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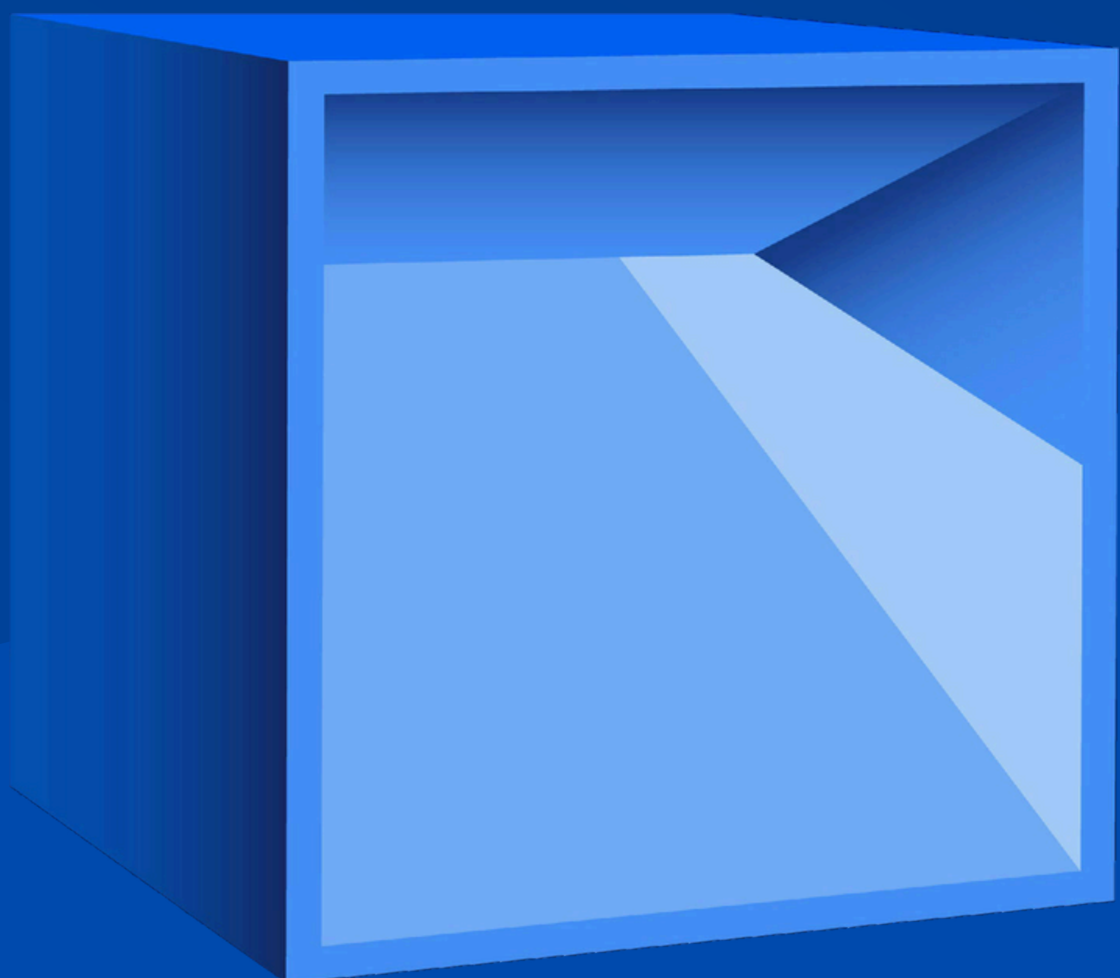

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SUMMARY

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Infinitesimally tight Lagrangian submanifolds in adjoint orbits: A classification of real forms

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ABSTRACT

In this paper, we study the geometry of real flag manifolds within complex flag manifolds, focusing on their Lagrangian properties. We prove that the natural immersion of real flag manifolds into their corresponding complex flag manifolds can be characterized as infinitesimally tight Lagrangian submanifolds with respect to the Kirillov-Kostant-Souriau (KKS) symplectic form. This property of tightness provides a significant geometric constraint, indicating that the submanifolds are locally minimal and cannot be deformed infinitesimally to reduce their volume further in the ambient space. We further provide a comprehensive classification of these immersions, detailing the conditions under which such submanifolds exist across various symmetric pairs. This classification elucidates the relationship between the structure of the real flags and the associated complex flags, contributing to a deeper understanding of the interplay between symplectic geometry and representation theory.

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RESUMEN

En este artículo, estudiamos la geometría de variedades bandera reales dentro de variedades bandera complejas, con foco en sus propiedades Lagrangianas. Demostramos que la inmersión natural de variedades bandera reales en sus correspondientes variedades bandera complejas puede caracterizarse como subvariedades Lagrangianas infinitesimalmente estrechas con respecto a la forma simpléctica de Kirillov-Kostant-Souriau (KKS). Esta propiedad de estrechez provee una restricción geométrica significativa, indicando que las subvariedades son localmente mínimas y no pueden deformarse infinitesimalmente para reducir aún más su volumen en el espacio ambiente. Además entregamos una clasificación completa de estas inmersiones, detallando las condiciones bajo las cuales tales subvariedades existen entre varios pares simétricos. Esta clasificación aclara la relación entre la estructura de las banderas reales y las banderas complejas asociadas, contribuyendo a un entendimiento más profundo de la interacción entre la geometría simpléctica y la teoría de representaciones.

Keywords and Phrases: Flag manifolds, homogeneous space, Lagrangian submanifolds, infinitesimally tight

2020 AMS Mathematics Subject Classification: 14M15, 17Bxx, 22Bxx, 22Cxx, 22F30, 53D12

1 Introduction

Lagrangian submanifolds in symplectic homogeneous spaces have been extensively studied, with significant contributions to their classification in various contexts. For instance, compact symplectic homogeneous manifolds have been classified in [24]. In this paper, we focus on the coadjoint orbits of semisimple Lie groups, exploring the applications of semisimple Lie theory to symplectic geometry, specifically in identifying Lagrangian submanifolds within adjoint orbits. Our motivation stems from the homological mirror symmetry conjecture and, in particular, from concepts in Fukaya–Seidel categories, where objects and morphisms are generated by Lagrangian vanishing cycles and their thimbles, exhibiting specific behaviors within symplectic fibrations (see [10] and [12]).

The primary objective of this paper is to investigate the locally, globally, and infinitesimally tight Lagrangian submanifolds on adjoint orbits, a concept first introduced by Y.-G. Oh in 1991 (see [17]). Oh defined tightness for closed Lagrangian submanifolds in compact Hermitian symmetric spaces as follows:

Definition 1.1. *Let (M, ω, J) be a Hermitian symmetric space of compact type and $\mathcal{L}$ a closed embedded Lagrangian submanifold of M. Then $\mathcal{L}$ is said to be* **globally tight** *(resp.* **locally tight***) if it satisfies*

$$\#(\mathcal{L} \cap g \cdot \mathcal{L}) = \mathrm{SB}(\mathcal{L}, \mathbb{Z}_2)$$

for any isometry $g \in G$ (resp. sufficiently close to the identity) such that $\mathcal{L}$ intersects $g \cdot \mathcal{L}$ transversely. Here, $\mathrm{SB}(\mathcal{L}, \mathbb{Z}_2)$ *denotes the sum of the $\mathbb{Z}_2$-Betti numbers of $\mathcal{L}$.*

In the same work, Oh demonstrated that the standard $\mathbb{RP}^n$ inside $\mathbb{CP}^n$ is tight and minimizes volume among all its Hamiltonian deformations (see [17]), linking tightness to Hamiltonian volume minimization (see [18]). This concept is further connected to the Arnold–Givental conjecture, which posits that the number of intersection points between a Lagrangian $\mathcal{L}$ and its image under a Hamiltonian flow $\phi(\mathcal{L})$ is bounded below by the sum of its $\mathbb{Z}_2$-Betti numbers:

$$\#(\mathcal{L} \cap \phi(\mathcal{L})) \geq \sum b_k(\mathcal{L}; \mathbb{Z}_2).$$

The study of tight Lagrangian submanifolds is therefore of significant interest in symplectic geometry. Oh also posed the open problem:

Problem 1.2. *Classify all possible tight Lagrangian submanifolds in other Hermitian symmetric spaces.*

By [17], Oh proposed the following conjecture:

Conjecture 1.3. *Are the real forms in these spaces the only possible tight Lagrangian submanifolds?*

While Oh's conjecture suggests that real forms may be the only possible tight Lagrangian submanifolds in Hermitian symmetric spaces, our study is restricted to the case of flag manifolds. In particular, we examine the natural immersion of real flag manifolds into their corresponding complex flag manifolds and demonstrate that they can be characterized as infinitesimally tight Lagrangian submanifolds with respect to the Kirillov-Kostant-Souriau (KKS) symplectic form. This characterization provides a significant geometric constraint, indicating that these submanifolds are locally minimal and cannot be deformed infinitesimally to further reduce their volume in the ambient space.

Furthermore, we provide a comprehensive classification of these immersions, detailing the conditions under which such submanifolds exist across various symmetric pairs. This classification elucidates the relationship between the structure of real flags and their associated complex flag manifolds, contributing to a deeper understanding of the interplay between symplectic geometry and representation theory.

In a similar vein, Iriyeh and Sakai classified tight Lagrangian submanifolds in $S^2 \times S^2$ (see [15]), showing that if $\mathcal{L}$ is a closed, embedded, tight Lagrangian surface in $S^2 \times S^2$, then $\mathcal{L}$ must be one of the following:

- $\mathcal{L} = \{(x, -x) \in S^2 \times S^2 : \ x \in S^2\}$ (global tight submanifold).
- $\mathcal{L} = S^1(a) \times S^1(b) \subset S^2 \times S^2$, where $S^1(a)$ is a round circle of radius $0 < a \leq 1$ (locally tight submanifold).

This classification forms a special case of tight submanifolds in products of flag manifolds, which were studied in [13]. There, the authors demonstrated that a product of flag manifolds $\mathbb{F}_{\Theta_1} \times \mathbb{F}_{\Theta_2}$ admits a Lagrangian orbit under the diagonal action (or shifted diagonal action) if and only if $\Theta_2 = \Theta_1^*$, where $\Theta_2 = \sigma\Theta_1$ with σ being the symmetry of the Dynkin diagram, given by $\sigma = -w_0$, and w_0 being the longest element of the Weyl group $\mathcal{W}$. Such a Lagrangian orbit is described by the graph of

$$-\operatorname{id} : \operatorname{Ad}(U)(iH) \to \operatorname{Ad}(U)(i\sigma(H)),$$

or by the graph of $-\operatorname{Ad}(m)$, where $m \in U$ for the shifted diagonal action.

A significant contribution of [13] was the introduction of the concept of infinitesimally tight submanifolds. The authors proved that Lagrangian orbits resulting from the diagonal (or shifted diagonal) action are infinitesimally tight. This notion is formally defined as follows:

Definition 1.4. *Let $\mathcal{L}$ be a submanifold of $M = G/H$. An element $X \in \mathfrak{g} = \mathrm{Lie}(G)$ is called* **transversal** *to $\mathcal{L}$ if it satisfies the following two conditions:*

(1) *For any $x \in \mathcal{L}$, if $\widetilde{X}(x) \in T_x\mathcal{L}$, then $\widetilde{X}(x) = 0$.*

(2) *The set*

$$f_{\mathcal{L}}(X) = \left\{x \in \mathcal{L} : \ 0 = \widetilde{X}(x) \in T_x\mathcal{L}\right\}$$

is finite.

That is, $\widetilde{X}$ is tangent to $\mathcal{L}$ only at finitely many points where it vanishes.

A Lagrangian submanifold $\mathcal{L}$ in $M = G/H$ is called **infinitesimally tight** if

$$\#\left(f_{\mathcal{L}}(X)\right) = \mathrm{SB}\left(\mathcal{L}, \mathbb{Z}_2\right)$$

for any $X \in \mathfrak{g}$ such that $\widetilde{X}$ is transversal to $\mathcal{L}$. Moreover, [13] presents the following theorem:

Theorem 1.5. *Let $M = G/H$ be a homogeneous space with a G-invariant symplectic form ω. Then a Lagrangian submanifold $\mathcal{L} \subset M$ is infinitesimally tight if and only if it is locally tight.*

As discussed in [6] and [13], isotropic submanifolds can be characterized through the moment map of a Hamiltonian action. In particular, Gorodski and Podestà [6] classified compact tight Lagrangian submanifolds in irreducible compact homogeneous Kähler manifolds that have the $\mathbb{Z}_2$-homology of a sphere. This classification is closely related to our study, as it provides structural constraints on the existence of tight Lagrangian submanifolds within compact homogeneous spaces. Our work builds upon these ideas by characterizing the complex flag manifolds that admit real flag manifolds as Lagrangian submanifolds.

To establish this characterization, we equip the complex flag manifolds with the Kirillov-Kostant-Souriau (KKS) symplectic form and consider the compact orbits of the real forms of the associated complex Lie group. This approach aligns with recent developments related to the Ph.D. thesis of Báez, where the author studied Lagrangian submanifolds of adjoint semisimple orbits. The results from this thesis are directly related to the findings presented in this paper, further reinforcing the connection between Lagrangian submanifolds and the geometry of adjoint orbits in semisimple Lie theory.

Regarding the work of Gorodski and Podestà [6], although our conclusions share similarities, the methodologies differ significantly. While their approach focuses on homogeneous Kähler manifolds with topological constraints on homology, our classification provides a systematic study of complex flag manifolds and their real forms that possess compact Lagrangian orbits. This classification is explicitly detailed in Table 1 at the end of Subsection 3.1, with a case-by-case proof given in Appendix A.

Specifically, in Section 4, we prove that real flag manifolds can be seen as infinitesimally tight submanifolds of the corresponding complex flag manifolds. This result establishes a direct link between the structure of flag manifolds, symplectic geometry, and representation theory, offering a broader perspective on the classification of Lagrangian orbits within homogeneous symplectic spaces.

2 Flag manifolds

Flag manifolds play a central role in the study of Lie groups and their geometric structures. However, their treatment varies significantly depending on whether they are considered within the framework of complex semisimple Lie groups or real semisimple Lie groups. This distinction is crucial, as notation and conventions often diverge in the literature, with most works focusing exclusively on either the real or the complex setting. To provide a unified perspective, this section introduces both real and complex flag manifolds, along with fundamental concepts such as Weyl chambers and Weyl groups. The goal is to establish a consistent notation and clarify potential ambiguities, ensuring that the reader can navigate seamlessly through subsequent discussions.

There exist several equivalent definitions of flag manifolds, and they are sometimes referred to as *generalized flag manifolds*. This terminology appears in various sources, with one of the most well-known references being Alekseevsky's work (see [1]), where these spaces are studied from a broader geometric perspective. A fundamental definition, which serves as a starting point for our discussion, is the following:

Definition 2.1. *Let $\mathfrak{g}$ be a semisimple non-compact Lie algebra, and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. The flag manifold $\mathbb{F}_H$ is the homogeneous space*

$$\mathbb{F}_H = G/P_H,$$

where P_H is a parabolic subgroup of G, determined by an element $H \in \mathfrak{g}$, which can be chosen within the closure of a positive Weyl chamber of $\mathfrak{g}$.

The construction of the parabolic subgroup P_H depends on whether $\mathfrak{g}$ is a real or complex Lie algebra. In what follows, we shall present these constructions using fundamental tools from semisimple Lie theory. Although different approaches provide valuable insights, in this work, we adopt the perspective that complex flag manifolds are most naturally understood as adjoint orbits of compact semi-simple Lie groups. This viewpoint not only highlights their intrinsic geometric structure but also establishes a direct connection with symplectic geometry and representation theory, which will be further explored in the subsequent discussion.

To avoid confusion, let us denote the following:

- The notation $\mathfrak{g}^{\mathbb{C}}$ will be used to explicitly indicate that $\mathfrak{g}$ is considered as a complex Lie algebra, and similarly, $G^{\mathbb{C}}$ will denote a complex Lie group when necessary. When this notation is omitted, $\mathfrak{g}$ and G should be understood in a general sense or as real structures, depending on the context.
- The notation $\mathfrak{g}_{\mathbb{C}}$ denotes the complexification of the Lie algebra $\mathfrak{g}$, which in this case is a real Lie algebra.

For a more detailed study of these flag manifolds, we recommend referring to [1–3,19]. Additionally, for further geometric insights, see [4,8].

2.1 Complex flag manifolds

Let $\mathfrak{g}^{\mathbb{C}}$ be a semisimple complex Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We define the following:

- $\Pi_{\mathbb{C}}$ is a root system, where for each $\alpha \in \Pi_{\mathbb{C}}$, there exists an element $H_\alpha \in \mathfrak{h}^{\mathbb{C}}$ such that

$$\alpha(H) = \langle H_\alpha, H\rangle, \quad \forall H \in \mathfrak{h}^{\mathbb{C}},$$

  where $\langle \cdot, \cdot \rangle$ denotes the Cartan–Killing form of $\mathfrak{g}^{\mathbb{C}}$.
- $\Sigma_{\mathbb{C}}$ is a simple root system, such that $\Pi_{\mathbb{C}}^{+}$ denotes the set of positive roots in $\Pi_{\mathbb{C}}$, and $\{H_\alpha : \alpha \in \Sigma_{\mathbb{C}}\}$ forms a basis of $\mathfrak{h}^{\mathbb{C}}$.
- $\mathfrak{a}^{+}$ is the corresponding positive Weyl chamber, given by

$$\mathfrak{a}^{+} = \{H \in \mathfrak{h}^{\mathbb{C}} : \alpha(H) > 0, \quad \forall \alpha \in \Sigma_{\mathbb{C}}\}.$$

Thus, we have the root space decomposition:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \Pi_{\mathbb{C}}} \mathfrak{g}_\alpha^{\mathbb{C}},$$

where each root space is given by

$$\mathfrak{g}_\alpha^{\mathbb{C}} = \{X \in \mathfrak{g}^{\mathbb{C}} : [H, X] = \alpha(H) \cdot X, \quad \forall H \in \mathfrak{h}^{\mathbb{C}}\}.$$

The *Borel subalgebra* $\mathfrak{b}$, which is the maximal solvable subalgebra, is defined as

$$\mathfrak{b} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \Pi_{\mathbb{C}}^{+}} \mathfrak{g}_{\alpha}^{\mathbb{C}}.$$

A subalgebra $\mathfrak{p}$ of $\mathfrak{g}^{\mathbb{C}}$ is called *parabolic* if it contains a Borel subalgebra. The parabolic subalgebra associated with an element H is defined as

$$\mathfrak{p}_H = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha(H) \geq 0} \mathfrak{g}_{\alpha}^{\mathbb{C}}. \tag{2.1}$$

Remark 2.2. *In some sources, the parabolic subalgebra defined in Equation* (2.1) *is denoted by* $\mathfrak{p}_{\Theta_H}$, *where* $\Theta_H = \{\alpha \in \Sigma_{\mathbb{C}} : \alpha(H) = 0\}$.

Let $G^{\mathbb{C}}$ be a connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. The *parabolic subgroup* P_H is the normalizer of $\mathfrak{p}_H$ in $G^{\mathbb{C}}$, given by

$$P_H = \{g \in G^{\mathbb{C}} : \mathrm{Ad}(g) \cdot \mathfrak{p}_H = \mathfrak{p}_H\}.$$

The **complex flag manifold** associated with H is then defined as the quotient space:

$$\mathbb{F}_H = G^{\mathbb{C}} / P_H.$$

Furthermore, we will see that the complex flag manifold can be seen as an adjoint orbit of a compact Lie group. For instance, choosing a Weyl basis given by H_α for $\alpha \in \Sigma_{\mathbb{C}}$ and $X_\alpha \in \mathfrak{g}_{\alpha}^{\mathbb{C}}$ for $\alpha \in \Pi_{\mathbb{C}}$, we have:

- $[X_\alpha, X_{-\alpha}] = H_\alpha$,
- $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$ with $m_{\alpha,\beta} \in \mathbb{R}$, where $m_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root and $m_{\alpha,\beta} = -m_{-\alpha,-\beta}$.

Defining $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = i(X_\alpha + X_{-\alpha})$, we obtain the compact real form:

$$\mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{iH_\alpha, A_\alpha, S_\alpha : \alpha \in \Pi_{\mathbb{C}}^{+}\}.$$

Let $U = \exp \mathfrak{u}$ be a compact real form of $G^{\mathbb{C}}$, and define

$$U_H = P_H \cap U.$$

The adjoint action of U is transitive on $\mathbb{F}_H$ with isotropy subgroup U_H at H, yielding

$$\mathbb{F}_H \simeq U/U_H \simeq \mathrm{Ad}(U) \cdot H.$$

Additionally, denoting $b_H = 1 \cdot U_H$ as the origin of $\mathbb{F}_H$, its tangent space at b_H is given by

$$T_{b_H}\mathbb{F}_H = \mathrm{span}_{\mathbb{R}}\{A_\alpha, S_\alpha : \alpha(H) > 0\} = \sum_{\alpha(H)>0} \mathfrak{u}_\alpha,$$

where $\mathfrak{u}_\alpha = \left(\mathfrak{g}_\alpha^{\mathbb{C}} \oplus \mathfrak{g}_{-\alpha}^{\mathbb{C}}\right) \cap \mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{A_\alpha, S_\alpha\}$.

Remark 2.3. *Given a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$, a real Lie algebra $\mathfrak{g}_0$ is called a* real form *of $\mathfrak{g}^{\mathbb{C}}$ if its complexification satisfies $\mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{g}^{\mathbb{C}}$. A real form of $\mathfrak{g}^{\mathbb{C}}$ can be either compact or non-compact. Additionally, all compact semisimple Lie algebras are real.*

2.2 Real flag manifolds

Let $\mathfrak{g}$ be a semisimple, non-compact real Lie algebra. To construct real flag manifolds, we introduce the following fundamental elements of real semisimple Lie theory:

- Let θ be a *Cartan involution*, that is, an involutive automorphism such that the associated bilinear form

  $$B_\theta(X, Y) = -\langle X, \theta Y\rangle, \quad X, Y \in \mathfrak{g}$$

  defines an inner product on $\mathfrak{g}$, where $\langle \cdot, \cdot \rangle$ denotes the Cartan–Killing form of $\mathfrak{g}$. The Cartan involution induces a *Cartan decomposition*

  $$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s},$$

  where

  $$\mathfrak{k} = \{X \in \mathfrak{g} : \theta X = X\}, \quad \text{and} \quad \mathfrak{s} = \{Y \in \mathfrak{g} : \theta Y = -Y\}.$$

  The subspaces $\mathfrak{k}$ and $\mathfrak{s}$ are orthogonal with respect to both B_θ and the Cartan–Killing form. Notably, $\mathfrak{k}$ is often referred to as the compact component of the Cartan decomposition, although it is not necessarily compact. Furthermore, we define the maps $\kappa : \mathfrak{g} \to \mathfrak{k}$ and $\sigma : \mathfrak{g} \to \mathfrak{s}$, given by

  $$\kappa(X) = \frac{X + \theta X}{2}, \quad \text{and} \quad \sigma(X) = \frac{X - \theta X}{2},$$

  which correspond to the parallel projections onto $\mathfrak{k}$ and $\mathfrak{s}$, respectively.

- Let $\mathfrak{a} \subset \mathfrak{s}$ be a *maximal Abelian subalgebra*. Then, there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that contains $\mathfrak{a}$. Given a pair $(\theta, \mathfrak{a})$, we denote by $\Pi_{\mathbb{R}}$ the set of roots associated with $(\theta, \mathfrak{a})$, where each root is a linear functional $\alpha : \mathfrak{a} \to \mathbb{R}$ satisfying

$$B_\theta(H_\alpha, H) = \alpha(H), \quad \forall H \in \mathfrak{a}.$$

These roots can be interpreted as restrictions of the roots of $\mathfrak{h}^{\mathbb{C}}$, the Cartan subalgebra of the complexification of $\mathfrak{g}$, denoted as $\mathfrak{g}_{\mathbb{C}}$.

- The *Weyl group* associated with $\mathfrak{a}$ is the finitely generated group of reflections across the hyperplanes defined by $\alpha = 0$ in $\mathfrak{a}$, for α in the root system of $\mathfrak{a}$. The generators of the Weyl group corresponding to these reflections are called *simple reflections*.

- The *Weyl chambers* associated with $(\theta, \mathfrak{a})$ are the connected components of

$$\{H \in \mathfrak{a} : \alpha(H) \neq 0, \quad \forall \alpha \in \Pi_{\mathbb{R}}\}.$$

Selecting one of these chambers as the *positive Weyl chamber* $\mathfrak{a}^+$, we define the set of positive roots as

$$\Pi_{\mathbb{R}}^+ = \{\alpha \in \Pi_{\mathbb{R}} : \alpha|_{\mathfrak{a}^+} > 0\}.$$

Consequently, we define

$$\mathfrak{n} = \sum_{\alpha \in \Pi_{\mathbb{R}}^+} \mathfrak{g}_\alpha, \quad \text{and} \quad \mathfrak{n}^- = \sum_{\alpha \in \Pi_{\mathbb{R}}^+} \mathfrak{g}_{-\alpha},$$

where $\theta \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$ and $\theta \mathfrak{n} = \mathfrak{n}^-$. Furthermore, there exists a simple root system $\Sigma_{\mathbb{R}}$ associated with $\mathfrak{a}^+$, such that $\{H_\alpha \in \mathfrak{a} : \alpha \in \Sigma_{\mathbb{R}}\}$ forms a basis of $\mathfrak{a}$.

Moreover, we obtain the B_θ-orthogonal decomposition

$$\mathfrak{s} = \mathfrak{a} \oplus \sigma(\mathfrak{n}).$$

The triplet $(\theta, \mathfrak{a}, \mathfrak{a}^+)$ is called an *admissible triple* of $\mathfrak{g}$, and it gives rise to the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

known as the *Iwasawa decomposition*. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. If K, A, and N are the connected subgroups generated by $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$, respectively, then G is diffeomorphic to $K \times A \times N$. This leads to the *global Iwasawa decomposition*:

$$G = KAN.$$

For $H \in \mathrm{cl}(\mathfrak{a}^+)$, we define

$$\mathfrak{n}_H^+ = \sum_{\alpha(H)>0} \mathfrak{g}_\alpha, \quad \text{and} \quad \mathfrak{n}_H^- = \sum_{\alpha(H)<0} \mathfrak{g}_\alpha.$$

Remark 2.4. *If $H \in \mathfrak{a}^+$, i.e., H is a regular element, then $\mathfrak{n} = \mathfrak{n}_H^+$ and $\mathfrak{n}^- = \mathfrak{n}_H^-$. In some literature, $\mathfrak{n}_H^+$ is denoted by $\mathfrak{n}_\Theta^+$, where $\Theta = \{\alpha \in \Sigma_{\mathbb{R}} : \alpha(H) = 0\}$.*

Given an admissible triple $(\theta, \mathfrak{a}, \mathfrak{a}^+)$, *the parabolic subalgebra associated with* $H \in \mathrm{cl}(\mathfrak{a}^+)$ is

$$\mathfrak{p}_H = \mathfrak{k}_H \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. The parabolic subgroup associated with H is defined as the normalizer of $\mathfrak{p}_H$ in G. By the global Iwasawa decomposition of G, we obtain:

$$K_H = \{k \in K : \mathrm{Ad}(k)|_{\mathfrak{a}_H} = \mathrm{id}_{\mathfrak{a}_H}\}$$

where $\mathfrak{a}_H = \mathfrak{a} \ominus \mathfrak{a}(H)$ and $\mathfrak{a}(H)$ be a subalgebra generated by $\{H_\alpha : \alpha(H) \neq 0\}$. Then, the parabolic subgroup P_H is given by:

$$P_H = K_H \cdot A \cdot N.$$

Consequently, we have the quotient structure:

$$G/P_H = \frac{K \cdot A \cdot N}{K_H \cdot A \cdot N} \simeq K/K_H,$$

and it follows that:

$$K/K_H \simeq \mathrm{Ad}(K) \cdot H$$

which represents the K-adjoint orbit passing through H, commonly known as the **real flag manifold**.

Remark 2.5. *Given $\widetilde{H} \in \mathfrak{s}$, we have that $\mathrm{Ad}(K) \cdot \widetilde{H} \cap \mathrm{cl}(\mathfrak{a}^+) \neq \varnothing$. Since the action of K is transitive, we can choose an element $H \in \mathrm{cl}(\mathfrak{a}^+)$ which determines the same manifold.*

Remark 2.6. *We denote by $\mathbb{F}_H$ the flag manifold passing through $H \in \mathrm{cl}(\mathfrak{a}^+)$ when there is no ambiguity regarding the compact group acting on it. Otherwise, we will specify it as an adjoint orbit. To maintain clarity, we will represent flag manifolds in terms of the adjoint action (as the orbit of U in the complex case and K in the real case).*

3 Lagrangian immersion of real flags on complex flag

In this section, we investigate the conditions under which a given real flag manifold can be realized as a Lagrangian submanifold within a complex flag manifold. Specifically, given an adjoint orbit $\mathrm{Ad}(K)\cdot H$ corresponding to a real flag manifold, we determine in which complex flag manifolds it can be immersed as a Lagrangian submanifold. Importantly, this classification depends on the choice of H, which we analyze using Satake diagrams, as well as the structural properties of K. Contrary to a universal embedding, our approach highlights the interplay between the choice of H and the ambient complex flag manifold.

As discussed in [3], given a compact semisimple Lie group U with Lie algebra $\mathfrak{u}$, the adjoint orbits of U in $\mathfrak{u}$ correspond to the flag manifolds of its complexified Lie group $U_{\mathbb{C}}$, whose Lie algebra is $\mathfrak{u}_{\mathbb{C}}$. These adjoint orbits naturally inherit a symplectic structure, providing a geometric foundation for our analysis.

The *Kostant–Kirillov–Souriau (KKS) symplectic form* on an adjoint orbit $\mathrm{Ad}(U)\cdot H$ is given by

$$\omega_x\left(\widetilde{X}(x),\widetilde{Y}(x)\right)=\langle x,[X,Y]\rangle_{\mathfrak{u}},\quad X,Y\in\mathfrak{u},\tag{3.1}$$

where $\langle\cdot,\cdot\rangle_{\mathfrak{u}}$ denotes the Cartan–Killing form on $\mathfrak{u}$, and $\widetilde{X}=\mathrm{ad}(X)$ represents the Hamiltonian vector field associated with the Hamiltonian function $H_X(x)=\langle x,X\rangle_{\mathfrak{u}}$. As a consequence, the moment map μ of the U-adjoint action is simply the identity map, which is inherently equivariant.

To identify specific *isotropic submanifolds* within $\mathrm{Ad}(U)\cdot H$, we rely on the following key result:

Proposition 3.1. *Let (M,ω) be a connected symplectic manifold equipped with a Hamiltonian action of a Lie group G, given by $G\times M\to M$, along with an equivariant moment map μ. Let $L\subset G$ be a Lie subgroup.*

Then, the orbit $L\cdot x$ is isotropic if and only if $\mu(x)$ belongs to the annihilator $(\mathfrak{l}')^0$ of the derived algebra $\mathfrak{l}'$ of $\mathfrak{l}$.

This proposition was established in [13] and [14] using distinct methodologies.

3.1 Lagrangian immersion of real flags

Let U be a compact semisimple Lie group with Lie algebra $\mathfrak{u}$, and let $\mathfrak{k}\subset\mathfrak{u}$ be a Lie subalgebra. The pair $(\mathfrak{u},\mathfrak{k})$ is called a **symmetric pair** if

$$[\mathfrak{k},\mathfrak{k}^\perp]\subset\mathfrak{k}^\perp,\quad\text{and}\quad[\mathfrak{k}^\perp,\mathfrak{k}^\perp]\subset\mathfrak{k},$$

where $\perp$ denotes the orthogonal complement with respect to the Cartan–Killing form on $\mathfrak{u}$.

For any symmetric pair $(\mathfrak{u}, \mathfrak{k})$, if we define $K = \langle \exp \mathfrak{k} \rangle$, then the quotient space U/K forms a symmetric space. The *dual symmetric pair* is given by $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{g}$ is a non-compact semisimple Lie algebra that serves as the real form of $\mathfrak{u}_{\mathbb{C}}$ and admits a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}, \quad \text{where} \quad \mathfrak{s} = i\mathfrak{k}^{\perp} \subset \mathfrak{u}_{\mathbb{C}}.$$

By construction, the orbits of the K-isotropy representation on $\mathfrak{s}$ (or equivalently on $\mathfrak{k}^{\perp}$) correspond to the flag manifolds of $\mathfrak{g}$.

Given $H \in \mathfrak{k}^{\perp}$, the Lagrangian immersion of real flag manifolds into their corresponding complex flag manifolds is constructed as follows: Let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal abelian subalgebra. Then, there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{h}$ and $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Consequently, for $H \in \mathfrak{a}$, we obtain

$$K/K_H = \mathrm{Ad}(K) \cdot H \hookrightarrow \mathrm{Ad}(U) \cdot iH = U/U_H = \mathbb{F}_H. \tag{3.2}$$

Thus, the flag manifolds of $\mathfrak{g}$ are determined by the adjoint action of K on H and are immersed in the flag manifolds of $\mathfrak{g}_{\mathbb{C}}$ (complexification), which are determined by the adjoint action of U on iH. Moreover, since $\mathfrak{u}$ is compact, the connected component of the identity of $\mathfrak{k}'^{\perp}$ corresponds to the orthogonal complement of $\mathfrak{k}'$ with respect to the invariant scalar product on $\mathfrak{u}$. Consequently, we arrive at the following proposition:

Proposition 3.2. *Given a symmetric pair* $(\mathfrak{u}, \mathfrak{k})$ *and an element* $H \in \mathfrak{a} \subset i\mathfrak{k}^{\perp}$*, the real flag manifold* $\mathrm{Ad}(K) \cdot H$ *is a Lagrangian submanifold of* $\mathbb{F}_H$ *with respect to the Kirillov-Kostant-Souriau (KKS) symplectic form.*

Proof. Since $\mathfrak{k}' \subset \mathfrak{k}$, then $\mathfrak{k}^{\perp} \subset (\mathfrak{k}')^{\perp}$ and $\mathrm{Ad}(K) \cdot H \subset \mathfrak{k}^{\perp} = i\mathfrak{s}$, then $\mathrm{Ad}(K) \cdot H \cap (\mathfrak{k}')^{\perp} \neq \varnothing$ and by Proposition 3.1, the adjoint K-orbit (real flag) is an isotropic submanifold.

Furthermore, if $b_H = 1 \cdot K$, we have that

$$\dim\left(T_{b_H} \mathrm{Ad}(K) \cdot H\right) = \dim\left(\sum_{\alpha(H)<0} \mathfrak{g}_{\alpha}\right) = \#\left\{\alpha \in \Pi_{\mathbb{C}} : \ \alpha(H) < 0\right\},$$

and as the root spaces of $\mathfrak{g}_{\mathbb{C}}$ are 1-dimensional complex spaces (*i.e.*, 2-dimensional real spaces), then

$$2 \dim_{\mathbb{R}}\left(\mathrm{Ad}(K) \cdot H\right) = \dim_{\mathbb{R}}(\mathbb{F}_H).$$

Hence $\mathrm{Ad}(K) \cdot H$ is a Lagrangian submanifold of $\mathbb{F}_H$. ☐

Remark 3.3. *Intuitively, one can observe that each complex root space effectively doubles its dimension when considered as a real vector space. However, this identification is purely at the level of vector spaces and does not yet take into account the underlying Lie algebraic or geometric structure. In [16, 21, 23], the authors provide a detailed exposition of this vector space approach, emphasizing how the real and complex structures relate in the context of flag manifolds.*

Our focus now shifts to identifying the complex flag manifolds of $\mathfrak{g}_{\mathbb{C}}$ (complexification of $\mathfrak{g}$ real non-compact semi-simple) that admit a real flag manifold, generated by the action of $K = \langle \exp \mathfrak{k} \rangle$ for the symmetric pair $(\mathfrak{u}, \mathfrak{k})$, as a Lagrangian submanifold. Consider a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{s}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{h}$. Let $\Pi_{\mathbb{C}}$ be the set of roots of $\mathfrak{h}_{\mathbb{C}}$, where the roots of $\mathfrak{a}$ correspond to their restrictions on $\mathfrak{h}_{\mathbb{C}}$.

If θ is a Cartan involution associated with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, then there exists an involutive extension of θ to $\mathfrak{g}_{\mathbb{C}}$, which we also denote by θ. As shown in [21], the restriction of $\Pi_{\mathbb{C}}$ to $\mathfrak{a}$ is given by

$$P = \frac{1}{2}(1 - \theta^*), \quad \text{where} \quad \theta^* \alpha = \alpha \circ \theta.$$

Define $\Pi_{\mathrm{im}} \subset \Pi_{\mathbb{C}}$ as the set of *imaginary roots*, where $\alpha \in \Pi_{\mathrm{im}}$ if and only if $P(\alpha) = 0$. Letting $\Pi_{\mathrm{co}} = \Pi_{\mathbb{C}} \setminus \Pi_{\mathrm{im}}$, the set of restricted roots is given by $P(\Pi_{\mathrm{co}})$.

Considering an appropriate ordering (such as the lexicographic order on $\mathfrak{a}^*$), let Σ_{im} denote the system of imaginary simple roots, and let Σ_{co} be its complement. The projection of Σ_{co} onto $\mathfrak{a}^*$ forms a *system of restricted roots* Σ, with $\mathfrak{a}^+$ denoting the positive Weyl chamber of $\mathfrak{g}$ determined by Σ.

For $H \in \mathrm{cl}(\mathfrak{a}^+)$, define

$$\Theta_H = \{\beta \in \Sigma : \beta(H) = 0\} \subset \Sigma.$$

Next, define $\widetilde{\Theta}_H \subset \Sigma_{\mathbb{C}}$ by

$$\widetilde{\Theta}_H = P^{-1}(\Theta_H) \cup \Sigma_{\mathrm{im}}, \tag{3.3}$$

which is determined by the **Satake diagram** of $\mathfrak{g}$ (see [16, 21]).

Remark 3.4. *In general, we select $H \in \mathrm{cl}(\mathfrak{a}^+)$ because for any other choice $H' \in \mathfrak{a}$ such that the orbits remain the same, there exists a Weyl conjugation σ satisfying $\sigma \cdot H = H'$. Consequently, the associated sets of admissible roots remain unchanged, i.e., $\widetilde{\Theta}_H = \widetilde{\Theta}_{H'}$.*

Proposition 3.5. $\widetilde{\Theta}_H = \{\alpha \in \Sigma_{\mathbb{C}} : \alpha(H) = 0\}$.

Proof. If $H \in \mathfrak{a}$, then for all $\alpha \in \Sigma_{\mathbb{C}}$

$$\theta^*\alpha(H) = \alpha \circ \theta(H) = -\alpha(H), \tag{3.4}$$

because $\theta|_{\mathfrak{s}} = -\,\mathrm{id}$. Also, if $\alpha \in \Sigma_{\mathrm{im}}$, then $\theta^*\alpha = \alpha$, and by (3.4) we have that $\alpha(H) = 0$, therefore it is enough to see for roots in Σ_{co}. If $\alpha \in P^{-1}(\Theta_H)$, then $(\alpha - \theta^*\alpha)(H) = 0$ implies that $\alpha(H) = \theta^*\alpha(H)$, and by (3.4) we have that $\alpha(H) = 0$. Thus $\widetilde{\Theta}_H \subseteq \{\alpha \in \Sigma_{\mathbb{C}} : \alpha(H) = 0\}$. Conversely, if $\alpha \in \Sigma_{\mathrm{co}}$ such that $\alpha(H) = 0$, then $\theta^*\alpha(H) = -\alpha(H) = 0$, thus $P(\alpha)(H) = 0$ and implies that $P(\alpha) \in \Theta_H$, *i.e.* $\alpha \in P^{-1}(\Theta_H)$. □

Therefore,

Theorem 3.6. *Given a symmetric pair* $(\mathfrak{u}, \mathfrak{k})$, *the complex flags of* $\mathfrak{u}_{\mathbb{C}}$ *of type* $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$ *admit, as Lagrangian submanifold, the real flag of* $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}^{\perp}$ *of type* $\Theta \subset \Sigma$ *if and only if*

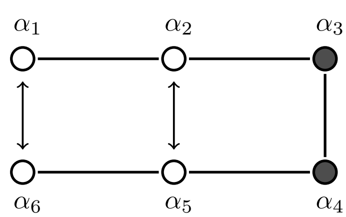
$$\widetilde{\Theta} = P^{-1}(\Theta) \cup \Sigma_{\mathrm{im}}.$$

That is, $\widetilde{\Theta}$ *is determined by the Satake diagram of* $\mathfrak{g}$.

In particular, we can conclude

Corollary 3.7. *A maximal flag* $\mathbb{F}$ *of* $\mathfrak{g}_{\mathbb{C}}$ *admits a real flag* $\mathrm{Ad}(K) \cdot H$ *as Lagrangian submanifold if and only if* $\Sigma_{\mathrm{im}} = \varnothing$ *and* $\varnothing = \Theta_H$.

Example 3.8. *Let* $\mathfrak{u} = \mathfrak{su}(7)$, $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(5))$ *and* $\mathfrak{g} = \mathfrak{su}(2,5)$ *that determine the symmetric pair* $(\mathfrak{u}, \mathfrak{k})$ *and its respective dual symmetric pair* $(\mathfrak{g}, \mathfrak{k})$. *The Satake diagram of* $\mathfrak{su}(2,5)$ *is*



By Theorem 3.6, the flags of type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$ *that admit as Lagrangian submanifold a real flag of type* $\Theta \subset \Sigma = \{\beta_1 = P(\alpha_1) = P(\alpha_6), \beta_2 = P(\alpha_2) = P(\alpha_5)\}$ *are*

- *If* $\Theta_0 = \varnothing$, *then* $\widetilde{\Theta}_0 = \Sigma_{\mathrm{im}} = \{\alpha_3, \alpha_4\}$.
- *If* $\Theta_1 = \{\beta_1\}$, *then* $\widetilde{\Theta}_1 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}$.
- *If* $\Theta_2 = \{\beta_2\}$, *then* $\widetilde{\Theta}_2 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

Analogously, this is equivalent to that given in the Table 1, for $n = 7$:

- $\widetilde{\Theta}_0 = \Sigma_{\mathbb{C}} \setminus \{\alpha_1, \alpha_2, \alpha_{n-2}, \alpha_{n-1}\}$,
- $\widetilde{\Theta}_1 = \Sigma_{\mathbb{C}} \setminus \{\alpha_2, \alpha_{n-2}\}$,
- $\widetilde{\Theta}_2 = \Sigma_{\mathbb{C}} \setminus \{\alpha_1, \alpha_{n-1}\}$.

Hence, using the Satake diagrams we can determine which are the complex flags of type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$, for which there exists Θ such that Theorem 3.6 is satisfied.

Corollary 3.9. *The complex flags of type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$ admits as Lagrangian submanifold a real flag given by the K-adjoint orbit if and only if $\widetilde{\Theta}$ appears in Table 1.*

Remark 3.10. *Corollary 3.9 states that given a complex flag manifold $\mathbb{F}_H$ associated with the semisimple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, we can determine which real flag manifolds of $\mathfrak{g}_0$ are Lagrangian submanifolds of $\mathbb{F}_H$ by analyzing the Satake diagram of $\mathfrak{g}_0$. Here, $\mathfrak{g}_0$ denotes a real form of $\mathfrak{g}^{\mathbb{C}}$.*

The proof of this result is given in the following subsection. For that we will use a convenient notation of partitioning an integer, that is, we define $\flat(n)$ for $n \in \mathbb{N}$, as the set of ordered l-tuples of integers $(n_1, \dots, n_l)$ such that $0 < n_1 < \dots < n_l \leq n$, for example:

$$\flat(3) = \{(1), (2), (3), (1,2), (1,3), (2,3), (1,2,3)\}.$$

Using this notation, we build the Table 1. The case-by-case analysis used to construct Table 1 is detailed in Appendix A.

4 Infinitesimally tight

In this section, we establish the main result of this paper. Specifically, we demonstrate that the Lagrangian submanifolds listed in Table 1 are infinitesimally tight. To achieve this, we compute the sum of the $\mathbb{Z}_2$-Betti numbers of the real flag manifolds and identify the transversal elements. To lay the groundwork for our proof, we first provide the necessary definitions to understand Schubert cells, which play a fundamental role in computing the homology of real flag manifolds. This exposition is based on [5] and [9].

Let $\mathfrak{g}$ be a semisimple non-compact real Lie algebra, and let $\mathcal{W}$ be the Weyl group associated with a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}$, with Σ denoting the corresponding system of simple roots.

Table 1: Complex flags that admit a Lagrangian immersion of the real flag determined by the action of $K = \exp \mathfrak{k}$.

	$\mathfrak{g}_{\mathbb{C}}$	$\mathfrak{g}$	$\mathfrak{k}$	Flags type $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$
A	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{so}(n)$	All possibilities
	$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(n)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2n_1},\dots,\alpha_{2n_j} : (n_1,\dots,n_j) \in \flat(n-1)\right\}$
	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(k,n-k)$, $k < n$	$\mathfrak{s}\left(\mathfrak{u}(k) \oplus \mathfrak{u}(n-k)\right)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j},\alpha_{n-n_j},\dots,\alpha_{n-n_1} : (n_1,\dots,n_j) \in \flat(k)\right\}$
	$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{su}(n,n)$	$\mathfrak{s}\left(\mathfrak{u}(n) \oplus \mathfrak{u}(n)\right)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j},\alpha_n,\alpha_{2n-n_j},\dots,\alpha_{2n-n_1} : (n_1,\dots,n_j) \in \flat(n-1)\right\}$, or $\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j},\alpha_{2n-n_j},\dots,\alpha_{2n-n_1} : (n_1,\dots,n_j) \in \flat(n-1)\right\}$
B	$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(n,n+1)$	$\mathfrak{so}(n) \oplus \mathfrak{so}(n+1)$	All possibilities
		$\mathfrak{so}(k,2n+1-k)$, $k \le n$	$\mathfrak{so}(k) \oplus \mathfrak{so}(2n+1-k)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j} : (n_1,\dots,n_j) \in \flat(k)\right\}$
C	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{u}(n)$	All possibilities
		$\mathfrak{sp}(k,n-k)$, $k \le n-k$	$\mathfrak{sp}(k) \times \mathfrak{sp}(n-k)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2n_1},\dots,\alpha_{2n_j} : (n_1,\dots,n_j) \in \flat(k)\right\}$
D	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(n,n)$	$\mathfrak{so}(n) \oplus \mathfrak{so}(n)$	All possibilities
		$\mathfrak{so}(k,2n-k)$, $k < n-1$	$\mathfrak{so}(k) \oplus \mathfrak{so}(2n-k)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j} : (n_1,\dots,n_j) \in \flat(k)\right\}$
		$\mathfrak{so}(n-1,n+1)$,	$\mathfrak{so}(n-1) \oplus \mathfrak{so}(n+1)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j},\alpha_{n-1},\alpha_n : (n_1,\dots,n_j) \in \flat(n-2)\right\}$, or $\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{n_1},\dots,\alpha_{n_j} : (n_1,\dots,n_j) \in \flat(n-2)\right\}$
		$\mathfrak{so}^*(2n)$, n even	$\mathfrak{u}(n)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2n_1},\dots,\alpha_{2n_j} : (n_1,\dots,n_j) \in \flat(k)\right\}$
		$\mathfrak{so}^*(2n)$, n odd	$\mathfrak{u}(n)$	$\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2n_1},\dots,\alpha_{2n_j},\alpha_{n-1},\alpha_n : (n_1,\dots,n_j) \in \flat\left(\frac{n-3}{2}\right)\right\}$, or $\Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2n_1},\dots,\alpha_{2n_j} : (n_1,\dots,n_j) \in \flat\left(\frac{n-3}{2}\right)\right\}$
E	E_6	E_6^6	$\mathfrak{sp}(4)$	All possibilities
		E_6^2	$\mathfrak{su}(2) \oplus \mathfrak{su}(6)$	See appendix A
		E_6^{-14}	$\mathfrak{so}(10) \oplus \mathbb{R}$	See appendix A
		E_6^{-26}	F_4	See appendix A
	E_7	E_7^7	$\mathfrak{su}(8)$	All possibilities
		E_7^{-5}	$\mathfrak{su}(2) \oplus \mathfrak{so}(12)$	See appendix A
		E_7^{-25}	$E_6 \oplus \mathbb{R}$	See appendix A
	E_8	E_8^8	$\mathfrak{so}(16)$	All possibilities
		E_8^{-24}	$\mathfrak{su}(2) \oplus E_7$	See appendix A
F	F_4	F_4^4	$\mathfrak{su}(2) \oplus \mathfrak{sp}(3)$	All possibilities
		F_4^{-20}	$\mathfrak{so}(9)$	See appendix A
G	G_2	G_2^2	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	All possibilities

- For $\Theta \subseteq \Sigma$, the subgroup $\mathcal{W}_\Theta$ of $\mathcal{W}$ is generated by the roots in Θ. This subgroup acts transitively on the cosets of $\mathcal{W}$.

