamma(\alpha)\rho^\beta}\int_a^x \psi'(s)[\psi(s)]^\eta \exp(-\gamma\psi(s))\big[\psi(x)-\psi(s)\big]^{\alpha-1} f(s)g(s)ds.
\end{aligned}$$

Thus,

$$\frac{1}{M}\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)g(x)\Big) \leq \frac{1}{(1+m)(1+M)}\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho}(f+g)^2(x)\Big) \leq \frac{1}{m}\Big(I^{\alpha,\beta;\psi}_{a+;\eta,k,\gamma,\rho} f(x)g(x)\Big). \qquad \square$$

6 Conclusion

Minkowski type inequalities play a crucial role in various fields of science. In recent years, these inequalities have been proved by numerous researchers using different fractional integrals. The aim of this work was to prove a generalized Minkowski type inequality which recovers most of the previous results. For this purpose, we defined a new generalized ψ fractional integral, which generalizes most of the pre-existing fractional integrals. Then, we gave some essential properties of this new operator and we presented some examples. As an application, we used this generalized ψ fractional integral to prove a Minkowski type inequality and several related ones. These inequalities recover a large number of a well known results. Many other interesting inequalities as Grüss-type, Hermite-Hadamard type or Čebyšev type inequalities can be proved using the newly defined integral operator. These questions will be discussed in a forthcoming paper.

Conflict of interest statement:

The author declares that he has no conflict of interest.

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Almost automorphic solutions for some nonautonomous evolution equations under the light of integrable dichotomy

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ABSTRACT

In this work, we prove the existence and uniqueness of μ-pseudo almost automorphic solutions for a class of semilinear nonautonomous evolution equations of the form: $u'(t) = A(t)u(t) + f(t, u(t)),\ t \in \mathbb{R}$ where $(A(t))_{t\in\mathbb{R}}$ is a family of closed linear operators acting in a Banach space X that generates an evolution family having an integrable dichotomy on $\mathbb{R}$ and $f : \mathbb{R} \times X \longrightarrow X$ is μ-pseudo almost automorphic with respect to t and Lipshitzian in the second variable. Moreover we provide an application illustrating our results.

RESUMEN

En este trabajo, demostramos la existencia y unicidad de soluciones μ-pseudo casi automorfas para una clase de ecuaciones de evolución semilineales no autónomas de la forma: $u'(t) = A(t)u(t)+f(t, u(t)),\ t \in \mathbb{R}$ donde $(A(t))_{t\in\mathbb{R}}$ es una familia de operadores lineales cerrados actuando en un espacio de Banach X que genera una familia de evolución que posee una dicotomía integrable en $\mathbb{R}$ y $f : \mathbb{R} \times X \longrightarrow X$ es μ-pseudo casi automorfa con respecto a t y Lipschitziana en la segunda variable. Más aún presentamos una aplicación ilustrando nuestros resultados.

Keywords and Phrases: Evolution family, delay evolution equations, exponential dichotomy, integrable dichotomy, μ-pseudo almost automorphic functions.

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1 Introduction

The current paper deals with the existence and uniqueness of μ-pseudo almost automorphic solutions for the following evolution equations:

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R} \tag{1.1}$$

and

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

and the perturbed delay system

$$u'(t) = A(t)u(t) + f(t, u(t), u(t - \tau)), \quad t \in \mathbb{R}, \tag{1.3}$$

where $(A(t), D(A(t)))$, $t \in \mathbb{R}$ is a family of closed linear operators that generates a strongly continuous evolution family $(U(t,s))_{t \geq s}$ on a Banach space X which has an integrable dichotomy on $\mathbb{R}$. The function f is μ-pseudo almost automorphic in t for each $x \in X$ and Lipschitzian with respect to the second and third arguments, $\tau > 0$ is a fixed constant . This work is a continuation of the works done in [21, 22].

In the theory of differential equations, exponential dichotomy is a classical concept and it plays a central role for getting important results. So, there exist many researchs on this topics see [15,20]. It is well-known that the concept of integrable dichotomy is a generalization of exponential dichotomy [1, 21, 22]. This concept was introduced by Pinto *et al.* [21], they proved the existence and uniqueness of bounded periodic solutions of nonlinear integro-differential equations with infinite delay. In [22], the authors proved the existence and uniqueness of almost periodic and pseudo-almost periodic mild solutions of equations (4.1) and (4.2) under the light of integrable bi-almost periodic Green's functions. In fact, the authors established some examples of purely integrale dichotomy (*i.e.*, which is not necessarily of exponential type). Recently, in [1], Abadias *et al.* investigate the semi-linear differential equation $x'(t) = A(t)x(t) + f(t, x(t), \varphi[\alpha(t, x(t))])$, $t \in \mathbb{R}$, where $(A(t), D(A(t)))$, $t \in \mathbb{R}$, generate an evolution family which has an integrable dichotomy. They obtained several results of existence and uniqueness of (ω, c)-periodic mild solutions under some assumptions on the nonlinear term. To our knowledge in the literature, there are few papers which deal with integrable dichotomy.

The concept of almost periodic functions is introduced by H. Bohr [12]. This notion has been much invested before being generalized by the concept of almost automorphic functions introduced by S. Bochner [8–11]. In [24], the authors introduced the notion of pseudo almost automorphic functions which is more general than the notion of almost automorphic functions. Moreover, they proved that the space $(PAA(\mathbb{R}, X), \|\cdot\|_0)$ is complete and they obtained an existence and uniqueness result of pseudo almost automorphic mild solutions to equation (4.1) in Banach spaces. In [4], Blot *et al.* introduced the notion of weighted pseudo almost automorphic functions which generalizes the concept of pseudo almost automorphic functions. For more details on these topics, one can see [19, 26]. More recently, the concept of μ-pseudo almost automorphy due to Ezzinbi *et al.* [5, 16] generalizes both notions of pseudo almost automorphy and weighted pseudo almost automorphy. For more details, one can see [4, 14, 17, 24].

In this work, our main results are Theorems 3.1 and 4.3. We show that equations (4.1) and (4.2) have respectively, unique bounded almost automorphic and μ-pseudo almost automorphic solutions. It should be noted that we obtained these results under light of integrable dichotomy, dominated convergence Theorem, Banach fixed point, standard and locally Lipschitz conditions. The nonlinear term f is in $PAA(\mathbb{R}, X, \mu)$.

The rest of this paper is organized as follows. Section 2 is devoted to some preliminaries. In sections 3 and 4, we present some criteria ensuring the existence of μ-pseudo almost automorphic mild solutions to equations (4.1) and (4.2). An example is given to illustrate our theoretical result in section 5.

2 Almost automorphic functions and integrable dichotomy

This section is concerned with some notations and preliminary facts that are used in the sequel of this work.

Definition 2.1 ([12])**.** *A continuous function $f : \mathbb{R} \to X$ is to be almost periodic if for every $\varepsilon > 0$, there exists $l_\varepsilon > 0$, such that for every $a \in \mathbb{R}$, there exists $\tau \in [a, a + l_\varepsilon]$ satisfying:*

$$\|f(t + \tau) - f(t)\| < \varepsilon \quad \text{for all } t \in \mathbb{R}$$

The space of all such functions is denoted by $\mathrm{AP}(\mathbb{R}, X)$.

Definition 2.2 ([9])**.** *A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for every sequence $(s'_n)_{n\geq 0}$ of real numbers, there exist a subsequence $(s_n)_{n\geq 0} \subset (s'_n)_{n\geq 0}$ and a measurable function $g : \mathbb{R} \to X$, such that*

$$g(t) = \lim_{n\to\infty} f(t + s_n) \quad \text{and} \quad f(t) = \lim_{n\to\infty} g(t - s_n) \quad \text{for all } t \in \mathbb{R}.$$

The space of all such functions is denoted by $AA(\mathbb{R}, X)$.

Remark 2.3 ([3]). *An almost automorphic function may not be uniformly continuous. Indeed, the real function* $f(t) = \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right)$ *for* $t \in \mathbb{R}$, *belongs to* $AA(\mathbb{R}, \mathbb{R})$, *but is not uniformly continuous. Hence,* f *does not belong to* $AP(\mathbb{R}, \mathbb{R})$.

Then, we have the following inclusions:

$$AP(\mathbb{R}, X) \subset AA(\mathbb{R}, X) \subset BC(\mathbb{R}, X).$$

Definition 2.4 ([3]). *A bounded continuous function* $f : \mathbb{R} \times X \to Y$ *is called almost automorphic if for each bounded set* $K \subset X$ *and for every sequence of real numbers* $\{\tau'_n\}_{n\geq 0}$, *there exist a subsequence* $\{\tau_n\}_{n\geq 0} \subset \{\tau'_n\}_{n\geq 0}$ *and a mesurable function* $\tilde{f} : \mathbb{R} \times X \to Y$, *such that*

$$\tilde{f}(t, x) = \lim_{n\to\infty} f(t + \tau_n, x) \text{ and } f(t, x) = \lim_{n\to\infty} \tilde{f}(t - \tau_n, x)$$

are well defined in $t \in \mathbb{R}$ *and* $x \in K \subset X$.

Definition 2.5 ([3]). *A continuous function* $F : \mathbb{R} \times \mathbb{R} \to X$ *is said to be bi-almost automorphic if for every sequence* $(s'_n)_{n\geq 0}$ *of real numbers, there exist a subsequence* $(s_n)_{n\geq 0} \subset (s'_n)_{n\geq 0}$ *and a measurable function* $G : \mathbb{R} \times \mathbb{R} \to X$, *such that*

$$G(t, s) = \lim_{n\to\infty} F(t + s_n, s + s_n) \quad \text{and} \quad F(t, s) = \lim_{n\to\infty} G(t - s_n, s - s_n) \quad \text{for all } t, s \in \mathbb{R}.$$

The space of all such functions is denoted by $bAA(\mathbb{R}, X)$.

2.1 μ-pseudo almost automorphic functions

This section is devoted to properties of μ-ergodic and μ-pseudo almost automorphic functions. In the sequel, we denote by $\mathcal{B}(\mathbb{R})$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}(\mathbb{R})$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$ for all $a, b \in \mathbb{R}$ with $(a \leq b)$, we denote also by Y any other Banach space. We assume the following hypothesis.

(M) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau : a \in A\}) \leq \beta\mu(A) \quad \text{where } A \in \mathcal{B}(\mathbb{R}) \text{ and } A \cap I = \emptyset.$$

Definition 2.6 ([6]). *Let* $\mu \in \mathcal{M}$. *A continuous bounded function* $f : \mathbb{R} \longrightarrow X$ *is called* μ*-ergodic, if*

$$\lim_{r\to+\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|f(t)\| d\mu(t) = 0.$$

The space of all such functions is denoted by $\mathcal{E}(\mathbb{R}, X, \mu)$.

Proposition 2.7 ([6]). *Let $\mu \in \mathcal{M}$. Then,*

(i) $(\mathcal{E}(\mathbb{R}, X, \mu), \|\cdot\|_\infty)$ *is a Banach space.*

(ii) If μ satisfies **(M)**, *then $\mathcal{E}(\mathbb{R}, X, \mu)$ is translation invariant.*

Example 2.8. *(1) An ergodic function in the sense of Zhang [25] is a μ-ergodic function in the particular case where the measure μ is the Lebesgue measure.*

(2) Let $\rho : \mathbb{R} \longrightarrow [0, +\infty)$ be a $\mathcal{B}(\mathbb{R})$-measurable function. We define the positive measure μ on $\mathcal{B}(\mathbb{R})$ by

$$\mu(A) = \int_A \rho(t)dt \quad \text{for } A \in \mathcal{B}(\mathbb{R}),$$

where dt denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. The measure μ is absolutely continuous with respect to dt and the function ρ is called the Radon-Nikodym derivative of μ with respect to dt. In this case $\mu \in \mathcal{M}$ if and only if the function ρ is locally Lebesgue-integrable on $\mathbb{R}$ and it satisfies

$$\int_{\mathbb{R}} \rho(t)dt = +\infty.$$

(3) In [18], the authors considered the space of bounded continuous functions $f : \mathbb{R} \longrightarrow X$ satisfying

$$\lim_{r\to+\infty} \frac{1}{2r} \int_{[-r,r]} \|f(t)\|dt = 0 \quad \text{and} \quad \lim_{N\to+\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \|f(n)\| = 0.$$

This space coincides with the space of μ-ergodic functions where μ is defined in $\mathcal{B}(\mathbb{R})$ by the sum $\mu(A) = \mu_1(A) + \mu_2(A)$ with μ_1 is the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and

$$\mu_2(A) = \begin{cases} card(A \cap \mathbb{Z}) & \text{if } A \cap \mathbb{Z} \text{ is finite,} \\ \infty & \text{if } A \cap \mathbb{Z} \text{ is infinite.} \end{cases}$$

Definition 2.9 ([5]). *Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \longrightarrow X$ is said to be μ-pseudo almost automorphic if f is written in the form:*

$$f = g + \varphi,$$

where $g \in AA(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$.

The space of all such functions is denoted by $PAA(\mathbb{R}, X, \mu)$.

Proposition 2.10 ([5])**.** *Let $\mu \in \mathcal{M}$ satisfy* (**M**)*. Then the following are true:*

(i) The decomposition of a μ-pseudo almost automorphic in the form $f = g + \varphi$ where $g \in AA(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$, is unique.

(ii) $PAA(\mathbb{R}, X, \mu)$ equipped with the supnorm is a Banach space.

Definition 2.11 ([7])**.** *A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is said to be almost automorphic in t uniformly with respect to $x \in X$ if the following two conditions hold:*

(i) For all $x \in X$, $f(\cdot, x) \in AA(\mathbb{R}, Y)$,

(ii) f is uniformly continuous on each compact $K \subset X$ with respect to the second variable x, namely, for each compact $K \subset X$, for all $\epsilon > 0$, there exists $\delta > 0$ such that all x_1,$x_2 \in K$, one has $\|x_1 - x_2\| \leq \delta \Rightarrow \sup_{t\in\mathbb{R}} \|f(t, x_1) - f(t, x_2)\| \leq \epsilon$.

Denote by $AAU(\mathbb{R} \times X, Y)$ the set of all such functions.

Definition 2.12. *Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is said to be μ-ergodic in t uniformly with with respect to $x \in X$, if the following two conditions hold:*

(i) For all $x \in X$, $f(\cdot, x) \in \mathcal{E}(\mathbb{R}, Y, \mu)$,

(ii) f is uniformly continuous on each compact $K \subset X$ with respect to the second variable x.

Denote by $\mathcal{E}U(\mathbb{R} \times X, Y, \mu)$ the set of all such functions.

Definition 2.13. *Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is said to be μ-pseudo almost automorphic in t uniformly with with respect to $x \in X$, if f is written in the form:*

$$f = g + h$$

where $g \in AAU(\mathbb{R} \times X, Y)$ and $h \in \mathcal{E}U(\mathbb{R} \times X, Y, \mu)$.

$PAAU(\mathbb{R} \times X, Y)$ denotes the set of such functions. We have

$$AAU(\mathbb{R} \times X, Y) \subset PAAU(\mathbb{R} \times X, Y).$$

Proposition 2.14 ([5])**.** *Let $\mu \in \mathcal{M}$ and $f : \mathbb{R} \times X \longrightarrow Y$ be a μ-pseudo almost automorphic in t uniformly with with respect to $x \in X$. Then*

(i) For all $x \in X$, $f(\cdot, x) \in PAA(\mathbb{R}, Y, \mu)$,

(ii) f is uniformly continuous on each compact $K \subset X$ with respect to the second variable x.

Theorem 2.15 ([5]). *Let $\mu \in \mathcal{M}$, $f \in PAAU(\mathbb{R} \times X, Y, \mu)$ and $x \in PAA(\mathbb{R}, X, \mu)$. Assume that the following hypothesis holds:*

(C) *For all bounded subset K of X, f is bounded on $\mathbb{R} \times K$.*

Then $[t \mapsto f(t, x(t))] \in PAA(\mathbb{R}, Y, \mu)$.

2.2 Integrable dichotomy

Let X and Y be any Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_Y$ respectively. Throughout this work we will assume that Y is densely and continuously imbedded in X *i.e.*, Y is a dense subspace of X and there is a constant C such that

$$\|\xi\| \leq C\|\xi\|_Y \quad \text{for } \xi \in Y.$$

Consider the following linear evolution equation:

$$\begin{cases} u'(t) = A(t)u(t), & t \geq s, \\ u(s) = x \in X, \end{cases} \tag{2.1}$$

The associated inhomogeneous equation is given by:

$$\frac{d}{dt}u(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{2.2}$$

where $f : \mathbb{R} \longrightarrow X$ is continuous and bounded.

Definition 2.16 ([20]). *Let X be a Banach space. The family $(A(t))_{t\geq 0}$ of infinitesimal generators of C_0-semigroup on X is called stable if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$(\omega, \infty) \subset \rho(A(t)) \quad \textit{for } t \geq 0$$

and

$$\left\| \prod_{j=1}^{k} R(\lambda, A(t_j)) \right\| \leq M(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and for every finite sequence $\{t\}_{j=1}^{k}$ with $0 \leq t_1 \leq \cdots \leq t_k < \infty$ and $k = 1, 2, \ldots$

Definition 2.17. *For each $t \in \mathbb{R}$, let $A(t)$ be the infinitesimal generator of a C_0 semigroup $T_t(s)$, $s \in \mathbb{R}$, on X. A subspace Y of X is called $A(t)$-admissible if it is an invariant subspace of $T_t(s)$, $s \in \mathbb{R}$, and the restriction of $T_t(s)$ to Y is a C_0 semigroup in Y (i.e. it is strongly continuous in the norm $\|\cdot\|_Y$).*

We will make the following assumptions.

(A$_1$) $(A(t))_{t\in\mathbb{R}}$ is a stable family with stability constants M, ω.

(A$_2$) Y is $A(t)$-admissible for $t \in \mathbb{R}$ and the family $(\tilde{A}(t))_{t\in\mathbb{R}}$ of parts $\tilde{A}(t)$ of $A(t)$ in Y, is a stable family in Y with stability constants $\tilde{M}$, $\tilde{\omega}$.

(A$_3$) For each $t \in \mathbb{R}$, $D(A(t)) \supset Y$, $A(t)$ is a bounded operator from Y into X and $t \to A(t)$ is continuous in the $B(Y,X)$ norm $\|\cdot\|_{Y\to X}$.

It is well known that if a family $(A(t))_{t\in\mathbb{R}}$ satisfies conditions (**A$_1$**)-(**A$_3$**), then one can associate a unique evolution family $(U(t,s))_{s\leq t}$ with the equation (2.1), (see [15, 20]). Throughout this work $(A(t), D(A(t)))$, $t \in \mathbb{R}$ satisfies conditions (**A$_1$**)-(**A$_3$**).

Definition 2.18 ([15, 20]). *An evolution family $(U(t,s))_{s\leq t}$ on a Banach space X is said to have an exponential dichotomy (or hyperbolic) in $\mathbb{R}$ if there exists a family of projections $P(t) \in \mathcal{L}(X)$, $t \in \mathbb{R}$, being strongly continuous with respect to t, and constants $\delta, M > 0$ such that*

(i) $U(t,s)P(s) = P(t)U(t,s)$,

(ii) $U(t,s) : Q(s)X \to Q(t)X$ *is invertible with the inverse* $\tilde{U}(t,s)$,

(iii) $\|U(t,s)P(s)\| \leq Me^{-\delta(t-s)}$ *and* $\|\tilde{U}(t,s)Q(t)\| \leq Me^{-\delta(t-s)}$,

for all $t, s \in \mathbb{R}$ with $s \leq t$, where, $Q(t) = I - P(t)$.

Definition 2.19. *Let $(U(t,s))_{s\leq t}$ have an exponential dichotomy. We define the Green function by:*

$$G(t,s) = \begin{cases} U(t,s)P(s), & t,s \in \mathbb{R}, \quad s \leq t \\ -\tilde{U}(t,s)Q(s), & t,s \in \mathbb{R}, \quad s > t. \end{cases}$$

For a given evolution family $(U(t,s))_{s\leq t}$ associated to equation (2.1), that has an dichotomy exponential, the Green function associated to the evolution family satisfies

$$\|G(t,s)\| = \begin{cases} Me^{-\delta(t-s)}, & \text{if } t \geq s \\ Me^{-\delta(s-t)}, & \text{if } s > t. \end{cases}$$

where $M > 0$ and $\delta > 0$ are positive constant.

Definition 2.20 ([22]). *We say that equation* (2.1) *has an integrable dichotomy with data* (λ, P) *if there are projections $P(t)$, $t \in \mathbb{R}$, uniformly bounded and strongly continuous in t satisfying* (i) *and* (ii), *with $Q(t) = I - P(t)$ and there exists a function $\lambda : \mathbb{R}^2 \to (0, \infty)$ such that*

$$\|G(t,s)\| \leq \lambda(t,s), \quad \textit{for all } t,s \in \mathbb{R}, \tag{2.3}$$

and

$$\sup_{t\in\mathbb{R}}\int_{\mathbb{R}}\lambda(t,s)ds \leq L < \infty. \tag{2.4}$$

In the pseudo almost automorphic context, we will make the following additional assumption for the function $\lambda(t,s)$ in Definition 2.20.

(A) Let $\lambda_1 : (-\infty,-T) \to (0,\infty)$ and $\lambda_2 : (T,\infty) \to (0,\infty)$ defined by $\lambda_1(s) = \int_{-T}^{T}\lambda(t,s)d\mu(t)$, $\lambda_2(s) = \int_{-T}^{T}\lambda(t,s)d\mu(t)$ for all $T > 0$. We assume that there exists a constant $C > 0$ such that for all $T > 0$,

$$\int_s^T \lambda(t,s)d\mu(t) \leq C, \text{ and } \int_{-T}^{s}\lambda(t,s)d\mu(t) \leq C, \tag{2.5}$$

$$\int_{-\infty}^{-T}\lambda_1(s)ds \leq C, \text{ and } \int_T^{\infty}\lambda_2(s)ds \leq C. \tag{2.6}$$

Remark 2.21. *We notice that some differences between exponential dichotomy and integrable dichotomy. In the case of exponential dichotomy, if we consider the Lebesgue mesure on $\mathcal{B}(\mathbb{R})$, the constante C quoted in* **(A)** *is equal to* $\max\{\frac{M}{\delta},\frac{M}{\delta^2}\}$ *and* $L = 2\frac{M}{\delta}$*. Indeed, for $T > 0$, we have*

$$\int_{\mathbb{R}} G(t,s)ds = M\int_{-\infty}^{t} e^{-\delta(t-s)}ds + M\int_t^{\infty} e^{-\delta(s-t)}ds = 2\frac{M}{\delta} = L, \tag{2.7}$$

$$\text{for } t \geq s, \quad M\int_s^T e^{-\delta(t-s)}dt = \frac{M}{\delta}\left[-e^{-\delta(T-s)}+1\right] \leq \frac{M}{\delta}, \tag{2.8}$$

$$\text{for } t \geq s, \quad M\int_{-\infty}^{-T}\int_{-\infty}^{T} e^{-\delta(t-s)}dtds = \frac{M}{\delta}\left(e^{\delta T}-e^{-\delta T}\right)\int_{-\infty}^{-T} e^{\delta s}ds \leq \frac{M}{\delta^2}. \tag{2.9}$$

If $t < s$, we obtain the same results. Moreover a system that admits integrable dichotomy is not necessarily exponentially stable what means that integrable dichotomy is more general than exponential dichotomy. For more details, one can see [13, 22].

Theorem 2.22 ([21])**.** *Assume that equation* (2.1) *has an integrable dichotomy and f is a bounded function. Then equation* (2.2) *has a unique bounded integral solution given by*

$$u(t) = \int_{\mathbb{R}} G(t,s)f(s)ds, \quad t \in \mathbb{R}. \tag{2.10}$$

3 Almost automorphic and pseudo almost automorphic solutions in the nonhomogeneous linear case

(H1) We assume that $(A(t))_{t\in\mathbb{R}}$ generates an evolution family $\{U(t,s)\}_{(s\leq t\in\mathbb{R})}$, on X *i.e.* $(A(t), D(A(t)))$, $t \in \mathbb{R}$ satisfy conditions $(\mathbf{A}_1)$-$(\mathbf{A}_3)$.

(H2) The evolution family $U(t,s)$ generated by $A(t)$ has an integrable dichotomy satisfying (2.3) with function λ, dichotomy projections $P(t)$, $t \in \mathbb{R}$, and Green's function $G(t,s)$.

(H3) The Green's function $G(t,s)x$ function is bi-almost automorphic in $t, s \in \mathbb{R}$, for all $x \in X$.

We first consider the nonhomogeneous linear case

$$u'(t) = A(t)u(t) + f(t), \tag{3.1}$$

where $f : \mathbb{R} \to X$ is a function.

3.1 Almost automorphic solutions of equation (3.1)

Theorem 3.1. *Assume that* **(H1)**, **(H2)** *hold and* $f \in AA(\mathbb{R}, X)$. *Then equation* (3.1) *has a unique almost automorphic mild solution given by*

$$u(t) = \int_{\mathbb{R}} G(t,s)f(s)ds, \quad t \in \mathbb{R}. \tag{3.2}$$

Proof. By the Theorem 2.22, u is a unique mild solution to equation (3.1). Now, it remains to show that $u \in AA(\mathbb{R}, X)$. Let $\{\tau_n'\}$ be a sequence of real numbers. Since $f \in AA(\mathbb{R}, X)$, there exists a subsequence $\{\tau_n\}$ of $\{\tau_n'\}$ such that

$$\lim_n G(t+\tau_n, s+\tau_n) = \tilde{G}(t,s), \text{ and } \lim_n \tilde{G}(t-\tau_n, s-\tau_n) = G(t,s),$$

$\tilde{f}(t) = \lim_{n\to\infty} f(t+s_n)$ and $f(t) = \lim_{n\to\infty} \tilde{f}(t-s_n)$ for each $t, s \in \mathbb{R}$. Now, we define

$$\tilde{u}(t) = \int_{\mathbb{R}} \tilde{G}(t,s)\tilde{f}(s)ds, \quad t \in \mathbb{R}.$$

Note that

$$\begin{aligned}\|u(t+\tau_n) - \tilde{u}(t)\| &= \left\| \int_{\mathbb{R}} G(t+\tau_n, s)f(s)ds - \int_{\mathbb{R}} \tilde{G}(t,s)\tilde{f}(s)ds \right\| \\ &= \left\| \int_{\mathbb{R}} G(t+\tau_n, s+\tau_n)f(s+\tau_n)ds - \int_{\mathbb{R}} \tilde{G}(t,s)\tilde{f}(s)ds \right\|\end{aligned}$$

$$\leq \int_{\mathbb{R}} \left\| G(t+\tau_n, s+\tau_n) \left[f(s+\tau_n) - \tilde{f}(s) \right] \right\| ds$$
$$+ \int_{\mathbb{R}} \left\| \left[G(t+\tau_n, s+\tau_n) - \tilde{G}(t,s) \right] \tilde{f}(s) \right\| ds.$$

Let

$$I_{1,n} := \int_{\mathbb{R}} G(t+\tau_n, s+\tau_n) \left[f(s+\tau_n) - \tilde{f}(s) \right] ds$$

and

$$I_{2,n} := \int_{\mathbb{R}} \left[G(t+\tau_n, s+\tau_n) - \tilde{G}(t,s) \right] \tilde{f}(s) ds.$$

We have

$$I_{1,n} \leq \int_{\mathbb{R}} \lambda(t,s) \left[f(s+\tau_n) - \tilde{f}(s) \right] ds.$$

Since $f \in AA(\mathbb{R}, X)$ and by the dominated convergence Theorem, it follows that $I_{1,n} \to 0$ as $n \to \infty$.

For $I_{2,n}$ since $G(t,s)$ is bi-almost automorphic, given $\varepsilon > 0$, there is $N > 0$ such that for $n \geq N$, we have

$$\|G(t+\tau_n, s+\tau_n)\tilde{f}(s) - \tilde{G}(t,s)\tilde{f}(s)\| < \varepsilon \|f\|_\infty, \quad t, s \in \mathbb{R},$$

so for $n \geq N$,

$$I_{2,n} \leq \int_{\mathbb{R}} \|G(t+\tau_n, s+\tau_n)\tilde{f}(s) - \tilde{G}(t,s)\tilde{f}(s)\| ds$$

Thus, by the dominated convergence Theorem we have that $I_{2,n} \to 0$ as $n \to \infty$. Thus $\lim_n u(t+\tau_n) = \tilde{u}(t)$. We can show in a similar way that $\lim_n \tilde{u}(t-\tau_n) = u(t)$. Hence, $\lim_n u(t+\tau_n) = \tilde{u}(t)$ and $\lim_n \tilde{u}(t-\tau_n) = u(t)$, for $t \in \mathbb{R}$. Therefore, we conclude that $u \in AA(\mathbb{R}, X)$. □

Theorem 3.2. *Let $\mu \in \mathcal{M}$. Assume that* **(H1)**-**(H3)** *are satisfied and $f \in PAA(\mathbb{R}, X, \mu)$. Let u be a bounded solution of equation* (3.1). *Then $u \in PAA(\mathbb{R}, X, \mu)$.*

Proof. Let $f = g + h \in PAA(\mathbb{R}, X, \mu)$, where $g \in AA(\mathbb{R}, X)$ and $h \in \mathcal{E}(\mathbb{R}, X, \mu)$. Then u has a unique decomposition:

$$u = u_1 + u_2$$

where, for all $t \in \mathbb{R}$, we have

$$u_1(t) = \int_{\mathbb{R}} G(t,s) g(s) ds$$

and

$$u_2(t) = \int_{\mathbb{R}} G(t,s) h(s) ds$$

Using Theorem 3.1, we obtain that $u_1 \in AA(\mathbb{R}, X)$. It remains to show that $u_2 \in \mathcal{E}(\mathbb{R}, X, \mu)$. Let

$r > 0$. Then,

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t)\|\,d\mu(t) &= \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{\mathbb{R}} G(t,s)h(s)ds\right\| d\mu(t)\\ &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{-\infty}^{t} G(t,s)h(s)ds\right\| d\mu(t)\\ &+ \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{t}^{\infty} G(t,s)h(s)ds\right\| d\mu(t).\end{aligned}$$

For any fixed $r > 0$, we have

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{-\infty}^{t} G(t,s)h(s)ds\right\| d\mu(t) &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{-\infty}^{-r}\|G(t,s)h(s)\|ds\,d\mu(t)\\ &+ \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{-r}^{t}\|G(t,s)h(s)\|ds\,d\mu(t)\\ &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{-\infty}^{-r}\lambda(t,s)\|h(s)\|ds\,d\mu(t)\\ &+ \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{-r}^{t}\lambda(t,s)\|h(s)\|ds\,d\mu(t).\end{aligned}$$

By assumption (**H3**) and by changing the order of integration, we have

$$\int_{-r}^{r}\int_{-\infty}^{-r}\lambda(t,s)\|h(s)\|dsd\mu(t) := \int_{-\infty}^{-r}\left(\int_{-r}^{r}\lambda(t,s)d\mu(t)\right)\|h(s)\|ds \leq \|h\|_\infty\int_{-\infty}^{-r}\lambda_1(s)ds \leq C\|h\|_\infty,$$

and

$$\int_{-r}^{r}\int_{-r}^{t}\lambda(t,s)\|h(s)\|ds\,d\mu(t) := \int_{-r}^{r}\left(\int_{t}^{r}\lambda(t,s)d\mu(t)\right)\|h(s)\|ds \leq C\int_{-r}^{r}\|h(s)\|ds.$$

By a similary way, we have

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{t}^{\infty} G(t,s)h(s)ds\right\| d\mu(t) &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{t}^{\infty}\|G(t,s)h(s)\|ds\,d\mu(t)\\ &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{t}^{\infty}\lambda(t,s)\|h(s)\|ds\,d\mu(t)\\ &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{t}^{r}\lambda(t,s)\|h(s)\|ds\,d\mu(t)\\ &+ \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{r}^{\infty}\lambda(t,s)\|h(s)\|ds\,d\mu(t).\end{aligned}$$

By assumption (**H3**) and by changing the order of integration, we have

$$\int_{-r}^{r}\int_{t}^{r}\lambda(t,s)\|h(s)\|ds\,d\mu(t) := \int_{-r}^{r}\left(\int_{-r}^{s}\lambda(t,s)d\mu(t)\right)\|h(s)\|ds \leq C\int_{-r}^{r}\|h(s)\|ds,$$

and

$$\int_{-r}^{r}\int_{r}^{\infty}\lambda(t,s)\|h(s)\|ds\,d\mu(t)=\int_{r}^{\infty}\left(\int_{-r}^{s}\lambda(t,s)d\mu(t)\right)\|h(s)\|ds\leq\|h\|_\infty\int_{r}^{\infty}\lambda_2(s)ds\leq C\|h\|_\infty.$$

Thus, we have

$$\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t)\|d\mu(t)\leq\frac{2C}{\mu([-r,r])}\left(\|h\|_\infty+\int_{-r}^{r}\|h(s)\|ds\right). \tag{3.3}$$

From (3.3), we claim that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t)\|d\mu(t)=0.$$

Hence, $u_2\in PAA(\mathbb{R},X,\mu)$. We obtain the proof of the theorem. □

4 μ-pseudo almost automorphic solutions of equations (4.1) and (4.2)

Let X and Y be Banach spaces and $BC(\mathbb{R}\times X,Y)$ be the Banach space of all bounded continuous functions from $\mathbb{R}\times X$ in Y with the supremum norm of $\|\cdot\|_\infty$. In this section, we consider the nonlinear differential equation (4.1), where $f:\mathbb{R}\times X\to X$ is a function under convenient conditions,

$$u'(t)=A(t)u(t)+f(t,u(t)),\quad t\in\mathbb{R}, \tag{4.1}$$

and we analyze the delay case, were $\tau>0$ is fixed,

$$u'(t)=A(t)u(t)+f(t,u(t),u(t-\tau)),\quad t\in\mathbb{R}. \tag{4.2}$$

Definition 4.1. *A bounded continuous function $u:\mathbb{R}\to X$ is called a mild solution of equation* (4.1) *if*

$$u(t)=\int_{\mathbb{R}}G(t,s)f(s,u(s),u(s-\tau))ds,\quad t\in\mathbb{R}. \tag{4.3}$$

Definition 4.2. *A bounded continuous function $u:\mathbb{R}\to X$ is called a mild solution of equation* (4.2) *if*

$$u(t)=\int_{\mathbb{R}}G(t,s)f(s,u(s))ds,\quad t\in\mathbb{R}. \tag{4.4}$$

4.1 Existence of almost automorphic solutions to equation (4.1)

We need the following additional assumption:

(H4) There exists $\kappa > 0$ constant such that

$$\|f(t,u_1) - f(t,u_2)\| \leq \kappa \|u_1 - u_2\|, \quad \text{for all } t \in \mathbb{R},\, u_1, u_2 \in X. \tag{4.5}$$

Theorem 4.3. *Let $\mu \in \mathcal{M}$ satisfy* **(M)**. *Asumme that* **(H1)**-**(H4)** *hold and $f \in PAA(\mathbb{R}\times X, X, \mu)$ with*

$$\kappa < \frac{1}{L}$$

Then, equation (4.1) *has a unique mild solution $u \in PAA(\mathbb{R}, X, \mu)$ given by*

$$u(t) = \int_{\mathbb{R}} G(t,s) f(s, u(s)) ds, \quad t \in \mathbb{R}.$$

Proof. Let define the functional Λ on $PAA(\mathbb{R}, X, \mu)$ by

$$(\Lambda\phi)(t) = \int_{\mathbb{R}} G(t,s) f(s, \phi(s)) ds, \quad t \in \mathbb{R}.$$

By the composition Theorem 2.15 and Theorem 3.2, one has $\Lambda(PAA(\mathbb{R}, X, \mu)) \subset PAA(\mathbb{R}, X, \mu)$. Moreover we prove existence and uniqueness of solution to equation (4.1). Considering the fact that $\|f\|_\infty < \infty$, for all $t \in \mathbb{R}$, we have

$$\|(\Lambda\phi)(t)\| \leq \int_{-\infty}^{\infty} \|G(t,s) f(s,\phi(s))\| ds \leq \int_{-\infty}^{\infty} \lambda(t,s) \|f(s,\phi(s))\| ds \leq \|f\|_\infty \int_{-\infty}^{\infty} \lambda(t,s) ds \leq L\|f\|_\infty.$$

This proves that $\Lambda\phi$ is bounded. Now, we will prove that Λ is a contraction.

$$\begin{aligned} \|(\Lambda\phi)(t) - (\Lambda\varphi)(t)\| &\leq \int_{-\infty}^{\infty} \|G(t,s)\| \|f(s,\phi(s)) - f(s,\varphi(s))\| ds \\ &\leq \int_{-\infty}^{\infty} \lambda(t,s) \|f(s,\phi(s)) - f(s,\varphi(s))\| ds \\ &\leq \kappa \|\phi - \varphi\|_\infty \int_{\mathbb{R}} \lambda(t,s) ds \leq \kappa L \|\phi - \varphi\|_\infty. \end{aligned}$$

Therefore, by the Banach fixed point theorem, Λ has a unique fixed point such that $\Lambda\phi = \phi$, which is a μ-pseudo almost automorphic mild solution of equation (4.1). □

4.2 Existence of almost automorphic solutions to equation (4.2)

We need the following additional assumption:

(H5) The function $f(t,u,v)$ is locally Lipschitz in $u,v \in X$ *i.e.* for each positive number θ, for all, u_1,u_2,v_1,v_2 with $\|u_i\| \leq \theta$, $\|v_i\| \leq \theta$, $i=1,2$

$$\|f(t,u_1,v_1)-f(t,u_2,v_2)\| \leq k_1(\theta)\|u_1-u_2\|+k_2(\theta)\|v_1-v_2\|, \tag{4.6}$$

where $k_1,k_2:[0,\infty)\to[0,\infty)$ are functions and there is a positive constant ρ, such that $2\max(k_1(\rho),k_2(\rho))<\frac{1}{L}$ and $\sup_{t\in\mathbb{R}}\|f(t,0,0)\| \leq \frac{\rho}{L}\left[1-2L\max(k_1(\rho),k_2(\rho))\right]$.