- Given $\Theta \subseteq \Sigma$, the *Bruhat decomposition* of the real flag manifold $\mathbb{F}_\Theta = G/P_\Theta$ expresses it as the disjoint union of N-orbits, where N is determined by the Iwasawa decomposition of $P_\Theta = K_\Theta AN$. That is,

$$\mathbb{F}_\Theta = \coprod_{w \in \mathcal{W}/\mathcal{W}_\Theta} N \cdot w b_\Theta,$$

  where the equivalence relation $N \cdot w_1 b_\Theta = N \cdot w_2 b_\Theta$ if $w_1 \cdot \mathcal{W}_\Theta = w_2 \cdot \mathcal{W}_\Theta$ holds.

- Each N-orbit passing through $w \in \mathcal{W}$ is diffeomorphic to a Euclidean space, and the orbit $N \cdot w b_\Theta$ is referred to as a *Bruhat cell.*

- Every Bruhat cell is open and dense in $\mathbb{F}_\Theta$.

- The *Schubert cell* associated with $w \in \mathcal{W}/\mathcal{W}_\Theta$ is denoted by S_w^Θ and defined as

$$S_w^\Theta = \mathrm{cl}(N \cdot w b_\Theta), \quad w \in \mathcal{W}/\mathcal{W}_\Theta.$$

Using the Schubert cells S_w^Θ, the authors of [20] introduced a boundary map ∂, which was employed to compute the homology of the real flag manifold $\mathbb{F}_\Theta$. In particular, for any $H \in \mathrm{cl}(\mathfrak{a}^+)$, there exists a subset $\Theta_H \subset \Sigma$ such that the $\mathbb{Z}_2$-homology of $\mathbb{F}_{\Theta_H} = \mathrm{Ad}(K) \cdot H$ is freely generated by the *Schubert cells* $\mathcal{S}_w^{\Theta_H}$, where $w \in \mathcal{W}/\mathcal{W}_{\Theta_H}$.

Therefore,

$$\mathrm{SB}\left(\mathrm{Ad}(K) \cdot H, \mathbb{Z}_2\right) = \#\left(\mathcal{W}/\mathcal{W}_{\Theta_H}\right), \tag{4.1}$$

That is, the cardinality of the quotient $\mathcal{W}/\mathcal{W}_\Theta$.

Since $\mathrm{Ad}(K) \cdot H \subseteq \mathfrak{s} = i\mathfrak{k}^\perp$, for $x \in \mathrm{Ad}(K) \cdot H$ we have:

$$T_x\left(\mathrm{Ad}(K) \cdot H\right) = \{\mathrm{ad}(A)(x) : A \in \mathfrak{k}\}.$$

Then,

- If $X \in \mathfrak{k}^\perp$, then $\widetilde{X} = \mathrm{ad}(X)$ is a Hamiltonian field of the function $H_X = \langle X, x\rangle$. Thus the singularities of X are the singularities of H_X, and their number is finite, if and only if X is regular.

  Therefore, the transversal elements are the regular elements X, and they satisfy

$$\#\left(f_{\mathrm{Ad}(K)\cdot H}(X)\right) = \#\left(\mathcal{W}/\mathcal{W}_{\Theta_H}\right).$$

- If $Y \in \mathfrak{k}$, then $\widetilde{Y}$ is tangent, thus it cannot be transversal.

- If $Z = X + Y$ for $X \in \mathfrak{k}^{\perp}$ and $Y \in \mathfrak{k}$, then $\widetilde{Z}(x) \notin T_x \operatorname{Ad}(K) \cdot H$ if $\widetilde{X}(x) \neq 0$, so for Z to have singularity in x we need that $\widetilde{X}(x) = \widetilde{Y}(x) = 0$ in a finite quantity. But this only happens for X regular, such that $[X, Y] = 0$. Thus:

$$\#\left(f_{\operatorname{Ad}(K)\cdot H}(Z)\right) = \#\left(\mathcal{W}/\mathcal{W}_{\Theta_H}\right).$$

Consequently,

Theorem 4.1. *The real flags are infinitesimally tight submanifolds of their corresponding complex flag manifolds, as listed in Table 1.*

As a result of Theorem 1.5, we have:

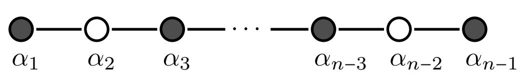
Corollary 4.2. *The real flags are locally tight submanifolds of their corresponding complex flag manifolds.*

A Appendix

In this appendix, we analyze each Satake diagram case by case to identify all complex flag manifolds that permit the Lagrangian immersion of the corresponding real flag, as determined by the possible symmetric pairs. This analysis culminates in the construction of Table 1, where for the classical cases AI, CI, G_2, F_4I, E_6I, E_7I, and E_8I, all possible sets $\widetilde{\Theta} \subset \Sigma_{\mathbb{C}}$ are admissible.

Type AII

In this case, we have $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{H})$, with $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2n, \mathbb{C})$. The Satake diagram is represented as:



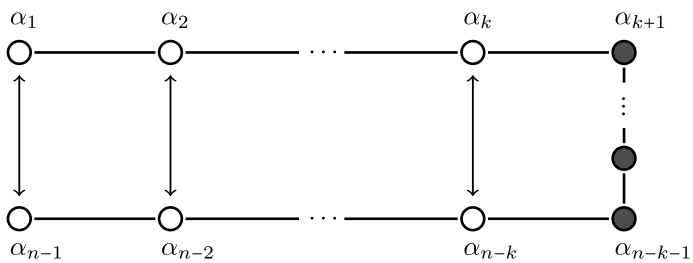
Here, $\Sigma_{\text{im}} = \{\alpha_{2j-1} : 1 \leq j \leq n\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) : 1 \leq j \leq n-1\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : (s_1, \ldots, s_l) \in \flat(n-1)\}. \tag{A.1}$$

Type *AIII*

For $\mathfrak{g} = \mathfrak{su}(k, n-k)$

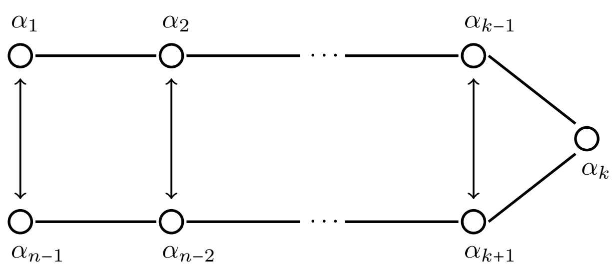
- If $k < n-k$, the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_j : k < j < n-k\}$ and $\Sigma = \{\beta_j = P(\alpha_j) = P(\alpha_{n-j}) : 1 \le j \le k\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_{n-s_l}, \ldots, \alpha_{n-s_1} : (s_1, \ldots, s_l) \in \flat(k)\}. \tag{A.2}$$

- If $k = n-k$, the Satake diagram is



As $\Sigma_{\text{im}} = \varnothing$ and $\Sigma = \{\beta_j = P(\alpha_j) = P(\alpha_{n-j}),\ \beta_k = P(\alpha_k) : 1 \le j \le k-1\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

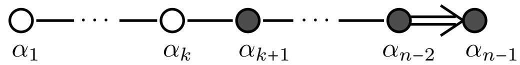
$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_{n-s_l}, \ldots, \alpha_{n-s_1} :\ (s_1, \ldots, s_l) \in \flat(k-1)\}, \tag{A.3}$$

or

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_k, \alpha_{n-s_l}, \ldots, \alpha_{n-s_1} : (s_1, \ldots, s_l) \in \flat(k-1)\}. \tag{A.4}$$

Type B

For $\mathfrak{g} = \mathfrak{so}(k, 2n+1-k)$, then the Satake diagram is



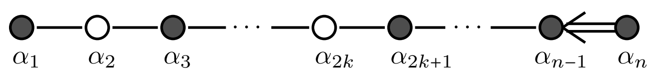
As $\Sigma_{\text{im}} = \{\alpha_j : k < j \leq n\}$ and $\Sigma = \{\beta_j = P(\alpha_j) : 1 \leq j \leq k\}$. If $k = n$ then $\mathfrak{g}$ is normal, and the sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \smallsetminus \{\alpha_{s_1}, \ldots, \alpha_{s_l} : (s_1, \ldots, s_l) \in \flat(k)\}. \tag{A.5}$$

Type CII

For $\mathfrak{g} = \mathfrak{sp}(k, n-k)$.

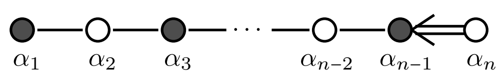
- If $k < n-k$, the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_{2j-1}, \alpha_q : 1 \leq j \leq k,\ q > 2k\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) : 1 \leq j \leq k\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \smallsetminus \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : (s_1, \ldots, s_l) \in \flat(k)\}. \tag{A.6}$$

- If $n = 2m$ and $k = m$, the Satake diagram is



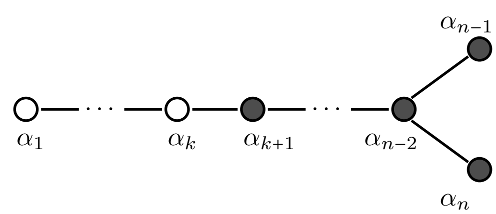
As $\Sigma_{\text{im}} = \{\alpha_{2j-1} :\ 1 \leq j \leq m\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) :\ 1 \leq j \leq m\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \smallsetminus \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : (s_1, \ldots, s_l) \in \flat(k)\}. \tag{A.7}$$

Type DI

For $\mathfrak{g} = \mathfrak{so}(k, 2n-k)$.

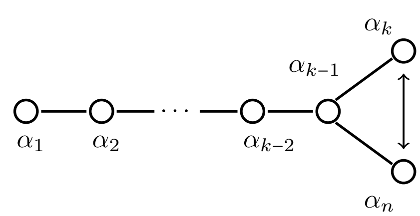
- If $k = n$ then $\mathfrak{g}$ is a normal form.
- If $k < n-1$ then the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_j : j > k\}$ and $\Sigma = \{\beta_j = P(\alpha_j) : 1 \leq j \leq k\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \dots, \alpha_{s_l} : (s_1, \dots, s_l) \in \flat(k)\}. \tag{A.8}$$

- If $k = n-1$ then the Satake diagram is



As $\Sigma_{\text{im}} = \varnothing$ and $\Sigma = \{\beta_j = P(\alpha_j),\ \beta_k = P(\alpha_k) = P(\alpha_n) :\ 1 \leq j < k\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \dots, \alpha_{s_l} : (s_1, \dots, s_l) \in \flat(k-1)\}, \tag{A.9}$$

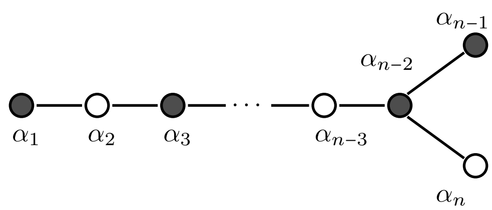
or

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{s_1}, \dots, \alpha_{s_l}, \alpha_k, \alpha_n : (s_1, \dots, s_l) \in \flat(k-1)\}. \tag{A.10}$$

Type DII

For $\mathfrak{g} = \mathfrak{so}^*(2n)$.

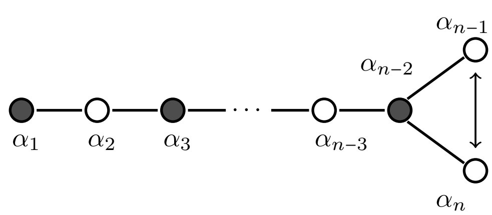
- If n is even, the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_j : j \text{ is odd}\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) : 1 \le j \le n\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : (s_1, \ldots, s_l) \in \flat(k)\}. \tag{A.11}$$

- If n is odd, the Satake diagram is



As $\Sigma_{\text{im}} = \{\alpha_j : j \text{ is odd and } j < n\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}), \beta_k = P(\alpha_{n-1}) = P(\alpha_n) : 1 \le j \le k,\ k = (n-1)/2\}$. The sets $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are given by:

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : (s_1, \ldots, s_l) \in \flat\left(\frac{n-3}{2}\right)\right\}, \tag{A.12}$$

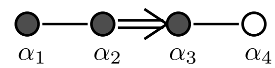
or

$$\widetilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2s_1}, \ldots, \alpha_{2s_l}, \alpha_{n-1}, \alpha_n : (s_1, \ldots, s_l) \in \flat\left(\frac{n-3}{2}\right)\right\}. \tag{A.13}$$

Exceptional cases

Type *F4II*

For $\mathfrak{g} = F_4^{-20}$, then the Satake diagram is

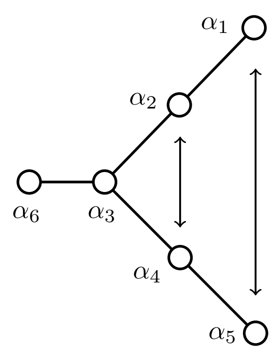


Therefore the only non-trivial possibility of $\widetilde{\Theta}$ that satisfy the Theorem 3.6 is

$$\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3\} = \Sigma_{\text{im}}. \tag{A.14}$$

Type *E6II*

For $\mathfrak{g} = E_6^2$, then the Satake diagram is

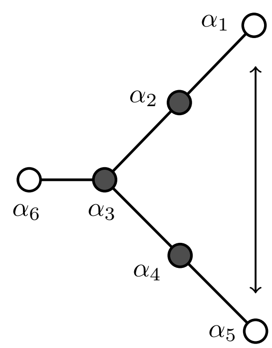


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy Theorem 3.6 are:

- $\widetilde{\Theta} = \varnothing$,
- $\widetilde{\Theta} = \{\alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_3\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_4\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_5\}$,
- $\widetilde{\Theta} = \{\alpha_3, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_5, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_5\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

Type $E6III$

For $\mathfrak{g} = E_6^{-14}$, then the Satake diagram is

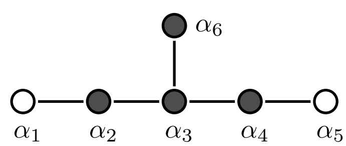


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are:

- $\widetilde{\Theta} = \{\alpha_2, \alpha_3\alpha_4\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \ \}$.

Type $E6IV$

For $\mathfrak{g} = E_6^{-26}$, then the Satake diagram is

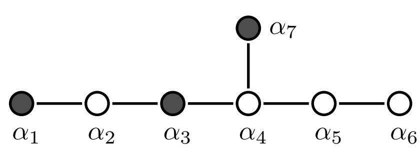


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are:

- $\widetilde{\Theta} = \{\alpha_2, \alpha_3\alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \ \}$.

Type $E7II$

For $\mathfrak{g} = E_7^{-5}$, then the Satake diagram is

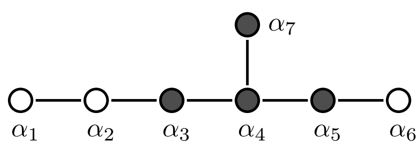


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are:

- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$.
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7\}$.

Type *E7III*

For $\mathfrak{g} = E_7^{-25}$, then the Satake diagram is

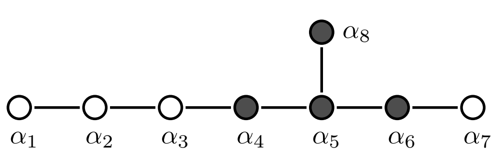


Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are:

- $\widetilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$.

Type *E8II*

For $\mathfrak{g} = E_8^{-24}$, then the Satake diagram is



Therefore the non-trivial possibilities for $\widetilde{\Theta}$ that satisfy the Theorem 3.6 are:

- $\widetilde{\Theta} = \{\alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$,
- $\widetilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$.

References

[1] D. V. Alekseevsky, "Flag manifolds," 1997, vol. 6(14), pp. 3–35, 11th Yugoslav Geometrical Seminar (Divčibare, 1996).

[2] D. Alekseevsky and A. Arvanitoyeorgos, "Riemannian flag manifolds with homogeneous geodesics," *Trans. Amer. Math. Soc.*, vol. 359, no. 8, pp. 3769–3789, 2007, doi: 10.1090/S0002-9947-07-04277-8.

[3] A. Arvanitoyeorgos, *An introduction to Lie groups and the geometry of homogeneous spaces*, ser. Student Mathematical Library. American Mathematical Society, Providence, RI, 2003, vol. 22, doi: 10.1090/stml/022.

[4] A. Arvanitoyeorgos, "Geometry of flag manifolds," *Int. J. of Geometric Methods in Modern Physics*, vol. 3, no. 5, pp. 957–974, 2006.

[5] J. Báez and L. A. B. San Martin, "Deformations of adjoint orbits for semisimple Lie algebras and Lagrangian submanifolds," *Differential Geom. Appl.*, vol. 75, 2021, Art. ID 101719, doi: 10.1016/j.difgeo.2021.101719.

[6] L. Bedulli and A. Gori, "Homogeneous Lagrangian submanifolds," *Comm. Anal. Geom.*, vol. 16, no. 3, pp. 591–615, 2008, doi: 10.4310/cag.2008.v16.n3.a5.

[7] I. N. Bernšteĭn, I. M. Gel'fand, and S. I. Gel'fand, "Schubert cells, and the cohomology of the spaces G/P," *Uspehi Mat. Nauk*, vol. 28, no. 3(171), pp. 3–26, 1973.

[8] A. L. Besse, *Einstein manifolds*, ser. Classics in Mathematics. Springer-Verlag, Berlin, 2008, reprint of the 1987 edition.

[9] L. Casian and R. J. Stanton, "Schubert cells and representation theory," *Invent. Math.*, vol. 137, no. 3, pp. 461–539, 1999, doi: 10.1007/s002220050334.

[10] E. Gasparim, L. Grama, and L. A. B. San Martin, "Symplectic Lefschetz fibrations on adjoint orbits," *Forum Math.*, vol. 28, no. 5, pp. 967–979, 2016, doi: 10.1515/forum-2015-0039.

[11] E. Gasparim, L. Grama, and L. A. B. San Martin, "Adjoint orbits of semi-simple Lie groups and Lagrangian submanifolds," *Proc. Edinb. Math. Soc. (2)*, vol. 60, no. 2, pp. 361–385, 2017, doi: 10.1017/S0013091516000286.

[12] E. Gasparim and L. A. B. San Martin, "Morse functions and real Lagrangian thimbles on adjoint orbits," *J. Topol. Anal.*, vol. 17, no. 3, pp. 653–679, 2025, doi: 10.1142/S1793525323500395.

[13] E. Gasparim, L. A. B. San Martin, and F. Valencia, "Infinitesimally tight Lagrangian orbits," *Math. Z.*, vol. 297, no. 3-4, pp. 1877–1898, 2021, doi: 10.1007/s00209-020-02583-9.

[14] C. Gorodski and F. Podestà, "Tight Lagrangian homology spheres in compact homogeneous Kähler manifolds," *Israel J. Math.*, vol. 206, no. 1, pp. 413–429, 2015, doi: 10.1007/s11856-014-1145-5.

[15] H. Iriyeh and T. Sakai, "Tight Lagrangian surfaces in $S^2 \times S^2$," *Geom. Dedicata*, vol. 145, pp. 1–17, 2010.

[16] N. Jacobson, *Lie algebras*. Dover Publications, Inc., New York, 1979, republication of the 1962 original.

[17] Y.-G. Oh, "Tight Lagrangian submanifolds in $\mathbf{C}\mathrm{P}^n$," *Math. Z.*, vol. 207, no. 3, pp. 409–416, 1991, doi: 10.1007/BF02571398.

[18] Y.-G. Oh, "Volume minimization of Lagrangian submanifolds under Hamiltonian deformations," *Math. Z.*, vol. 212, no. 2, pp. 175–192, 1993, doi: 10.1007/BF02571651.

[19] Y. Ohnita, "Geometry of R-spaces canonically embedded in Kähler C-spaces as Lagrangian submanifolds," in *Proceedings of the 22nd International Workshop on Differential Geometry of Submanifolds in Symmetric Spaces & the 17th RIRCM-OCAMI Joint Differential Geometry Workshop*, 2019, pp. 115–132.

[20] L. Rabelo and L. A. B. San Martin, "Cellular homology of real flag manifolds," *Indag. Math. (N.S.)*, vol. 30, no. 5, pp. 745–772, 2019.

[21] L. A. B. San Martin, *Álgebras de Lie*, 2nd ed. Campinas, SP, Brasil: Editora da Unicamp, 2010.

[22] L. A. B. San Martin, *Grupos de Lie*. Campinas, SP, Brasil: Editora da Unicamp, 2016.

[23] J.-P. Serre, *Complex semisimple Lie algebras*, ser. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001, doi: 10.1007/978-3-642-56884-8.

[24] P. B. Zwart and W. M. Boothby, "On compact, homogeneous symplectic manifolds," *Ann. Inst. Fourier (Grenoble)*, vol. 30, no. 1, pp. 129–157, 1980, doi: 10.5802/aif.778.

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Zeros of cubic polynomials in zeon algebra

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ABSTRACT

It is well known that every cubic polynomial with complex coefficients has three not necessarily distinct complex zeros. In this paper, zeros of cubic polynomials over complex zeons are considered. In particular, a monic cubic polynomial with zeon coefficients may have three spectrally simple zeros, uncountably many zeros, or no zeros at all. A classification of zeros is developed based on an extension of the cubic discriminant to zeon polynomials. In indeterminate cases, sufficient conditions are provided for existence of spectrally nonsimple zeon zeros. We also show that when considering zeros of cubic polynomials over the finite-dimensional complex zeon algebra $\mathbb{C}\mathfrak{Z}_2$, there are no indeterminate cases.

RESUMEN

Es bien sabido que todo polinomio cúbico con coeficientes complejos tiene tres ceros complejos no necesariamente distintos. En este artículo consideramos los ceros de polinomios cúbicos sobre los complejos zeones. En particular, un polinomio cúbico mónico con coeficientes zeones puede tener tres ceros espectralmente simples, una cantidad no numerable de ceros, o no tener ceros. Desarrollamos una clasificación de ceros en base a una extensión del discriminante cúbico a polinomios zeones. En casos indeterminados, entregamos condiciones suficientes para la existencia de ceros zeones espectralmente no simples. También mostramos que cuando consideramos ceros de polinomios cúbicos sobre el álgebra de complejos zeones finito-dimensional $\mathbb{C}\mathfrak{Z}_2$, no hay casos indeterminados.

Keywords and Phrases: Zeons, polynomials, cubic formula, symbolic computation

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1 Introduction

The *n-particle (real) zeon algebra* is a commutative $\mathbb{R}$-algebra generated by a fixed collection $\{\zeta_{\{i\}} : 1 \leq i \leq n\}$ and the scalar identity $1 = \zeta_\varnothing$, whose generators satisfy the zeon commutation relations

$$\zeta_{\{i\}}\zeta_{\{j\}} + \zeta_{\{j\}}\zeta_{\{i\}} = \begin{cases} 2\zeta_{\{i\}}\zeta_{\{j\}} & i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

We denote this algebra by $\mathbb{Z}_n$. Like fermions the algebra has null-square generators; like bosons, the generators commute. Hence the name "zeon algebra", first suggested by Feinsilver [2].

Combinatorial properties of zeons have proven useful in problems ranging from enumerating paths and cycles in finite graphs to routing problems in communication networks. Where classical approaches to routing problems require construction of trees and the use of heuristics to prevent combinatorial explosion, the zeon algebraic approach avoids tree constructions and heuristics. Much of the essential background on algebraic and combinatorial properties and applications of zeons is summarized in the books [9] and [13]. Other works involving zeons include combinatorial identities developed by Neto [5–8] and first and second order differential equations considered by Mansour and Schork [4].

Polynomials over the n-particle complex zeon algebra, denoted by $\mathbb{C}\mathfrak{Z}_n$, were first considered in [11]. We extend the finite-dimensional zeon algebras to the infinite-dimensional complex zeon algebra $\mathbb{C}\mathfrak{Z}$ and focus on zeros of cubic polynomials over $\mathbb{C}\mathfrak{Z}$. Our study is restricted to monic polynomials of the form $\varphi(u) = u^3 + \alpha u^2 + \beta u + \gamma \in \mathbb{C}\mathfrak{Z}[u]$, which generalize naturally to non-monic cubic polynomials with invertible leading coefficients. Observing that

$$\varphi\left(u - \frac{\alpha}{3}\right) = u^3 + 3qu - 2r,$$

where $q = \frac{1}{3}\beta - \frac{1}{9}\alpha^2$ and $r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3$, our work is further simplified by focusing on solutions of the *depressed cubic equation* $u^3 + 3qu - 2r = 0$.

Traditionally, the cubic discriminant $\Delta_f = 18abc - 4a^3b + a^2b^2 - 4b^3 - 27c^2$ is used to classify the zeros of the real monic cubic function $f(x) = x^3 + ax^2 + bx + c \in \mathbb{R}[x]$. In particular, $\Delta_f = 0$ implies that the polynomial has a repeated zero, $\Delta_f < 0$ implies distinct real zeros, and $\Delta_f > 0$ indicates that the polynomial has one real zero and a conjugate pair of complex zeros.

To classify the zeon zeros of monic zeon cubic function $\varphi(u)$, we define the zeon cubic discriminant by $\Delta_\varphi = q^3 + r^2$. When Δ_φ is invertible, the zeon cubic φ has three spectrally simple zeon zeros. If q is also invertible, the zeros can be obtained from the depressed zeon cubic formula (or general extension thereof). If q is nilpotent, zeros can be obtained using the spectrally simple zeros algorithm recalled in Section 2. By contrast, when Δ_φ is not invertible, the zeon cubic φ either has no zeros or uncountably many of them. Some examples and special cases are considered in detail in Section 4.

We proceed as follows. Terminology, notational conventions, and essential results on kth roots of complex zeons are established in Subsections 1.1 and 1.4. Essential background on zeon polynomials is recalled in Section 2.

Main results appear in Sections 3 and 4, where depressed and general cubic formulas are presented and a classification of zeros based on the cubic discriminant is established. Beginning with Theorem 3.2, we show that a depressed cubic $\varphi(u) = u^3 + 3qu - 2r \in \mathbb{C}\mathfrak{Z}[u]$ with invertible q has zeon zeros given by $u = A^{1/3} - qA^{-1/3}$ for the cube roots of $A = r \pm \sqrt{q^3 + r^2}$ with either choice of sign, provided $q^3 + r^2$ has square roots. The restrictions are relaxed to allow nilpotent q in Theorem 3.5, where we find that if r is invertible, then $\varphi(u)$ has three spectrally simple zeros, while if r is nilpotent, then φ has either no zeros or uncountably many nilpotent zeros. Section 3 concludes with the establishment of a general cubic formula for zeon polynomials in Theorem 3.16.

In Section 4, our attention turns to classification via the cubic discriminant. In Theorem 4.1, we consider zeon cubic $\varphi(u) = u^3 + \alpha u^2 + \beta u + \gamma \in \mathbb{C}\mathfrak{Z}[u]$, and define the discriminant $\Delta_\varphi = q^3 + r^2$, where $q = \frac{1}{3}\beta - \frac{1}{9}\alpha^2$, and $r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3$. We show that if Δ_φ is invertible, then φ has three spectrally simple zeros. On the other hand, if Δ_φ is nilpotent, then φ either has no zeros or has uncountably many zeros. Section 4 concludes with a discussion of classification of cubic polynomials over the finite-dimensional zeon algebra $\mathbb{C}\mathfrak{Z}_2$.

Examples appearing throughout the paper have been computed using *Mathematica* with the "Zeon Essentials" package freely available online via the "Research" link at `https://www.siue.edu/~sstaple`.

1.1 Preliminaries

Throughout the paper $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ represent the natural numbers (*i.e.*, positive integers), real numbers, and complex numbers, respectively.

Let $\mathbb{C}\mathfrak{Z}$ denote the infinite-dimensional complex Abelian algebra generated by a fixed collection $\{\zeta_{\{i\}} : i \in \mathbb{N}\}$ along with the scalar $1 = \zeta_\varnothing$ subject to the zeon commutation relation (ZCR):

$$\{\zeta_{\{i\}}, \zeta_{\{j\}}\} = \zeta_{\{i\}}\zeta_{\{j\}} + \zeta_{\{j\}}\zeta_{\{i\}} = 2\delta_{ij}\zeta_{\{i\}}\zeta_{\{j\}} := 2\delta_{ij}\zeta_{\{i,j\}},$$

where we employ multi-index notation for the final equality. For each finite subset I of $\mathbb{N}$, define $\zeta_I = \prod_{\iota \in I} \zeta_\iota$. Letting the finite subsets of positive integers be denoted by $[\mathbb{N}]^{<\omega}$, the algebra $\mathbb{C}\mathfrak{Z}$ has a canonical basis of the form $\{\zeta_I : I \in [\mathbb{N}]^{<\omega}\}$. Elements of this basis are referred to as the *basis blades* of $\mathbb{C}\mathfrak{Z}$. The algebra $\mathbb{C}\mathfrak{Z}$ is called the *(complex) zeon algebra*.

While nonzero scalar multiples of generators also generate the algebra $\mathbb{C}\mathfrak{Z}$, nontrivial linear combinations of generators are *not* generators. For example, $i \neq j$ and $a, b \neq 0$ imply $(a\zeta_{\{i\}} + b\zeta_{\{j\}})^2 = 2ab\zeta_{\{i,j\}}$, which is not a generator of the algebra. Hence, the representation is unique up to generator labeling and scaling.

By the null-square property of the generators $\{\zeta_i : i \in \mathbb{N}\}$, the basis blade product satisfies

$$\zeta_I \zeta_J = \begin{cases} \zeta_{I\cup J} & I \cap J = \varnothing, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

An element $u \in \mathbb{C}\mathfrak{Z}$ has canonical expansion $u = \sum_I u_I \zeta_I$, where each I is a finite subset of $\mathbb{N}$, $u_I \in \mathbb{C}$, and only finitely many of the coefficients u_I are nonzero. Two elements u, v are equal if and only if $u_I = v_I$ for every multi-index in the canonical expansions.

We note that $\mathbb{C}\mathfrak{Z}$ is graded. For non-negative integer k, the *grade-k part* of element $u = \sum_I u_I \zeta_I$ is defined as

$$\langle u \rangle_k = \sum_{\{I:|I|=k\}} u_I \zeta_I. \tag{1.2}$$

The mapping $\langle \cdot \rangle_k : \mathbb{C}\mathfrak{Z} \to \mathbb{C}\mathfrak{Z}$ is clearly a projection onto the subspace of $\mathbb{C}\mathfrak{Z}$ spanned by $\{\zeta_I : |I| = k\}$.

Given $z \in \mathbb{C}\mathfrak{Z}$ we write $\mathfrak{C}z = \langle z \rangle_0$ for the *complex (scalar) part* of z, and $\mathfrak{D}z = z - \mathfrak{C}z$ for the *dual part* of z. Here, the term "dual" is motivated by regarding zeons as higher-dimensional dual numbers.

Remark 1.1. *The algebra $\mathbb{C}\mathfrak{Z}$ can be regarded as the algebra of polynomials in commuting null-square variables $\zeta_{\{1\}}, \zeta_{\{2\}}, \ldots$ Equivalently, $\mathbb{C}\mathfrak{Z} \cong \mathbb{C}[z_1, z_2, \ldots]/\langle {z_1}^2, {z_2}^2, \ldots \rangle$, the quotient of the algebra of complex polynomials in commuting variables z_i by the ideal generated by squares of variables. The basis blades of $\mathbb{C}\mathfrak{Z}$ correspond to basis monomials of the polynomial algebra.*

Definition 1.2. *The* minimal grade *of $u \in \mathbb{C}\mathfrak{Z}$ is defined by*

$$\natural u = \begin{cases} \min\{k \in \mathbb{N} : \langle \mathfrak{D}u \rangle_k \neq 0\} & \mathfrak{D}u \neq 0, \\ 0 & u = \mathfrak{C}u. \end{cases} \tag{1.3}$$

We emphasize that $\natural u = 0$ if and only if u is a scalar, in which case u is said to be trivial. *As it is often useful to refer to the minimal grade part of an element $u \in \mathbb{C}\mathfrak{Z}$, we further define the following notation:*

$$u_\natural := \langle u \rangle_{\natural u}.$$

Example 1.3. *Let* $u = 3 - \zeta_{\{2\}} + 5\zeta_{\{3\}} - 12\zeta_{\{1,2,3\}}$. *We are looking for the minimal grade and the minimal grade part of* u. *Appealing to* (1.2), *we see that* u *has nonzero grade-*k *parts for* $k \in \{0, 1, 3\}$. *In particular,*

$$\begin{aligned}\langle u\rangle_0 &= 3,\\ \langle u\rangle_1 &= -\zeta_{\{2\}} + 5\zeta_{\{3\}},\\ \langle u\rangle_3 &= -12\zeta_{\{1,2,3\}}.\end{aligned}$$

Hence, by Definition 1.3, the minimal grade of u *is* $\natural u = 1$ *and the minimal grade part of* u *is* $u_\natural = \langle u\rangle_1 = -\zeta_{\{2\}} + 5\zeta_{\{3\}}$.

Finally, we note that the nilpotent elements of $\mathbb{C}\mathfrak{Z}$ form a maximal ideal, which we denote by

$$\mathbb{C}\mathfrak{Z}^\circ = \{u \in \mathbb{C}\mathfrak{Z} : \mathfrak{C}u = 0\}.$$

The invertible elements form a multiplicative abelian group denoted by

$$\mathbb{C}\mathfrak{Z}^\times = \mathbb{C}\mathfrak{Z} \setminus \mathbb{C}\mathfrak{Z}^\circ = \{u \in \mathbb{C}\mathfrak{Z} : \mathfrak{C}u \neq 0\}.$$

1.2 Finite-dimensional complex zeon algebras

Letting $[n]$ denote the n-set $\{1, \ldots, n\}$, the complex zeon algebra generated by $\{\zeta_{\{i\}} : i \in [n]\}$ along with the unit scalar 1 is denoted by $\mathbb{C}\mathfrak{Z}_n$. As a vector space over $\mathbb{C}$, $\mathbb{C}\mathfrak{Z}_n$ has dimension 2^n.

Given any zeon $u \in \mathbb{C}\mathfrak{Z}$, we define the *maximum index* of u to be the least positive integer n such that

$$u \in \mathbb{C}\mathfrak{Z}_n \subset \mathbb{C}\mathfrak{Z}_{n+1} \subset \mathbb{C}\mathfrak{Z}_{n+2} \subset \cdots .$$

Equivalently, we have the following definition.

Definition 1.4. *The* maximum index *of* $u \in \mathbb{C}\mathfrak{Z}$ *is the unique positive integer* n *such that* $u \in \mathbb{C}\mathfrak{Z}_n$ *and* $u \notin \mathbb{C}\mathfrak{Z}_{n-1}$.

For example, if $u = 1 + 3\zeta_{\{1,4\}} - 2\zeta_{\{1,3,5\}}$, the maximum index of u is $n = 5$.

1.3 Multiplicative properties of zeons

The elements of $\mathbb{C}\mathfrak{Z}$ form a multiplicative semigroup, and it is not difficult to establish convenient formulas for expanding products of zeons. Moreover, $u \in \mathbb{C}\mathfrak{Z}$ is invertible if and only if $\mathfrak{C}u \neq 0$. The following result is recalled from [1] for reference.

Proposition 1.5. *Let $u \in \mathbb{C}\mathfrak{Z}$, and let κ denote the index of nilpotency*[1] *of $\mathfrak{D}u$. It follows that u is uniquely invertible if and only if $\mathfrak{C}u \neq 0$, and the inverse is given by*

$$u^{-1} = \frac{1}{\mathfrak{C}u} \sum_{j=0}^{\kappa-1} (-1)^j (\mathfrak{C}u)^{-j} (\mathfrak{D}u)^j. \tag{1.4}$$

One way to see Proposition 1.5 is to first recall that if the geometric series $\sum_{j=0}^{\infty} x^j$ converges, its limit is $\frac{1}{1-x}$. Again letting $a = \mathfrak{C}u \neq 0$ and writing $u = a + \mathfrak{D}u$, we see that

$$u^{-1} = (a + \mathfrak{D}u)^{-1} = a^{-1} \frac{1}{1 - (-a\mathfrak{D}u)} = a^{-1} \sum_{j=0}^{\kappa-1} (-1)^j a^{-j} (\mathfrak{D}u)^j,$$

where nilpotency of $\mathfrak{D}u$ reduces the infinite series to a finite sum, eliminating any concern about lack of convergence.

1.3.1 Products and partitions

For convenience, we recall without proof the multinomial theorem. Let $\{x_1, \ldots, x_m\}$ be a collection of commuting variables. For any positive integer m and any nonnegative integer n, one has

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{\substack{k_1 + \cdots + k_m = n \\ k_1, k_2, \ldots, k_m \geq 0}} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{\ell=1}^{m} {x_\ell}^{k_\ell}, \tag{1.5}$$

where

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}$$

is a multinomial coefficient. We further take $x^0 = 1$ even when $x = 0$.

When $n = 2$, (1.5) reduces to the more commonly seen binomial theorem. The importance of the multinomial theorem when considering powers of zeons becomes evident when one realizes that the nonnegative integers $k_1, \ldots, k_m$ are restricted to values 0 or 1 when $x_1, \ldots, x_m$ are zeon basis blades.

For an immediate consequence, let $u, v \in \mathbb{C}\mathfrak{Z}$, write $u = \sum_I u_I \zeta_I$ and $v = \sum_I v_I \zeta_I$, and let the product $w = uv$ be written $w = \sum_I w_I \zeta_I$. Then for fixed multi-index I, the corresponding coefficient of ζ_I in w is given by

$$w_I = \sum_{K \subseteq I} u_K v_{I \setminus K}.$$

Extending to powers of zeons, let $u = \sum_I u_I \zeta_I \in \mathbb{C}\mathfrak{Z}$. For positive integer k, let $w = u^k$ be written

[1] In particular, κ is the least positive integer such that $(\mathfrak{D}u)^\kappa = 0$.

$w = \sum_I w_I \zeta_I$. For any fixed multi-index I, the corresponding coefficient of ζ_I in w is given by

$$w_I = \sum_{j=0}^{k} \frac{k!}{j!} {u_\varnothing}^j \sum_{\substack{\pi \in \mathcal{P}(I) \\ |\pi| = k-j}} u_\pi.$$

Here, $\mathcal{P}(I)$ denotes the collection of partitions of the multi-index I. When $\pi \in \mathcal{P}(I)$ is a partition, $|\pi|$ denotes the number of *blocks* (nonempty subsets of I) in the partition π and $u_\pi := \prod_{b\in\pi} u_b$; *i.e.*, the product of coefficients u_b in the expansion of u corresponding to blocks b in the partition π. Note that the scalar part of u is $\mathfrak{C}u = u_\varnothing$. By convention, we define ${u_\emptyset}^0 = 1$ when $u_\emptyset = 0$.

1.4 Complex zeon roots: Existence and recursive formulations

Invertible zeons have roots of all positive integer orders. Generalizing the result established in [1] for $\mathbb{Z}_n$, their existence is established recursively as follows.

Theorem 1.6. *Let $w \in \mathbb{C}\mathfrak{Z}^\times$, and let $k \in \mathbb{N}$. Then, there exists some $z \in \mathbb{C}\mathfrak{Z}^\times$ such that $z^k = w$. Further, writing $w = u + v\zeta_{\{n\}}$, where $u, v \in \mathbb{C}\mathfrak{Z}_{n-1}$, z is computed recursively by*

$$z = w^{1/k} = u^{1/k} + \frac{1}{k} u^{-(k-1)/k} v \zeta_{\{n\}}.$$

Proof. Proof is by induction on the maximum index n of w. When $n = 1$, let $w = w_\varnothing + b\zeta_{\{1\}}$, where $w_\varnothing = \mathfrak{C}w \neq 0$ and $b \in \mathbb{C}$. Applying the binomial theorem and null-square properties of zeon generators, one finds

$$\left({w_\varnothing}^{1/k} + \frac{b}{k {w_\varnothing}^{(k-1)/k}} \zeta_{\{1\}} \right)^k = w_\varnothing + k {w_\varnothing}^{(k-1)/k} \frac{b}{k {w_\varnothing}^{(k-1)/k}} \zeta_{\{1\}} = w_\varnothing + b\zeta_{\{1\}}.$$

Next, suppose the result holds for some $n-1 \geq 1$ and let $w \in \mathbb{C}\mathfrak{Z}_n$ be written $w = u + v\zeta_{\{n\}}$, where $u, v \in \mathbb{C}\mathfrak{Z}_{n-1}$. In particular, this implies $u \in \mathbb{C}\mathfrak{Z}_n{}^\times$. Let $\alpha = u^{1/k}$, and let $z = \alpha + \frac{1}{k}\zeta_{\{n\}}\alpha^{-(k-1)}v$. Then

$$z^k = \left(\alpha + \frac{1}{k} \alpha^{-(k-1)} v \zeta_{\{n\}} \right)^k = u + k\alpha^{(k-1)} \frac{1}{k} \alpha^{-(k-1)} v \zeta_{\{n\}} = u + v\zeta_{\{n\}} = w. \qquad \square$$

Theorem 1.6 establishes the existence of kth roots of invertible zeons. The following corollary shows that for each kth root of $\mathfrak{C}w$, there exists exactly one zeon kth root of w.

Corollary 1.7. *Let $w \in \mathbb{C}\mathfrak{Z}^{\times}$, and let $k \in \mathbb{N}$. Then, w has exactly k distinct kth roots; i.e., $\sharp\{u : u^k = w\} = k$.*

Proof. Given any invertible zeon w, the nonzero scalar part $\mathfrak{C}w$ has precisely k distinct kth roots in $\mathbb{C}$. We claim that for each of these scalars λ, there is precisely one zeon z satisfying $\mathfrak{C}z = \lambda$ and $z^k = w$.

To see this, suppose $u^k = w = v^k$, where $\mathfrak{C}u = \mathfrak{C}v = \lambda$ and observe that $u - v$ is nilpotent because $u = \lambda + \mathfrak{D}u$ and $v = \lambda + \mathfrak{D}v$. Note that the product $w\delta$ of invertible w and nilpotent δ, is zero if and only if $\delta = 0$, since $0 = w^{-1}0 = \delta$. With the assumption $u^k = v^k$, we then have

$$\begin{aligned} u^k - v^k &= (u-v)(u^{k-1} + u^{k-2}v + \cdots + v^{k-1}) \\ &= (u-v)\left[(\lambda^{k-1} + \delta_1) + (\lambda^{k-1} + \delta_2) + \cdots + (\lambda^{k-1} + \delta_k)\right] \\ &= (u-v)\left[k\lambda^{k-1} + \delta\right], \end{aligned}$$

where $\delta = \delta_1 + \cdots + \delta_k$ is nilpotent because $\mathbb{C}\mathfrak{Z}^{\circ}$ is an ideal. It is clear that $k\lambda^{k-1} + \delta$ is invertible, so $(u-v)(k\lambda^{k-1} + \delta) = 0$ implies $(u - v) = 0$. □

Given invertible $u \in \mathbb{C}\mathfrak{Z}$ and positive integer k, the *principal kth root* of u is defined to be the zeon kth root of u whose scalar part is the principal kth root of $\mathfrak{C}u \in \mathbb{C}$.

1.4.1 Roots of nilpotent zeons

Generally, for positive integer $k \geq 2$, a nilpotent zeon has either no kth roots or uncountably many of them. We restrict our attention to square roots and cube roots here because these are the only roots of interest when dealing with cubic polynomials.

An element $u = \sum_{\{I \in \mathbb{N}^{\leq \omega}\}} u_I \zeta_I$ has a square root $w = \sum_J w_J \zeta_J$ if for each coefficient u_I in the expansion of u, the coefficients of w satisfy

$$\sum_{K \subset I} w_K w_{I \setminus K} = u_I. \tag{1.6}$$

For each nonempty multi index I, (1.6) is an equation in $2^{|I|} - 1$ variables. Letting n denote the smallest positive integer such that $u \in \mathbb{C}\mathfrak{Z}_n{}^{\circ}$, and observing that squares of elements in the maximal ideal $\mathbb{C}\mathfrak{Z}^{\circ}$ always have minimal grade greater than 1, it follows that there are $2^n - n - 1$ such equations to consider. The resulting underdetermined system of $2^n - n - 1$ equations in $2^n - 2$ variables then has either no solution or uncountably many solutions.

Example 1.8. *Consider the nilpotent zeon* $u = 4\zeta_{\{1,2\}} - 5\zeta_{\{1,3\}} - 10\zeta_{\{2,3\}} - 5\zeta_{\{1,2,3\}}$. *A square root* $w = \sum_I w_I\zeta_I$ *of* u *must satisfy the following system of equations:*

$$\begin{aligned} w_{\{1\}}w_{\{2\}} &= 2, \\ w_{\{1\}}w_{\{3\}} &= -\frac{5}{2}, \\ w_{\{2\}}w_{\{3\}} &= -5, \\ w_{\{3\}}w_{\{1,2\}} + w_{\{2\}}w_{\{1,3\}} + w_{\{1\}}w_{\{2,3\}} &= -\frac{5}{2}. \end{aligned}$$

One such solution is

$$w = -\zeta_{\{1\}} - 2\zeta_{\{2\}} + \frac{5}{2}\zeta_{\{3\}} + \zeta_{\{1,2\}} + \zeta_{\{1,3\}} + 3\zeta_{\{2,3\}}.$$

Similarly, a nilpotent zeon of minimal grade 3 or more having expansion $u = \sum_{\{I\in\mathbb{N}^{\leq\omega}:|I|\geq 3\}} u_I\zeta_I$ has cube root $w = \sum_J w_J\zeta_J$ if for each coefficient u_I, the coefficients of w satisfy

$$\sum_{\{K,L\subset I:K\cap L=\varnothing\}} w_K w_L w_{I\setminus(K\cup L)} = u_I.$$

This leads to an underdetermined system of $2^n - \binom{n}{2} - n - 1$ equations in $2^n - 2$ variables with either no solution or uncountably many solutions.

We turn now to a simple special case for which symbolic computation is straightforward.

1.4.2 Fundamental roots of nonzero null monomials

In this section we consider kth roots of $a\zeta_I$ for $a \in \mathbb{C}^\times$ and nonempty $I \subset \mathbb{N}$. Such elements are referred to as *nonzero null monomials*[2] of $\mathbb{C}\mathfrak{Z}$.

Remark 1.9. *Nonzero null monomials are square roots of zero. It follows that every* k*th root of a nonzero null monomial is a* $2k$*th root of zero.*

Definition 1.10. *Given a nonzero null monomial* $w = a\zeta_I$ *and a* k*-block partition* π *of* I, *a* fundamental kth root *of* w *is any nilpotent zeon of the form*

$$u_\pi = \sum_{J\in\pi} u_J\zeta_J, \tag{1.7}$$

satisfying $(u_\pi)^k = w$.

[2] In particular, $a\zeta_I$ is a zero of the monomial $\varphi(u) = u^2$ for any $I \neq \varnothing$.

For purposes of symbolic computation, two forms of roots are particularly convenient. Roots of the form (1.8) are referred to as *flat form* fundamental kth roots of u, while roots of the form (1.9) will be referred to as *spike form* fundamental kth roots of u.

Lemma 1.11. *Given a nonzero null monomial $w = a\zeta_I$, a nilpotent zeon of the form*

$$u_\pi = \sqrt[k]{\frac{a}{k!}} \sum_{J\in\pi} \zeta_J \tag{1.8}$$

satisfies ${u_\pi}^k = w$ for any k-block partition π of the multi index I and any complex kth root of $\frac{a}{k!}$. Moreover,

$$u_{\pi,M} = \sum_{J\in\pi\setminus M} \zeta_J + \frac{a}{k!}\zeta_M \tag{1.9}$$

satisfies ${u_\pi}^k = w$ for any fixed block M of the k-block partition π of the multi index I.

Proof. By direct computation via the multinomial theorem,

$$\left(\sqrt[k]{\frac{a}{k!}} \sum_{J\in\pi} \zeta_J\right)^k = \frac{a}{k!} k! \prod_{\ell=1}^{k} \zeta_{I_\ell} = a\zeta_I = \left(\sum_{J\in\pi\setminus M} \zeta_J + \frac{a}{k!}\zeta_M\right)^k. \qquad \square$$

Hence, the result.

Example 1.12. *The flat form fundamental square roots of $a\zeta_{\{1,2,3\}}$ are*

$$u_{1|23} = \pm\sqrt{\frac{a}{2}}(\zeta_{\{1\}} + \zeta_{\{2,3\}}), \qquad u_{2|13} = \pm\sqrt{\frac{a}{2}}(\zeta_{\{2\}} + \zeta_{\{1,3\}}),$$
$$u_{3|12} = \pm\sqrt{\frac{a}{2}}(\zeta_{\{3\}} + \zeta_{\{1,2\}}),$$

and the spike form fundamental square roots are

$$\begin{aligned} u_{1|23,\{2,3\}} &= \left(\zeta_{\{1\}} + \frac{a}{2}\zeta_{\{2,3\}}\right), & u_{2|13,\{1,3\}} &= \left(\zeta_{\{2\}} + \frac{a}{2}\zeta_{\{1,3\}}\right), \\ u_{3|12,\{1,2\}} &= \left(\zeta_{\{3\}} + \frac{a}{2}\zeta_{\{1,2\}}\right), & u_{1|23,\{1\}} &= \left(\frac{a}{2}\zeta_{\{1\}} + \zeta_{\{2,3\}}\right), \\ u_{2|13,\{2\}} &= \left(\frac{a}{2}\zeta_{\{2\}} + \zeta_{\{1,3\}}\right), & u_{3|12,\{3\}} &= \left(\frac{a}{2}\zeta_{\{3\}} + \zeta_{\{1,2\}}\right). \end{aligned}$$

Notation. The numbers of k-block partitions of sets containing m elements are given by Stirling numbers of the second kind, denoted $\left\{ {m \atop k} \right\}$.

Lemma 1.13. *The number of fundamental kth roots of a null monomial of grade $m \geq k$ is $k\left\{ {m \atop k} \right\}$.*

Proof. Each partition of I into k nonempty subsets $\{I_\ell : 1 \leq \ell \leq \left\{ {m \atop k} \right\}\}$ gives a principal kth root of $a\zeta_I$ since

$$(a^{1/k}\zeta_{I_\ell})^k = k!(a^{1/k})^k \prod_{\ell=1}^{k} \zeta_{I_\ell} = a\zeta_I.$$

Each $a \in \mathbb{C}^\times$ has k distinct complex kth roots, so there are k zeon kth roots of the form seen in (1.8) for each k-block partition π of I. □

2 Zeon polynomials

Let $f(z) = a_m z^m + \cdots + a_1 z + a_0$ $(a_m \neq 0)$ be a polynomial function with complex coefficients, and recall that by the Fundamental Theorem of Algebra, $f(z)$ has exactly m complex zeros. If $f(z)$ can be written in the form $f(z) = (z-r)^\ell g(z)$, where $\ell \in \mathbb{N}$ and $g(r) \neq 0$, then r is said to be a *zero of multiplicity* ℓ of $f(z)$. For convenience, $\mu_f(r)$ will denote the multiplicity of r as a zero of $f(z)$.

On the other hand, if $\varphi(u) = \alpha_m u^m + \cdots + \alpha_1 u + \alpha_0 \in \mathbb{C}\mathfrak{Z}[u]$ is a polynomial with zeon coefficients, it is not obvious how many zeros this polynomial may have in $\mathbb{C}\mathfrak{Z}$. For example, $\varphi(u) = u^2 - \zeta_{\{1\}}$ has no zeon zeros because $\zeta_{\{1\}}$ has no square root.

2.1 Spectrally simple zeros of zeon polynomials

Given a complex zeon polynomial $\varphi(u) = \alpha_m u^m + \cdots + \alpha_1 u + \alpha_0$, a *complex polynomial* $f_\varphi : \mathbb{C} \to \mathbb{C}$ is induced by

$$f_\varphi(z) = \sum_{\ell=0}^{m} (\mathfrak{C}\alpha_\ell) z^\ell.$$

It follows that

$$f_\varphi(\mathfrak{C}u) = \sum_{\ell=0}^{m} (\mathfrak{C}\alpha_\ell)(\mathfrak{C}u)^\ell = \mathfrak{C}(\varphi(u)),$$

so that $f_\varphi \circ \mathfrak{C} = \mathfrak{C} \circ \varphi$.

We restrict our attention to zeon polynomials with invertible leading coefficients because when α_m is nilpotent, the induced polynomial $f_\varphi(z)$ is of lower degree than $\varphi(u)$. Moreover, as a matter of convenience the zeros of $\varphi(u)$ are exactly the zeros of the monic polynomial ${\alpha_m}^{-1}\varphi(u)$.

The zeon extension of the Fundamental Theorem of Algebra developed in [11] shows that $\varphi(u)$ has a simple zeon zero if the complex polynomial $f_\varphi(z)$ has a simple complex zero.

Let $\varphi(u)$ be a nonconstant monic zeon polynomial. A zeon $\lambda \in \mathbb{C}\mathfrak{Z}$ is said to be a *simple* zero of φ if $\varphi(u) = (u-\lambda)g(u)$ for some zeon polynomial g satisfying $g(\lambda) \neq 0$.

The spectrum of an element u in a unital algebra is the collection of scalars λ for which $u - \lambda$ is not invertible. Hence, the spectrum of $u \in \mathbb{C}\mathfrak{Z}$ is the singleton $\{\lambda = \mathfrak{C}u\}$, motivating the next definition.

Definition 2.1. *A simple zero $\lambda_0 \in \mathbb{C}\mathfrak{Z}$ of $\varphi(u)$ is said to be a* spectrally simple *if $\mathfrak{C}\lambda_0$ is a simple zero of the complex polynomial $f_\varphi(z)$.*

2.1.1 Fundamental theorem of zeon algebra

The *Fundamental Theorem of Zeon Algebra* presented in [11] for the finite dimensional zeon algebra $\mathbb{C}\mathfrak{Z}_n$ shows that a zeon polynomial $\varphi(u) \in \mathbb{C}\mathfrak{Z}_n[u]$ has a spectrally simple zero $\lambda = \lambda_0 + \mathfrak{D}\lambda$ whenever the complex polynomial $f_\varphi(z) \in \mathbb{C}[z]$ has a simple zero $\lambda_0 \in \mathbb{C}$. The theorem also holds also for a polynomial over $\mathbb{C}\mathfrak{Z}$ by first defining the *maximum index of a zeon polynomial* φ to be the least positive integer n such that $\varphi(u) \in \mathbb{C}\mathfrak{Z}_n[u]$ and proceeding as in the finite-dimensional zeon algebra.

For reference, the theorem is recalled here without proof. We note that it also provides a method for calculating spectrally simple zeros of any zeon polynomial.

Theorem 2.2 (Fundamental Theorem of Zeon Algebra)**.** *Let $\varphi(u) \in \mathbb{C}\mathfrak{Z}[u]$ be a monic zeon polynomial of degree m and having maximum index n, and let $f_\varphi(z) \in \mathbb{C}[z]$ be induced by φ. If $\lambda_0 \in \mathbb{C}$ is a simple zero of $f_\varphi(z)$, let g be the unique complex polynomial such that $f_\varphi(\mathfrak{C}u) = (\mathfrak{C}u - \lambda_0)g(\mathfrak{C}(u))$. Then $\varphi(u)$ has a simple zero λ such that $\mathfrak{C}\lambda = \lambda_0$. Letting n denote the maximum index of $\varphi(u)$, for $1 \leq k \leq n$, the grade-k part of λ (denoted λ_k) is given by*

$$\lambda_k = -\frac{1}{g(\lambda_0)}\left\langle \varphi\left(\sum_{i=0}^{k-1}\lambda_i\right)\right\rangle_k .$$

Moreover, such a zero λ is unique.

The idea behind the proof is that when λ_0 is a simple zero of $f_\varphi(z)$, the remainder $\varphi(\lambda_0)$ of $\varphi(u)$ when divided by $u - \lambda_0$ has zero scalar part. The minimal grade part of the remainder $w = \varphi(\lambda_0)$ can then be utilized to construct a new zeon element $\lambda_0 + \lambda_{\natural w}$ having the property that $\varphi(\lambda_0 + \lambda_{\natural w})$ has higher minimal grade than $\varphi(\lambda_0)$. Grades of all remainders will be at most n (the maximum index of $\varphi(u)$), so the process terminates in a finite number of steps.

Of particular significance, Theorem 2.2 provides an algorithm by which a spectrally simple zeon zero can be calculated. Algorithm 1 returns the spectrally simple zeon zero λ of φ whose scalar part λ_0 satisfies $f_\varphi(\lambda_0) = 0$.

Algorithm 1: Compute spectrally simple zeon zero.

input : Zeon polynomial $\varphi(u)$ over $\mathbb{C}\mathfrak{Z}_n$ and a simple nonzero root λ_0 of the associated complex polynomial $\mathfrak{C}(\varphi(u))$.
output: Zeon zero λ of $\varphi(u)$ with $\mathfrak{C}\lambda = \lambda_0$.

Initialize complex polynomial $g(\mathfrak{C}u)$.

$g(\mathfrak{C}u) \leftarrow \dfrac{\mathfrak{C}(\varphi(u))}{\mathfrak{C}u - \lambda_0}$;

Note $g(\mathfrak{C}u)$ satisfies $\mathfrak{C}(\varphi(u)) = (\mathfrak{C}u - \lambda_0)g(\mathfrak{C}u)$, where $g(\lambda_0) \neq 0$.

$\xi \leftarrow \varphi(\lambda_0)_\natural / g(\lambda_0)$;
$\lambda \leftarrow \lambda_0 - \xi$;

while $0 < \natural\xi \leq n$ **do**
  $\xi \leftarrow \varphi(\lambda)_\natural / g(\lambda_0)$;
  $\lambda \leftarrow (\lambda - \xi)$;

return λ;

When $\varphi(u) \in \mathbb{C}\mathfrak{Z}[u]$ is of degree $m \geq 1$ and the zeros of $f_\varphi(z)$ are all simple, we see that $\varphi(u)$ has exactly m complex zeon zeros. For example, when $\alpha \in \mathbb{C}\mathfrak{Z}^\times$, $\varphi(u) = u^k + \alpha$ has exactly k distinct complex zeon zeros.

2.2 Spectrally nonsimple zeon zeros

Algorithm 1 is useful for computing spectrally simple zeros of $\varphi(u)$, but it is not applicable to any zero w whose scalar part $\mathfrak{C}w$ is a multiple zero of the induced complex polynomial f_φ satisfying $\mathfrak{C}(\varphi(u)) = f_\varphi(\mathfrak{C}u)$. These spectrally nonsimple zeros are considered next.

A zero $\lambda_0 \in \mathbb{C}\mathfrak{Z}$ of $\varphi(u) \in \mathbb{C}\mathfrak{Z}[u]$ is said to be *spectrally nonsimple* if $\mathfrak{C}\lambda_0$ is a multiple zero of the induced complex polynomial f_φ. We note that zeon zeros of multiplicity greater than one are included among spectrally nonsimple zeros.

It was shown in [11] that if a monic polynomial $\varphi(u) \in \mathbb{C}\mathfrak{Z}[u]$ has distinct complex zeon zeros w_1, w_2 satisfying $\mathfrak{C}w_1 = \mathfrak{C}w_2 = w_\varnothing$, then $\varphi(u)$ has uncountably many zeros of the form $w = w_\varnothing + \mathfrak{D}w$. As a consequence, if $\varphi \in \mathbb{C}\mathfrak{Z}[u]$ has a zero $z \in \mathbb{C}\mathfrak{Z}$ of multiplicity two or greater, then φ has uncountably many zeros $w \in \mathbb{C}\mathfrak{Z}$ satisfying $\mathfrak{C}w = \mathfrak{C}z$.

Lacking an algorithm for computing spectrally nonsimple zeros of zeon polynomials, our attention turns to zeon extensions of well-known special cases: quadratic and cubic polynomials.

2.2.1 The zeon quadratic formula

We close this review of zeon polynomials by recalling a basic result concerning zeros of quadratic zeon polynomials. A zeon quadratic polynomial has solutions if and only if its discriminant has a square root [3].

Theorem 2.3 (Zeon Quadratic Formula). *Let $\varphi(u) = \alpha u^2 + \beta u + \gamma$ be a quadratic function with zeon coefficients from $\mathbb{C}\mathfrak{Z}$, where $\mathfrak{C}\alpha \neq 0$. Let $\Delta_\varphi = \beta^2 - 4\alpha\gamma$ denote the zeon discriminant of φ. The zeros of φ are given by*

$$\varphi^{-1}(0) = \left\{ \frac{\alpha^{-1}}{2}(w - \beta) : w^2 = \beta^2 - 4\alpha\gamma \right\}.$$

In particular,

(1) *When $\Delta_\varphi = 0$, the zeros of φ are given by $u = -\alpha^{-1}\beta/2 + \eta$ for any $\eta \in \mathbb{C}\mathfrak{Z}$ satisfying $\eta^2 = 0$.*

(2) *When $\mathfrak{C}\Delta_\varphi \neq 0$, $\varphi(u) = 0$ has two distinct solutions.*

(3) *If $\Delta_\varphi \neq 0$ is nilpotent and $\varphi(u) = 0$ has a solution, then it has uncountably many solutions.*

To see the result, begin by writing $\alpha u^2 + \beta u + \gamma = \frac{\alpha^{-1}}{4}((2\alpha u + \beta)^2 - (\beta^2 - 4\alpha\gamma))$ and expand. This reduces the problem to seeking square roots of the zeon discriminant. We are now ready to turn our attention to cubic polynomials over $\mathbb{C}\mathfrak{Z}$.

3 Cubic polynomials with zeon coefficients

Beginning with the general zeon cubic equation $z^3 + \alpha z^2 + \beta z + \gamma = 0$, where $\alpha, \beta, \gamma \in \mathbb{C}\mathfrak{Z}$ and $\alpha \neq 0$, the *depressed cubic equation* is obtained via the substitution $z = u - \alpha/3$. In particular,

$$\begin{aligned} 0 &= \left(u - \frac{\alpha}{3}\right)^3 + \alpha\left(u - \frac{\alpha}{3}\right)^2 + \beta\left(u - \frac{\alpha}{3}\right) + \gamma = u^3 + \left(\beta - \frac{\alpha}{3}\right)u + \frac{2\alpha^3}{27} - \frac{\alpha\beta}{3} + \gamma \\ &= u^3 + 3\left(\frac{\beta}{3} - \frac{\alpha^2}{9}\right)u - 2\left(\frac{-\alpha^3}{27} + \frac{\alpha\beta}{6} - \frac{\gamma}{2}\right) = u^3 + 3qu - 2r \end{aligned}$$

where $q = \frac{1}{3}\beta - \frac{1}{9}\alpha^2$ and $r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3$. It follows that depressed cubics are sufficient for our purposes.

We note that any monic cubic polynomial having a spectrally simple zero λ can be reduced via polynomial division to the product $\varphi(u) = (u - \lambda)\psi(u)$, where $\psi(\lambda) \neq 0$ is a quadratic polynomial over $\mathbb{C}\mathfrak{Z}$. The remaining zeros of $\varphi(u)$ can then be classified by the zeon quadratic formula of Theorem 2.3.

Example 3.1. *To motivate our discussion, consider the depressed zeon cubic equation* $\varphi(u) = 0$ *where*

$$\varphi(u) = u^3 + u\left(-18\zeta_{\{1,2,3\}} - 6\zeta_{\{1\}} + 9\zeta_{\{2\}} - 9\right) - 10\zeta_{\{1,2\}} - 6\zeta_{\{1,2,3\}}. \tag{3.1}$$

The induced scalar cubic polynomial is $f_\varphi(z) = z^3 - 9z$, *which has simple zeros* $\{-3, 0, 3\}$. *Consequently,* $\varphi(u)$ *has three spectrally simple zeon zeros, each of which can be found by applying Algorithm 1. Applying the algorithm with simple zero* $\lambda_0 = -3$ *of* $f_\varphi(z)$, *we obtain the first zero:*

$$u_1 = -3 + \frac{1}{18}\zeta_{\{1,2\}} - \frac{8}{3}\zeta_{\{1,2,3\}} - \zeta_{\{1\}} + \frac{3\zeta_{\{2\}}}{2}.$$

At this point, we may either repeat the algorithm with the other two zeros of $f_\varphi(z)$ *or we may perform polynomial division to write* $\varphi(u) = (u - u_1)\psi(u)$ *and apply the zeon quadratic formula to* $\psi(u)$ *to obtain the remaining zeros. In the latter method, we apply the quadratic formula to*

$$\psi(u) = u^2 + u\left(\frac{1}{18}\zeta_{\{1,2\}} - \frac{8}{3}\zeta_{\{1,2,3\}} - \zeta_{\{1\}} + \frac{3\zeta_{\{2\}}}{2} - 3\right) - \frac{10}{3}\zeta_{\{1,2\}} - 2\zeta_{\{1,2,3\}},$$

which yields the remaining zeros:

$$\begin{aligned} u_2 &= -\frac{10}{9}\zeta_{\{1,2\}} - \frac{2}{3}\zeta_{\{1,2,3\}}, \\ u_3 &= 3 + \frac{19}{18}\zeta_{\{1,2\}} + \frac{10}{3}\zeta_{\{1,2,3\}} + \zeta_{\{1\}} - \frac{3}{2}\zeta_{\{2\}}. \end{aligned}$$

We point out that the approach taken in Example 3.1 involves the application of Algorithm 1 once, followed by polynomial division and an application of the zeon quadratic formula. Alternatively, since the zeros of $\varphi(u)$ were all spectrally simple, we could have applied Algorithm 1 three times.

We further point out that when the scalar polynomial $f_\varphi(z)$ has a single zero of multiplicity three, the approach taken in Example 3.1 fails completely.

To treat such cases as well as to gain deeper insight on zeros of zeon cubics for all cases, we now consider a zeon extension of the cubic formula. The complex zeon result below is based on Cardano's approach to cubic polynomials with real coefficients, as presented in [10].

Theorem 3.2 (Depressed Zeon Cubic Formula)**.** *Let* $\varphi(u) = u^3 + 3qu - 2r \in \mathbb{C}\mathfrak{Z}[u]$, *where* $\mathfrak{C}q \neq 0$ *and square roots of* $q^3 + r^2$ *are assumed to exist. The zeon zeros of* $\varphi(u)$ *are given by* $u = A^{1/3} - qA^{-1/3}$, *for the cube roots of* $A = r \pm \sqrt{q^3 + r^2}$ *with either choice of sign.*

Proof. Note that A is invertible if and only if $\mathfrak{C}q \neq 0$, since $\mathfrak{C}A = 0$ if and only if $\mathfrak{C}r = \mp\mathfrak{C}\left(\sqrt{q^3 + r^2}\right)$. Squaring both sides yields $\mathfrak{C}q^3 = 0$. Proof is then by direct substitution, where all necessary cube roots, square roots, and inverses exist. Assuming $A = r + \sqrt{q^3 + r^2}$, it follows that

$$\begin{aligned}
\varphi(A^{\frac{1}{3}} - qA^{-\frac{1}{3}}) &= (A^{\frac{1}{3}} - qA^{-\frac{1}{3}})^3 + 3q(A^{\frac{1}{3}} - qA^{-\frac{1}{3}}) - 2r \\
&= A - 3A^{\frac{2}{3}}qA^{-\frac{1}{3}} + 3A^{\frac{1}{3}}q^2A^{-\frac{2}{3}} - q^3A^{-1} + 3qA^{\frac{1}{3}} - 3q^2A^{-\frac{1}{3}} - 2r \\
&= A - 3qA^{\frac{1}{3}} + 3q^2A^{-\frac{1}{3}} - q^3A^{-1} + 3qA^{\frac{1}{3}} - 3q^2A^{-\frac{1}{3}} - 2r \\
&= A - q^3A^{-1} - 2r \\
&= (r^2 + 2r\sqrt{q^3+r^2} + q^3 + r^2 - q^3)(r + \sqrt{q^3+r^2})^{-1} - 2r \\
&= 2r(r + \sqrt{q^3+r^2})(r + \sqrt{q^3+r^2})^{-1} - 2r = 2r - 2r = 0.
\end{aligned}$$

Similar calculations establish the result for $A = r - \sqrt{q^3 + r^2}$. □

Since A is assumed to be invertible in Theorem 3.2, there are three distinct zeon cube roots of A for any square root of $q^3 + r^2$.

Example 3.3 ($\mathfrak{C}q \neq 0$, $q^3 + r^2$ invertible)**.** *Consider the zeon cubic $\varphi(u) = u^3 + 3qu - 2r$ defined by*

$$\varphi(u) = u^3 + u\left(6\zeta_{\{1,2\}} - 12\zeta_{\{2,3\}} - 3\zeta_{\{3\}} - 36\right) + 6\zeta_{\{2\}} - 4\zeta_{\{3\}}.$$

We note that $\mathfrak{C}q \neq 0$, since

$$q = -12 + 2\zeta_{\{1,2\}} - 4\zeta_{\{2,3\}} - \zeta_{\{3\}}.$$

Further, $q^3 + r^2$ is invertible since r is clearly nilpotent. The zeros of $\varphi(u)$ are then found via Theorem 3.2:

$$\begin{aligned}
u_1 &= -6 + \frac{1}{2}\zeta_{\{1,2\}} - \frac{215}{216}\zeta_{\{2,3\}} - \frac{5}{432}\zeta_{\{1,2,3\}} - \frac{\zeta_{\{2\}}}{12} - \frac{7}{36}\zeta_{\{3\}}, \\
u_2 &= -\frac{1}{72}\zeta_{\{2,3\}} - \frac{1}{54}\zeta_{\{1,2,3\}} + \frac{\zeta_{\{2\}}}{6} - \frac{\zeta_{\{3\}}}{9}, \\
u_3 &= 6 - \frac{1}{2}\zeta_{\{1,2\}} + \frac{109}{108}\zeta_{\{2,3\}} + \frac{13}{432}\zeta_{\{1,2,3\}} - \frac{1}{12}\zeta_{\{2\}} + \frac{11}{36}\zeta_{\{3\}}.
\end{aligned}$$

When $q^3 + r^2 \in \mathbb{C}\mathfrak{Z}$ is nilpotent and has a square root, uncountably many square roots exist. In this case, the associated cubic equation has infinitely many solutions.

Example 3.4 ($\mathfrak{C}q \neq 0$, $q^3 + r^2 \in \mathbb{C}\mathfrak{Z}^{\circ}$)**.** *Consider the depressed zeon cubic $\varphi(u) = u^3 + 3qu - 2r$, where*

$$\begin{aligned}
q &= -\zeta_{\{1,2\}} + \zeta_{\{1,3\}} + \zeta_{\{2,3\}} + 2\zeta_{\{1\}} - 2\zeta_{\{2\}} - 1, \\
r &= -3\zeta_{\{1\}} + 3\zeta_{\{2\}} + 1.
\end{aligned}$$

The nilpotent element $q^3 + r^2 = 3\zeta_{\{1,2\}} + 3\zeta_{\{1,3\}} + 3\zeta_{\{2,3\}}$ *has uncountably many square roots; for example,* $\rho = \sqrt{\frac{3}{2}}\left(\zeta_{\{1\}} + \zeta_{\{2\}} + \zeta_{\{3\}}\right)$. *It follows that* $\varphi(u)$ *has uncountably many zeros of the form* $(r+\rho)^{1/3} - q(r+\rho)^{-1/3}$. *In particular,*

$$u_0 = 2 - 2\zeta_{\{1\}} + 2\zeta_{\{2\}} + \frac{10}{3}\zeta_{\{1,2\}} - \frac{2}{3}\zeta_{\{1,3\}} - \frac{2}{3}\zeta_{\{2,3\}}$$

satisfies $\varphi(u_0) = 0$.

Next we consider the depressed cubic $\varphi(u) = u^3 + 3qu - 2r$, where $\mathfrak{C}q = 0$. It follows that the complex polynomial induced by φ is $f_\varphi(z) = z^3 - 2\mathfrak{C}r$. If $\mathfrak{C}r = 0$, then $f_\varphi(z) = z^3$ has one zero 0 of multiplicity three. Hence, if φ has zeros, there are uncountably many and they are all nilpotent. On the other hand, if $\mathfrak{C}r \neq 0$, then $f_\varphi(z)$ has exactly three distinct complex zeros, so that φ has three spectrally simple zeros. Thus, we have derived the following theorem.

Theorem 3.5 (Depressed Cubic Zeros II)**.** *Let* $\varphi(u) = u^3 + 3qu - 2r \in \mathbb{C}\mathfrak{Z}[u]$*, where* $\mathfrak{C}q = 0$. *Then the following are true.*

(1) *If* $\mathfrak{C}r \neq 0$*, then* $\varphi(u)$ *has three spectrally simple zeros.*

(2) *If* $\mathfrak{C}r = 0$*, then* φ *has either no zeros or uncountably many nilpotent zeros.*

We illustrate Theorem 3.5 with the following example.

Example 3.6. *The case* $\mathfrak{C}q = 0$ *is illustrated by the zeon cubic polynomial*

$$\varphi(u) = u^3 + \left(-\frac{2}{3}\zeta_{\{1,2\}} + \frac{4}{3}\zeta_{\{1,3\}} + \frac{4}{3}\zeta_{\{2,3\}} - \frac{8}{3}\zeta_{\{1,2,3\}}\right)u - \frac{8}{9}\zeta_{\{1,2,3\}}.$$

In this example $r = \frac{4}{9}\zeta_{\{1,2,3\}}$*, so that* $\varphi(u)$ *either has no zeros or uncountably many. Letting* $s = \zeta_{\{1\}} + \zeta_{\{2\}} - \zeta_{\{3\}} + \zeta_{\{1,2\}} - \zeta_{\{1,3\}}$*, it is seen that* $\varphi(s) = 0$. *Moreover,* $\varphi(s + a\zeta_{\{1,2,3\}}) = 0$ *for any* $a \in \mathbb{C}$.

3.1 Special case: $\varphi(u) = u^3 + 3qu$

Note that if $r = 0$, the zeros of $\varphi(u)$ include $\{0, \pm\sqrt{-3q}\}$, provided the square roots exist. When q is invertible (*i.e.*, $\mathfrak{C}q \neq 0$), these are the three distinct zeros of $\varphi(u)$. When $\mathfrak{C}q = 0$, $\varphi(u) = 0$ has uncountably many solutions.

Our goal in this subsection is to describe some of the zeros of $u^3 + 3qu$ when q is nilpotent.

Definition 3.7. *Let* $q = \sum_I q_I\zeta_I \in \mathbb{C}\mathfrak{Z}$. *The* index support *of* q *is defined to be*

$$[q] = \bigcup_{\{I : q_I \neq 0\}} I. \tag{3.2}$$

The index support of a nilpotent q is used to obtain a null monomial that "annihilates" q; *i.e.*, $q\zeta_{[q]} = 0$. For this reason, $\zeta_{[q]}$ will be referred to as an *annihilator* of $q \in \mathbb{C}\mathfrak{Z}^\circ$. More generally, $q\zeta_{[q]} = (\mathfrak{C}q)\zeta_{[q]}$ for arbitrary $q \in \mathbb{C}\mathfrak{Z}$, so that $\zeta_{[q]}$ is an annihilator of $\mathfrak{D}q$.

Example 3.8. *Let $q = 3 + 4\zeta_{\{2\}} - 5\zeta_{\{1,3,4\}}$. Then $[q] = \{1, 2, 3, 4\}$ and*

$$q\zeta_{[q]} = (3 + 4\zeta_{\{2\}} - 5\zeta_{\{1,3,4\}})\zeta_{\{1,2,3,4\}} = 3\zeta_{\{1,2,3,4\}}.$$

While it is clear that when q is nilpotent, $q\zeta_I = 0$ for all $I \supseteq [q]$, a nilpotent q may also be annihilated by a basis blade ζ_I for one or more $I \subsetneq [q]$. Letting $\mathcal{N}_q = \{I \subseteq [q] : q\zeta_I = 0\}$, it follows that w

$$q \sum_{I\in\mathcal{N}_q} a_I\zeta_I = 0$$

for any linear combination of basis blades indexed by $\mathcal{N}_q$. The resulting subspace of $\mathbb{C}\mathfrak{Z}$ is denoted by $\mathrm{Ann}_{\mathfrak{Z}}(q)$.

It is clear that $\mathrm{Ann}_{\mathfrak{Z}}(u) \cap \mathrm{Ann}_{\mathfrak{Z}}(v) \subseteq \mathrm{Ann}_{\mathfrak{Z}}(u + v)$ because $z \in \mathrm{Ann}_{\mathfrak{Z}}(u) \cap \mathrm{Ann}_{\mathfrak{Z}}(v)$ implies $z(u + v) = zu + zv = 0$. However, the reverse inclusion need not hold, as illustrated in Example 3.9.

Example 3.9. *Let $u = \zeta_{\{1\}} + \zeta_{\{2\}}$, $v = -\zeta_{\{2\}} \in \mathbb{C}\mathfrak{Z}^\circ$. Letting $z = \zeta_{\{1\}}$, we see that*

$$z(u + v) = \zeta_{\{1\}}(\zeta_{\{1\}} + \zeta_{\{2\}} - \zeta_{\{2\}}) = {\zeta_{\{1\}}}^2 = 0,$$

so that $z \in \mathrm{Ann}_{\mathfrak{Z}}(u + v)$ even though $z \notin \mathrm{Ann}_{\mathfrak{Z}}(u)$ and $z \notin \mathrm{Ann}_{\mathfrak{Z}}(v)$.

With the concept of zeon annihilators in hand, we are ready to present our result on zeros of $\varphi(u) = u^3 + 3qu$ when q is nilpotent.

Theorem 3.10 (Zeros of $\varphi(u) = u^3 + 3qu$ when $\mathfrak{C}q = 0$)**.** *Let $\varphi(u) = u^3 + 3qu \in \mathbb{C}\mathfrak{Z}[u]$, where $q \neq 0$ and $\mathfrak{C}q = 0$. Then,*

(1) *$\varphi(z) = 0$ for any $z \in \mathrm{Ann}_{\mathfrak{Z}}(q)$ satisfying $\kappa(z) \leq 3$; and*

(2) *if q has square roots, then $\varphi(z) = 0$ for any $z \in \{\pm\sqrt{-3q}\}$.*

In particular, $\varphi(a\zeta_{[q]}) = 0$ for $a \in \mathbb{C}$.

Proof. First, for any $z \in \mathrm{Ann}_{\mathfrak{Z}}(q)$ satisfying $\kappa(z) \leq 3$,

$$\varphi(z) = z^3 + 3qz = 0 + 0 = 0.$$

Second, let $(-3q)^{1/2} = \{z \in \mathbb{C}\mathfrak{Z} : z^2 = -3q\}$ and recall that this set has infinite cardinality when it is nonempty. It follows that for each $z \in (-3q)^{1/2}$,

$$\varphi(z) = z(z^2 + 3q) = z(-3q + 3q) = 0.$$

Finally, $\zeta_{[q]} \in \mathrm{Ann}_{\mathfrak{Z}}(q)$ satisfies $\kappa(\zeta_{[q]}) = 2$, so $\varphi(a\zeta_{[q]}) = 0$ for all $a \in \mathbb{C}$. □

Theorem 3.10 does not characterize all zeros of the cubic $\varphi(u) = u^3 + 3qu$, as illustrated by Example 3.11.

Example 3.11. *Consider the cubic $\varphi(u) = u^3 + 3qu$, where*

$$q = \frac{1}{3}\zeta_{\{1,2,3\}} - \frac{2}{3}\zeta_{\{1,2\}} - \frac{2}{3}\zeta_{\{1,3\}} - \frac{2}{3}\zeta_{\{2,3\}}.$$

Letting $z = \zeta_{\{1\}} + \zeta_{\{2\}} + \zeta_{\{3\}}$, one finds that $z^2 = 2(\zeta_{\{1,2\}} + \zeta_{\{1,3\}} + \zeta_{\{2,3\}})$, $z^2 + 3q = \zeta_{\{1,2,3\}}$, and $z^3 = 6\zeta_{\{1,2,3\}}$, so that $\kappa(z) > 3$ and $z \notin (-3q)^{1/2}$. Further, $z \notin \mathrm{Ann}_{\mathfrak{Z}}(q)$ because

$$qz = \frac{1}{3}(\zeta_{\{1,2,3\}} - 2\zeta_{\{1,2\}} - 2\zeta_{\{1,3\}} - 2\zeta_{\{2,3\}})(\zeta_{\{1\}} + \zeta_{\{2\}} + \zeta_{\{3\}}) = -2\zeta_{\{1,2,3\}}.$$

Clearly, z fails to satisfy the sufficient conditions described in Theorem 3.10. However, $z \in \varphi^{-1}(0)$ since

$$\varphi(z) = z^3 + 3qz = 6\zeta_{\{1,2,3\}} - 6\zeta_{\{1,2,3\}} = 0.$$

Corollary 3.12. *Let $\varphi(u) = u^3 - a\zeta_I u \in \mathbb{C}\mathfrak{Z}[u]$, where $a \neq 0$ and $|I| \geq 2$. Then $\varphi(u) = 0$ has $\left\{ {|I| \atop 2} \right\}$ flat form solutions of the form*

$$u_\pi = \sqrt{\frac{a}{2}} \sum_{J \in \pi} \zeta_J,$$

where π ranges over the 2-block partitions of the multi index I.

Proof. Note that $u^3 - a\zeta_I u = u(u^2 - a\zeta_I) = 0$. Let π be a 2-block partition of I. Let K be one block of the partition. It follows that

$$u_\pi = \sqrt{\frac{a}{2}}(\zeta_K + \zeta_{I \setminus K}),$$

so that

$$\begin{aligned}\varphi(u_\pi) = u_\pi({u_\pi}^2 - a\zeta_I) &= u_\pi \left(\left(\sqrt{\frac{a}{2}}(\zeta_K + \zeta_{I\setminus K}) \right)^2 - a\zeta_I \right) \\ &= u_\pi \frac{a}{2}(2a\zeta_I - a\zeta_I) = u_\pi(a\zeta_I - a\zeta_I) = 0.\end{aligned}$$

The number of two block partitions π of I is $\left\{{|I| \atop 2}\right\}$, so the result follows from Lemma 1.13. $\square$

3.2 Special case: $q = 0$

Lemma 3.13 (Depressed cubics: $q = 0$)**.** *Let $\varphi(u) = u^3 - 2r \in \mathbb{C}\mathfrak{Z}[u]$. Then the following are true.*

(1) *If $r = 0$, then $\varphi^{-1}(0) = \{\eta \in \mathbb{C}\mathfrak{Z}^\circ : \kappa(\eta) \leq 3\}$.*

(2) *If $\mathfrak{C}r \neq 0$, then φ has three spectrally simple zeros: $\varphi^{-1}(0) = (2r)^{1/3}$.*

(3) *If $r \neq 0$ and $\mathfrak{C}r = 0$, then φ has either no zeros or uncountably many zeros; in particular, $\varphi^{-1}(0) = \{\omega : \omega^3 = 2r\}$.*

Proof. Consider the zeon cubic $\varphi(u) = u^3 - 2r$.

(1) Clearly $\varphi(\eta) = \eta^3 = 0$ if and only if η is nilpotent of index 3 or less.

(2) If r is invertible, then $u^3 - 2r = 0$ if and only if u is a cube root of $2r$. There are three such zeros, one for each complex cube root of $\mathfrak{C}2r$.

(3) When r is nonzero and nilpotent, the zeros of $\varphi(u)$ are precisely the nilpotent cube roots of $2r$. As seen in Section 1.4.1, $2r$ has either no cube roots or uncountably many of them. $\square$

Corollary 3.14. *Let $\varphi(u) = u^3 - a\zeta_I \in \mathbb{C}\mathfrak{Z}[u]$, where $a \neq 0$ and $|I| \geq 3$. It follows that $\varphi(u) = 0$ has $\left\{{|I| \atop 3}\right\}$ flat form solutions of the form*

$$u_\pi = \sqrt[3]{\frac{a}{6}} \sum_{J\in\pi} \zeta_J,$$

where π ranges over the 3-block partitions of the multi index I.

Proof. Proceeding as in the proof of Corollary 3.12, let π be a 3-block partition of I. Let J, K, L be the blocks of partition π. It follows that

$$u_\pi = \sqrt[3]{\frac{a}{6}}(\zeta_J + \zeta_K + \zeta_L),$$

so that

$$\varphi(u_\pi) = \left(\sqrt[3]{\frac{a}{6}}(\zeta_J + \zeta_K + \zeta_L)\right)^3 - a\zeta_I = \frac{a}{6}6\zeta_J\zeta_K\zeta_L - a\zeta_I = a\zeta_I - a\zeta_I = 0.$$

The number of three block partitions π of I is $\left\{{|I| \atop 3}\right\}$, so the result follows from Lemma 1.13. $\square$

Example 3.15. *Consider the cubic polynomial*

$$\varphi(u) = u^3 + u^2\left(3 - \zeta_{\{1\}}\right) + u\left(-\zeta_{\{1,2\}} - 2\zeta_{\{1\}} + 3\right) + 1 - \zeta_{\{1\}} - \zeta_{\{1,2\}}.$$

Writing $\varphi(u) = u^3 + \alpha u^2 + \beta u + \gamma$, let $q = \left(\frac{\beta}{3} - \frac{\alpha^2}{9}\right)$ and let $r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3$, so that $\varphi(u) = u^3 + 3qu - 2r$. It follows that $\zeta_{[q]} = \zeta_{\{1,2\}}$ and that q has spike form fundamental square roots $\zeta_{\{1\}} - \frac{3}{2}\zeta_{\{2\}}$ and $\zeta_{\{2\}} - \frac{3}{2}\zeta_{\{1\}}$, the (uncountably many) zeros of $\varphi(u)$ include the following:

$$\begin{aligned}
u_1 &= -1 + \frac{1}{3}\zeta_{\{1\}} + \zeta_{\{1,2\}},\\
u_2 &= -1 + \frac{4}{3}\zeta_{\{1\}} - \frac{3}{2}\zeta_{\{2\}},\\
u_3 &= -1 - \frac{7}{6}\zeta_{\{1\}} + \zeta_{\{2\}}.
\end{aligned}$$

These zeros are easily confirmed by evaluating the polynomial.

3.3 A general cubic formula

For convenience in symbolic computation, a general cubic formula is now obtained as a corollary of Theorem 3.2.

Theorem 3.16 (General Zeon Cubic Formula)**.** *Let $\varphi(u) = u^3 + \alpha u^2 + \beta u + \gamma \in \mathbb{C}\mathfrak{Z}[u]$, let $q = \frac{1}{3}\beta - \frac{1}{9}\alpha^2$ and let $r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3$. Suppose $\mathfrak{C}q \neq 0$ and set $\Delta_\varphi = q^3 + r^2$. Suppose Δ_φ has a square root δ. Letting $s_1 \in (r+\delta)^{1/3}$ and $s_2 \in (r-\delta)^{1/3}$, it follows that $\varphi(u)$ has zeros given by*

$$\begin{aligned}
u_1 &= (s_1 + s_2) - \frac{\alpha}{3},\\
u_2 &= -\frac{1}{2}(s_1 + s_2) - \frac{\alpha}{3} + \frac{i\sqrt{3}}{2}(s_1 - s_2),\\
u_3 &= -\frac{1}{2}(s_1 + s_2) - \frac{\alpha}{3} - \frac{i\sqrt{3}}{2}(s_1 - s_2).
\end{aligned}$$

Proof. First, the general cubic equation $\varphi(u) = u^3 + \alpha u^2 + \beta u + \gamma = 0$ is depressed by the substitution $u \mapsto z - \alpha/3$ as follows

$$\begin{aligned}
\varphi(z - \alpha/3) &= (z - \alpha/3)^3 + \alpha(z - \alpha/3)^2 + \beta(z - \alpha/3) + \gamma\\
&= z^3 + \left(\beta - \frac{\alpha^2}{3}\right)z + \frac{2\alpha^3}{27} - \frac{\alpha\beta}{3} + \gamma\\
&= z^3 + \left(\beta - \frac{\alpha^2}{3}\right)z - 2\left(\frac{\alpha\beta}{6} - \frac{\alpha^3}{27} - \frac{\gamma}{2}\right).
\end{aligned}$$

Since $\mathfrak{C}q \neq 0$, the zeros of $\varphi(z - \alpha/3)$ are given by the depressed cubic formula of Theorem 3.2. In particular, the zeros are given by

$$z = A^{1/3} - \left(\frac{\beta}{3} - \frac{\alpha^2}{9}\right) A^{-1/3},$$

corresponding to the cube roots of

$$A = \left(\frac{\alpha\beta}{6} - \frac{\gamma}{2} - \frac{\alpha^3}{27}\right) \pm \sqrt{\left(\frac{\beta}{3} - \frac{\alpha^2}{9}\right)^3 + \left(\frac{\alpha\beta}{6} - \frac{\alpha^3}{27} - \frac{\gamma}{2}\right)^2}.$$

Letting $q = \beta/3 - \alpha^2/9$ and $r = (\alpha\beta - 3\gamma)/6 - \alpha^3/27$, we have $z = A^{1/3} - qA^{-1/3}$, where

$$A = r \pm \sqrt{q^3 + r^2}.$$

Letting δ be a square root of $\Delta_\varphi = q^3 + r^2$, we have $A = r \pm \delta$. Next, observe that $(r+\delta)(r-\delta) = r^2 - \delta^2 = -q^3$, so that

$$(r+\delta)^{-1} = -(r-\delta)q^{-3}.$$

It follows that $qA^{-1/3} = -(r-\delta)^{1/3}$. Hence, the first zero of the depressed cubic is $z_1 = s_1 + s_2$, where $s_1 = (r+\delta)^{1/3}$ and $s_2 = (r-\delta)^{1/3}$. Letting x_0 be a fixed cube root of A, it follows that $e^{i2\pi/3}x_0$ and $e^{i4\pi/3}x_0$ are the remaining cube roots, where $e^{i4\pi/3} = (e^{i2\pi/3})^{-1}$. Thus, the remaining zeros of the depressed cubic are

$$z_2 = e^{i2\pi/3}s_1 + e^{i4\pi/3}s_2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)s_1 + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)s_2 = -\frac{1}{2}(s_1+s_2) + i\frac{\sqrt{3}}{2}(s_1 - s_2)$$

and

$$z_3 = e^{i4\pi/3}s_1 + e^{i2\pi/3}s_2 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)s_1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)s_2 = -\frac{1}{2}(s_1+s_2) - i\frac{\sqrt{3}}{2}(s_1 - s_2).$$

Translating by $\alpha/3$ gives the zeros $u_j = z_j - \alpha/3$ of the general cubic for $j = 1, 2, 3$. □

4 Classification

As we have seen since beginning with Example 3.1, there can be multiple possible approaches to finding solutions of zeon cubic equations. It would be helpful to have a method for determining which methods are appropriate for a given zeon cubic. For that, we turn to a zeon extension of the cubic discriminant.

We recall that given a monic cubic polynomial (with real coefficients) $f(x) = x^3+ax^2+bx+c \in \mathbb{R}[x]$, the *cubic discriminant* of $f(x)$ is defined to be

$$\Delta_f = 18abc - 4a^3b + a^2b^2 - 4b^3 - 27c^2. \tag{4.1}$$

Letting $q = \frac{b}{3} - \frac{a^2}{9}$ and $r = \frac{1}{6}(ab - 3c) - \frac{a^3}{27}$, the discriminant is given by

$$\Delta_f = -4(3q)^3 - 27(2r)^2 = -108(q^3 + r^2).$$

Traditionally, the cubic discriminant is used to characterize the zeros of $f(x)$. In particular, the following properties are well known.

- When $\Delta_f = 0$, the cubic has a repeated root.
- When $\Delta_f < 0$, the cubic has three distinct real roots.
- When $\Delta_f > 0$, the cubic has one real root and a conjugate pair of complex roots.

We extend the cubic discriminant to zeon cubic polynomials by defining $\Delta_\varphi = q^3 + r^2$. In view of Theorems 3.2, and 3.5, the following classification is sensible for cubic polynomials over $\mathbb{C}\mathfrak{Z}$.

Theorem 4.1 (Classification)**.** *Let $\varphi(u) = u^3 + \alpha u^2 + \beta u + \gamma \in \mathbb{C}\mathfrak{Z}[u]$. Let $\Delta_\varphi = q^3 + r^2$, where*

$$q = \frac{1}{3}\beta - \frac{1}{9}\alpha^2, \quad r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3.$$

Then the following hold.

(1) *If $\mathfrak{C}\Delta_\varphi \neq 0$, then φ has three spectrally simple zeros. When $\mathfrak{C}q \neq 0$, the zeros are given by the cubic formula of Theorem 3.16. When $\mathfrak{C}q = 0$, the zeros are obtained from Algorithm 1 using the scalar zeros of f_φ.*

(2) *If $\mathfrak{C}\Delta_\varphi = 0$, then φ either has no zeros or has uncountably many zeros.*

Proof. Observing that $\Delta_{f_\varphi} = -108\mathfrak{C}\Delta_\varphi$, we see that the scalar polynomial f_φ has three distinct complex zeros when the discriminant is nonzero. Hence, φ has three spectrally simple zeon zeros when $\mathfrak{C}\Delta_\varphi \neq 0$.