Theorem 4.4. *Assume that* **(H1)-(H3)** *and f hold* **(H5)**. *Then, equation* (4.2) *has a unique bounded solution $u(t)$, $t\in\mathbb{R}$, with $\|u\|_\infty \leq \rho$.*

Proof. Let $G(t,s)$ be the Green's function associated with the equation (4.2) and we define the functional on X by

$$(\Gamma\phi)(t)=\int_{-\infty}^{\infty} G(t,s)f(s,\phi(s),\phi(s-\tau))ds, \quad t\in\mathbb{R}.$$

We show that Γ has a fixed point. First, we prove that Γ is bounded. There are ρ constant positive and a ball $\overline{\mathcal{B}}(0,\rho)$ which satisfies assumption (**H5**). Thus, we have,

$$\begin{aligned}
\|(\Gamma\phi)(t)\| &\leq \int_{-\infty}^{\infty}\|G(t,s)f(s,\phi(s),\phi(s-\tau))\|ds \leq \int_{-\infty}^{\infty}\lambda(t,s)\|f(s,\phi(s),\phi(s-\tau))\|ds\\
&\leq (k_1(\rho)+k_2(\rho))\int_{-\infty}^{\infty}\lambda(t,s)\|\phi(s)\|ds+\int_{-\infty}^{\infty}\lambda(t,s)\|f(s,0,0)\|ds\\
&\leq L(k_1(\rho)+k_2(\rho))\|\phi\|_\infty + L\sup_{t\in\mathbb{R}}\|f(t,0,0)\|\\
&\leq 2L\max(k_1(\rho),k_2(\rho))\rho+\rho\left[1-2L\max(k_1(\rho),k_2(\rho))\right]\leq\rho
\end{aligned}$$

This proves that $\Gamma\phi\in\overline{\mathcal{B}}(0,\rho)$ for all $\phi\in\overline{\mathcal{B}}(0,\rho)$. Finally, we prove that Γ is a contraction in $\overline{\mathcal{B}}(0,\rho)$. In fact,

$$\begin{aligned}
\|(\Gamma\phi)(t)-(\Gamma\varphi)(t)\| &\leq \int_{-\infty}^{\infty}\|G(t,s)\|\|f(s,\phi(s),\phi(s-\tau))-f(s,\varphi(s),\varphi(s-\tau))\|ds\\
&\leq \int_{-\infty}^{\infty}\lambda(t,s)\|f(s,\phi(s),\phi(s-\tau))-f(s,\varphi(s),\varphi(s-\tau))\|ds\\
&\leq L\int_{-\infty}^{\infty}\|k_1(\rho)\|\phi(s)-\varphi(s)\|+k_2(\rho)\|\phi(s-\tau)-\varphi(s-\tau)\|ds\\
&\leq L(k_1(\rho)+k_2(\rho))\|\phi-\varphi\|_\infty.
\end{aligned}$$

Using Banach fixed point Theorem, we deduce by **(H5)** that Γ has a fixed point ϕ. $\square$

Now, we will prove that equation (4.2) has an almost automorphic solution.

Theorem 4.5. *Assume that* **(H1)-(H3)** *and* **(H5)** *hold and* $f \in AA(\mathbb{R} \times X \times X, X)$*. Then, equation* (4.2) *has a unique almost automorphic mild solution* $u(t)$*,* $t \in \mathbb{R}$*, with* $\|u\|_\infty \leq \rho$*.*

Proof. We define the functional on X as in Theorem 4.4 by

$$(\Gamma\phi)(t) = \int_{-\infty}^{\infty} G(t,s) f(s,\phi(s),\phi(s-\tau))ds, \quad t \in \mathbb{R}.$$

We show that $\Gamma(AA(\mathbb{R},X)) \subset AA(\mathbb{R},X)$. Since $f \in AA(\mathbb{R} \times X \times X, X)$, and for each $u \in \overline{\mathcal{B}}(0,\rho)$ there exists a subsequence $\{\tau_n\}$ of $\{\tau_n'\}$ such that

$$\lim_n G(t+\tau_n, s+\tau_n)x - \tilde{G}(t,s)x = 0, \quad \text{and} \quad \lim_n \tilde{G}(t-\tau_n, s-\tau_n)x - G(t,s)x = 0,$$

$$\tilde{f}(t,u(t),u(t-\tau)) = \lim_{n\to\infty} f(t+s_n, u(t+s_n), u(t+s_n-\tau))$$

and

$$f(t) = \lim_{n\to\infty} \tilde{f}(t-s_n, u(t-s_n), u(t-s_n-\tau))$$

for each $t,s \in \mathbb{R}$, $x \in K$. Thus, we have

$$\tilde{\Gamma u}(t) = \int_{\mathbb{R}} \tilde{G}(t,s)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))ds, \quad t \in \mathbb{R}.$$

Note that

$$\begin{aligned}
\|\Gamma u(t+\tau_n) - \tilde{\Gamma u}(t)\| &= \left\| \int_{\mathbb{R}} G(t+\tau_n, s) f(s,u(s),u(s-\tau))ds - \int_{\mathbb{R}} \tilde{G}(t,s)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))ds \right\| \\
&= \left\| \int_{\mathbb{R}} G(t+\tau_n, s+\tau_n) f(s+\tau_n, u(s+s_n), u(s+s_n-\tau))ds \right. \\
&\quad \left. - \int_{\mathbb{R}} \tilde{G}(t,s)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))ds \right\| \\
&\leq \int_{\mathbb{R}} \left\| G(t+\tau_n, s+\tau_n) \left[f(s+\tau_n, u(s+s_n), u(s+s_n-\tau)) - \tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau)) \right] \right\| ds \\
&\quad + \int_{\mathbb{R}} \left\| \left[G(t+\tau_n, s+\tau_n) - \tilde{G}(t,s) \right] \tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau)) \right\| ds.
\end{aligned}$$

Let

$$J_{1,n} := \int_{\mathbb{R}} G(t+\tau_n, s+\tau_n) \left[f(s+\tau_n, u(s+s_n), u(s+s_n-\tau)) - \tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau)) \right] ds$$

and

$$J_{2,n} := \int_{\mathbb{R}} \left[G(t+\tau_n, s+\tau_n) - \tilde{G}(t,s) \right] \tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))ds.$$

We have

$$J_{1,n} \leq \int_{\mathbb{R}} \lambda(t,s) \left[f(s+\tau_n, u(s+s_n), u(s+s_n-\tau)) - \tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau)) \right] ds.$$

Since $f \in AA(\mathbb{R}\times X\times X, X)$ and by the dominated convergence theorem, it follows that $J_{1,n} \to 0$ as $n \to \infty$.

For $J_{2,n}$ since $G(t,s)$ is bi-almost automorphic, given $\varepsilon > 0$, there is $N > 0$ such that for $n \geq N$, we have

$$\|G(t+\tau_n, s+\tau_n)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau) - \tilde{G}(t,s)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))\| < \varepsilon \|f\|_\infty, \quad t,s \in \mathbb{R},$$

so for $n \geq N$,

$$J_{2,n} \leq \int_{\mathbb{R}} \|G(t+\tau_n, s+\tau_n)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau) - \tilde{G}(t,s)\tilde{f}(s,\tilde{u}(s),\tilde{u}(s-\tau))\| ds.$$

Thus, by the dominated convergence theorem we have that $J_{2,n} \to 0$ as $n \to \infty$. Thus $\lim_n \Gamma u(t+\tau_n) = \tilde{\Gamma u}(t)$. We can show in a similar way that $\lim_n \tilde{\Gamma u}(t-\tau_n) = \Gamma u(t)$. Hence, $\lim_n \Gamma u(t+\tau_n) = \tilde{\Gamma u}(t)$ and $\lim_n \tilde{\Gamma u}(t-\tau_n) = \Gamma u(t)$, for $t \in \mathbb{R}$. By Theorem 3.2, equation (4.2) has a unique bounded mild solution $u(t)$, $t \in \mathbb{R}$, with $\|u\|_\infty \leq \rho$ and $u \in AA(\mathbb{R}, X)$. □

Theorem 4.6. *Let $\mu \in \mathcal{M}$ and μ satisfy* **(M)**. *Assume that* **(H1)**-**(H3)** *and* **(H5)** *hold and $f \in PAA(\mathbb{R}\times X\times X, X, \mu)$. Then, equation (4.2) has a unique μ-pseudo almost automorphic mild solution $u(t)$, $t \in \mathbb{R}$, with $\|u\|_\infty \leq \rho$.*

Proof. We define the functional on X as in Theorem 4.4 by

$$\Gamma\phi(t) = \int_{-\infty}^{\infty} G(t,s) f(s,\phi(s),\phi(s-\tau)) ds, \quad t \in \mathbb{R}.$$

By Theorem 4.4, equation (4.2) has a unique bounded mild solution $u(t)$, $t \in \mathbb{R}$, with $\|u\|_\infty \leq \rho$. Let $f = g+h \in PAA(\mathbb{R}\times X\times X, X, \mu)$ where $g \in AA(\mathbb{R}\times X\times X, X)$ and $h \in \mathcal{E}(\mathbb{R}\times X\times X, X, \mu)$. Thus, $\Gamma\phi$ has a unique decomposition:

$$\Gamma\phi(t) = u_1(t) + u_2(t)$$

where, for all $t \in \mathbb{R}$, we have

$$u_1(t) = \int_{\mathbb{R}} G(t,s) g(s,u(s),u(s-\tau)) ds$$

and

$$u_2(t) = \int_{\mathbb{R}} G(t,s) h(s,u(s),u(s-\tau)) ds.$$

Using Theorem 4.5, we obtain that $u_1 \in AA(\mathbb{R}, X)$. It remains to show that $u_2 \in \mathcal{E}(\mathbb{R}, X, \mu)$. Let $r > 0$. Then,

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t)\|\,d\mu(t) &= \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{\mathbb{R}} G(t,s)h(s,u(s),u(s-\tau))ds\right\|d\mu(t)\\ &\leq \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{-\infty}^{t} G(t,s)h(s,u(s),u(s-\tau))ds\right\|d\mu(t)\\ &+\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left\|\int_{t}^{\infty} G(t,s)h(s,u(s),u(s-\tau))ds\right\|d\mu(t)\end{aligned}$$

For any fixed $r > 0$, by calculations similar as to the Theorem 3.2, we have

$$\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t)\|d\mu(t) \leq \frac{2C}{\mu([-r,r])}\left(\|h\|_\infty + \int_{-r}^{r}\|h(s)\|ds\right) \tag{4.7}$$

From (4.7), we claim that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|u_2(t)\|d\mu(t) = 0$$

Hence, $u_2 \in PAA(\mathbb{R}, X, \mu)$. We obtain the proof of the Theorem. □

5 Applications

In the next example, we show that integrable dichotomy is a generalization of exponential dichotomy.

Example 5.1. *We give an example of family of operators $(A(t))_{t\in\mathbb{R}}$ that generates an evolution family with an integrable dichotomy. Let $\{b_k\}_{k\in\mathbb{Z}}$ be a positive Riemamn sequence such that $b_k = \frac{1}{k^2+1}$. Let $J_k := [k - b_k^2, k + b_k^2]$, for $k \in \mathbb{Z}$. Let $\ell : \mathbb{R} \to (0, \infty)$ be continuously differentiable function given by $\ell(t) = 1$, if $t \notin J_k$ and in J_k, $\ell(t) \in \left[\frac{1}{k^2+1}, 1\right]$ where $\ell(k) = b_k$. We have*

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\int_{J_k}\ell^{-1}(s)ds &= \sum_{k\in\mathbb{Z}}\int_{k-\frac{1}{(k^2+1)^2}}^{k+\frac{1}{(k^2+1)^2}}(k^2+1)ds = 2\sum_{k\in\mathbb{Z}}\frac{k^2+1}{(k^2+1)^2}\\ &\leq 2\left(1+2\sum_{k=1}^{\infty}\frac{1}{k^2}\right) \leq 2\left(\frac{\pi^2}{3}+1\right) < \infty.\end{aligned}$$

Consider the scalar differential equation

$$u'(t) = a(t)u(t), \quad a(t) = -\alpha + \ell'(t)\ell(t)^{-1}, \quad \alpha > 0, \tag{5.1}$$

one has

$$u(t) = u_0 e^{-\alpha t}\ell(t) \text{ where } u_0 \text{ is the initial data .}$$

It is well-known that the evolution family of the equation (5.1) *with projections* $P(t) = I$, $t \in \mathbb{R}$ *is given by* $U(t,s) = e^{-\alpha(t-s)}\frac{\ell(t)}{\ell(s)}$. *We have Green's function* $G(t,s) = U(t,s)$ *has an integrable dichotomy. Indeed,*

$$\int_{-\infty}^{t} U(t,s)ds \leq \int_{-\infty}^{t} e^{-\alpha(t-s)} + \sum_{k=-\infty}^{[t]+2} \int_{J_k} \ell^{-1}(s)ds \leq \frac{1}{\alpha} + 2\left(\frac{\pi^2}{3} + 1\right) < \infty.$$

Condition (2.4) *is satisfied with* $L = \frac{1}{\alpha} + 2\left(\frac{\pi^2}{3} + 1\right)$. *The equation* (5.1) *is not exponentialy stable. In fact,*

$$U(k + b_k^2, k) = (k^2+1)e^{-\frac{\alpha}{(k^2+1)^2}} \to \infty, \quad \text{as } k \to \infty.$$

Thus integrable dichotomy is more general than the exponential dichotomy. Note that

$$\mid U(t,s) \mid \leq e^{-\alpha(t-s)} + \lambda_0(s), \quad s \leq t$$

with

$$\lambda_0(s) = \sum_{k\in\mathbb{Z}} \ell^{-1}(s)\chi_{J_k}(s),$$

where χ_{J_k} *is the characteristic function on* J_k. *It is clear that* $\lambda_0 \in L^1(\mathbb{R})$. *Then equation* (5.1) *has an integrable dichotomy with* $\lambda(t,s) = e^{-\alpha(t-s)} + \lambda_0(s)$, $s \leq t$ *satisfying*

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \lambda(t,s)ds \leq L. \tag{5.2}$$

In a similar way, we can prove that

$$\sup_{t\in\mathbb{R}} \int_{t}^{\infty} U(t,s)ds \leq L, \tag{5.3}$$

but the evolution family is not exponentially stable at $-\infty$. *Let the diagonal matrix*

$$A(t) = diag(b_1(t), b_2(t), \ldots, b_n(t))$$

with each b_i *satisfying* (5.2) *for* $i = 1, \ldots, k$ *and satisfying* (5.3) *for* $i = k+1, \ldots, n$ *(*$k > 0$*). Then, this construction yields the linear system*

$$x' = A(t)x$$

which has an integrable dichotomy with

$$\lambda(t,s) = e^{-|t-s|} + \lambda_0(s), \quad t,s \in \mathbb{R},$$

λ_0 *integrable in* $\mathbb{R}$. *We consider the projections*

$$P(t) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad Q(t) = I - P(t) = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix},$$

where I_r *and* I_{n-r} *are identity matrix of order respectively* r *and* $n-r$. *Finally, one extend the diagonal and integrable caracter of the dichotomy of* $A(t)$ *to a diagonal infinite dimensional.*

Example 5.2. *Let* μ *be a mesure with a Radon-Nikodym derivative* ρ *defined by:*

$$\rho(t) = \begin{cases} e^t, & t \leq 0 \\ 1, & t > 1. \end{cases} \tag{5.4}$$

We consider the existence and uniqueness of a μ*-pseudo almost automorphic solutions for the following system:*

$$\begin{cases} \dfrac{\partial u(t,\xi)}{\partial t} = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + \alpha(t)u(t,\xi) + g(t, u(t,\xi)), & t \in \mathbb{R}, \quad \xi \in [0,\pi], \\ u'(t,0) = u'(t,\pi) = 0, & t \in \mathbb{R}, \end{cases} \tag{5.5}$$

where $\alpha(t) = \frac{1}{2}\sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right) \in AA(\mathbb{R}, X)$. *Take* $X = L^2[0,\pi]$ *with norm* $\|\cdot\|$ *and inner product* $(\cdot,\cdot)_2$. $g : \mathbb{R} \times L^2[0,\pi] \to L^2[0,\pi]$ *is* μ*-pseudo almost automorphic with*

$$g(t,\xi) = e^{-|t|}\psi(\xi),$$

where $t \mapsto e^{-|t|}$ *belongs to* $\mathcal{E}(\mathbb{R},\mathbb{R},\mu)$. *The function* ψ *is Lipschitzian. Let* $\kappa > 0$

$$|\psi(x) - \psi(y)| \leq \kappa|x - y|.$$

Let $f : \mathbb{R} \times L^2[0,\pi] \to L^2[0,\pi]$ *be a function defined by*

$$f(t,v)(x) = e^{-|t|}\psi(v(x)).$$

We define $A : D(A) \subset X \to X$ *by*

$$A\phi = \phi'' \ \ \textit{for} \ \ \phi(\cdot) \in D(A),$$

with domain

$$D(A) = \{u \in H^2(0,\pi) : u'(0) = u'(\pi) = 0\}.$$

It is well-known that the operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X such that $\|T(t)\| \leq 1$ for $t \geq 0$. Moreover, we have

$$T(t)\phi = \sum_{n=0}^{\infty} e^{-n^2 t}(\phi, e_n)_2 e_n, \quad \textit{for all} \quad t \geq 0, \ \phi \in X,$$

with $e_n(t) = \sqrt{\frac{2}{\pi}} \cos(nt)$ *for each* $n \in \mathbb{N}$. *Define a family of linear operators* $A(t)$ *by:*

$$A(t) = \frac{\partial^2}{\partial x^2} + \alpha(t)I = A + \alpha(t)I \quad \textit{for } t \in \mathbb{R},$$

with domain

$$D(A(t)) = D(A) = \{u \in H^2(0,\pi) : u'(0) = u'(\pi) = 0\}.$$

It is easy to see that the family of linear operators $A(t)$ *satisfy assumptions* **$(\mathbf{A}_1)$-$(\mathbf{A}_3)$**. *Indeed, just take* $Y = X$, $M = 1$ *and* $\omega = \frac{1}{2}$.

Let $v(t) = u(t, \cdot)$. *Then* (5.5) *becomes*

$$\frac{d}{dt}v(t) = A(t)v(t) + f(t, v(t)).$$

The operators $A(t)$ *generate an evolution family* $(U(t,s))_{t\geq s}$ *given by:*

$$U(t,s)\phi = \sum_{n=0}^{\infty} e^{\int_s^t [\alpha(\tau) - n^2] d\tau}(\phi, e_n)_2 e_n, \quad \textit{for all } t \geq s, \ \phi \in X.$$

Lemma 5.3. *The evolution family has an integrable dichotomy with data* (λ, P).

Proof. We divide the series in two parts *i.e.*, thus

$$U(t,s)\phi = e^{\int_s^t [\alpha(\tau)-1] d\tau}(\phi, e_0)_2 e_0 + \sum_{n=1}^{\infty} e^{\int_s^t [\alpha(\tau) - n^2] d\tau}(\phi, e_n)_2 e_n, \quad \text{for all} \ \ t \geq s, \ \phi \in X.$$

For $t \geq s$ and $\phi \in Vect\{e_0\}$,

$$|U(t,s)\phi| = |e^{\int_s^t \alpha(\tau) d\tau}(\phi, e_0)_2 e_0| \leq e^{\frac{1}{2}(t-s)}|\phi|.$$

Let $\phi \in Vect\{e_n; n = 1, 2, \dots\}$,

$$|U(t,s)\phi| = \left| \sum_{n=1}^{\infty} e^{\int_s^t [\alpha(\tau) - n^2] d\tau}(\phi, e_n)_2 e_n \right| \leq e^{\int_s^t [\alpha(\tau)-1] d\tau} \left| \sum_{n=1}^{\infty} (\phi, e_n)_2 e_n \right| \leq e^{-\int_s^t [1-\alpha(\tau)] d\tau}|\phi|.$$

Let $I-P = diag(1,0,\dots,0,0,0,\dots)$ and $P = diag(0,1,1,\dots)$ be projections with $Rank(I-P)=1$ and $Rank(P)=\infty$. Thus, the Green function is defined by

$$G(t,s) = \begin{cases} U(t,s)P = \sum_{n=1}^{\infty} e^{\int_s^t [\alpha(\tau)-n^2]d\tau} e_n, & \text{if } t \geq s, \\ -\tilde{U}(t,s)(I-P) = -e^{-\int_s^t \alpha(\tau)d\tau} e_0, & \text{if } t < s. \end{cases}$$

Then, $u'(t) = A(t)u(t)$ has an integrable dichotomy with data (λ, P), where λ is given by:

$$\lambda(t,s) = \begin{cases} e^{-\int_s^t [1-\alpha(\tau)]d\tau}, & \text{if } t \geq s, \\ e^{-\int_s^t \alpha(\tau)d\tau}, & \text{if } t < s. \end{cases}$$

Let us calculate L and C as mentioned in Definition 2.20. Let $t \in \mathbb{R}$, using the fact that $-\frac{1}{2} \leq \alpha(\tau) \leq \frac{1}{2}$, one obtain

$$\begin{aligned} \sup_{t\in\mathbb{R}} \int_{\mathbb{R}} \lambda(t,s)ds &= \sup_{t\in\mathbb{R}} \left(\int_{-\infty}^{t} e^{-\int_s^t [1-\alpha(\tau)]d\tau} ds + \int_t^{\infty} e^{-\int_s^t \alpha(\tau)d\tau} ds \right) \\ &\leq \left(\int_{-\infty}^{t} e^{-\frac{1}{2}(t-s)} ds + \int_t^{\infty} e^{\frac{1}{2}(t-s)} ds \right) = 4 = L. \end{aligned}$$

Now, let us verify hypothesis (**A**). Let $T > 0$, we have

$$\begin{aligned} \int_T^{\infty} \int_{-T}^{T} \lambda(t,s)d\mu(t)ds &= \int_T^{\infty} \left(\int_{-T}^{0} e^t e^{\frac{1}{2}(t-s)} dt + \int_0^T e^{\frac{1}{2}(t-s)} dt \right) ds \\ &\leq \left(\frac{2}{3} + 2e^{\frac{1}{2}T} \right) \int_T^{\infty} e^{-\frac{1}{2}s} ds \leq \frac{16}{3} = C. \end{aligned}$$

In a similar way, we can show that

$$\int_s^T \lambda(t,s)d\mu(t) \leq C, \int_{-T}^{s} \lambda(t,s)d\mu(t) \leq C, \quad \text{and} \quad \int_{-\infty}^{-T} \int_{-T}^{T} \lambda(t,s)d\mu(t)ds \leq C.$$ □

Hence, **(H1)** *and* **(H2)** *hold.*

Lemma 5.4. *The Green's function is bi-almost automorphic.*

Proof. Let $\alpha \in AA(\mathbb{R}, X)$, then, for every sequence $(s'_k)_{k\geq 0}$ of real numbers, there exists a subsequence $(s_k)_{k\geq 0} \subset (s'_k)_{k\geq 0}$ and a measurable function $\tilde{\alpha}$, such that

$$\lim_k \alpha(\tau + s_k) = \tilde{\alpha}(\tau) \quad \text{and} \quad \lim_k \tilde{\alpha}(\tau - s_k) = \alpha(\tau) \quad \text{for all } \tau \in \mathbb{R}.$$

Let us define, $\tilde{U}(t,s)\phi = T(t-s)e^{\int_s^t \tilde{\alpha}(\tau)d\tau}\phi$, for all $t \geq s$, $\phi \in X$. Since U is bi-almost automorphic, we have

$$\begin{aligned}\lim_k U(t+s_k, s+s_k)\phi - \tilde{U}(t,s)\phi &\leq \lim_k \left\| T(t-s)e^{\int_{s+s_k}^{t+s_k} \alpha(\tau)d\tau}\phi - T(t-s)e^{\int_s^t \tilde{\alpha}(\tau)d\tau}\phi \right\| \\ &\leq \lim_k \left\| T(t-s)\left(e^{\int_{s+s_k}^{t+s_k} \alpha(\tau)d\tau} - e^{\int_s^t \tilde{\alpha}(\tau)d\tau} \right)\phi \right\| \\ &\leq \lim_k \left\| T(t-s)\left(e^{\int_s^t \alpha(\tau - s_k)d\tau} - e^{\int_s^t \tilde{\alpha}(\tau)d\tau} \right)\phi \right\| \\ &\leq \lim_k \left\| T(t-s)e^{\int_s^t \tilde{\alpha}(\tau)d\tau}\left(e^{\int_s^t |\alpha(\tau - s_k) - \tilde{\alpha}(\tau)|d\tau} - 1 \right)\phi \right\|\end{aligned}$$

As $\alpha \in AA(\mathbb{R}, X)$, we have

$$\left| e^{\int_s^t |\alpha(\tau - s_k) - \tilde{\alpha}(\tau)|d\tau} - 1 \right| \to 0 \quad \text{as} \quad k \to \infty.$$

Then

$$\lim_k U(t+s_k, s+s_k)\phi - \tilde{U}(t,s)\phi = 0.$$

In a similar way, we can prove that $\lim_k \tilde{U}(t-s_k, s-s_k)\phi - U(t,s)\phi = 0$. Then, U is bi-almost automorphic. □

Consequently, all assumptions of the Theorem 4.3 are satisfied. We can deduce by the Theorem 4.3 that the problem (4.1) *has an unique μ-pseudo almost automorphic mild solution on $\mathbb{R}$, under the condition κ small enough.*

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Retraction Note: Heisenberg-type uncertainty principle for the second q-Bargmann transform on the unit disk

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The Editor-in-Chief and the Managing Editor of CUBO, A Mathematical Journal, have decided to retract this article, in accordance with the Ethics Statement of the journal. The article was submitted on September 8th, 2024, and it was scheduled to appear in vol. 27, no. 1, pp. 55–73, 2025.

After this manuscript was already been published online in CUBO's webpage, the original referee of the article informed the editorial team that a very similar article had been recently published on the Communications of the Korean Mathematical Society, vol. 40, no. 2, pp. 291–302, 2025. DOI: https://doi.org/10.4134/CKMS.c240150. After a careful analysis of both manuscripts, the editorial team concluded that this was an instance of self-plagiarism. Even though the titles are different, the results, techniques and discussion in both manuscripts are almost identical.

It is important to stress that the manuscript was accepted for publication in CUBO on March 20th, 2025, and it was accepted in the CKMS on February 18th, 2025. This situation is completely anomalous. The Editor-in-Chief of the CKMS has been informed.

The author has not responded to us regarding this retraction.

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Estimating the remainder of an alternating p-series revisited

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ABSTRACT

For the nth remainder $R_n(p) := \sum_{k=n+1}^{\infty}(-1)^{k+1}k^{-p}$ of an alternating p-series, several asymptotic estimates are presented. For example, for any integer $n \geq 3$, and $p \in \mathbb{R}^+$, we have

$$R_n(p) = \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{p}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+1}} + \varepsilon_n^*(p)$$

and

$$|\varepsilon_n^*(p)| < \frac{p(p+1)}{5\,(n-2)^{p+2}},$$

where $\lfloor x \rfloor$ denotes the integer part (the floor) of x.

RESUMEN

Para el residuo n-ésimo $R_n(p) := \sum_{k=n+1}^{\infty}(-1)^{k+1}k^{-p}$ de una p-serie alternante, se presentan diversas estimaciones asintóticas. Por ejemplo, para cualquier entero $n \geq 3$ y $p \in \mathbb{R}^+$, tenemos

$$R_n(p) = \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{p}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+1}} + \varepsilon_n^*(p)$$

y

$$|\varepsilon_n^*(p)| < \frac{p(p+1)}{5\,(n-2)^{p+2}},$$

donde $\lfloor x \rfloor$ denota la parte entera (el piso) de x.

Keywords and Phrases: Alternating generalized harmonic number, alternating p-series, approximation, Dirichlet's eta function, estimate, remainder, slow convergence.

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1 Introduction

In [5] it was shown that the best constants a and b such that inequalities

$$\frac{1}{2n+a} \leq \left| \sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{1}{k} \right| < \frac{1}{2n+b} \tag{1.1}$$

hold for every $n \geq 1$ are $a = \frac{1}{1-\ln 2} - 2 \approx 1.25\,8891$ and $b = 1$.

In the paper [1] it was proved for the nth remainder $R_n(p)$,

$$R_n(p) := \sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}}{k^p}, \tag{1.2}$$

for alternating p-series (for Dirichlet eta function $\eta(p)$, *i.e.* for the Riemann alternating zeta function),

$$\eta(p) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^p}\,, \tag{1.3}$$

the relations

$$a(n,p) := \frac{1}{2(n+1)^p - \rho(p)} \leq |R_n(p)| \leq \frac{1}{2n^p + \sigma(p)} =: b(n,p), \tag{1.4}$$

true for integers $n \geq 1$ and $p \geq 2$ and with (the best) constants

$$\rho(p) := 2^{p+1} - \frac{1}{1-(1-2^{1-p})\zeta(p)} \quad \text{and} \quad \sigma(p) := \frac{1}{1-(1-2^{1-p})\zeta(p)} - 2. \tag{1.5}$$

Accuracy or sharpness of the double inequality $A(x) \leq F(x) \leq B(x)$ at the point x we define as the difference $B(x) - A(x)$, *i.e.* as the width of the interval $[A(x), B(x)]$. For example the double inequality (1.1) has the sharpness equal to $\frac{a-b}{(2n+a)(2n+b)}$, *i.e.* $\mathcal{O}(\frac{1}{n^2})$, using the Landau big O notation. Similarly, the double inequality (1.4) has the sharpness $\mathcal{O}(\frac{1}{n^{p+1}})$.

Motivated by [3, 4] and [5], and especially by [1], where the validity of (1.4) is based on the supposition that p is a positive integer different from 1, we shall provide some estimates of the remainder $R_n(p)$, which are close to the relation (1.4) and are valid for any $p \in \mathbb{R}^+$.

2 Background

We shall use the results from the paper [2], where appear special sums[1]

$$\sigma_q^*(x,p) := \sum_{i=1}^{\lfloor q/2 \rfloor} \left(4^i - 1\right) \frac{B_{2i} \cdot p^{(2i-1)}}{x^{p+2i-1} \cdot (2i)!} \quad (q \in \mathbb{N},\, p, x \in \mathbb{R}), \tag{2.1}$$

[1] By definition, $\sum_{i=m}^{n} x_i = 0$ if $m > n$.

where B_k denotes the kth Bernoulli coefficient (or Bernoulli number)[2], defined by the identity $\frac{t}{e^t-1} \equiv \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$ ($|t| < 2\pi$), where the symbol $x^{(k)}$ designates the upper (rising) Pochhammer product defined as

$$x^{(0)} := 1, \;\; x^{(k)} := \prod_{i=0}^{k-1}(x+i) = x(x+1)\cdots(x+k-1) \quad (x \in \mathbb{R}, \; k \in \mathbb{N}), \tag{2.2}$$

and where the symbol $\lfloor x \rfloor$ denotes the integer part (the floor) of any $x \in \mathbb{R}^+$.

We will use the following lemma.

Lemma 2.1 ([2, Theorem 1])**.** *For $p \in \mathbb{R}^+$ and every $k, n, q \in \mathbb{N}$, with $n \geq 2k+1 \geq 3$, the nth remainder $R_n(p) := \eta(p) - \sum_{j=1}^{n} (-1)^{j+1} \frac{1}{j^p}$ is given in the form*

$$R_n(p) = \Delta_q(n,p) + \delta_q(k,p),$$

with

$$\Delta_q(n,p) = \frac{(-1)^n}{2\left(2\lfloor \frac{n+1}{2} \rfloor\right)^p} - \sigma_q^*\left(2\lfloor \tfrac{n+1}{2} \rfloor, p\right),$$

and

$$|\delta_q(k,p)| < \frac{5p^{(q-1)}}{3\pi^{q-1}} \cdot \frac{1}{(2k)^{p+q-1}} \,.$$

3 Asymptotic estimates of the remainder $R_n(p)$

Now, for any integer $n \geq 3$, the floor (the integer part) $\nu := \left\lfloor \frac{n-1}{2} \right\rfloor$ is a positive integer estimated as $\frac{n-1}{2} - 1 < \nu \leq \frac{n-1}{2}$. Consequently $n - 3 < 2\nu \leq n-1$, that is

$$n - 2 \leq 2\nu \leq n - 1. \tag{3.1}$$

Therefore, using $k = \nu$ in Lemma 2.1, together with the new naming $\varepsilon_n(p,q) := \delta_q(\nu, p)$, we obtain the next result.

Proposition 3.1. *For integers $n \geq 3$ and $q \geq 1$, and for $p \in \mathbb{R}^+$, we have*

$$R_n(p) = \frac{(-1)^n}{2\left(2\lfloor \frac{n+1}{2} \rfloor\right)^p} - \sum_{i=1}^{\lfloor q/2 \rfloor} \left(4^i - 1\right) \frac{B_{2i} \cdot p^{(2i-1)}}{\left(2\left\lfloor \frac{n+1}{2} \right\rfloor\right)^{p+2i-1} \cdot (2i)!} + \varepsilon_n(p,q),$$

[2] $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = B_5 = B_7 = \cdots = 0$, $B_4 = B_8 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, ...

where

$$|\varepsilon_n(p,q)| < \frac{5p^{(q-1)}}{3\pi^{q-1}} \cdot \frac{1}{\left(2\lfloor\frac{n-1}{2}\rfloor\right)^{p+q-1}} \le \frac{5p^{(q-1)}}{3\pi^{q-1}} \cdot \frac{1}{(n-2)^{p+q-1}}\,.$$

Here q is a parameter controlling the magnitude of the error term $\varepsilon_n(p,q)$.

Using $q=1$ in Proposition 3.1, we obtain the first corollary.

Corollary 3.2. *For an integer $n \ge 3$ and $p \in \mathbb{R}^+$ there hold the following estimates:*

$$\frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{5}{3\left(n-2\right)^p} < R_n(p) < \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} + \frac{5}{3\left(n-2\right)^p}$$

and

$$|R_n(p)| < \frac{1}{2\left(n-1\right)^p} + \frac{5}{3\left(n-2\right)^p}\,.$$

Putting $q=3$ in Proposition 3.1, we get the following corollary.

Corollary 3.3. *For $p \in \mathbb{R}^+$ and every integer $n \ge 3$, the formulas*

$$R_n(p) = \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{p}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+1}} + \varepsilon_n(p,3),$$

hold true, where

$$|\varepsilon_n(p,3)| < \frac{5}{3\pi^2} \cdot \frac{p(p+1)}{\left(2\lfloor\frac{n-1}{2}\rfloor\right)^{p+2}} < \frac{p(p+1)}{5\,(n-2)^{p+2}}$$

and

$$\left|\,\Big|R_n(p)\Big| - \left|\frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{p}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+1}}\right|\,\right| < \frac{p(p+1)}{5\,(n-2)^{p+2}}\,.$$

Setting $q=5$ in Proposition 3.1, we provide the following result.

Corollary 3.4. *For every integer $n \ge 3$ and $p \in \mathbb{R}^+$, there holds the equality*

$$R_n(p) = \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{p}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+1}} + \frac{p(p+1)(p+2)}{48\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+3}} + \varepsilon_n(p,5)$$

with the estimate

$$|\varepsilon_n(p,5)| < \frac{5}{3\pi^4} \cdot \frac{p(p+1)(p+2)(p+3)}{\left(2\lfloor\frac{n-1}{2}\rfloor\right)^{p+4}} < \frac{p(p+1)(p+2)(p+3)}{58\,(n-2)^{p+4}}\,.$$

4 Approximations of $\left|R_n(p)\right|$

Using the Landau big O notation, the relation (1.4) means that $\left|R_n(p)\right| = \mathcal{O}\left(\frac{1}{n^p}\right)$ as $n \to \infty$, for integers $p \geq 2$. However, the next Proposition 4.1 improves this result.

Proposition 4.1. *For integers $n \geq 3$ and $q \geq 1$, for any $p \in \mathbb{R}^+$, and for* $S_n(p,q) := \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \sum_{i=1}^{\lfloor q/2\rfloor} (4^i - 1) \frac{B_{2i} \cdot p^{(2i-1)}}{\left(2\left\lfloor\frac{n+1}{2}\right\rfloor\right)^{p+2i-1} \cdot (2i)!}$*, we have* $\left|R_n(p)\right| = \mathcal{O}\left(\frac{1}{n^{p+q-1}}\right)$ *as $n \to \infty$; more precisely*[3]

$$\left|S_n(p,q)\right| - \frac{5p^{(q-1)}}{3\pi^{q-1}(n-2)^{p+q-1}} \leq \left|R_n(p)\right| \leq \left|S_n(p,q)\right| + \frac{5p^{(q-1)}}{3\pi^{q-1}(n-2)^{p+q-1}}.$$

Proof. Thanks to Proposition 3.1, using the triangle inequalities, we have

$$\begin{aligned}\left|R_n(p)\right| &= \Big|\left|S_n(p,q)\right| - \left(\left|S_n(p,q)\right| - \left|R_n(p)\right|\right)\Big| \geq \left|S_n(p,q)\right| - \Big|\left|S_n(p,q)\right| - \left|R_n(p)\right|\Big| \\ &\geq \left|S_n(p,q)\right| - \left|S_n(p,q) - R_n(p)\right| = \left|S_n(p,q)\right| - \left|\varepsilon_n(p,q)\right|\end{aligned}$$

and

$$\begin{aligned}\left|R_n(p)\right| &= \Big|\left|S_n(p,q)\right| - \left(\left|S_n(p,q)\right| - \left|R_n(p)\right|\right)\Big| \leq \left|S_n(p,q)\right| + \Big|\left|S_n(p,q)\right| - \left|R_n(p)\right|\Big| \\ &\leq \left|S_n(p,q)\right| + \left|S_n(p,q) - R_n(p)\right| = \left|S_n(p,q)\right| + \left|\varepsilon_n(p,q)\right|.\end{aligned}$$

□

Numerical experiment. *Using the Mathematica computer system [6] and considering* (1.4), *together with Proposition 3.1, we obtain for functions*

$$A(n,p,q) := \left|\frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \sum_{i=1}^{\lfloor q/2\rfloor} \frac{(4^i-1)B_{2i} \cdot p^{(2i-1)}}{\left(2\left\lfloor\frac{n+1}{2}\right\rfloor\right)^{p+2i-1} \cdot (2i)!}\right| - \frac{5p^{(q-1)}}{3\pi^{q-1}n^{p+q-1}}$$

and

$$B(n,p,q) := \left|\frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \sum_{i=1}^{\lfloor q/2\rfloor} \frac{(4^i-1)B_{2i} \cdot p^{(2i-1)}}{\left(2\left\lfloor\frac{n+1}{2}\right\rfloor\right)^{p+2i-1} \cdot (2i)!}\right| + \frac{5p^{(q-1)}}{3\pi^{q-1}n^{p+q-1}}$$

the following estimates:

$$A(n,3,3) > a(n,3), \quad \textit{for } 6 \leq n \leq 100,$$
$$B(n,3,3) < b(n,3), \quad \textit{for } 4 \leq n \leq 100,$$
$$B(n,3,3) - A(n,3,3) < b(n,3) - a(n,3), \quad \textit{for } 5 \leq n \leq 100.$$

[3] At $q = 1$, the given lower bound for $\left|R_n(p)\right|$ is negative.

Similarly, we get

$$A(n,3,5) > a(n,3), \quad \textit{for } 4 \leq n \leq 100,$$

$$B(n,3,5) < b(n,3), \quad \textit{for } 3 \leq n \leq 100,$$

$$B(n,3,5) - A(n,3,5) < b(n,3) - a(n,3), \quad \textit{for } 3 \leq n \leq 100.$$

These inequalities are illustrated in Figures 1–3, where the graphs of the functions $n \mapsto A(n,3,q)/a(n,3)$, $n \mapsto B(n,3,q)/b(n,3)$ *and* $n \mapsto \big(B(n,3,q) - A(n,3,q)\big)/\big(b(n,3) - a(n,3)\big)$, *having* $q \in \{3,5\}$, *are plotted using the Mathematica software [6]. Thus, numerical examples confirm that our estimates of* $|R_n(p)|$, *given in Proposition 4.1, are more accurate, for* $n \geq 5$ *and* $q \geq 3$, *than that given in* (1.4). *This is consistent with the fact that the sharpness of the estimates for* $|R_n(p)|$ *given in Proposition 4.1, is equal to* $\mathcal{O}\left(\frac{1}{n^{p+q-1}}\right)$.

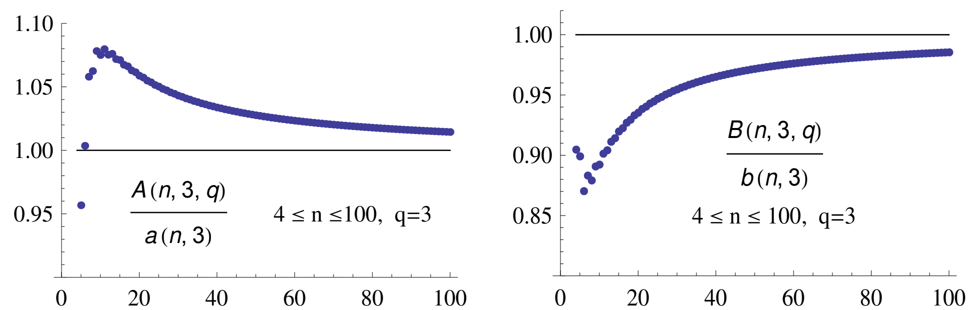

Figure 1: The graphs of the sequences $n \mapsto A(n,3,3)/a(n,3)$ (left) and $n \mapsto B(n,3,3)/b(n,3)$, (right).

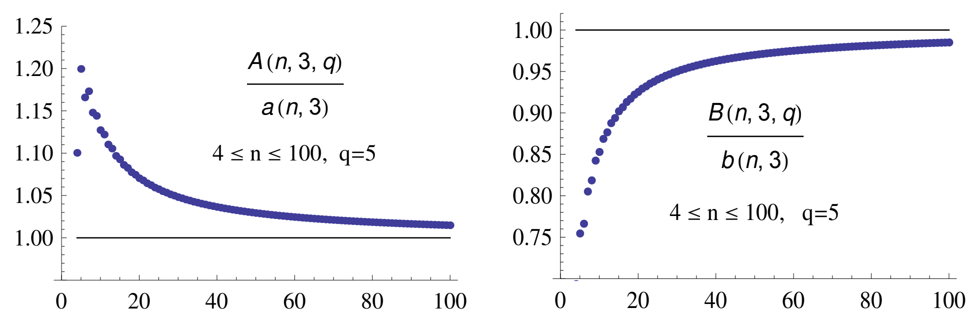

Figure 2: The graphs of the sequences $n \mapsto A(n,3,5)/a(n,3)$ (left) and $n \mapsto B(n,3,5)/b(n,3)$, (right).

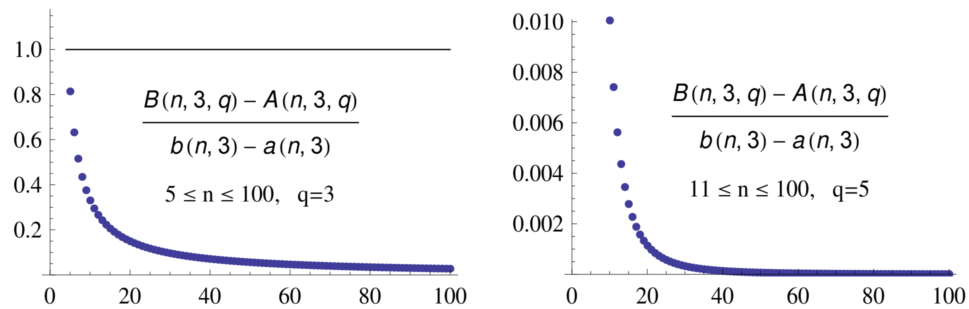

Figure 3: The graphs of the sequences $n \mapsto \frac{B(n,3,3)-A(n,3,3)}{b(n,3)-a(n,3)}$ (left) and $n \mapsto \frac{B(n,3,5)-A(n,3,5)}{b(n,3)-a(n,3)}$ (right).

4.1 Conclusion

The paper easily provides several asymptotic estimates of a remainder of an alternating p-series for all $p \in \mathbb{R}^+$. The presented relations supplement the double inequality for a remainder, given in the paper [1], which works only for integers $p \geq 2$. In addition, the derived estimates are very useful even in the case of $p \approx 0$.

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Congruences of infinite semidistributive lattices

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ABSTRACT

Not every finite distributive lattice is isomorphic to the congruence lattice of a finite semidistributive lattice. This note provides a construction showing that many of these finite distributive lattices are isomorphic to congruence lattices of infinite semidistributive lattices.

RESUMEN

No todo reticulado distributivo finito es isomorfo al reticulado de congruencia de un reticulado finito semidistributivo. Esta nota proporciona una construcción mostrando que muchos de estos reticulados finitos distributivos son isomorfos a reticulados de congruencia de reticulados infinitos semidistributivos.

Keywords and Phrases: Distributive lattice, semidistributive lattice, congruence lattice.

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1 Introduction

Congruence lattices of lattices are distributive, and every finite distributive lattice is isomorphic to the congruence lattice of a finite lattice. We would like to know more about: *Which finite distributive lattices are the congruence lattice of some semidistributive lattice?*

Not every finite distributive lattice D is isomorphic to $\operatorname{Con} L$ for a *finite* semidistributive lattice L. There are two known restrictions [2, 9]: if D is the congruence lattice of a finite semidistributive lattice, then considering D as the lattice $\mathcal{O}(P)$ of order ideals of an ordered set, neither $\mathbf{2}$ nor Y (Figure 1) can be an order filter in P. An equivalent formulation is that neither a 3-element chain nor $(B_2)_{++} := \mathbf{2} + \mathbf{2}^2$ can be a filter in D. There may be other restrictions.

This note presents a construction to show that many finite distributive lattices with $\mathbf{3}$ or $(B_2)_{++}$ as a filter are isomorphic to the congruence lattice of an *infinite* semidistributive lattice.

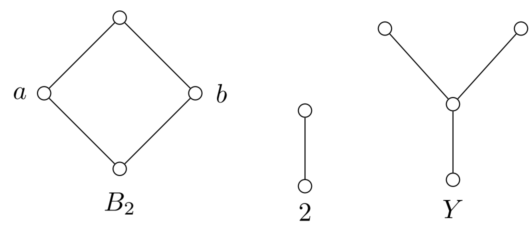

Figure 1: Ordered sets referred to in the text: B_2, $\mathbf{2}$, Y

2 Background

The join-semidistributive law for lattices is

$$\text{(JSD)} \qquad x \vee y = x \vee z \text{ implies } x \vee y = x \vee (y \wedge z).$$

Its dual is the meet-semidistributive law, (MSD). Lattices that satisfy both are called *semidistributive*, abbreviated SD. The semidistributive laws were found by B. Jónsson as a property of free lattices; see [6–8] and the survey [1].

Finite distributive lattices are isomorphic to the lattice of order ideals (downsets) of an ordered set. In fact, $D \cong \mathcal{O}(P)$, where $P = (\mathrm{J}(D), \leq)$ is the set of join-irreducible elements of D. This reflects the fact that join-irreducible elements in a distributive lattice are join-prime. Our results are formulated in terms of this duality.

Let $\mathbf{n}$ denote an n-element chain, A_n an n-element antichain, and B_n the boolean lattice with n atoms. For the ordered sets, P and Q, the ordered set $P \mathbin{\dot\cup} Q$ has the elements of P and Q incomparable, while the ordered set $P + Q$ has every element of P below every element of Q. For the lattices K and L with 0 and 1, let $K\#L$ denote the glued sum, where $1_K = 0_L$.

Lemma 2.1. *Let P and Q be ordered sets. If $\mathcal{O}(P) = K$ and $\mathcal{O}(Q) = L$, then*

(1) $\mathcal{O}(P \mathbin{\dot{\cup}} Q) = K \times L$,

(2) $\mathcal{O}(P + Q) = K\#L$.

If L is a lattice, then L_+ denotes the lattice obtained by adjoining a new zero element, that is, $L_+ = \mathbf{1} + L$. Thus $L_{++} = \mathbf{2} + L$. Likewise, L^+ is the lattice obtained by adjoining a new top element, that is, $L^+ = L + \mathbf{1}$.

The congruence lattice of a finite lattice is a finite distributive lattice. There are two restrictions mentioned in the introduction: if $\mathcal{O}(P) \cong \operatorname{Con} K$ for a *finite semidistributive* lattice, then neither $\mathbf{2}$ nor Y can be an order filter (upset) of P. Note that $\mathcal{O}(\mathbf{2}) = \mathbf{3}$ and $\mathcal{O}(Y) = (B_2)_{++}$; remember to include the empty order ideal. The following elementary technical observation [9] then shows that neither $\mathbf{3}$ nor $(B_2)_{++}$ is a filter of $\mathcal{O}(P)$.

Lemma 2.2. *Let S and P be finite ordered sets. Then $\mathcal{O}(S)$ is isomorphic to a filter of $\mathcal{O}(P)$ if and only if S is an order filter of P.*

Now $\mathbf{2}$ is the only finite simple SD lattice. Indeed, if L is JSD and has a largest element 1, then it has a prime ideal, and hence L has $\mathbf{2}$ as a homomorphic image. There are however *infinite* simple SD lattices [4].

The original lattices in [4] contained no completely doubly irreducible (c.d.i.) elements, that is, elements that are completely join-irreducible and completely meet-irreducible. A straightforward modification of the construction yields infinite simple semidistributive lattices containing infinitely many c.d.i. elements; see [3]. (Replace the defining relations (7) and (8) in [4] by $b_i < b_{i+1}$ and $d_i < d_{i+1}$; these are slightly stronger.)

The infinite simple SD lattices constructed in [3, 4], containing an infinite chain of c.d.i. elements, are called FN lattices. The letter F will denote an arbitrary FN lattice with c.d.i. elements. An infinite simple semidistributive lattice necessarily has neither 0 nor 1. We will use FN lattices as the building blocks for our constructions.

The least congruence on a lattice is denoted by Δ, and the greatest congruence ∇. In this note we are dealing with infinite lattices that have finite congruence lattices. Of course, that is not always the case.

3 Direct products

The first operation for building new representations from existing ones is the direct product.

Lemma 3.1. *If K and L are lattices, then* $\mathrm{Con}(K \times L) \cong \mathrm{Con}\ K \times \mathrm{Con}\ L$*. For ordered sets, this translates to the disjoint union, that is, if* $\mathrm{Con}\ K \cong \mathcal{O}(P)$ *and* $\mathrm{Con}\ L \cong \mathcal{O}(Q)$*, then*

$$\mathrm{Con}(K \times L) \cong \mathcal{O}(P) \times \mathcal{O}(Q) = \mathcal{O}(P \dot\cup Q).$$

The lemma allows us to represent $B_m = \mathcal{O}(A_m)$ as $\mathrm{Con}\ 2^m$ or $\mathrm{Con}\ F^m$ where F is an FN lattice.

The following properties will play a role later.

IGD(K) The congruence generated by collapsing any nonempty ideal of K is ∇.

FGD(K) The congruence generated by collapsing any nonempty filter of K is ∇.

A lattice satisfying both IGD and FGD is called *half-simple*, and FN lattices (being simple) clearly are half-simple. Half-simple lattices can have neither 0 nor 1.

Lemma 3.2. *A finite direct product of lattices with* FGD *has* FGD*. Likewise, for* IGD *and half-simple.*

Proof. Let $L = K_1 \times \cdots \times K_n$, with each K_j having FGD, and let G be a nonempty filter of L. Let θ denote the congruence on L obtained by collapsing G. We want to show that $\theta = \nabla_L$.

Lattices have factorable congruences, as a consequence of congruence distributivity. This means that there exist congruences $\theta_i \in \mathrm{Con}\ K_i$ such that, for x, $y \in L$, we have $x\,\theta\,y$ iff $x_i\,\theta_i\,y_i$ for all $1 \le i \le n$. But each θ_i is the congruence generated on K_i by the projection of the filter G onto K_i, which is a nonempty filter. Since K_i has FGD, this implies that $\theta_i = \nabla_{K_i}$, whence $\theta = \nabla_L$. □

4 Replacing a c.d.i. element with a half-simple lattice

Let d be a c.d.i. element in a lattice K, and let H be half-simple. The lattice $K(d \hookrightarrow H)$ is the set $(K - \{d\}) \dot\cup H$ with the natural order, that is, for $k \in K - \{d\}$ and $h \in H$, $k \le h$ iff $k \le d$, and $k \ge h$ iff $k \ge d$. Joins and meets are well-defined in $K(d \hookrightarrow H)$, because d is doubly irreducible. Indeed, $K - \{d\}$ and H are sublattices, while

$$k \vee h = \begin{cases} h, & \text{if } k \le d; \\ k \vee d, & \text{otherwise;} \end{cases}$$

and dually. Thus, $K(d \hookrightarrow H)$ is semidistributive, if both K and H are semidistributive. The construction is illustrated schematically in Figure 2.

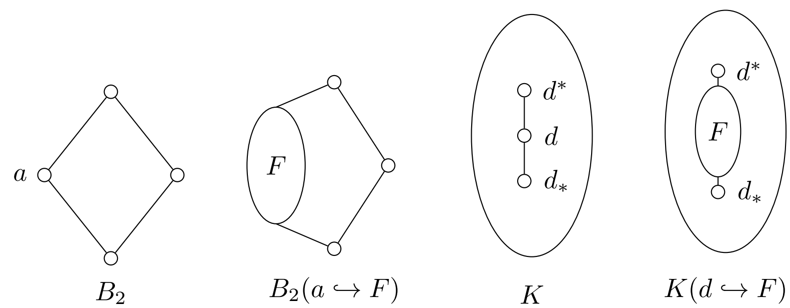

Figure 2: Schematic representation of the construction, replacing a c.d.i. element with an FN lattice F

One can also replace multiple c.d.i. elements independently, forming $K(d_1 \hookrightarrow H_1, \ldots, d_n \hookrightarrow H_n)$.

Let us now analyze $\mathrm{Con}\,(K(d \hookrightarrow H))$.

For any element $u \in K$, considering how joins of congruences work, there is a unique largest congruence ζ_u in $\mathrm{Con}\,K$ such that the congruence class $[u]_\theta$ is a singleton, that is, $[u]_\theta = \{u\}$ iff $\theta \leq \zeta_u$. Note that when $\mathrm{Con}\,K \cong \mathcal{O}(P)$, the congruence ζ_u corresponds to an order ideal of P, which we also denote ζ_u.

Theorem 4.1. *Let K be a lattice with a c.d.i. element d, and let H be a half-simple lattice. Form $L = K(d \hookrightarrow H)$. Then*

$$\mathrm{Con}\,L \cong \{(\theta, \alpha) \in \mathrm{Con}\,K \times \mathrm{Con}\,H \colon \theta \nleq \zeta_d \ \rightarrow\ \alpha = \nabla_H\}.$$

In terms of ordered sets, if $\mathrm{Con}\,K \cong \mathcal{O}(P)$ and $\mathrm{Con}\,H \cong \mathcal{O}(Q)$, then $\mathrm{Con}\,L \cong \mathcal{O}(R)$ where $R = Q \cup P$ with the order $q \leq p$ iff $p \notin \zeta_d$ for $p \in P$, $q \in Q$.

Figure 3 illustrates how Theorem 4.1 applies to $N_5(c \hookrightarrow F)$ and $\zeta_c > \Delta$.

Proof. Let φ be the congruence on $L = K(d \hookrightarrow H)$ that collapses H back to a single point, so that $L/\varphi \cong K$. By the isomorphism theorems, $\uparrow\varphi$ in $\mathrm{Con}\,L$ is isomorphic to $\mathrm{Con}\,K$. Explicitly, if $f \colon L \twoheadrightarrow K$ with $\ker f = \varphi$ and $\psi \geq \varphi$, then $k\, f(\psi)\, k'$ if and only if there exist x, x' in L with $k = f(x)$, $k' = f(x')$, and $x\,\psi\,x'$. Equivalently, in view of $\psi \geq \varphi$, for all x, x' in L, we have that $f(x)\, f(\psi)\, f(x')$ if and only if $x\,\psi\,x'$.

Let $\mathcal{S}$ be the sublattice of $\mathrm{Con}\,K \times \mathrm{Con}\,H$ given in the theorem. We establish inverse lattice homomorphisms $\sigma\colon \mathrm{Con}\,L \to \mathcal{S}$ and $\tau\colon \mathcal{S} \to \mathrm{Con}\,L$.

For $\psi \in \text{Con } L$, let $\sigma(\psi) = (f(\psi \vee \varphi), \psi|_H)$. For $(\theta, \alpha) \in \mathcal{S}$ and k, $k' \in K - \{d\}$, h, $h' \in H$, let

$$k\,\tau(\theta,\alpha)\,k' \text{ iff } k\,\theta\,k',$$
$$h\,\tau(\theta,\alpha)\,h' \text{ iff } h\,\alpha\,h',$$
$$k\,\tau(\theta,\alpha)\,h \text{ iff } k\,\theta\,d.$$

The crucial observations are these.

- If $f(\psi \vee \varphi) \nleq \zeta_d$, then $k\,f(\psi \vee \varphi)\,d$ for some $k \in K - \{d\}$. Hence $k\,\psi\,h$ for some $h \in H$ (as $f(h) = d$).
- If $k\,\psi\,h$ for some $k \in K - \{d\}$ and $h \in H$, then ψ collapses either an ideal or a filter of H (or both). Because H is half-simple, this implies $\psi|_H = \nabla_H$.
- The condition $\psi|_H = \nabla_H$ is equivalent to $\psi \geq \varphi$.

On the other hand, if $\theta \in \text{Con } K$ with $\theta \leq \zeta_d$, and $\alpha \in \text{Con } H$, let ξ be the relation on L such that $\xi|_{K-\{d\}} = \theta|_{K-\{d\}}$, $\xi|_H = \alpha$, and ξ contains no pairs of the form (k,h) or (h,k). Then ξ is a congruence on L and $\xi = \tau(\theta,\alpha)$. The remaining details are left as an exercise to the reader. □

Corollary 4.2. *Let K be a lattice with a c.d.i. element d, and let H be a half-simple lattice. If $\zeta_d = \Delta_K$, then*

$$\text{Con } K(d \hookrightarrow H) \cong \text{Con } H \,\#\, \text{Con } K.$$

In particular, with an FN lattice,

$$\text{Con } K(d \hookrightarrow F) \cong \mathbf{1} + \text{Con } K = (\text{Con } K)_+ \textit{ when } \zeta_d = \Delta_K,$$
$$\text{Con } F(d \hookrightarrow H) \cong \text{Con } H + \mathbf{1} = (\text{Con } H)^+ \textit{ when } H \textit{ is half-simple.}$$

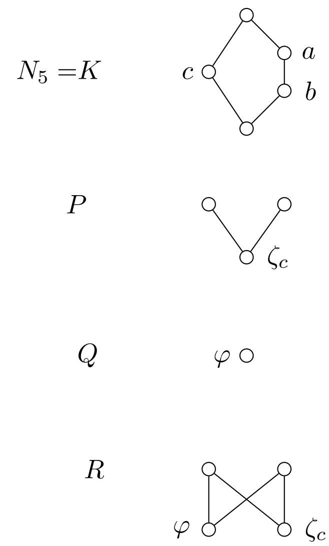

Figure 3: Example $N_5(c \hookrightarrow F)$ for Theorem 4.1. Note $\zeta_c = \mathrm{Cg}(a, b)$.

Recall that for $n \geq 3$, the n-element chain is not the congruence lattice of a finite semidistributive lattice (or even a finite join-semidistributive lattice [2]).

Corollary 4.3. *For every $n \geq 2$, the n-element chain $\mathbf{n}$ can be represented as the congruence lattice of an infinite semidistributive lattice.*

$$\begin{array}{ll} \mathbf{2} = \mathcal{O}(\mathbf{1}) & F \\ \mathbf{3} = \mathcal{O}(\mathbf{2}) & F\langle d_1 \hookrightarrow F\rangle \\ \mathbf{4} = \mathcal{O}(\mathbf{3}) & F\langle d_1 \hookrightarrow F\langle d_2 \hookrightarrow F\rangle\rangle \\ \mathbf{5} = \mathcal{O}(\mathbf{4}) & F\langle d_1 \hookrightarrow F\langle d_2 \hookrightarrow F\langle d_3 \hookrightarrow F\rangle\rangle\rangle \end{array}$$

etc.

As an application of direct products (Lemma 3.1):

Corollary 4.4. *For positive integers $n_1, \ldots, n_k$, the lattices $\mathbf{n}_1 \times \cdots \times \mathbf{n}_k$ are congruence lattices of infinite SD lattices.*

If any $n_j \geq 3$, then $\mathbf{n}_1 \times \cdots \times \mathbf{n}_k$ is not the congruence lattice of a finite SD lattice.

The lattice $\mathcal{O}(Y) = (B_2)_{++}$ is the other lattice minimally not representable as the congruence lattice of a finite SD lattice. However, $\mathcal{O}(Y) \cong \mathrm{Con}\, K$ for both of the following infinite SD lattices:

- $K_1 = B_2(a \hookrightarrow F(d \hookrightarrow F))$

- $K_2 = N_5(a_0 \hookrightarrow F)$ where $N_5 = B_2[a]$, doubling an atom.

These lattices are drawn schematically in Figure 4.

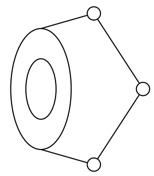

Figure 4: Schematic representation of lattices K_j with Con $K_j = \mathcal{O}(Y)$.

One can just as easily use $K = B_n$ and one of its atoms to represent $(B_n)_{++}$ for any $n \geq 2$ as the congruence lattice of an infinite SD lattice, generalizing either of the representations K_1 or K_2.

A *dual tree* is a connected finite ordered set such that every element has at most one cover. A *dual forest* is a disjoint union of finitely many dual trees. When P is a dual forest, there is a straightforward way to represent $\mathcal{O}(P)$ as a congruence lattice. For branching in the dual tree, we replace multiple c.d.i. elements.

Theorem 4.5. *If P is a finite dual forest, then $\mathcal{O}(P)$ is the congruence lattice of an infinite SD lattice.*

Proof. Without loss of generality P is a dual tree, as we can use direct products for a dual forest.

Let $u \succ v_1, \ldots, v_n$ in P, and assume inductively that each $\mathcal{O}(\downarrow v_j) \cong$ Con H_j for a half-simple SD lattice H_j. Let F be an FN lattice, and choose distinct c.d.i. elements $d_1, \ldots, d_n \in F$. Form $L = F(d_1 \hookrightarrow H_1, \ldots, d_n \hookrightarrow H_n)$. Then L is half-simple, and Con $L \cong \mathcal{O}(\downarrow u)$ by the straightforward extension of Corollary 4.2 for multiple substitutions. $\square$

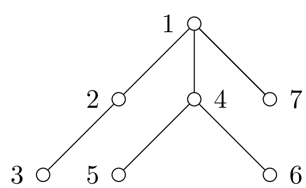

Figure 5: Dual tree example

The method is best illustrated by an example. Let P be the dual tree in Figure 5. To find an infinite SD lattice K with Con $K \cong \mathcal{O}(P)$, we use $K = F_1\langle b_1 \hookrightarrow H_1, b_2 \hookrightarrow H_2, b_3 \hookrightarrow H_3\rangle$ where

$$H_1 = F_2\langle b_4 \hookrightarrow F_3\rangle$$
$$H_2 = F_4\langle b_5 \hookrightarrow F_5, b_6 \hookrightarrow F_6\rangle$$
$$H_3 = F_7.$$

Also observe that Theorem 4.5 includes $\mathcal{O}(A_n + \mathbf{k}) = B_n^{+\cdots+}$ with k "+" signs.

5 Conclusion

We have shown that many finite distributive lattices that are not the congruence lattice of a *finite* semidistributive lattice, are the congruence lattice of an *infinite* semidistributive lattice. Some of these examples were included in an earlier version of this note [5].

This suggests two problems.

Question 1. *Are there additional restrictions on congruence lattices of finite SD lattices?*

Question 2. *Is every finite distributive lattice the congruence lattice of an infinite SD lattice?*

We conjecture that the answers are NO and YES, respectively.

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On the Φ-Hilfer iterative fractional differential equations

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ABSTRACT

To avoid studying iterative differential equations with distinct fractional order derivatives it is essential to treat them with a broad fractional derivative, which leaves other fractional derivatives as a special case. In this way, we study an initial value problem for non-linear iterative fractional differential equations involving Φ-Hilfer fractional derivative. We establish the existence and uniqueness of the solution through fixed point theorems. We prove results concerning the dependence of solution and Ulam-Hyers stability of the problem. Finally, we present an example for illustration to demonstrate our outcome.

RESUMEN

Para evitar estudiar ecuaciones diferenciales iterativas con dereivadas fraccionarias de distintos órdenes, es esencial tratarlas a través de una derivada fraccionaria amplia, que deje otras derivadas fraccionarias como un caso especial. De este modo, estudiamos un problema de valor inicial para ecuaciones diferenciales fraccionarias iterativas no-lineales que involucra la derivada fraccionaria Φ-Hilfer. Establecemos la existencia y unicidad de la solución a través de teoremas de punto fijo. Demostramos resultados relacionados a la dependencia de la solución y la estabilidad de Ulam-Hyers del problema. Finalmente, presentamos un ejemplo para ilustrar lo obtenido.

Keywords and Phrases: Iterative fractional differential equations, Φ-Hilfer derivative, fixed point theorems, existence and uniqueness, data dependency, Ulam-Hyers stability.

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1 Introduction

Fractional calculus is a branch of mathematics in which we obtain definitions of derivatives and integrals with arbitrary positive real order so that the classical derivative can act as a special case. There are many more definitions of fractional derivatives, see the monographs [17,28,29]. It is worth obtaining the most generalized fractional differential operator to unify all these definitions. Later, Sousa and Oliveira (2018) [35] investigated the most generalized Φ-Hilfer fractional derivative. In [14, 21, 22, 27], significant theoretical advancements concerning various forms of nonlinear Φ-Hilfer fractional differential equations and several important properties of their solutions are examined. Development of theory after the proposal of the Φ-Hilfer fractional derivative, other versions of fractional operators were studied. For example, a work that addresses the fractional derivative in variable order with respect to the Φ function [38] and the work on calculus of Φ-Hilfer fractional derivative with an additional parameter $k > 0$ and associated fractional differential equations [15, 20].

We note that fractional calculation has been extensively studied and its theory, although well consolidated, still new versions of fractional operators are presented and, certainly interesting and important applications arising from them, will be discussed in the near future. On the other hand, we can also highlight problems of fractional differential equations with p-Laplacian, which have been attracting the attention of researchers. In 2022, Sousa *et al.* [39] first work on variational problems using the Φ-Hilfer fractional derivative was presented. In the work, the authors presented a new fractional Sobolev space for the Φ-Hilfer fractional derivative, and built a variational structure so that it was possible to investigate the existence of weak solutions to a fractional p-Laplacian problem via Nehari manifold [33, 34, 37].

The differential equations which involves the iterates of unknown function is called as Iterative differential equations (IDEs). IDEs are especially useful for simulating real-world systems where the rate of change is dependent on both the function and the number of times the unknown function is applied. They extend traditional differential equations to capture more complex, nonlinear, and self-referential dynamics, with applications across various fields, including biology, physics, and engineering. Examples include infectious disease models [45], the motion of charged particles with retarded interaction [11], insect population dynamics [2], and Nicholson's blowflies model [16]. Due to their wide range of applications, IDEs are an essential area of study.

Eder [7] studied the IDEs of the form

$$u'(t) = u(u(t)),$$

and showed that every solution either identically vanishes or is strictly monotonic. Feckan [8] investigated the functional differential equation

$$u'(t) = h(u(u(t))), \quad u(0) = 0.$$

Vasile Bernide [1] proved convergence theorems under weaker conditions than those suggested by A. Buica [3] and proved the existence of solutions for first-order iterative differential equations.

Iterative fractional differential equations (FDEs) deals iterative differential equations associated with various types of fractional derivatives. They serve as powerful tools for modeling complex systems that exhibit memory effects, non-local interactions, and long-term dependencies. Here, we highlight a few significant studies on iterative FDEs.

Ibrahim [11] investigated the existence and approximation of solution for the iterative Riemann-Liouville FDEs of the form

$$D^{\xi}u(t) = h(t, u(t), u(u(t))), \quad u(0) = u_0.$$

Damag *et al.* [4] proved the existence of solution for the iterative FDEs

$$D^{\xi}u(t) = h(t, u(t), u(u(t)), u'(t)), \quad u(t_0) = u_0, \quad t_0 \in J,$$

by applying non-expansive operator method and Browder-Ghode-Kirk fixed point theorem. Guerfi and Ardjouni [9] investigated existence, uniqueness, continuous dependence and Ulam-Hyers stability of mild solution for the Caputo iterative FDEs of the form

$$\begin{aligned} &{}^{C}D^{\xi}_{0+}u(t) = h\left(u^{[0]}(t), u^{[1]}(t), \dots, u^{[n]}(t)\right), \\ &u(0) = u'(0) = 0. \end{aligned}$$

Existence and approximation problems for the iterative differential equations are solved in [5, 6, 12, 24, 44–46, 48]. Also, iterative integro-differential equations are studied [10, 13, 18, 32]. For further development of iterative differential equations see [26, 31, 41, 42] and the references therein.

Vivek *et al.* [40] examined the class of Φ-Riemann-Liouville iterative fractional differential equation with non-local condition

$$\begin{aligned} &D^{\xi;\Phi}u(t) = h(t, u(u(t))), \quad 0 < \xi < 1, \\ &u(0) + f(u) = u_0. \end{aligned}$$

Motivated by interesting work mentioned above on iterative differential equations we consider the non-linear iterative FDEs of the form

$$\begin{aligned} {}^{H}D_{0+}^{\xi,\,\eta;\,\Phi}u(t) = h\Big(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[0]}(t), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[1]}(t), \\ \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[n]}(t)\Big),\ t\in J, \end{aligned} \tag{1.1}$$

$$I_{0+}^{1-\zeta;\,\Phi}u(0) = u_0, \quad u_0 \geq 0, \quad \zeta = \xi + \eta(1-\xi), \tag{1.2}$$

where $J=[0,T]$, Φ is an increasing function on J such that $\Phi \in C^1(J)$ and $\Phi^{'}(t)\neq 0$, for all $t\in J$, ${}^{H}D_{0+}^{\xi,\,\eta;\,\Phi}(\cdot)$ is the Φ-Hilfer derivative of order $\xi\in(0,1)$ and type $\eta\in[0,1]$. Further,

$$\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(t) = t, \tag{1.3}$$

$$\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[j]}(t) = (\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[j-1]}(t)\right), \quad j=1,\ldots n, \tag{1.4}$$

are the iterates of the function $(\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u$ and $h\in C(J^{n+1},\mathbb{R})$ is a positive non-linear function that fulfills a few other requirements, which are detailed subsequently.

We believe that the main results of this paper are best presented as follows:

(1) Before attacking the main results, it was necessary to discuss some properties for the space with weight $C_{1-\zeta;\Phi}(J,\mathbb{R},M)$.

(2) The first contribution of the paper was to investigate the existence and uniqueness of solutions to the problem (1.1)-(1.2) through the theory of fixed points.

(3) In addition to the above, we investigated the continuous dependence and Ulam-Hyers stability.

(4) Finally, we present an example, in order to elucidate the results discussed.

We analyzed iterative differential equations associated Φ-Hilfer fractional derivative for the existence and qualitative properties of solutions in the space of weighted Lipschitz functions.

The iterates of unknown functions defined by (1.3) and (1.4) that appears in the equations (1.1)-(1.2) make the study challenging as it requires domain and codomain of unknown functions should be same and hence appropriate solutions space is required to deal with the solutions of iterative FDEs (1.1)-(1.2). In this context the two weighted spaces are defined. The weighted space $C_{1-\zeta;\Phi}(J,\mathbb{R},L)$ ensures that that the iterates are well defined and $C_{1-\zeta;\Phi}(J,\mathbb{R},M)$ ensures the existence of solution for the iterative FDEs.

The Φ-Hilfer fractional derivative is the most generalized form of fractional derivatives, encompassing various fractional differential operators described in [35] as special cases for varying values of

η and different choices of the function Φ. In this context, the Φ-Hilfer fractional derivative serves as a powerful tool in fractional calculus that unifies the study of fractional differential equations (FDEs) under a single framework. As a result, it is no longer necessary to conduct independent analyses of FDEs using various fractional derivative operators.

This paper is organized as follows. In Section 2, we discuss about Φ-fractional calculus, define some weighted spaces that required for further calculation. Section 3 deals with the properties of weighted space. In Section 4, we investigate existence via fixed point theorem and uniqueness result. Further Section 5 includes continuous dependence, Ulam-Hyers and generalized Ulam-Hyers stability of solution. In Section 6, example is provided to illustrate our results.

2 Preliminaries

In this section, we provide definitions and few basic results pertaining to Φ-fractional calculus. Further, we provide the suitable weighted space to deal with solutions of iterative FDEs.

2.1 Φ-fractional calculus

Definition 2.1 ([17])**.** *The Φ-Riemann-Liouville fractional integral of order $\xi > 0$ $(\xi \in \mathbb{R})$ of the function $u \in C([a,b],\mathbb{R})$ is given by*

$$I_{a^+}^{\xi\,;\Phi} u(t) = \frac{1}{\Gamma(\xi)} \int_a^t \Phi'(s) (\Phi(t) - \Phi(s))^{\xi - 1} u(s)\, ds. \tag{2.1}$$

Definition 2.2 ([35])**.** *The Φ-Hilfer fractional derivative of a function $u \in C^m([a,b],\mathbb{R})$ of order $m-1 < \xi < m$ and type $\eta \in [0,1]$, is defined by*

$${}^H D_{a^+}^{\xi,\,\eta;\,\Phi} u(t) = I_{a^+}^{\eta(m-\xi);\,\Phi} \left(\frac{1}{\Phi'(t)} \frac{d}{dt} \right)^m I_{a^+}^{(1-\eta)(m-\xi);\,\Phi} u(t), \quad t \in (a,b].$$

Lemma 2.3 ([35])**.** *Let $m-1 < \xi < m \in \mathbb{N}$, $u \in (C^m[a,b],\mathbb{R})$ and $\eta \in [0,1]$. Then*

(i) $I_{a^+}^{\xi\,;\Phi}\, {}^H D_{a^+}^{\xi,\,\eta;\,\Phi} u(t) = u(t) - \sum_{k=1}^{m} \frac{(\Phi(t)-\Phi(a))^{\zeta-k}}{\Gamma(\xi-k+1)} u_\Phi^{[m-k]} I_{a^+}^{(1-\eta)(m-\xi)\,;\Phi} u(a)$, *where* $u_\Phi^{[m-k]} u(t) = \left(\frac{1}{\Phi'(t)}\frac{d}{dt}\right)^{m-k} u(t)$,

(ii) ${}^H D_{a^+}^{\xi,\,\eta;\,\Phi} I_{a^+}^{\xi\,;\Phi} u(t) = u(t)$,

where $\zeta = \xi + \eta(m-\xi)$.

2.2 Weighted spaces

Consider the weighted space

$$C_{1-\zeta;\,\Phi}(J,\mathbb{R}) = \{u : (0,T] \to \mathbb{R} |\, (\Phi(t)-\Phi(0))^{1-\zeta}\, u(t) \in C[0,T]\}$$

with the norm

$$||u||_{C_{1-\zeta\,;\Phi}(J,\mathbb{R})} = \sup_{t\in J} \left| (\Phi(t)-\Phi(0))^{1-\zeta}\, u(t) \right|, \quad 0<\zeta\leq 1.$$

Then the space $\left(C_{1-\zeta;\,\Phi}(J,\mathbb{R}), ||\cdot||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}\right)$ is Banach space.

For $0 < L \leq T$ and $M > 0$, we define the following sets

$$C_{1-\zeta\,;\Phi}(J,\mathbb{R};L) = \left\{ u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R}) : 0 \leq (\Phi(t)-\Phi(0))^{1-\zeta}\, u(t) \leq L \right\},$$

and

$$C_{1-\zeta;\Phi}(J,\mathbb{R};M) = \Big\{ u \in C_{1-\zeta;\Phi}(J,\mathbb{R};L) : \left| (\Phi(t_2)-\Phi(0))^{1-\zeta}\, u(t_2) - (\Phi(t_1)-\Phi(0))^{1-\zeta}\, u(t_1) \right| \leq M\, |t_2-t_1|\,, t_1,t_2 \in J \Big\}.$$

If $\zeta = 1$ then above weighted spaces reduces respectively to

$$C(J,\mathbb{R};L) = \{u \in C(J,\mathbb{R}) : 0 \leq u(t) \leq L,\, \forall t \in J\}$$

and

$$C(J,\mathbb{R};M) = \{u \in C(J,\mathbb{R};L) : |u(t_2)-u(t_1)| \leq M|t_2-t_1|,\, \forall t_1,t_2 \in J\}, \quad M>0$$

which are defined in [9].

Lemma 2.4 ([48]). *If $u_1, u_2 \in C(J,\mathbb{R};M)$, then*

$$\left\| u_1^{[n]} - u_2^{[n]} \right\|_{C(J)} \leq \sum_{j=0}^{n-1} M^j \left\| u_1 - u_2 \right\|_{C(J)}, \quad n = 1,2,\ldots$$

where $C(J,\mathbb{R}) = \{u | u : J \to \mathbb{R}$ is continuous$\}$ is Banach space with the supremum norm.