It is clear that $\mathfrak{C}\Delta_\varphi = 0$ implies $\mathfrak{C}q \neq 0 \Leftrightarrow \mathfrak{C}r \neq 0$. It follows that the induced complex polynomial $f_\varphi(z - \mathfrak{C}\alpha/3) = z^3 + 3\mathfrak{C}qz - 2\mathfrak{C}r$ has a repeated root, λ_0. Thus, φ has no zeros or uncountably many zeros. If the repeated root λ_0 has multiplicity 2, there exists a spectrally simple zero μ of $\varphi(u)$ and uncountably many other zeros having common scalar part $\lambda_0 - \mathfrak{C}\alpha/3$. If λ_0 is a zero of multiplicity three and φ has zeros, then all zeros of $\varphi(u)$ have common scalar part $\lambda_0 - \mathfrak{C}\alpha/3$. $\square$

Example 4.2. *Consider the zeon cubic polynomial* $\varphi(u) = u^3 + \zeta_{\{1,2\}}u - (1 + 3\zeta_{\{2\}})$. *The zeon cubic discriminant of* φ *is* $\Delta_\varphi = \frac{1}{4} + \zeta_{\{2,3\}}$, *which is invertible. Hence,* φ *has three spectrally simple zeros. However, since* $q = \frac{1}{3}\zeta_{\{1,2\}}$ *is nilpotent, the cubic formula of Theorem 3.2 fails.*

The scalar zeros of $f_\varphi(z) = z^3 - 1$ *are* $\left\{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i\right\}$. *Applying Algorithm 1, rational approximations of the spectrally simple zeros are as follows:*

$$\begin{aligned}
\lambda_1 &= 1 - \frac{1}{3}\zeta_{\{1,2\}} + \frac{2}{3}\zeta_{\{2,3\}},\\
\lambda_2 &= \left(-\frac{1}{2} + \frac{181i}{209}\right) + \left(\frac{1}{6} + \frac{125i}{433}\right)\zeta_{\{1,2\}} - \left(\frac{1}{3} - \frac{153i}{265}\right)\zeta_{\{2,3\}},\\
\lambda_3 &= \left(-\frac{1}{2} - \frac{181i}{209}\right) + \left(\frac{1}{6} - \frac{153i}{530}\right)\zeta_{\{1,2\}} - \left(\frac{1}{3} + \frac{153i}{265}\right)\zeta_{\{2,3\}}.
\end{aligned}$$

4.1 Cubic polynomials over $\mathbb{C}\mathfrak{Z}_2$

In this section, the special case of cubic polynomials over $\mathbb{C}\mathfrak{Z}_2$ are considered. When φ is a cubic polynomial in $\mathbb{C}\mathfrak{Z}_2[u]$, there are no indeterminate cases.

Proposition 4.3. *Let* $\varphi(u) = u^3 + 3qu - 2r \in \mathbb{C}\mathfrak{Z}_2[u]$. *Let* $\Delta_\varphi = q^3 + r^2$, *where*

$$q = \frac{1}{3}\beta - \frac{1}{9}\alpha^2, \quad r = \frac{1}{6}(\beta\alpha - 3\gamma) - \frac{1}{27}\alpha^3.$$

(1) *If* Δ_φ *is invertible, then* $\varphi(u)$ *has three spectrally simple zeon zeros. The zeros are given by the cubic formula of Theorem 3.2 if* $\mathfrak{C}q \neq 0$. *Otherwise, the zeros are obtained from Algorithm 1 using the scalar zeros of* f_φ.

(2) *If* Δ_φ *is a nonzero null monomial of grade 2, then*

  (a) $\varphi(u)$ *has one spectrally simple zero and a set of spectrally non-simple zeros if* q *is invertible;*

  (b) $\varphi(u)$ *has no zeros if* q *is a nonzero nilpotent in* $\mathbb{C}\mathfrak{Z}_2$.

(3) *If* Δ_φ *is a nonzero nilpotent of minimal grade 1, then* $\varphi(u)$ *has no zeros.*

(4) *If* $\Delta_\varphi = 0$, *then*

  (a) $\varphi(u)$ *has a spectrally simple zeon zero and a set of spectrally non-simple zeon zeros if* r *is invertible;*

  (b) $\varphi(u)$ *has a set of spectrally non-simple zeros if* $r = a\zeta_{[2]}$ *for* $a \in \mathbb{C}$;

  (c) $\varphi(u)$ *has no zeros if* $r \neq 0$ *is nilpotent and not a null monomial of grade 2.*

Proof. The results follow from Theorems 3.2 and 3.5 along with the following observations.

(1) Nilpotent cube roots do not exist in $\mathbb{C}\mathfrak{Z}_2$.

(2) In $\mathbb{C}\mathfrak{Z}_2$, nilpotent square roots only exist for null monomials $a\zeta_{[2]}$.

To prove 2(b), suppose $\Delta_\varphi = a\zeta_{[2]}$ for nonzero $a \in \mathbb{C}$. If q is a nonzero nilpotent in $\mathbb{C}\mathfrak{Z}_2$, then $q^3 = 0$ so that $\Delta_\varphi = r^2$. It follows that $r = b\zeta_{\{1\}} + c\zeta_{\{2\}}$ for nonzero $b, c \in \mathbb{C}$. Any zeros of φ must also be nilpotent. Hence, any zero $z \in \mathbb{C}\mathfrak{Z}_2$ must satisfy

$$\varphi(z) = z^3 + 3qz - 2r = 3qz - 2(b\zeta_{\{1\}} + c\zeta_{\{2\}}) = 0,$$

where the minimal grade of qz is either 0 or 2. In either case, we have a contradiction.

Part 3 follows from the fact that a nilpotent of minimal grade 1 has no square roots.

Next, 4(b) is established as follows. If $r = a\zeta_{[2]}$ and $\Delta_\varphi = 0$, then $\mathfrak{C}q = 0$ so that $q^3 = 0$. If $q = s\zeta_i$ for any nonzero s, then

$$\varphi\left(\frac{2a}{3s}\zeta_{[2]\setminus\{i\}}\right) = \left(\frac{2a}{3}\zeta_{[2]\setminus\{i\}}\right)^3 + 3s\zeta_{\{i\}}\left(\frac{2a}{3s}\zeta_{[2]\setminus\{i\}}\right) - 2a\zeta_{[2]} = 0 + 2a\zeta_{[2]} - 2a\zeta_{[2]} = 0.$$

Turning to 4(c), suppose $r = a\zeta_{\{1\}} + b\zeta_{\{2\}}$ where $a, b \in \mathbb{C}$ are not both zero. If a and b are both nonzero, then $r^2 = 2ab\zeta_{[2]}$. Thus $\Delta_\varphi = 0$ requires $q^3 = -r^2$, which is impossible in $\mathbb{C}\mathfrak{Z}_2$. We conclude that $r = a\zeta_{\{i\}}$ for nonzero $a \in \mathbb{C}$ and $i \in \{1, 2\}$; further, we see that q is nilpotent. Hence, if $z \in \mathbb{C}\mathfrak{Z}_2$ is a zero of φ, it follows that

$$\varphi(z) = z^3 + 3qz - 2r = 3qz - 2a\zeta_{\{i\}} = 0,$$

where the minimal grade of qz is either 0 or 2. Again, we have a contradiction. □

5 Conclusion & avenues for further research

Zeros of cubic polynomials over $\mathbb{C}\mathfrak{Z}$ have been classified up to two indeterminate cases. In those indeterminate cases, sufficient conditions have been provided for existence of spectrally nonsimple zeon zeros. In the special case of cubic polynomials over $\mathbb{C}\mathfrak{Z}_2$, the zeros have been completely classified.

One obvious goal of future work is the consideration of zeros of quartic zeon polynomials over $\mathbb{C}\mathfrak{Z}$, particularly since the quartic is the highest order polynomial equation that can be solved by radicals in the general case. Based on existing results, a quartic polynomial $\varphi(u) = u^4 + \alpha u^3 + \beta u^2 + \gamma u + \delta$ having one spectrally simple zeon zero ω can be reduced by polynomial division to the product

$(u-\omega)\psi(u)$, where $\psi(u)$ is a monic cubic polynomial in $\mathbb{C}\mathfrak{Z}[u]$. The classification of cubic zeros established here can then be applied to $\psi(u)$. If $\varphi(u)$ has two simple zeros, the zeon quadratic formula can be applied to the remaining factor. If $\varphi(u)$ splits, all zeros can be found using Algorithm 1. If all zeros of $\varphi(u)$ are spectrally nonsimple, additional tools are needed: either an effective algorithm for computing spectrally nonsimple zeros or a zeon extension of the quartic formula.

More broadly, zeros of zeon polynomials are essential for considering spectral properties of zeon matrices. Letting Ψ denote an $m \times m$ matrix with entries from $\mathbb{C}\mathfrak{Z}$, eigenvalues of Ψ are spectrally simple zeon zeros of the characteristic polynomial of Ψ. Here Ψ is appropriately regarded as a $\mathbb{C}\mathfrak{Z}$-linear operator on the module $\mathbb{C}\mathfrak{Z}^m$. The zeon combinatorial Laplacian has recently been shown to enumerate paths and cycles in finite graphs, so its spectral properties are of particular interest [12]. With zeon eigenvalues in hand Putzer's theorem can also be useful for computing zeon matrix exponentials.

References

[1] L. M. Dollar and G. S. Staples, “Zeon roots,” *Adv. Appl. Clifford Algebr.*, vol. 27, no. 2, pp. 1133–1145, 2017, doi: 10.1007/s00006-016-0732-4.

[2] P. Feinsilver, “Zeon algebra, Fock space, and Markov chains,” *Commun. Stoch. Anal.*, vol. 2, no. 2, pp. 263–275, 2008, doi: 10.31390/cosa.2.2.06.

[3] E. Haake and G. S. Staples, “Zeros of zeon polynomials and the zeon quadratic formula,” *Adv. Appl. Clifford Algebr.*, vol. 29, no. 1, 2019, Art. ID 21, doi: 10.1007/s00006-019-0938-3.

[4] T. Mansour and M. Schork, “On the differential equation of first and second order in the zeon algebra,” *Adv. Appl. Clifford Algebr.*, vol. 31, no. 2, 2021, Art. ID 21, doi: 10.1007/s00006-021-01126-7.

[5] A. F. Neto, “Higher order derivatives of trigonometric functions, Stirling numbers of the second kind, and Zeon algebra,” *J. Integer Seq.*, vol. 17, no. 9, 2014, Art. 14.9.3.

[6] A. F. Neto, “Carlitz’s identity for the Bernoulli numbers and Zeon algebra,” *J. Integer Seq.*, vol. 18, no. 5, 2015, Art. ID 15.5.6.

[7] A. F. Neto, “A note on a theorem of Guo, Mező, and Qi,” *J. Integer Seq.*, vol. 19, no. 4, 2016, Art. ID 16.4.8.

[8] A. F. Neto and P. H. R. dos Anjos, “Zeon algebra and combinatorial identities,” *SIAM Rev.*, vol. 56, no. 2, pp. 353–370, 2014, doi: 10.1137/130906684.

[9] R. Schott and G. S. Staples, *Operator calculus on graphs.* Imperial College Press, London, 2012, doi: 10.1142/p843.

[10] T. Shifrin, *Abstract algebra: A geometric approach.* Prentice Hall Englewood Cliffs, New York, NY, 1995.

[11] G. S. Staples, “Spectrally simple zeros of zeon polynomials,” *Adv. Appl. Clifford Algebr.*, vol. 31, no. 4, 2021, Art. ID 66, doi: 10.1007/s00006-021-01167-y.

[12] G. S. Staples, “Zeon matrix inverses and the zeon combinatorial Laplacian,” *Adv. Appl. Clifford Algebr.*, vol. 31, no. 3, 2021, Art. ID 40, doi: 10.1007/s00006-021-01152-5.

[13] G. S. Staples, *Clifford algebras and zeons.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020.

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A note on constructing sine and cosine functions in discrete fractional calculus

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ABSTRACT

In this paper, we introduce two sets of linear fractional order h-difference equations and derive their solutions. These solutions, referred to as trigonometric functions of fractional h-discrete calculus, are proven to have properties similar to sine and cosine functions on $\mathbb{R}$. The illustrated graphs confirm these similarities.

RESUMEN

En este artículo, introducimos dos conjuntos de ecuaciones de h-diferencias lineales de orden fraccionario y derivamos sus soluciones. Probamos que estas soluciones, referidas como funciones trigonométricas del cálculo fraccionario h-discreto, tienen propiedades similares a las funciones seno y coseno en $\mathbb{R}$. Las gráficas ilustradas confirman estas similaridades.

Keywords and Phrases: Discrete trigonometric functions, nabla operator, fractional h-discrete calculus, fractional difference equations, Picard's iteration.

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1 Introduction

The linear second order differential equation

$$y''(t) + \omega^2 y(t) = 0,$$

where $t \in \mathbb{R}$ and ω is a nonzero real number, produces two linearly independent solutions. They are well-known trigonometric functions, $\sin(\omega t) = \sum_{n=0}^{\infty}(-1)^n \omega^{2n+1} \frac{t^{2n+1}}{(2n+1)!}$ and $\cos(\omega t) = \sum_{n=0}^{\infty}(-1)^n \omega^{2n} \frac{t^{2n}}{(2n)!}$. Picard's iteration method is one of the fundamental methods in applied mathematics to construct these infinite series. Motivated by this construction technique, we will use calculus on the discrete time domain $h\mathbb{N}_a = \{a, a+h, a+2h, \dots\}$, where $a \in \mathbb{R}$ and $h \in \mathbb{R}^+$, to derive corresponding sum representations for sine and cosine functions of h-discrete fractional calculus.

Discrete fractional calculus, also known as non-integer order calculus on a discrete domain, has garnered significant attention from mathematicians over the past decade. It offers a novel approach to analyzing differences (derivatives) and sums (integrals) of arbitrary (non-integer) orders within discrete settings. A recent book by Goodrich and Peterson [6] provides a comprehensive collection of pioneering results for discrete fractional calculus, with a particular focus on the case where $h = 1$. In particular, the results obtained on the domain $h\mathbb{N}_a$ extend the findings of fractional discrete calculus on $\mathbb{N}_a$. This generalization provides a more comprehensive framework for understanding and applying fractional calculus within discrete settings. For further reading on this generalized domain, we refer the reader to the following papers [1–3, 7–15].

In this article, our goal is to introduce and derive solutions for the following two sets of linear fractional order h-difference equations

$$\nabla^{\alpha}_{h,a} y(t) + \omega^2 y(t-h) = 0, \tag{1.1}$$

and

$$\nabla^{\alpha}_{h,a} y(t) + \omega^2 y(t) = 0, \tag{1.2}$$

where $t \in h\mathbb{N}_a$, $1 < \alpha < 2$. The equation (1.1) includes a time delay, while the equation (1.2) is formulated without a time delay. We also note that in [1] the authors proved that $\nabla^{\alpha}_{h} y(t)$ is getting close to $y''(t)$ when $\alpha \to 2$ and $h \to 0$. Hence in the limit position, the Eq. (1.2) is approximating to the second order differential equation, $y''(t) + \omega^2 y(t) = 0$.

The following theorem, found in [3], presents the solution to the fractional h-difference equation in terms of Mittag-Leffler type functions. A natural question arises: Do sine and cosine functions appear in this solution when $k = 2$? To explore this, we apply Picard's iteration method to derive the sine and cosine functions of fractional h-discrete calculus.

Theorem 1.1 ([3])**.** *Let $\lambda \in \mathbb{R}$, $h > 0$, $k \in N$, and $\alpha \in (k-1, k)$. The general solution of the following problem*

$$\nabla^{\alpha}_{h,0} y(t) = \lambda y(t-h), \quad t \in h\mathbb{N}_{kh}, \tag{1.3}$$

is given by

$$y(t) = C_1 \tilde{E}^{h}_{\lambda,\alpha,\alpha-1}(t,0) + C_2 \tilde{E}^{h}_{\lambda,\alpha,\alpha-2}(t,0) + \cdots + C_k \tilde{E}^{h}_{\lambda,\alpha,\alpha-k}(t,0),$$

where $C_1, C_2, \ldots, C_k$ are constants.

This paper is organized by following the outline given below. In order to make our calculations easy to follow, we provide basic definitions in h-discrete fractional calculus and related results in the preliminary section. We use Riemann-Liouville definition for the fractional derivative. Additionally, we develop techniques to convert Eqs. (1.1) and (1.2) into sum equations to apply Picard's iteration. Section 3 focuses on Eq. (1.1), where we define a iteration formula and derive two finite sums as solutions, illustrating their graphs for various values of α between one and two. Building on Section 3, we define two infinite series and state a theorem that outlines their properties and shows them as solutions to Eq. (1.2) in Section 4. Finally, we give a short concluding remark.

2 Preliminaries

Let h be any positive real number and a be any real number. We define $h\mathbb{N}_a$ to be the set $\{a, a+h, a+2h, \ldots\}$. Suppose $F : h\mathbb{N}_a \to \mathbb{R}$ is a function.

Definition 2.1 ([5])**.** *The forward and the backward h-difference operator are defined by*

$$\Delta_h F(t) = \frac{F(t+h) - F(t)}{h}, \quad t \in h\mathbb{N}_a,$$

and

$$\nabla_h F(t) = \frac{F(t) - F(t-h)}{h}, \quad t \in h\mathbb{N}_{a+h},$$

respectively.

Remark 2.2. *Throughout this paper, we suggest that the reader considers the following:*

(i) if $h = 1$, we have the backward difference operator, or nabla operator (∇)

$$\nabla F(t) = F(t) - F(t-1), \quad t \in \mathbb{N}_{a+1};$$

(ii) if $\lim\limits_{h\to 0} \dfrac{F(t) - F(t-h)}{h}$ exists, then we have $\lim\limits_{h\to 0} \nabla_h F(t) = F'(t)$.

Definition 2.3 ([3]). *For any t, $r \in \mathbb{R}$, the h-rising factorial function is defined by*

$$t_h^{\overline{r}} = h^r \frac{\Gamma(\frac{t}{h}+r)}{\Gamma(\frac{t}{h})},$$

whenever the quotient is well-defined. Here $\Gamma(\cdot)$ denotes the Euler gamma function.

Definition 2.4 ([3]). *Let $a \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. The nabla h-fractional sum of order α is defined by*

$$\nabla_{h,a}^{-\alpha} F(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a/h}^{t/h} (t-\rho(sh))_h^{\overline{\alpha-1}} F(sh)h, \quad t \in h\mathbb{N}_a,$$

where $h \in \mathbb{R}^+$ and $\rho(t) = t - h$.

Definition 2.5 ([3]). *The nabla h-fractional difference of order α in the sense of Riemann–Liouville is defined by*

$$\nabla_{h,a}^{\alpha} F(t) := \nabla_h^n \nabla_{h,a}^{-(n-\alpha)} F(t), \quad t \in h\mathbb{N}_{a+nh},$$

where $a \in \mathbb{R}$, $n-1 < \alpha \leq n$, and $n \in \mathbb{N}$.

Lemma 2.6 ([3]). *Let $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$ such that $\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}$ and $\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}$ are defined. Then we have that*

(i) $\nabla_{h,a}^{-\alpha}(t-\rho(a))_h^{\overline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-\rho(a))_h^{\overline{\beta+\alpha}}, \quad t \in h\mathbb{N}_a.$

(ii) $\nabla_{h,a}^{\alpha}(t-\rho(a))_h^{\overline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-\rho(a))_h^{\overline{\beta-\alpha}}, \quad t \in h\mathbb{N}_a.$

In the following sections, we use Lemma 2.6 as one of the main tools to obtain some important identities. We want to note that $\dfrac{1}{\Gamma(-n)}$ for $n \in \mathbb{N}_0$ is considered as zero. The proof of the next lemma is elementary. We omit the proof.

Lemma 2.7. *Let $b \in hN_a$. The following is valid:*

$$\nabla_h (b-t)_h^{\overline{\beta}} = -\beta(b-\rho(t))_h^{\overline{\beta-1}}, \quad t \in h\mathbb{N}_a.$$

The following equality is known as Leibniz's rule for the nabla difference operator. The proof can be adapted from its proof in time scales calculus [5].

Lemma 2.8. *For a function $G : h\mathbb{N}_a \times \mathbb{N} \to \mathbb{R}$, the following is valid:*

$$\nabla_h \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} G(t, sh)h = \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} \nabla_h G(t, sh)h + G(\rho(t), t),$$

where $t \in h\mathbb{N}_a$.

Next, we demonstrate how ∇_h and $\nabla_{h,a}^{-\alpha}$ commute in a theorem.

Theorem 2.9. *For any positive real number α, the following equality holds:*

$$\nabla_{h,a+h}^{-\alpha}\nabla_h f(t) = \nabla_h \nabla_{h,a}^{-\alpha} f(t) - \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a).$$

Proof. Using Lemma 2.7 and the summation by parts formula in h-discrete calculus, we have

$$\begin{aligned}
\nabla_{h,a+h}^{-\alpha}\nabla_h f(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=\frac{a+h}{h}}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{\alpha-1}} \nabla_h f(sh)h \\
&= \frac{(t-\rho(sh))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(sh)\Big|_{\frac{a}{h}}^{\frac{t}{h}} + \frac{(\alpha-1)}{\Gamma(\alpha)} \sum_{s=\frac{a+h}{h}}^{\frac{t}{h}} (t+h-\rho(sh))_h^{\overline{\alpha-2}} f(\rho(sh))h \\
&= h^{\alpha-1} f(t) - \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) + \frac{1}{\Gamma(\alpha-1)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-1} (t-\rho(sh))_h^{\overline{\alpha-2}} f(sh)h \\
&= -\frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) + \frac{1}{\Gamma(\alpha-1)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{\alpha-2}} f(sh)h \\
&= \nabla_h \nabla_{h,a}^{-\alpha} f(t) - \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a).
\end{aligned}$$

□

This result can be generalized for the operator ∇_h^n using the principle of mathematical induction.

Theorem 2.10. *Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$. The following equality holds.*

$$\nabla_{h,a+nh}^{-\alpha}\nabla_h^n f(t) = \nabla_h^n \nabla_{h,a}^{-\alpha} f(t) - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \nabla_h^{n-k-1} (t-kh-\rho(a))_h^{\overline{\alpha-1}} \nabla_h^k f(t)\Big|_{t=a+kh}.$$

Proof. The proof of the equality follows from Theorem 2.9 for $n = 1$ and the induction assumption for $n > 1$

$$\nabla_{h,a+nh}^{-\alpha}\nabla_h^n f(t) = \nabla_h^n \nabla_{h,a}^{-\alpha} f(t) - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \nabla_h^{n-k-1} (t-kh-\rho(a))_h^{\overline{\alpha-1}} \nabla_h^k f(t)\Big|_{t=a+kh}.$$

For $n+1$, we have

$$\begin{aligned}
\nabla_{h,a+(n+1)h}^{-\alpha}\nabla_h^{n+1} f(t) &= \nabla_{h,a+nh+h}^{-\alpha}\nabla_h \nabla_h^n f(t) \\
&= \nabla_h \nabla_{h,a+nh}^{-\alpha} \nabla_h^n f(t) - \frac{(t-nh-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_h^k f(t)\Big|_{t=a+nh} \\
&= I.
\end{aligned}$$

Next we use the induction assumption on the quantity $\nabla_h \nabla_{h,a+nh}^{-\alpha} \nabla_h^n f(t)$ to obtain

$$\begin{aligned} I &= \nabla_h \left[\nabla_h^n \nabla_{h,a}^{-\alpha} f(t) - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \nabla_h^{n-k-1} (t-kh-\rho(a))_h^{\overline{\alpha-1}} \nabla_h^k f(t)\Big|_{t=a+kh} \right] \\ &\quad - \frac{(t-nh-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_h^k f(t)\Big|_{t=a+nh} \\ &= \nabla_h^{n+1} \nabla_{h,a}^{-\alpha} f(t) - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \nabla_h^{n-k} (t-kh-\rho(a))_h^{\overline{\alpha-1}} \nabla_h^k f(t)\Big|_{t=a+kh} \\ &\quad - \frac{(t-nh-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_h^k f(t)\Big|_{t=a+nh} \\ &= \nabla_h^{n+1} \nabla_{h,a}^{-\alpha} f(t) - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \nabla_h^{n-k} (t-kh-\rho(a))_h^{\overline{\alpha-1}} \nabla_h^k f(t)\Big|_{t=a+kh}. \end{aligned}$$

This completes the proof. □

We close the preliminary section with the following lemma. The identities we have in this lemma will be used in the following sections to shortened the quantities in our calculations.

Lemma 2.11. *Let $1 < \alpha < 2$. The following are valid.*

(i) $\nabla_{h,a}^{-(2-\alpha)} f(t)\Big|_{t=a} = h^{2-\alpha} f(a)$,

(ii) $\nabla_h \nabla_{h,a}^{-(2-\alpha)} f(t)|\Big|_{t=a+h} = h^{1-\alpha} \left[f(a+h) + (1-\alpha) f(a) \right].$

Proof. The proof of the part (i) follows from the definition of the fractional h-difference operator. Indeed we have,

$$\begin{aligned} \nabla_{h,a}^{-(2-\alpha)} f(t)\Big|_{t=a} &= \frac{1}{\Gamma(2-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{2-\alpha-1}} f(sh)h\Big|_{t=a} \\ &= \frac{h}{\Gamma(2-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{a}{h}} (a-\rho(sh))_h^{\overline{1-\alpha}} f(sh) \\ &= \frac{h}{\Gamma(2-\alpha)} (a-\rho(a))_h^{\overline{1-\alpha}} f(a) = h^{2-\alpha} f(a). \end{aligned}$$

For the proof of the part (ii), we use Lemma 2.8 as a tool. Hence we have

$$\nabla_h \nabla_{h,a}^{-(2-\alpha)} f(t)|\Big|_{t=a+h} = \nabla_h \left[\frac{1}{\Gamma(2-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{1-\alpha}} f(sh)h \right]\Big|_{t=a+h}$$

$$= \frac{1}{\Gamma(2-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}} \nabla_h [(t-\rho(sh))_h^{\overline{1-\alpha}}] f(sh)h \Big|_{t=a+h} +$$
$$(t-h-\rho(t))_h^{\overline{1-\alpha}}] f(t) \Big|_{t=a+h}$$
$$= \frac{h(1-\alpha)}{\Gamma(2-\alpha)} \sum_{s=\frac{a}{h}}^{\frac{a}{h}+1} (a+h-\rho(sh))_h^{\overline{-\alpha}} f(sh)$$
$$= \frac{h(1-\alpha)}{\Gamma(2-\alpha)} \Big[(2h)_h^{\overline{-\alpha}} f(a) + (h)_h^{\overline{-\alpha}} f(a+h)| \Big]$$
$$= h^{1-\alpha} \left[f(a+h) + (1-\alpha) f(a) \right]. \qquad \square$$

3 A fractional order h-difference equation with delay

Here we consider the following α-th order linear fractional h-difference equation

$$\nabla_h^\alpha y(t) + \omega^2 y(t-h) = 0, \tag{3.1}$$

where $1 < \alpha < 2$ and $\omega \in \mathbb{R}$.

Apply the operator $\nabla_{h,a+2h}^{-\alpha}$ to each side of the equation (3.1) to obtain

$$\nabla_{h,a+2h}^{-\alpha} \nabla_h^2 \nabla_{h,a}^{-(2-\alpha)} y(t) + \nabla_{h,a+2h}^{-\alpha} \omega^2 y(t-h) = 0.$$

Apply Theorem 2.10 to obtain

$$\nabla_h^2 \nabla_{h,a}^{-\alpha} \nabla_{h,a}^{-(2-\alpha)} y(t) - \sum_{k=0}^{1} \frac{\nabla_h^{1-k} (t-kh-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_h^k \nabla_{h,a}^{-(2-\alpha)} y(t)|_{t=a+kh} + \nabla_{h,a+2h}^{-\alpha} \omega^2 y(t-h) = 0.$$

Hence we have

$$\nabla_h^2 \nabla_{h,a}^{-\alpha} \nabla_{h,a}^{-(2-\alpha)} y(t) = \frac{(t-\rho(a))_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} \nabla_{h,a}^{-(2-\alpha)} y(t)|_{t=a} + \frac{(t-h-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_h \nabla_{h,a}^{-(2-\alpha)} y(t)|_{t=a+h}$$
$$- \omega^2 \nabla_{h,a+2h}^{-\alpha} y(t-h).$$

It follows from Lemma 2.11 and the composition property for the fractional sum operators (Lemma 2 in [2]), we have

$$y(t) = \frac{(t-\rho(a))_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} h^{2-\alpha} y(a) + \frac{(t-h-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h^{1-\alpha} \left[(1-\alpha) y(a) + y(a+h) \right]$$
$$- \omega^2 \nabla_{h,a+2h}^{-\alpha} y(t-h). \tag{3.2}$$

Conversely, assume that y has the representation (3.2). We first note that

$$\nabla_{h,a+2h}^{-\alpha}y(t)=\nabla_{h,a}^{-\alpha}y(t)-\frac{1}{\Gamma(\alpha)}(t-\rho(a+h)_h^{\overline{\alpha-1}}y(a+h)h-\frac{1}{\Gamma(\alpha)}(t-\rho(a))_h^{\overline{\alpha-1}}y(a)h.$$

In addition to the above equality, we use the power rule (Lemma 2.6) and the composition property for the fractional sum operators (Lemma 2 in [2]) to derive from (3.2) by applying the operator ∇_h^α to the each side of the equation to obtain

$$\begin{aligned}\nabla_h^\alpha y(t)=&\nabla_h^\alpha\left[\frac{(t-\rho(a))_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}h^{2-\alpha}y(a)+\frac{(t-h-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}h^{1-\alpha}[(1-\alpha)y(a)+y(a+h)]\right.\\&\left.-\omega^2\nabla_{h,a+2h}^{-\alpha}y(t-h)\right]\\&=\nabla_h^\alpha\left[\frac{(t-\rho(a))_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}h^{2-\alpha}y(a)+\frac{(t-h-\rho(a))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}h^{1-\alpha}\left[(1-\alpha)y(a)+y(a+h)\right]\right]\\&-\omega^2\nabla_h^\alpha\left[\nabla_{h,a}^{-\alpha}y(t-h)-\frac{1}{\Gamma(\alpha)}(t-h-\rho(a+h))_h^{\overline{\alpha-1}}y(a+h)h\right.\\&\left.-\frac{1}{\Gamma(\alpha)}(t-h-\rho(a))_h^{\overline{\alpha-1}}y(a)h\right]=-\omega^2y(t-h).\end{aligned}$$

Thus, we have proved the following lemma.

Lemma 3.1. *y is a solution of the problem* (3.1)*, if and only if, y has the representation* (3.2)*.*

Next, for the simplicity in our calculations, we consider $a=0$. We define a sequence of functions on $h\mathbb{N}_0$ as follows:

$$\begin{aligned}y_0(t)&=\frac{(t+h)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}h^{2-\alpha}y(0)+\frac{(t)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}h^{1-\alpha}\left[(1-\alpha)y(0)+y(h)\right],\\y_n(t)&=-\omega^2\nabla_{h,2h}^{-\alpha}y_{n-1}(t-h),\end{aligned}$$

for $n\in\mathbb{N}_1$.

Using this iteration formula along with the power rule (Lemma 2.6), we observe that $\sum_{n=0}^{\infty}y_n(t)$ truncates to the following finite sum

$$\begin{aligned}&h^{2-\alpha}y(0)\sum_{n=0}^{\frac{t}{h}}(-1)^n\omega^{2n}\frac{(t-(n-1)h)_h^{\overline{(n+1)\alpha-2}}}{\Gamma((n+1)\alpha-1)}\\&\qquad+h^{1-\alpha}\left[(1-\alpha)y(0)+y(h)\right]\sum_{n=0}^{\frac{t}{h}}(-1)^n\omega^{2n}\frac{(t-nh)_h^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)}.\end{aligned}$$

To convince the reader, we present a few elements in the sequence $< y_n(t) >$ and explain why we have finite sum instead of infinite sum at the end of this process.

$$\begin{aligned} y_1(t) &= -\omega^2 \nabla_{h,2h}^{-\alpha} y_0(t-h) \\ &= -\omega^2 \nabla_{h,2h}^{-\alpha} \left[\frac{(t)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} h^{2-\alpha} y(0) + \frac{(t-h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h^{1-\alpha} \left[(1-\alpha)y(0)+y(h)\right] \right] \\ &= -\omega^2 \left[\frac{(t)_h^{\overline{2\alpha-2}}}{\Gamma(2\alpha-1)} h^{2-\alpha} y(0) + \frac{(t-h)_h^{\overline{2\alpha-1}}}{\Gamma(2\alpha)} h^{1-\alpha} \left[(1-\alpha)y(0)+y(h)\right] \right]. \end{aligned}$$

We repeat this calculation for y_2 to obtain the general term y_n of the sequence. Our main tool is the power rule (Lemma 2.6).

$$\begin{aligned} y_1(t) &= -\omega^2 \nabla_{h,2h}^{-\alpha} y_1(t-h) \\ &= -\omega^2 \nabla_{h,2h}^{-\alpha} \left[-\omega^2 \left[\frac{(t-h)_h^{\overline{2\alpha-2}}}{\Gamma(2\alpha-1)} h^{2-\alpha} y(0) + \frac{(t-2h)_h^{\overline{2\alpha-1}}}{\Gamma(2\alpha)} h^{1-\alpha} \left[(1-\alpha)y(0)+y(h)\right] \right] \right] \\ &= \omega^4 \left[\frac{(t-h)_h^{\overline{3\alpha-2}}}{\Gamma(3\alpha-1)} h^{2-\alpha} y(0) + \frac{(t-2h)_h^{\overline{3\alpha-1}}}{\Gamma(3\alpha)} h^{1-\alpha} \left[(1-\alpha)y(0)+y(h)\right] \right]. \end{aligned}$$

From this, the general term $y_n(t)$ follows.

$$y_n(t) = (-1)^n \omega^{2n} \left[h^{2-\alpha} y(0) \frac{(t-(n-1)h)_h^{\overline{(n+1)\alpha-2}}}{\Gamma((n+1)\alpha-1)} + h^{1-\alpha} \left[(1-\alpha)y(0)+y(h)\right] \frac{(t-nh)_h^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)} \right].$$

When we consider the infinite sum $\sum_{n=0}^{\infty} y_n(t)$, the terms with h-rising factorial powers become zero for $n > \frac{t}{h}$. Hence we obtain

$$\begin{aligned} h^{2-\alpha} y(0) \sum_{n=0}^{\frac{t}{h}} (-1)^n \omega^{2n} \frac{(t-(n-1)h)_h^{\overline{(n+1)\alpha-2}}}{\Gamma((n+1)\alpha-1)} \\ + h^{1-\alpha} \left[(1-\alpha)y(0)+y(h)\right] \sum_{n=0}^{\frac{t}{h}} (-1)^n \omega^{2n} \frac{(t-nh)_h^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)}. \end{aligned}$$

If we look closely at this sum which is the general solution of the fractional difference equation (3.1), we observe that there are two linearly independent solutions. Hence we define these two linearly independent solutions as sine and cosine functions.

We define

$$C_h(t,\alpha,\omega) = \sum_{n=0}^{\frac{t}{h}} (-1)^n \omega^{2n} \frac{(t+h-nh)_h^{\overline{(n+1)\alpha-2}}}{\Gamma((n+1)\alpha-1)}$$

and

$$S_h(t,\alpha,\omega)=\sum_{n=0}^{\frac{t}{h}}(-1)^n\omega^{2n+1}\frac{(t-nh)_h^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)},$$

$t\in h\mathbb{N}_0$. It turns out that $\tilde{E}^h_{-\omega^2,\alpha,\alpha-2}(t+h,0)=C_h(t,\alpha,\omega)$ and $\tilde{E}^h_{-\omega^2,\alpha,\alpha-1}(t,0)=S_h(t,\alpha,\omega)$ when we compare the above solutions with the solution representation in Theorem 1.1.

Next we list some properties of these functions.

Theorem 3.2. *The following equalities are valid.*

(i) $\Delta_h S_h(t,\alpha,\omega)=\omega C_h(t,\alpha,\omega)$.

(ii) $h^{2-\alpha}C_h(0,\alpha,\omega)=1, \qquad S_h(0,\alpha,\omega)=0$.

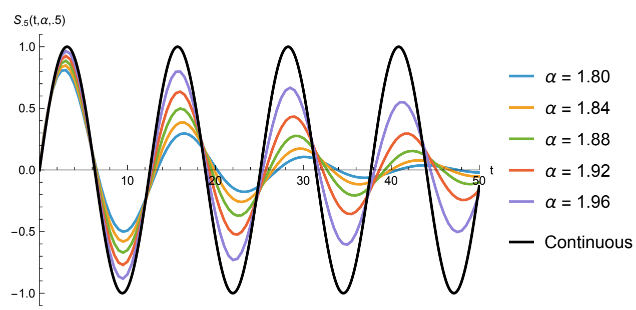
(iii) $\nabla^\alpha_{h,a}C_h(t,\alpha,\omega)+\omega^2 C_h(t-h,\alpha,\omega)=0$.

(iv) $\nabla^\alpha_{h,a}S_h(t,\alpha,\omega)+\omega^2 S_h(t-h,\alpha,\omega)=0$.

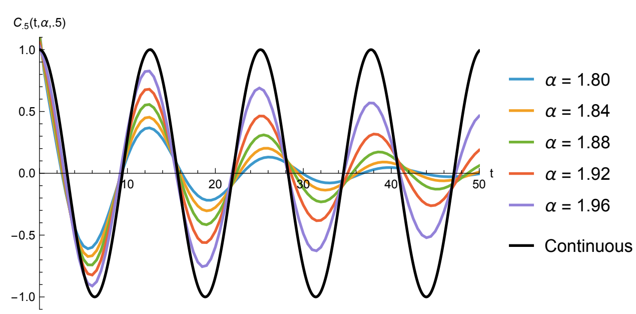
Proof. The proofs of (i) and (ii) are straightforward from the definitions of the functions C_h and S_h. The proofs of (iii) and (iv) can be found at Theorem 3.6 in [3]. □

Remark 3.3. *In Figure 1, we illustrate the graphs of $C_h(t,\alpha,\omega)$ and $S_h(t,\alpha,\omega)$ for a small value of h and for several α values between one and two.*

Figure 1: Family of graphs of $S_{.5}(t,\alpha,.5)$ and $C_{.5}(t,\alpha,.5)$.



(a) $S_{.5}(t,\alpha,.5)$



(b) $C_{.5}(t,\alpha,.5)$

4 A fractional order h-difference equation without delay

Here we consider the following α-th order linear fractional h-difference equation

$$\nabla_{h,a}^{\alpha} y(t) + \omega^2 y(t) = 0, \tag{4.1}$$

where $1 < \alpha < 2$ and $\omega \in \mathbb{R}$. We assume that $\omega^2 h^\alpha < 1$.

Here we define

$$Cos_h(t, \alpha, \omega) = (1 + \omega^2 h^\alpha) \sum_{n=0}^{\infty} (-1)^n \omega^{2n} \frac{(t+h)_h^{\overline{(n+1)\alpha - 2}}}{\Gamma((n+1)\alpha - 1)}$$

and

$$Sin_h(t, \alpha, \omega) = (1 + \omega^2 h^\alpha) \sum_{n=0}^{\infty} (-1)^n \omega^{2n+1} \frac{(t)_h^{\overline{(n+1)\alpha - 1}}}{\Gamma((n+1)\alpha)},$$

$t \in h\mathbb{N}_0$. These series are convergent when $\omega^2 h^\alpha < 1$.

Next we list some properties of these functions. We omit their proof since they are mainly relying on the power rule (Lemma 2.6).

Theorem 4.1. *The following equalities are valid.*

(i) $\Delta_h Sin_h(t, \alpha, \omega) = \omega Cos_h(t, \alpha, \omega)$.

(ii) $h^{2-\alpha} Cos_h(0, \alpha, \omega) = 1, \quad Sin_h(0, \alpha, \omega) = 0$.

(iii) $\nabla_{h,a}^{\alpha} Cos_h(t, \alpha, \omega) + \omega^2 Cos_h(t, \alpha, \omega) = 0$.

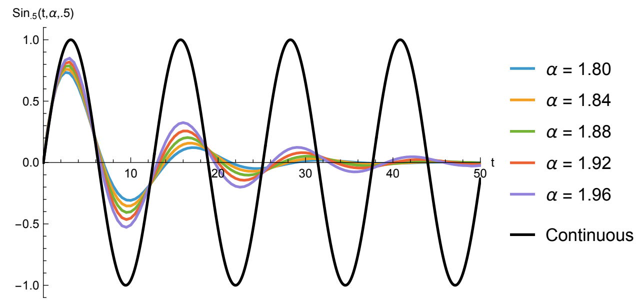
(iv) $\nabla_{h,a}^{\alpha} Sin_h(t, \alpha, \omega) + \omega^2 Sin_h(t, \alpha, \omega) = 0$.

Remark 4.2. *In Figure 2, we illustrate the graphs of $Cos_h(t, \alpha, \omega)$ and $Sin_h(t, \alpha, \omega)$ for a small value of h and for several α values between one and two.*

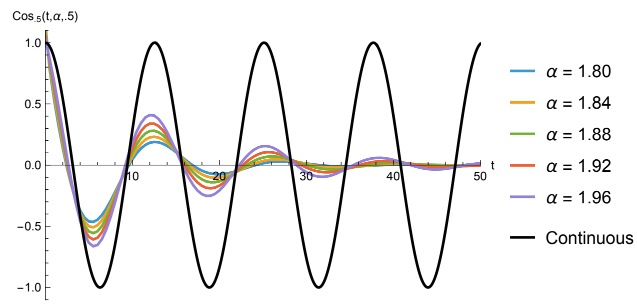
5 A concluding remark

The development of fractional calculus on the set $h\mathbb{N}_a = \{a, a+h, a+2h, \dots\}$ has shown promising results. In a recent paper [4], the pharmacokinetic (PK)-pharmacodynamic (PD) model was formulated on this time domain, with the PK component defined on an hourly basis and the PD component on a daily basis. h-discrete calculus offers the flexibility to select the right h values, enabling the construction of such an advanced model. Continuous improvement in existing models, whether in science, technology, or any other field, often hinges on the development of new theories and the refinement of analytical methods. Such a development of the theory starts with construction of the basic functions.

Figure 2: Family of graphs of $Sin_{.5}(t, \alpha, .5)$ and $Cos_{.5}(t, \alpha, .5)$.



(a) $Sin_{.5}(t, \alpha, .5)$



(b) $Cos_{.5}(t, \alpha, .5)$

In this article, we employed the widely recognized applied mathematics technique, Picard's iteration, to develop sine and cosine like functions within the framework of h-discrete fractional calculus. We constructed these functions as solutions to some linear fractional h-difference equations and illustrated their graphs. Sine and cosine functions as infinite sums can be calculated using a similar matrix method as in [2]. All these functions are potential candidates for application in various areas of mathematics. Deriving their analytical properties is just one of many open problems to explore.

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References

[1] F. M. Atıcı, S. Chang, and J. Jonnalagadda, “Grünwald-Letnikov fractional operators: from past to present,” *Fract. Differ. Calc.*, vol. 11, no. 1, pp. 147–159, 2021.

[2] F. M. Atıcı, S. Chang, and J. M. Jonnalagadda, “Mittag–leffler functions in discrete time,” *Fractal and Fractional*, vol. 7, no. 3, 2023, Art. ID 254, doi: 10.3390/fractalfract7030254.

[3] F. M. Atıcı, F. M. Dadashova, and K. Jonnalagadda, “Linear fractional order h-difference equations,” *Int. J. Difference Equ.*, vol. 15, no. 2, pp. 281–300, 2020.

[4] F. M. Atıcı, N. Nguyen, K. Dadashova, S. E. Pedersen, and G. Koch, “Pharmacokinetics and pharmacodynamics models of tumor growth and anticancer effects in discrete time,” *Comput. Math. Biophys.*, vol. 8, pp. 114–125, 2020, doi: 10.1515/cmb-2020-0105.

[5] M. Bohner and A. Peterson, *Dynamic equations on time scales.* Birkhäuser Boston, Inc., Boston, MA, 2001, doi: 10.1007/978-1-4612-0201-1.

[6] C. Goodrich and A. C. Peterson, *Discrete fractional calculus.* Springer, Cham, 2015, doi: 10.1007/978-3-319-25562-0.

[7] J. W. He, C. Lizama, and Y. Zhou, “The Cauchy problem for discrete time fractional evolution equations,” *J. Comput. Appl. Math.*, vol. 370, 2020, Art. ID 112683, doi: 10.1016/j.cam.2019.112683.

[8] B. Jia, F. Du, L. Erbe, and A. Peterson, “Asymptotic behavior of nabla half order h-difference equations,” *J. Appl. Anal. Comput.*, vol. 8, no. 6, pp. 1707–1726, 2018, doi: 10.11948/2018.1707.

[9] X. Liu, F. Du, D. Anderson, and B. Jia, “Monotonicity results for nabla fractional h-difference operators,” *Math. Methods Appl. Sci.*, vol. 44, no. 2, pp. 1207–1218, 2021, doi: 10.1002/mma.6823.

[10] X. Liu, A. Peterson, B. Jia, and L. Erbe, “A generalized h-fractional Gronwall’s inequality and its applications for nonlinear h-fractional difference systems with ‘maxima’,” *J. Difference Equ. Appl.*, vol. 25, no. 6, pp. 815–836, 2019, doi: 10.1080/10236198.2018.1551382.

[11] D. Mozyrska and E. Girejko, “Overview of fractional h-difference operators,” in *Advances in harmonic analysis and operator theory*, ser. Oper. Theory Adv. Appl. Birkhäuser/Springer Basel AG, Basel, 2013, vol. 229, pp. 253–268, doi: 10.1007/978-3-0348-0516-2_14.

[12] D. Mozyrska, E. Girejko, and M. Wyrwas, “Comparison of h-difference fractional operators,” in *Advances in the theory and applications of non-integer order systems*, ser. Lect. Notes Electr. Eng. Springer, Cham, 2013, vol. 257, pp. 191–197, doi: 10.1007/978-3-319-00933-9_17.

[13] S. Shaimardan, "Fractional order Hardy-type inequality in fractional h-discrete calculus," *Math. Inequal. Appl.*, vol. 22, no. 2, pp. 691–702, 2019, doi: 10.7153/mia-2019-22-47.

[14] I. Suwan, S. Owies, and T. Abdeljawad, "Monotonicity results for h-discrete fractional operators and application," *Adv. Difference Equ.*, 2018, Art. ID 207, doi: 10.1186/s13662-018-1660-5.

[15] M. Wyrwas, E. Pawluszewicz, and E. Girejko, "Stability of nonlinear h-difference systems with n fractional orders," *Kybernetika (Prague)*, vol. 51, no. 1, pp. 112–136, 2015, doi: 10.14736/kyb-2015-1-0112.