3 Properties of weighted space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$.

To prove existence of solution of iterative FDEs (1.1)-(1.2) we use the following Schauder's fixed point theorem.

Theorem 3.1 (Schauder's fixed point theorem [30])**.** *Let U be a non-empty compact convex subset of Banach space $(B, ||\cdot||)$ and $A : U \to U$ is a continuous mapping. Then A has a fixed point.*

In the view of Theorem 3.1, we have to prove that the space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is non-empty, convex and compact subset of a Banach space $C_{1-\zeta;\,\Phi}(J,\mathbb{R})$, and the proof of the same is provided in following theorems.

Theorem 3.2. *For $0 < L \leq T$ and $M > 0$, the weighted space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is non-empty, closed and convex subset of $C_{1-\zeta;\,\Phi}(J,\mathbb{R})$.*

Proof. Define $v : (0,T] \to \mathbb{R}$ by $v(t) = (\Phi(t)-\Phi(0))^{\zeta-1}\,L$, $t \in (0,T]$. Then $(\Phi(t)-\Phi(0))^{1-\zeta}\,v(t) = L \in C(J,\mathbb{R})$. Therefore $v \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};L)$. Further for any $t_1, t_2 \in J$, we have

$$\begin{aligned}
&\left|(\Phi(t_2)-\Phi(0))^{1-\zeta}\,v(t_2) - (\Phi(t_1)-\Phi(0))^{1-\zeta}\,v(t_1)\right| \\
&\quad = \left|(\Phi(t_2)-\Phi(0))^{1-\zeta}\,(\Phi(t_2)-\Phi(0))^{\zeta-1}\,L - (\Phi(t_1)-\Phi(0))^{1-\zeta}\,(\Phi(t_1)-\Phi(0))^{\zeta-1}\,L\right| \\
&\quad = 0 \leq M|t_2-t_1|.
\end{aligned}$$

From above discussion it follows that $v \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$.

Let $\{u_n\}_{n=1}^{\infty}$ be any sequence in $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ and $u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R})$ is such that

$$\lim_{n\to\infty} ||u_n - u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} = 0. \tag{3.1}$$

Note that

$$\begin{aligned}
0 &\leq \left|(\Phi(t)-\Phi(0))^{1-\zeta}\,(u_n(t)-u(t))\right| \\
&\leq \sup_{t\in J}\left|(\Phi(t)-\Phi(0))^{1-\zeta}\,(u_n(t)-u(t))\right| = ||u_n - u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}.
\end{aligned} \tag{3.2}$$

Using squeeze theorem for sequences from (3.1) and (3.2) it follows that

$$\lim_{n\to\infty}\left|(\Phi(t)-\Phi(0))^{1-\zeta}\,u_n(t) - (\Phi(t)-\Phi(0))^{1-\zeta}\,u(t)\right| = 0. \tag{3.3}$$

Further if $u_n \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ then $u_n \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};L)$ for all n. Thus

$$0 \leq (\Phi(t)-\Phi(0))^{1-\zeta}\,u_n(t) \leq L, \quad \text{for all } n \text{ and } t \in J. \tag{3.4}$$

Taking limit as $n \to \infty$ in inequality (3.4) and using the continuity of modulus and the limit (3.3), we have

$$0 \leq (\Phi(t) - \Phi(0))^{1-\zeta} u(t) \leq L, \quad \text{for all } t \in J.$$

Therefore $u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; L)$.

Consider for $t_1, t_2 \in J$,

$$\begin{aligned}
&\left|(\Phi(t_2) - \Phi(0))^{1-\zeta} u(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta} u(t_1)\right| \\
&\quad \leq \left|(\Phi(t_2) - \Phi(0))^{1-\zeta} (u_n(t_2) - u(t_2))\right| + \left|(\Phi(t_1) - \Phi(0))^{1-\zeta} (u_n(t_1) - u(t_1))\right| \\
&\quad + \left|(\Phi(t_2) - \Phi(0))^{1-\zeta} u_n(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta} u_n(t_1)\right| \\
&\quad \leq 2 \left\|u_n - u\right\|_{C_{1-\zeta;\Phi}(J,\mathbb{R})} + M|t_2 - t_1|.
\end{aligned}$$

Letting $n \to \infty$ we get, $\left|(\Phi(t_2) - \Phi(0))^{1-\zeta} u(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta} u(t_1)\right| \leq M|t_2 - t_1|$. Thus $u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}, M)$.

Consider any $v, w \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}, M)$ and $s \in [0, 1]$. Then $(\Phi(t) - \Phi(0))^{1-\zeta} v(t)$ and $(\Phi(t) - \Phi(0))^{1-\zeta} w(t)$ are continuous on J hence $(\Phi(t) - \Phi(0))^{1-\zeta} (sv + (1-s)w)(t)$ is continuous on J. This gives $sv + (1-s)w \in C_{1-\zeta;\,\Phi}(J)$. Since $v, w \in C_{1-\zeta;\Phi}(J, \mathbb{R}, L)$ we have $0 \leq (\Phi(t) - \Phi(0))^{1-\zeta} v(t) \leq L$ and $0 \leq (\Phi(t) - \Phi(0))^{1-\zeta} w(t) \leq L$. Therefore for any $t \in J$, yields that

$$\begin{aligned}
0 &\leq (\Phi(t) - \Phi(0))^{1-\zeta} (sv + (1-s)w)(t) \\
&= s\,(\Phi(t) - \Phi(0))^{1-\zeta} v(t) + (\Phi(t) - \Phi(0))^{1-\zeta} w(t) - s\,(\Phi(t) - \Phi(0))^{1-\zeta} w(t) \\
&\leq sL + L - sL = L.
\end{aligned}$$

This proves $sv + (1-s)w \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}, L)$. Consider any $t_1, t_2 \in J$, then

$$\begin{aligned}
&\left|(\Phi(t_2) - \Phi(0))^{1-\zeta} (sv + (1-s)w)(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta} (sv + (1-s)w)(t_1)\right| \\
&\quad = s\left|(\Phi(t_2) - \Phi(0))^{1-\zeta} v(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta} v(t_1)\right| \\
&\quad + (1-s)\left|(\Phi(t_2) - \Phi(0))^{1-\zeta} w(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta} w(t_1)\right| \\
&\quad \leq sM|t_2 - t_1| + (1-s)M|t_2 - t_1| = M|t_2 - t_1|.
\end{aligned}$$

From above discussion it follows that $sv + (1-s)w \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ for any $s \in [0, 1]$. Thus proof of $C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ is non-empty, closed and convex subset of $C_{1-\zeta;\,\Phi}(J, \mathbb{R})$ is completed. □

Theorem 3.3. *For $0 < L \leq T$ and $M > 0$, the weighted space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is uniformly bounded and equicontinuous.*

Proof. Let any $u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ then $u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};L)$. Hence

$$0 \leq (\Phi(t) - \Phi(0))^{1-\zeta}\, u(t) \leq L, \quad \text{for all } t \in J.$$

This gives $||u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} \leq L$, for all $u \in C_{1-\zeta\,;\Phi}(J,\mathbb{R};M)$. This proves $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is uniformly bounded.

Let any $u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$. Then $(\Phi(t) - \Phi(0))^{1-\zeta} u(t)$ is continuous for each $t \in J$. Further, we have

$$\left|(\Phi(t_2) - \Phi(0))^{1-\zeta}\, u(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta}\, u(t_1)\right| \leq M|t_2 - t_1|, \quad \text{for all } t_1, t_2 \in J.$$

Let any $\epsilon > 0$. Define $\delta = \frac{\epsilon}{M}$. Then $t_1, t_2 \in J$, $|t_2 - t_1| < \delta$ implies

$$\left|(\Phi(t_2) - \Phi(0))^{1-\zeta}\, u(t_2) - (\Phi(t_1) - \Phi(0))^{1-\zeta}\, u(t_1)\right| < \epsilon.$$

This proves $C_{1-\zeta;\Phi}(J,\mathbb{R};M)$ is equicontinuous. This completes the proof of $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is uniformly bounded and equicontinuous. □

Remark 3.4. *From* Theorem 3.3 *and Arzela-Ascoli theorem it follows that $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is relatively compact. But $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is also closed subset of $C_{1-\zeta;\,\Phi}(J,\mathbb{R})$ and hence $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is compact subspace of $C_{1-\zeta;\,\Phi}(J,\mathbb{R})$.*

4 Existence and uniqueness results

Theorem 4.1. *Assume that the function $h : J^{n+1} \to [0,\infty)$ satisfies the Lipschitz type condition*

$$|h(t, u_1, u_2, \ldots, u_n) - h(t, v_1, v_2, \ldots, v_n)| \leq \sum_{i=1}^{n} c_i|u_i - v_i|, \quad \textit{where } c_i > 0. \tag{4.1}$$

Then, the iterative FDEs (1.1)-(1.2) *has at least one solution in the weighted space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$, provided*

$$\frac{u_0}{\Gamma(\zeta)} + \frac{\rho^*}{\Gamma(\xi+1)}(\Phi(T) - \Phi(0))^{\xi-\zeta+1} \leq L, \tag{4.2}$$

and

$$\frac{\rho^*}{\Gamma(\xi+1)}\left|(\xi - \zeta + 1)\,(\Phi(c) - \Phi(0))^{\xi-\zeta}\, \Phi'(c)\right| \leq M, \quad \textit{for some } c \in (0,T), \tag{4.3}$$

where

$$\rho = \sup_{t\in J}\{h(t,0,0,\ldots,0)\} \quad \textit{and} \quad \rho^* = \rho + L\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j.$$

Proof. Considering equivalent fractional integral equation [36] to the iterative FDEs (1.1)-(1.2), we define an operator A on $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ by

$$(Au)(t) = \frac{(\Phi(t)-\Phi(0))^{\zeta-1}}{\Gamma(\zeta)}u_0 + \frac{1}{\Gamma(\xi)}\int_0^t \Phi'(\tau)\,(\Phi(t)-\Phi(\tau))^{\xi-1} \times h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[n]}(\tau)\right)d\tau, \tag{4.4}$$

where $t\in(0,T]$. In the view of Schauder's fixed point theorem, we have to show that the mapping $A : C_{1-\zeta;\,\Phi}(J,\mathbb{R};M) \to C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ is well defined and continuous. Proof of the same is given in several steps.

Since h is continuous on J we have $h \in C_{1-\zeta;\,\Phi}(J)$. Further, $I_{0+}^{\xi;\,\Phi}$ is bounded from $C_{1-\zeta;\,\Phi}(J)$ to $C_{1-\zeta;\,\Phi}(J)$ implies $I_{0+}^{\xi;\,\Phi}h \in C_{1-\zeta;\,\Phi}(J)$. This gives $Au \in C_{1-\zeta;\,\Phi}(J)$, for all $u \in C_{1-\zeta;\,\Phi}(J)$. Thus the mapping A is well defined.

Now, we show that the mapping A is continuous. Using Lipschitz condition on h, for any $t\in J$, one has

$$\begin{aligned}
&\left|(\Phi(t)-\Phi(0))^{1-\zeta}(Au-Av)(t)\right| \\
&\le \frac{(\Phi(t)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)}\int_0^t \Phi'(\tau)\,(\Phi(t)-\Phi(\tau))^{\xi-1} \\
&\quad\times\left|h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[0]}(\tau),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[n]}(\tau)\right)\right. \\
&\quad\left.-\,h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}v\right)^{[0]}(\tau),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}v\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}v\right)^{[n]}(\tau)\right)\right|d\tau \\
&\le \frac{(\Phi(t)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)}\int_0^t \Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\xi-1} \\
&\quad\times\sum_{i=1}^{n} c_i\left|\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[i]}(\tau)-\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[i]}(\tau)\right|d\tau \\
&\le \frac{(\Phi(t)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)} \\
&\quad\times\sum_{i=1}^{n} c_i\left\|\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[i]}-\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[i]}\right\|_{C(J,\mathbb{R})}\int_0^t \Phi'(\tau)\,(\Phi(t)-\Phi(\tau))^{\xi-1}\,d\tau \\
&\le \frac{(\Phi(t)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\sum_{i=1}^{n} c_i\sum_{j=0}^{i-1}M^j\left\|(\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u-(\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right\|_{C(J,\mathbb{R})} \\
&= \frac{(\Phi(t)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\sum_{i=1}^{n} c_i\sum_{j=0}^{i-1}M^j\left\|(\Phi(\cdot)-\Phi(0))^{1-\zeta}\,(u-v)\right\|_{C(J,\mathbb{R})} \\
&= \sum_{i=1}^{n} c_i\sum_{j=0}^{i-1}M^j\,\frac{(\Phi(t)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\,||u-v||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}.
\end{aligned}$$

Therefore, we get

$$\|Au - Av\|_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} \leq \frac{1}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j (\Phi(T) - \Phi(0))^{\xi-\zeta+1} ||u - v||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}. \tag{4.5}$$

Let any $\epsilon > 0$. Define

$$\delta = \frac{\epsilon\Gamma(\xi+1)}{(\Phi(T) - \Phi(0))^{\xi-\zeta+1} \sum\limits_{i=1}^{n} c_i \sum\limits_{j=0}^{i-1} M^j}.$$

Then for any $u, v \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ and $||u-v||_{C_{1-\zeta,\,\Phi}(J,\mathbb{R})} < \delta$ we have $||Au-Av||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} < \epsilon$. This proves A is continuous mapping. Next we prove that

$$A\left(C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)\right) \subseteq C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M).$$

Let any $u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$. Then,

$$\begin{aligned}\left|(\Phi(t) - \Phi(0))^{1-\zeta}(Au)(t)\right| &\leq \frac{u_0}{\Gamma(\zeta)} + \frac{(\Phi(t) - \Phi(0))^{1-\zeta}}{\Gamma(\xi)} \int_0^t \Phi'(\tau)(\Phi(t) - \Phi(\tau))^{\xi-1} \\ &\times \left| h\left(\left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[0]}(\tau), \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[n]}(\tau) \right) \right| d\tau.\end{aligned} \tag{4.6}$$

Using Lipschitz condition on h for any $u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$, it follows that

$$\begin{aligned}&\left| h\left(\left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[0]}(\tau), \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[n]}(\tau) \right) \right| \\ &\leq \left| h\left(\left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[0]}(\tau), \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[n]}(\tau) \right) \right. \\ &\left. - h(\tau, 0, \ldots, 0) \right| + |h(\tau, 0, \ldots, 0)| \\ &\leq \sum_{i=1}^{n} c_i \left| \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[i]}(\tau) \right| + \rho \leq \sum_{i=1}^{n} c_i \left\| \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[i]} \right\|_{C(J,\mathbb{R})} + \rho.\end{aligned}$$

Using the inequality in Lemma 2.4, we obtain

$$\begin{aligned}&\left| h\left(\left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[0]}(\tau), \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[n]}(\tau) \right) \right| \\ &\leq \rho + \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \left\| (\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right\|_{C(J,\mathbb{R})}.\end{aligned}$$

Using the definition of space $C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$, we get

$$\begin{aligned}&\left| h\left(\left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[0]}(\tau), \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u\right)^{[n]}(\tau) \right) \right| \\ &\leq \rho + \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j L = \rho^*, \quad u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M).\end{aligned} \tag{4.7}$$

Using inequality (4.7) in (4.6) for any $u \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$, we obtain

$$\begin{aligned}\left|(\Phi(t)-\Phi(0))^{1-\zeta}(Au)(t)\right| &\le \frac{u_0}{\Gamma(\zeta)}+\frac{\rho^*}{\Gamma(\xi)}(\Phi(t)-\Phi(0))^{1-\zeta}\int_0^t \Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\xi-1}d\tau\\ &\le \frac{u_0}{\Gamma(\zeta)}+\frac{\rho^*}{\Gamma(\xi+1)}(\Phi(t)-\Phi(0))^{\xi-\zeta+1}\le \frac{u_0}{\Gamma(\zeta)}+\frac{\rho^*}{\Gamma(\xi+1)}(\Phi(T)-\Phi(0))^{\xi-\zeta+1}\\ &\le L.\end{aligned}$$

Therefore

$$0\le (\Phi(t)-\Phi(0))^{1-\zeta}(Au)(t)\le \left|(\Phi(t)-\Phi(0))^{1-\zeta}(Au)(t)\right|\le L,\ u\in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M). \tag{4.8}$$

This proves $Au \in C_{1-\zeta;\,\Phi}(J,\mathbb{R};L)$.

Further, for any $t_1,t_2\in J$ with $t_1<t_2$, using inequality (4.7), we have

$$\begin{aligned}&\left|(\Phi(t_2)-\Phi(0))^{1-\zeta}(Au)(t_2)-(\Phi(t_1)-\Phi(0))^{1-\zeta}(Au)(t_1)\right| = \Bigg|\frac{(\Phi(t_2)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)}\int_0^{t_2}\Phi'(\tau)(\Phi(t_2)-\Phi(\tau))^{\xi-1}\\ &\times\left|h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[0]}(\tau),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[n]}(\tau)\right)\right|\\ &-\frac{(\Phi(t_1)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)}\int_0^{t_1}\Phi'(\tau)(\Phi(t_1)-\Phi(\tau))^{\xi-1}\\ &\times\left|h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[0]}(\tau),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}u\right)^{[n]}(\tau)\right)\right|d\tau\Bigg|\\ &\le\left|\frac{\rho^*}{\Gamma(\xi)}(\Phi(t_2)-\Phi(0))^{1-\zeta}\frac{(\Phi(t_2)-\Phi(0))^{\xi}}{\xi}-\frac{\rho^*}{\Gamma(\xi)}(\Phi(t_1)-\Phi(0))^{1-\zeta}\frac{(\Phi(t_1)-\Phi(0))^{\xi}}{\xi}\right|\\ &=\left|\frac{\rho^*}{\Gamma(\xi+1)}\left[(\Phi(t_2)-\Phi(0))^{\xi-\zeta+1}-(\Phi(t_1)-\Phi(0))^{\xi-\zeta+1}\right]\right|.\end{aligned}$$

Define $g(t)=(\Phi(t)-\Phi(0))^{\xi-\zeta+1}$, $t\in[0,T]$. Then clearly g is continuous on $[t_1,t_2]$ and differentiable on (t_1,t_2) for any $t_1,t_2\in J$ with $t_1<t_2$. Therefore using mean value theorem there exists $c\in(0,T)$ such that

$$g'(c)=\frac{g(t_2)-g(t_1)}{t_2-t_1}.$$

Using definition of function g, it follows that

$$(\Phi(t_2)-\Phi(0))^{\xi-\zeta+1}-(\Phi(t_1)-\Phi(0))^{\xi-\zeta+1}=\{(\xi-\zeta+1)\,(\Phi(c)-\Phi(0))^{\xi-\zeta}\,\Phi'(c)\}(t_2-t_1).$$

Therefore, using condition (4.3), one has

$$\begin{aligned}&\left|(\Phi(t_2)-\Phi(0))^{1-\zeta}(Au)(t_2)-(\Phi(t_1)-\Phi(0))^{1-\zeta}(Au)(t_1)\right|\\ &\qquad\le\frac{\rho^*}{\Gamma(\xi+1)}\left|\{(\xi-\zeta+1)\,(\Phi(c)-\Phi(0))^{\xi-\zeta}\,\Phi'(c)\}\right|(t_2-t_1)\le M|t_2-t_1|.\end{aligned} \tag{4.9}$$

From inequalities (4.8) and (4.9) , it follows that $(Au)\in C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$. This completes the proof of $A\left(C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)\right)\subseteq C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$.

We have proved that A fulfills all the conditions of Schauder's fixed point theorem. Therefore, A has at least one fixed point which is the solution of the iterative FDEs (1.1)-(1.2). $\square$

Theorem 4.2. *Suppose that all conditions of Theorem 4.1 hold. Then the problem* (1.1)-(1.2) *has a unique solution in* $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$ *provided*

$$\frac{(\Phi(T)-\Phi(0))^{\xi+1-\zeta}}{\Gamma(\xi+1)}\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j < 1. \tag{4.10}$$

Proof. If possible the iterative FDEs (1.1)-(1.2) has two distinct solution v_1 and v_2 in the weighted space $C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)$. Then in view of equivalent fractional integral equation to the iterative FDEs (1.1)-(1.2) and the operator A defined in (4.4), we have $Av_1 = v_1$ and $Av_2 = v_2$.

Therefore

$$||v_1 - v_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} = ||Av_1 - Av_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}.$$

Proceeding as in the proof of Theorem 4.1, we obtain the estimation on the line of equation (4.5), as follows

$$\begin{aligned}||v_1 - v_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} &= ||Av_1 - Av_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}\\ &\leq \frac{1}{\Gamma(\xi+1)}\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j (\Phi(T)-\Phi(0))^{\xi-\zeta+1}||v_1 - v_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}.\end{aligned}$$

Using condition (4.10), in above estimation, we obtain

$$||v_1 - v_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} < ||v_1 - v_2||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})},$$

which is not possible. Therefore iterative FDEs (1.1)-(1.2) has a unique solution. $\square$

5 Continuous dependence and stability results

5.1 Continuous dependence results

To investigate the data dependency of solution of the nonlinear iterative FDEs (1.1)-(1.2), we consider the another nonlinear iterative FDEs of the form

$${}^{H}D_{0+}^{\xi,\,\eta;\,\Phi}\tilde{u}(t) = \tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\tilde{u}\right)^{[0]}(t), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\tilde{u}\right)^{[1]}(t), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\tilde{u}\right)^{[n]}(t)\right), \quad t \in J, \tag{5.1}$$

$$I_{0+}^{1-\zeta,\,\Phi}\tilde{u}(0) = \tilde{u}_0, \quad \tilde{u}_0 \geq 0, \quad \zeta = \xi + \eta(1-\xi), \tag{5.2}$$

where $\tilde{h}$ is a function different from h that satisfies all the assumptions of h.

Theorem 5.1. *Suppose that all the assumptions of Theorem 4.2 hold. Then, solution u of iterative FDEs* (1.1)-(1.2) *and solution $\tilde{u}$ of iterative FDEs* (5.1)-(5.2) *satisfies the inequality*

$$||\tilde{u}-u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} \leq \frac{\frac{1}{\Gamma(\zeta)}}{1-\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j}|\tilde{u}_0-u_0| + \frac{\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}}{1-\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j}\left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})}. \tag{5.3}$$

Proof. Using equivalent fractional integral of iterative FDE (1.1)-(1.2) and (5.1)-(5.2), for any $t \in J$ we have

$$\begin{aligned}
&\left|(\Phi(t)-\Phi(0))^{1-\zeta}(\tilde{u}(t)-u(t))\right| \leq \left|\frac{\tilde{u}_0}{\Gamma(\zeta)}-\frac{u_0}{\Gamma(\zeta)}\right| + \frac{(\Phi(t)-\Phi(0))^{1-\zeta}}{\Gamma(\xi)}\int_0^t \Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\xi-1} \\
&\times\left|\tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[n]}(\tau)\right)\right. \\
&\left.-h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)\right|d\tau.
\end{aligned} \tag{5.4}$$

Next, using Lipschitz condition on $\tilde{h}$, for any $\tau \in J$, we have

$$\begin{aligned}
&\left|\tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[n]}(\tau)\right)\right. \\
&\quad\left.-h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)\right| \\
&\leq \left|\tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[n]}(\tau)\right)\right. \\
&\quad\left.-\tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)\right| \\
&\quad+\left|\tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)\right. \\
&\quad\left.-h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau), \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)\right| \\
&\leq \sum_{i=1}^{n} c_i\left|\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[i]}(\tau)-\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[i]}(\tau)\right| + \left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})} \\
&\leq \left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})} + \sum_{i=1}^{n} c_i\left\|\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[i]}-\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[i]}\right\|_{C(J,\mathbb{R})}.
\end{aligned}$$

Using Lemma 2.4, for any $\tau \in J$, we obtain

$$\begin{aligned}
&\Big|\tilde{h}\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[0]}(\tau),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[1]}(\tau),\dots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}\right)^{[n]}(\tau)\right)\\
&\quad - h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau),\dots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)\Big|\\
&\le \left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})}+\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}M^j\left\|(\Phi(\cdot)-\Phi(0))^{1-\zeta}\,\tilde{u}-(\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right\|_{C(J,\mathbb{R})}\\
&\le \left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})}+\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}M^j||\tilde{u}-u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}.
\end{aligned}\tag{5.5}$$

Using estimation (5.5) in the inequality (5.4), for any $t \in J$, we have

$$\begin{aligned}
&\left|(\Phi(t)-\Phi(0))^{1-\zeta}(\tilde{u}(t)-u(t))\right|\\
&\quad\le \frac{|\tilde{u}_0-u_0|}{\Gamma(\zeta)}+\frac{(\Phi(t)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\left[\left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})}+\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}M^j\left\|\tilde{u}-u\right\|_{C_{1-\zeta;\,\Phi}(J)}\right]\\
&\quad\le \frac{|\tilde{u}_0-u_0|}{\Gamma(\zeta)}+\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\left[\left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})}+\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}M^j||\tilde{u}-u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})}\right]
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
||\tilde{u}-u||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R})} &\le \frac{\frac{1}{\Gamma(\zeta)}}{1-\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\sum\limits_{i=1}^{n}c_i\sum\limits_{j=0}^{i-1}M^j}|\tilde{u}_0-u_0|\\
&\quad+\frac{\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}}{1-\frac{(\Phi(T)-\Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}\sum\limits_{i=1}^{n}c_i\sum\limits_{j=0}^{i-1}M^j}\left\|\tilde{h}-h\right\|_{C(J,\mathbb{R})}. \qquad \square
\end{aligned}$$

Remark 5.2. (1) *Theorem 5.1 gives the continuous dependence of the solution of the problem* (1.1)-(1.2) *on the initial condition as well as on the nonlinear functions.*

(2) *If* $h = \tilde{h}$ *in* (5.3) *then* Theorem 5.1 *gives the dependency of the solution of* (1.1)-(1.2) *on initial condition.*

(3) *If* $u_0 = \tilde{u}_0$ *in* (5.3) *then* Theorem 5.1 *gives the dependency of the solution of* (1.1)-(1.2) *on the nonlinear functions.*

(4) *If* $h = \tilde{h}$ *and* $u_0 = \tilde{u}_0$ *in* (5.3), Theorem 5.1 *gives the uniqueness of solution of the problem* (1.1)-(1.2).

5.2 Stability results

To discuss the Ulam-Hyers stablity results we need the following definitions.

Definition 5.3 ([19]). *The iterative FDEs* (1.1)-(1.2) *is said to be Ulam-Hyers stable if there exists a real number $K > 0$ such that for each $\epsilon > 0$ and for each solution $v \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ of the inequality*

$$\Big|{}^{H}D^{\xi,\,\eta;\,\Phi}_{0+}v(t) \\ -h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[0]}(t),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[1]}(t),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[n]}(t)\right)\Big| \leq \epsilon, \quad (5.6)$$

with $I^{1-\zeta;\,\Phi}_{o+}v(0) = u_0$, there exists a solution $u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ of problem (1.1)-(1.2) *that satisfy*

$$||u - v||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)} \leq K\epsilon, \quad t \in J.$$

Definition 5.4 ([19]). *The iterative FDEs* (1.1)-(1.2) *is said to be generalized Ulam-Hyers stable if there exists $\chi \in C(J, \mathbb{R}^+)$ with $\chi(0) = 0$ such that for each $\epsilon > 0$ and for each solution $v \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ of the inequality* (5.6) *with $I^{1-\zeta;\,\Phi}_{o+}v(0) = u_0$, there exists a solution $u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ of the problem* (1.1)-(1.2) *satisfying*

$$||u - v||_{C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)} \leq \chi(\epsilon), \quad t \in J.$$

Theorem 5.5. *Assume all the assumptions of* Theorem 4.2 *hold. Then the problem* (1.1)-(1.2) *is Ulam-Hyers stable.*

Proof. Consider $v \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ be a function such that $I^{1-\zeta;\,\Phi}_{o+}v(0) = u_0$, that satisfy the inequality (5.6). Then integrating it, we obtain

$$\Big|v(t) - \frac{(\Phi(t)-\Phi(0))^{\zeta-1}}{\Gamma(\zeta)}u_0 - \frac{1}{\Gamma(\xi)}\int_0^t \Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\xi-1} \\ \times h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[0]}(\tau),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,v\right)^{[n]}(\tau)\right)d\tau\Big| \\ \leq I^{\xi;\,\Phi}_{0+}\epsilon = \frac{\epsilon}{\Gamma(\xi+1)}(\Phi(t)-\Phi(0))^{\xi}, \quad t \in (0,T].$$

If $u \in C_{1-\zeta;\,\Phi}(J, \mathbb{R}; M)$ is the solution of the iterative FDEs (1.1)-(1.2) then using Lipschitz condition of h, we obtain

$$|v(t) - u(t)| \\ = \Big|v(t) - \frac{(\Phi(t)-\Phi(0))^{\zeta-1}}{\Gamma(\zeta)}u_0 - \frac{1}{\Gamma(\xi)}\int_0^t \Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\xi-1} \\ \times h\left(\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[0]}(\tau),\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[1]}(\tau),\ldots,\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}\,u\right)^{[n]}(\tau)\right)d\tau\Big|$$

$$\leq \left| v(t) - \frac{(\Phi(t)) - \Phi(0))^{\zeta-1}}{\Gamma(\zeta)} u_0 - \frac{1}{\Gamma(\xi)} \int_0^t \Phi'(\tau)(\Phi(t) - \Phi(\tau))^{\xi-1} \right.$$
$$\left. \times h\left(\left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[0]}(\tau), \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[n]}(\tau) \right) d\tau \right|$$
$$+ \left| \frac{1}{\Gamma(\xi)} \int_0^t \Phi'(\tau)(\Phi(t) - \Phi(\tau))^{\xi-1} \right.$$
$$\times \left[h\left(\left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[0]}(\tau), \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[n]}(\tau) \right) \right.$$
$$\left. \left. - h\left(\left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right)^{[0]}(\tau), \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right)^{[1]}(\tau), \ldots, \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right)^{[n]}(\tau) \right) \right] d\tau \right|$$
$$\leq \frac{\epsilon}{\Gamma(\xi+1)} (\Phi(t) - \Phi(0))^{\xi} + \frac{1}{\Gamma(\xi)} \int_0^t \Phi'(\tau)(\Phi(t) - \Phi(\tau))^{\xi-1}$$
$$\times \sum_{i=1}^{n} c_i \left| \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[i]}(\tau) - \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right)^{[i]}(\tau) \right| d\tau.$$

Using the inequality in the Lemma 2.4, we have

$$|v(t) - u(t)|$$
$$\leq \frac{\epsilon}{\Gamma(\xi+1)} (\Phi(t) - \Phi(0))^{\xi} + \frac{(\Phi(t) - \Phi(0))^{\xi}}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \left\| \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} v \right)^{[i]} - \left((\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right)^{[i]} \right\|_{C(J)}$$
$$\leq \frac{\epsilon}{\Gamma(\xi+1)} (\Phi(t) - \Phi(0))^{\xi} + \frac{(\Phi(t) - \Phi(0))^{\xi}}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \left\| (\Phi(\cdot) - \Phi(0))^{1-\zeta} v - (\Phi(\cdot) - \Phi(0))^{1-\zeta} u \right\|_{C(J)}$$
$$= \frac{\epsilon}{\Gamma(\xi+1)} (\Phi(t) - \Phi(0))^{\xi} + \frac{(\Phi(t) - \Phi(0))^{\xi}}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|v - u\|_{C_{1-\zeta;\,\Phi}(J)}, \quad t \in J.$$

Therefore consider for all $t \in J$,

$$\|v - u\|_{C_{1-\zeta;\,\Phi}(J,\mathbb{R};M)} = \sup_{t\in J} \left| (\Phi(t) - \Phi(0))^{1-\zeta} (v(t) - u(t)) \right|$$
$$\leq \frac{\epsilon}{\Gamma(\xi+1)} (\Phi(T) - \Phi(0))^{\xi-\zeta+1} + \frac{(\Phi(T) - \Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j \|v - u\|_{C_{1-\zeta;\,\Phi}(J)}.$$

This gives

$$\|v - u\|_{C_{1-\zeta;\,\Phi}(J)} \leq \frac{\frac{\epsilon}{\Gamma(\xi+1)} (\Phi(T) - \Phi(0))^{\xi-\zeta+1}}{1 - \frac{1}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j (\Phi(T) - \Phi(0))^{\xi-\zeta+1}}.$$

Define $K = \dfrac{\frac{(\Phi(T) - \Phi(0))^{\xi-\zeta+1}}{\Gamma(\xi+1)}}{1 - \frac{1}{\Gamma(\xi+1)} \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} M^j (\Phi(T) - \Phi(0))^{\xi-\zeta+1}}$. Then $K > 0$ and we have

$$\|v - u\|_{C_{1-\zeta;\,\Phi}(J)} \leq K\epsilon.$$

This proves iterative FDEs (1.1)-(1.2) is Ulam-Hyers stable. □

Corollary 5.6. *Suppose all the assumptions of* Theorem 5.5 *are satisfied then the iterative FDEs* (1.1)-(1.2) *is generalized Ulam-Hyers stable.*

Proof. Follows by taking $\phi(\epsilon) = K\epsilon$. □

6 Examples

Example 6.1. *Consider the following initial value problem for iterative fractional differential equations*

$$\begin{aligned}
{}^{H}D_{0+}^{\frac{1}{2},\eta;\Phi}v(t) &= \frac{(\Phi(t)-\Phi(0))^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{1}{100}(\Phi(t)-\Phi(0))^{2-\zeta} \\
&\quad + \frac{1}{200}(\Phi(t)-\Phi(0))^{1-\zeta}\left(\Phi\left(\frac{(\Phi(t)-\Phi(0))^{2-\zeta}}{2}\right)-\Phi(0)\right) - \frac{1}{50}\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}v\right)^{[1]}(t) \\
&\quad - \frac{1}{100}\left((\Phi(\cdot)-\Phi(0))^{1-\zeta}v\right)^{[2]}(t), \qquad (6.1)\\
I_{0+}^{1-\zeta;\Phi}v(0) &= 0, \quad t\in\tilde{J}=[0,1]. \qquad (6.2)
\end{aligned}$$

Define the function $h : \tilde{J}^3 \to [0,\infty)$ by,

$$\begin{aligned}
h(t,u,v) &= \frac{(\Phi(t)-\Phi(0))^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{1}{100}(\Phi(t)-\Phi(0))^{2-\zeta} \\
&\quad + \frac{1}{200}(\Phi(t)-\Phi(0))^{1-\zeta}\left(\Phi\left(\frac{(\Phi(t)-\Phi(0))^{2-\zeta}}{2}\right)-\Phi(0)\right) - \frac{1}{50}u - \frac{1}{100}v.
\end{aligned}$$

Then for any $t\in\tilde{J}$ and $u_i, v_i \in \tilde{J}, (i=1,2)$, we have

$$|h(t,u_1,u_2)-h(t,v_1,v_2)| \le \frac{1}{50}|u_1-v_1| + \frac{1}{100}|u_2-v_2|.$$

This shows h satisfies Lipschitz type condition (4.1) with $c_1 = \frac{1}{50}$ and $c_2 = \frac{1}{100}$. We have $T=1$, choose $L=1$ then the condition $0 < L \le T$ hold. Further, in the view of condition (4.2) and (4.3) choose $c=\frac{1}{3}$, $M>0$ and the function Φ such that

$$\frac{2\rho^*}{\sqrt{\pi}}\left|\left(\frac{3}{2}-\zeta\right)\left(\Phi\left(\frac{1}{3}\right)-\Phi(0)\right)^{\frac{1}{2}-\zeta}\Phi'\left(\frac{1}{3}\right)\right| \le M, \qquad (6.3)$$

and

$$\frac{2\rho^*}{\sqrt{\pi}}(\Phi(1)-\Phi(0))^{\frac{3}{2}-\zeta} \le 1, \qquad (6.4)$$

where

$$\begin{aligned}\rho &= \sup_{t\in[0,1]} \{h(t,0,0)\} \\ &= \frac{(\Phi(1)-\Phi(0))^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{(\Phi(1)-\Phi(0))^{2-\zeta}}{100} + \frac{(\Phi(1)-\Phi(0))^{1-\zeta}}{200}\left(\Phi\left(\frac{(\Phi(1)-\Phi(0))^{2-\zeta}}{2}\right) - \Phi(0)\right),\end{aligned} \tag{6.5}$$

$$\rho^* = \rho + c_1 + c_2(1+M) = \rho + \frac{1}{50} + \frac{1}{100}(1+M). \tag{6.6}$$

With the choices of constant M and the function Φ that satisfies conditions (6.3) and (6.4), all the assumptions of Theorem 4.1 are satisfied. Thus Schauder's fixed point Theorem 3.1 guarantee the at least one solution of the iterative FDEs (6.1)-(6.2) in the weighted space $C_{1-\zeta;\Phi}(\tilde{J},\mathbb{R};M)$. By actual substitution one can verify that

$$v(t) = \frac{\Phi(t)-\Phi(0)}{2}, \quad t\in[0,1], \tag{6.7}$$

is the solution of the iterative FDEs (6.1)-(6.2). Further in addition to the conditions (6.3) and (6.4), if the constant M and the function Φ satisfy the condition

$$\frac{2(\Phi(1)-\Phi(0))^{\frac{3}{2}-\zeta}}{\sqrt{\pi}}\left(\frac{1}{50} + \frac{1}{100}(1+M)\right) < 1, \tag{6.8}$$

the problem (6.1)-(6.2) has unique solution in the weighted space $C_{1-\zeta;\Phi}(\tilde{J},\mathbb{R};M)$.

Note that the function v defined in (6.7) is the unique solution of the problem (6.1)-(6.2). If we take $\Phi(t)=t$, $t\in[0,1]$ and $\eta=1$ the problem (6.1)-(6.2) involving Φ-Hilfer fractional derivative reduces to the following initial value problem for iterative FDEs of the form

$${}^{C}D^{\frac{1}{2}}_{0+}v(t) = \frac{t^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{1}{100}t + \frac{1}{400}t - \frac{1}{50}v^{[1]}(t) - \frac{1}{100}v^{[2]}(t) \tag{6.9}$$

$$v(0) = 0. \tag{6.10}$$

In this case

$$\rho = \frac{1}{\sqrt{\pi}} + \frac{1}{100} + \frac{1}{400} = 0.5766.$$

If we choose $M=1$ then

$$\rho^* = 0.5766 + \frac{1}{50} + \frac{2}{100} = 0.6166.$$

Further, the conditions (6.3), (6.4) and (6.8) reduce respectively to

$$\frac{2\rho^*}{\sqrt{\pi}}\left|\left(\frac{3}{2}-\zeta\right)\left(\Phi\left(\frac{1}{3}\right)-\Phi(0)\right)^{\frac{1}{2}-\zeta}\Phi'\left(\frac{1}{3}\right)\right| = \frac{0.6166\times 2}{\sqrt{\pi}}\left|\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{-1}{2}}\right| = 0.6025 < 1 \tag{6.11}$$

$$\frac{2\rho^*}{\sqrt{\pi}}\left(\Phi(1)-\Phi(0)\right)^{\frac{3}{2}-\zeta} = \frac{0.6166\times 2}{\sqrt{\pi}} = 0.6957 < 1 \tag{6.12}$$

and

$$\frac{2(\Phi(1)-\Phi(0))^{\frac{3}{2}-\zeta}}{\sqrt{\pi}}\left(\frac{1}{50}+\frac{1}{100}(1+M)\right) = \frac{2}{\sqrt{\pi}}\left(\frac{1}{50}+\frac{2}{100}\right) = 0.0451 < 1. \tag{6.13}$$

Note that all the conditions of Theorem 4.2 are satisfied. Therefore the initial value problem for Caputo iterative FDEs (6.9)-(6.10) has a unique solution in the space $C(\tilde{J},\mathbb{R};1)$. By actual substitution, one can verify that

$$v(t) = \frac{t}{2}, \quad t \in [0,1], \tag{6.14}$$

is the unique solution of the problem (6.9)-(6.10).

We remark that the constants c_1 and c_2 appears naturally as h satisfy Lipschitz condition. $T = 1$ is the end point of the interval on which the problem (6.1)-(6.2) is considered. The constant L $(0 < L \leq T)$, $c \in (0, T)$ and $M > 0$ one choose in the view of condition (4.2) and (4.3). These constants depends on the choice of function Φ.

7 Conclusion

Through analytical approaches we examine the nonlinear iterative FDEs with Φ-Hilfer fractional derivative for existence, uniqueness, stability and dependency of solutions. The conditions (4.2) and (4.3) required to prove the existence and uniqueness results Theorem 4.1 and Theorem 4.2 are strong. Achieving the same kind of outcomes by removing the restrictions in (4.2) and (4.3) will be very interesting. We have given specific examples to demonstrate our findings. Investigating alternative conditions with weaker constraints is essential for ensuring the existence and uniqueness of solutions for iterative Φ-Hilfer fractional differential equations (FDEs). In this context, one can analyze iterative Φ-Hilfer FDEs under various types of initial and boundary conditions to study their existence, uniqueness, different forms of stability, and other qualitative properties. Further, the work explored in [23, 25, 43, 47, 49] can be analyzed by integrating the iterates of unknown function and the fractional calculus.

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Author contributions

The author whose name appears on the submission contributed equally to this work, solely responsible for the conception of the work and its final form.

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Compactness of the difference of weighted composition operators between weighted l^p spaces

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ABSTRACT

This paper investigates the properties of weighted composition operators acting between different weighted l^p spaces. Inspired by recent advancements in the field, we explore criteria for the continuity and compactness of these operators. Specifically, we provide simple conditions, in terms of normalized canonical sequences, for the continuity and compactness of the difference between two weighted composition operators, $W_{\varphi,u}$ and $W_{\psi,v}$. Furthermore, we calculate the essential norm of these operators. Our results extend and generalize previous works, offering new insights into the behavior of weighted composition operators in Banach sequence spaces. The findings contribute to the understanding of these operators' topological properties, particularly their applications in sequence spaces and functional analysis.

RESUMEN

Este artículo investiga las propiedades de operadores de composición con peso actuando entre diferentes espacios l^p con pesos. Inspirados por avances recientes en el área, exploramos criterios para la continuidad y compacidad de estos operadores. Específicamente, entregamos condiciones simples, en términos de sucesiones canónicas normalizadas, para la continuidad y compacidad de la diferencia entre dos operadores de composición con peso, $W_{\varphi,u}$ y $W_{\psi,v}$. Más aún, calculamos la norma esencial de estos operadores. Nuestros resultados extienden y generalizan trabajos previos, ofreciendo nuevas formas de entender el comportamiento de operadores de composición con peso en espacios de Banach de sucesiones. Los hallazgos contribuyen a la comprensión de las propiedades topológicas de estos operadores, particularmente sus aplicaciones a espacios de sucesiones y análisis funcional.

Keywords and Phrases: Banach sequence spaces, weighted composition operators, compactness.

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1 Introduction

The study of the properties of weighted composition operators has captivated numerous researchers worldwide. These operators play a significant role in various areas of functional analysis and have applications in sequence spaces and spaces of analytic functions. Specifically, in the context of Banach sequence spaces, weighted composition operators are useful for studying processes where the inputs are infinite collections of data $\{x(k)\}$ that undergo an organization and selection process, and are finally assigned a weight to obtain an output, similar to creating frequency tables in statistics. Organizing a sequence $\boldsymbol{x} = \{x(k)\}$ involves defining a function $\varphi : \mathbb{N} \to \mathbb{N}$, while assigning weights involves multiplying by a sequence $\boldsymbol{u} = \{u(k)\}$. This leads to the definition of the weighted composition operator $W_{\varphi,u}$ by

$$W_{\varphi,u}(\boldsymbol{x}) := \boldsymbol{u} \cdot (\boldsymbol{x} \circ \varphi).$$

The operator $W_{\varphi,u}$ can be seen as a composition of two important classical transformations: the multiplication operator M_u and the composition operator C_φ. In fact, when φ is the identity, $W_{\varphi,u}$ becomes M_u, and when $u(n) = 1$ for all n, it becomes C_φ. The properties of these operators have been widely studied in various contexts, including weighted sequence spaces [5, 9, 14, 15], which we define in the next paragraph.

Throughout the development of this document, p represents a fixed parameter in $[1, \infty)$. A numerical sequence $\boldsymbol{x} = \{x(k)\}$ is said to belong to the weighted l^p space, denoted as $\boldsymbol{x} \in l^p(\boldsymbol{r})$, if

$$\|\boldsymbol{x}\|_{l^p(\boldsymbol{r})} = \left(\sum_{k=1}^{\infty} |x(k)|^p \, r(k)^p \right)^{1/p} < \infty, \tag{1.1}$$

where $\boldsymbol{r} = \{r(k)\}$ is a weight, that is, $r(k) > 0$ for all $k \in \mathbb{N}$. The pair $\left(l^p(\boldsymbol{r}), \|\cdot\|_{l^p(\boldsymbol{r})} \right)$ constitutes a Banach space. These kinds of spaces naturally appear in the literature when studying properties of some operators in sequence spaces. For instance, for $p > 1$, the Cesàro space ces_p is contained in $l^p\left(k^{1-p}\right)$, indicating that every evaluation functional on ces_p is continuous.

Inspired by the work of Carpintero *et al.* [5], where they explored in detail the properties of weighted composition operators acting on weighted $\ell^\infty(\boldsymbol{r})$ sequence spaces, and as a continuation of the recent work of Cardona-Gutierrez *et al.* [4], which characterized the functions u and φ that define weighted composition operators with closed ranges when acting between two different weighted l^p spaces and analyzed when this operator is upper or lower semi-Fredholm, we aim to give simple criteria in terms of the normalized canonical sequences for the continuity and compactness of the difference of two weighted composition operators $W_{\varphi,u} - W_{v,\psi}$ acting between two different weighted l^p spaces. An important consequence of our results is the computation of the essential norm of the weighted composition operators $W_{\varphi,u}$ acting between two distinct weighted l^p spaces.

Our findings significantly extend and generalize previous works, such as those by [8, 10], which analyzed the case of weighted ℓ^2, and more recently, the work by Albanese and Mele [2], where the continuity and compactness of $W_{\varphi,u}$ between two different weighted l^p spaces were characterized.

In this article, we are particularly interested in knowing when $W_{\varphi,u}(\boldsymbol{x}) \in l^p(\boldsymbol{s})$ for all $\boldsymbol{x} \in l^p(\boldsymbol{r})$ (continuity problem) and in establishing other topological properties such as the compactness of the difference of two weighted composition operators (compactness problem). These problems have been widely studied in the context of holomorphic function spaces (see [7, 11, 13] and references therein), but in the context of Banach sequence spaces, they are still under development. Specifically, we shall prove the following properties:

(1) The operator $W_{\varphi,u}$ is continuous from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ if and only if

$$L_{\varphi,u} = \sup_{n\in\mathbb{N}} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} < \infty.$$

In this case, $\|W_{\varphi,u}\| = L_{\varphi,u}$.

(2) The difference of weighted composition operators $W_{\varphi,u} - W_{\psi,v}$ from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ is compact if and only if

$$\lim_{n\to\infty} \frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = 0.$$

(3) The essential norm of $W_{\varphi,u}$ from $l^p(\boldsymbol{r})$ to $l^p(\boldsymbol{s})$ is computed by

$$\|W_{\varphi,u}\|_e = \limsup_{n\to\infty} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}}.$$

This last result extends a result by Castillo *et al.* in [6].

The problem (1), was recently solved by Albanese and Mele [2]; however, in Section 2, for the sake of completeness and to benefit the reader, we provide a simple proof.

Additionally, in Section 3, we establish a very general criteria for the compactness of pointwise continuous operators acting between different weighted l^p spaces, which allows us to characterize the compactness of the difference of two weighted composition operators (see Theorem 3.1).

Finally, in this article, we use $\boldsymbol{x} = \{x(k)\}$ to denote a numerical complex sequence, while $(\boldsymbol{x}_n)$ denotes a sequence of sequences. Also, for a fixed $n \in \mathbb{N}$, we consider the canonical sequence $\boldsymbol{e}_n$, defined as $e_n(k) = 1$ if $k = n$ and $e_n(k) = 0$ otherwise.

2 Continuity of the weighted composition operators on $l^p(\boldsymbol{r})$

In this section we characterize all continuous weighted composition operators between two different weighted l^p spaces in terms of the norm of the images of the normalized canonical sequence. From now, for a function $\varphi : \mathbb{N} \to \mathbb{N}$, it is convenient to define

$$\operatorname{Ran}(\varphi) = \{n \in \mathbb{N} : n = \varphi(k) \text{ for some } k \in \mathbb{N}\}.$$

We can see that

$$\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}^p = \sum_{k:\varphi(k)=n} |u(k)|^p \, s(k)^p$$

and $\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})} = 0$ whenever $n \notin \operatorname{Ran}(\varphi)$. The following result is due to Albanese and Mele [2], and we include a brief proof for the benefit of the reader.

Theorem 2.1 ([2])**.** *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights. Suppose that $\boldsymbol{u} = \{u(k)\}$ is a complex sequence and let $\varphi : \mathbb{N} \to \mathbb{N}$ be a function. The operator $W_{\varphi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is continuous if and only if*

$$L_{\varphi,u} = \sup_{n\in\mathbb{N}} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} < \infty. \tag{2.1}$$

In this case, $\|W_{\varphi,u}\|_{op} = L_{\varphi,u}$.

Proof. Since $\boldsymbol{e}_n \in l^p(\boldsymbol{r})$ for all $n \in \mathbb{N}$, the condition (2.1) holds when we suppose that the operator $W_{\varphi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is continuous. Conversely, if there exists $L_{\varphi,u} > 0$ such that

$$\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})} \le L_{\varphi,u} \|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})} = L_{\varphi,u} r(n),$$

and we fix any $\boldsymbol{x} = \{x(k)\} \in l^p(\boldsymbol{r})$, then we have

$$\begin{aligned} \|W_{\varphi,u}(\boldsymbol{x})\|_{l^p(\boldsymbol{s})}^p &= \sum_{k=1}^{\infty} |u(k)|^p \, |x(\varphi(k))|^p s(k)^p \le \sum_{n\in\varphi(\mathbb{N})} |x(n)|^p L_{\varphi,u}^p r(n)^p \\ &\le L_{\varphi,u}^p \sum_{n=1}^{\infty} |x(n)|^p r(n)^p = L_{\varphi,u}^p \|\boldsymbol{x}\|_{l^p(\boldsymbol{r})}^p, \end{aligned}$$

and the operator $W_{\varphi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is continuous. The above argument also proves that

$$\|W_{\varphi,u}\|_{op} = \sup_{n\in\mathbb{N}} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = L_{\varphi,u}.$$

This proves the result. □

Since weighted composition operators generalize multiplication and composition operators, we have the following two important consequences:

Corollary 2.2. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights and suppose that $\boldsymbol{u} = \{u(k)\}$ is a complex sequence. The multiplication operator $M_u : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is continuous if and only if*

$$\sup_{n\in\mathbb{N}} \frac{\|M_u(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = \sup_{n\in\mathbb{N}} \frac{s(n)}{r(n)}|u(n)| < \infty.$$

Proof. It follows from Theorem 2.1 with $\varphi = \mathrm{Id}$, the identity function on $\mathbb{N}$, and by recalling that $W_{\mathrm{Id},u} = M_u$. □

Corollary 2.3. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights and let $\varphi : \mathbb{N} \to \mathbb{N}$ be a function. The composition operator $C_\varphi : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is continuous if and only if*

$$\sup_{n\in\varphi(\mathbb{N})} \frac{\|C_\varphi(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = \sup_{n\in\varphi(\mathbb{N})} \frac{1}{r(n)} \left(\sum_{k:\varphi(k)=n} s(k)^p \right)^{1/p} < \infty.$$

Proof. It follows from Theorem 2.1 with $\boldsymbol{u}(n) = 1$ for all $n \in \mathbb{N}$, a constant function on $\mathbb{N}$, and by recalling that, in this case, $W_{\varphi,u} = C_\varphi$. □

Similar results were obtained in [3] in the context of analytic functions (see also [12]).

3 On the compactness

In this section we shall obtain a characterization for the compactness of the difference operator $W_{\varphi,u} - W_{\psi,v} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ in terms of the canonical sequences. We said that a linear operator $K : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is pointwise continuous if for each sequence $(\boldsymbol{x}_n) \subseteq l^p(\boldsymbol{r})$ such that $\boldsymbol{x}_n \to 0$ pointwise ($\lim_{n\to\infty} x_n(m) = 0$ for all $m \in \mathbb{N}$), we have

$$\lim_{n\to\infty} (K(\boldsymbol{x}_n))(m) = 0$$

for all $m \in \mathbb{N}$. Clearly, the difference between two weighted composition operators is pointwise continuous. For this kind of operators, we have the following result which could have some interest by itself. A much more general result can be found in [1]. We include a proof for benefit of the reader.

Theorem 3.1. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights and suppose that $K : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is a pointwise continuous operator. The operator $K : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is compact if and only if for each norm-bounded sequence $(\boldsymbol{x}_n) \subseteq l^p(\boldsymbol{r})$ such that $\boldsymbol{x}_n \to 0$ pointwise, we have*

$$\lim_{n\to\infty} \|K(\boldsymbol{x}_n)\|_{l^p(\boldsymbol{s})} = 0. \tag{3.1}$$

Proof. Let us suppose first that $K : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is a compact operator. Let $(\boldsymbol{x}_n) \subseteq l^p(\boldsymbol{r})$ be any norm-bounded sequence such that $\boldsymbol{x}_n \to 0$ pointwise and suppose that the condition (3.1) is false. Then, there exists an $\epsilon > 0$ and a subsequence $(\boldsymbol{x}_{n_k})$ of $(\boldsymbol{x}_n)$ such that

$$\|K(\boldsymbol{x}_{n_k})\|_{l^p(\boldsymbol{s})} \geq \epsilon \tag{3.2}$$

for all $k \in \mathbb{N}$. Thus, the compactness of K implies that, by passing to a subsequence, if necessary, we can suppose that $(K(\boldsymbol{x}_{n_k}))$ converges to $\boldsymbol{y} \in l^p(\boldsymbol{s})$. That is,

$$\lim_{k\to\infty} \|K(\boldsymbol{x}_{n_k}) - \boldsymbol{y}\|_{l^p(\boldsymbol{s})} = 0. \tag{3.3}$$

We shall prove that $\boldsymbol{y} = \boldsymbol{0}$ (the null sequence). Indeed, for $m \in \mathbb{N}$ arbitrary but fixed, we have

$$|y_{n_k}(m) - y(m)|^p \leq \frac{1}{s(m)^p} \|K(\boldsymbol{x}_{n_k}) - \boldsymbol{y}\|^p_{l^p(\boldsymbol{s})},$$

with $\boldsymbol{y}_{n_k} = K(\boldsymbol{x}_{n_k})$. Thus, since K is pointwise continuous, we can write

$$|y(m)|^p = \lim_{k\to\infty} |y_{n_k}(m) - y(m)|^p \leq \lim_{k\to\infty} \frac{1}{s(m)^p} \|K(\boldsymbol{x}_{n_k}) - \boldsymbol{y}\|^p_{l^p(\boldsymbol{s})} = 0.$$

This last fact produces a contradiction between (3.2) and (3.3). Therefore

$$\lim_{n\to\infty} \|K(\boldsymbol{x}_n)\|_{l^p(\boldsymbol{s})} = 0,$$

and the implication is proved.

Next, we suppose that for all norm-bounded sequence $(\boldsymbol{x}_n) \subseteq l^p(\boldsymbol{r})$ such that $\boldsymbol{x}_n \to 0$ pointwise, we have

$$\lim_{n\to\infty} \|K(\boldsymbol{x}_n)\|_{l^p(\boldsymbol{s})} = 0.$$

We are going to show that the operator $K : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is compact. To see this, we fix any $(\boldsymbol{y}_n) \subseteq l^p(\boldsymbol{r})$ such that $\|\boldsymbol{y}_n\|_{l^p(\boldsymbol{r})} \leq 1$ for all $n \in \mathbb{N}$. The numerical sequence $\{y_n(1)\}$ of all first components is bounded since

$$|y_n(1)|^p\, r(1)^p \leq \sum_{k=1}^{\infty} |y_n(k)|^p\, r(k)^p = \|\boldsymbol{y}_n\|^p_{l^p(\boldsymbol{r})} \leq 1.$$

Hence, the Bolzano–Weierstrass theorem guarantees that there exists a convergent subsequence $\left\{y_n^{(1)}(1)\right\}$ of $\{y_n(1)\}$ and we can find $y(1) \in \mathbb{C}$ such that

$$\lim_{n\to\infty} |y_n^{(1)}(1) - y(1)| = 0.$$

Hence, we obtain a subsequence $\left(\boldsymbol{y}_n^{(1)}\right)$ of $(\boldsymbol{y}_n)$ whose first component is a convergent numerical sequence.

Arguing similarly, we have $\left|y_n^{(1)}(2)\right| r(2)^p \leq 1$, so there is a $y(2) \in \mathbb{C}$ and a subsequence $\left(\boldsymbol{y}_n^{(2)}\right)$ of $\left(\boldsymbol{y}_n^{(1)}\right)$ such that

$$\lim_{n\to\infty} \left|y_n^{(2)}(2) - y(2)\right| = 0.$$

Furthermore, we also have

$$\lim_{n\to\infty} \left|y_n^{(2)}(1) - y(1)\right| = 0.$$

Thus, by repeating this process, we obtain a subsequence $\left(\boldsymbol{y}_{n_k}\right)$ of $(\boldsymbol{y}_n)$ and a numerical sequence $\boldsymbol{y} = \{y(j)\}$ such that $\boldsymbol{y}_{n_k} \to \boldsymbol{y}$ pointwise. Also, for $H \in \mathbb{N}$ fixed we have

$$\sum_{j=1}^{H} |y(j)|^p r(j)^p = \limsup_{k\to\infty} \sum_{j=1}^{H} |y_{n_k}(j)|^p r(j)^p \leq \limsup_{k\to\infty} \left\|\boldsymbol{y}_{n_k}\right\|_{l^p(\boldsymbol{r})}^p \leq 1$$

and $\boldsymbol{y} \in l^p(\boldsymbol{r})$. Thus, applying the hypothesis to the sequence $\boldsymbol{x}_k = \boldsymbol{y}_{n_k} - \boldsymbol{y}$ which converges to zero pointwise, we conclude that

$$\lim_{k\to\infty} \|K(\boldsymbol{x}_k)\|_{l^p(\boldsymbol{s})} = \lim_{k\to\infty} \left\|K\left(\boldsymbol{y}_{n_k}\right) - K(\boldsymbol{y})\right\|_{l^p(\boldsymbol{s})} = 0$$

and the operator $K : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is compact. □

As an important consequence of the above result we have:

Theorem 3.2. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights, suppose that $\boldsymbol{u} = \{u(k)\}$ and $\boldsymbol{v} = \{v(k)\}$ are complex sequences, $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ are functions and $W_{\varphi,u}, W_{\psi,v} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ are continuous operators. The difference $W_{\varphi,u} - W_{\psi,v}$ from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ is compact if and only if*

$$\lim_{n\to\infty} \frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = 0. \tag{3.4}$$

Proof. Let us suppose first that the difference $W_{\varphi,u} - W_{\psi,v} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is a compact operator. For each $n \in \mathbb{N}$, we set

$$\boldsymbol{x}_n = \frac{\boldsymbol{e}_n}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}}.$$

Then $(\boldsymbol{x}_n)$ is a norm-bounded sequence which converges pointwise to the null sequence. Hence, Theorem 3.1 implies that the expression (3.4) holds.

Assume now that (3.4) holds and suppose that $(\boldsymbol{x}_n)$ is any bounded sequence in $l^p(\boldsymbol{r})$ such that

$$\lim_{n\to\infty} x_n(m) = 0$$

for all $m \in \mathbb{N}$. We shall prove that

$$\lim_{n\to\infty} \|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{x}_n)\|_{l^p(\boldsymbol{s})} = 0.$$

We can write

$$\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{x}_n)\|^p_{l^p(\boldsymbol{s})} = S_1(n) + S_2(n),$$

where

$$S_1(n) = \sum_{k:\varphi(k)=\psi(k)} |u(k) - v(k)|^p |x_n(\varphi(k))|^p s(k)^p,$$

$$S_2(n) = \sum_{k:\varphi(k)\neq\psi(k)} |u(k) x_n(\varphi(k)) - v(k) x_n(\psi(k))|^p s(k)^p.$$

For the first sum we have

$$S_1(n) \leq \sum_{m=1}^{\infty} |x_n(m)|^p r(m)^p \frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_m)\|^p_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_m\|^p_{l^p(\boldsymbol{r})}}.$$

While for the second sum we can see that

$$S_2(n) \leq 2^p \sum_{k:\varphi(k)\neq\psi(k)} (|u(k) x_n(\varphi(k))|^p + |v(k) x_n(\psi(k))|^p) s(k)^p \leq S_3(n) + S_4(n)$$

with

$$S_3(n) = 2^p \sum_{m\in\varphi(\mathbb{N})} |x_n(m)|^p \sum_{l\in\psi(\mathbb{N})-\{m\}} \sum_{k\in\varphi^{-1}(\{m\})\cap\psi^{-1}(\{l\})} |u(k)|^p s(k)^p,$$

$$S_4(n) = 2^p \sum_{l\in\psi(\mathbb{N})} |x_n(l)|^p \sum_{m\in\varphi(\mathbb{N})-\{l\}} \sum_{k\in\varphi^{-1}(\{m\})\cap\psi^{-1}(\{l\})} |v(k)|^p s(k)^p.$$

But, if $k \in \varphi^{-1}(\{m\}) \cap \psi^{-1}(\{l\})$, then $\varphi(k) = m$ and $\psi(k) = l \neq m$ and thus $e_m(\varphi(k)) = 1$, $e_m(\psi(k)) = 0$ and the third sum on the right of $S_3(n)$ can be written as

$$\sum_{k\in\varphi^{-1}(\{m\})\cap\psi^{-1}(\{l\})} |u(k) e_m(\varphi(k)) - v(k) e_m(\psi(k))|^p s(k)^p.$$

Thus

$$S_3(n) \leq 2^p \sum_{m=1}^{\infty} |x_n(m)|^p r(m)^p \frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_m)\|^p_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_m\|^p_{l^p(\boldsymbol{r})}}$$

and the same is also true for $S_4(n)$. Therefore,

$$\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{x}_n)\|^p_{l^p(\boldsymbol{s})} \leq 2^{p+2} \sum_{m=1}^{\infty} |x_n(m)|^p \, r(m)^p \frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_m)\|^p_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_m\|^p_{l^p(\boldsymbol{r})}}.$$

Finally, by hypothesis, for any $\varepsilon > 0$, we can find $m_0 \in \mathbb{N}$ such that

$$\frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_m)\|^p_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_m\|^p_{l^p(\boldsymbol{r})}} < \frac{\varepsilon}{2^{p+2}}$$

for all $m \geq m_0$. Also, there exists $M > 0$ such that $\|\boldsymbol{x}_n\|_{l^p(\boldsymbol{r})} \leq M$. Thus, we can write

$$\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{x}_n)\|^p_{l^p(\boldsymbol{s})} \leq 2^{p+2} \sum_{m=1}^{m_0} |x_n(m)|^p \, r(m)^p \frac{\|(W_{\varphi,u} - W_{\psi,v})(\boldsymbol{e}_m)\|^p_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_m\|^p_{l^p(\boldsymbol{r})}} + \varepsilon M^p$$

and the result follows from Theorem 3.1 since $(\boldsymbol{x}_n)$ converges pointwise to zero as $n \to \infty$. □

As an immediate consequence, we have:

Corollary 3.3. *Let* $\boldsymbol{r}, \boldsymbol{s}$ *be two weights, suppose that* $\boldsymbol{u} = \{u(k)\}$ *is a complex sequence and* $\varphi : \mathbb{N} \to \mathbb{N}$ *is a function.*

(1) *The operator* $W_{\varphi,u}$ *from* $l^p(\boldsymbol{r})$ *into* $l^p(\boldsymbol{s})$ *is compact if and only if*

$$\lim_{n\to\infty} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = 0.$$

This result was recently obtained by Albanese and Mele in [2, Theorem 3.12].

(2) *The multiplication operator* M_u*, as defined in the proof of Corollary 2.2, from* $l^p(\boldsymbol{r})$ *into* $l^p(\boldsymbol{s})$ *is compact if and only if*

$$\lim_{n\to\infty} \frac{\|M_u(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = \lim_{n\to\infty} \frac{s(n)}{r(n)} |u(n)| = 0.$$

(3) *The composition operator* C_φ*, as defined in the proof of Corollary 2.3, from* $l^p(\boldsymbol{r})$ *into* $l^p(\boldsymbol{s})$ *is compact if and only if*

$$\lim_{n\to\infty} \frac{\|C_\varphi(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = \lim_{n\to\infty} \frac{1}{r(n)} \left(\sum_{k:\varphi(k)=n} s(k)^p \right)^{1/p} = 0.$$

4 On the essential norm of $W_{\phi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$

In this section we calculate the essential norm of weighted composition operators acting between weighted l^p spaces in terms of canonical basis. We recall that if X and Y are Banach spaces and $\mathcal{K}(X,Y)$ denotes the set of all compact operators from X into Y, then the essential norm of T is denoted by $\|T\|_e$ and it is the distance of T to $\mathcal{K}(X,Y)$. That is,

$$\|T\|_e = \inf\{\|T-K\|_{op} : K \in \mathcal{K}(X,Y)\}.$$

It is clear that $T : X \to Y$ is compact if and only if $\|T\|_e = 0$. Hence, in virtue of Corollary 3.3 (1), the following result is expected.

Theorem 4.1. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights, suppose that $\boldsymbol{u} = \{u(k)\}$ is a complex sequence, $\varphi : \mathbb{N} \to \mathbb{N}$ is a function and suppose that the operator $W_{\varphi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is continuous. Then*

$$\|W_{\varphi,u}\|_e = \limsup_{n\to\infty} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}}.$$

Proof. It is convenient to consider, for $\epsilon > 0$ fixed, the following set

$$S_\epsilon = \left\{ n \in \mathbb{N} : \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} \geq \epsilon \right\}.$$

Then

$$\limsup_{n\to\infty} \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = \inf\{\epsilon > 0 : S_\epsilon \text{ is finite}\}.$$

Clearly $S_\epsilon \subseteq \varphi(\mathbb{N})$ for all $\epsilon > 0$ and $S_{\epsilon_1} \subseteq S_{\epsilon_2}$ whenever $\epsilon_1 > \epsilon_2$. The set

$$S = \{\epsilon > 0 : S_\epsilon \text{ is finite}\}$$

is bounded from below by zero, hence we can consider the number

$$\eta = \inf\{\epsilon > 0 : S_\epsilon \text{ is finite}\}.$$

We have two case: $\eta = 0$ and $\eta > 0$. We are going to prove that in both of the cases we can conclude $\|W_{\varphi,u}\|_e = \eta$.

Suppose first that $\eta = 0$. Then S_ϵ is finite for all $\epsilon > 0$. We shall prove that the operator $W_{\varphi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is compact. Indeed, if $W_{\varphi,u} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is not a compact operator, then by Corollary 3.3, we can find an $\epsilon_0 > 0$ and an unbounded and increasing sequence $\{n_k\} \subseteq \mathbb{N}$ such that

$$\frac{\|W_{\varphi,u}(\boldsymbol{e}_{n_k})\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}} \geq \epsilon_0$$

for all $k \in \mathbb{N}$. This means that S_{ϵ_0} is an infinite set and it is a contradiction to the fact that $\eta = 0$.

Suppose now that $\eta > 0$. Consider $\epsilon > 0$ such that $\eta - \frac{1}{2}\epsilon > 0$. Then by definition of infimum, $\eta - \frac{1}{2}\epsilon \notin S$, the set

$$S_{\eta - \frac{1}{2}\epsilon} = \left\{ n \in \mathbb{N} : \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} \geq \eta - \frac{\epsilon}{2} \right\}$$

is infinite and we can find an unbounded and increasing sequence $\{n_k\}$ of positive integers contained in $S_{\eta - \frac{1}{2}\epsilon}$. Hence, the sequence $(\boldsymbol{x}_k)$ defined by

$$\boldsymbol{x}_k = \frac{\boldsymbol{e}_{n_k}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}}$$

is bounded in $l^p(\boldsymbol{r})$, it converges pointwise to zero as $k \to \infty$ and therefore, Theorem 3.1 allows us to say that

$$\lim_{k \to \infty} \left\| K \left(\frac{\boldsymbol{e}_{n_k}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}} \right) \right\|_{l^p(\boldsymbol{s})} = 0$$

for any compact operator K from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$. Thus, for any $K \in \mathcal{K}(l^p(\boldsymbol{r}), l^p(\boldsymbol{s}))$ we have

$$\begin{aligned}
\|W_{\varphi,u} - K\| &\geq \left\| (W_{\varphi,u} - K) \left(\frac{\boldsymbol{e}_{n_k}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}} \right) \right\|_{l^p(\boldsymbol{s})} \\
&\geq \left\| W_{\varphi,u} \left(\frac{\boldsymbol{e}_{n_k}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}} \right) \right\|_{l^p(\boldsymbol{s})} - \left\| K \left(\frac{\boldsymbol{e}_{n_k}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}} \right) \right\|_{l^p(\boldsymbol{s})} \\
&\geq \eta - \frac{1}{2}\epsilon - \left\| K \left(\frac{\boldsymbol{e}_{n_k}}{\|\boldsymbol{e}_{n_k}\|_{l^p(\boldsymbol{r})}} \right) \right\|_{l^p(\boldsymbol{s})}
\end{aligned}$$

for all $n_k \in S_{\eta - \frac{1}{2}\epsilon}$. Taking $k \to \infty$, we obtain

$$\|W_{\varphi,u} - K\| \geq \eta - \frac{1}{2}\epsilon$$

and since $K \in \mathcal{K}(l^p(\boldsymbol{r}), l^p(\boldsymbol{s}))$ and $\epsilon > 0$ are arbitrary, we really have $\|W_{\varphi,u}\|_e \geq \eta$.

Next, we shall prove that $\|W_{\varphi,u}\|_e \leq \eta$. By definition of infimum, for any $\epsilon > 0$, the number $\eta + \epsilon$ is not a lower bound, hence the set

$$S_{\eta + \epsilon} = \left\{ n \in \mathbb{N} : \frac{\|W_{\varphi,u}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} \geq \eta + \epsilon \right\}$$

is finite. We set the symbol v by

$$v(k) = \begin{cases} u(k), & \text{if } \varphi(k) \in S_{\eta+\epsilon}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $S_{\eta+\epsilon}$ is finite, it is clear that

$$\lim_{n\to\infty} \frac{\|W_{\varphi,v}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = 0.$$

Indeed, for $n > \max S_{\eta+\epsilon}$, we have

$$\|W_{\varphi,v}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}^p = \sum_{k=1}^{\infty} |v(k)|^p \, |e_n(\varphi(k))|^p \, s(k)^p = \sum_{k:\varphi(k)\in S_{\eta+\epsilon}} |u(k)|^p \, |e_n(\varphi(k))|^p \, s(k)^p = 0.$$

In particular, $W_{\varphi,v} : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$ is a compact operator (see Corollary 3.3 (1)). Hence, the definition of essential norm of $W_{\varphi,u}$ allow us to write

$$\begin{aligned}
\|W_{\varphi,u}\|_e \le \|W_{\varphi,u} - W_{\varphi,v}\| &= \sup\left\{ \frac{\|W_{\varphi,u-v}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}}, n \in \mathbb{N} \right\} \\
&= \sup\left\{ \frac{\|W_{\varphi,u-v}(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}}, n \in \mathbb{N}\setminus S_{\eta+\epsilon} \right\} \le \eta + \epsilon,
\end{aligned}$$

since $u(k) - v(k) = 0$ when $\varphi(k) \in S_{\eta+\epsilon}$. Hence, we conclude that $\|W_{\varphi,u}\|_e \le \eta$ and the proof of theorem is complete. □

Remark 4.2. *From the proof of the above theorem, we can see that $\eta = 0$ when $\varphi(\mathbb{N})$ is finite. Hence, any weighted composition operator $W_{\varphi,u}$ from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ in which the symbol φ is a bounded function is a compact operator. Furthermore, since a linear operator $K : X \to Y$ is compact if and only if its essential norm is zero, an immediate consequence of our Theorem 4.1 is Theorem 3.12 in [2], which is stated in Corollary 3.3 (1).*

Corollary 4.3. *Let $\boldsymbol{r}, \boldsymbol{s}$ be two weights, suppose that $\boldsymbol{u} = \{u(k)\}$ is a complex sequence and $\varphi : \mathbb{N} \to \mathbb{N}$ is a function.*

(1) *Suppose that the multiplication operator $M_u : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$, as defined in the proof of Corollary 2.2, is continuous. The essential norm of this operator M_u from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ is computed by*

$$\limsup_{n\to\infty} \frac{\|M_u(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = \limsup_{n\to\infty} \frac{s(n)}{r(n)} |u(n)|.$$

(2) *Suppose that the composition operator $C_\varphi : l^p(\boldsymbol{r}) \to l^p(\boldsymbol{s})$, as defined in the proof of Corollary 2.3, is continuous. The essential norm of this operator C_φ from $l^p(\boldsymbol{r})$ into $l^p(\boldsymbol{s})$ is computed by*

$$\limsup_{n\to\infty} \frac{\|C_\varphi(\boldsymbol{e}_n)\|_{l^p(\boldsymbol{s})}}{\|\boldsymbol{e}_n\|_{l^p(\boldsymbol{r})}} = \limsup_{n\to\infty} \frac{1}{r(n)} \left(\sum_{k:\varphi(k)=n} s(k)^p \right)^{1/p}.$$

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Canonical metrics and ambiKähler structures on 4-manifolds with $U(2)$ symmetry

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ABSTRACT

For $U(2)$-invariant 4-metrics, we show that the B^t-flat metrics are very different from the other canonical metrics (Bach-flat, Einstein, extremal Kähler, etc.) We show every $U(2)$-invariant metric is conformal to two separate Kähler metrics, leading to ambiKähler structures. Using this observation we find new complete extremal Kähler metrics on the total spaces of $\mathcal{O}(-1)$ and $\mathcal{O}(+1)$ that are conformal to the Taub-bolt metric. In addition to its usual hyperKähler structure, the Taub-NUT's conformal class contains two additional complete Kähler metrics that make up an ambi-Kähler pair, making five independent compatible complex structures for the Taub-NUT, each of which is conformally Kähler.

RESUMEN

Para 4-métricas $U(2)$-invariantes, mostramos que las métricas B^t-planas son muy diferentes de las otras métricas canónicas (Bach-planas, Einstein, Kähler extremas, etc.) Mostramos que toda métrica $U(2)$-invariante es conforme a dos métricas Kähler separadas, lo que nos lleva a estructuras ambiKähler. Usando esta observación encontramos nuevas métricas Kähler extremas completas en los espacios totales de $\mathcal{O}(-1)$ y $\mathcal{O}(+1)$ que son conformes a la métrica Taub-bolt. Adicionalmente a su estructura usual hiperKähler, la clase conforme de Taub-NUT contiene dos métricas Kähler completas adicionales que hacen un par ambi-Kähler, lo que genera cinco estructuras complejas compatibles independientes para el Taub-NUT, cada una de las cuales es conformemente Kähler.

Keywords and Phrases: Cohomogeneity-1 metric, canonical metric, Taub-NUT, Bach tensor

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1 Introduction

Cohomogeneity-1 metrics with $U(2)$ symmetry have the form

$$g = A(r)\,dr^2 + B(r)\,(\eta^1)^2 + C(r)\left((\eta^2)^2 + (\eta^3)^2\right) \tag{1.1}$$

where η^1, η^2, η^3 are the usual left-invariant covector fields on $\mathbb{S}^3$. Naively the topology is $\mathbb{R}\times\mathbb{S}^3$, but there could be a quotient on the $\mathbb{S}^2$ factor, and topological changes occur at locations where B or C reach zero. We classify canonical metrics of this form including the B^t-flat metrics, and create new explicit examples of canonical metrics using the ambiKähler techniques of [2]. This project began as a way to develop supporting examples for other work, and treads such familiar ground that we expected few surprises. But we did find surprises, two of which we feel worth reporting to the wider community.

The first is how the B^t-flat metrics fit among the other canonical metrics. The space of $U(2)$-invariant extremal Kählers is rather small—up to homothety the moduli space is 3-dimensional—and except for the B^t flat metrics there are basically no other canonical metrics. Up to a choice of conformal factor, the Bach-flat metrics are a 2-parameter subspace of the extremal[1] metrics. The Einstein and harmonic-curvature metrics [14] are identical, and up to conformal factors are exactly the Bach-flat metrics. Half-conformally flat metrics are conformally extremal, and up to conformal factors the metrics with $W^+=0$ (or $W^-=0$) form a 1-parameter subspace of the Bach-flat metrics. The KE metrics and the Ricci-flat metrics are each a 1-parameter subclass of the Bach-flats. Up to homothety there are exactly three complete Ricci-flat KE metrics: flat $\mathbb{R}^4$, the Eguchi-Hanson, and the Taub-NUT. The Taub-NUT is extraordinary; see Proposition 2.5 and Section 4.