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A sub-elliptic system with strongly coupled critical terms and concave nonlinearities

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ABSTRACT

In this work, we study the Nehari manifold and its application to the following sub-elliptic system involving strongly coupled critical terms and concave nonlinearities:

$$\begin{cases} -\Delta_{\mathbb{G}} u = \dfrac{\eta_1\alpha_1}{2^*}\,|u|^{\alpha_1-2}|v|^{\beta_1}u + \dfrac{\eta_2\alpha_2}{2^*}\,|u|^{\alpha_2-2}|v|^{\beta_2}u & \\ \qquad + \lambda\, g(z)\,|u|^{q-2}u, & z\in\Omega, \\ -\Delta_{\mathbb{G}} v = \dfrac{\eta_1\beta_1}{2^*}\,|u|^{\alpha_1}|v|^{\beta_1-2}v + \dfrac{\eta_2\beta_2}{2^*}\,|u|^{\alpha_2}|v|^{\beta_2-2}v & \\ \qquad + \mu\, h(z)\,|v|^{q-2}v, & z\in\Omega, \\ \quad u = v = 0, & z\in\partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{G}$ with smooth boundary, $-\Delta_{\mathbb{G}}$ is the sub-Laplacian on a Carnot group $\mathbb{G}$; η_1, η_2, λ, μ, are positive, $\alpha_1+\beta_1 = 2^*$, $\alpha_2+\beta_2 = 2^*$, $1<q<2$, $2^* = \frac{2Q}{Q-2}$ is the critical Sobolev exponent, and Q is the homogeneous dimension of $\mathbb{G}$. By exploiting the Nehari manifold and variational methods, we prove that the system has at least two positive solutions.

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RESUMEN

En este trabajo, estudiamos la variedad de Nehari y su aplicación al siguiente sistema sub-elíptico que involucra términos críticos fuertemente acoplados y nolinealidades cóncavas:

$$\begin{cases} -\Delta_{\mathbb{G}} u = \dfrac{\eta_1 \alpha_1}{2^*} |u|^{\alpha_1 - 2}|v|^{\beta_1} u + \dfrac{\eta_2 \alpha_2}{2^*} |u|^{\alpha_2 - 2}|v|^{\beta_2} u \\ \qquad + \lambda\, g(z)\, |u|^{q-2} u, & z \in \Omega, \\ -\Delta_{\mathbb{G}} v = \dfrac{\eta_1 \beta_1}{2^*} |u|^{\alpha_1}|v|^{\beta_1 - 2} v + \dfrac{\eta_2 \beta_2}{2^*} |u|^{\alpha_2}|v|^{\beta_2 - 2} v \\ \qquad + \mu\, h(z)\, |v|^{q-2} v, & z \in \Omega, \\ \quad u = v = 0, & z \in \partial\Omega, \end{cases}$$

donde Ω es un conjunto abierto acotado de $\mathbb{G}$ con frontera suave, $-\Delta_{\mathbb{G}}$ es el sub-Laplaciano en un grupo de Carnot $\mathbb{G}$; η_1, η_2, λ, μ, son positivas, $\alpha_1 + \beta_1 = 2^*$, $\alpha_2 + \beta_2 = 2^*$, $1 < q < 2$, $2^* = \frac{2Q}{Q-2}$ es el exponente crítico de Sobolev, y Q es la dimensión homogénea de $\mathbb{G}$. Usando la variedad de Nehari y métodos variacionales, demostramos que el sistema tiene al menos dos soluciones positivas.

Keywords and Phrases: Sub-Laplacian, concave-convex nonlinearities, strongly coupled critical terms, Nehari manifold.

2020 AMS Mathematics Subject Classification: 35J60, 47J30.

1 Introduction

In this paper, we are concerned with the sub-Laplacian system involving strongly coupled critical terms and concave nonlinearities on the Carnot group $\mathbb{G}$ given below

$$\begin{cases} -\Delta_{\mathbb{G}} u = \frac{\eta_1 \alpha_1}{2^*} |u|^{\alpha_1 - 2} |v|^{\beta_1} u + \frac{\eta_2 \alpha_2}{2^*} |u|^{\alpha_2 - 2} |v|^{\beta_2} u + \lambda g(z) |u|^{q-2} u, & z \in \Omega, \\ -\Delta_{\mathbb{G}} v = \frac{\eta_1 \beta_1}{2^*} |u|^{\alpha_1} |v|^{\beta_1 - 2} v + \frac{\eta_2 \beta_2}{2^*} |u|^{\alpha_2} |v|^{\beta_2 - 2} v + \mu h(z) |v|^{q-2} v, & z \in \Omega, \\ u = v = 0, & z \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{G}$ with smooth boundary, $-\Delta_{\mathbb{G}}$ is the sub-Laplacian on a Carnot group $\mathbb{G}$. λ, μ, are positive, $2^* = \frac{2Q}{Q-2}$ is the critical Sobolev exponent, and Q is the homogeneous dimension of $\mathbb{G}$. We consider the following conditions:

$(\mathcal{A}_0)$ $Q \geq 4$, $1 < q < 2$, $0 < \eta_i < \infty$, $\alpha_i, \beta_i > 1$ and $\alpha_i + \beta_i = 2^*$ $(i = 1, 2)$,

and we give the following assumptions on the weight functions g and h:

$(\mathcal{A}_1)$ $g, h \in L^{\frac{2^*}{2^*-q}}(\Omega)$, $g^{\pm} = \max\{\pm g, 0\} \neq 0$ in $\bar{\Omega}$ and $h^{\pm} = \max\{\pm h, 0\} \neq 0$ in $\bar{\Omega}$.

$(\mathcal{A}_2)$ There exist $a_0, r_0 > 0$ such that $B_d(0, r_0) \subset \Omega$ and $g(z), h(z) \geq a_0$ for all $z \in B_d(0, r_0)$.

Here $B_d(z, r)$ denotes the quasi-ball with center at z and radius r with respect to the gauge d.

$|u|^{\alpha_i - 2} u |v|^{\beta_i}$ and $|u|^{\alpha_i} |v|^{\beta_i - 2} v$, $i = 1, 2$ are called strongly coupled terms. We now recall some known results concerning the elliptic system involving the strongly coupled critical terms. When $\mathbb{G}$ is the ordinary Euclidean space $(\mathbb{R}^N, +)$, $\eta_1 = \eta_2 = 1$, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $g = h \equiv 1$, problem (1.1) becomes the following Laplacian elliptic system:

$$\begin{cases} -\Delta u = \frac{2\alpha}{\alpha+\beta} |u|^{\alpha-2} |v|^{\beta} u + \lambda |u|^{q-2} u, & \text{in } \Omega, \\ -\Delta v = \frac{\beta}{\alpha+\beta} |u|^{\alpha} |v|^{\beta-2} v + \mu |v|^{q-2} v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The authors in [10] proved that the system (1.2) admits at least two positive solutions. Later, Hsu [9] obtained the same results for the p-Laplacian elliptic system. There are other multiplicity results or critical elliptic equations involving concave–convex nonlinearities, see for example [1, 2]. Contrary to the nonlinear elliptic problem with the Laplacian or p-Laplacian in Euclidean space that have been widely investigated, the situation seems to be in a developing state for the sub-Laplacian problem on Carnot groups. Recently, great attention has been devoted to nonlinear elliptic problems involving critical nonlinearities, in the context of Carnot group, see for example [11, 13, 20] and references therein. To the best of our knowledge, there is no result so far concerning sub-elliptic system involving strongly coupled critical terms nonlinearities with sign-changing weight functions on Carnot group.

We look for weak solutions of (1.1) in the product space $\mathcal{H} := S_0^1(\Omega) \times S_0^1(\Omega)$, endowed with the norm

$$\|(u,v)\|_{\mathcal{H}} = \left(\|u\|^2_{S_0^1(\Omega)} + \|v\|^2_{S_0^1(\Omega)}\right)^{\frac{1}{2}}, \quad \forall (u,v) \in \mathcal{H},$$

where the Folland-Stein space $S_0^1(\Omega) = \{u \in L^2(\Omega) : \int_\Omega |\nabla_{\mathbb{G}} u|^2 dz < \infty\}$, is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\left(\|u\|_{S_0^1(\Omega)}\right) = \left(\int_\Omega |\nabla_{\mathbb{G}} u|^2 dz\right)^{\frac{1}{2}}, \quad \forall u \in S_0^1(\Omega).$$

By using the Nehari manifold and fibering map analysis, we establish the existence of at least two positive solutions for a sub-elliptic system (1.1) when (λ, μ) belongs to certain subset of $\mathbb{R}^2_+$. Since the embedding $S_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact, then the corresponding energy functional does not satisfy the Palais-Smale condition in general. Therefore, it is difficult to obtain the critical points of energy functional by simple arguments, which are based on the compactness of the Sobolev embedding. To overcome this difficulty, we extract a Palais-Smale sequence in the Nehari manifold and show that the weak limit of this sequence is the required solution of problem (1.1). The best constant of the Sobolev inequality was studied on graded groups in [15]. But in that paper, the best constant was expressed in variational form.

We consider the following scalar critical equation:

$$-\Delta_{\mathbb{G}} u = |u|^{2^*-2} u \quad \text{in } \mathbb{G}. \tag{1.3}$$

For equation (1.3), it is well known (see *e.g.* [3, 11]) that positive solutions have the following decay:

$$U(z) \sim \frac{C}{d(z)^{Q-2}} \quad \text{as } d(z) \to \infty, \tag{1.4}$$

where d is the gauge norm on $\mathbb{G}$. This result applies, in particular, to the extremals of the Sobolev inequality on Carnot groups (whose existence was proved in [8, 17], *i.e.*, to the functions U that achieve the best constant for the embedding $S_0^1(\mathbb{G}) \hookrightarrow L^{2^*}(\mathbb{G})$, that is,

$$S_{\mathbb{G}} := \inf_{u \in S_0^1(\mathbb{G}) \setminus \{0\}} \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 dz}{\left(\int_{\mathbb{G}} |u|^{2^*} dz\right)^{\frac{2}{2^*}}} = \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} U|^2 dz}{\left(\int_{\mathbb{G}} |U|^{2^*} dz\right)^{\frac{2}{2^*}}}.$$

We underline that the knowledge of the exact asymptotic behavior of Sobolev minimizers turns out to be a crucial ingredient in order to obtain existence results for Brézis-Nirenberg type problems, whenever the explicit form of Sobolev minimizers is not known, as in the present Carnot case. The knowledge of the behavior of Sobolev minimizers turns out to be crucial also for the system, due to the relation between the extremals for the best constant $S_{\eta,\alpha,\beta}$ associated to the system and the Sobolev constant $S_{\mathbb{G}}$ (see Theorem 2.1 below).

The energy functional $I_{\eta,\alpha,\beta} : \mathcal{H} \longrightarrow \mathbb{R}$ associated to (1.1) is given by

$$I_{\eta,\alpha,\beta}(u,v) = \frac{1}{2}\|(u,v)\|^2_{\mathcal{H}} - \frac{1}{2^*}K_\eta(u,v) - \frac{1}{q}\Psi_{\lambda,\mu}(u,v), \quad \forall (u,v) \in \mathcal{H},$$

where

$$K_\eta(u,v) = \int_\Omega \left(\eta_1|u|^{\alpha_1}|v|^{\beta_1} + \eta_2|u|^{\alpha_2}|v|^{\beta_2}\right) dz, \quad \Psi_{\lambda,\mu}(u,v) = \int_\Omega \left(\lambda g(z)|u|^q + \mu h(z)|v|^q\right) dz.$$

It is easy to check that $I_{\eta,\alpha,\beta} \in C^1(\mathcal{H},\mathbb{R})$ and the critical point of $I_{\eta,\alpha,\beta}$ is the weak solution of (1.1). We call a solution (u,v) positive if both u and v are positive, (u,v) is nontrivial if $u \not\equiv 0$ or $v \not\equiv 0$.

Definition 1.1. *A pair of functions $(u,v) \in \mathcal{H}$ is said to be a weak solution of problem (1.1) if*

$$\begin{aligned}\int_\Omega (\nabla u \nabla \phi + \nabla v \nabla \psi)\, \mathrm{d}x = &\int_\Omega \left(\frac{\eta_1\alpha_1}{2^*}|u|^{\alpha_1-2}|v|^{\beta_1}u\phi + \frac{\eta_2\alpha_2}{2^*}|u|^{\alpha_2-2}|v|^{\beta_2}u\phi\right) dx \\ &+ \int_\Omega \left(\frac{\eta_1\beta_1}{2^*}|u|^{\alpha_1}|v|^{\beta_1-2}v\psi + \frac{\eta_2\beta_2}{2^*}|u|^{\alpha_2}|v|^{\beta_2-2}v\psi\right) dx \\ &+ \int_\Omega \left(\lambda g(x)|u|^{q-2}u\phi + \mu h(x)|v|^{q-2}v\psi\right) dx \quad \text{for all } (\phi,\psi) \in \mathcal{H}.\end{aligned} \tag{1.5}$$

Define the set

$$\begin{aligned}\mathfrak{D}_\sigma &:= \left\{(\lambda,\mu) \in \mathbb{R}^+ \times \mathbb{R}^+ \backslash \{(0,0)\} : 0 < \mu\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \sigma\right\}, \quad \text{and} \\ \Lambda &:= \frac{2^*-2}{2^*-q}\left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)}\right)^{\frac{2-q}{2^*-2}} S_{\mathbb{G}}^{\frac{2^*-q}{2^*-2}}.\end{aligned} \tag{1.6}$$

So, the main result of this paper can be included in the following theorem.

Theorem 1.2. *Let $\mathbb{G}$ be a Carnot group. Assume that $(\mathcal{A}_0)$, $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold. Then, we have the following results:*

(i) *If $(\lambda,\mu) \in \mathfrak{D}_\Lambda$, then (1.1) has at least one positive solution in $\mathcal{H}$.*

(ii) *There exists a constant $\Lambda_* > 0$ such that system (1.1) has at least two distinct positive solutions in $\mathcal{H}$ for all $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_*}$.*

The paper is organized into three sections. In Section 2, we recall some basic definitions of Sobolev space on Carnot groups and we give some useful auxiliary lemmas. In Section 3, we investigate the Palais-Smale condition for the energy functional $I_{\eta,\alpha,\beta}$. Finally, the proof of Theorem 1.2 is given in Sections 4 and 5.

2 Preliminaries

In this section we recall some basic facts on the Carnot groups. For a complete treatment, we refer to the classical papers [6, 7]. We also quote for an overview on general homogeneous Lie group.

Let $\mathbb{G} = \left(\mathbb{R}^N, \circ\right)$ be a homogeneous group, *i.e.*, a Lie group equipped with a family $\{\delta_\gamma\}_{\gamma>0}$ of dilations, acting on $z \in \mathbb{R}^N$ as follows

$$\delta_\gamma\left(z^{(1)}, \ldots, z^{(r)}\right) = \left(\gamma^1 z^{(1)}, \gamma^2 z^{(2)}, \ldots, \gamma^r z^{(r)}\right),$$

where $z^{(k)} \in \mathbb{R}^{N_k}$ for every $k \in \{1, \ldots, r\}$ and $N = \sum_{k=1}^{r} N_k$. Then, the structure $\mathbb{G} := \left(\mathbb{R}^N, \circ, \{\delta_\gamma\}_{\gamma>0}\right)$ is called a homogeneous group with homogeneous dimension

$$Q := \sum_{k=1}^{r} k \cdot N_k.$$

Note that the number Q is naturally associated to the family $\{\delta_\gamma\}_{\gamma>0}$ since, for every $\gamma > 0$, the Jacobian of the map $z \mapsto \delta_\gamma(z)$ equals γ^Q. From now on, we shall assume throughout that $Q \geq 3$. We remark that, if $Q \leq 3$, then $\mathbb{G}$ is necessarily the ordinary Euclidean space $\mathbb{G} = \left(\mathbb{R}^Q, +\right)$.

Let $\mathfrak{g}$ be the Lie algebra of left invariant vector fields on $\mathbb{G}$ and assume that $\mathfrak{g}$ is stratified, *i.e.*, $\mathfrak{g} = \bigoplus_{k=1}^{r} V_k$ with $[V_1, V_k] = V_{k+1}$, for $1 \leq k \leq r-1$ and $[V_1, V_r] = \{0\}$. Under these assumptions, we call $\mathbb{G}$ a Carnot group. Here the integer r is called the step of $\mathbb{G}$, $\dim(V_k) = N_k$ and the symbol $[V_1, V_k]$ denotes the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$, where $X \in V_1$ and $Y \in V_k$. Let $X = \{X_1, X_2, \ldots, X_m\}$ be a basis of V_1 with $m = \dim(V_1)$. From Proposition 1.2.29 of [14], the left invariant vector field X_i $(k = 1, \ldots, m)$ has an explicit form as follows:

$$X_i = \frac{\partial}{\partial x_i^{(1)}} + \sum_{l=2}^{k} \sum_{r=1}^{\dim(V_l)} a_{i,r}^{(l)}\left(x^{(1)}, \ldots, x^{(l-1)}\right) \frac{\partial}{\partial x_r^{(l)}},$$

where $a_{i,r}^{(l)}$ is a homogeneous (with respect to δ_γ) polynomial function of degree $l-1$. Then, once a basis $X_1, X_2, \ldots, X_m$ of the horizontal layer is fixed, we define, for any function $u : \mathbb{G} \to \mathbb{R}$ for which the partial derivatives $X_j u$ exist, the horizontal gradient of u, denoted by $\nabla_{\mathbb{G}} u$, as the horizontal section

$$\nabla_{\mathbb{G}} u := \left(X_1 u, X_2 u, \ldots, X_m u\right).$$

Moreover, if $\phi = (\phi_1, \phi_2, \ldots, \phi_m)$ is an horizontal section such that $X_j \phi_j \in L^1_{loc}(\mathbb{G})$ for $j = 1, \ldots, m$, we define $\operatorname{div}_{\mathbb{G}} \phi$ as the real-valued function

$$\operatorname{div}_{\mathbb{G}}(\phi) := -\sum_{j=1}^{m} X_j^* \phi_j = \sum_{j=1}^{m} X_j \phi_j.$$

From the above results, the second-order differential operator

$$\Delta_{\mathbb{G}} := \sum_{j=1}^{m} X_j^2.$$

is called the (canonical) sub-Laplacian on $\mathbb{G}$. The sub-Laplacian $\Delta_{\mathbb{G}}$ is a left invariant homogeneous hypoelliptic differential operator, thanks to Hörmander's theorem, and $\Delta_{\mathbb{G}} u = \operatorname{div}_{\mathbb{G}}(\nabla_{\mathbb{G}} u)$. In addition, we can check that $\nabla_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}$ are left-translation invariant with respect to the group action τ_z and δ_γ-homogeneous, respectively, of degree one and two, that is, $\nabla_{\mathbb{G}}(u \circ \tau_z) = \nabla_{\mathbb{G}} u \circ \tau_z, \nabla_{\mathbb{G}}(u \circ \delta_\gamma) = \gamma \nabla_{\mathbb{G}} u \circ \delta_\gamma$, and $\Delta_{\mathbb{G}}(u \circ \tau_z) = \Delta_{\mathbb{G}} u \circ \tau_z$, $\Delta_{\mathbb{G}}(u \circ \delta_\gamma) = \gamma^2 \Delta_{\mathbb{G}} u \circ \delta_\gamma$, where the left translation $\tau_z : \mathbb{G} \to \mathbb{G}$ is defined as

$$x \mapsto \tau_z x := z \circ x, \quad \forall x, z \in \mathbb{G}.$$

Moreover, there exists a homogeneous norm d on $\mathbb{G}$ such that

$$\Gamma(z) = \frac{C}{d(z)^{Q-2}}, \quad \forall z \in \mathbb{G},$$

is a fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at 0, for a suitable constant $C > 0$. By definition, the homogeneous norm d on $\mathbb{G}$ is a continuous smooth function, away from the origin, such that $d(\delta_\gamma(z)) = \gamma d(z)$ for every $\gamma > 0$ and $z \in \mathbb{G}, d\left(z^{-1}\right) = d(z)$ and $d(z) = 0$ iff $z = 0$.

We will give some results which will be used to prove the existence in multiple critical cases. Let U be a fixed positive minimizer for the best constant $S_{\mathbb{G}}$ and define the family

$$U_\varepsilon(z) = \varepsilon^{\frac{2-Q}{2}} U\left(\delta_{\frac{1}{\varepsilon}}(z)\right), \quad \forall \varepsilon > 0. \tag{2.1}$$

The functions U_ε are also minimizers for $S_{\mathbb{G}}$ and, up to a normalization, they satisfy

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} U_\varepsilon|^2 dz = \int_{\mathbb{G}} |U_\varepsilon(z)|^{2^*} dz = S_{\mathbb{G}}^{\frac{Q}{2}}, \quad \forall \varepsilon > 0.$$

For any $0 < \eta_i < \infty$ $(i = 1, 2)$, $\alpha_i, \beta_i > 1$ with $\alpha_i + \beta_i = 2^*$, by the Young inequality, the following best Sobolev-type constants are well defined and crucial for the study of (1.1):

$$\begin{aligned} S_{\eta,\alpha,\beta} :&= \inf_{(u,v)\in\mathcal{H}\setminus\{(0,0)\}} \frac{\int_{\mathbb{G}} \left(|\nabla_{\mathbb{G}} u|^2 + |\nabla_{\mathbb{G}} v|^2\right) dz}{\left(\int_{\mathbb{G}} \left(\eta_1 |u|^{\alpha_1} |v|^{\beta_1} + \eta_2 |u|^{\alpha_2} |v|^{\beta_2}\right) dx\right)^{2/2^*}} \\ &= \inf_{(u,v)\in\mathcal{H}\setminus\{(0,0)\}} \|(u,v)\|^2 \left(\int_{\mathbb{G}} \left(\eta_1 |u|^{\alpha_1} |v|^{\beta_1} + \eta_2 |u|^{\alpha_2} |v|^{\beta_2}\right) dx\right)^{-2/2^*}. \end{aligned} \tag{2.2}$$

For any $t \geq 0$, we define the function

$$\mathfrak{h}(t) := \frac{1+t^2}{(\eta_1 t^{\beta_1} + \eta_2 t^{\beta_2})^{\frac{2}{2^*}}}. \tag{2.3}$$

Since $\mathfrak{h}$ is continuous on $(0,\infty)$ such that $\lim_{t\to 0^+} \mathfrak{h}(t) = \lim_{t\to+\infty} \mathfrak{h}(t) = +\infty$, then there exists $t_0 > 0$ a minimal point of function $\mathfrak{h}$, that is,

$$\mathfrak{h}(t_0) = \min_{t\geq 0} \mathfrak{h}(t) > 0. \tag{2.4}$$

Summarizing, we have the following relationship between $S_{\mathbb{G}}$ and $S_{\eta,\alpha,\beta}$.

Theorem 2.1. *Assume that $(\mathcal{A}_0)$ hold, then*

(i) $S_{\eta,\alpha,\beta} = \mathfrak{h}(t_0)\, S_{\mathbb{G}}$.

(ii) $S_{\eta,\alpha,\beta}$ *has the minimizers* $\big(U_\varepsilon(z),\, t_0 U_\varepsilon(z)\big)$, *for* $\varepsilon > 0$, *where* $U_\varepsilon(z)$ *are defined as in* (2.1).

Proof. Suppose $\kappa \in S_0^1(\mathbb{G})$. Choosing $(u,v) = (\kappa, t_0\kappa)$ in (2.2) we have

$$\frac{1+t_0^2}{\left(\eta_1 t_0^{\beta_1} + \eta_2 t_0^{\beta_2}\right)^{\frac{2}{2^*}}} \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\kappa|^2 dz}{\left(\int_{\mathbb{G}} |\kappa|^{2^*} dz\right)^{2/2^*}} \geq S_{\eta,\alpha,\beta}. \tag{2.5}$$

Taking the infimum as $\kappa \in S_0^1(\mathbb{G})$ in (2.5), we have

$$\mathfrak{h}(t_0) S_{\mathbb{G}} \geq S_{\eta,\alpha,\beta}. \tag{2.6}$$

Let $\{(u_n, v_n)\} \subset \mathcal{H}$ be a minimizing sequence of $S_{\eta,\alpha,\beta}$ and define $w_n = s_n v_n$, where

$$s_n := \left(\left(\int_{\mathbb{G}} |v_n|^{2^*}\ \mathrm{d}z\right)^{-1} \int_{\mathbb{G}} |u_n|^{2^*}\ \mathrm{d}z\right)^{\frac{1}{2^*}}.$$

Then

$$\int_{\mathbb{G}} |w_n|^{2^*}\ \mathrm{d}z = \int_{\mathbb{G}} |u_n|^{2^*}\ \mathrm{d}z. \tag{2.7}$$

From the Young inequality and (2.6) it follows that

$$\begin{aligned} \int_{\mathbb{G}} |u_n|^{\alpha_i} |w_n|^{\beta_i}\ \mathrm{d}z &\leq \frac{\alpha_i}{2^*} \int_{\mathbb{G}} |u_n|^{2^*}\ \mathrm{d}z + \frac{\beta_i}{2^*} \int_{\mathbb{G}} |w_n|^{2^*}\ \mathrm{d}z \\ &= \int_{\mathbb{G}} |u_n|^{2^*}\ \mathrm{d}z = \int_{\mathbb{G}} |w_n|^{2^*}\ \mathrm{d}z, \quad i = 1,2. \end{aligned} \tag{2.8}$$

Consequently,

$$\frac{\|(u_n,v_n)\|^2}{\left(\int_{\mathbb{G}}\left(\eta_1|u_n|^{\alpha_1}|v_n|^{\beta_1}+\eta_2|u_n|^{\alpha_2}|v_n|^{\beta_2}\right)dx\right)^{2/2_s^*}} \geq$$

$$\frac{\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u_n|^2dz}{\left(\left(\eta_1 s_n^{-\beta_1}+\eta_2 s_n^{-\beta_2}\right)\int_{\mathbb{G}}|u_n|^{2^*}\right)^{\frac{2}{2^*}}}+\frac{s_n^{-2}\int_{\mathbb{G}}|\nabla_{\mathbb{G}}w_n|^2dz}{\left(\left(\eta_1 s_n^{-\beta_1}+\eta_2 s_n^{-\beta_2}\right)\int_{\mathbb{G}}|w_n|^{2^*}\,\mathrm{d}z\right)^{\frac{2}{2^*}}} \geq \mathfrak{h}(s_n^{-1})S_{\mathbb{G}} \geq \mathfrak{h}(t_0)S_{\mathbb{G}}.$$

As $n\to\infty$ we have

$$S_{\eta,\alpha,\beta} \geq \mathfrak{h}(t_0)S_{\mathbb{G}},$$

which together with (2.6) implies that

$$S_{\eta,\alpha,\beta} = \mathfrak{h}(t_0)S_{\mathbb{G}}.$$

By (2.2) and (2.1), $S_{\eta,\alpha,\beta}$ has the minimizers $(U_\varepsilon(x), t_0U_\varepsilon(x))$. □

Let $R>0$ be such that $B_d(0,R)\subset\Omega$ (we can suppose $0\in\Omega$, due to the group translation invariance) and let a cut-off function $\varphi\in C_0^\infty(B_d(0,R))$, $0\le\varphi\le 1$, $\varphi=1$ in $B_d\left(0,\frac{R}{2}\right)$ and $\varphi=0$ in $\mathbb{G}\backslash B_d(0,R)$. Set

$$u_\varepsilon(z)=\varphi(z)U_\varepsilon(z).$$

Then, from [11, Lemma 3.3], we obtain the required results.

Lemma 2.2. *The functions u_ε satisfy the following estimates, as $\varepsilon\to 0$:*

$$\int_\Omega|\nabla_{\mathbb{G}}u_\varepsilon|^2dz = S_{\mathbb{G}}^{\frac{Q}{2}}+O\left(\varepsilon^{Q-2}\right),\quad \int_\Omega|u_\varepsilon|^{2^*}dz = S_{\mathbb{G}}^{\frac{Q}{2}}+O\left(\varepsilon^{Q}\right),$$

and

$$\int_\Omega|u_\varepsilon|^2dz=\begin{cases}C\varepsilon^2+O\left(\varepsilon^{Q-2}\right), & \text{if } Q>4,\\ C\varepsilon^2|\ln\varepsilon|+O\left(\varepsilon^2\right), & \text{if } Q=4.\end{cases}$$

Moreover, similarly as the proof of [12, Lemma 6.1], we get the following results.

Lemma 2.3. *The following estimates hold as $\varepsilon\to 0$:*

$$\int_\Omega|u_\varepsilon|^q dz \geq \begin{cases}O\left(\varepsilon^{Q+\frac{(2-Q)q}{2}}\right), & \text{if } \frac{Q}{Q-2}<q<2,\\ O\left(\varepsilon^{Q+\frac{(2-Q)q}{2}}|\ln(\varepsilon)|\right), & \text{if } q=\frac{Q}{Q-2},\\ O\left(\varepsilon^{\frac{(Q-2)}{2}}\right), & \text{if } 1\le q<\frac{Q}{Q-2}.\end{cases}$$

3 The Palais-Smale condition

In this section, we use the second concentration-compactness principle and concentration-compactness principle at infinity to prove that the $(PS)_c$ condition holds.

Definition 3.1. *Let $c \in \mathbb{R}$ and $I_{\eta,\alpha,\beta} \in C^1(\mathcal{H}, \mathbb{R})$.*

(i) *A sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is called a Palais–Smale sequence at the level c $(PS)_c$-sequence, in short, for the functional $I_{\eta,\alpha,\beta}$ if $I_{\eta,\alpha,\beta}(u_n, v_n) \to c$ and $I'_{\eta,\alpha,\beta}(u_n, v_n) \to 0$ as $n \to \infty$.*

(ii) *We say that $I_{\eta,\alpha,\beta}$ satisfies the $(PS)_c$ condition if any $(PS)_c$-sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ for $I_{\eta,\alpha,\beta}$ has a convergent subsequence in E.*

Since $g, h \in L^{\frac{2^*}{2^*-q}}(\Omega)$, we obtain from the Hölder and Sobolev inequalities that, for all $u \in S_0^1(\Omega)$,

$$\int_\Omega g(z)|u|^q dz \leq \left(\int_\Omega |g(z)|^{\frac{2^*}{2^*-q}} dz\right)^{\frac{2^*-q}{2^*}} \left(\int_\Omega |u|^{2^*} dz\right)^{\frac{q}{2}} \leq \|g\|_{L^{\frac{2^*}{2^*-q}}} S_{\mathbb{G}}^{-\frac{q}{2}} \|u\|^q_{S_0^1(\Omega)}. \tag{3.1}$$

Similarly, one can get

$$\int_\Omega h(z)|v|^q dz \leq \left(\int_\Omega |h(z)|^{\frac{2^*}{2^*-q}} dz\right)^{\frac{2^*-q}{2^*}} \left(\int_\Omega |v|^{2^*} dz\right)^{\frac{\psi}{2}} \leq \|h\|_{L^{\frac{2^*}{2^*-q}}} S_{\mathbb{G}}^{-\frac{q}{2}} \|v\|^q_{S_0^1(\Omega)}. \tag{3.2}$$

Hence, in view of (3.1) and (3.2), we can obtain

$$\Psi_{\lambda,\mu}(u, v) \leq \left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right) S_{\mathbb{G}}^{-\frac{q}{2}} \|(u, v)\|^q_{\mathcal{H}}. \tag{3.3}$$

Moreover, the Young inequality and (3.1), (3.2) imply that

$$\begin{aligned}\Psi_{\lambda,\mu}(u, v) \leq{}& \frac{1}{Q}\frac{2^*q}{2^*-q}\|(u, v)\|_{\mathcal{H}} \\ &+ \frac{2-q}{2} S_{\mathbb{G}}^{-\frac{q}{2-q}} \left(\frac{2^*-q}{2^*-2}\right)^{\frac{q}{2-q}} \left[\left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}} + \left(\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}}\right].\end{aligned} \tag{3.4}$$

Lemma 3.2. *Let $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ be a $(PS)_c$-sequence of $I_{\eta,\alpha,\beta}$ with $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$. Then $I'_{\eta,\alpha,\beta}(u, v) = 0$ and*

$$I_{\eta,\alpha,\beta}(u, v) \geq -\frac{(2^*-q)(2-q)}{2q2^*} S_{\mathbb{G}}^{-\frac{q}{2-q}} \left(\frac{2^*-q}{2^*-2}\right)^{\frac{q}{2-q}} \left[\left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}} + \left(\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}}\right].$$

Proof. Since $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a $(PS)_c$-sequence of $I_{\eta,\alpha,\beta}$ with $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$, it is easy to check that $I'_{\eta,\alpha,\beta}(u, v) = 0$, and then $\left\langle I'_{\eta,\alpha,\beta}(u, v), (u, v)\right\rangle = 0$, that is,

$$\|(u, v)\|_{\mathcal{H}} = K_\eta(u, v) + \Psi_{\lambda,\mu}(u, v).$$

Then from (3.4), we have

$$\begin{aligned} I_{\eta,\alpha,\beta}(u, v) &= \frac{1}{Q}\|(u, v)\|_{\mathcal{H}} - \frac{2^* - q}{2^* q}\Psi_{\lambda,\mu}(u, v) \\ &\geq -\frac{(2^* - q)(2 - q)}{2q2^*} S_{\mathbb{G}}^{-\frac{q}{2-q}} \left(\frac{2^* - q}{2^* - 2}\right)^{\frac{q}{2-q}} \left[\left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}} + \left(\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}}\right]. \end{aligned}$$

This ends the proof of lemma. □

Lemma 3.3. *Assume that $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a $(PS)_c$-sequence of $I_{\eta,\alpha,\beta}$ and the condition $(\mathcal{A}_1)$ holds. Then $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$.*

Proof. Assume by contradiction that $\|(u_n, v_n)\|_{\mathcal{H}} \to +\infty$. Set

$$(\tilde{u}_n, \tilde{v}_n) = \left(\frac{u_n}{\|(u_n, v_n)\|_{\mathcal{H}}}, \frac{v_n}{\|(u_n, v_n)\|_{\mathcal{H}}}\right).$$

Then, $\|(\tilde{u}_n, \tilde{v}_n)\|_{\mathcal{H}} = 1$, and

$$\begin{cases} (\tilde{u}_n, \tilde{v}_n) \rightharpoonup (u, v) \text{ weakly in } \mathcal{H}, \\ (\tilde{u}_n, \tilde{v}_n) \to (u, v) \text{ strongly in } \left(L^r(\Omega)\right)^2, \forall r \in [1, 2^*), \\ (\tilde{u}_n(z), \tilde{v}_n(z)) \to (u(z), v(z)) \text{ a.e. in } \Omega. \end{cases} \tag{3.5}$$

Set $\bar{u}_n := \tilde{u}_n - u$, $\bar{v}_n := \tilde{v}_n - v$, there exists a positive constant $C > 0$ such that

$$\int_\Omega |\bar{u}_n|^{2^*} dz < C, \quad \int_\Omega |\bar{v}_n|^{2^*} dz < C, \tag{3.6}$$

and by (3.5), one has that for any $\varepsilon > 0$, there exists $r_0 > 0$ such that

$$\int_{B_d(0,r_0)} |\bar{u}_n|^{2^*} dz < \varepsilon, \quad \int_{B_d(0,r_0)} |\bar{v}_n|^{2^*} dz < \varepsilon, \tag{3.7}$$

for n large enough, where $B_d(0, r_0) = \{z \in \mathbb{G} : d(0, z) \leq r_0\}$ is a ball with center at 0 and radius r_0 with respect to the gauge d. Moreover, since $g, h \in L^{\frac{2^*}{2^*-q}}(\Omega)$, for the above constant r_0, we have

$$\int_{\Omega\setminus B_d(0,r_0)} |g(z)|^{\frac{2^*}{2^*-q}} dz < \varepsilon, \quad \int_{\Omega\setminus B_d(0,r_0)} |h(z)|^{\frac{2^*}{2^*-q}} dz < \varepsilon. \tag{3.8}$$

Then, by (3.6), (3.7), (3.8) and Hölder inequality, we get

$$\begin{aligned}\Psi_{\lambda,\mu}(\bar{u}_n,\bar{u}_n) &= \int_{\Omega\setminus B_d(0,r_0)} (\lambda g(z)|\bar{u}_n|^q + \mu h(z)|\bar{v}_n|^q)\,dz + \int_{B_d(0,r_0)} (\lambda g(z)|\bar{u}_n|^q + \mu h(z)|\bar{v}_n|q^q)\,dz \\ &\leq \lambda\left(\int_{\Omega\setminus B_d(0,r_0)} |g|^{\frac{2^*}{2^*-q}}\,dz\right)^{\frac{2^*-q}{2^*}} \left(\int_{\Omega\setminus B_d(0,r_0)} |\bar{u}_n|^{2^*}\,dz\right)^{\frac{q}{2^*}} \\ &+ \mu\left(\int_{\Omega\setminus B_d(0,r_0)} |h|^{\frac{2^*}{2^*-q}}\,dz\right)^{\frac{2^*-q}{2^*}} \left(\int_{\Omega\setminus B_d(0,u_0)} |\bar{v}_n|^{2^*}\,dz\right)^{\frac{q}{2^*}} \\ &+ \lambda\left(\int_{B_d(0,r_0)} |g|^{\frac{2^*}{2^*-q}}\,dz\right)^{\frac{2^*-q}{2^*}} \left(\int_{B_d(0,r_0)} |\bar{u}_n|^{2^*}\,dz\right)^{\frac{q}{2^*}} \\ &+ \mu\left(\int_{B_d(0,r_0)} |h|^{\frac{2^*}{2^*-q}}\,dz\right)^{\frac{2^*-q}{2^*}} \left(\int_{B_d(0,u_0)} |\bar{v}_n|^{2^*}\,dz\right)^{\frac{q}{2^*}} \\ &\leq C_1\varepsilon^{\frac{2^*-q}{2^*}} + C_2 2\varepsilon^{\frac{q}{2^*}},\end{aligned}$$

which yields that $\Psi_{\lambda,\mu}(\bar{u}_n,\bar{v}_n)\to 0$ as $n\to\infty$. Consequently,

$$\lim_{n\to\infty}\Psi_{\lambda,\mu}(\tilde{u}_n,\tilde{v}_n) = \lim_{n\to\infty}\Psi_{\lambda,\mu}(\bar{u}_n,\bar{v}_n) + \Psi_{\lambda,\mu}(u,v) = \Psi_{\lambda,\mu}(u,v). \tag{3.9}$$

On the other hand, since $\{(u_n,v_n)\}_{n\in\mathbb{N}}\subset\mathcal{H}$ is a $(PS)_c$-sequence of $I_{\eta,\alpha,\beta}$ and $u_n = \|(u_n,v_n)\|_{\mathcal{H}}\cdot\tilde{u}_n$, $v_n = \|(u_n,v_n)\|_{\mathcal{H}}\cdot\tilde{v}_n$, we deduce that

$$\begin{aligned}\frac{1}{2}\|(u_n,v_n)\|_{\mathcal{H}}\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} &= \frac{1}{2^*}\|(u_n,v_n)\|_{\mathcal{H}}^{2^*} K_\eta(\tilde{u}_n,\tilde{v}_n) \\ &+ \frac{1}{q}\|(u_n,v_n)\|_{\mathcal{H}}^{q}\Psi_{\lambda,\mu}(\tilde{u}_n,\tilde{v}_n) + o_n(1),\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\|(u_n,v_n)\|_{\mathcal{H}}\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} &= \|(u_n,v_n)\|_{\mathcal{H}}^{2^*} K_\eta(\tilde{u}_n,\tilde{v}_n) \\ &+ \|(u_n,v_n)\|_{\mathcal{H}}^{q}\Psi_{\lambda,\mu}(\tilde{u}_n,\tilde{v}_n) + o_n(1).\end{aligned} \tag{3.11}$$

From (3.9), (3.10), (3.11), $1<q<2$ and $\|(u_n,v_n)\|_{\mathcal{H}}\to+\infty$, one has

$$\lim_{n\to\infty}\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} = \frac{2(2^*-q)}{q(2^*-2)}\lim_{n\to\infty}\frac{\Psi_{\lambda,\mu}(\bar{u}_n,\bar{v}_n)}{\|(u_n,v_n)\|_{\mathcal{H}}^{2-q}} = 0,$$

which contradicts $\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} = 1$. Therefore, $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. □

Lemma 3.4. *$\mathcal{I}_{\lambda,\alpha,\beta}$ satisfies the $(PS)_c$ condition in $\mathcal{H}$, with c satisfying*

$$0 < c < c_\infty := \frac{1}{Q} S_{\eta,\alpha,\beta}^{\frac{Q}{2}} - C_0 \left[\left(\lambda \|g\|_{L^{\frac{2^*}{2^*-q}}} \right)^{\frac{2}{2-q}} + \left(\mu \|h\|_{L^{\frac{2^*}{2^*-q}}} \right)^{\frac{2}{2-q}} \right] \tag{3.12}$$

where $C_0 = C_0(q,Q) := \frac{(2^*-q)(2-q)}{2q2^*} S_{\mathbb{G}}^{-\frac{q}{2-q}} \left(\frac{2^*-q}{2^*-2}\right)^{\frac{q}{2-q}}$ is a positive constant depending only on q, Q and $S_{\mathbb{G}}$.

Proof. Let $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ be a $(PS)_c$-sequence for $I_{\eta,\alpha,\beta}$ with $c \in (0, c_\infty)$. It follows from Lemma 3.3 that $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. Then, there exists a subsequence still denoted by $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ and $(u,v) \in \mathcal{H}$ such that $(u_n,v_n) \rightharpoonup (u,v)$ weakly in $\mathcal{H}$, and

$$\begin{cases} u_n \rightharpoonup u, \quad v_n \rightharpoonup v \text{ weakly in } L^{2^*}(\Omega), \\ u_n \to u, \quad v_n \to v \text{ strongly in } L^r(\Omega),\ \forall\, 1 \le r < 2^*, \\ u_n(z) \to u(z), \quad v_n(z) \to v(z) \text{ a.e. in } \Omega. \end{cases} \tag{3.13}$$

Hence, from (3.13), it is easy to verify that $I'_{\eta,\alpha,\beta}(u,v) = 0$ and

$$\lim_{n\to\infty} \Psi_{\lambda,\mu}(u_n,v_n) = \Psi_{\lambda,\mu}(u,v). \tag{3.14}$$

Set $\tilde{u}_n = u_n - u, \tilde{v}_n = v_n - v$. By Brézis-Lieb lemma [18], we get

$$\|(u_n,v_n)\|_{\mathcal{H}} = \|(u,v)\|_{\mathcal{H}} + \|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} + o_n(1), \tag{3.15}$$

$$\int_\Omega |u_n|^{2^*} dz = \int_\Omega |u|^{2^*} dz + \int_\Omega |\tilde{u}_n|^{2^*} dz + o_n(1), \tag{3.16}$$

$$\int_\Omega |v_n|^{2^*} dz = \int_\Omega |v|^{2^*} dz + \int_\Omega |\tilde{v}_n|^{2^*} dz + o_n(1), \tag{3.17}$$

and

$$\int_\Omega |u_n|^{\alpha_i}|v_n|^{\beta_i} dz = \int_\Omega |u|^{\alpha_i}|v|^{\beta_i} dz + \int_\Omega |\tilde{u}_n|^{\alpha_i}|\tilde{v}_n|^{\beta_i} dz + o_n(1). \tag{3.18}$$

So, (3.16), (3.17) and (3.18) yield

$$K_\eta(u_n,v_n) = K_\eta(u,v) + K_\eta(\tilde{u}_n,\tilde{v}_n) + o_n(1). \tag{3.19}$$

Then, using (3.14), (3.15) and (3.19), we have

$$c = \frac{1}{2}\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} - \frac{1}{2_s^*} K_\eta(\tilde{u}_n,\tilde{v}_n) + I_{\eta,\alpha,\beta}(u,v) + o_n(1), \tag{3.20}$$

and

$$o_n(1) = \|(\bar{u}_n,\bar{v}_n)\|_{\mathcal{H}} - K_\eta(\bar{u}_n,\bar{v}_n). \tag{3.21}$$

We may assume that

$$\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} \to l, \quad K_\eta(\tilde{u}_n,\tilde{v}_n) \to l \geq 0 \quad \text{as } n\to\infty.$$

If $l=0$, the proof is completed. Assume that $l>0$, then from (3.21), we have

$$S_{\eta,\alpha,\beta} l^{\frac{2}{2^*}} = S_{\eta,\alpha,\beta}\left(\lim_{n\to\infty} K_\eta(\tilde{u}_n,\tilde{v}_n)\right)^{\frac{2}{2^*}} \leq \lim_{n\to\infty}\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}} = l,$$

which implies that $l \geq S_{\eta,\alpha,\beta}^{\frac{Q}{2}}$. Hence, from (3.20) and Lemma 3.2, we have

$$\begin{aligned} c &= I_{\eta,\alpha,\beta}(u_n,v_n) + o_n(1) = \left(\frac{1}{2}-\frac{1}{2^*}\right) l + I_{\eta,\alpha,\beta}(u,v) + o_n(1) \\ &\geq \frac{1}{Q} S_{\eta,\alpha,\beta}^{\frac{Q}{2}} - C_0\left[\left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}} + \left(\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right)^{\frac{2}{2-q}}\right], \end{aligned} \tag{3.22}$$

which contradicts $c < c_\infty$. The proof is completed. □

4 Nehari manifold

Now we focus our attention on Problem (1.1) by using the Nehari manifold approach. For this reason, we introduce the Nehari manifold

$$\mathcal{N}_{\eta,\alpha,\beta} = \left\{w \in \mathcal{H}\backslash\{0\} : \left\langle I'_{\eta,\alpha,\beta}(w), w\right\rangle = 0\right\}.$$

where $w=(u,v)$ and $\|w\|_{\mathcal{H}} = \|(u,v)\|_{\mathcal{H}}$. Note that $\mathcal{N}_{\eta,\alpha,\beta}$ contains all nonzero solution of (1.1), and $w\in\mathcal{N}_{\eta,\alpha,\beta}$ if and only if

$$\|w\|_{\mathcal{H}} = K_\eta(w) + \Psi_{\lambda,\mu}(w). \tag{4.1}$$

Lemma 4.1. *$I_{\eta,\alpha,\beta}$ is coercive and bounded below on $\mathcal{N}_{\eta,\alpha,\beta}$.*

Proof. Let $w\in\mathcal{N}_{\eta,\alpha,\beta}$ by (3.3) and (4.1). We find

$$\begin{aligned} I_{\eta,\alpha,\beta}(w) &= \frac{2^*-2}{22^*}\|w\|_{\mathcal{H}} - \frac{2^*-2}{q2^*}\Psi_{\lambda,\mu}(w) \\ &\geq \frac{2^*-2}{22^*}\|w\|_{\mathcal{H}} - \frac{2^*-q}{q2^*}\left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*-q}{2^*}}}\right) S_{\mathbb{G}}^{-\frac{q}{2}}\|w\|_{\mathcal{H}}^q. \end{aligned} \tag{4.2}$$

Since $1<q<2$, we see that $I_{\eta,\alpha,\beta}$ is coercive and bounded below on $\mathcal{N}_{\eta,\alpha,\beta}$. This achieves the proof of the lemma. □

Define $\Phi(w) := \left\langle I'_{\eta,\alpha,\beta}(w), w \right\rangle$, then for all $w = (u, v) \in \mathcal{N}_{\eta,\alpha,\beta}$, we have

$$\begin{aligned} \langle \Phi'(w), w \rangle &= 2\|w\|_{\mathcal{H}} - 2^* K_\eta(w) - q\Psi_{\lambda,\mu}(w) \\ &= (2-q)\|w\|_{\mathcal{H}} - (2^* - q)\, K_\eta(w) \\ &= (2-2^*)\,\|w\|_{\mathcal{H}} + (2^*-q)\,\Psi_{\lambda,\mu}(w). \end{aligned} \tag{4.3}$$

Now, similar to the method used in [16], we split $\mathcal{N}_{\eta,\alpha,\beta}$ into three disjoint parts:

$$\begin{aligned} \mathcal{N}^+_{\eta,\alpha,\beta} &:= \left\{ w \in \mathcal{N}_{\eta,\alpha,\beta} : \langle \Phi'(w), w \rangle > 0 \right\}, \\ \mathcal{N}^0_{\eta,\alpha,\beta} &:= \left\{ w \in \mathcal{N}_{\eta,\alpha,\beta} : \langle \Phi'(w), w \rangle = 0 \right\}, \\ \mathcal{N}^-_{\eta,\alpha,\beta} &:= \left\{ w \in \mathcal{N}_{\eta,\alpha,\beta} : \langle \Phi'(w), w \rangle < 0 \right\}. \end{aligned} \tag{4.4}$$

Note that $\mathcal{N}_{\eta,\alpha,\beta}$ contains every nonzero solution of problem (1.1). In order to study the properties of Nehari manifolds. We now present some properties of $\mathcal{N}^+_{\eta,\alpha,\beta}, \mathcal{N}^0_{\eta,\alpha,\beta}$ and $\mathcal{N}^-_{\eta,\alpha,\beta}$ to state our main results.