The B^t-flat metrics of [25] are exceptions to this framework. A B^t-flat metric is a metric satisfying the Euler-Lagrange equations of the functional

$$B^t = \int |W|^2 + t\int s^2 \tag{1.2}$$

where $t\in(-\infty,\infty]$, and we set $B^\infty = \int s^2$. The B^0 extremals are the Bach-flat metrics, and the B^∞ extremals are either scalar-flat or Einstein (see [5] for stable points of the $\int s^2$ functional). For $t\neq 0,\infty$ the B^t Euler-Lagrange equations are an overdetermined 8^{th} order system. After an appropriate reduction we find a 5-dimensional moduli space of B^t-flat metrics up to homothety. If the constant scalar curvature (CSC) condition is imposed, the CSC B^t-flat metrics constitute a 4-parameter family up to homothety. Intuitively, as t varies in $[0,\infty]$, the B^t-flat metrics would seem to interpolate between the Bach-flat metrics at $t=0$ and the Einstein metrics at $t=\infty$. As we pointed out, up to conformal factors these are exactly the same class, so it would stand

[1] We will use *extremal* to mean *extremal Kähler*, and *KE* to mean *Kähler-Einstein*.

to reason that the B^t-flat metrics would stay within this class. We find this is not the case; see Theorem 1.4.

The second surprise has to do with the global nature of certain complete ambiKähler pairs. Any metric (1.1) is automatically compatible with two complex structures which give opposite orientations that are both conformally Kähler—in short, each Kähler metric of the form (1.1) is a partner in an ambiKähler pair [2]. In Section 4 we consider four examples: an ambiKähler pair conformal to the classic Taub-NUT, and an ambiKähler pair conformal to the classic Taub-bolt. The two metrics conformal to the Taub-NUT are complete extremal Kähler metrics, one of which has zero scalar curvature (ZSC) and is 2-ended, and the other of which is one-ended and strictly extremal. The two metrics conformal to the Taub-bolt are complete extremal metrics, and exist on two different underlying complex surfaces, $\mathcal{O}(-1)$ and $\mathcal{O}(+1) \approx \mathbb{C}P^2 \setminus \{pt\}$. The metric on $\mathcal{O}(+1)$ is the only complete extremal Kähler metric, known to the authors, with a curve of positive self intersection. For instance the Eguchi-Hanson [17] and LeBrun metrics [29] lie on the total spaces of various $\mathcal{O}(k)$ with $k < 0$.

Placing the metric (1.1) in a more useful form, we solve $dz = \frac{2\sqrt{AB}}{C}dr$ for z to obtain

$$g = C\left(\frac{1}{4F}dz^2 + F(\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2\right) \tag{1.3}$$

where we have abbreviated $F = \frac{B}{C}$, now a function of z. If $f = f(z)$ is any function and $\{e_1, e_1, e_3\}$ is the $\mathbb{S}^3$ frame dual to $\{\eta^1, \eta^2, \eta^3\}$, then

$$J_f = -2f\frac{\partial}{\partial z} \otimes \eta^1 + \frac{1}{2f}e_1 \otimes dz - e_2 \otimes \eta^3 + e_3 \otimes \eta^2 \tag{1.4}$$

is a complex structure; see Lemma 2.1. Setting $f = \pm F$, the two complex structures $J^\pm = J_{\pm F}$ are compatible with g, and produce opposite orientations. The (1,1) forms are

$$\omega^\pm = g(J^\pm\cdot, \cdot) = \pm\frac{1}{2}C dz \wedge \eta^1 + C\eta^2 \wedge \eta^3. \tag{1.5}$$

From $d\eta^i = -\epsilon^i{}_{jk}\eta^j \wedge \eta^k$ we have $d\omega^\pm = (\pm C + C_z)dz \wedge \eta^2 \wedge \eta^3$, so a $U(2)$-invariant metric g is always conformally Kähler, and is Kähler when the conformal factor is $C = C_0 e^{\mp z}$, respectively.

The following linear operators appear frequently:

$$\mathcal{L}^+ = \left(-\frac{1}{2}\frac{d}{dz} + 1\right)\left(-\frac{d}{dz} + 1\right), \quad \mathcal{L}^- = \left(\frac{1}{2}\frac{d}{dz} + 1\right)\left(\frac{d}{dz} + 1\right) \tag{1.6}$$

as does the 4^{th} order linear operator $\mathcal{L}^+ \circ \mathcal{L}^- = \frac{1}{4}\frac{\partial^4}{\partial z^4} - \frac{5}{4}\frac{\partial^2}{\partial z^2} + 1$. The third-order nonlinear operator $\mathcal{B}$ also appears:

$$\mathcal{B}(F,F) = \left(-\frac{1}{2}F_{zz} + \frac{3}{2}F_z + F - 1\right)(\mathcal{L}^+(F) - 1) + F_z\big(\mathcal{L}^+(F)\big)_z. \tag{1.7}$$

This is a bit messy, but $\mathcal{B}$ can be understood as a first integral of the inhomogeneous operator $F \mapsto \mathcal{L}^+(\mathcal{L}^-(F)) - 1$; see equation (3.15). We will often use $\{\sigma^0, \sigma^1, \sigma^2, \sigma^3\}$, where $\sigma^0 = \frac{1}{|dz|}dz$ and $\sigma^i = \frac{1}{|\eta^i|}\eta^i$, to mean the orthonormal frame found by normalizing orthogonal frame $\{dz, \eta^1, \eta^2, \eta^3\}$.

Proposition 1.1. *The metric* (1.3) *has scalar curvature*

$$s = -4C^{-1}\left(\frac{\partial^2 F}{\partial z^2} + \frac{1}{2}F - 2\right) - 24C^{-\frac{3}{2}}\frac{\partial}{\partial z}\left(F\,\frac{\partial C^{\frac{1}{2}}}{\partial z}\right) \tag{1.8}$$

and trace-free Ricci tensor

$$\begin{aligned} \mathring{\mathrm{Ric}} = {} & \frac{4F}{\sqrt{C}}\left(\frac{\partial^2}{\partial z^2}\frac{1}{\sqrt{C}} - \frac{1}{4}\frac{1}{\sqrt{C}}\right)\cdot\big((\sigma^0)^2 - (\sigma^1)^2\big) \\ & + 2\left(\frac{1}{\sqrt{C}}\frac{\partial}{\partial z}\left(F\frac{\partial}{\partial z}\frac{1}{\sqrt{C}}\right) - \frac{1}{C}\left(\frac{1}{2}\frac{\partial^2 F}{\partial z^2} - \frac{3}{4}F + 1\right)\right)\cdot\big((\sigma^0)^2 + (\sigma^1)^2 - (\sigma^2)^2 - (\sigma^3)^2\big). \end{aligned} \tag{1.9}$$

The Weyl curvatures and their divergences are

$$\begin{aligned} W^{\pm} &= -\frac{1}{C}(\mathcal{L}^{\pm}(F) - 1)\left(\omega^{\pm}\otimes\omega^{\pm} - \frac{2}{3}Id_{\bigwedge^{\pm}}\right) \\ \delta W^{\pm} &= W^{\pm}\left(\nabla\log\left|e^{\pm\frac{3}{2}z}(\mathcal{L}^{\pm}(F) - 1)\sqrt{C}\right|, \cdot, \cdot, \cdot\right). \end{aligned} \tag{1.10}$$

The Bach tensor is

$$\begin{aligned} Bach = {} & \frac{16}{3C^2}\cdot F\cdot\big(\mathcal{L}^-(\mathcal{L}^+(F)) - 1\big)\cdot\Big(-2(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\Big) \\ & + \frac{8}{3C^2}\cdot\mathcal{B}(F,F)\cdot\Big(-(\sigma^0)^2 - (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\Big). \end{aligned} \tag{1.11}$$

If the metric is Kähler with respect to J^+, then the scalar curvature and Ricci form are

$$\begin{aligned} s &= -\frac{8}{C}\left(\mathcal{L}^+(F) - 1\right), \quad \textit{and} \\ \rho &= -\frac{2}{C}\left(\mathcal{L}^+(F) - 1\right)\omega^+ - \frac{2}{C}\left(\left(-\frac{1}{2}\frac{\partial}{\partial z} + 1\right)\left(\frac{\partial}{\partial z} + 1\right)F - 1\right)\omega^-. \end{aligned} \tag{1.12}$$

We remark that the $U(2)$-ansatz linearizes the Bach-flat equations $Bach = 0$, reducing them to $\mathcal{L}^+\circ\mathcal{L}^-(F) - 1 = 0$. The equation $\mathcal{B}(F,F) = 0$ is then an algebraic restriction on initial conditions.

When studying metrics—rather than just solutions of ODEs—it is useful to reduce the metrics by homothetic equivalence. In our setting this reduces the dimension of the solution space by two: one dimension for translation in z and one for multiplication of g by a positive constant.

Proposition 1.2 (Extremal and Bach-flat metrics)**.** *The metric* (1.3) *is extremal with complex structure* J^+ *if and only if* $C = C_0 e^{-z}$ *and* $\mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$*, meaning*

$$F(z) = 1 + \frac{1}{2}C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2}C_4 e^{2z}. \tag{1.13}$$

Such a metric is Bach-flat if and only if, in addition to (1.13)*, also* $C_1C_4 - C_2C_3 = 0$*.*

Consequently, up to homothety, the extremal metrics form a 3*-parameter family, and up to homothety and conformal factors the Bach-flat metrics constitute a* 2*-parameter subfamily of the extremal metrics.*

A metric is said to have *harmonic curvature* if $\delta\,\mathrm{Rm} = 0$, which is equivalent to $\delta W = 0$ and $s = const$; see [7, 14]. In the $U(2)$-invariant case $\delta W = 0$ actually implies $s = const$.

Proposition 1.3 (Einstein and harmonic-curvature metrics)**.** *For the metric* (1.3) *the following are equivalent: 1)* $\delta W = 0$*, 2)* $\delta\,\mathrm{Rm} = 0$*, 3) the metric is Einstein:* $\mathring{\mathrm{Ric}} = 0$*, 4)* F *and* C *satisfy*

$$F = 1 + \frac{1}{2}C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2}C_4 e^{2z}, \quad C = \frac{e^{-z}}{(C_5 + C_6 e^{-z})^2}, \tag{1.14}$$

with the two relations $C_1C_5 - C_2C_6 = 0$ *and* $C_3C_5 - C_4C_6 = 0$*. Given* (1.14)*, scalar curvature is the constant* $s = -24(C_2C_5^2 - 2C_5C_6 + C_3C_6^2)$*.*

A $U(2)$-invariant metric is Bach-flat if and only if it is conformally Einstein. The metric (1.14) is KE with respect to J^+ if and only if $C_6 = 0$ (so also $C_1 = C_3 = 0$), and KE with respect to J^- if and only if $C_5 = 0$ (so also $C_2 = C_4 = 0$). Up to homothety, there is a 1-parameter family of Ricci-flat metrics, and exactly three complete Ricci-flat KE metrics: the flat metric, the Taub-NUT metric, and the Eguchi-Hanson metric. See Propositions 3.2 and 3.5.

Theorem 1.4. *In the* $U(2)$*-invariant case, the space of solutions to the* B^t*-flat equations is 7-dimensional. Up to homothety, these constitute a 5-parameter family of metrics and the CSC* B^t*-flat metrics constitute a 4-parameter family. When* $t \neq 0, \infty$*, there exist CSC* B^t*-flat metrics that are not conformal to any extremal metric.*

The overdetermined 8^{th} order B^t-flat system is complicated, but appears explicitly in Lemma 3.8. In Section 4 we discuss the ambiKähler transform, and examine complete extremal metrics conformal to the classic Taub-NUT and -bolt metrics.

2 Complex structures, metrics, and topology

The metric (1.3), complex structures $J^{\pm}$, and $(1,1)$ forms $\omega^{\pm} = g(J^{\pm}\cdot, \cdot)$ are

$$\begin{aligned} g &= C\Big(\frac{1}{4F}dz^2 + F(\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2\Big) \\ J^{\pm} &= \mp 2F\frac{\partial}{\partial z}\otimes\eta^1 \pm \frac{1}{2F}e_1\otimes dz - e_2\otimes\eta^3 + e_3\otimes\eta^2 \\ \omega^{\pm} &= \pm\frac{1}{2}Cdz\wedge\eta^1 + C\eta^2\wedge\eta^3. \end{aligned} \tag{2.1}$$

In Section 2.1 we study the complex structures. In Section 2.2 we compute curvature quantities up through the Bach tensor. In Section 2.3 we examine the topology and asymptotics which the $U(2)$ ansatz may produce.

2.1 The complex structures

Here we check the integrability of the left-invariant almost complex structures J_f. We also study the *right*-invariant compatible complex structures that we call $I^{\pm}$.

Lemma 2.1. *Given any $f = f(z) \neq 0$, the complex structure J_f is integrable.*

Proof. The splitting $\bigwedge^1_{\mathbb{C}} = \bigwedge^{1,0}\oplus\bigwedge^{0,1}$ into $\pm\sqrt{-1}$ eigenspaces of J_f gives

$$\bigwedge{}^{0,1} = span_{\mathbb{C}}\left\{\frac{1}{2f}dz - \sqrt{-1}\eta^1,\ \ \eta^2 - \sqrt{-1}\eta^3\right\}. \tag{2.2}$$

On bases we compute

$$\begin{aligned} &d\left(\frac{1}{2f}dz - \sqrt{-1}\eta^1\right) = -2\sqrt{-1}\eta^2\wedge\eta^3 = 2\eta^2\wedge\left(\eta^2 - \sqrt{-1}\eta^3\right), \\ &d\left(\eta^2 - \sqrt{-1}\eta^3\right) = 2\eta^1\wedge\eta^3 + 2\sqrt{-1}\eta^1\wedge\eta^2 = 2\sqrt{-1}\eta^1\wedge\left(\eta^2 - \sqrt{-1}\eta^3\right). \end{aligned} \tag{2.3}$$

Therefore $d\bigwedge^{0,1}\subset\bigwedge^1\wedge\bigwedge^{0,1} = \bigwedge^{1,1}\oplus\bigwedge^{0,2}$ so we conclude that J_f is integrable. □

Lemma 2.2. *The complex structures $J^{\pm}$ are metric compatible. Their $(1,1)$ forms $\omega^{\pm} = g(J^{\pm}\cdot,\cdot)$ are closed if and only if $C = C_0e^{\mp z}$, respectively.*

Proof. Checking compatibility with the metric is an elementary computation which we omit. From (1.5), $d\omega^{\pm} = 0$ if and only if $C = C_0e^{\mp z}$. □

To create right-invariant complex structures and relate them to the metric (which is left-invariant) we require background coordinates. Polar coordinates on $\mathbb{R}^4 \approx \mathbb{C}^2$ are

$$(r,\psi,\theta,\varphi) \longmapsto \left(r\cos(\theta/2)e^{-\frac{i}{2}(\psi+\varphi)},\ r\sin(\theta/2)e^{-\frac{i}{2}(\psi-\varphi)}\right). \tag{2.4}$$

The three "Euler coordinates" (ψ,θ,φ) have ranges $|\psi\pm\varphi|<2\pi$ and $\theta\in[0,\pi]$. The transitions between the coordinate framing and the left-invariant framing are

$$\begin{array}{ll}\eta^0=dz\;=\;\frac{\sqrt{F}}{2\sqrt{C}}dr & e_0=\frac{\partial}{dz}\;=\;\frac{\sqrt{F}}{2\sqrt{C}}\frac{\partial}{\partial r}\\ \eta^1=\frac{1}{2}(d\psi+\cos\theta\,d\varphi) & e_1=2\frac{\partial}{\partial\psi}\\ \eta^2=\frac{1}{2}(\sin\psi\,d\theta-\cos\psi\sin\theta\,d\varphi) & e_2=2\left(\cos\psi\cot\theta\frac{\partial}{\partial\psi}+\sin\psi\frac{\partial}{\partial\theta}-\cos\psi\csc\theta\frac{\partial}{\partial\varphi}\right)\\ \eta^3=\frac{1}{2}\left(\cos\psi\,d\theta+\sin\psi\sin\theta\,d\varphi\right) & e_3=2\left(-\sin\psi\cot\theta\frac{\partial}{\partial\psi}+\cos\psi\frac{\partial}{\partial\theta}+\sin\psi\csc\theta\frac{\partial}{\partial\varphi}\right).\end{array}\tag{2.5}$$

To create the right-invariant frames we apply quaterionic conjugation $T(z,w)=(\bar{z},-w)$ to $\mathbb{C}^2$, which changes the parameterization of $\mathbb{C}^2$ to

$$(r,\psi,\theta,\varphi)\;\longmapsto\;\left(r\cos(\theta/2)e^{\frac{i}{2}(\varphi+\psi)},\;-r\sin(\theta/2)e^{\frac{i}{2}(\varphi-\psi)}\right).\tag{2.6}$$

In coordinates, T is $T(r,\psi,\theta,\varphi)=(r,-\varphi,-\theta,-\psi)$. The left-invariant forms η^i pull back to right-invariant forms $\bar{\eta}^i=T^*(\eta^i)$. In the bases $\{\eta^i\}$, $\{\bar{\eta}^i\}$, the linear map $T^*:\bigwedge^1\to\bigwedge^1$ is

$$T^*=\begin{pmatrix}1&0&0&0\\0&-\cos\theta&\cos\psi\sin\theta&-\sin\psi\sin\theta\\0&-\sin\theta\cos\varphi&-\cos\psi\cos\theta\cos\varphi+\sin\psi\sin\varphi&\sin\psi\cos\theta\cos\varphi+\cos\psi\sin\varphi\\0&-\sin\theta\sin\varphi&-\cos\psi\cos\theta\sin\varphi-\sin\psi\cos\varphi&\sin\psi\cos\theta\sin\varphi-\cos\psi\cos\varphi\end{pmatrix}.\tag{2.7}$$

In the bases $\{e_i\}$, $\{\bar{e}_i\}$ we have that $T_*:TM\to TM$ is the transpose $T_*=(T^*)^T$. One can check directly that $T^*,T_*\in O(4)$.

Let $\sigma^i=\frac{1}{|\eta^i|}\eta^i$ be the unit length forms

$$\sigma^0=\sqrt{\frac{C}{4F}}dz,\quad \sigma^1=\sqrt{CF}\eta^1,\quad \sigma^2=\sqrt{C}\eta^2,\quad \sigma^3=\sqrt{C}\eta^3\tag{2.8}$$

and let $\{f_i\}=\frac{1}{|e_i|}e_i$ be the corresponding frame. Then the complex structures $J^\pm$ are

$$J^\pm\;=\;\mp f_0\otimes\sigma^1\pm f_1\otimes\sigma^0-f_2\otimes\sigma^3+f_3\otimes\sigma^2.\tag{2.9}$$

Under T, $J^\pm$ are conjugate to *right*-invariant complex structures $I^\mp$, given by $T_*\circ I^\pm\circ T_*=J^\mp$. Because $I^\mp$ are isomorphic to $J^\pm$ under a diffeomorphism on M^4 (the $\mathbb{S}^3$ antipodal map), I^+ and I^- are integrable. We summarize this in the following lemma.

Lemma 2.3. *The structures $I^\pm$ are integrable, right-invariant, and g-compatible. The structures J^+,I^+ produce a common orientation, with corresponding $(1,1)$-forms $\omega^+,\omega_I^+\in\bigwedge^+$. Similarly J^-,I^- produce a common orientation, and $\omega^-,\omega_I^-\in\bigwedge^-$.*

The Hermitian structures $(g, J^{\pm})$ produce a very flexible array of Kähler metrics, as F may be chosen freely. By contrast, the Kähler conditions for $(g, I^{\pm})$ are far more restrictive. This is because the left-action of $SU(2)$ fixes g but permutes $I^{\pm}$ among an $\mathbb{S}^2$ worth of complex structures; therefore if $\omega_I^{\pm}$ is Kähler, it is part of a hyperKähler structure. In particular $d\omega_I^{\pm} = 0$ forces $\mathrm{Ric}_g = 0$.

Proposition 2.4. *Letting $\omega_I^- = g(I^- \cdot, \cdot)$, then $d\omega_I^- = 0$ if and only if*

$$F \;=\; (1 + C_1 e^z)^2 \quad \textit{and} \quad C \;=\; \frac{C_0 e^z}{(1 + C_1 e^z)^2}. \tag{2.10}$$

Any such metric is Ricci-flat. The same holds for ω_I^+ after replacing z by $-z$ in (2.10).

Proof. We may compute $d\omega_I^-$ explicitly using the matrices for T^* in (2.7) and its transpose T_*. The computation is tedious but completely elementary, and works out to be

$$\begin{aligned} *d\omega_I^- \;=\; \frac{2}{\sqrt{C}} \Bigg(& \cos\theta \left((-2 + F^{\frac{1}{2}}) + F^{\frac{1}{2}} \frac{\partial}{\partial z} \log C \right) \eta^1 \\ & - F^{-\frac{1}{2}} \sin\theta \cos\psi \left(2F^{\frac{1}{2}} - 2F \frac{\partial}{\partial z} \log C - \frac{\partial}{\partial z} F \right) \eta^2 \\ & - F^{-\frac{1}{2}} \sin\theta \sin\psi \left(2F^{\frac{1}{2}} - 2F \frac{\partial}{\partial z} \log C - \frac{\partial}{\partial z} F \right) \eta^3 \Bigg). \end{aligned} \tag{2.11}$$

Setting this to zero gives the partially decoupled system

$$\frac{\partial}{\partial z} F^{\frac{1}{2}} = \left(-1 + F^{\frac{1}{2}} \right), \quad \frac{\partial}{\partial z} \log C \;=\; \left(-1 + 2F^{-\frac{1}{2}} \right) \tag{2.12}$$

which has general solution $F = (1 + C_1 e^z)^2$, $C = \frac{C_0 e^z}{(1 + C_1 e^z)^2}$. Ricci-flatness follows from the general fact that any hyperKähler metric is Ricci flat [5], or from Proposition 3.2 below. ☐

Proposition 2.4 gives a two parameter family of solutions. Up to homothety we have two metrics.

Proposition 2.5. *Up to homothety, there are exactly two metrics g of the form* (1.3) *for which I^- is a Kähler structure. The first is*

$$F = (1 - e^z)^2 \quad \textit{and} \quad C = \frac{e^z}{(1 - e^z)^2}. \tag{2.13}$$

This hyperKähler metric has an ALF end at $z = 0$ a nut at $z = +\infty$. The second is

$$F = (1 + e^z)^2 \quad \textit{and} \quad C = \frac{e^z}{(1 + e^z)^2}. \tag{2.14}$$

This metric is incomplete, with a nut at $z = -\infty$ and a curvature singularity at $z = +\infty$.

For an analysis of the nut-like topology see Section 2.3.1 and for ALF ends see Section 2.3.2. To verify the claim that (2.14) has a curvature singularity as $z \to +\infty$, we may use (2.27) below to find $|W^+|^2 = 384(-1+e^z)^6$. The metric (2.13) is the Euclidean Taub-NUT; see Section 4.

2.2 Curvature quantities

It is useful to place the metric (2.1) into LeBrun ansatz form [30]. Referring to the polar coordinates of (2.4), from $(r, \varphi, \theta, \psi)$ we change to (Z, τ, x, y) where $x = \log\tan\frac{\theta}{2}$, $y = \varphi$, $\tau = \psi$, and Z solves $dZ = \frac{1}{4}C dz$. Then $(\eta^2)^2 + (\eta^3)^2 = \frac{1}{4}(d\theta^2 + \sin^2\theta\, d\varphi^2) = \frac{1}{4\cosh^2 x}(dx^2+dy^2)$ and the metric is

$$g = \frac{C}{4\cosh^2 x}(dx^2+dy^2) \; + \; \frac{FC}{4}\left(d\tau - \tanh(x)dy\right)^2 + \frac{4}{FC}dZ^2. \tag{2.15}$$

Written this way, the metric (2.15) is precisely in the form of Proposition 1 of [30]—the LeBrun ansatz—where $w = \frac{4}{FC}$ and $e^u = \frac{FC^2}{16\cosh^2 x}$. The complex structures in these coordinates are

$$J^{\pm}(dZ) = \mp 2FC\eta^1, \quad J^{\pm}(dx) = -dy. \tag{2.16}$$

We record the useful fact that $\eta^2 \wedge \eta^3 = \frac{1}{4\cosh^2(x)}dx \wedge dy$.

Proposition 2.6 (Ricci Curvature in the Kähler case)**.** *If g is Kähler with respect to J^+, its Ricci form $\rho = Ric(J\cdot,\cdot)$ and scalar curvature are*

$$\rho = -\frac{2}{C}\left(\mathcal{L}^+(F) - 1\right)\omega^+ - \frac{2}{C}\left[\left(-\frac{1}{2}\frac{\partial}{\partial z}+1\right)\left(\frac{\partial}{\partial z}+1\right)F - 1\right]\omega^-, \tag{2.17}$$

$$s = -\frac{8}{C}\left(\mathcal{L}^+(F) - 1\right). \tag{2.18}$$

Proof. Setting $C = C_0 e^{-z}$ we follow the computation in [30]. From that paper, the Ricci form is $\rho = -i\partial\bar\partial u = \frac{1}{2}d(Jdu)$ where in our case $u = \log(FC^2) - \log(16\cosh^2(x))$, as we found in (2.15). Using coordinates (z, τ, x, y) (specifically using z, not Z from (2.15)), we have $J(dz) = -2F\eta^1$ and $J(dx) = -dy$ from (1.4) and (2.16). Using also $dx \wedge dy = 4\cosh^2(x)\eta^2 \wedge \eta^3$ and $d\eta^1 = -2\eta^2 \wedge \eta^3$,

$$\begin{aligned} u &= \log F - 2z + 2\log C_0 - 2\log(4\cosh x) \\ du &= (F_z F^{-1} - 2)dz \; - \; 2\tanh(x)dx \\ Jdu &= (-2F_z + 4F)\eta^1 + 2\tanh(x)dy \\ dJdu &= (-2F_{zz} + 4F_z)dz \wedge \eta^1 + (-4F_z - 8F + 8)\eta^2 \wedge \eta^3 \end{aligned} \tag{2.19}$$

From (2.1), $dz \wedge \eta^1 = C^{-1}(\omega^+ - \omega^-)$ and $\eta^2 \wedge \eta^3 = \frac{1}{2}C^{-1}(\omega^+ + \omega^-)$. Therefore

$$\rho = \frac{2}{C}\left(-\frac{1}{2}F_{zz} + \frac{3}{2}F_z - F + 1\right)\omega^+ + \frac{2}{C}\left(\frac{1}{2}F_{zz} - \frac{1}{2}F_z - F + 1\right)\omega^- \tag{2.20}$$

as claimed. Scalar curvature for any Kähler metric is $s = 2*(\omega^+ \wedge \rho)$, so (2.17) along with the facts $\omega^+ \wedge \omega^- = 0$ and $*(\omega^+ \wedge \omega^+) = 2$ gives (2.18). □

Proposition 2.7 (Ricci curvature, general case)**.** *Scalar curvature is*

$$s \;=\; -4C^{-1}\left(\frac{\partial^2 F}{\partial z^2} + \frac{1}{2}F - 2\right) - 24C^{-\frac{3}{2}}\frac{\partial}{\partial z}\left(F\frac{\partial}{\partial z}C^{\frac{1}{2}}\right). \tag{2.21}$$

Using the unit frames σ^i of (2.8) *the trace-free Ricci curvature is*

$$\begin{aligned} \mathring{\mathrm{Ric}} = {}& 4FC^{-\frac{1}{2}}\left(\frac{\partial^2}{\partial z^2}C^{-\frac{1}{2}} - \frac{1}{4}C^{-\frac{1}{2}}\right)\cdot\left((\sigma^0)^2 - (\sigma^1)^2\right) \\ &+ 2\left(C^{-\frac{1}{2}}\frac{\partial}{\partial z}\left(F\frac{\partial}{\partial z}C^{-\frac{1}{2}}\right) - C^{-1}\left(\frac{1}{2}\frac{\partial^2 F}{\partial z^2} - \frac{3}{4}F + 1\right)\right) \\ &\cdot\left((\sigma^0)^2 + (\sigma^1)^2 - (\sigma^2)^2 - (\sigma^3)^2\right). \end{aligned} \tag{2.22}$$

Proof. We use the conformal change formulas from [5]. The scalar curvature (2.21) follows from (2.18) along with the formula $\tilde{s} = U^{-2}(s - 6U^{-1}\triangle_g U)$ when $\tilde{g} = U^2 g$. In the Kähler metric where $C = e^{-z}$, the Laplacian $\triangle_g$ acting on any $U = U(z)$ is $\triangle_g U = 4e^{2z}\frac{\partial}{\partial z}\left(e^{-z}F\frac{\partial U}{\partial z}\right)$. To obtain (2.21), use $U = e^{\frac{1}{2}z}C^{\frac{1}{2}}$.

To compute $\mathring{\mathrm{Ric}}$, again we start with the Kähler case; (2.17) gives

$$\mathring{\mathrm{Ric}}_g = 2e^z\left(\frac{1}{2}F_{zz} - \frac{1}{2}F_z - F + 1\right)\left(-(\sigma^0)^2 - (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\right) \tag{2.23}$$

The trace-free Ricci conformally changes by $\mathring{\mathrm{Ric}}_{\tilde{g}} = \mathring{\mathrm{Ric}}_g + 2U(\nabla_g^2 U^{-1} - \frac{1}{4}(\triangle_g U^{-1})g)$. Then using

$$\begin{aligned} 2U\Big(\nabla_g^2 U^{-1} - \frac{1}{4}(\triangle_g U^{-1})g\Big) = {}& -4UF(e^z(U^{-1})_z)_z\left(-(\sigma^0)^2 + (\sigma^1)^2\right) \\ &- 2U(e^zF(U^{-1})_z)_z\big(-(\sigma^0)^2 - (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\big) \end{aligned} \tag{2.24}$$

and $U = e^{\frac{1}{2}z}C^{\frac{1}{2}}$, we add (2.24) to (2.23) to give (2.22). □

Proposition 2.8. *The metric* (2.1) *has Weyl curvatures*

$$W^{\pm} \;=\; -C^{-1}\left(\mathcal{L}^{\pm}(F) - 1\right)\left(\omega^{\pm}\otimes\omega^{\pm} - \frac{2}{3}Id_{\bigwedge^{\pm}}\right). \tag{2.25}$$

Proof. We use Derdzinski's Theorem (see [15, Section 3, Proposition 2]) to find W^+ in the Kähler case, then conformally change to the arbitrary case. By Derdzinski's Theorem $W^+ = \frac{s}{12}\left(\frac{3}{2}\omega\otimes\omega - Id_{\bigwedge^+}\right)$ where ω is a Kähler form. When $C = e^{-z}$, ω^+ is Kähler and Proposition 2.6 gives

$$W^+ = -\frac{2}{3}\,e^z\left(\mathcal{L}^+(F) - 1\right)\left(\frac{3}{2}\omega^+\otimes\omega^+ - Id_{\bigwedge^+}\right). \tag{2.26}$$

Conformally changing from $C = e^{-z}$ to any $C = C(z)$ gives (2.25). Computing W^- is the same, after setting $C = e^z$ to make ω^- rather than ω^+ into a Kähler form. $\square$

From Proposition 2.8, $|W^\pm|^2$ and $|W^\pm|^2 dVol$ are

$$\begin{aligned} |W^\pm|^2 &= \frac{32}{3C^2}\left(\mathcal{L}^\pm(F) - 1\right)^2 \qquad \text{and} \\ |W^\pm|^2 dVol &= \frac{16}{3}\left(\mathcal{L}^\pm(F) - 1\right)^2 dz \wedge \eta^1 \wedge \eta^2 \wedge \eta^3. \end{aligned} \tag{2.27}$$

We compute the divergences $\delta W^\pm$ and the Bach tensor.

Proposition 2.9. *For the metric* (2.1)*,*

$$\delta W^\pm \;=\; W^\pm\left(\nabla \log\left| e^{\pm\frac{3}{2}z}(\mathcal{L}^\pm(F) - 1)\sqrt{C}\right|,\ \cdot,\ \cdot,\ \cdot\right). \tag{2.28}$$

Proof. Again we first conformally change the metric so it is Kähler. By Lemma (2.4) the metric $\tilde{g} = e^{-z}C^{-1}g$ is Kähler and the form $\widetilde{\omega} = \tilde{g}(J^+\cdot,\cdot)$ is closed. Then $\tilde{\delta}\widetilde{\omega} = -*d\widetilde{\omega} = 0$ so also $\tilde{\delta}(\widetilde{\omega}\otimes\widetilde{\omega}) = 0$, and $\tilde{\delta}(Id_{\bigwedge^+}) = 0$ because $Id_{\bigwedge^+}$ is covariant-constant. Therefore (2.25) gives

$$\begin{aligned} \tilde{\delta}\,\widetilde{W}^+(\cdot,\cdot,\cdot) &= \tilde{\delta}\left(-e^z(\mathcal{L}^+(F) - 1)\left(\widetilde{\omega}\otimes\widetilde{\omega} - \frac{2}{3}Id_{\bigwedge^+}\right)\right)(\cdot,\cdot,\cdot) \\ &= -\left(\widetilde{\omega}\otimes\widetilde{\omega} - \frac{2}{3}Id_{\bigwedge^+}\right)\left(\widetilde{\nabla}(e^z(\mathcal{L}^+(F) - 1)),\ \cdot,\ \cdot,\ \cdot\right) \\ &= \widetilde{W}^+\left(\widetilde{\nabla}\log\left|e^z(\mathcal{L}^+(F) - 1)\right|,\ \cdot,\ \cdot,\ \cdot\right) \\ &= W^+\left(\nabla\log\left|e^z(\mathcal{L}^+(F) - 1)\right|,\ \cdot,\ \cdot,\ \cdot\right). \end{aligned} \tag{2.29}$$

Derdzinski's conformal change formula, equation (19) of [15], is

$$\tilde{\delta}\,\widetilde{W}^+ = \delta W^+ - \frac{1}{2}W^+\left(\nabla\log\left(e^z C\right),\ \cdot,\ \cdot,\ \cdot\right) \tag{2.30}$$

so changing the metric back with conformal factor $e^z C$, (2.29) and (2.30) give

$$\delta W^+ = W^+\left(\nabla\log\left|e^{\frac{3}{2}z}(\mathcal{L}^+(F) - 1)\sqrt{C}\right|,\ \cdot,\ \cdot,\ \cdot\right). \tag{2.31}$$

The argument for δW^- is entirely the same, after conformally changing so $\tilde{\omega}^-$ not $\tilde{\omega}$ is closed. $\square$

Proposition 2.10 (The Bach Tensor)**.** *The Bach tensor of* (2.1) *is*

$$\begin{aligned} Bach = &\frac{16}{3C^2}F\big(\mathcal{L}^-(\mathcal{L}^+(F)) - 1\big)\cdot\Big(-2(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\Big) \\ &+ \frac{8}{3C^2}\mathcal{B}(F,F)\cdot\Big(-(\sigma^0)^2 - (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\Big). \end{aligned} \tag{2.32}$$

Proof. In the Kähler case we decompose the Bach tensor into its J-invariant and J-anti-invariant parts $Bach^+$, $Bach^-$ respectively. It is known that $Bach^+ = \frac{1}{3}(\nabla^2 s)^+_0 + \frac{1}{6} s \,\mathring{\mathrm{Ric}}$ and $Bach^- = -\frac{1}{6}(\nabla^2 s)^-$; see Eq. (39) of [15], Eq. (20) of [1] or Lemma 6 of [10]. We have

$$\begin{aligned} \nabla^2 s &= \frac{4F}{C} s_{zz}\sigma^0 \otimes \sigma^0 + s_z \nabla dz \\ &= \left(\frac{4F}{C} s_{zz} - \frac{2F^2}{C^2} s_z \left(F^{-1}C\right)_z \right) (\sigma^0)^2 + \frac{2}{C^2} s_z \left(FC\right)_z (\sigma^1)^2 + \frac{2F}{C^2} s_z C_z \left((\sigma^2)^2 + (\sigma^3)^2 \right). \end{aligned} \tag{2.33}$$

In the Kähler case where $C = e^{-z}$ and $s = -8e^z(\mathcal{L}^+(F) - 1)$, we compute

$$\begin{aligned} (\nabla^2 s)^- &= -32 e^{2z} F \left(\mathcal{L}^-(\mathcal{L}^+)(F) - 1 \right) \left((\sigma^0)^2 - (\sigma^1)^2 \right) \\ (\nabla^2 s)^+ &= -16 e^{2z} \left(2F(\mathcal{L}^-(\mathcal{L}^+(F) - 1) - F_z \mathcal{L}^+(F_z + F) - 1 \right) \left((\sigma^0)^2 + (\sigma^1)^2 \right) \\ &\quad + 16 e^{2z} F \left(\mathcal{L}^+(F_z + F) - 1 \right) \left((\sigma^0)^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \right) \\ \triangle s &= 4 e^{-2z} \left(FC\right) \left(FC s_z\right)_z. \end{aligned} \tag{2.34}$$

Then $(\nabla s)^+_0 = (\nabla^2 s)^+ - \frac{1}{4}(\triangle s) g$ and using the expression for $\mathring{\mathrm{Ric}}$ of (2.23),

$$\begin{aligned} Bach^+ &= \frac{16 e^{2z}}{3} \left(\frac{1}{2} \mathcal{B}(F,F) + F \cdot \left(\mathcal{L}^-(\mathcal{L}^+(F)) - 1 \right) \right) \cdot \left(-(\sigma^0)^2 - (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \right) \\ Bach^- &= \frac{16 e^{2z}}{3} \cdot F \cdot \left(\mathcal{L}^-(\mathcal{L}^+(F)) - 1 \right) \cdot \left((\sigma^0)^2 - (\sigma^1)^2 \right) \end{aligned} \tag{2.35}$$

Conformally changing from $C = e^{-z}$ to arbitrary C, we obtain (2.32). □

Compare also with Proposition 14 of [1].

Compare equation (2.32) with (3.3) of [34]; after substituting $C = 1$, $F = f^2$ and $dz = 2f dt$ the expression here and the expression there are identical.

2.3 Topology: "nuts", "bolts", and asymptotics

Here we discuss global aspects of $U(2)$-invariant metrics. Ostensibly the metric (2.1) is well defined on $\mathbb{R} \times \mathbb{S}^3$ but topology changes occur if F or C attain 0 somewhere. If F reaches zero, the metric most naturally lives on a quotient $I \times (\mathbb{S}^3/\Gamma)/\sim$ where Γ is some discrete subgroup of $SU(2)$, and $\sim$ identifies some 3-sphere to a 2-sphere, via the Hopf map. Where F or C is infinite, there is a (possibly incomplete) manifold end.

2.3.1 Bolts, Nuts

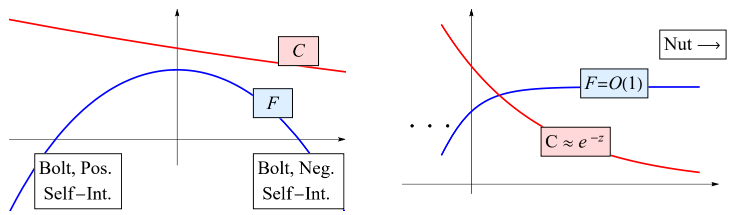

Figure 1: *A compact manifold with a bolt of positive and of negative self-intersection. A nut at* $z = +\infty$.

The first kind of topology change occurs when the Hopf fiber collapses but the conformal factor remains non-zero, meaning F but not C reaches zero. When $F(z_0) = 0$, the locus $z = z_0$ is not a 3-sphere but a 2-sphere, colloquially known as a "bolt" [21] (see also [17, 29, 32]).

As this is well known, we describe it only briefly. Recalling the coordinates of Section 2.1, transversals to the bolt are 2-dimensional submanifolds locally given by $\theta = const$, $\varphi = const$, and the metric is smooth at the bolt provided it is smooth on such transversals. The inherited metric on the transversal is $\hat{g}_2 = \frac{1}{4F}dz^2 + \frac{F}{4}d\psi^2$ with $\psi \in [-2\pi, 2\pi)$, which we write $\hat{g}_2 = dr^2 + (\sqrt{F}d(\frac{1}{2}\psi))^2$ by solving $dr = \frac{1}{\sqrt{4F}}dz$ with $r = 0$ at $z = z_0$. If $\sqrt{F} = kr + O(r^2)$, where $k \neq 0$, then $(\sqrt{F}d(\frac{1}{2}\psi))^2 \approx r^2(d(\frac{k}{2}\psi))^2$ so the metric $\hat{g}_2$ will be conical at $r = 0$ with cone angle $2\pi|k|$ (so smooth if and only if $k = \pm 1$). If $k \in \mathbb{Z} \setminus \{0\}$ however, we can obtain a smooth metric on the quotient $I \times \mathbb{S}^3/\Gamma$ where Γ is a cyclic subgroup of order $|k|$ of the Hopf action. From $\sqrt{F} = kr + O(r^2)$ we have $k = \frac{d\sqrt{F}}{dr}$, and because $\frac{d}{dr} = 2\sqrt{F}\frac{d}{dz}$, $k = \frac{dF}{dz}$. We summarize this in the following Proposition.

Proposition 2.11 (The "bolting condition")**.** *Assume* $z = z_0$ *is a zero of* $F(z)$ *but not* $C(z)$*. If*

$$\left.\frac{dF}{dz}\right|_{z=z_0} = k \tag{2.36}$$

where $k \neq 0$ *then we may identify the locus* $\{z = z_0\}$ *with a 2-sphere (a "bolt"). Assuming* $k \in \mathbb{Z} \setminus \{0\}$*, then taking the* $|k|$*-to-1 quotient of the* $\mathbb{S}^3$ *factor, the metric is smooth near* $\{z = z_0\}$ *and the "bolt" is a 2-sphere of self intersection number* k*.*

It is possible that two bolts occur, one at z_0 and one at z_1 where $z_0 < z_1$, as in Figure 1. We certainly must have $\frac{dF}{dz} \geq 0$ at z_0 and $\frac{dF}{dz} \leq 0$ at z_1, so the bolts, assuming they are both smooth after resolution, must have self-intersection numbers k and $-k$ where $k \in \mathbb{Z} \setminus \{0\}$. With either complex structure J^+ or J^-, this is the "odd" Hirzebruch surface Σ_{2k-1}; see [33].

A nut, by contrast, occurs when the $\mathbb{S}^3$ factor contracts to a point; the nearby topology is that of a ball in $\mathbb{R}^4$. This occurs when C becomes zero but F remains finite. When ω is Kähler and

$C = C_0 e^{-z}$, a nut may occur at $z = +\infty$; this is depicted in Figure 1. When ω^- is Kähler and $C = C_0 e^{-z}$ a nut may occur at $z = -\infty$.

Proposition 2.12 (The Nut condition at $z = \infty$)**.** *Assume $C = O(e^{-z})$ and $F = 1 + O(e^{-z})$ as $z \to \infty$. Adding a point at $z = \infty$, this point is a finite distance away and has a neighborhood with bounded curvature and the topology of a ball.*

2.3.2 ALE, ALF, and cusp-like ends

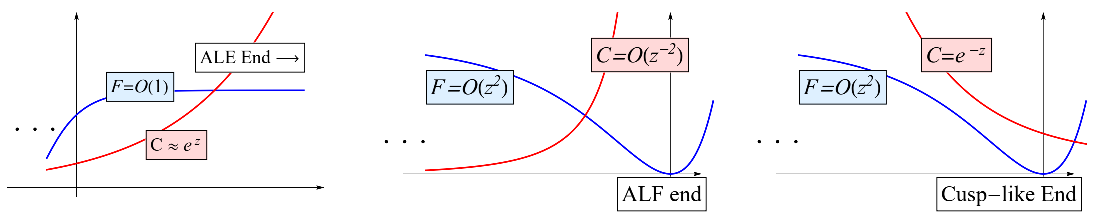

Figure 2: *ALE, ALF, and cusp-like ends in the $U(2)$ ansatz.*

If g is Kähler with respect to J^- so $C = C_0 e^z$, an ALE end can occur as $z \to \infty$, as depicted in Figure 2. If instead g is Kähler with respect to J^+ then replacing z by $-z$, Figure 2 is flipped and an ALE end occurs as $z \to -\infty$.

Proposition 2.13. *Assume g is Kähler with respect to J^+, so $C = e^{-z}$. If $F = 1 + O(z^{-2})$ as $z \to -\infty$, the metric is ALE with better-than-quadratically decaying curvature.*

Proof. Letting r be the distance function that solves $dr = \frac{1}{2}\sqrt{C/F}dz = \frac{1}{2}e^{-\frac{1}{2}z}(1 + O(z^{-2}))dz$, by assumption we have $r = e^{-\frac{1}{2}z} + O(z^{-1})$. Then $C = e^{-z} = r^2 + O(r^{-4})$, so the metric is $g \approx dr^2 + (r^2 + O(r^{-4}))d\sigma_{\mathbb{S}^3}$ as $r \to \infty$, so it is ALE. To check curvature decay, Proposition 2.6 gives

$$\rho = -2C^{-1}\left(\frac{1}{2}F_{zz} - \frac{3}{2}F_z + F - 1\right)\omega + 2C^{-1}\left(\frac{1}{2}F_{zz} - \frac{1}{2}F_z - F + 1\right)\omega^- \tag{2.37}$$

so asymptotically $\rho \approx e^z O(z^{-2})\omega + e^z O(z^{-2})\omega^- = o(r^{-2})$. The expressions for $|W^+|$, $|W^-|$ in (2.27) give the same decay rates. Thus the Riemann tensor decays like $|\,\mathrm{Rm}\,| = o(r^{-2})$. □

The ALF end has cubic volume growth, cubic curvature decay, and $\mathbb{R}^3$ tangent cone at infinity. See for example [13, 16, 18, 26]. By a “cusp-like” end, we mean an end that locally resembles a Riemannian product of a tractrix of revolution (sometimes called a pseudosphere) with a sphere. Toward infinity the scalar and Weyl curvatures decrease rapidly, whereas the Ricci curvature approaches a constant bilinear form of signature $(-,-,+,+)$. These two kinds of ends are conformal to each other: we have $C = \frac{e^z}{(1-e^z)^2}$ in the ALF case and $C = e^{-z}$ or $C = e^z$ in the cusp-like case. In both cases F has a second-order zero at $z = 0$. See Figure 2.

Proposition 2.14. *Assume $F = z^2 + O(z^3)$ near $z = 0$.*

If C remains finite then the manifold forms a complete, cusp-like end near $z = 0$. Asymptotically the Hopf fiber shrinks to zero and the metric has the local geometry of the product of a psuedosphere times a sphere.

If $C = O(z^{-2})$ then the the metric forms an ALF end near $z = 0$.

Proof. The distance function r satisfies $dr = \frac{1}{2}\sqrt{\frac{C}{F}}dz$ so in the cusp-like case, where C remains finite, then $\sqrt{F} = O(z)$ gives $r \approx \frac{1}{2}\log|z|$ near 0 and indeed the distance to 0 is infinite so the metric is complete. From $\omega \wedge \omega = -C^2 dz \wedge \eta^1 \wedge \eta^2 \wedge \eta^3$, we see the volume is finite. Checking the tensors $W^{\pm}$, from $F = z^2 + O(z^3)$ we find that $\mathcal{L}^{\pm}(F) - 1 = O(z)$ and so $|W^{\pm}| \searrow 0$ as $z \to 0$. In the Kähler case ρ is a multiple of ω added to a multiple of ω^-. The multiple on ω is also $O(z)$, but the multiple on ω^-, by (2.17), approaches $4C^{-1}$. This justifies the assertion that, in the Kähler case, the local geometry approaches a $+1$ times a -1 curvature surface. In the non-Kähler case, the usual conformal change formulas for Ricci curvature shows this remains true.

Next we verify that when $C = z^{-2} + O(1)$ near $z = 0$, the metric has an ALF end. Then $dr = \frac{1}{2}\sqrt{\frac{C}{F}}dz = \left(\frac{1}{2}z^{-2} + O(1)\right)dz$ so $r = z^{-1} + O(z)$ near $z = 0$. To compute volume, we use $C^{\frac{3}{2}} = O(r^3)$ and $F^{\frac{1}{2}} = O(z) = O(r^{-1})$, so we have

$$dVol = -C^{\frac{3}{2}}F^{\frac{1}{2}}dr \wedge d\sigma_{\mathbb{S}^3} \approx r^2 dr \wedge d\sigma_{\mathbb{S}^3}. \tag{2.38}$$

Integrating (2.38) and noting that r is a distance function, indeed we observe cubic volume growth. Next we check curvature decay. From (2.27) we have $\mathcal{L}^{\pm}(F) - 1 = O(1)$ so that $|W^+| \approx \frac{32}{3}C^{-2} = O(z^2) = O(r^{-2})$ and similarly for $|W^-|$. Inserting F, C into the Ricci form ρ from (2.19), we see Ricci curvature decays quadratically. □

We close by noting that ALE ends are conformal to nuts and vice-versa—by changing between $C = e^{-z}$ and $C = e^{z}$—and similarly ALF ends and cusp-like ends are conformal to each other.

3 Special Metrics

We use the computations of Section 2.2 to determine what conditions are needed to make a $U(2)$-invariant metric special or canonical.

3.1 Scalar Curvature

From (2.21) of Proposition 2.7, specifying scalar curvature is equivalent to

$$sC^{\frac{3}{2}} + 4C^{\frac{1}{2}}\left(\frac{\partial^2 F}{\partial z^2} + \frac{1}{2}F - 2\right) + 24\frac{\partial}{\partial z}\left(F\frac{\partial}{\partial z}C^{\frac{1}{2}}\right) = 0, \tag{3.1}$$

for given $s = s(z)$. This underdetermined equation is linear in F. Imposing the Kähler condition $C = C_0 e^{\pm z}$ creates a critically determined linear equation.

3.2 Extremal Kähler metrics

A Kähler metric is extremal if the functional $g \mapsto \int s^2 dVol$ is stable under those perturbations of g that preserve the Kähler class. From [9] the Euler-Lagrange equations are that the gradient ∇s is a holomorphic vector field, but there are several ways to assess whether (1.3) is extremal. In our context we are less concerned with global functionals such as $\int s^2$. We use the local condition that a Kähler metric is extremal if and only if $J\nabla s$ is Killing.

Proposition 3.1 (The extremal condition)**.** *The metric* (2.1) *with complex structure* J^+ *is extremal Kähler if and only if* $C = C_0 e^{-z}$ *and* $\mathcal{L}^-(\mathcal{L}^+(F)) = 1$*, which is*

$$F = 1 + \frac{1}{2}C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2}C_4 e^{2z}. \tag{3.2}$$

Its scalar curvature is $s = -\frac{24}{C_0}(C_1 e^{-z} + C_2)$*.*

Likewise, the metric with complex structure J^- *is extremal Kähler if and only if* $C = C_0 e^{z}$ *and again* $\mathcal{L}^-(\mathcal{L}^+(F)) = 1$*. Its scalar curvature is* $s = -\frac{24}{C_0}(C_3 + C_4 e^{z})$*.*

Proof. From (2.1) and (2.5), we have $\nabla z = 4\frac{F}{C}\frac{\partial}{\partial z} = \frac{4}{C}J\frac{\partial}{\partial \psi}$. Because the coordinate field $\frac{\partial}{\partial \psi}$ is itself a Killing field and because $s = s(z)$ is a function of z alone, the extremal condition is $\nabla s = -4\alpha J\frac{\partial}{\partial \psi} = -\alpha e^{z}\nabla z = \nabla\left(\alpha e^{-z}\right)$ where α is a constant. Therefore $s = \alpha e^{-z} + \beta$ where β is another constant. Using $s = -8C_0^{-1}e^{z}(\mathcal{L}^+(F) - 1)$, from (2.18) we obtain

$$-8C_0^{-1}e^{z}\left(\frac{1}{2}\frac{\partial^2 F}{\partial z^2} - \frac{3}{2}\frac{\partial F}{\partial z} + F - 1\right) = \alpha e^{-z} + \beta. \tag{3.3}$$

After setting $C_1 = -\frac{1}{24}\alpha C_0$ and $C_2 = -\frac{1}{24}\beta C_0$ we obtain (3.2).

For J^- in place of J^+, reverse the sign on z in all computations. □

3.3 Einstein metrics

By (2.22), $\mathring{\mathrm{Ric}} = 0$ if and only if

$$\frac{\partial^2}{\partial z^2} C^{-\frac{1}{2}} = \frac{1}{4} C^{-\frac{1}{2}} \quad \text{and} \quad C^{\frac{1}{2}} \frac{\partial}{\partial z}\left(F \frac{\partial}{\partial z} C^{-\frac{1}{2}} \right) = \left(\frac{1}{2}\frac{\partial^2 F}{\partial z^2} - \frac{3}{4} F + 1 \right). \tag{3.4}$$

This is critically determined and partly decoupled. It is 4^{th} order in total so we will have a 4-parameter solution space. The general solution is

$$\begin{aligned} &F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2} C_4 e^{2z}, \quad C = \frac{e^{-z}}{(C_5 + C_6 e^{-z})^2}, \\ &\text{where} \quad C_1 C_5 - C_2 C_6 = 0, \quad \text{and} \quad C_3 C_5 - C_4 C_6 = 0. \end{aligned} \tag{3.5}$$

With six constants and two algebraic relations we have the expected four-parameter family of solutions. Compare with Proposition 2.4. The algebraic relations on the C_i are equivalent to the pairs (C_1, C_2), (C_3, C_4), and (C_5, C_6) being proportional to each other. These imply also that $C_1 C_4 - C_2 C_3 = 0$, so we recover the fact that Einstein metrics are Bach-flat; see (3.14) below. By Lemma 2.2 the metric is Kähler when $C_6 = 0$ (for J^+) or $C_5 = 0$ (for J^-).

To be Ricci-flat, C and F require, in addition to (3.4), that $s = 0$. This third relation appears to make the overall system overdetermined, but it does not, for the reason that s is a first integral for the system (3.4) so only contributes an algebraic relation. From (3.1),

$$s = -24(C_2 C_5^2 - 2C_5 C_6 + C_3 C_6^2). \tag{3.6}$$

Proposition 3.2 (The Einstein conditions)**.** *The metric* (1.3) *is Einstein if and only if*

$$\begin{aligned} &F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + C_3 e^{z} + \frac{1}{2} C_4 e^{2z}, \quad C = \frac{e^{-z}}{(C_5 + C_6 e^{-z})^2}, \\ &C_1 C_5 - C_2 C_6 = 0, \quad \textit{and} \quad C_3 C_5 - C_4 C_6 = 0. \end{aligned} \tag{3.7}$$

Its scalar curvature is the constant $s = -24(C_2 C_5^2 - 2C_5 C_6 + C_3 C_6^2)$.

Up to homothety, there is a 2-dimensional family of Einstein metrics. Up to homothety, there is a 1-dimensional family of Ricci-flat metrics, a 1-dimensional family of KE metrics with respect to J^+, and a 1-dimensional family of KE metrics with respect to J^-. Up to homothety and biholomorphism, there are exactly five Ricci-flat Kähler metrics, three of which are complete.

Proof. We have proven everything except the final assertion, that exactly five metrics of the form (1.3) are Ricci-flat Kähler, up to homothety. We prove this regardless of the complex structure, whether one of the structures considered here or not. A $U(2)$-invariant metric is Einstein if and only if it has the form (3.7). By Derdzinski's theorem [15], if a scalar-flat metric is Kähler—

regardless of the complex structure—then it is half-conformally flat. In particular $C_1 = C_2 = 0$ or $C_3 = C_4 = 0$.

So assume $C_3 = C_4 = 0$; the case $C_1 = C_2 = 0$ is identical under the isomorphism $z \mapsto -z$. We have four remaining variables C_1, C_2, C_5, C_6 and two relations: $C_1C_5 - C_2C_6 = 0$ from (3.5) and $C_2C_5^2 - 2C_5C_6 = 0$ from (3.6). If in addition to $C_3 = C_4 = 0$ we have both $C_1 = C_2 = 0$ then either $C_5 = 0$ or else $C_6 = 0$ and in either case we have the flat metric: $F = 1$ and $C = C_0e^{\pm z}$.

Suppose $C_1 = 0$ but $C_2 \neq 0$; then the two relations force $C_5 = C_6 = 0$, an impossibility. Suppose $C_1 \neq 0$ but $C_2 = 0$; then the relations force $C_6 = 0$ so

$$F = 1 + \frac{1}{2}C_1e^{-2z}, \quad C = \frac{1}{C_5^2}e^{-z} \tag{3.8}$$

which is Kähler with respect to J^+. Up to homothety, there are exactly two such metrics: the first is given by $F = 1 - e^{-2z}$, $C = e^{-z}$, which is the Eguchi-Hanson metric, and the second is given by

$$F = 1 + e^{-2z}, \quad C = e^{-z} \tag{3.9}$$

which is incomplete and has a curvature singularity at $z = -\infty$.

Lastly it is possible that neither C_1 nor C_2 are zero. The two relations now give $\frac{C_6}{C_5} = \frac{C_1}{C_2}$ and $\frac{C_6}{C_5} = \frac{C_2}{2}$, so $C_1 = \frac{1}{2}C_2^2$. Therefore the metric is

$$F = 1 + \frac{1}{4}C_2^2e^{-2z} + C_2e^{-z} = \left(1 + \frac{1}{2}C_2e^{-z}\right)^2, \quad C = \frac{C_5^2e^{-z}}{\left(1 + \frac{1}{2}C_2e^{-z}\right)^2}. \tag{3.10}$$

Under the isomorphism $z \mapsto -z$ this is the Kähler metric of Proposition 2.4 which is Kähler with respect to the complex structure I^-; therefore the metric (3.10) is Kähler with respect to the complex structure I^+. As in Proposition 2.5 there are two such metrics: one where $C_2 < 0$ (which is the Taub-NUT metric) and one where $C_2 > 0$ (which has a curvature singularity). □

3.4 Half-conformally flat, half-harmonic, and Bach-flat metrics

Proposition 3.3. *The metric* (1.3) *has* $W^\pm = 0$ *if and only if* $\mathcal{L}^\pm(F) - 1 = 0$, *meaning*

$$F = 1 + C_3e^z + \frac{1}{2}C_4e^{2z} \quad \text{or} \quad F = 1 + \frac{1}{2}C_1e^{-2z} + C_2e^{-z}, \tag{3.11}$$

respectively. Up to homothety, each case constitutes a 1-parameter family of such metrics, each a subspace of the 2-parameter family of Bach-flat metrics.

In the case g *is Kähler with respect to* J^+ *so* $C = C_0e^{-z}$, *then* $W^+ = 0$ *implies* $s = 0$, *and* $W^- = 0$ *implies* $s = -\frac{24}{C_0}(C_1e^{-z} + C_2)$.

The half-harmonic condition $\delta W^+ = 0$ (or $\delta W^- = 0$) is underdetermined, and requires an additional condition to be critically determined. Three possibilities are $s = const$, the Kähler condition, and both $\delta W^\pm = 0$.

Proposition 3.4. *The metric* (1.3) *has* $\delta W^+ = 0$ *if and only if a constant* k_1 *exists so* $e^{\frac{3}{2}z}\left(\mathcal{L}^+(F) - 1\right) C = k_1$*, and* $\delta W^- = 0$ *if and only if* $e^{-\frac{3}{2}z}\left(\mathcal{L}^-(F) - 1\right) C = k_2$ *for some* $k_2 \in \mathbb{R}$*.*

Assume (2.1) *is Kähler with respect to* J^+*, meaning* $C = C_0 e^{-z}$*. Then*

- *a)* $\delta W^+ = 0$ *if and only if* $F = 1 + C_2 e^{-z} + C_3 e^z + \frac{1}{2} C_4 e^{2z}$*. In particular* $s = \frac{-24C_2}{C_0}$ *is constant.*
- *b)* $\delta W^- = 0$ *if and only if* $F = 1 + \frac{1}{2} C_1 e^{-2z} + C_2 e^{-z} + \frac{1}{2} C_4 e^{2z}$*. In particular the metric is extremal and* $s = -24\frac{1}{C_0}(C_1 e^{-z} + C_2)$*.*

Proof. For $\delta W^+ = 0$ this follows from Proposition 2.9 with $C = C_0 e^{-z}$, $e^{\frac{3}{2}z}(\mathcal{L}^+(F) - 1)\sqrt{C} = k_1$ and finding the general solution. In the Kähler case, *a)* and *b)* follow from Proposition 3.1. ☐

In the $U(2)$-invariant case, $\delta W = 0$ is equivalent to the Einstein condition.

Proposition 3.5 (Harmonic curvature)**.** *The metric* (2.1) *has* $\delta W = 0$ *if and only if* g *is Einstein.*

Proof. Because $\delta W^+ \in T^*M \otimes \bigwedge^+$ and $\delta W^- \in T^*M \otimes \bigwedge^-$, we have $\delta W = 0$ if and only if δW^+ and δW^- are both zero. Then by Lemma 2.9 constants k_1, k_2 exist so

$$e^{\frac{3}{2}z}(\mathcal{L}^+(F) - 1)\sqrt{C} = k_1 \quad \text{and} \quad e^{-\frac{3}{2}z}(\mathcal{L}^-(F) - 1)\sqrt{C} = k_2. \tag{3.12}$$

Eliminating C, we obtain $k_2 e^{\frac{3}{2}z}(\mathcal{L}^+(F) - 1) = k_1 e^{-\frac{3}{2}z}(\mathcal{L}^-(F) - 1)$ which has solution

$$F = 1 + k_1\left(\frac{1}{2}C_1 e^{-2z} + C_2 e^{-z}\right) + k_2\left(C_1 e^z + \frac{1}{2}C_2 e^{2z}\right). \tag{3.13}$$

Using either equation in (3.12), $C = \frac{C_0 e^{-z}}{(C_2 + C_1 e^{-z})^2}$. By Proposition 3.2, the metric is Einstein. ☐

Next we consider the case of Bach-flat metrics. By Proposition 2.10, F solves the fourth order linear equation $\mathcal{L}^-(\mathcal{L}^+(F)) - 1 = 0$ and the third order non-linear equation $\mathcal{B}(F, F) = 0$. This seems to be overdetermined, but due to (3.15) the two equations are not independent.

Lemma 3.6. *If* F *solves* $\mathcal{L}^+(\mathcal{L}^-(F)) - 1$ *then* $\mathcal{B}(F, F) = const$*. If* F *solves* $\mathcal{B}(F, F) = 0$*, then* $\mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$*. Lastly* $\mathcal{B}(F, F) = \mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$ *if and only if*

$$F = 1 + \frac{1}{2}C_1 e^{-2z} + C_2 e^{-z} + C_3 e^z + \frac{1}{2}C_4 e^{2z} \quad \textit{and} \quad C_1 C_4 - C_2 C_3 = 0. \tag{3.14}$$

Proof. A tedious but completely elementary computation shows

$$\frac{\partial}{\partial z}\mathcal{B}(F,F) = 2\left(\mathcal{L}^+(\mathcal{L}^-(F)) - 1\right)\frac{\partial F}{\partial z}. \tag{3.15}$$

Therefore $\mathcal{B}(F,F)$ is indeed constant on solutions of $\mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$. Next, $\mathcal{B}(F,F) = 0$ implies either $F = const$ or $\mathcal{L}^+(\mathcal{L}^-(F)) = 1$. By direct computation the only constant that satisfies $\mathcal{B}(F,F) = 0$ is $F = 1$, which indeed solves $\mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$. We conclude that $\mathcal{B}(F,F) = 0$ implies $\mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$.

The general solution of $\mathcal{L}^+(\mathcal{L}^-(F)) = 1$ is $F = 1 + \frac{1}{2}C_1e^{-2z} + C_2e^{-z} + C_3e^z + \frac{1}{2}C_4e^{2z}$, and in this case direct computation shows that $\mathcal{B}(F,F) = 3(C_2C_3 - C_1C_4)$. Therefore the general solution of $\mathcal{L}^+(\mathcal{L}^-(F)) = 1$, $\mathcal{B}(F,F) = 0$ is the three parameter family of (3.14). ☐

Proposition 3.7. *The metric* (2.1) *is Bach-flat if and only if*

$$F = 1 + \frac{1}{2}C_1e^{-2z} + C_2e^{-z} + C_3e^z + \frac{1}{2}C_4e^{2z} \quad \textit{and} \quad C_1C_4 - C_2C_3 = 0. \tag{3.16}$$

In particular g is Bach-flat if and only if it is conformally Einstein. Up to conformal factors and translation in z, the Bach-flat metrics constitute a 2-parameter family of metrics.

Proof. The metric g is Bach-flat if and only if $\mathcal{L}^+(\mathcal{L}^-(F)) - 1 = 0$ and $\mathcal{B}(F,F) = 0$. From Lemma 3.6, this holds if and only if $F = 1 + \frac{1}{2}C_1e^{-2z} + C_2e^{-z} + C_3e^z + \frac{1}{2}C_4e^{2z}$ and $C_1C_4 - C_2C_3 = 0$, giving a 3-parameter family of solutions. Factoring out by translation in z, this is a 2-parameter family, as claimed. To see that any Bach-flat metric is conformal to an Einstein metric, simply let C be a conformal factor from Proposition 3.2. ☐

3.5 B^t-flat metrics

The B^t-flat metrics [25] extremize the functional $B^t(g) = \int |W|^2 + t\int s^2$, where we take $B^\infty = \int s^2$. The Euler-Lagrange equations of this functional [25] are

$$-4Bach + t\,\mathcal{C} = 0 \tag{3.17}$$

where $\mathcal{C} = 2\left(\nabla^2 s - (\triangle s)g - s\,\mathring{\mathrm{Ric}}\right)$. The Bach tensor is always trace-free and $Tr(\mathcal{C}) = -6\triangle s$, so tracing the B^t-flat condition (3.17) gives $\triangle s = 0$. Then we can rewrite the B^t-flat condition as the two equations $2Bach + t(s\,\mathring{\mathrm{Ric}} - \nabla^2 s) = 0$ and $\triangle s = 0$. We can express these as an ODE system.

Lemma 3.8 (The unreduced B^t-flat equations)**.** *In the metric* (2.1) *the B^t-flat equations $\triangle s = 0$, $2Bach + t(s\,\mathring{\mathrm{Ric}} - \nabla^2 s) = 0$ are equivalent to*

$$\frac{\partial}{\partial z}\left(CF\frac{\partial s}{\partial z}\right) = 0, \quad \mathcal{F}_1(F,C) = 0, \quad \mathcal{F}_2(F,C) = 0, \quad \mathcal{T}(F,C) = 0 \tag{3.18}$$

where $\mathcal{F}_1$, $\mathcal{F}_2$ *and* $\mathcal{T}$ *are the operators*

$$\begin{aligned}
\mathcal{F}_1(F,C) &= 24\frac{\partial}{\partial z}\left(F\frac{\partial}{\partial z}C^{\frac{1}{2}}\right)+4C^{\frac{1}{2}}\left(\frac{\partial^2 F}{\partial z^2}+\frac{1}{2}F-2\right)+s\,C^{\frac{3}{2}}\\
\mathcal{F}_2(F,C) &= \frac{8}{3}\left(\mathcal{L}^+(\mathcal{L}^-(F))-1\right)+t\,s\,C^{\frac{3}{2}}\left(\frac{\partial^2}{\partial z^2}C^{-\frac{1}{2}}-\frac{1}{4}C^{-\frac{1}{2}}\right)+\frac{t}{2}\frac{C}{F}\frac{\partial F}{\partial z}\frac{\partial s}{\partial z}+t\frac{\partial C}{\partial z}\frac{\partial s}{\partial z}\\
\mathcal{T}(F,C) &= 16\,\mathcal{B}(F,F)-18tF\frac{\partial C}{\partial z}\frac{\partial s}{\partial z}-6tC\frac{\partial F}{\partial z}\frac{\partial s}{\partial z}\\
&\quad -\frac{3}{4}tsC^{-1}\left(C^2(-16+4F+Cs)+12F\left(\frac{\partial C}{\partial z}\right)^2+8C\frac{\partial C}{\partial z}\frac{\partial F}{\partial z}\right)
\end{aligned}\tag{3.19}$$

and $\mathcal{B}$ *is the operator from* (1.7).

Proof. In coordinates, $\triangle=\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(\sqrt{\det g}\,g^{ij}\frac{\partial}{\partial x^j}\right)$. Using (Z,τ,x,y)-coordinates of (2.15) we have $det\,g=\frac{1}{16\cosh^2(x)}C^2$ and $g^{11}=4FC$. Because $s=s(Z)$ is a function of Z alone, then $0=\triangle s$ is

$$0=\frac{4\cosh^2(x)}{C}\frac{\partial}{\partial Z}\left(\frac{C}{4\cosh^2(x)}4FC\frac{\partial s}{\partial Z}\right)=\frac{4}{C}\frac{\partial}{\partial Z}\left(FC^2\frac{\partial s}{\partial Z}\right).\tag{3.20}$$

The coordinate change from z to Z of (2.15) gives $C\frac{\partial}{\partial Z}=\frac{\partial}{\partial z}$, so we obtain the first equation of (3.18). The second equation $\mathcal{F}_1(F,C)=0$ is precisely the scalar curvature equation (3.1). With $\triangle s=0$ the Hessian $\nabla^2 s$ is trace-free; then a straightforward computation gives

$$\nabla^2 s=-2C^{-4}\frac{\partial s}{\partial z}\frac{\partial(FC^3)}{\partial z}(\sigma^0)^2+2C^{-2}\frac{\partial s}{\partial z}\frac{\partial(FC)}{\partial z}(\sigma^1)^2+2FC^{-2}\frac{\partial s}{\partial z}\frac{\partial C}{\partial z}\left((\sigma^2)^2+(\sigma^3)^2\right).\tag{3.21}$$

Now for the third and fourth equations we use (2.22), (2.32), and (3.21). We expect precisely two additional relations, due to the fact that each of the tensors $Bach$, $\mathring{\mathrm{Ric}}$, and $\nabla^2 s$ have four non-zero components, but also the two algebraic relations of being trace-free, and having identical $(3,3)$ and $(4,4)$ entries. We take one relation from $2(Bach_{00}+Bach_{22})+t(s\,\mathring{\mathrm{Ric}}_{00}+s\,\mathring{\mathrm{Ric}}_{22}-s_{,00}-s_{,22})=0$. Using (1.9), (1.11), and (3.21), this is

$$\frac{8}{3}\left(\mathcal{L}^-(\mathcal{L}^+(F))-1\right)+tsC^{\frac{3}{2}}\left(\frac{\partial^2\frac{1}{\sqrt{C}}}{\partial z^2}-\frac{1}{4}\frac{1}{\sqrt{C}}\right)+\frac{t}{2}\frac{C}{F}\frac{\partial F}{\partial z}\frac{\partial s}{\partial z}+t\frac{\partial C}{\partial z}\frac{\partial s}{\partial z}=0\tag{3.22}$$

which is $\mathcal{F}_2(C,F)=0$. We take another relation from $2Bach_{00}+t(s\,\mathring{\mathrm{Ric}}_{00}-s_{,00})=0$, which is

$$\begin{aligned}
0&=16\mathcal{B}(F,F)-18tF\frac{\partial C}{\partial z}\frac{\partial s}{\partial z}-6tC\frac{\partial F}{\partial z}\frac{\partial s}{\partial z}\\
&\quad-\frac{3}{4}tsC^{-1}\left(C^2(-16+4F+sC)+12F\left(\frac{\partial C}{\partial z}\right)^2+8C\frac{\partial C}{\partial z}\frac{\partial F}{\partial z}\right)\\
&\quad+\frac{3}{4}tsC^{\frac{1}{2}}\left(4C^{\frac{1}{2}}\left(\frac{\partial^2 F}{\partial z^2}+\frac{1}{2}F-2\right)+24\frac{\partial}{\partial z}\left(F\frac{\partial C^{\frac{1}{2}}}{\partial z}\right)+sC^{\frac{3}{2}}\right).
\end{aligned}\tag{3.23}$$

Using (3.1) to eliminate the last term, this is $\mathcal{F}_1(F,C)=0$. □

The equations (3.18) give four equations for the three unknowns s, F, C, so the system appears to be overdetermined. But the equations of (3.18) are not independent.

Lemma 3.9. *We have the following relation:*

$$\frac{\partial \mathcal{T}}{\partial z} = \frac{-3t}{2\sqrt{C}} \frac{\partial (sC)}{\partial z} \mathcal{F}_1 + 12 \frac{\partial F}{\partial z} \mathcal{F}_2 - 6t \frac{\partial \log(C^3 F)}{\partial z} \frac{\partial}{\partial z} \left(CF \frac{\partial s}{\partial z} \right). \tag{3.24}$$

In particular $\mathcal{T}(F,C)$ is constant along solutions of the system $\mathcal{F}_1(F,C) = \mathcal{F}_2(F,C) = \triangle s = 0$.

Proof. This follows from a tedious but completely elementary computation. □

Lemma 3.10. *At all points where $C \neq 0$ and $F \neq 0$, the 8^{th} order system*

$$\frac{\partial}{\partial z} \left(CF \frac{\partial s}{\partial z} \right) = 0, \quad \mathcal{F}_1(F,C) = 0, \quad \mathcal{F}_2(F,C) = 0 \tag{3.25}$$

is critically determined, $\mathcal{T}$ is a constant of the motion, and (3.25) combined with the restraint $\mathcal{T}(F,C) = 0$ admits a 7-parameter family of solutions.

Up to homothety, in the $U(2)$-invariant setting the B^t-flat metrics form a 5-parameter family, and the CSC B^t-flat metrics form a 4-parameter family.

Proof. To ascertain whether the system (3.25) is critically determined, we examine the coefficients on the derivatives of s, F, and C. These coefficients of the form FC, CF^{-1}, $C^{\frac{1}{2}}$, $C^{-\frac{1}{2}}$, $FC^{-\frac{3}{2}}$ and so on. Provided F and C remain bounded away from 0 and $+\infty$, we have a non-singular principal symbol. We conclude that the system (3.25), which has three unknowns and three equations, remains critically determined when F and C remain positive.

We count the degrees of freedom in the solution space. The equations $\frac{\partial}{\partial z}\left(CF\frac{\partial s}{\partial z}\right) = 0$, $\mathcal{F}_1 = 0$, and $\mathcal{F}_2 = 0$ are fourth order in F, second order in C, and second order in s, which makes eight derivatives in total, requiring eight initial conditions. Then we restrict to $\mathcal{T} = 0$. From Lemma 3.9, $\mathcal{T}$ is constant along solutions so is completely determined by the system's initial conditions. $\mathcal{T}(F,C)$ is third order in F, second order in C, and first order in s, so $\mathcal{T} = 0$ is a single algebraic relationship among the initial conditions, and reduces the solution space from eight dimensions to seven. Up to homothety the solution space is therefore 5-dimensional. Finally, requiring $s = const$ is the same as imposing an initial condition of $s_z = 0$, so the CSC B^t-flat solution space is 4-dimensional up to homothety. □

Theorem 3.11. *The ZSC B^t-flat metrics, $t \neq \infty$, are the ZSC Bach-flat metrics.*

Assume g is B^t-flat, conformally extremal, and $t \neq 0, \infty$. Then it is CSC if and only if it is ZSC or Einstein.

If $t \neq 0, \frac{1}{3}, \infty$ there exist CSC B^t-flat solutions that are not conformally extremal.

Proof. The CSC B^t-flat equations are (3.18) with initial condition $s_z = 0$. As discussed above, this is a system with 6 degrees of freedom (4 up to homothety). First we examine the ZSC case, where $s = 0$. In this case $\mathcal{T} = 16\mathcal{B}$, so $\mathcal{B}(F, F) = 0$ and so the metric is Bach-flat. Thus F lies in the 3-parameter family given by Lemma 3.6. Fixing F, $\mathcal{F}_1 = 0$ gives a 2-parameter family of solutions for C and we obtain the expected 5-parameter solution space of ZSC Bach-flat metrics (which has 3 parameters up to homothety).