Lemma 4.2. *Assume that $w_0 = (u_0, v_0)$ is a local minimizer for $I_{\eta,\alpha,\beta}$ on the set $\mathcal{N}_{\eta,\alpha,\beta} \setminus \mathcal{N}^0_{\eta,\alpha,\beta}$. Then $I'_{\eta,\alpha,\beta}(w_0) = 0$ in $\mathcal{H}^{-1}$, where $\mathcal{H}^{-1}$ denotes the dual space of the space $\mathcal{H}$.*

Proof. The proof is similar as that of [21, Lemma 3.4] and the details are omitted. □

Lemma 4.3. *$\mathcal{N}^0_{\eta,\alpha,\beta} = \emptyset$ for all $(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+$ with*

$$0 < \lambda \|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu \|h\|_{L^{\frac{2^*}{2^*-q}}} < \Lambda$$

where Λ is given in (1.6).

Proof. We argue by contradiction. Assume that there exist $\lambda, \mu \in (0, +\infty)$ with

$$0 < \lambda \|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu \|h\|_{L^{\frac{2^*}{2^*-q}}} < \Lambda$$

such that $\mathcal{N}^0_{\eta,\alpha,\beta} \neq \emptyset$. Then, for $w \in \mathcal{N}^0_{\eta,\alpha,\beta}$, by (4.3), we have

$$\|w\|_{\mathcal{H}} = \frac{2^*-q}{2-q} K_\eta(w) \tag{4.5}$$

and

$$\|w\|_{\mathcal{H}} = \frac{2^*-q}{2^*-2} \Psi_{\lambda,\mu}(w). \tag{4.6}$$

From the Young inequality, we have that

$$K_\eta(w) \leq (\eta_1 + \eta_2)\, S_{\mathbb{G}}^{-\frac{2^*}{2}} \|w\|^{2^*}_{\mathcal{H}},$$

and (4.5) yields

$$\|w\|_{\mathcal{H}} \geq \left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)} S_{\mathbb{G}}^{\frac{2^*}{2}}\right)^{\frac{1}{2^*-2}}. \tag{4.7}$$

On the other hand, from (3.3) and (4.6), it follows that

$$\|w\|_{\mathcal{H}} \leq \left(\frac{2^*-q}{2^*-2}\left(\lambda\|g\|_{L^{\frac{2*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right) S_{\mathbb{G}}^{\frac{-q}{2}}\right)^{\frac{1}{2-q}}. \tag{4.8}$$

Therefore, in view of (4.7) and (4.8), we obtain

$$\lambda\|g\|_{L^{\frac{2*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} \geq \frac{2^*-2}{2^*-q}\left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)}\right)^{\frac{2-q}{2^*-2}} S_{\mathbb{G}}^{\frac{2^*-q}{2^*-2}} := \Lambda,$$

which is a contradiction. This completes the proof of Lemma. □

By Lemmas 4.2 and 4.3 , for $(\lambda,\mu) \in \mathfrak{D}_\Lambda$, we can write $\mathcal{N}_{\eta,\alpha,\beta} = \mathcal{N}^+_{\eta,\alpha,\beta} \cup \mathcal{N}^-_{\eta,\alpha,\beta}$ and define

$$c_{\eta,\alpha,\beta} = \inf_{w\in\mathcal{N}_{\eta,\alpha,\beta}} I_{\eta,\alpha,\beta}(w); \quad c^+_{\eta,\alpha,\beta} = \inf_{w\in\mathcal{N}^+_{\eta,\alpha,\beta}} I_{\eta,\alpha,\beta}(w); \quad c^-_{\eta,\alpha,\beta} = \inf_{w\in\mathcal{N}^-_{\eta,\alpha,\beta}} I_{\eta,\alpha,\beta}(w).$$

Lemma 4.4. *Assume that $(\mathcal{A}_0)$, hold. Then, we have the following results:*

(i) $c_{\eta,\alpha,\beta} \leq c^+_{\eta,\alpha,\beta} < 0$ *for all* $(\lambda,\mu) \in \mathfrak{D}_\Lambda$.

(ii) *There exists a constant* $C_0 = C_0(\lambda, q, Q, S_{\mathbb{G}}, \Lambda) > 0$ *such that* $c^-_{\eta,\alpha,\beta} \geq C_0 > 0$, *for all* $(\lambda,\mu) \in \mathfrak{D}_{\frac{q}{2}\Lambda}$.

Proof. (i) For $w \in \mathcal{N}^+_{\eta,\alpha,\beta} \subset \mathcal{N}_{\eta,\alpha,\beta}$, by (4.3), we have

$$\|w\|_{\mathcal{H}} > \frac{2^*-q}{2-q} K_\eta(w),$$

and so

$$\begin{aligned} I_{\eta,\alpha,\beta}(w) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|w\|_{\mathcal{H}} - \left(\frac{1}{2^*}-\frac{1}{q}\right)K_\eta(w) \\ &\leq \left(\frac{q-2}{2q} + \frac{2^*-q}{2^*q}\frac{2-q}{2^*-q}\right)\|w\|_{\mathcal{H}} = -\frac{(2-q)(2^*-2)}{22^*q}\|w\|_{\mathcal{H}} < 0. \end{aligned}$$

Thus, from the definition of $c_{\eta,\alpha,\beta}$ and $c^+_{\eta,\alpha,\beta}$, we can deduce that $c_{\eta,\alpha,\beta} \leq c^+_{\eta,\alpha,\beta} < 0$.

(ii) For $w \in \mathcal{N}^-_{\eta,\alpha,\beta}$, similar to (4.7), we have

$$\|w\|_{\mathcal{H}} > \left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)} S_{\mathbb{G}}^{\frac{2^*}{2}}\right)^{\frac{1}{2^*-2}}. \tag{4.9}$$

In view of (4.2) and (4.9), we get

$$
\begin{aligned}
I_{\eta,\alpha,\beta}(w) &\geq \|w\|_{\mathcal{H}}^{q}\left(\frac{2^*-2}{22^*}\|w\|_{\mathcal{H}}^{2-q}-\frac{2^*-q}{q2^*}\left(\lambda\|g\|_{L^{\frac{2}{2^*-q}}}+\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right)S_{\mathbb{G}}^{-\frac{q}{2}}\right)\\
&\geq \|w\|_{\mathcal{H}}^{q}\left(\frac{2^*-2}{22^*}\left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)}\right)^{\frac{2-q}{2^*-2}}S_{\mathbb{G}}^{\frac{2^*(2-q)}{2(2^*-2)}}\right.\\
&\quad\left.-\frac{2^*-q}{q2^*}\left(\lambda\|g\|_{L^{\frac{2}{2^*-q}}}+\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}\right)S_{\mathbb{G}}^{-\frac{q}{2}}\right).
\end{aligned}
$$

So, if namely,

$$
0<\lambda\|g\|_{L^{\frac{2}{2^*-q}}}+\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}<\frac{q}{2}\frac{2^*-2}{2^*-q}\left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)}\right)^{\frac{2-q}{2^*-2}}S_{\mathbb{G}}^{\frac{2^*-q}{2^*-2}}=\frac{q}{2}\Lambda,
$$

we get

$$
\begin{aligned}
I_{\eta,\alpha,\beta}(w) \geq &\left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)}S_{\mathbb{G}}^{\frac{2^*}{2}}\right)^{\frac{q}{2^*-2}}\left(\frac{2^*-2}{22^*}\left(\frac{2-q}{(\eta_1+\eta_2)(2^*-q)}\right)^{\frac{2-q}{2*-2}}S_{\mathbb{G}}^{\frac{2^*(2-q)}{2(2^*-2)}}\right.\\
&\left.-\frac{2^*-q}{q2^*}\left(\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}}+\mu\|h\|_{L^{\frac{2^*}{2^*-4}}}\right)S_{\mathbb{G}}^{-\frac{q}{2}}\right):=C_0\left(\lambda,q,Q,S_{\mathbb{G}},\Lambda\right)>0,
\end{aligned}
$$

and this completes the proof. □

For each $w\in\mathcal{H}\backslash\{0\}$, we have $K_\eta(w)>0$ and let

$$
t_{\max}=\left(\frac{(2-q)\|w\|_{\mathcal{H}}}{(2^*-q)\,K_\eta(w)}\right)^{\frac{1}{2^*-2}}>0.
$$

So, we get the following result.

Lemma 4.5. *Let $(\lambda,\mu)\in\mathfrak{D}_\Lambda$. For every $w\in\mathcal{H}$ with $K_\eta(w)>0$, the following results hold:*

(i) *If $\Psi_{\lambda,\mu}(w)\leq 0$, then there is a unique $t^->t_{\max}$ such that $(t^-w)\in\mathcal{N}^-_{\eta,\alpha,\beta}$ and*

$$
I_{\eta,\alpha,\beta}(t^-w)=\sup_{t\geq 0}I_{\eta,\alpha,\beta}(tw).
$$

(ii) *If $\Psi_{\lambda,\mu}(w)>0$, then there exist unique t^+ and t^- with $0<t^+<t_{\max}<t^-$ such that $(t^+w)\in\mathcal{N}^+_{\eta,\alpha,\beta}$ and $(t^-w)\in\mathcal{N}^-_{\eta,\alpha,\beta}$. Moreover,*

$$
I_{\eta,\alpha,\beta}(t^+w)=\inf_{0\leq t\leq t_{\max}}I_{\eta,\alpha,\beta}(tw),\quad I_{\eta,\alpha,\beta}(t^-w)=\sup_{t\geq 0}I_{\eta,\alpha,\beta}(tw).
$$

Proof. The proof is similar to [5, Lemma 2.6], and is omitted here. □

5 Proof of the main results

In this section, we provide the proofs of the main results of this work. Before giving the proof of Theorem 1.2, we need the following lemma.

Lemma 5.1. *Assume that $(\mathcal{A}_0)$, hold. Then, we have the following results:*

(i) *If $(\lambda,\mu)\in\mathfrak{D}_\Lambda$, then there exists a $(PS)_{c_{\eta,\alpha,\beta}}$-sequence $\{(u_n,v_n)\}_{n\in\mathbb{N}}\subset\mathcal{N}_{\eta,\alpha,\beta}$ for $I_{\eta,\alpha,\beta}$.*

(ii) *If $(\lambda,\mu)\in\mathfrak{D}_{\frac{q}{2}\Lambda}$, then there exists a $(PS)_{c^-_{\eta,\alpha,\beta}}$-sequence $\{(u_n,v_n)\}_{n\in\mathbb{N}}\subset\mathcal{N}^-_{\eta,\alpha,\beta}$ for $I_{\eta,\alpha,\beta}$.*

Proof. The proof is almost the same as Proposition 9 in [19]. □

Now we establish the existence of a local minimizer of $I_{\eta,\alpha,\beta}$ on $\mathcal{N}^+_{\eta,\alpha,\beta}$.

Theorem 5.2. *Assume that $(\mathcal{A}_0)$, hold. If $(\lambda,\mu)\in\mathfrak{D}_\Lambda$, then $I_{\eta,\alpha,\beta}$ has a minimizer $(u_1,v_1)\in\mathcal{N}^+_{\eta,\alpha,\beta}$ such that (u_1,v_1) is a nonnegative solution of (1.1) and*

$$I_{\eta,\alpha,\beta}(u_1,v_1)=c_{\eta,\alpha,\beta}=c^+_{\eta,\alpha,\beta}<0.$$

Proof. In view of the Lemma 5.1 (i), there exists a minimizing sequence $\{(u_n,v_n)\}_{n\in\mathbb{N}}\subset\mathcal{N}_{\eta,\alpha,\beta}$ such that

$$\lim_{n\to\infty} I_{\eta,\alpha,\beta}(u_n,v_n)=c_{\eta,\alpha,\beta}\quad\text{and}\quad\lim_{n\to\infty} I'_{\eta,\alpha,\beta}(u_n,v_n)=0. \tag{5.1}$$

Since $I_{\eta,\alpha,\beta}$ is coercive on $\mathcal{N}_{\eta,\alpha,\beta}$, we get that $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. Passing to a subsequence, still denoted by $\{(u_n,v_n)\}_{n\in\mathbb{N}}$, we can assume that there exists $(u_1,v_1)\in\mathcal{H}$ such that $(u_n,v_n)\rightharpoonup(u_1,v_1)$ weakly in $\mathcal{H}$ and

$$\begin{cases} u_n\rightharpoonup u_1,\quad v_n\rightharpoonup v_1 \text{ weakly in } L^{2^*}(\Omega),\\ u_n\to u_1,\quad v_n\to v_1 \text{ strongly in } L^r(\Omega),\ \forall r\in[1,2^*),\\ u_n(z)\to u_1(z),\quad v_n(z)\to v_1(z) \text{ a.e. in } \Omega.\end{cases} \tag{5.2}$$

By the proof of Lemma 3.3 and (5.2), we get

$$\lim_{n\to\infty}\Psi_{\lambda,\mu}(u_n,v_n)=\Psi_{\lambda,\mu}(u_1,v_1). \tag{5.3}$$

From (5.1), (5.2) and (5.3), it is easy to prove that (u_1,v_1) is a weak solution of (1.1). Moreover, the fact that $(u_n,v_n)\in\mathcal{N}_{\eta,\alpha,\beta}$ implies that

$$\Psi_{\lambda,\mu}(u_n,v_n)=\frac{q(2^*-2)}{2(2^*-q)}\|(u_n,v_n)\|_{\mathcal{H}}-\frac{q2^*}{2^*-q}I_{\eta,\alpha,\beta}(u_n,v_n). \tag{5.4}$$

Let $n\to\infty$ in (5.4), by (5.3) and $c_{\eta,\alpha,\beta}<0$, we deduce that

$$\Psi_{\lambda,\mu}(u_1,v_1)\geq-\frac{q2^*}{2^*-q}c_{\eta,\alpha,\beta}>0,$$

which implies that $(u_1, v_1) \in \mathcal{H}$ is a nontrivial solution of (1.1).

Now, we prove that $(u_n, v_n) \to (u_1, v_1)$ strongly in $\mathcal{H}$ and that $I_{\eta,\alpha,\beta}(u_1, v_1) = c_{\eta,\alpha,\beta}$. By applying Fatou's lemma and $(u_1, v_1) \in \mathcal{N}_{\eta,\alpha,\beta}$, one has

$$\begin{aligned} c_{\eta,\alpha,\beta} \le I_{\eta,\alpha,\beta}(u_1, v_1) &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\|(u_1, v_1)\|_{\mathcal{H}} - \frac{2^* - q}{q2^*}\Psi_{\lambda,\mu}(u_1, v_1) \\ &\le \liminf_{n\to\infty}\left[\left(\frac{1}{2} - \frac{1}{2^*}\right)\|(u_n, v_n)\|_{\mathcal{H}} - \frac{2^* - q}{q2^*}\Psi_{\lambda,\mu}(u_n, v_n)\right] \le \lim_{n\to\infty} I_{\eta,\alpha,\beta}(u_n, v_n) = c_{\eta,\alpha,\beta}. \end{aligned}$$

This yields $I_{\eta,\alpha,\beta}(u_1, v_1) = c_{\eta,\alpha,\beta}$ and $\lim_{n\to\infty}\|(u_n, v_n)\|_{\mathcal{H}} = \|(u_1, v_1)\|_{\mathcal{H}}$. The standard argument shows that $(u_n, v_n) \to (u_1, v_1)$ strongly in $\mathcal{H}$.

Next, we claim that $(u_1, v_1) \in \mathcal{N}^+_{\eta,\alpha,\beta}$. In fact, if $(u_1, v_1) \in \mathcal{N}^-_{\eta,\alpha,\beta}$, by Lemma 4.5 (ii), there are unique $t_1^+$and $t_1^- > 0$ such that $\left(t_1^+ u_1, t_1^+ v_1\right) \in \mathcal{N}^+_{\eta,\alpha,\beta}$, $\left(t_1^- u_1, t_1^- v_1\right) \in \mathcal{N}^-_{\eta,\alpha,\beta}$ and $t_1^+ < t_1^- = 1$. Since $\frac{d}{dt}I_{\eta,\alpha,\beta}\left(t_1^+ u_1, t_1^+ v_1\right) = 0$ and $\frac{d^2}{dt^2}I_{\eta,\alpha,\beta}\left(t_1^+ u_1, t_1^+ v_1\right) > 0$, there exists $t_1^* \in \left(t_1^+, t_1^-\right)$such that $I_{\eta,\alpha,\beta}\left(t_1^+ u_1, t_1^+ v_1\right) < I_{\eta,\alpha,\beta}(t_1^* u_1, t_1^* v_1)$. By Lemma 4.5, it follows that

$$I_{\eta,\alpha,\beta}\left(t_1^+ u_1, t_1^+ v_1\right) < I_{\eta,\alpha,\beta}(t_1^* u_1, t_1^* v_1) \le I_{\eta,\alpha,\beta}\left(t_1^- u_1, t_1^- v_1\right) = I_{\eta,\alpha,\beta}(u_1, v_1),$$

which contradicts $I_{\eta,\alpha,\beta}(u_1, v_1) = c_{\eta,\alpha,\beta}$. Moreover, since $I_{\eta,\alpha,\beta}(u_1, v_1) = I_{\eta,\alpha,\beta}(|u_1|, |v_1|)$ and $(|u_1|, |v_1|) \in \mathcal{N}^+_{\eta,\alpha,\beta}$, we may assume that (u_1, v_1) is a nonnegative nontrivial solution of system (1.1). By means of Bony's maximum principle [4], such solution turn out to be strictly positive. □

Now we establish the existence of a local minimizer of $I_{\eta,\alpha,\beta}$ on $\mathcal{N}^-_{\eta,\alpha,\beta}$.

Lemma 5.3. *Assume that* $(\mathcal{A}_0)$ *hold. Then, there exist* $(u_0, v_0) \in \mathcal{H}\backslash\{(0,0)\}$ *and* $\Lambda_5 > 0$ *such that for all* $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_5}$*, the following holds:*

$$\sup_{t\ge 0} I_{\eta,\alpha,\beta}(tu_0, tv_0) < c_\infty, \tag{5.5}$$

where c_∞ *is a constant given in* (3.12). *In particular,* $c^-_{\eta,\alpha,\beta} < c_\infty$ *for all* $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_5}$.

Proof. Without loss of generality, we assume that $0 \in \Omega$. Let $R \in (0, r_0)$ be such that the quasi-ball $B_d(0, R) \subset \Omega$, and let a cut-off function $\varphi \in C_0^\infty(B_d(0, R))$ satisfying $0 \le \varphi \le 1$, $\varphi = 1$ in $B_d\left(0, \frac{R}{2}\right)$ and $\varphi = 0$ in $\mathbb{G}\backslash B_d(0, R)$. Here r_0 is given in $(\mathcal{A}_2)$. Now, let $u_\varepsilon(z) = \varphi(z)U_\varepsilon(z)$ and consider the function

$$J_\eta(t) = \frac{t^2}{2}\left(1 + t_0^2\right)\|u_\varepsilon\|^2_{S_0^1(\Omega)} - \frac{t^{2^*}}{2^*}\left(\eta_1 t^{\beta_1} + \eta_2 t^{\beta_2}\right)\int_\Omega |u_\varepsilon|^{2^*} dz, \tag{5.6}$$

where t_0 be given in Theorem 2.1. By Lemma 2.2 and the definition of $S_{\eta,\alpha,\beta}$, we obtain that

$$\begin{aligned}\sup_{t\geq 0} J_\eta(t) &\leq \left(\frac{1}{2}-\frac{1}{2^*}\right)\left(\frac{\left(1+t_0^2\right)\|u_\varepsilon\|^2_{S_0^1(\Omega)}}{\left(\eta_1 t^{\beta_1}+\eta_2 t^{\beta_2}\right)^{\frac{2}{2^*}}\left(\int_\Omega |u_\varepsilon|^{2^*}dz\right)^{\frac{2}{2^*}}}\right)^{\frac{2^*}{2^*-2}} \\ &\leq \frac{1}{Q}\left(\mathfrak{h}\left(t_0\right)\frac{\|u_\varepsilon\|^2_{S_0^1(\Omega)}}{\left(\int_\Omega |u_\varepsilon|^{2^*}dz\right)^{\frac{2}{2^*}}}\right)^{\frac{\rho}{2}} = \frac{1}{Q}\left(\mathfrak{h}\left(t_0\right)\frac{S_{\mathbb{G}}^{\frac{Q}{2}}++O(\varepsilon^{Q-2})}{\left(S_{\mathbb{G}}^{\frac{Q}{2}}+O(\varepsilon^{Q})\right)^{\frac{2}{2^*}}}\right)^{\frac{Q}{2}} \\ &= \frac{1}{Q}\left(\mathfrak{h}\left(t_0\right)S_{\mathbb{G}}\right)^{\frac{Q}{2}}+c_1\varepsilon^{Q-2} = \frac{1}{Q}S_{\eta,\alpha,\beta}^{\frac{Q}{2}}+c_1\varepsilon^{Q-2},\end{aligned} \tag{5.7}$$

where c_1 is a positive constant and the following fact has been used:

$$\sup_{t\geq 0}\left(\frac{t^2}{2}A-\frac{t^{2^*}}{2^*}B\right)=\frac{1}{Q}\left(\frac{A}{B^{\frac{Q-2}{Q}}}\right)^{\frac{Q}{2}}, \quad \forall A,B>0.$$

Choosing $\Lambda_1>0$ such that $0<\lambda\|g\|_{L^{\frac{2^*-q}{2^*}}}+\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}<\Lambda_1$, by the definitions of $I_{\eta,\alpha,\beta}$, there exists $t_m\in(0,1)$ such that

$$I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right)\leq \frac{t^2}{2}\left(1+t_0^2\right)\|u_\varepsilon\|^2_{S_0^1(\Omega)}<c_\infty, \quad \forall t<t_m,$$

and one has

$$\sup_{0\leq t<t_m} I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right)<c_\infty, \tag{5.8}$$

for all $\lambda,\mu\in(0,+\infty)$ with

$$0<\lambda\|g\|_{L^{\frac{2^*}{2^*-q}}}+\mu\|h\|_{L^{\frac{2^*}{2^*-q}}}<\Lambda_1.$$

Moreover, by the definitions of $I_{\eta,\alpha,\beta}$ and $(u_\varepsilon, t_0u_\varepsilon)$, using the condition $(\mathcal{A}_2)$, Lemma 2.3 and (5.7), we have

$$\begin{aligned}\sup_{t\geq t_m} I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right) &= \sup_{t\geq t_m}\left(J_\lambda(t)-\frac{t^q}{q}\int_\Omega\left(\lambda g(z)+\mu h(z)t_0^q\right)|u_\varepsilon|^q dz\right) \\ &\leq \frac{1}{Q}S_{\eta,\alpha,\beta}^{\frac{Q}{2}}+c_1\varepsilon^{Q-2}-\frac{t_m^q}{q}a_0\left(\lambda+\mu t_0^q\right)\int_\Omega |u_\varepsilon|^q dz \\ &\leq \frac{1}{Q}S_{\eta,\alpha,\beta}^{\frac{Q}{2}}+c_1\varepsilon^{Q-2} \\ &\quad -\frac{t_m^q}{q}a_0\left(\lambda+\mu t_0^q\right)\begin{cases} c_2\varepsilon^{Q-\frac{(Q-2)q}{2}}, & \text{if } q>\frac{Q}{Q-2}, \\ c_3\varepsilon^{Q-\frac{(Q-2)q}{2}}|\ln\varepsilon|, & \text{if } q=\frac{Q}{Q-2}, \\ c_4\varepsilon^{\frac{(Q-2)q}{2}}, & \text{if } q<\frac{Q}{Q-2}, \end{cases}\end{aligned} \tag{5.9}$$

where c_2, c_3, c_4 are positive constants.

(i) If $1 < q < \frac{Q}{Q-2}$, then by $Q \geq 4$ one can get that $q\frac{Q-2}{2} < \frac{Q}{2} \leq Q-2$. Thus, for $\varepsilon > 0$ small enough, we can choose $\Lambda_2 > 0$ such that

$$\sup_{t\geq t_m} I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right) \leq \frac{1}{Q}S_{\eta,\alpha,\beta}^{\frac{q}{2}} + c_1\varepsilon^{Q-2} - \frac{t_0^q}{q}a_0c_4\varepsilon^{\frac{(Q-2)q}{2}} < c_\infty,$$

for all $\lambda, \mu \in (0, +\infty)$, with $0 < \lambda\|g\|_{L^{\frac{2}{2^*-q}}} + \mu\|h\|_{L^{\frac{2}{2^*-q}}} < \Lambda_2$.

(ii) If $\frac{Q}{Q-2} \leq q < 2$, we have $Q > 4$ and $q \geq \frac{Q}{Q-2} > \frac{4}{Q-2}$, which implies that

$$Q - \frac{(Q-2)q}{2} - (Q-2) = 2 - \frac{(Q-2)q}{2} = \frac{4-(Q-2)q}{2} = \frac{(Q-2)\left(\frac{4}{Q-2} - q\right)}{2} < 0.$$

Then for ε small enough, by a similar argument in (i), we can choose $\Lambda_3 > 0$ such that

$$\sup_{t\geq t_m} I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right) < c_\infty,$$

for all $\lambda, \mu \in (0, +\infty)$ with $0 < \lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \Lambda_3$.

Set $\Lambda_4 = \min\{\Lambda_2, \Lambda_3\}$, from cases (i) and (ii), for all $\lambda, \mu \in (0, +\infty)$ with

$$0 < \sup_{t\geq t_m} I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right) < c_\infty. \tag{5.10}$$

Thus, taking $\Lambda_5 = \min\{\Lambda_1, \Lambda_4\}$, (5.8) and (5.10) induce that $\sup_{t\geq 0} I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right) < c_\infty$ holds for all $\lambda, \mu \in (0, +\infty)$ with $0 < \lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \Lambda_5$.

Finally, we prove that $c^-_{\eta,\alpha,\beta} < c_\infty$ for all $\lambda, \mu \in (0, +\infty)$ with $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_5}$. Recall that $(u_0, v_0) := (u_\varepsilon, t_0u_\varepsilon)$. It is easy to see that $K_\eta\left(u_\varepsilon, t_0u_\varepsilon\right) > 0$. Then, combining (5.5) with Lemma 4.5, and using the definition of $c^-_{\eta,\alpha,\beta}$, we obtain that there exists $t_2^- > 0$ such that $\left(t_2^-u_0, t_2^-v_0\right) \in \mathcal{N}^-_{\eta,\alpha,\beta}$ and

$$c^-_{\eta,\alpha,\beta} \leq I_{\eta,\alpha,\beta}\left(t_2^-u_0, t_2^-v_0\right) \leq \sup_{t\geq 0} I_{\eta,\alpha,\beta}\left(tu_0, tv_0\right) < c_\infty,$$

for all $\lambda, \mu \in (0, +\infty)$ with $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_5}$. The proof is now complete. □

Theorem 5.4. *Under the assumptions of Theorem 1.2. If $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_*}$, then the functional $I_{\eta,\alpha,\beta}$ has a minimizer $(u_2, v_2) \in \mathcal{N}^-_{\eta,\alpha,\beta}$ and it satisfies $I_{\eta,\alpha,\beta}(u_2, v_2) = c^-_{\eta,\alpha,\beta}$, and (u_2, v_2) is a positive solution of* (1.1)*, where $\Lambda_* = \min\left\{\Lambda_5, \frac{q}{2}\Lambda\right\}$.*

Proof. By Lemma 5.1 (ii), there exists a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\eta,\alpha,\beta}$ in $\mathcal{H}$ for $I_{\eta,\alpha,\beta}$, for all $(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+$satisfying

$$0 < \lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \frac{q}{2}\Lambda.$$

In the light of Lemmas 5.3, 3.4 and 5.1 (ii), for $0 < \lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \Lambda_*$, the functional $I_{\eta,\alpha,\beta}$ satisfies $(PS)_{c^-_{\eta,\alpha,\beta}}$ condition for $c^-_{\eta,\alpha,\beta} > 0$. Since $I_{\eta,\alpha,\beta}$ is coercive on $\mathcal{N}_{\eta,\alpha,\beta}$, we can deduce that $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{N}_{\eta,\alpha,\beta}$ and $\mathcal{H}$. So, there exists a subsequence still denoted by $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ and $(u_2, v_2) \in \mathcal{N}^-_{\eta,\alpha,\beta}$ such that $(u_n, v_n) \to (u_2, v_2)$ strongly in $\mathcal{H}$, and $I_{\eta,\alpha,\beta}(u_2, v_2) = c^-_{\eta,\alpha,\beta} > 0$, $I'_{\eta,\alpha,\beta}(u_2, v_2) = 0$ for all $(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+$ with

$$0 < \lambda\|g\|_{L^{\frac{2^*}{2^*-q}}} + \mu\|h\|_{L^{\frac{2^*}{2^*-q}}} < \Lambda_*.$$

Finally, arguing as in the proof of Theorem 5.2, we have that (u_2, v_2) is a positive solution of the system (1.1). □

Proof of Theorem 1.2. By Theorem 5.2, we obtain that for all $(\lambda, \mu) \in \mathfrak{D}_\Lambda$, Problem (1.1) has a positive solution $(u_1, v_1) \in \mathcal{N}^+_{\eta,\alpha,\beta}$. By Theorem 5.4, we obtain a second positive solution $(u_2, v_2) \in \mathcal{N}^-_{\eta,\alpha,\beta}$ for all $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_*} \subset \mathfrak{D}_\Lambda$. Since $\mathcal{N}^+_{\eta,\alpha,\beta} \cap \mathcal{N}^-_{\eta,\alpha,\beta} = \emptyset$, this implies that (u_1, v_1) and (u_2, v_2) are distinct.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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References

[1] C. O. Alves and Y. H. Ding, "Multiplicity of positive solutions to a p-Laplacian equation involving critical nonlinearity," *J. Math. Anal. Appl.*, vol. 279, no. 2, pp. 508–521, 2003, doi: 10.1016/S0022-247X(03)00026-X.

[2] S. Benmouloud, R. Echarghaoui, and S. M. Sbaï, "Multiplicity of positive solutions for a critical quasilinear elliptic system with concave and convex nonlinearities," *J. Math. Anal. Appl.*, vol. 396, no. 1, pp. 375–385, 2012, doi: 10.1016/j.jmaa.2012.05.078.

[3] A. Bonfiglioli and F. Uguzzoni, "Nonlinear Liouville theorems for some critical problems on H-type groups," *J. Funct. Anal.*, vol. 207, no. 1, pp. 161–215, 2004, doi: 10.1016/S0022-1236(03)00138-1.

[4] J.-M. Bony, "Principe du maximum, inégalite de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés," *Ann. Inst. Fourier (Grenoble)*, vol. 19, pp. 277–304 xii, 1969.

[5] K. J. Brown and T.-F. Wu, "A semilinear elliptic system involving nonlinear boundary condition and sign-changing weight function," *J. Math. Anal. Appl.*, vol. 337, no. 2, pp. 1326–1336, 2008, doi: 10.1016/j.jmaa.2007.04.064.

[6] G. B. Folland, "Subelliptic estimates and function spaces on nilpotent Lie groups," *Ark. Mat.*, vol. 13, no. 2, pp. 161–207, 1975, doi: 10.1007/BF02386204.

[7] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, ser. Mathematical Notes. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982, vol. 28.

[8] N. Garofalo and D. Vassilev, "Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups," *Math. Ann.*, vol. 318, no. 3, pp. 453–516, 2000, doi: 10.1007/s002080000127.

[9] T.-S. Hsu, "Multiple positive solutions for a critical quasilinear elliptic system with concave-convex nonlinearities," *Nonlinear Anal.*, vol. 71, no. 7-8, pp. 2688–2698, 2009, doi: 10.1016/j.na.2009.01.110.

[10] T.-S. Hsu and H.-L. Lin, "Multiple positive solutions for a critical elliptic system with concave-convex nonlinearities," *Proc. Roy. Soc. Edinburgh Sect. A*, vol. 139, no. 6, pp. 1163–1177, 2009, doi: 10.1017/S0308210508000875.

[11] A. Loiudice, "Semilinear subelliptic problems with critical growth on Carnot groups," *Manuscripta Math.*, vol. 124, no. 2, pp. 247–259, 2007, doi: 10.1007/s00229-007-0119-x.

[12] A. Loiudice, "Critical growth problems with singular nonlinearities on Carnot groups," *Nonlinear Anal.*, vol. 126, pp. 415–436, 2015, doi: 10.1016/j.na.2015.06.010.

[13] A. Loiudice, "Local behavior of solutions to subelliptic problems with Hardy potential on Carnot groups," *Mediterr. J. Math.*, vol. 15, no. 3, 2018, Art ID 81, doi: 10.1007/s00009-018-1126-8.

[14] M. Ruzhansky and D. Suragan, *Hardy inequalities on homogeneous groups*, ser. Progress in Mathematics. Birkhäuser/Springer, Cham, 2019, vol. 327, doi: 10.1007/978-3-030-02895-4.

[15] M. Ruzhansky, N. Tokmagambetov, and N. Yessirkegenov, "Best constants in Sobolev and Gagliardo-Nirenberg inequalities on graded groups and ground states for higher order nonlinear subelliptic equations," *Calc. Var. Partial Differential Equations*, vol. 59, no. 5, 2020, Art. ID 175, doi: 10.1007/s00526-020-01835-0.

[16] G. Tarantello, "On nonhomogeneous elliptic equations involving critical Sobolev exponent," *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, vol. 9, no. 3, pp. 281–304, 1992, doi: 10.1016/S0294-1449(16)30238-4.

[17] D. Vassilev, "Existence of solutions and regularity near the characteristic boundary for sub-Laplacian equations on Carnot groups," *Pacific J. Math.*, vol. 227, no. 2, pp. 361–397, 2006, doi: 10.2140/pjm.2006.227.361.

[18] M. Willem, *Minimax theorems*, ser. Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston, Inc., Boston, MA, 1996, vol. 24, doi: 10.1007/978-1-4612-4146-1.

[19] T.-F. Wu, "On semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight function," *J. Math. Anal. Appl.*, vol. 318, no. 1, pp. 253–270, 2006, doi: 10.1016/j.jmaa.2005.05.057.

[20] J. Zhang, "Sub-elliptic problems with multiple critical Sobolev-Hardy exponents on Carnot groups," *Manuscripta Math.*, vol. 172, no. 1-2, pp. 1–29, 2023, doi: 10.1007/s00229-022-01406-x.

[21] J. Zhang and T.-S. Hsu, "Nonlocal elliptic systems involving critical Sobolev-Hardy exponents and concave-convex nonlinearities," *Taiwanese J. Math.*, vol. 23, no. 6, pp. 1479–1510, 2019, doi: 10.11650/tjm/190109.

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Vector-valued algebras and variants of amenability

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ABSTRACT

Let $\{A_x : x \in X\}$ be a collection of complex Banach algebras indexed by the compact Hausdorff space X. We investigate the weak- and pseudo-amenability of certain algebras $\mathcal{A}$ of A_x-valued functions in relation to the corresponding properties of the A_x.

RESUMEN

Sea $\{A_x : x \in X\}$ una colección de álgebras de Banach complejas indexadas por un espacio de Hausdorff compacto X. Investigamos la amenabilidad débil y la seudo-amenabilidad de ciertas álgebras $\mathcal{A}$ de funciones con valores en A_x en relación a las propiedades correspondientes de los A_x.

Keywords and Phrases: Function algebra, weak amenability, pseudo-amenability.

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1 Preliminaries

Suppose that $\mathcal{P}$ is a property which a Banach space A might possess. A reasonable question to ask about $\mathcal{P}$ is of the sort "What constructions on Banach spaces preserve $\mathcal{P}$?" To clarify this, we take a specific example: Let A be a complex Banach algebra and suppose that we are interested in amenability. It is then well known that $\mathcal{P}$, the property of being amenable, is preserved by quotients: If A is an amenable Banach algebra, and $I \subset A$ is a closed ideal, then A/I is also amenable. It is also well-known that amenability is preserved by projective tensor products: if A and B are both amenable, then so is $A\widehat{\otimes}B$. (See *e.g.* the survey paper [15, Prop. 2.3.2].) Loosely speaking, we can ask whether $\mathcal{P}$ is preserved "downwards" (through quotients), or "sideways" (through tensor products); amenability is preserved in both of these directions. We could also ask whether $\mathcal{P}$ is preserved by actions in two directions: *e.g.* if for a closed ideal I of a Banach algebra A, both I and A/I satisfy $\mathcal{P}$, does A also satisfy $\mathcal{P}$? (This is the 3-space problem.)

This short discussion leads us to ask the following: Can a property $\mathcal{P}$ be preserved "upwards"? We make this airy question more explicit: Let X be a set (usually, a topological space), and let $\{A_x : x \in X\}$ be a collection of Banach spaces indexed by X, over a common scalar field $\mathbb{K}$, either $\mathbb{R}$ or $\mathbb{C}$. Suppose that A_x possesses property $\mathcal{P}$ for each $x \in X$. Let $\mathcal{A} \subset \prod\{A_x : x \in X\}$ be a Banach space of functions under the pointwise operations, and let $\sigma \in \mathcal{A}$, so that $\sigma(x) \in A_x$ for all $x \in X$. What conditions on X, $\mathcal{A}$, and the A_x (aside from possessing $\mathcal{P}$) might be sufficient to insure that $\mathcal{A}$ also has $\mathcal{P}$? Again using amenability as an example, we see that if X is an infinite compact Hausdorff space, then $\mathcal{A} = \ell^1(X)$, the space of absolutely summable complex-valued functions on X under the pointwise operations, is not amenable: The amenability of each $A_x = \mathbb{C}$ is not passed on to $\mathcal{A}$, since $\mathcal{A}$ does not have a bounded approximate identity. On the other hand, $\mathcal{A} = c_0(X)$, the closure in the sup-norm of the space of $\mathbb{C}$-valued functions with finite support, is amenable. This suggests that, at the very least, $\mathcal{A}$ should have some conditions on it, and perhaps also that the collection $\{A_x : x \in X\}$ should satisfy some additional unifying property, aside from just having all A_x possess $\mathcal{P}$. We might also want X to satisfy some reasonable conditions.

Obviously, there are many ways in which it might be possible to go "up" in this sense. In this paper, we focus on one type of vector-valued function space $\mathcal{A}$. We will assume unless otherwise specified that X is a compact Hausdorff space and that $\{A_x : x \in X\}$ is a collection of complex Banach spaces; we take $\mathcal{A} \subset \prod\{A_x : x \in X\}$ to be a Banach space of functions under the pointwise operations which satisfies the conditions:

C1) For each $x \in X$, $A_x = p_x(\mathcal{A}) = \{\sigma(x) : \sigma \in \mathcal{A}\}$, that is, $\mathcal{A}$ is said to be full; p_x is the evaluation map at x, so that $p_x(\sigma) = \sigma(x)$.

C2) For each $\sigma \in \mathcal{A}$, the norm map $x \mapsto \|\sigma(x)\|$ is upper semicontinuous on X; hence σ is bounded and norm-attaining, with

$$\|\sigma\| = \sup\{\|\sigma(x)\| : x \in X\} = \|\sigma(x_0)\|$$

for some $x_0 \in X$.

C3) $\mathcal{A}$ is a $C(X)$-module under the pointwise operations.

C4) $\mathcal{A}$ is complete in the sup-norm.

We will call a space $\mathcal{A}$ which satisfies C1) - C4) an upper semicontinuous function space with fibers A_x, and abbreviate it to "function space".

If in addition to C1) - C4) it is also the case that

C5) Each A_x is a Banach algebra, and $\mathcal{A}$ is closed under pointwise multiplication (so that $\mathcal{A}$ is a Banach algebra). We call such an $\mathcal{A}$ a *function algebra*. Evidently, a function algebra $\mathcal{A}$ is commutative if and only if each fiber A_x is commutative.

Examples of such function spaces (algebras) can be found in [11, Section 2], and also (using the language of section spaces of bundles of Banach spaces and Banach algebras) in [5] and [14]. In particular, if A is a Banach algebra and $\{A_x : x \in X\}$ is a collection of Banach algebras, such function algebras include $C(X, A)$, the space of continuous A-valued functions on X, and $c_0(X, \{A_x\})$, the closure in the sup norm of the functions $\sigma \in \prod\{A_x : x \in X\}$ with finite support. A brief, and quite incomplete, bibliographical note on such function spaces can be found at the end of [11, Section 2].

For a more general setting, the reader may also wish to consult [1]. Using slightly different language, that paper studies algebras $\mathcal{A}$ of vector-valued functions over a completely regular Hausdorff space X. These functions take their values in associative topological algebras $\{A_x : x \in X\}$, and the algebra $\mathcal{A}$ is assumed to satisfy C1) and C3) above, without the completeness or norm conditions of C2), C4), or C5).

Heritability has been explored previously (using either the language of function spaces or section spaces of bundles of Banach spaces), for example in [4] (where $\mathcal{A}$ is simply a function space), [7], and [8]. Of particular interest here are papers concerned with how some variants of amenability can be inherited by function algebras $\mathcal{A}$, *e.g.* amenability itself ([9]), module amenability ([11]), and character amenability ([12]); in all these papers, appropriate uniform boundedness conditions were shown sufficient to guarantee that the property under consideration was preserved by $\mathcal{A}$. By using the existence of certain conditions intrinsic to Banach algebras sufficient to establish the various properties $\mathcal{P}$ of interest, it was possible to avoid the homological definitions of the properties. In this paper we will investigate the preservation to $\mathcal{A}$ of weak amenability in the A_x by employing a similar work-around. We will also investigate the preservation to $\mathcal{A}$ of pseudo-amenability; the results in this case are not so satisfying.

We note several important properties of function spaces (algebras) $\mathcal{A}$.

I) The evaluation map $\sigma \mapsto \sigma(x)$ from $\mathcal{A}$ to A_x is a quotient map. Indeed, we have $A_x \simeq \mathcal{A}/\overline{I_x\mathcal{A}}$, where $I_x \subset C(X)$ is the maximal ideal of functions f such that $f(x) = 0$, and $\overline{I_x\mathcal{A}}$ is the closed span in $\mathcal{A}$ of elements of the form $f\sigma$ ($\sigma \in \mathcal{A}$, $f \in I_x$). The correspondence is given by $\sigma(x) \leftrightarrow \sigma + \overline{I_x\mathcal{A}}$.

II) Let $\mathcal{B}$ be a function subspace of $\mathcal{A}$, *i.e.* a closed subspace of $\mathcal{A}$ which is also a $C(X)$-module, and set $B_x = p_x(\mathcal{B}) = \{\sigma(x) : \sigma \in \mathcal{B}\} \subset A_x$. Then $B_x \subset A_x$ is a closed subspace, and $\mathcal{B}_x = \{\sigma \in \mathcal{A} : \sigma(x) \in B_x\}$ is a function subspace (necessarily full) of $\mathcal{A}$; $\mathcal{B}_x$ has fibers B_x and A_y (if $y \neq x$). Moreover, (*) $\mathcal{B} = \bigcap\{\mathcal{B}_x : x \in X\} = \{\sigma \in \mathcal{A} : \sigma(x) \in B_x \text{ for all } x \in X\}$ and $p_x(\mathcal{B}) = B_x$. In particular, if $\mathcal{B}$ and $\mathcal{C}$ are function subspaces of $\mathcal{A}$ such that $B_x = C_x$ for all x, then $\mathcal{B} = \mathcal{C}$. [Two caveats: 1) We need $\mathcal{B}$ and $\mathcal{C}$ to be subspaces of a common function space $\mathcal{A}$; it is not enough to have function spaces $\mathcal{B}$ and $\mathcal{C}$ over X which have fibers $B_x = C_x$ for all $x \in X$, as the example $\mathcal{B} = C(X)$ and $\mathcal{C} = c_0(X)$ shows. 2) In order for (*) to hold, we also need to specify $\mathcal{B}$, and hence both its fibers B_x and the function subspaces $\mathcal{B}_x \subset \mathcal{A}$. Merely specifying some subspaces $B_x \subset A_x$ is insufficient if we wish the fibers of $\mathcal{B} = \bigcap\{\mathcal{B}_x : x \in X\}$ to be the B_x. For example, consider the case $X = [0,1]$, $\mathcal{B} = C(X)$, and $B_x = 0$ if x is rational, and $B_x = \mathbb{C}$ otherwise. Then $\mathcal{B}_x = \{f \in \mathcal{B} : f(x) = 0\}$ if x is rational, and $\mathcal{B}_x = \mathcal{B}$ otherwise. But $\bigcap\{\mathcal{B}_x : x \in X\} = \{0\} \subset \mathcal{B}$.]

III) Let $\mathcal{A}$ be a function algebra with fibers A_x. Then $\mathcal{A}$ is a $C(X)$-(bi)module, and we let $J = J_{\mathcal{A}} \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ be the closed span of elements of the form $(f\sigma \otimes \tau) - (\sigma \otimes f\tau) = [(f \otimes \overline{\mathbf{1}}) - (\overline{\mathbf{1}} \otimes f)](\sigma \otimes \tau)$, where $\overline{\mathbf{1}} \in C(X)$ is the identity, *i.e.* the function with constant value 1. We call J the $C(X)$-balanced kernel in $\mathcal{A}\widehat{\otimes}\mathcal{A}$. It is easy to check that J is both an ideal and a $C(X)\widehat{\otimes}C(X)$ submodule in $\mathcal{A}\widehat{\otimes}\mathcal{A}$. Then there is a function algebra $\mathcal{A} \otimes_X \mathcal{A}$ with fibers $A_x\widehat{\otimes}A_x$ and a $C(X)$-isometric isomorphism $q : (\mathcal{A}\widehat{\otimes}\mathcal{A})/J \to \mathcal{A} \otimes_X \mathcal{A}$, where $[q(\sigma \otimes \tau + J)](x) = (\sigma \odot \tau)(x) = \sigma(x) \otimes \tau(x)$. The isometry is given by

$$\left\|\sum_k \sigma_k \odot \tau_k\right\|_{\mathcal{A}\otimes_X\mathcal{A}} = \sup_{x\in X}\left\|\sum_k \sigma_k(x) \otimes \tau_k(x)\right\|_{A_x\widehat{\otimes}A_x} = \left\|\sum_k \sigma_k \otimes \tau_k + J\right\|_{(\mathcal{A}\widehat{\otimes}\mathcal{A})/J}.$$

Of these properties, I) and II) can be found in various locations in [5, Chap. 9]; and III) is [16, Thm. 1.2 and Prop. 1.5].

We now proceed to our studies of weak amenability and pseudo-amenability.

2 Weak amenability and heritability

Recall that if A is a Banach algebra and M is a Banach A-bimodule, then M^* can be made into Banach A-bimodule in a standard fashion via the actions

$$\langle m^* a, m\rangle = \langle m^*, am\rangle \quad \text{and} \quad \langle am^*, m\rangle = \langle m^*, ma\rangle ,$$

where $a \in A, m \in M$, and $m^* \in M^*$. A derivation $D : A \to M$ is a continuous linear map such that $D(ab) = aD(b) + D(a)b$. The derivation $D : A \to M$ is said to be *inner* if

$$D(a) = \delta_m(a) = am - ma$$

for some $m \in M$.

Definition 2.1. *Let A be a complex Banach algebra. Say that A is weakly amenable if each derivation $D : A \to A^*$ is inner.*

Recalling that A is said to be amenable if, for any Banach A-bimodule M, each derivation $D : A \to M^*$ is inner [13], it is clear that an amenable algebra is also weakly amenable. If A is commutative, then $Fa = aF$ for each $a \in A$ and $F \in A^*$, so for a commutative algebra A to be weakly amenable is to say that A has no non-zero derivations to A^*.

Note that amenability and weak amenability can also be expressed in homological terms, the details of which are not necessary here. In the event that A is commutative there are, however, conditions intrinsic to A which are equivalent to its weak amenability. For the remainder of this section we will assume (at the possible loss of some unnecessary generality) that all algebras are commutative, and employ these conditions to investigate the heritability of weak amenability for function algebras $\mathcal{A}$.

We first note a necessary condition for $\mathcal{A}$ to be weakly amenable.

Proposition 2.2. *Suppose that the function algebra $\mathcal{A}$, defined over X, is weakly amenable. Then each fiber A_x is weakly amenable.*

Proof. Recall that $A_x \simeq \mathcal{A}/\overline{I_x\mathcal{A}}$, and use the fact that quotients of weakly amenable algebras by closed ideals are themselves weakly amenable. (See [6, Prop. 2.1] or [15, Prop. 2.5.3].) □

To exhibit conditions on the fibers A_x sufficient to make the function algebra $\mathcal{A}$ weakly amenable, we start with some notation modified from [6]: If A is a complex commutative Banach algebra, then we take $A^\#$ to be A with its standard adjunction of identity. (So, $A^\# = A \oplus \mathbb{C}\mathbf{1}$, with the ℓ^1-norm, and multiplication $(a \oplus \lambda\mathbf{1})(b \oplus \mu\mathbf{1}) = (ab + \lambda b + \mu a) \oplus \lambda\mu\mathbf{1}$. We will abuse notation only slightly and write $a + \lambda = a + \lambda\mathbf{1} \in A^\#$.) Let $K_A^\# \subset A^\#\widehat{\otimes}A^\#$ be the closed ideal which is the kernel of the multiplication map $\pi^{\#} : A^{\#}\widehat{\otimes}A^{\#} \to A^{\#}, (a+\lambda)\otimes(b+\mu) \mapsto ab+\lambda b+\mu a+\lambda\mu$, and let $\pi : A\widehat{\otimes}A \to A$ be the restriction of $\pi^{\#}$ to $A\widehat{\otimes}A \subset A^{\#}\widehat{\otimes}A^{\#}$. We set $K_A^0 = K_A^{\#} \cap (A\widehat{\otimes}A)$; note that this makes sense since A is complemented in $A^{\#}$ by a projection of norm 1, and hence $A\widehat{\otimes}A$ is an actual subset of $A^{\#}\widehat{\otimes}A^{\#}$ ([17, Prop. 2.4]). Since $A\widehat{\otimes}A$ is an ideal in $A^{\#}\widehat{\otimes}A^{\#}$, we have $K_A^0 \subset A\widehat{\otimes}A$ is also a closed ideal. In particular, for $u = \sum_k (a_k+\lambda_k)\otimes(b_k+\mu_k) \in A\widehat{\otimes}A$, we have $u \in K_A^{\#}$ if and only if

$$\sum_k (a_kb_k + \lambda_kb_k + \mu_ka_k + \lambda_k\mu_k) = 0 \in A^{\#}.$$

Especially, we have $\sum_k \lambda_k\mu_k = 0 \in \mathbb{C}$. Likewise, an element $u \in K_A^0$ is of the form $\sum_k a_k \otimes b_k$, with $\sum_k a_kb_k = 0 \in A$.

For later use, we note the following:

Lemma 2.3. *Let A be a Banach algebra. Then $\overline{(K_A^0)^2} \subset \overline{K_A^{\#}(A\widehat{\otimes}A)}$.*

Proof. By definition, K_A^0 is a subset of both $K_A^{\#}$ and $A\widehat{\otimes}A$, so that if $z, z' \in K_A^0$, we consider that $z \in K_A^{\#}$ and $z' \in A\widehat{\otimes}A$, so that $zz' \in \left(K_A^0\right)^2 \subset K_A^{\#}(A\widehat{\otimes}A)$; now use linearity and density. □

The following result characterizes the weak amenability of a commutative Banach algebra A. Recall that a Banach algebra is said to be essential provided that $A^2 = span\{ab : a, b \in A\}$ is dense in A.

Theorem 2.4 ([6, Thm. 3.2])**.** *Let A be a commutative complex Banach algebra. Then the following are equivalent:*

1) *A is weakly amenable;*
2) *A is essential and $\overline{(K_A^0)^2} = \overline{K_A^{\#}(A\widehat{\otimes}A)}$.*

Note that there are other equivalences established in the cited theorem; the one here is sufficient for our purpose.

Suppose now that $\mathcal{A}$ is a function algebra such that each A_x is weakly amenable. We will show that $\mathcal{A}$ is also weakly amenable. The first task is to show that $\mathcal{A}$ is essential.

Proposition 2.5. *Let $\mathcal{A}$ be a function algebra over X, and suppose that each A_x is essential. Then $\mathcal{A}$ is essential, and conversely.*

Proof. This result (a Stone-Weierstrass theorem for function algebras) is a variant of [5, Cor. 4.3], and can also be found in [11]. However, it is worth looking at a proof, using our current language.

Let $\sigma \in \mathcal{A}$, and let $\varepsilon > 0$. For each $x \in X$, we can find $t_x = \sum_{k=1}^{m_x} a_{x,k}b_{x,k} \in A_x^2$ such that $\|\sigma(x) - t_x\| < \varepsilon$. From condition C1), above, we can choose $\tau_{x,k}, \tau'_{x,k} \in \mathcal{A}$ such that $\tau_{x,k}(x) = a_{x,k}$, $\tau'_{x,k}(x) = b_{x,k}$. Set $\nu_x = \sum_{k=1}^{m_x} \tau_{x,k}\tau'_{x,k} \in \mathcal{A}^2$. Since $\|\sigma(x) - \nu_x(x)\| < \varepsilon$, it follows from C2) that there is a neighborhood V_x of x such that whenever $y \in V_x$, we have $\|\sigma(y) - \nu_x(y)\| < \varepsilon$. Take a finite subcover $\{V_j\} = \{V_{x_j} : j = 1, \ldots, n\}$ of the V_x, and let $\{f_j : j = 1, \ldots, n\} \subset C(X)$ be a partition of unity subordinate to the V_j, so that for each $j = 1, \ldots, n$, we have: $0 \leq f_j \leq 1$, f_j is supported on V_j, and $\sum_j f_j = \mathbf{\overline{1}}$.

For any $y \in X$, we then have

$$\left\|\sigma(y) - \sum_j f_j(y)\nu_j(y)\right\| = \left\|\sum_j f_j(y)\left[\sigma(y) - \nu_j(y)\right]\right\| \leq \sum_{j \text{ s.t. } y \in V_j} f_j(y)\left\|\sigma(y) - \nu_j(y)\right\| < \varepsilon,$$

so that $\left\|\sigma - \sum_j f_j\nu_j\right\| < \varepsilon$, and therefore σ is in the closure of $\mathcal{A}^2$.

The converse is an immediate consequence of C1). □

This shows, in particular, that the property of being essential is preserved by function algebras. Especially, if each A_x has an approximate identity, so also does $\mathcal{A}$; if the approximate identities in the fibers A_x of $\mathcal{A}$ are uniformly bounded, then the approximate identity in $\mathcal{A}$ is bounded. (See [9] and [12].)

It is straightforward to check that both $\overline{(K^0_{\mathcal{A}})^2}$ and $\overline{K^{\#}_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})}$ are closed $C(X)\widehat{\otimes}C(X)$-submodules of, and ideals in, $\mathcal{A}\widehat{\otimes}\mathcal{A}$.

Lemma 2.6. *Let $\mathcal{A}$ be a function algebra such that each A_x is weakly amenable, and let $J \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ be the $C(X)$-balanced kernel. Then $J \subset \overline{(K^0_{\mathcal{A}})^2}$.*

Proof. Note that since each A_x is weakly amenable, we have each A_x^2 is dense in A_x, so that $\mathcal{A}^2$ is dense in $\mathcal{A}$. By definition, J is the closed span in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ of elements of the form $[(\mathbf{\overline{1}} \otimes f) - (f \otimes \mathbf{\overline{1}})](\sigma \otimes \tau)$. But since $\mathcal{A}^2$ is dense in $\mathcal{A}$, $\sigma \otimes \tau$ can be written as a limit of elements of form $(\sum_k \sigma'_k\sigma''_k) \otimes \left(\sum_j \tau'_j\tau''_j\right) = \sum_{k,j} \sigma'_k\sigma''_k \otimes \tau'_j\tau''_j = \sum_{j,k}\left(\sigma'_k \otimes \tau'_j\right)\left(\sigma''_k \otimes \tau''_j\right)$. Restricting ourselves for the moment to elements of the form $\sigma \otimes \tau = \sigma_1\sigma_2 \otimes \tau_1\tau_2 = (\sigma_1 \otimes \tau_1)(\sigma_2 \otimes \tau_2) \in \mathcal{A}\widehat{\otimes}\mathcal{A}$, and noting that $(\mathbf{\overline{1}} \otimes f) - (f \otimes \mathbf{\overline{1}})$ is in the kernel of the multiplication map $f \otimes g \mapsto fg$ from $C(X)\widehat{\otimes}C(X)$ to $C(X)$, we can write

$$\left[(\mathbf{\overline{1}} \otimes f) - (f \otimes \mathbf{\overline{1}})\right](\sigma \otimes \tau) = \lim_\mu \left[(\mathbf{\overline{1}} \otimes f) - (f \otimes \mathbf{\overline{1}})\right](\sigma_1 \otimes \tau_1)h_\mu(\sigma_2 \otimes \tau_2),$$

where $\{h_\mu\}$ is a bounded approximate identity for $\ker \pi_{C(X)}$. (Such an $\{h_\mu\}$ exists because $C(X)$ is amenable, so that the above-mentioned kernel J has a bounded approximate identity; see [10, p. 254].)

It is evident that both $\left[(\mathbf{\overline{1}} \otimes f) - (f \otimes \mathbf{\overline{1}})\right](\sigma_1 \otimes \tau_1) \in K^0_{\mathcal{A}}$, and $h_\mu(\sigma_2 \otimes \tau_2) \in K^0_{\mathcal{A}}$, so that $\left[(\mathbf{\overline{1}} \otimes f) - (f \otimes \mathbf{\overline{1}})\right](\sigma_1 \otimes \tau_1)h_\mu(\sigma_2 \otimes \tau_2) \in \overline{(K^0_{\mathcal{A}})^2}$. The rest follows by linearity, density, and the boundedness of $\{h_\mu\}$. □

It follows from Lemma 2.3 that we also have $J \subset \overline{K^{\#}_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})}$.

We obtain from the above that $\mathcal{G} = \overline{(K^0_{\mathcal{A}})^2}/J$ and $\mathcal{H} = \overline{K^{\#}_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})}/J$ are function subalgebras of $\mathcal{A} \otimes_X \mathcal{A}$, with fibers G_x, $H_x \subset A_x\widehat{\otimes}A_x$, respectively.

Now, consider $\mathcal{H} = \overline{K^{\#}_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})}/J$. Recalling the discussion preceding Lemma 2.3 about the multiplication maps $\pi^{\#}$ and π, we see that a typical element $u \in \overline{K^{\#}_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})}$ is a limit of sums of elements of the form

$$\left[\sum_k (\sigma_k + \lambda_k) \otimes (\tau_k + \mu_k)\right]\left[\sum_j \alpha_j \otimes \beta_j\right] \in \mathcal{A}\widehat{\otimes}\mathcal{A},$$

(where the first sum is in $K^{\#}_{\mathcal{A}}$ and the second is in $\mathcal{A}\widehat{\otimes}\mathcal{A}$), with $\pi^{\#}(u) = \pi(u) = 0 \in \mathcal{A}$. If u is such a limit, the image of u in $\overline{K^{\#}_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})}/J$ under the quotient map is therefore a (uniform) limit of sums of functions of the type

$$\left[\sum_k (\sigma_k + \lambda_k) \odot (\tau_k + \mu_k)\right]\left[\sum_j \alpha_j \odot \beta_j\right] = \sum_{k,j} (\sigma_k\alpha_j + \lambda_k\alpha_j) \odot (\tau_k\beta_j + \mu_k\beta_j) \in \mathcal{A}\widehat{\otimes}_X\mathcal{A},$$

where

$$\begin{aligned}\pi\left(\left[\sum_{k,j} (\sigma_k\alpha_j + \lambda_k\alpha_j) \odot (\tau_k\beta_j + \mu_k\beta_j)\right](x)\right) &= \pi\left(\sum_{k,j} [\sigma_k(x)\alpha_j(x) + \lambda_k\alpha_j(x)] \otimes [\tau_k(x)\beta_j(x) + \mu_k\beta_j(x)]\right)\\ &= \pi\left(\left[\sum_k (\sigma_k(x) + \lambda_k) \otimes (\tau_k(x) + \mu_k)\right]\left[\sum_j \alpha_j(x) \otimes \beta_j(x)\right]\right)\\ &= 0 \in A_x,\end{aligned}$$

for each $x \in X$. Thus, $(u + J)(x) \in \overline{K^{\#}_x(A_x\widehat{\otimes}A_x)}$, for each $x \in X$, where $K^{\#}_x = K^{\#}_{A_x}$, and so the fibers $H_x = p_x(\overline{K^{\#}_x(A_x\widehat{\otimes}A_x)}/J) \subset p_x(\mathcal{A} \otimes_X \mathcal{A})$ of $\mathcal{H}$ are subspaces of the $\overline{K^{\#}_x(A_x\widehat{\otimes}A_x)}$ for each $x \in X$.

On the other hand, an element $v \in \overline{K^{\#}_x(A_x\widehat{\otimes}A_x)}$ is the limit in $A_x\widehat{\otimes}A_x$ of sums of elements of the form

$$\left[\sum_k (a_k + \lambda_k) \otimes (b_k + \mu_k)\right]\left[\sum_j c_j \otimes d_j\right] = \sum_{k,j} (a_kc_j + \lambda_kc_j) \otimes (b_kd_j + \mu_kd_j) \in K^{\#}_x(A_x\widehat{\otimes}A_x).$$

For each such element v, we can choose $\alpha_{k,j}, \beta_{k,j} \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\alpha_{k,j}(x) = a_kc_j + \lambda_kc_j$, $\|\alpha_{k,j}\| = \|a_kc_j + \lambda_kc_j\|$, and $\beta_{k,j}(x) = b_kd_j + \mu_kd_j$, $\|\beta_{kj}\| = \|b_kd_j + \mu_kd_j\|$ (see [14, Prop. 1.1]).

Note that by the definition of the norm in projective tensor products, we have

$$\left\|\sum_k (a_k+\lambda_k)\otimes(b_k+\mu_k)\right\| \le \sum_k \|a_k+\lambda_k\|\,\|b_k+\mu_k\| < \infty,$$

and similarly for $\left\|\sum_j c_j\otimes d_j\right\|$. Then

$$\begin{aligned}\left\|\sum_{k,j}\alpha_{k,j}\otimes\beta_{k,j}\right\| &\le \sum_{k,j}\|\alpha_{k,j}\|\,\|\beta_{k,j}\| = \sum_{k,j}\|a_kc_j+\lambda_kc_j\|\,\|b_kd_j+\mu_kd_j\|\\ &\le \sum_{k,j}\|a_k+\lambda_k\|\,\|c_j\|\,\|b_k+\mu_k\|\,\|d_j\|\\ &= \left(\sum_k \|(a_k+\lambda_k)\otimes(b_k+\mu_k)\|\right)\left(\sum_j\|c_j\otimes d_j\|\right) < \infty\end{aligned}$$

and so we have $\sum_{k,j}\alpha_{k,j}\otimes\beta_{k,j}\in\mathcal{A}\widehat{\otimes}\mathcal{A}$. Moreover,

$$\begin{aligned}\pi\left[\sum_{k,j}\alpha_{k,j}\otimes\beta_{k,j}+J\right](x) &= \pi\left[\sum_{k,j}\alpha_{k,j}\odot\beta_{k,j}\right](x)\\ &= \pi\left[\sum_{k,j}(a_kc_j+\lambda_kc_j)\otimes(b_kd_j+\mu_kd_j)\right] = 0\in A_x\end{aligned}$$

so that $\sum_{k,j}\alpha_{k,j}\odot\beta_{k,j}\in\mathcal{H}_x$. Thus, $p_x(\mathcal{H}_x)=H_x$ is dense in $\overline{K_x^{\#}(A_x\widehat{\otimes}A_x)}$; coupled with the preceding, we have $H_x=\overline{K_x^{\#}(A_x\widehat{\otimes}A_x)}$.

By similar arguments, we have $p_x(\mathcal{G})=G_x=\overline{(K_x^0)^2}$, so that $\mathcal{G}=\overline{(K_{\mathcal{A}}^0)^2}/J=\bigcap_x\{z\in(\mathcal{A}\widehat{\otimes}\mathcal{A})+J : z(x)\in\overline{(K_x^0)^2}=G_x\}=\bigcap\{\mathcal{G}_x : x\in X\}$.

But now, since A_x is weakly amenable for each x, the fibers G_x of $\overline{(K_{\mathcal{A}}^0)^2}/J\subset\mathcal{A}\otimes_X\mathcal{A}$ and H_x of $\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}/J\subset\mathcal{A}\otimes_X\mathcal{A}$ are identical, so that $\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}/J=\overline{(K_{\mathcal{A}}^0)^2}/J\subset\mathcal{A}\otimes_X\mathcal{A}$.

We have shown:

Lemma 2.7. *Let $\mathcal{A}$ be a commutative function algebra such that each fiber A_x is weakly amenable. Then the quotient algebras $\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}/J\subset\mathcal{A}\otimes_X\mathcal{A}$ and $\overline{(K_{\mathcal{A}}^0)^2}/J\subset\mathcal{A}\otimes_X\mathcal{A}$ are identical.*

Corollary 2.8. *Let $\mathcal{A}$ be a commutative function algebra with weakly amenable fibers A_x. Then $\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}=\overline{(K_{\mathcal{A}}^0)^2}$.*

Proof. Elementary algebra: Let $z\in\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}$. Then from the preceding Lemma, there exists $w\in\overline{(K_{\mathcal{A}}^0)^2}$ such that $z+J=w+J$. Hence $z-w\in J\subset\overline{(K_{\mathcal{A}}^0)^2}$, so that $z\in\overline{(K_{\mathcal{A}}^0)^2}+w=\overline{(K_{\mathcal{A}}^0)^2}$. Similarly, $\overline{(K_{\mathcal{A}}^0)^2}\subset\overline{K_{\mathcal{A}}^{\#}(\mathcal{A}\widehat{\otimes}\mathcal{A})}$. □

Theorem 2.9. *Suppose that $\mathcal{A}$ is a commutative function algebra such that each fiber A_x is weakly amenable. Then $\mathcal{A}$ is weakly amenable.*

Proof. Apply Theorem 2.4 to the preceding results. □

Corollary 2.10. *Suppose that X is a compact Hausdorff space, and that A and $\{A_x : x \in X\}$ are commutative and weakly amenable. Then so are $C(X, A)$ and $c_0(X, \{A_x\})$. If X is locally compact and Hausdorff, and A is commutative and weakly amenable, then so is $C_0(X, A)$, the space of continuous A-valued functions disappearing at infinity.*

Proof. We need only address the last assertion. Let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of X. Then $C_0(X, A)$ is $C(X)$-isometrically isomorphic to the function algebra $\overline{I_\infty C(X_\infty, A)}$, where I_∞ is the ideal in $C(X_\infty)$ of functions which disappear at ∞; and it is easily checked that $\overline{I_\infty C(X_\infty, A)}$ is a function algebra with fibers $A_x = A$, if $x \neq \infty$, $A_\infty = \{0\}$. □

Corollary 2.11. *Let $\mathcal{A}$ be a function algebra all of whose fibers are commutative C^*-algebras. Then $\mathcal{A}$ is weakly amenable.*

Proof. A C^*-algebra is weakly amenable [2, Thm. 5.6.77]. □

To the authors' knowledge, it is an open question as to whether a function algebra $\mathcal{A}$ with fibers that are all C^*-algebras is itself a C^*-algebra. That is easily seen to be the case if $\mathcal{A}$ is adjoint-closed, but the conclusion is not apparent if $\mathcal{A}$ is not assumed to be adjoint-closed.

We note that, in say [9], and similarly in [11] and [12], in order to induce amenability of its fibers A_x upward to a function algebra $\mathcal{A}$, we had to find someway of spreading the necessary boundedness conditions on each A_x across X to all of $\mathcal{A}$. In [9], for instance, we accomplished this by assuming that the bounded approximate identities on each A_x were uniformly bounded across the A_x. In the present situation, a necessary (and sufficient) condition for weak amenability of the fibers A_x of $\mathcal{A}$ is that each fiber be essential and that $\overline{K_x^{\#}(A_x \widehat{\otimes} A_x)} = \overline{(K_x^0)^2}$. And, as it turns out, Proposition 2.5 and the passing from $\mathcal{A}\widehat{\otimes}\mathcal{A}$ to the quotient $\mathcal{A} \otimes_X \mathcal{A}$ are the tools which spread that property across X to all of $\mathcal{A}$.

3 Pseudo-amenability and heritability

In the preceding section we mentioned the presence of conditions involving boundedness or essentialness of fibers which were sufficient to induce the heritability of the relevant conditions from fibers upward to function algebras. What happens if we eliminate boundedness conditions from the fibers? The answer, as we see in the following, is not nearly so satisfactory, at least as far as we are able to demonstrate. In this section, we make no assumptions about commutativity.

Recall the definition:

Definition 3.1 ([3]). *A (complex) Banach algebra A is said to be pseudo-amenable if there is a net $\{u_\lambda\} \subset A\widehat{\otimes}A$ (called an approximate diagonal) such that for each $a \in A$ we have $\|u_\lambda a - a u_\lambda\| \to 0$ (in $A\widehat{\otimes}A$) and $\|\pi(u_\lambda)a - a\| \to 0$ (in A), where $\pi : A\widehat{\otimes}A \to A$ is the multiplication map.*

As an immediate consequence of this definition, we note:

Proposition 3.2. *Let $\mathcal{A}$ be a pseudo-amenable function algebra. Then each fiber A_x is pseudo-amenable.*

Proof. Note that $A_x \simeq \mathcal{A}/\overline{I_x\mathcal{A}}$ and that pseudo-amenability is preserved by quotients (see [3, Prop. 2.2]). □

If $\{A_x : x \in X\}$ is a collection of pseudo-amenable algebras over the compact Hausdorff space X, it is shown in [3, Prop. 2.1] that each of the algebras $c_0(X, \{A_x\})$ and $\ell^p(X, \{A_x\})$, $1 \le p < \infty$, is pseudo-amenable, where $\ell^p(X, \{A_x\}) \subset \prod\{A_x : x \in X\}$ is the space of choice functions σ over X such that $\|\sigma\| = \left(\sum_x \|\sigma(x)\|^p\right)^{1/p} < \infty$. While all of these are algebras and $C(X)$-modules, of course, only $c_0(X, \{A_x\})$ is a function algebra in our sense. Can we extend the pseudo-amenability result for $c_0(X, \{A_x\})$ to arbitrary function algebras with values in the A_x?

We obtain a partial answer. Recall that an elementary member of $\mathcal{A} \otimes_X \mathcal{A}$ is of the form $x \mapsto (\sigma \odot \tau)(x) = \sigma(x) \otimes \tau(x)$, where $\sigma, \tau \in \mathcal{A}$. Recall also that $J \subset \ker \pi$, where $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ is the multiplication map; thus (again abusing notation only slightly) $\pi : \mathcal{A} \otimes_X \mathcal{A} \to \mathcal{A}$ is well-defined.

Definition 3.3. *Let $\mathcal{A}$ be a function algebra over the compact Hausdorff space X with fibers A_x. Say that $\mathcal{A}$ is quotient pseudo-amenable if there exists a net $\{\nu_\lambda\} \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that for any $\sigma \in \mathcal{A}$ we have both*

$$\|(\nu_\lambda + J)\sigma - \sigma(\nu_\lambda + J)\|_{\mathcal{A}\otimes_X\mathcal{A}} = \sup_x \|\nu_\lambda(x)\sigma(x) - \sigma(x)\nu_\lambda(x)\| \to 0$$

and

$$\|\pi(\nu_\lambda + J)\sigma - \sigma\|_{\mathcal{A}} = \sup_x \|[\pi(\nu_\lambda)](x)\sigma(x) - \sigma(x)\| \to 0.$$

There is a slightly stronger version of Proposition 3.2:

Proposition 3.4. *Suppose that the function algebra $\mathcal{A}$ is quotient pseudo-amenable. Then each fiber A_x is pseudo-amenable.*

Proof. Let $\{\nu_\lambda\} \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ be a net which makes $\mathcal{A}$ pseudo-amenable. For any $x_0 \in X$ and $\sigma \in \mathcal{A}$, we have

$$\|(\nu_\lambda + J)\sigma - \sigma(\nu_\lambda + J)\| = \sup_{x\in X} \|[\nu_\lambda(x)]\sigma(x) - \sigma(x)[\nu_\lambda(x)]\| \ge \|[\nu_\lambda(x_0)]\sigma(x_0) - \sigma(x_0)[\nu_\lambda(x_0)]\| \to 0$$

and similarly for the other necessary convergence. □

Before we proceed to the next result, we gather some notation. Suppose that $\mathcal{A}$ is a function algebra with pseudo-amenable fibers $\{A_x : x \in X\}$ and respective approximate diagonals $\{u_{\lambda_x} : \lambda_x \in \Lambda_x\}$. Set $\Lambda = \prod\{\Lambda_x : x \in X\}$, and write $\lambda(x) = \lambda_x$ (to avoid having subscripts be nested too deeply). Order Λ pointwise, *i.e.* $\lambda' \geq \lambda$ if and only if $\lambda'(x) \geq \lambda(x)$ for each $x \in X$. Given $\lambda \in \Lambda$, for each $x \in X$ we can choose and fix $\nu_{\lambda(x)} \in \mathcal{A} \otimes_X \mathcal{A}$ such that $\nu_{\lambda(x)}(x) = u_{\lambda(x)}$ and such that $\|\nu_{\lambda(x)}(x)\| = \|u_{\lambda(x)}\|$; again, the existence of such $\nu_{\lambda(x)}$ is guaranteed by Prop. 1.1 of [14]. Then by the definition of pseudo-amenability, for each $x \in X$ and $\sigma \in \mathcal{A}$, we have

$$\|\nu_{\lambda(x)}(x)\sigma(x) - \sigma(x)\nu_{\lambda(x)}(x)\| = \|u_{\lambda(x)}\sigma(x) - \sigma(x)u_{\lambda(x)}\| \to 0$$

and

$$\|[\pi(\nu_{\lambda(x)})](x)\sigma(x) - \sigma(x)\| = \|\pi(u_{\lambda(x)})\sigma(x) - \sigma(x)\| \to 0,$$

both as $\lambda(x)$ increases in Λ_x.

Theorem 3.5. *Let $\mathcal{A}$ be a function algebra over the compact Hausdorff space X with fibers A_x, and suppose that each A_x is pseudo-amenable. Then $\mathcal{A}$ is quotient pseudo-amenable.*

Proof. We use the methods of Lemma 4 and Cor. 3 of [11]. Let $F = \{\sigma_k : k = 1, \ldots, n\} \in \mathcal{A}$ and $m \in \mathbb{N}$ be given. Fix $\sigma = \sigma_k \in F$ and $x \in X$. Choose $\nu_{\lambda(x)}$ as above, and choose $\lambda_{m,k}(x) \in \Lambda_x$ such that if $\lambda(x) \geq \lambda_{m,k}(x)$, then both

$$\|\nu_{\lambda(x)}(x)\sigma(x) - \sigma(x)\nu_{\lambda(x)}(x)\| = \|u_{\lambda(x)}\sigma(x) - \sigma(x)u_{\lambda(x)}\| < 1/m$$

and

$$\|[\pi(\nu_{\lambda(x)})](x)\sigma(x) - \sigma(x)\| = \|\pi(u_{\lambda(x)})\sigma(x) - \sigma(x)\| < 1/m.$$

Then if $\lambda_m \in \Lambda$ is such that $\lambda_m \geq \max\{\lambda_{m,k} : k = 1, \ldots, n\}$ (*i.e.* $\lambda_m(x) \geq \max\{\lambda_{m,k}(x) : k = 1, \ldots, n\}$ for each $x \in X$), the above inequalities hold (for λ_m) for each $\sigma \in F$ and $x \in X$.

We now employ the upper semicontinuity of the norm functions in both $\mathcal{A}$ and $\mathcal{A} \otimes_X \mathcal{A}$. For $x \in X$, choose a neighborhood $V_x(F, m)$ such that if $y \in V_x(F, m)$ then both

$$\|\nu_{\lambda(x)}(x)\sigma(x) - \sigma(x)\nu_{\lambda(x)}(x)\| < 1/m$$

and

$$\|[\pi(\nu_{\lambda(x)})](x)\sigma(x) - \sigma(x)\| < 1/m$$

for all $\sigma \in F$.

Now, X is compact, so we can choose $\{x_j : j = 1, \ldots, s\}$ such that $\{V_j\} = \{V_{x_j}(F, m) : j = 1, \ldots, s\}$ also covers X. As in Proposition 2.5, let $\{f_j : j = 1, \ldots, s\}$ a partition of unity subordinate to the

V_j, and define $\xi = \xi(F, m)$ by $\xi = \sum_{j=1}^{s} f_j \nu_{\lambda_m(x_j)} \in \mathcal{A} \otimes_X \mathcal{A}$. Then for $y \in X$ and $\sigma \in F$, and setting $p = \|\xi(y)\sigma(y) - \sigma(y)\xi(y)\|$, we have

$$\begin{aligned} p &= \left\| \sum_{j \text{ s.t. } y \in V_j} f_j(y)[\nu_{\lambda_m(x_j)}(y)\sigma(y) - \sigma(y)\nu_{\lambda_m(x_j)}(y)] \right\| \\ &\leq \sum_{j \text{ s.t. } y \in V_j} f_j(y) \left\| \nu_{\lambda_m(x_j)}(y)\sigma(y) - \sigma(y)\nu_{\lambda_m(x_j)}(y) \right\| < \sum_{j \text{ s.t. } y \in V_j} f_j(y) \cdot 1/m \leq 1/m, \end{aligned}$$

so that $\|\xi\sigma - \sigma\xi\| = \sup_y \left\| \nu_{\lambda_m(x_j)}(y)\sigma(y) - \sigma(y)\nu_{\lambda_m(x_j)}(y) \right\| < 1/m$ (in $\mathcal{A} \otimes_X \mathcal{A}$).

Similarly, we have $\|\pi(\xi)\sigma - \sigma\| = \sup_x \|[\pi(\xi)](y)\sigma(y) - \sigma(y)\| < 1/m$ (in $\mathcal{A}$).

Finally, set $\Psi = \{(F, m) : F \subset \mathcal{A} \text{ is finite and } m \in \mathbb{N}\}$, and order Ψ by $(F', m') > (F, m)$ if $F' \supset F$ and $m' > m$. By the preceding, for each $(F, m) \in \Psi$ there exists $\xi = \xi(F, m) \in \mathcal{A} \otimes_X \mathcal{A}$ such that for each $\sigma \in F$ we have both $\|\xi\sigma - \sigma\xi\| < 1/m$ and $\|\pi(\xi)\sigma - \sigma\| < 1/m$. In particular, for a given $\sigma_0 \in \mathcal{A}$ and $m_0 \in \mathbb{N}$, there exists $(F_0, m_0) \in \Psi$, with $\sigma_0 \in F_0$, such that if $(F', m') > (F_0, m_0)$ then both $\|\xi'\sigma_0 - \sigma_0\xi'\| < 1/m' < 1/m_0$ and $\|\pi(\xi')\sigma_0 - \sigma_0\| < 1/m'$, where $\xi' = \xi'(F', m')$ is constructed as above. Therefore $\{\xi = \xi(F, m) : F \subset \mathcal{A} \text{ is finite and } m \in \mathbb{N}\}$ is an approximate diagonal for $\mathcal{A}$. □

Thus, $\mathcal{A}$ is quotient pseudo-amenable if and only if each A_x is itself pseudo-amenable.

Proposition 3.6. *Suppose that $\mathcal{A}$ is a function algebra over X, and that each fiber A_x is abelian and pseudo-amenable. Then $\mathcal{A}$ is weakly amenable.*

Proof. An abelian pseudo-amenable algebra is weakly amenable [3, Cor. 3.7]. Therefore $\mathcal{A}$ is weakly amenable; see Theorem 2.9. □

Naturally, Theorem 3.5 is a weaker result than we would like, especially given other amenability results on function algebras. We suspect that the main obstacle in general is that for pseudo-amenability we can not employ any boundedness conditions. (Indeed, in [15], pseudo-amenability is introduced as "amenability without boundedness.") The reader will note that in the proof of pseudo-amenability of $c_0(X, \{A_x\})$ (and the other spaces $\ell^p(X, \{A_x\})$) in [3], crucial use is made of the facts that elements $\sigma \in c_0(X, \{A_x\})$ with finite support are dense in the space and that there are projections from $c_0(X, \{A_x\})$ into its subspaces consisting of functions with finite support. This of course need not be the case for general function algebras.

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References

[1] M. Abel, M. Abel, and P. Tammo, "Closed ideals in algebras of sections," *Rend. Circ. Mat. Palermo (2)*, vol. 59, no. 3, pp. 405–418, 2010, doi: 10.1007/s12215-010-0031-1.

[2] H. G. Dales, *Banach algebras and automatic continuity*, ser. London Mathematical Society Monographs. New Series. The Clarendon Press, Oxford University Press, New York, 2000, vol. 24.

[3] F. Ghahramani and Y. Zhang, "Pseudo-amenable and pseudo-contractible Banach algebras," *Math. Proc. Cambridge Philos. Soc.*, vol. 142, no. 1, pp. 111–123, 2007, doi: 10.1017/S0305004106009649.

[4] G. Gierz, "Representation of spaces of compact operators and applications to the approximation property," *Arch. Math. (Basel)*, vol. 30, no. 6, pp. 622–628, 1978, doi: 10.1007/BF01226110.

[5] G. Gierz, *Bundles of topological vector spaces and their duality*, ser. Queen's Papers in Pure and Applied Mathematics. Springer-Verlag, Berlin-New York, 1982, vol. 57.

[6] N. Groenbaek, "A characterization of weakly amenable Banach algebras," *Studia Math.*, vol. 94, no. 2, pp. 149–162, 1989, doi: 10.4064/sm-94-2-149-162.

[7] T. Hõim and D. A. Robbins, "Some extremal properties of section spaces of Banach bundles and their duals. II," *Quaest. Math.*, vol. 26, no. 1, pp. 57–65, 2003, doi: 10.2989/16073600309486043.

[8] T. Hõim and D. A. Robbins, "Spectral synthesis and other results in some topological algebras of vector-valued functions," *Quaest. Math.*, vol. 34, no. 3, pp. 361–376, 2011, doi: 10.2989/16073606.2011.622899.

[9] T. Hõim and D. A. Robbins, "Amenability as hereditary property in some algebras of vector-valued functions," in *Function spaces in analysis*, ser. Contemp. Math. Amer. Math. Soc., Providence, RI, 2015, vol. 645, pp. 135–144, doi: 10.1090/conm/645/12927.

[10] A. Y. Helemskii, *The homology of Banach and topological algebras*, ser. Mathematics and its Applications (Soviet Series). Kluwer Academic Publishers Group, Dordrecht, 1989, vol. 41, doi: 10.1007/978-94-009-2354-6.

[11] T. Hill and D. A. Robbins, "Module bundles and module amenability," *Acta Comment. Univ. Tartu. Math.*, vol. 25, no. 1, pp. 119–141, 2021, doi: 10.12697/acutm.2021.25.08.

[12] T. Hill and D. A. Robbins, "Character amenability of vector-valued algebras," *Acta Comment. Univ. Tartu. Math.*, vol. 27, no. 2, pp. 257–268, 2023.

[13] B. E. Johnson, *Cohomology in Banach algebras*, ser. Memoirs of the American Mathematical Society. American Mathematical Society, Providence, RI, 1972, vol. 127.

[14] J. W. Kitchen and D. A. Robbins, "Gel'fand representation of Banach modules," *Dissertationes Math. (Rozprawy Mat.)*, vol. 203, p. 47, 1982.

[15] O. T. Mewomo, "Various notions of amenability in Banach algebras," *Expo. Math.*, vol. 29, no. 3, pp. 283–299, 2011, doi: 10.1016/j.exmath.2011.06.003.

[16] W. Paravicini, "A note on Banach $C_0(X)$-modules," *Münster J. Math.*, vol. 1, pp. 267–278, 2008.

[17] R. A. Ryan, *Introduction to tensor products of Banach spaces*, ser. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2002, doi: 10.1007/978-1-4471-3903-4.

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Existence and stability of solutions of totally nonlinear neutral Caputo q-fractional difference equations

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ABSTRACT

This paper investigates the existence and stability of solutions for a class of totally nonlinear neutral Caputo q-fractional difference equations of order $0 < \alpha < 1$. By transforming the equation into an equivalent integral equation and leveraging the Krasnoselskii-Burton fixed point theorem, we establish sufficient conditions for the existence of solutions. The methodology involves decomposing the integral operator into a sum of a compact operator and a large contraction. Furthermore, suitable conditions for the stability of these solutions are derived. Our theoretical results extend and generalize previous findings in the literature. An illustrative example is provided to demonstrate the applicability of the main theorems.

RESUMEN

Este artículo investiga la existencia y estabilidad de soluciones para una clase de ecuaciones en diferencias Caputo q-fraccionarias neutrales totalmente no lineales de orden $0 < \alpha < 1$. Transformando la ecuación en una ecuación integral equivalente y aprovechando el teorema de punto fijo de Krasnoselskii-Burton, establecemos condiciones suficientes para la existencia de soluciones. La metodología involucra descomponer el operador integral en una suma de operadores compactos y una contracción grande. Más aún, derivamos condiciones apropiadas para la estabilidad de estas soluciones. Nuestros resultados teóricos extienden y generalizan hallazgos previos en la literatura. Se entrega un ejemplo ilustrativo para demostrar la aplicabilidad de los teoremas principales.

Keywords and Phrases: Existence, stability, q-fractional difference equations, Krasnoselskii-Burton fixed point, large contraction, Arzela-Ascoli's theorem.

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1 Introduction

The realm of q-calculus, also known as quantum calculus, offers a fascinating extension of classical calculus, operating without the conventional concept of limits. Its genesis can be traced back to the early twentieth century with the pioneering work of F. H. Jackson [21]. This framework provides a robust mathematical toolkit for analyzing functions that may exhibit non-smooth behavior. Subsequent developments by numerous researchers have significantly enriched the theoretical underpinnings of q-calculus and broadened its applicability [5, 17–20].

In recent decades, the intersection of q-calculus with fractional calculus has given rise to the vibrant field of q-fractional calculus, leading to the study of q-fractional difference equations. These equations have garnered considerable attention due to their capacity to model complex systems with memory and hereditary properties [2, 8–10, 17, 22–24, 30]. Fixed point theorems have emerged as indispensable tools in the analysis of q-fractional difference equations, instrumental not only in establishing the existence and uniqueness of solutions but also in examining crucial stability properties [6, 12–16, 25–28]. The work of Mesmouli, Ardjouni, and collaborators [25–28] is particularly relevant, addressing various forms of nonlinear neutral Caputo q-fractional difference equations. The Caputo q-fractional derivative, introduced by Abdeljawad and Baleanu [3], alongside supporting theoretical work [1, 7], provides essential tools for such investigations.

For $0 < q < 1$, define the time scale $\mathbb{T}_q = \{q^n, n \in \mathbb{Z}\} \cup \{0\}$, where $\mathbb{Z}$ is the set of integers. For $a = q^{n_0}$ and $n_0 \in \mathbb{Z}$, denote $\mathbb{T}_a = [a, \infty)_q = \{q^i a, i = 0, 1, 2, \dots\}$. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space and define $\mathbb{I}_\tau = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, \dots, a\}$ and $\mathbb{T}_{\tau a} = [\tau a, \infty)q = \{q^{-i}\tau a, i = 0, 1, 2, \dots\}$, where $\tau = q^d \in \mathbb{T}_q$, $d \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\mathbb{I}_\tau = \{a\}$ with $d = 0$, is the non-delay case.

Recently, Abdeljawad, Alzabut and Zhou in [2] studied the existence of solutions for the q-fractional difference equation

$$\begin{cases} {}_qC_a^\alpha x(t) = f(t, x(t), x(\tau t)), & t \in \mathbb{T}_a, \\ x(t) = \phi(t), & t \in \mathbb{I}_\tau, \end{cases} \tag{1.1}$$

where $f : \mathbb{T}_a \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and ${}_qC_a^\alpha$ represents Caputo's q-fractional difference of order $\alpha \in (0, 1)$. By employing the Krasnoselskii fixed point theorem, the authors obtained existence results.

Moreover, Mesmouli and Ardjouni in [25] studied the existence, uniqueness and stability of solutions for nonlinear neutral q-fractional difference equation

$$\begin{cases} {}_qC_a^\alpha (x(t) - g(t, x(\tau t)) = f(t, x(t), x(\tau t)), & t \in \mathbb{T}_a, \\ x(t) = \psi(t), & t \in \mathbb{I}_\tau, \end{cases} \tag{1.2}$$

where $f : \mathbb{T}_a \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{T}_a \times \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{I}_\tau \to \mathbb{R}$ and ${}_qC_a^\alpha$ represents Caputo's q-fractional difference of order $\alpha \in (0,1)$. To establish the results, the authors applied Krasnoselskii's and Banach's fixed point theorems, as well as Arzela–Ascoli's theorem.

Motivated by [2] and [25], we study the existence and stability of solutions for the totally nonlinear neutral q-fractional difference equation

$$\begin{cases} {}_qC_a^\alpha(h(x(t)) - g(t, x(\tau t))) = f(t, x(t), x(\tau t)), & t \in \mathbb{T}_a, \\ x(t) = \psi(t), & t \in \mathbb{I}_\tau, \end{cases} \tag{1.3}$$

where $h : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{T}_a \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{T}_a \times \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{I}_\tau \to \mathbb{R}$ and ${}_qC_a^\alpha$ represents Caputo's q-fractional difference of order $\alpha \in (0,1)$. To prove our main results, we employ the Krasnoselskii-Burton fixed point theorem.

The paper is structured as follows: Section 2 provides essential preliminaries, including definitions and lemmas from q-calculus and fractional difference calculus, the inversion of Equation (1.3) to its integral form, and the statement of the Krasnoselskii-Burton fixed point theorem. Section 3 is dedicated to proving the existence of solutions for Equation (1.3) under derived conditions. Section 4 presents results on the stability of these solutions. Section 5 offers an illustrative example. Finally, Section 6 presents concluding remarks.

2 Preliminaries

In this section, we give some basic notations, definitions, and properties of q-calculus and fractional difference calculus, which are used throughout this paper; see [2] and [25].

Definition 2.1 ([3]). *For a function $f : \mathbb{T}_q \to \mathbb{R}$, its nabla q-derivative of f is defined as*

$${}_q\nabla f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in \mathbb{T}_q - \{0\}. \tag{2.1}$$

Definition 2.2 ([3]). *For a function $f : \mathbb{T}_q \to \mathbb{R}$, the nabla q-integral of f is defined as*

$$\int_0^t f(s)\nabla_q s = (1-q)t \sum_{i=0}^{\infty} q^i f\left(q^i t\right). \tag{2.2}$$

For $a \in \mathbb{T}_q$, (2.2) becomes

$$\int_a^t f(s)\nabla_q s = \int_0^t f(s)\nabla_q s - \int_0^a f(s)\nabla_q s. \tag{2.3}$$

Definition 2.3 ([1,3]). *The q-factorial function for $n \in \mathbb{N}$ is given by*

$$(t-s)_q^n = \prod_{i=0}^{n-1} \left(t - q^i s\right). \tag{2.4}$$

In case α is a non-positive integer, the q-factorial function is given by

$$(t-s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \frac{\left(1-\frac{s}{t}q^i\right)}{\left(1-\frac{s}{t}q^{i+\alpha}\right)}. \tag{2.5}$$

In the following Lemma, we present some properties of q-factorial functions.

Lemma 2.4 ([9]). *For $\alpha, \beta, a \in \mathbb{R}$, we have*

(i) $(t-s)_q^{\alpha+\beta} = (t-s)_q^\alpha (t-q^\alpha s)_q^\beta$.

(ii) $(at-as)_q^\alpha = a^\alpha (t-s)_q^\alpha$.