Next assume the metric is CSC B^t-flat, $s \neq 0$, and g conformally extremal. By Proposition 3.1, $F = \frac{1}{2}C_1e^{-2z} + C_2e^{-z} + C_3e^z + \frac{1}{2}C_4e^{2z}$. Plugging in this, along with $\frac{\partial s}{\partial z} = 0$ into $\mathcal{F}_2 = 0$, we obtain

$$\left(\frac{\partial^2}{\partial z^2}C^{-\frac{1}{2}} - \frac{1}{4}C^{-\frac{1}{2}}\right) = 0. \tag{3.26}$$

Therefore $C = \frac{e^{-z}}{(C_5+C_6e^{-z})^2}$. Plugging this into $\mathcal{F}_1 = 0$ provides

$$0 = C_5(C_1C_5 - C_2C_6)e^{-z} + \left(-\frac{s}{24} + C_2C_5^2 - 2C_5C_6 - C_3C_6^2\right) + C_6(C_4C_6 - C_3C_5)e^z. \tag{3.27}$$

We have the seven unknown constants $C_1, C_2, C_3, C_4, C_5, C_6$, and s, and (3.27) contributes three relations so we have a 4-parameter solution space. We consider the possibilities. First, the expression for C makes it impossible that C_5 and C_6 are both zero. If $C_5 \neq 0, C_6 = 0$ then $C = C_5^{-2}e^{-z}$ so the metric is Kähler with respect to J^+, and (3.27) forces $C_1 = 0$, $C_2 = \frac{s}{24C_5^2}$. Then $0 = \mathcal{T}$ is

$$0 = -\frac{1}{2}e^{2z}s\left(3st - 4e^{2z}(1-3t)C_3C_5^2\right), \tag{3.28}$$

and because $t \neq 0$, this forces $s = 0$, contradicting the assumption $s \neq 0$. (Similarly assuming $C_5 = 0, C_6 \neq 0$ also gives $s = 0$, again contradicting $s \neq 0$.)

Therefore both $C_5, C_6 \neq 0$. Then (3.27) forces $C_1C_5 - C_2C_6 = 0$, $C_4C_6 - C_3C_5 = 0$, and by Proposition 3.2 the metric is Einstein. We conclude that if a CSC B^t-flat metric is conformally extremal, it is ZSC or Einstein.

Finally we prove that some CSC B^t-flat metrics are not conformally extremal. The family of Einstein solutions is 4-dimensional, and therefore, by what we just proved, the family of CSC B^t-flat that are conformally extremal is also 4-dimensional. But the space of CSC B^t-flat metrics is 6-dimensional. We conclude that some CSC B^t-flat metrics fail to be conformally extremal. □

4 AmbiKähler Pairs

AmbiKähler pairs are from [2]. An *ambiKähler structure* on a manifold is a pair of Kähler manifolds (M^n, J_1, g_1) and (M^n, J_2, g_2) where the complex stuctures J_1 and J_2 produce opposite orientations and the Kähler metrics g_1 and g_2 are conformal. Either member of the pair can be called the *ambiKähler transform* of the other. From Lemma 2.2, every $U(2)$-invariant metric on a 4-manifold has an ambiKähler structure using $J^{\pm}$, conformally related by letting C be e^{+z} or e^{-z}.

Consequently the classic $U(2)$-invariant Kähler metrics all have ambiKähler transforms. Most of these ambiKähler transforms produce nothing interesting. The ambiKähler transform of the Burns metric is the Fubini-study metric, for example, and the transforms of the other LeBrun instanton metrics are extremal Kähler metrics on weighted projective spaces—these are Bochner-flat metrics found by Bryant in [8, Section 2.2], although their conformal relationship with the LeBrun instantons was not discussed there. The transform of an odd Hirzebruch surface is precisely itself. The transforms of the Taub-NUT-Λ and Eguchi-Hanson-Λ metrics have curvature singularities.

The Taub-NUT and Taub-bolt cases, however, are more interesting. The Taub-NUT is hyperKähler with its family of complex structures being I^- and its left-translates. By Propositions 2.4 and 2.5

$$F = (1 - e^{-z})^2, \quad C = \frac{C_0 e^{-z}}{(1 - e^{-z})^2} \tag{4.1}$$

with coordinate range $z \in (0, \infty]$. The nut is located at $z = \infty$, and the ALF end is at $z = 0$; see Section 2.3 and Figure 3. Separate from the hyperKähler structure an ambiKähler structure exists, given by complex structures J^- and J^+ and conformal factors $C = C_0 e^z$, $C = C_0 e^{-z}$. Thus the conformal orbit of the Taub-NUT meets three complete canonical metrics: itself which is hyperKähler, a 2-ended ZSC Kähler metric, and a 1-ended extremal Kähler metric. We call the latter two the *modified Taub-NUT metrics of the first and second kinds.*

The modified Taub-NUT of the first kind has complex structure J^- and conformal factor $C = C_0 e^z$, which gives it the same orientation as the original Taub-NUT. This metric is two-ended: the nut at $z = -\infty$ becomes an ALE end, and the ALF end at $z = 0$ becomes a cusp-like end. This complete, 2-ended metric is scalar flat by Proposition 1.1. Letting J^+ be the complex structure with conformal factor $C = C_0 e^{-z}$ produces the modified Taub-NUT of the second kind. This metric is one-ended: it still has a nut at $z = \infty$, but the conformal change turns the ALF end into a cusp-like end. By Theorem 3.1 it is extremal Kähler. It has scalar curvature $s = 48(1 - e^{-z})$, which is positive and approaches 0 asymptotically along the cusp.

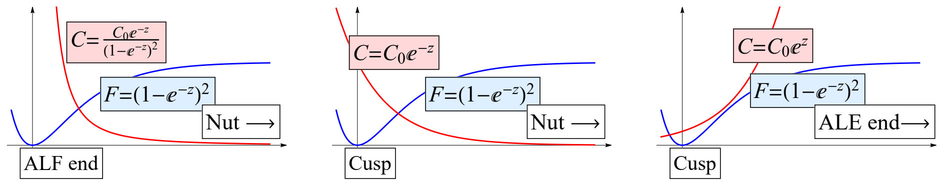

Figure 3: *The Taub-NUT and modified Taub-NUTs of the first and second kinds.*

The modified Taub-NUT of the first kind on $\mathbb{C}^2 \setminus \{(0,0)\}$ is the ZSC Kähler metric of [19] for $n = 2$, and the modified Taub-NUT of the second kind is a complete Bochner-flat metric from [8, Section 2.2] (see also [39]) and is explored in [20].

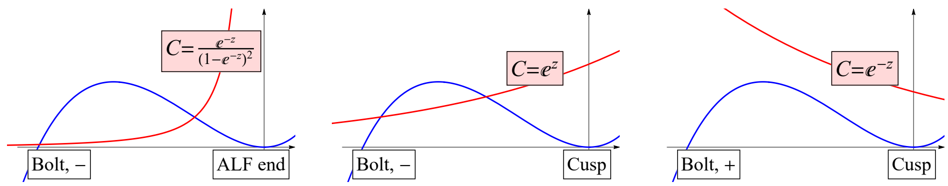

Figure 4: *The Taub-bolt, and the modified Taub-bolts of the first and second kinds.*

The classic Taub-bolt is Ricci-flat but not Kähler (and certainly not hyperKähler) with respect to any complex stucture[2]. The Taub-bolt metric is

$$C = \frac{C_0 e^{-z}}{(1-e^{-z})^2}, \quad F = 1 - \frac{1}{8}e^{-2z} + \frac{1}{4}e^{-z} - \frac{9}{4}e^{z} + \frac{9}{8}e^{2z} \tag{4.2}$$

on $z \in [-\log(3), 0)$. This metric is complete, Ricci-flat, Bach-flat, but not half-conformally flat: both W^+ and W^- are non-zero by Proposition 3.3; see [35, 36]. It has an ALF end at $z = 0$ and a bolt of self-intersection -1 at $z = -\log(3)$. The underlying manifold is the total space of $\mathcal{O}(-1)$. It is conformally Kähler with respect to either J^- or J^+, creating an ambiKähler pair—the *modified Taub-bolt metrics of the first and second kinds,* respectively. Changing between J^- and J^+ reverses the orientation, so changes the self-intersection number of the bolt from -1 to $+1$.

With the complex structure J^- and conformal factor $C = C_0 e^z$ we obtain an extremal Kähler metric we call the *modified Taub-bolt of the first kind.* This metric continues to have a bolt of self-intersection -1 at $z = -\log(3)$, but the ALF end at $z = 0$ has been transformed into a cusp-like end. The scalar curvature is $s = 54C_0^{-1}(1 - e^z)$, which is positive and approaches 0 along the cusp. Its underlying complex manifold is the total space of $\mathcal{O}(-1)$. Its ambiKähler transform has complex structure J^+ and conformal factor $C = C_0 e^{-z}$; we call this extremal Kähler metric the *modified Taub-bolt of the second kind.* The orientation has been reversed and the bolt has self-intersection $+1$ at $z = -\log(3)$. The ALF end at $z = 0$ has again been transformed into a cusp-like end. The scalar curvature is $s = 6C_0^{-1}(-1 + e^{-z})$, which again is positive and approaches

[2]If it were Kähler with respect to *any* complex structure, whether a complex structure considered here or not, Derdzinski's theorem would imply it is half-conformally flat which it is not.

zero asymptotically along the cusp. Its underlying complex manifold is the total space of $\mathcal{O}(+1)$, which is $\mathbb{C}P^2 \setminus \{pt\}$.

Like the Taub-NUT, the Taub-bolt's conformal orbit meets three canonical metrics: itself, which is Ricci flat, and two extremal Kähler metrics. See also [6] which explores the Taub-bolt among other topics (electronically released almost simultaneously with this paper). Neither of the modified Taub-bolts is Bochner-flat or half-conformally flat.

Notable is the presence of a rational curve of positive self intersection in the modified Taub-bolt of the second kind. This is the only example of a complete extremal Kähler metric with a curve of positive self-intersection, that is known to the authors. By contrast there are many examples with curves of zero or negative self intersection. These include the Burns, Eguchi-Hanson, and LeBrun metrics which are all Kähler metrics on $\mathcal{O}(k)$ with $k < 0$ [29]; the Chen-Teo metrics [11, 12] and conformally related Kähler metrics [6] which are on surfaces with rational curves of non-positive self-intersection; and the extremal Kähler "asymptotically equivariantly $\mathbb{R}^2 \times \mathbb{S}^2$" [40, 41] metrics which all have rational curves of non-positive self-intersection.

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Rational approximation of the finite sum of some sequences

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ABSTRACT

In this paper, we give some rational approximations of $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$ by the multiple-correction method and present the bounds of its error.

RESUMEN

En este artículo, entregamos algunas aproximaciones racionales de $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$ por el método de corrección múltiple y presentamos las cotas de su error.

Keywords and Phrases: Rational approximation, continued fraction, inequalities, multiple-correction method.

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1 Introduction

It is well known that we often need to deal with the problem of approximating the factorial function $n!$, and its extension to real numbers called the gamma function, defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \mathrm{Re}(x) > 0,$$

and the logarithmic derivative of $\Gamma(x)$ called the psi-gamma function, denoted by

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

For $x > 0$, the derivative $\psi'(x)$ is called the tri-gamma functions, while the derivatives $\psi^{(k)}(x)$, $k = 1, 2, 3, \ldots$ are called the poly-gamma functions.

In recent years, some authors paid attention to giving increasingly better approximations for the gamma function using continued fractions. For detailed information, please refer to the papers [1,2, 9,11,12] and references cited therein. In fact, it is quite well-known in the theory the algorithm for transforming every formal power series into an associated continued fraction, see [6]. In particular, there are certain methods of transforming the power series $\sum_{n=0}^{\infty} c_n x^{-n-1}$ into continued fractions, see [10, Section III].

For any integer i and $x > 0$, we have

$$\psi^{(i)}(x+1) - \psi^{(i)}(x) = (-1)^i \frac{i!}{x^{i+1}},$$

and when $i = 0$, it yields

$$\psi(x+1) - \psi(x) = \frac{1}{x}.$$

By adding equalities of the form

$$\psi(j+1) - \psi(j) = \frac{1}{j}$$

from $j = n^2 + 1$ to $j = n^2 + n$, we get

$$\psi(n^2 + n + 1) - \psi(n^2 + 1) = \sum_{j=1}^{n} \frac{1}{n^2 + j} = S(n) \tag{1.1}$$

Graham, Knuth and Patashnik [5] proposed the problem of obtaining the asymptotic value of the finite sum

$$S(n) = \sum_{j=1}^{n} \frac{1}{n^2 + j} = \frac{1}{n^2 + 1} + \frac{1}{n^2 + 2} + \cdots + \frac{1}{n^2 + n} \tag{1.2}$$

with a given absolute error.

In this paper, we handle the problem with the aid of the multiple-correction method [3, 4, 13]. We will give some rational approximations of $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$ by the multiple-correction method, and prove some inequalities for the upper and lower bounds. Throughout the paper, the notation $P(x; k)$ means a polynomial of degree k in x, which may be different at each occurrence.

2 Some lemmas

The following lemma gives a method for measuring the rate of convergence, for its proof see Mortici [7, 8].

Lemma 2.1. *If the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to zero and there exists the limit*

$$\lim_{n\to+\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty], \tag{2.1}$$

with $s > 1$, then

$$\lim_{n\to+\infty} n^{s-1}x_n = \frac{l}{s-1}. \tag{2.2}$$

We also need the following intermediary result.

Lemma 2.2. *For every positive integer k, we define*

$$f_k(x) = \ln x + \cfrac{s_1}{x + t_1 + \cfrac{s_2}{x+t_2+\cdots+\cfrac{s_k}{x+t_k}}},$$

where $s_1 = -\frac{1}{2}$, $t_1 = -\frac{1}{6}$; $s_2 = \frac{1}{36}$, $t_2 = -\frac{13}{30}$; $s_3 = \frac{9}{25}$, $t_3 = -\frac{17}{630}$; $s_4 = \frac{6241}{15876}$, $t_4 = -\frac{417941}{786366}$; ...

Then for $x > 1$, we have

$$f_2(x+1) - f_2(x) < \frac{1}{x} < f_3(x+1) - f_3(x). \tag{2.3}$$

Proof. We will apply the multiple-correction method [3, 4, 13] to study the two-sided inequality (2.3) as follows.

(Step 1) The initial-correction. Since $(\ln x)' = \frac{1}{x}$, so we choose $f_0(x) = \ln x$ and develop $F_0(x) := f_0(x+1) - f_0(x) - \frac{1}{x}$ into power series expansion in $\frac{1}{x}$, we have

$$F_0(x) = f_0(x+1) - f_0(x) - \frac{1}{x} = -\frac{1}{2}\frac{1}{x^2} + \frac{1}{3}\frac{1}{x^3} + O\left(\frac{1}{x^4}\right). \tag{2.4}$$

(Step 2) The first-correction. Let $f_1(x) = \ln x + \frac{s_1}{x+t_1}$ and develop $F_1(x) := f_1(x+1) - f_1(x) - \frac{1}{x}$ into power series expansion in $\frac{1}{x}$, we have

$$F_1(x) = \left(-\frac{1}{2} - s_1\right)\frac{1}{x^2} + \left(\frac{1}{3} + s_1 + 2s_1t_1\right)\frac{1}{x^3} + O\left(\frac{1}{x^4}\right). \tag{2.5}$$

Then let the coefficients of $\frac{1}{x^2}$ and $\frac{1}{x^3}$ in (2.5) equal zero, we have $s_1 = -\frac{1}{2}$, $t_1 = -\frac{1}{6}$ and

$$F_1(x) = \frac{1}{24}\frac{1}{x^4} + O\left(\frac{1}{x^5}\right). \tag{2.6}$$

(Step 3) The second-correction. Let $f_2(x) = \ln x + \frac{s_1}{x+t_1+\frac{s_2}{x+t_2}}$ and develop $F_2(x) := f_2(x+1) - f_2(x) - \frac{1}{x}$ into power series expansion in $\frac{1}{x}$, it can be derived that

$$F_2(x) = \left(\frac{1}{24} - \frac{3s_2}{2}\right)\frac{1}{x^4} + \left(-\frac{11}{270} + \frac{7s_2}{3} + 2s_2t_2\right)\frac{1}{x^5} + O\left(\frac{1}{x^6}\right). \tag{2.7}$$

Then let the coefficients of $\frac{1}{x^4}$ and $\frac{1}{x^5}$ in (2.7) equal zero, we have $s_2 = \frac{1}{36}$, $t_2 = -\frac{13}{30}$ and

$$F_2(x) = f_2(x+1) - f_2(x) - \frac{1}{x} = -\frac{1}{40}\frac{1}{x^6} + O\left(\frac{1}{x^7}\right). \tag{2.8}$$

Furthermore, we obtain

$$F_2'(x) = \frac{P(x)}{3x^2(1+x)(1-6x+10x^2)^2(5+14x+10x^2)^2},$$

where $P(x) = 75 - 480x - 508x^2 + 3680x^3 + 4500x^4$.

As all coefficients of $P(x+1) = 7267 + 27544x + 37532x^2 + 21680x^3 + 4500x^4$ are positive, which implies that $F_2(x)$ is strictly increasing. Since $F_2(\infty) = 0$, it can be found that $F_2(x) < 0$ on $x > 1$. This finishes the proof of the left-hand inequality in (2.3).

(Step 4) The third-correction. Similarly, let $f_3(x) = \ln x + \frac{s_1}{x+t_1+\frac{s_2}{x+t_2+\frac{s_3}{x+t_3}}}$ and develop $F_3(x) := f_3(x+1) - f_3(x) - \frac{1}{x}$ into power series expansion in $\frac{1}{x}$, we have

$$F_3(x) = \left(-\frac{1}{40} + \frac{5s_3}{72}\right)\frac{1}{x^6} + \frac{802 - 2275s_3 - 1750s_3t_3}{21000}\frac{1}{x^7} + O\left(\frac{1}{x^8}\right). \tag{2.9}$$

Then let the coefficients of $\frac{1}{x^6}$ and $\frac{1}{x^7}$ in (2.9) equal zero, we have $s_3 = \frac{9}{25}$, $t_3 = -\frac{17}{630}$ and

$$F_3(x) = f_3(x+1) - f_3(x) - \frac{1}{x} = \frac{6241}{453600}\frac{1}{x^8} + O\left(\frac{1}{x^9}\right). \tag{2.10}$$

Furthermore, we obtain

$$F_3'(x) = \frac{Q(x)}{3x^2(1+x)(-79+600x-790x^2+1260x^3)^2(991+2800x+2990x^2+1260x^3)^2},$$

where $Q(x) = 18387502563 - 175398675600\,x - 226510750180\,x^2 - 500966546560x^3 - 1400497343100x^4 - 1903983580800x^5 - 832289774400x^6$.

As all coefficients of $Q(x+1) = -5021259168077 - 22246965738440x - 41656576872460x^2 - 41788587214960x^3 - 23404761863100x^4 - 6897722227200x^5 - 832289774400x^6$ are negative, which implies that $F_3(x)$ is strictly decreasing. Since $F_3(\infty) = 0$, it can be found that $F_3(x) > 0$ on $x > 1$. This finishes the proof of the right-hand inequality in (2.3).

The proof of Lemma 2.2 is completed. □

3 Main results

By adding inequalities (2.3) of the form

$$f_2(x+1) - f_2(x) < \frac{1}{x} < f_3(x+1) - f_3(x)$$

from $x = n^2+1$ to $x = n^2+n$, we get

$$f_2(n^2+n+1) - f_2(n^2+1) < \sum_{j=1}^{n} \frac{1}{n^2+j} < f_3(n^2+n+1) - f_3(n^2+1). \tag{3.1}$$

This two-sided inequalities give the estimate of $\sum_{j=1}^{n} \frac{1}{n^2+j}$. So we have

Theorem 3.1. *For positive integer $n > 1$,*

$$\ln\left(1+\frac{n}{n^2+1}\right) + \frac{P(n;5)}{3P_1(n;4)P_2(n;4)} < \sum_{j=1}^{n} \frac{1}{n^2+j} < \ln\left(1+\frac{n}{n^2+1}\right) + \frac{5P(n;9)}{3P_1(n;6)P_2(n;6)}, \tag{3.2}$$

where

$$\begin{aligned}
P(n;5) &= 44n + 85n^2 + 170n^3 + 150n^4 + 150n^5,\\
P_1(n;4) &= 5 + 14n^2 + 10n^4,\\
P_2(n;4) &= 5 + 14n + 24n^2 + 20n^3 + 10n^4,\\
P(n;9) &= 387838n + 655457n^2 + 1744984n^3 + 1983990n^4 + 2717310n^5\\
&\quad + 2199960n^6 + 1942920n^7 + 952560n^8 + 476280n^9,\\
P_1(n;6) &= 991 + 2800n^2 + 2990n^4 + 1260n^6,\\
P_2(n;6) &= 991 + 2800n + 5790n^2 + 7240n^3 + 6770n^4 + 3780n^5 + 1260n^6.
\end{aligned}$$

Proof. The double inequality (3.1) can be equivalently written as (3.2). □

Theorem 3.1 gives an asymptotic formula of the sum $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$, but we want to obtain the rational approximation. It ensures the following approximation formula as $n \to \infty$, $\ln\left(1+\frac{n}{n^2+1}\right) \sim \frac{n}{n^2+1}$, but the rate of convergence is not satisfied. Now we estimate the function $\ln\left(1+\frac{n}{n^2+1}\right)$ as following.

Theorem 3.2. *For positive integer $n > 1$, we have*

$$\frac{n^2+\frac{133}{109}n-\frac{769}{6540}}{n^3+\frac{375}{218}n^2+\frac{768}{545}n+\frac{2401}{2180}} < \ln\left(1+\frac{n}{n^2+1}\right) < \frac{n-\frac{1}{22}}{n^2+\frac{5}{11}n+\frac{59}{66}}. \tag{3.3}$$

Proof. Developing the function $\ln\left(1+\frac{n}{n^2+1}\right) - \frac{s_2n^2+s_1n+s_0}{n^3+t_2n^2+t_1n+t_0}$ into power series expansion in $\frac{1}{n}$, we have

$$\begin{aligned}
&\ln\left(1+\frac{n}{n^2+1}\right) - \frac{s_2n^2+s_1n+s_0}{n^3+t_2n^2+t_1n+t_0} \qquad (3.4)\\
&= (1-s_2)\frac{1}{n} + \left(-\frac{1}{2}-s_1+s_2t_2\right)\frac{1}{n^2} + \left(-\frac{2}{3}-s_0+s_2t_1+s_1t_2-s_2t_2^2\right)\frac{1}{n^3}\\
&+ \left(\frac{3}{4}+s_2t_0+s_1t_1+s_0t_2-2s_2t_1t_2-s_1t_2^2+s_2t_2^3\right)\frac{1}{n^4}\\
&+ \left(\frac{1}{5}+s_1t_0+s_0t_1-s_2t_1^2-2s_2t_0t_2-2s_1t_1t_2-s_0t_2^2+3s_2t_1t_2^2+s_1t_2^3-s_2t_2^4\right)\frac{1}{n^5}\\
&+ \left(-\frac{2}{3}+s_0t_0-2s_2t_0t_1-s_1t_1^2-2s_1t_0t_2-2s_0t_1t_2+3s_2t_1^2t_2+3s_2t_0t_2^2+3s_1t_1t_2^2\right.\\
&\left.+s_0t_2^3-4s_2t_1t_2^3-s_1t_2^4+s_2t_2^5\right)\frac{1}{n^6} + O\left(\frac{1}{n^7}\right).
\end{aligned}$$

According to Lemma 2.1, to get the highest rate of convergence, we have $s_2 = 1$, $s_1 = \frac{133}{109}$, $s_0 = -\frac{769}{6540}$, $t_2 = \frac{375}{218}$, $t_1 = \frac{768}{545}$, $t_0 = \frac{2401}{2180}$ and

$$\ln\left(1+\frac{n}{n^2+1}\right) - \frac{s_2n^2+s_1n+s_0}{n^3+t_2n^2+t_1n+t_0} = \frac{31721}{305200}\frac{1}{n^7} + O\left(\frac{1}{n^8}\right).$$

Furthermore, we denote $G_1(x) = \ln\left(1+\frac{x}{x^2+1}\right) - \frac{x^2+\frac{133}{109}x-\frac{769}{6540}}{x^3+\frac{375}{218}x^2+\frac{768}{545}x+\frac{2401}{2180}}$, then we can get

$$G_1'(x) = -\frac{1409315+4813232x+3457589x^2}{(1+x^2)(1+x+x^2)(2401+3072x+3750x^2+2180x^3)^2} < 0,$$

which implies that $G_1(x)$ is strictly decreasing. Since $G_1(\infty) = 0$, it can be found that $G_1(n) > 0$ for every positive integer n. Then we have

$$\frac{n^2+\frac{133}{109}n-\frac{769}{6540}}{n^3+\frac{375}{218}n^2+\frac{768}{545}n+\frac{2401}{2180}} < \ln\left(1+\frac{n}{n^2+1}\right). \tag{3.5}$$

This finishes the proof of the left-hand inequality in (3.3).

Similarly, developing the function $\ln\left(1+\frac{n}{n^2+1}\right)-\frac{u_1n+u_0}{n^2+v_1n+v_0}$ into power series expansion in $\frac{1}{n}$, we have

$$\begin{aligned}
&\ln\left(1+\frac{n}{n^2+1}\right)-\frac{u_1n+u_0}{n^2+v_1n+v_0} \qquad (3.6)\\
&=(1-u_1)\frac{1}{n}+\left(-\frac{1}{2}-u_0+u_1v_1\right)\frac{1}{n^2}+\left(-\frac{2}{3}+u_1v_0+u_0v_1-u_1v_1^2\right)\frac{1}{n^3}\\
&+\left(\frac{3}{4}+u_0v_0-2u_1v_0v_1-u_0v_1^2+u_1v_1^3\right)\frac{1}{n^4}+O\left(\frac{1}{n^5}\right).
\end{aligned}$$

According to Lemma 2.1, to get the highest rate of convergence, we have $u_1=1$, $u_0=-\frac{1}{22}$, $v_1=\frac{5}{11}$, $v_0=\frac{59}{66}$ and

$$\ln\left(1+\frac{n}{n^2+1}\right)-\frac{u_1n+u_0}{n^2+v_1n+v_0}=-\frac{109}{1980}\frac{1}{n^5}+O\left(\frac{1}{n^6}\right).$$

Furthermore, we denote $G_2(x)=\ln\left(1+\frac{x}{x^2+1}\right)-\frac{x-\frac{1}{22}}{x^2+\frac{5}{11}x+\frac{59}{66}}$, then we can get

$$G_2'(x)=\frac{-503-840x+1199x^2}{(1+x^2)(1+x+x^2)(59+30x+66x^2)^2}>0$$

when $x>1$, which implies that $G_2(x)$ is strictly increasing. Since $G_2(\infty)=0$, it can be found that $G_2(n)>0$ for positive integer $n>1$. Then we have

$$\ln\left(1+\frac{n}{n^2+1}\right)<\frac{n-\frac{1}{22}}{n^2+\frac{5}{11}n+\frac{59}{66}}. \qquad (3.7)$$

This finishes the proof of the right-hand inequality in (3.3).

The proof of Theorem 3.2 is completed. □

Combining (3.2) and (3.3), we have

Theorem 3.3. *As $n\to\infty$,*

$$\frac{P(n;10)}{3P(n;3)P_1(n;4)P_2(n;4)}<\sum_{j=1}^{n}\frac{1}{n^2+j}<\frac{P(n;13)}{3P(n;2)P_1(n;6)P_2(n;6)}, \qquad (3.8)$$

where

$$\begin{aligned}
P(n;10)&=-19225+251314n+915243n^2+2580666n^3+4566456n^4+6735890n^5\\
&\quad+7304720n^6+6514900n^7+4331300n^8+2106000n^9+654000n^{10},\\
P(n;3)&=2401+3072n+3750n^2+2180n^3,
\end{aligned}$$

$$
\begin{aligned}
P(n;13) = {} & -8838729 + 283891048n + 724331705n^2 + 2291454430n^3 + 3803306340n^4 \\
& + 6508603530n^5 + 7775628660n^6 + 9153584460n^7 + 8099239500n^8 + 6891737400n^9 \\
& + 4319179200n^{10} + 2549232000n^{11} + 928746000n^{12} + 314344800n^{13}, \\
P(n;2) = {} & 59 + 30n + 66n^2.
\end{aligned}
$$

So we can get the rational approximation $\frac{P(n;10)}{3P(n;3)P_1(n;4)P_2(n;4)}$ of the finite sum $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$, and the error can be bounded as following,

Theorem 3.4. *As $n \to \infty$, we have*

$$
\sum_{j=1}^{n} \frac{1}{n^2+j} \sim T(n) = \frac{P(n;10)}{3P(n;3)P_1(n;4)P_2(n;4)}. \tag{3.9}
$$

Furthermore, we can give the bounds of the error estimation,

$$
0 < \sum_{j=1}^{n} \frac{1}{n^2+j} - T(n) < \frac{109}{1980}\frac{1}{n^5}. \tag{3.10}
$$

Proof. Set $D = \frac{109}{1980}$, from (3.8) we can get

$$
\begin{aligned}
& \frac{P(n;13)}{3P(n;2)P_1(n;6)P_2(n;6)} - T(n) - \frac{D}{n^5} \\
& = -\frac{P(n;24)}{1980n^5 P(n;2)P_1(n;3)P_1(n;4)P_2(n;4)P_1(n;6)P_2(n;6)} < 0,
\end{aligned} \tag{3.11}
$$

where

$$
\begin{aligned}
P(n;24) = {} & 379103668732775 + 2810435887808320n + 14242250073272280n^2 \\
& + 52307052296627116n^3 + 157936445498291068n^4 + 399973820542120296n^5 \\
& + 882209143385828432n^6 + 1711892774844546448n^7 + 2970795182632943800n^8 \\
& + 4635720249539129840n^9 + 6558910458343361680n^{10} + 8434105620517736160n^{11} \\
& + 9897520754047548080n^{12} + 10594749646379864160n^{13} + 10355798883536793600n^{14} \\
& + 9208131536164270400n^{15} + 7433462344335679600n^{16} + 5402752686291200000n^{17} \\
& + 3514488757828417600n^{18} + 2012863116859364800n^{19} + 1001770606450320000n^{20} \\
& + 417999105909504000n^{21} + 141577633391040000n^{22} + 34754556120480000n^{23} \\
& + 5414684436000000n^{24}.
\end{aligned}
$$

Proof of Theorem 3.4 is completed. □

Remark 3.5. *As $n \to \infty$, we also can get the rational approximation*

$$W(n) = \frac{P(n;13)}{3P(n;2)P_1(n;6)P_2(n;6)} \tag{3.12}$$

of the finite sum $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$.

Remark 3.6. *Using the Maclaurin series of the left and right hand of (3.2), we obtain*

$$\frac{29}{440}\frac{1}{n^{11}} + \frac{1}{30}\frac{1}{n^{12}} + O\left(\frac{1}{n^{13}}\right) \le \sum_{j=1}^{n} \frac{1}{n^2+j} - U(n) \le \frac{1}{11}\frac{1}{n^{11}} - \frac{1}{24}\frac{1}{n^{12}} + O\left(\frac{1}{n^{13}}\right). \tag{3.13}$$

So we have another approximation, as $n \to \infty$,

$$\sum_{j=1}^{n} \frac{1}{n^2+j} \sim U(n) = \frac{1}{n} - \frac{1}{2}\frac{1}{n^2} - \frac{1}{6}\frac{1}{n^3} + \frac{1}{4}\frac{1}{n^4} - \frac{2}{15}\frac{1}{n^5} + \frac{1}{12}\frac{1}{n^6} - \frac{1}{42}\frac{1}{n^7} - \frac{1}{24}\frac{1}{n^8} + \frac{7}{90}\frac{1}{n^9} - \frac{1}{10}\frac{1}{n^{10}}. \tag{3.14}$$

Furthermore, we denote $H_1(x) = \ln\left(1 + \frac{x}{x^2+1}\right) + \frac{P(x;5)}{3P_1(x;4)P_2(x;4)} - U(x) - \frac{29}{440}\frac{1}{x^{11}}$, then we can get $H_1'(x) = \frac{P(x;19)}{120x^{12}(1+x^2)(1+x+x^2)P_1^2(x;4)P_2^2(x;4)}$, where

$$\begin{aligned} P(x;19) = {} & 54375 + 283875x + 1223550x^2 + 3541475x^3 + 8928955x^4 + 18003620x^5 \\ & + 32386512x^6 + 48945976x^7 + 66608504x^8 + 76840064x^9 + 79734920x^{10} \\ & + 68524380x^{11} + 52231532x^{12} + 29887232x^{13} + 14214864x^{14} + 1988640x^{15} \\ & - 1179920x^{16} - 2468400x^{17} - 927200x^{18} - 480000x^{19}. \end{aligned}$$

As all coefficients of

$$\begin{aligned} P(x+3;19) = {} & -1095798626414130 - 6922138869735924x - 20458381656316617x^2 \\ & - 37730683241040109x^3 - 48798043215225557x^4 - 47107553905950172x^5 \\ & - 35247917132102064x^6 - 20940823139217776x^7 - 10032400214888248x^8 \\ & - 3912613116855772x^9 - 1247976394963924x^{10} - 325701204911892x^{11} \\ & - 69291596265604x^{12} - 11915674458880x^{13} - 1632596145936x^{14} - 174202919520x^{15} \\ & - 13962062720x^{16} - 791257200x^{17} - 28287200x^{18} - 480000x^{19} \end{aligned}$$

are negative, which implies that $H_1(x)$ is strictly decreasing on $x > 3$. Since $H_1(\infty) = 0$, it can be found that $H_1(n) > 0$ for positive integer $n > 3$. Then we have

$$\frac{29}{440}\frac{1}{n^{11}} \le \sum_{j=1}^{n} \frac{1}{n^2+j} - U(n). \tag{3.15}$$

Similarly, we denote $H_2(x) = \ln\left(1+\frac{x}{x^2+1}\right) + \frac{5P(x;9)}{3P_1(x;6)P_2(x;6)} - U(x) - \frac{1}{11}\frac{1}{x^{11}}$, *then we can get* $H_2'(x) = \frac{P(x;27)}{30x^{12}(1+x^2)(1+x+x^2)P_1^2(x;6)P_2^2(x;6)}$, *where*

$$\begin{aligned}P(x;27) = {} & 28934492716830 + 163504701528000x + 781783155292011x^2 \\ & + 2561640891519341x^3 + 7366886663076127x^4 + 17465244022945601x^5 \\ & + 37293047508784116x^6 + 69715428169427545x^7 + 119236982847280685x^8 \\ & + 183471922929904370x^9 + 260745743812768040x^{10} + 338060035189670685x^{11} \\ & + 406969201616917085x^{12} + 450014549032420100x^{13} + 463005366631670400x^{14} \\ & + 438405464461473000x^{15} + 385877522700724000x^{16} + 311756448527065800x^{17} \\ & + 233075982007921000x^{18} + 158623848613552500x^{19} + 98916577490962500x^{20} \\ & + 55177732215522000x^{21} + 27657182228634000x^{22} + 11962175918742000x^{23} \\ & + 4459721484330000x^{24} + 1316647483200000x^{25} + 296695768320000x^{26} \\ & + 37807106400000x^{27}.\end{aligned}$$

As all coefficients of $P(x;27)$ *are positive, which implies that* $H_2(x)$ *is strictly increasing. Since* $H_2(\infty) = 0$, *it can be found that* $H_2(n) < 0$ *for every positive integer* n. *Then we have*

$$\sum_{j=1}^{n} \frac{1}{n^2+j} - U(n) \le \frac{1}{11}\frac{1}{n^{11}}. \tag{3.16}$$

So we can give the upper and lower bounds as follow, for positive integer $n > 3$,

$$\frac{29}{440}\frac{1}{n^{11}} \le \sum_{j=1}^{n} \frac{1}{n^2+j} - U(n) \le \frac{1}{11}\frac{1}{n^{11}}. \tag{3.17}$$

4 Some new estimates and double side inequalities

In order to prove the announced inequalities, we use the direct consequence of Theorem 8 of Alzer [2] who proved that the double-sided inequalities for the function of arbitrary accuracies

$$\ln x - \frac{1}{2x} - \sum_{i=1}^{2n-1} \frac{B_{2i}}{2ix^{2i}} < \psi(x) < \ln x - \frac{1}{2x} - \sum_{i=1}^{2n} \frac{B_{2i}}{2ix^{2i}}, \quad (x > 0,\ n \in N), \tag{4.1}$$

where B_j, $j \ge 0$ denote the Bernoulli numbers which may be generated by

$$\frac{z}{e^z - 1} = \sum_{j=1}^{\infty} B_j \frac{z^j}{j!}.$$

In particular, for $i = 2$, we deduce that:

$$\ln x - \frac{1}{2x} - Q_6(x) < \psi(x) < \ln x - \frac{1}{2x} - Q_8(x), \tag{4.2}$$

where $Q_6(x) = \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6}$, $Q_8(x) = \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6} - \frac{1}{240x^8}$. Combining (1.1) and (4.2), we get

$$\begin{aligned}\ln \frac{n^2+n+1}{n^2+1} + \frac{P_1(n;25)}{5040(n^2+1)^8(n^2+n+1)^6} < S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j} \\ = \psi(n^2+n+1) - \psi(n^2+1) < \ln \frac{n^2+n+1}{n^2+1} + \frac{P_2(n;25)}{5040(n^2+1)^8(n^2+n+1)^6},\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}P_1(n;25) &= -21 + 3186n + 15651n^2 + 69238n^3 + 202356n^4 + 529934n^5 \\ &+ 1122353n^6 + 2160262n^7 + 3588004n^8 + 5473222n^9 + 7408367n^{10} \\ &+ 9267866n^{11} + 10416693n^{12} + 10852108n^{13} + 10193994n^{14} + 8875980n^{15} \\ &+ 6943146n^{16} + 5020008n^{17} + 3220812n^{18} + 1898232n^{19} + 966000n^{20} \\ &+ 446880n^{21} + 167580n^{22} + 56280n^{23} + 12600n^{24} + 2520n^{25},\end{aligned}$$

and

$$\begin{aligned}P_2(n;25) &= 21 + 3312n + 22842n^2 + 105784n^3 + 354605n^4 + 972552n^5 \\ &+ 2229004n^6 + 4439168n^7 + 7749915n^8 + 12075104n^9 + 16850506n^{10} \\ &+ 21261744n^{11} + 24267221n^{12} + 25182808n^{13} + 23708364n^{14} + 20294352n^{15} \\ &+ 15714090n^{16} + 11002824n^{17} + 6899676n^{18} + 3862152n^{19} + 1894620n^{20} \\ &+ 808080n^{21} + 287700n^{22} + 84000n^{23} + 17640n^{24} + 2520n^{25}.\end{aligned}$$

So we can immediately obtain the new estimates of the finite sum $S(n) = \sum_{j=1}^{n} \frac{1}{n^2+j}$ as following,

Theorem 4.1. *As $n \to \infty$, we have*

$$\sum_{j=1}^{n} \frac{1}{n^2+j} \sim V(n) = \ln \frac{n^2+n+1}{n^2+1} + \frac{2520n^{25}}{5040(n^2+1)^8(n^2+n+1)^6}. \tag{4.4}$$

Remark 4.2. *If we select a lager n in the double-sided inequalities* (4.1)*, we can get others double-sided rational estimates for the considered function S_n with arbitrary accuracies.*

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