(iii) The nabla q-derivative of the q-factorial function with respect to t is

$$\nabla_q (t-s)_q^\alpha = \frac{1-q^\alpha}{1-q}(t-s)_q^{\alpha-1}. \tag{2.6}$$

(iv) The nabla q-derivative of the q-factorial function with respect to s is

$$\nabla_q (t-s)_q^\alpha = \frac{1-q^\alpha}{1-q}(t-qs)_q^{\alpha-1}. \tag{2.7}$$

Definition 2.5 ([3,7]). *For a function $f : \mathbb{T}_q \to \mathbb{R}$, the left q-fractional integral ${}_q\nabla_a^{-\alpha}$ of order $\alpha \neq 0, -1, -2, \ldots$ and starting at $a = q^{n_0} \in \mathbb{T}_q, n_0 \in \mathbb{Z}$, is defined by*

$${}_q\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} f(s) \nabla_q s = \frac{1-q}{\Gamma_q(\alpha)} \sum_{i=n}^{n0-1} q^i \left(q^n - q^{i+1}\right)_q^{\alpha-1} f\left(q^i\right), \tag{2.8}$$

where

$$\Gamma_q(\alpha+1) = \frac{1-q^\alpha}{1-q}\Gamma_q(\alpha), \quad \Gamma_q(1) = 1, \quad \alpha > 1. \tag{2.9}$$

Remark 2.6. *The left q-fractional integral ${}_q\nabla_a^{-\alpha}$ maps functions defined on $\mathbb{T}_q$ to functions defined on $\mathbb{T}_q$.*

Definition 2.7 ([3]). *Let $0 < \alpha \notin \mathbb{N}$. Then*

(i) the left Caputo q-fractional derivative of order α of a function f defined on $\mathbb{T}_q$ is defined by

$${}_qC_a^\alpha f(t) = \nabla_a^{-(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma_q(n-\alpha)} \int_a^t (t-qs)_q^{n-\alpha-1} \nabla_q^n f(s) \nabla_q s \tag{2.10}$$

where $n = [\alpha] + 1$. *In case* $\alpha \in \mathbb{N}$, *then* ${}_qC_a^{\alpha} f(t) = \nabla_q^n f(t)$.

(ii) The left Riemann q-fractional derivative is defined by $\left({}_q\nabla_a^{\alpha} f\right)(t) = \left(\nabla_q \nabla_a^{-(n-\alpha)} f\right)(t)$.

(iii) In virtue of [3], the Riemann and Caputo q-fractional derivatives are related by

$$\left({}_qC_a^{\alpha} f\right)(t) = \left({}_q\nabla_a^{\alpha} f\right)(t) - \frac{(t-a)_q^{-\alpha}}{\Gamma_q(1-\alpha)} f(a). \tag{2.11}$$

Lemma 2.8 ([3]). *Let* $\alpha > 0$ *and* f *be defined in a suitable domain. Then*

$${}_q\nabla_a^{-\alpha} \left({}_qC_a^{\alpha} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a), \tag{2.12}$$

and if $0 < \alpha \leq 1$ *we have*

$${}_q\nabla_a^{-\alpha} \left({}_qC_a^{\alpha} f\right)(t) = f(t) - f(a). \tag{2.13}$$

The following identity is crucial in solving the linear q-fractional equations

$${}_q\nabla_a^{-\alpha} (x-a)_q^{\mu} = \frac{\Gamma_q(\mu+1)}{\Gamma(\alpha+\mu+1)} (x-a)_q^{\mu+\alpha}, \quad (0 < a < x < b), \tag{2.14}$$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$.

We give the equivalence of Equation (1.3). So, the solvability of this equivalent equation implies the existence and stability of solutions to Equation (1.3).

Lemma 2.9. $x(t)$ *is a solution of* (1.3) *if and only if it admits the following representation*

$$\begin{aligned} x(t) = \psi(a) - H(\psi(a)) - g(a, \psi(\tau a)) + H(x(t)) + g(t, x(\tau t)) \\ + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} f(s, x(s), x(\tau s)) \nabla_q s, \quad t \in \mathbb{T}_a \end{aligned} \tag{2.15}$$

where

$$H(x(t)) = x(t) - h(x(t)). \tag{2.16}$$

Proof. Let

$$z(t) = h(x(t)) - g(t, x(\tau t)).$$

Then, we can write (3) as

$${}_qC_a^{\alpha} z(t) = f(t, x(t), x(\tau t)).$$

By the same way used in [2] and [25], we obtain for $t \in \mathbb{T}_{a\tau}$, the initial value problem for Equation (1.3) is equivalent to the following equation

$$z(t) = z(a) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f(s, x(s), x(\tau s)) \nabla_q s. \tag{2.17}$$

So

$$x(t) = \psi(a) - H(\psi(a)) - g(a, \psi(\tau a)) + H(x(t)) + g(t, x(\tau t)) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f(s, x(s), x(\tau s)) \nabla_q s.$$

The proof is complete. □

The space l_∞ denotes the set of real bounded sequences with respect to the usual supremum norm. We recall that l_∞ is a Banach space.

Definition 2.10. *A set $\mathbb{M}$ of sequences in l_∞ is uniformly Cauchy if for every $\epsilon > 0$, there exists an integer $\mathbb{N}^*$ such that $|x(t) - x(s)| < \epsilon$ whenever $t, s > \mathbb{N}^*$ for any $x = \{x(n)\}$ in $\mathbb{M}$.*

The following discrete version of Arzela–Ascoli's theorem has a crucial role in the proof of our main theorem.

Definition 2.11 ([29, Arzela-Ascoli])**.** *A bounded, uniformly Cauchy subset $\mathbb{M}$ of $l_\infty(\mathbb{T}_a)$ (all bounded real-valued sequences with domain $\mathbb{T}_a$) is relatively compact.*

Definition 2.12 ([11, Large contraction])**.** *Let $(\mathbb{M}, d)$ be a metric space and $B : \mathbb{M} \to \mathbb{M}$. B is said to be a large contraction if for each pair $x, y \in \mathbb{M}$ with $x \neq y$ then $d(Bx, By) < d(x, y)$ and if for each $\varepsilon > 0$ there exists $\delta < 1$ such that*

$$[x, y \in \mathbb{M}, d(x, y) \geq \varepsilon] \Rightarrow d(Bx, By) < \delta d(x, y).$$

Theorem 2.13 ([11, Krasnoselskii-Burton])**.** *Let $\mathbb{M}$ be a closed convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map $\mathbb{M}$ into $\mathbb{M}$ such that*

(i) for all $x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,

(ii) A is continuous and $A\mathbb{M}$ is contained in a compact subset of $\mathbb{M}$,

(iii) B is a large contraction.

Then there is a $z \in \mathbb{M}$ with $z = Az + Bz$.

We will use the next theorem to show the existence of solutions for Equation (1.3).

Theorem 2.14 ([4]). *Let $\|\cdot\|$ be the supremum norm, $\mathbb{M} = \{x \in \mathbb{C}(\mathbb{T}, \mathbb{R}) : \|x\| \leq R\}$, where R is a positive constant. Suppose that h is satisfying the following conditions*

(H1) h *is continuous on* $U_R = [-R, R]$.

(H2) h *is strictly increasing on* U_R.

(H3) $\sup_{s \in U_R \cap \mathbb{T}_a}\, {}_qC_a^{\alpha} h(s) \leq 1$.

(H4) $(s-r)\Big\{\sup_{i \in U_R \cap \mathbb{T}_a}\, {}_qC_a^{\alpha} h(i)\Big\} \geq h(s) - h(r) \geq (s-r)\Big\{\inf_{i \in U_R \cap \mathbb{T}_a}\, {}_qC_a^{\alpha} h(i)\Big\} \geq 0$ *for* $s, r \in U_R$ *with* $s \geq r$.

Then, the mapping H defined by Equation (2.16) *is a large contraction on* $\mathbb{M}$.

Let $\mathbb{T} = [\tau a, T_1]_q = \{q^{-i}\tau a, i = 0, 1, \ldots, n_1 + d\}$ where $T_1 = q^{-n_1-d}\tau a$ with $n_1 \in [d+3, \infty) \cap \mathbb{Z}$, and $\mathbb{C}(\mathbb{T}, \mathbb{R})$ be the set of all real bounded sequences. $\mathbb{C}(\mathbb{T}, \mathbb{R})$ is a Banach space endowed with the norm

$$\|x\| = \sup_{t \in \mathbb{T}} |x(t)|.$$

Define the set

$$\mathbb{M} = \{x \in \mathbb{C}(\mathbb{T}, \mathbb{R}) : x(t) = \psi(t) \text{ for } t \in \mathbb{I}_\tau \text{ and } \|x\| \leq R\}, \tag{2.18}$$

a non-empty bounded closed and convex subset of $\mathbb{C}(\mathbb{T}, \mathbb{R})$.

3 Existence of solutions

We prove our main results under the following assumptions:

- There exists a constant $L_f > 0$ such that for all $t \in \mathbb{T}_a$, and for all $x, y, z, w \in \mathbb{R}$,

$$|f(t, x, z) - f(t, y, w)| \leq L_f(\|x - y\| + \|z - w\|). \tag{3.1}$$

- There exists a constant $L_g > 0$ such that for all $t \in \mathbb{T}_a$, and for all $x, y \in \mathbb{R}$,

$$|g(t, x) - g(t, y)| \leq L_g\|x - y\|. \tag{3.2}$$

- There exists a constant $R > 0$, satisfying the inequality,

$$J\left[|\psi(a)| + |H(\psi(a))| + |g(a, \psi(\tau a))| + RL_g + \sigma_g + \frac{(2RL_f + \sigma_f)C(\alpha)}{\Gamma_q(\alpha)}\right] \leq R, \tag{3.3}$$

  where $C(\alpha) = \frac{(1-q)(T_1 - a)_q^{\alpha}}{(1-q^{\alpha})}$ is a positive constant depending on α and T_1, with $\sigma_f = \sup_{t \in \mathbb{T}_a} |f(t, 0, 0)|$, $\sigma_g = \sup_{t \in \mathbb{T}_a} |g(t, 0)|$ and $J \geq 3$ is a constant.

Define a mapping $S : \mathbb{M} \to \mathbb{C}$ by

$$(Sx)(t) = \psi(a) - H(\psi(a)) - g(a, \psi(\tau a)) + H(x(t)) + g(t, x(\tau t)) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f(s, x(s), x(\tau s)) \nabla_q s. \quad (3.4)$$

We express (3.4) as

$$(Sx)(t) = (Ax)(t) + (Bx)(t),$$

where the operators $A, B : \mathbb{M} \to \mathbb{C}$ are defined by

$$(Ax)(t) = \psi(a) - H(\psi(a)) - g(a, \psi(\tau a)) + g(t, x(\tau t)) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f(s, x(s), x(\tau s)) \nabla_q s, \quad (3.5)$$

and

$$(Bx)(t) = H(x(t)). \quad (3.6)$$

Lemma 3.1. *Assume that conditions* (3.1), (3.2) *and* (3.3) *hold. Then, the operator* $A : \mathbb{M} \to \mathbb{M}$ *defined in Equation* (3.5) *is compact and continuous.*

Proof. Let A be defined by Equation (3.5). In view of conditions (3.1) and (3.2), we arrive at

$$\begin{aligned} |f(t, x(t), x(\tau t))| &= |f(t, x(t), x(\tau t)) - f(t, 0, 0) + f(t, 0, 0)| \\ &\le |f(t, x(t), x(\tau t)) - f(t, 0, 0)| + |f(t, 0, 0)| \le 2L_f \|x\| + \sigma_f. \end{aligned}$$

and

$$|g(t, x(\tau t))| = |g(t, x(\tau t)) - g(t, 0) + g(t, 0)| \le |g(t, x(\tau t)) - g(t, 0)| + |g(t, 0)| \le L_g \|x\| + \sigma_g.$$

We have

$$\begin{aligned} |(Ax(t))| &= \left| \psi(a) - H(\psi(a)) - g(a, \psi(\tau a)) + g(t, x(\tau t)) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f(s, x(s), x(\tau s)) \nabla_q s \right| \\ &\le |\psi(a)| + |H(\psi(a))| + |g(a, \psi(\tau a))| + |g(t, x(\tau t))| + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} |f(s, x(s), x(\tau s))| \nabla_q s \\ &\le |\psi(a)| + |H(\psi(a))| + |g(a, \psi(\tau a))| + L_g \|x\| + \sigma_g + \frac{2L_f \|x\| + \sigma_f}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} \nabla_q s \\ &\le |\psi(a)| + |H(\psi(a))| + |g(a, \psi(\tau a))| + RL_g + \sigma_g + \frac{2RL_f + \sigma_f}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} \nabla_q s. \end{aligned}$$

By the relations (2.9), (2.14) and the fact that $(t-a)_q^0 = 1$, we have

$$\frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}(t-a)_q^0 \nabla_q s = {}_q\nabla_a^\alpha (t-a)_q^0 = \frac{\Gamma_q(1)(t-a)_q^\alpha}{\Gamma_q(\alpha+1)}$$
$$\leq \frac{(T_1-a)_q^\alpha}{\Gamma_q(\alpha+1)} = \frac{(1-q)(T_1-a)_q^\alpha}{(1-q^\alpha)\Gamma_q(\alpha)}, \quad t < T_1.$$

Then

$$|(Ax(t))| \leq |\psi(a)| + |H(\psi(a))| + |g(a,\psi(\tau a))| + RL_g + \sigma_g + \frac{(2RL_f+\sigma_f)C(\alpha)}{\Gamma_q(\alpha)}.$$

Thus

$$\|Ax\| \leq \frac{R}{J} \leq R.$$

Hence, $A : \mathbb{M} \to \mathbb{M}$ which implies $A(\mathbb{M})$ is uniformly bounded.

To prove the continuity of A, we consider a sequence (x_n) which converges to x such that

$$\begin{aligned}|(Ax_n)(t) - (Ax)(t)| &\leq |g(t, x_n(\tau t)) - g(t, x(\tau t))| \\ &\quad + \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}\Big|f(s,x_n(s),x_n(\tau s)) - f(s,x(s),x(\tau s))\Big|\nabla_q s \\ &\leq L_g\|x_n - x\| + \frac{L_f}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}\|x_n-x\|\nabla_q s \\ &\leq L_g\|x_n - x\| + \frac{2L_fC(\alpha)}{\Gamma_q(\alpha)}\|x_n-x\| \leq \left(L_g + \frac{2L_fC(\alpha)}{\Gamma_q(\alpha)}\right)\|x_n - x\|.\end{aligned}$$

From the above analysis, it implies that

$$\|(Ax_n)(t) - (Ax)(t)\| \leq \left(L_g + \frac{2L_fC(\alpha)}{\Gamma_q(\alpha)}\right)\|x_n - x\|.$$

Hence whenever $x_n \to x$, $Ax_n \to Ax$. This shows the continuity of A.

To prove that A is compact. We will prove that $A(\mathbb{M})$ is equicontinuous. Let $x \in \mathbb{M}$, then for any $t_1, t_2 \in \mathbb{T}_a$ with $0 \leq t_1 \leq t_2 \leq \mathbb{T}_1$, we have

$$\begin{aligned}|(Ax)(t_2) - (Ax)(t_1)| &\leq |g(t_2, x(\tau t_2))| + |g(t_1, x(\tau t_1))| \\ &\quad + \frac{1}{\Gamma_q(\alpha)}\Big|\int_a^{t_2} (t_2-qs)_q^{\alpha-1} f(s,x(s),x(\tau s))\nabla_q s \\ &\quad - \int_a^{t_1} (t_1-qs)_q^{\alpha-1} f(s,x(s),x(\tau s))\nabla_q s\Big|\end{aligned}$$

$$\leq |g(t_2, x(\tau t_2))| + |g(t_1, x(\tau t_1))| + \frac{1}{\Gamma_q(\alpha)} \int_a^{t_1} \left|(t_2 - qs)_q^{\alpha-1} - (t_1 - qs)_q^{\alpha-1}\right| |f(s, x(s), x(\tau s))| \nabla_q s + \int_{t_1}^{t_2} (t_2 - qs)_q^{\alpha-1} |f(s, x(s), x(\tau s))| \nabla_q s.$$

By the assumptions (3.1), (3.3), and Lemma 2.9, we obtain

$$|(Ax)(t_2) - (Ax)(t_1)| \leq |g(t_2, x(\tau t_2))| + |g(t_1, x(\tau t_1))| + \Big(2RL_f + \sigma_f\Big) \left[\frac{1}{\Gamma_q(\alpha)} \int_a^{t_1} \left|(t_2 - qs)_q^{\alpha-1} - (t_1 - qs)_q^{\alpha-1}\right| \nabla_q s + \frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - qs)_q^{\alpha-1} \nabla_q s\right].$$

By using (2.8), we obtain

$$|(Ax)(t_2) - (Ax)(t_1)| \leq |g(t_2, x(\tau t_2))| + |g(t_1, x(\tau t_1))| + (2RL_f + \sigma_f) \left[{}_q\nabla_a^{-\alpha} \left((t_2 - a)_q^0 - (t_1 - a)_q^0\right) +_q \nabla_{t_1}^{-\alpha} (t_2 - t_1)_q^0\right].$$

From (2.14), it follows that

$$|(Ax)(t_2) - (Ax)(t_1)| \leq |g(t_2, x(\tau t_2))| + |g(t_1, x(\tau t_1))| + \frac{(2RL_f + \sigma_f)}{\Gamma_q(\alpha + 1)} \Big[(t_2 - a)_q^\alpha - (t_1 - a)_q^\alpha + (t_2 - t_1)_q^\alpha\Big].$$

Hence it follows that $|(Ax)(t_2) - (Ax)(t_1)| \to 0$ as $t_1 \to t_2$. Thus that $A(\mathbb{M})$ is equicontinuous. So, the compactness of A follows by the Ascoli-Arzela theorem. □

The next Lemma, gives a relationship between the mappings H and B in the sense of large contraction.

Lemma 3.2. *Let B be defined by* (3.6). *Suppose that*

$$\max\Big(|H(-R)|, |H(R)|\Big) \leq \frac{(J-1)}{J} R, \tag{3.7}$$

and all conditions of Theorem 2.14 hold. Then $B : \mathbb{M} \to \mathbb{M}$ is a large contraction.

Proof. We will first show that B maps $\mathbb{M}$ into itself. Let $x \in \mathbb{M}$, then by (3.7) we have

$$|(Bx)(t)| = |(Hx)(t)| \leq \max\Big\{|H(-R)|, |H(R)|\Big\} \leq \frac{(J-1)}{J} R \leq R.$$

Thus

$$\|Bx\| \leq R.$$

That is $Bx \in \mathbb{M}$ and consequently, we have $B : \mathbb{M} \to \mathbb{M}$.

We next show that B is a large contraction. By Theorem 2.14, if H is a large contraction on $\mathbb{M}$, then for any $x, y \in \mathbb{M}$ with $x \neq y$, we have $\|Hx - Hy\| \leq \|x - y\|$. This implies that

$$|(Bx)(t) - (By)(t)| = |(Hx)(t) - (Hy)(t)| \leq \|x - y\|.$$

Thus

$$\|Bx - By\| \leq \|x - y\|.$$

In a similar manner, one could also show that

$$\|Bx - By\| \leq \delta\|x - y\|,$$

holds if we know the existence of a $\delta \in (0, 1)$ and that for all $\epsilon > 0$,

$$[x, y \in \mathbb{M}, \|x - y\| > 0] \Rightarrow \|Hx - Hy\| \leq \delta\|x - y\|.$$

The proof is complete. □

Theorem 3.3. *Suppose the hypotheses of Lemmas 3.1 and 3.2 hold. Let* $\mathbb{M}$ *defined by* (2.18). *Then Equation* (1.3) *has a solution in* $\mathbb{M}$.

Proof. By Lemma 3.1, $A : \mathbb{M} \to \mathbb{M}$ is continuous and compact. Also, from Lemma 3.2, the mapping $B : \mathbb{M} \to \mathbb{M}$ is a large contraction. Next, we prove that if $x, y \in \mathbb{M}$, we have $\|Ax + By\| \leq R$. Let $x, y \in \mathbb{M}$ with $\|x\|, \|y| \leq R$. By (3.3) and (3.7), we obtain

$$\begin{aligned} \|Ax + By\| &\leq \|Ax\| + \|By\| \\ &\leq \left[|\psi(a)| + |H(\psi(a))| + |g(a, \psi(\tau a))| + RL_g + \sigma_g + \frac{(2RL_f + \sigma_f)C(\alpha)}{\Gamma_q(\alpha)}\right] + \frac{(J-1)R}{J} \\ &\leq \frac{R}{J} + \frac{(J-1)R}{J} = R. \end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii-Burton theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = Az + Bz$. By Lemma 2.9, this fixed point is a solution of Equation (1.3). Hence Equation (1.3) has a solution. This completes the proof. □

4 Stability

Now, we show that the solutions of Equation (1.3) are stable by giving sufficient conditions.

Theorem 4.1. *Assume that conditions* (3.1) *and* (3.2) *hold. Also, suppose that*

$$c = \left(k + L_g + \frac{2L_f C(\alpha)}{\Gamma_q(\alpha)}\right) < 1, \tag{4.1}$$

and all conditions of Theorem 2.14 hold. Moreover, for $\epsilon > 0$, there exists

$$\delta = \frac{1-c}{1+k+L_g}\epsilon.$$

Then, the solutions of Equation (1.3) *are stable.*

Proof. Let x be a solution of Equation (1.3) and $\widehat{x}$ be a solution of Equation (1.3) satisfying the initial function $\widehat{x}(t) = \widehat{\psi}(t)$ on $\mathbb{I}_\tau$. For $t \in \mathbb{T}_a$, applying conditions (3.1), (3.2), (4.1) and all conditions of Theorem 2.14, yields

$$\begin{aligned}|x(t) - \widehat{x}(t)| &\leq \left|\psi(a) - \widehat{\psi}(a)\right| + \left|H(\psi(a)) - H(\widehat{\psi}(a))\right| + \left|H(x(t)) - H(\widehat{x}(t))\right| \\ &+ \left|g(a, \psi(\tau a)) - g(a, \widehat{\psi}(\tau a))\right| + \left|g(t, x(\tau t)) - g(t, \widehat{x}(\tau t))\right| \\ &+ \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} \left|f(s, x(s), x(\tau s)) - f(s, \widehat{x}(s), \widehat{x}(\tau s))\right| \nabla_q s \\ &\leq (1+k+L_g)\|\psi - \widehat{\psi}\| + \left(k + L_g + \frac{2L_f C(\alpha)}{\Gamma_q(\alpha)}\right)\|x - \widehat{x}\| \\ &\leq (1+k+L_g)\|\psi - \widehat{\psi}\| + c\|x - \widehat{x}\|.\end{aligned}$$

Hence

$$\|x - \widehat{x}\| \leq \frac{1+k+L_g}{1-c}\|\psi - \widehat{\psi}\|$$

Then, for any $\epsilon > 0$, let $\delta = \frac{1-c}{1+k+L_g}\epsilon$, so for $\|\psi - \widehat{\psi}\| < \delta$ there is $\|x - \widehat{x}\| < \epsilon$. Therefore, the solutions of Equation (1.3) are stable. The proof is complete. □

5 Illustrative example

In this section we provide an example. Specifically, we apply Theorems 3.3 and 4.1 to the equation

$$\begin{cases} {}_{\frac{2}{3}}C_1^{\frac{3}{4}}\Big(\big(\frac{1}{10}\sin(x(t))+\frac{9}{10}x(t)\big)-\frac{1}{50}\cos(t)\arctan\big(x\big(\frac{2t}{3}\big)\big)\Big)=\frac{1}{100}\Big(\sin(x(t))+\cos\big(x\big(\frac{2t}{3}\big)\big)\Big)e^{-t}, \\ t\in[1,9/4]_{\frac{2}{3}}, \\ x(t)=0.02\cos(\pi t), \quad t\in\{2/3,1\}. \end{cases} \tag{5.1}$$

It follows from the equation that $q=2/3$, $\alpha=3/4$, $a=1$, $\tau=2/3$, $h(x)=\frac{1}{10}\sin(x)+\frac{9}{10}x$, which yields $H(x)=\frac{1}{10}(x-\sin(x))$.

Also,

$$g(t,x)=\frac{1}{50}\cos(t)\arctan(x), \quad f(t,x,z)=\frac{1}{100}(\sin(x)+\cos(z))e^{-t},$$

and

$$\psi(t)=0.02\cos(\pi t).$$

We define the set $\mathbb{M}=\{x\in\mathbb{C}:\|x\|\leq R\}$ with $R=0.5$.

Now on the domain $\mathbb{M}_R=[-0.5,0.5]$, $h(x)$ is strictly increasing since

$$h'(x)=\frac{1}{10}\cos(x)+\frac{9}{10}\geq\frac{1}{10}\cos(0.5)+0.9\approx 0.987>0.$$

It can be verified that conditions $(H3)$-$(H4)$ also hold, making $H(x)$ a large contraction.

The Lipschitz constant for $H(x)$ is

$$k=\sup_{x\in U_R}|H'(x)|=\sup_{x\in U_R}\Big|\frac{1}{10}(1-\cos(x))\Big|\leq\frac{1}{10}(1-\cos(0.5))\approx 0.001224.$$

Also,

$$|g(t,x)-g(t,y)|\leq\frac{1}{50}|x-y|, \quad |f(t,x,z)-f(t,y,w)|\leq\frac{1}{100}(|x-y|+|z-w|)$$

Thus, $L_g=0.02$ and $L_f=1/100=0.01$.

It must also be noted that $\sigma_g=\sup|g(t,0)|=0$ and $\sigma_f=\sup|f(t,0,0)|=\frac{1}{100}e^{-1}\approx 0.00368$, $\psi(1)=-0.02\Longrightarrow H(\psi(1))\approx 0$ and $g(1,\psi(2/3))\approx -0.000108$.

To verify the main conditions, we must select an endpoint T_1 for the time scale. Let us choose $T_1=9/4$. A rigorous numerical calculation using the definitions of the q-Gamma function and q-power function yields the q-integral bound

$$K_A=\frac{(T_1-1)_q^{\alpha}}{\Gamma_q(\alpha+1)}=\frac{(9/4-1)_{2/3}^{3/4}}{\Gamma_{2/3}(7/4)}\approx 1.4331.$$

It must also be noted that $|H(0.5)| \approx 0.00206$ and with $J = 5$

$$\frac{J-1}{J}R = \frac{4}{5}(0.5) = 0.4.$$

Hence, showing that Lemma 3.2 holds. Moreover, to verify condition (3.3), we have

$$\begin{aligned}
|\psi(a)| + |H(\psi(a))| + |g(a, \psi(\tau a))| + RL_g + \sigma_g + (2RL_f + \sigma_f)K_A \\
&= 0.02 + 0 + 0.000108 + (0.5)(0.02) + 0 + (2(0.5)(0.01) + 0.003679)(1.4331) = 0.0497 \\
&\leq 0.1.
\end{aligned}$$

Thus, condition (3.3) hold. It therefore follows from Theorem 3.3 that Equation (5.1) has at least one solution in $\mathbb{M}$.

To verify the stability of solutions we verify condition (4.1). Thus,

$$k + L_g + 2L_f K_A = 0.01224 + 0.02 + 2(0.01)(1.4331) = 0.03224 + 0.02866 \leq 1.$$

Thus, by Theorem 4.1 the solutions of Equation (5.1) are stable.

6 Conclusion

This paper has established sufficient conditions for the existence and stability of solutions to a class of totally nonlinear neutral Caputo q-fractional difference equations. The Krasnoselskii-Burton fixed point theorem was a key tool in proving existence, by decomposing the solution operator into a compact part and a large contraction. The stability analysis provides criteria based on the Lipschitz constants of the involved functions and the bound on the q-integral operator. The presented theoretical framework generalizes existing results by considering a more comprehensive nonlinear and neutral structure. The illustrative example demonstrates the method of verifying the derived conditions. Future work could explore specific applications of these equations or investigate uniqueness conditions and other qualitative properties.

References

[1] T. Abdeljawad, J. Alzabut, and D. Baleanu, "A generalized q-fractional Gronwall inequality and its applications to nonlinear delay q-fractional difference systems," *J. Inequal. Appl.*, 2016, Art. ID 240, doi: 10.1186/s13660-016-1181-2.

[2] T. Abdeljawad, J. Alzabut, and H. Zhou, "A Krasnoselskii existence result for nonlinear delay Caputo q-fractional difference equations with applications to Lotka-Volterra competition model," *Appl. Math. E-Notes*, vol. 17, pp. 307–318, 2017.

[3] T. Abdeljawad and D. Baleanu, "Caputo q-fractional initial value problems and a q-analogue Mittag-Leffler function," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 16, no. 12, pp. 4682–4688, 2011, doi: 10.1016/j.cnsns.2011.01.026.

[4] M. Adıvar and Y. N. Raffoul, "Existence of periodic solutions in totally nonlinear delay dynamic equations," *Electron. J. Qual. Theory Differ. Equ.*, 2009, Art. ID 1, doi: 10.14232/ejqtde.2009.4.1.

[5] W. A. Al-Salam, "Some fractional q-integrals and q-derivatives," *Proc. Edinburgh Math. Soc. (2)*, vol. 15, pp. 135–140, 1966/67, doi: 10.1017/S0013091500011469.

[6] J. Alzabut, T. Abdeljawad, and D. Baleanu, "Nonlinear delay fractional difference equations with applications on discrete fractional Lotka-Volterra competition model," *J. Comput. Anal. Appl.*, vol. 25, no. 5, pp. 889–898, 2018.

[7] M. H. Annaby and Z. S. Mansour, *q-fractional calculus and equations*, ser. Lecture Notes in Mathematics. Springer, Heidelberg, 2012, vol. 2056, doi: 10.1007/978-3-642-30898-7.

[8] A. Ardjouni, H. Boulares, and A. Djoudi, "Stability of nonlinear neutral nabla fractional difference equations," *Communications in Optimization Theory*, vol. 2018, 2018, Art. ID 8, doi: 10.23952/cot.2018.8.

[9] F. M. Atici and P. W. Eloe, "Fractional q-calculus on a time scale," *J. Nonlinear Math. Phys.*, vol. 14, no. 3, pp. 333–344, 2007, doi: 10.2991/jnmp.2007.14.3.4.

[10] E. A. Az-Zo'bi, W. A. AlZoubi, L. Akinyemi, M. Şenol, I. W. Alsaraireh, and M. Mamat, "Abundant closed-form solitons for time-fractional integro–differential equation in fluid dynamics," *Optical and Quantum Electronics*, vol. 53, 2021, Art. ID 132, doi: 10.1007/s11082-021-02782-6.

[11] T. A. Burton, *Stability by fixed point theory for functional differential equations.* Dover Publications, Inc., Mineola, NY, 2006.

[12] R. I. Butt, T. Abdeljawad, M. A. Alqudah, and M. u. Rehman, “Ulam stability of Caputo q-fractional delay difference equation: q-fractional Gronwall inequality approach,” *J. Inequal. Appl.*, 2019, Art. ID 305, doi: 10.1186/s13660-019-2257-6.

[13] R. I. Butt, T. Abdeljawad, and M. u. Rehman, “Stability analysis by fixed point theorems for a class of non-linear Caputo nabla fractional difference equation,” *Adv. Difference Equ.*, 2020, Art. ID 209, doi: 10.1186/s13662-020-02674-1.

[14] F. Chen, “Fixed points and asymptotic stability of nonlinear fractional difference equations,” *Electron. J. Qual. Theory Differ. Equ.*, 2011, Art. ID 39, doi: 10.14232/ejqtde.2011.1.39.

[15] F. Chen and Z. Liu, “Asymptotic stability results for nonlinear fractional difference equations,” *J. Appl. Math.*, 2012, Art. ID 879657.

[16] J. Dong, Y. Feng, and J. Jiang, “A note on implicit fractional differential equations,” *Mathematica Aeterna*, vol. 7, no. 3, pp. 261–267, 2017.

[17] T. Ernst, “A method for q-calculus,” *J. Nonlinear Math. Phys.*, vol. 10, no. 4, pp. 487–525, 2003, doi: 10.2991/jnmp.2003.10.4.5.

[18] T. Ernst, *A comprehensive treatment of q-calculus.* Birkhäuser/Springer Basel AG, Basel, 2012, doi: 10.1007/978-3-0348-0431-8.

[19] R. Floreanini and L. Vinet, “Symmetries of the q-difference heat equation,” *Lett. Math. Phys.*, vol. 32, no. 1, pp. 37–44, 1994, doi: 10.1007/BF00761122.

[20] R. Floreanini and L. Vinet, “Quantum symmetries of q-difference equations,” *J. Math. Phys.*, vol. 36, no. 6, pp. 3134–3156, 1995, doi: 10.1063/1.531017.

[21] F. H. Jackson, “On q-functions and a certain difference operator,” *Transactions of the Royal Society of Edinburgh*, vol. 46, no. 2, pp. 253–281, 1909, doi: 10.1017/S0080456800002751.

[22] F. Jarad, T. Abdeljawad, E. Gündoğdu, and D. Baleanu, “On the Mittag-Leffler stability of q-fractional nonlinear dynamical systems,” *Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci.*, vol. 12, no. 4, pp. 309–314, 2011.

[23] Z. S. I. Mansour, “Linear sequential q-difference equations of fractional order,” *Fract. Calc. Appl. Anal.*, vol. 12, no. 2, pp. 159–178, 2009.

[24] M. J. Mardanov, N. I. Mahmudov, and Y. A. Sharifov, “Existence and uniqueness results for q-fractional difference equations with p-Laplacian operators,” *Adv. Difference Equ.*, 2015, Art. ID 185, doi: 10.1186/s13662-015-0532-5.

[25] M. B. Mesmouli and A. Ardjouni, “Stability in nonlinear neutral Caputo q-fractional difference equations,” *Mathematics*, vol. 10, no. 24, 2022.

[26] M. B. Mesmouli, A. Ardjouni, and A. Djoudi, "Nonlinear neutral Caputo-fractional difference equations with applications to Lotka-Volterra neutral model," *Facta Univ. Ser. Math. Inform.*, vol. 35, no. 5, pp. 1475–1488, 2020, doi: 10.22190/fumi2005475m.

[27] M. B. Mesmouli, A. Ardjouni, and N. Iqbal, "Existence and asymptotic behaviors of nonlinear neutral Caputo nabla fractional difference equations," *Afr. Mat.*, vol. 33, no. 3, 2022, Art. ID 83, doi: 10.1007/s13370-022-01020-w.

[28] M. B. Mesmouli, A. Ardjouni, and N. Touafek, "Stability of advanced nonlinear difference equations," *Nonlinear Stud.*, vol. 29, no. 3, pp. 927–934, 2022.

[29] H. L. Royden, *Real analysis.* The Macmillan Company, New York; Collier Macmillan Ltd., London, 1963.

[30] Y. Zhao, H. Chen, and Q. Zhang, "Existence results for fractional q-difference equations with nonlocal q-integral boundary conditions," *Adv. Difference Equ.*, 2013, Art. ID 48, doi: 10.1186/1687-1847-2013-48.

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A note on Krein-Milman theorem

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ABSTRACT

In this note, we generalize a well-known theorem of Krein and Milman concerning the closed convex envelope of the extremal points of a compact convex set in a topological vector space to the case of an abstract convexity notion in a topological space with a convex structure defined on it. We also introduce some notions that will be useful to describe these generalizations.

RESUMEN

En esta nota generalizamos un teorema bien conocido de Krein y Milman respecto de la envolvente convexa cerrada de los puntos extremos de un conjunto compacto convexo en un espacio vectorial topológico al caso de una noción abstracta de convexidad en un espacio topológico con una estructura convexa definida en él. También introducimos algunas nociones que serán útiles para describir estas generalizaciones.

Keywords and Phrases: Convex sets, extremal points, Krein-Milman theorem.

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1 Introduction and results

The main pourpose of this paper is to prove a generalization of Krein and Milman Theorem (Theorem 1.14). We start with some definitions and properties which will be useful for our pourpose.

Definition 1.1. *Let X be a topological space. We say that a family $\mathcal{F} = \{\tau_{(x,y)} : [0,1] \to X : x, y \in X\}$ introduces a convex structure on X if:*

1. *The functions $\tau_{(x,y)}$ are continuous.*
2. *$\tau_{(x,y)}(0) = x$, $\tau_{(x,y)}(1) = y$.*

The space X with the a such family $\mathcal{F}$ we call a *space with convex structure.*

Definition 1.2. *Let X be a space with a convex structure. A subset A of X is said to be $\mathcal{F}-$convex if for all $x, y \in A$ we have that $\tau_{(x,y)}([0,1]) \subset A$.*

It is easy to see that the space X, the empty and the intersection of two $\mathcal{F}-$convex sets is an $\mathcal{F}-$convex set. Also the union of a monotone family of $\mathcal{F}-$convex sets is again an $\mathcal{F}-$convex set.

Example 1.3. *Let $X = S^2$ be a two dimensional sphere and let $\tau_{(x,y)} : [0,1] \to S^2$ be a parametrization of the geodesic line which joins the points x and y on this sphere. Then the family $\mathcal{F} = \{\tau_{(x,y)} : [0,1] \to X : x, y \in X\}$ introduces a convex structure on S^2. Particulary the spherical triangles are $\mathcal{F}-$convex sets.*

Example 1.4. *Let X be a topological vector space or an abstract topological cone. Then the family $\mathcal{F}_p = \{\tau_{(x,y)} : \tau_{(x,y)}(t) = ty + (1 - t^p)^{\frac{1}{p}} x,\ t \in [0,1],\ x, y \in X\}$ introduces a convex structure on X for all $p > 0$.*

Definition 1.5. *Let X be space with a convex structure and A be a subset of X. We say that $B \subset A$ is an extremal subset of A if the condition $\tau_{(x,y)}(t) \in B$ for some $x, y \in A$ and $t \in (0,1)$ implies that $x, y \in B$.*

Lemma 1.6. *Let X be a topological space with a convex structure $\mathcal{F}$. Then every compact $\mathcal{F}-$convex subset A of X contains an $\mathcal{F}-$convex extreme B which is minimal with respect to inclusion.*

Proof. Let $\mathcal{M}$ be the family of all $\mathcal{F}-$convex closed extreme subsets of A ordered by inclusion. It is easy to observe that $\mathcal{M}$ is nonempty since $A \in \mathcal{M}$ and the intersection of any chain of elements of $\mathcal{M}$ also belongs to $\mathcal{M}$. Hence by the Kuratowski-Zorn Lemma there exists an minimal element of $\mathcal{M}$. □

Similarly we can prove the following:

Definition 1.7. *Let X be a topological space. We say that the family $\mathcal{G} = \{g_t : X \to \mathbb{R} : t \in T\}$ separates points of the space X, if for all $x, y \in X$, $x \neq y$ there exists $t \in T$ such that $g_t(x) \neq g_t(y)$.*

Definition 1.8. *We say that a family $\mathcal{G} = \{g_t : X \to \mathbb{R} : t \in T\}$ exposes faces of a compact $\mathcal{F}-$convex subset A if for any $g \in \mathcal{G}$ the set*

$$H_g^A = \left\{ x \in A : g(x) = \sup_{t \in A} g(t) \right\}$$

is an extremal subset of A.

Definition 1.9. *Let A be an $\mathcal{F}-$convex subset of X. We say that $x \in A$ is an extreme point of A if the set $\{x\}$ is an extreme subset of A. The set of all extreme points of the set A is denoted by Ext(A).*

Definition 1.10. *Let X be a topological space with a convex structure $\mathcal{F}$ and the family $\mathcal{G}$ of functions which separates points of X. We say that the family $\mathcal{G}$ is compatible with the family $\mathcal{F}$ on the class of $\mathcal{F}-$compact convex sets if for any compact $\mathcal{F}-$convex set A the set H_g^A is $\mathcal{F}-$convex for all $g \in \mathcal{G}$.*

Proposition 1.11. *Let X be a topological space with a convex structure $\mathcal{F}$. Assume that there exists a family $\mathcal{G}$ compatible with the family $\mathcal{F}$ on the class of $\mathcal{F}-$compact convex sets which separates points of X and exposes faces of compact $\mathcal{F}-$convex sets. Then every compact $\mathcal{F}-$convex subset A of X has extreme point.*

Proof. By Lemma 1.6 there exists a minimal extreme $\mathcal{F}-$convex subset B of the set A. Suppose that $x, y \in B$ then there exists $g \in \mathcal{G}$ such that $g(x) \neq g(y)$ but in this case the set H_g^B is also extreme $\mathcal{F}-$convex subset of A which is included in B and at least one of x, y does not belong to B. Contradiction. ☐

Definition 1.12. *Let X be a topological space with the convex structure $\mathcal{F}$. We say that the family $\mathcal{G}$ of real functions defined on X separates compact $\mathcal{F}-$convex sets from points if for any compact $\mathcal{F}-$convex set A and any $b \notin A$ there exists $g \in \mathcal{G}$ such that $g(x) < g(b)$ for all $x \in A$.*

Definition 1.13. *Let A be a subset of a topological space X with the convex structure $\mathcal{F}$. The $\mathcal{F}-$convex hull of the set A is defined as the intersection of all $\mathcal{F}-$convex subsets of X which contain the set A and we denote it by $conv_{\mathcal{F}}(A)$.*

Analogously the closed $\mathcal{F}-$convex hull of the set A is defined as the intersection of all $\mathcal{F}-$convex closed subsets of X which contains the set A and we denote it by $\overline{conv}_{\mathcal{F}}(A)$.

Clearly the $\mathcal{F}-$convex hull of any subset is $\mathcal{F}-$convex set and the closed $\mathcal{F}-$convex hull of any subset is closed and $\mathcal{F}-$convex set. Now we are prove theorem which generalizes the classical Krein-Milman theore.

Theorem 1.14. *Let X be a topological space with a convex structure $\mathcal{F}$ on X. Suppose that there exists a family $\mathcal{G}$ on X which is compatible with $\mathcal{F}$ on $\mathcal{F}-$compact convex subsets of X such that separates $\mathcal{F}-$convex compact subsets of X from points of X and exposes faces of compact $\mathcal{F}-$convex sets. Then every $\mathcal{F}-$convex compact subset A of X is equal to closed $\mathcal{F}-$convex hull of its extremal points. Symbolically*

$$A = \overline{conv}_{\mathcal{F}}(Ext\ (A)).$$

Proof. From Proposition 1.11 we have that the set $\mathrm{Ext}(A)$ is not empty. Obviously

$$K = \overline{\mathrm{conv}}_{\mathcal{F}}(\mathrm{Ext}(A)) \subset A.$$

Assume that $A \setminus K \neq \emptyset$ and let $x \in A \setminus K$. Since the set K is a closed $\mathcal{F}-$convex subset of the compact set A it is also compact. Now since the family $\mathcal{G}$ separates points from compact $\mathcal{F}-$convex sets then there exists $g \in \mathcal{G}$ such that $\sup_{t\in K} g(t) < g(x)$. Since the family exposes faces of compact $\mathcal{F}-$convex sets therefore the set H_g^A is extreme subset of A. From the compatibility of the family $\mathcal{G}$ with $\mathcal{F}$ on the compact $\mathcal{F}-$convex subsets we obtain that the set H_g^A is itself a compact $\mathcal{F}-$convex set and hence the set $\mathrm{Ext}(H_g^A)$ is not empty. Hence

$$\mathrm{Ext}(H_g^A) \subset \mathrm{Ext}(A) \subset K,$$

but this gives a contradiction since for $y \in \mathrm{Ext}(H_g^A)$ we have

$$g(y) = \sup_{t\in A} g(t) \geq g(x) > \sup_{t\in K} g(t) \geq g(y)$$

which ends the proof. □

Extremal points play an important role in mathematics and its applications. As was shown in [4] it plays an crucial role in proving continuity of convex functions. Hence the above theorem may be a possible tool for examining the continuty of some wider class of convex functions (*i.e.* convex functions defined by using abstract convex structure).

Remark 1.15. *If X is a locally convex topological vector space then the family*

$$\mathcal{F} = \{I_{xy} : [0,1] \to X : I_{xy}(t) = (1-t)x + ty,\ x, y \in X\}$$

defines a convex structure on X. It is clear that $\mathcal{F}-$convex sets are usual convex sets in this case. Denote by X' the topological dual of X i.e. the space of all real continous linear functionals *defined on* X*. From the geometric form of Hahn-Banach theorem [1] it follows that the family* $\mathcal{G} = X'$ *separates points from compact convex subsets of* X*. Moreover it is easy that the family* $\mathcal{G}$ *is compatible with* $\mathcal{F}$ *on convex subsets of* X *and the family* $\mathcal{G}$ *exposes faces of compact convex sets. Hence the assumptions of Theorem 1.14 are satisfied and from this theorem we obtain a classical version of Krein-Milman Theorem ([2, 3]) i.e.*

Theorem 1.16 (Krein-Milman theorem)**.** *Let* X *be a locally convex topological vector space and let* A *be a compact convex subset of* X*. Then* A *is equal to the closed convex envelope of the set of its extreme points. Symbolically,*

$$A = \overline{conv}(Ext(A)).$$

References

[1] N. Bourbaki, *Topological vector spaces. Chapters 1–5*, ser. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1987, doi: 10.1007/978-3-642-61715-7.

[2] M. Krein and D. Milman, "On extreme points of regular convex sets," *Studia Math.*, vol. 9, pp. 133–138, 1940, doi: 10.4064/sm-9-1-133-138.

[3] D. Mil'man, "Characteristics of extremal points of regularly convex sets," *Doklady Akad. Nauk SSSR (N.S.)*, vol. 57, pp. 119–122, 1947.

[4] H. Przybycień, "A note on continuity of convex functions," *Novi Sad Journal of Mathematics*, doi: 10.30755/NSJOM.16194.

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Comparing the real genus and the symmetric crosscap number of a group

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ABSTRACT

Given a finite group G, there exist Klein surfaces, bordered X and unbordered non-orientable S, such that G acts as an automorphism group of X and of S. The minimum algebraic genus $\rho(G)$ of the surfaces X is called the real genus of G, and the minimal topological genus $\tilde{\sigma}(G)$ of the surfaces S is the symmetric crosscap number of G. In this work we study the relation between the real genus and the symmetric crosscap number of a group G and how both parameters can be compared. For instance, we see that there exist groups G such that the difference $\tilde{\sigma}(G) - \rho(G) = t$ for all even negative numbers t. In order to get it, we correct some inaccuracies in previous works, on these parameters for the groups $C_m \times D_n$ and $D_m \times D_n$. On the other hand, for some important families of groups, we prove that $\tilde{\sigma}(G) = \rho(G) + 1$. We use it to eliminate possible gaps in the symmetric crosscap spectrum, enforcing the conjecture that 3 is in fact the unique gap.

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RESUMEN

Dado un grupo finito G, existen superficies de Klein, con borde X y sin borde no-orientables S, tales que G actúa como un grupo de automorfismos de X y de S. El género algebraico mínimo $\rho(G)$ de las superficies X se llama el género real de G, y el género topológico mínimo $\tilde{\sigma}(G)$ de las superficies S es el "symmetric crosscap number" de G, que llamaremos género imaginario aunque no es una denominación estándar. En este trabajo, estudiamos la relación entre el género real y el imaginario de un grupo G y cómo se pueden comparar ambos parámetros. Por ejemplo, vemos que existen grupos G tales que la diferencia $\tilde{\sigma}(G) - \rho(G) = t$ para todos los números negativos pares t. Para ello, corregimos algunas inexactitudes en trabajos previos sobre estos parámetros para los grupos $C_m \times D_n$ y $D_m \times D_n$. Por otra parte, para algunas familias importantes de grupos, demostramos que $\tilde{\sigma}(G) = \rho(G) + 1$. Esto lo utilizamos para eliminar posibles huecos en el espectro simétrico imaginario, dando evidencia adicional a la conjetura de que 3 es, de hecho, el único hueco posible.

Keywords and Phrases: Real genus, symmetric crosscap number, Klein surfaces.

2020 AMS Mathematics Subject Classification: 30F50, 14H37, 14H55, 20H15.

1 Introduction and preliminaries

A Klein surface X is a compact surface endowed with a dianalytic structure. Klein surfaces may be seen as a generalization of Riemann surfaces, including bordered and non-orientable surfaces. An orientable unbordered Klein surface is a Riemann surface.

In the study of Klein surfaces and their automorphism groups, the non-Euclidean crystallographic groups (NEC groups, in short) play an essential role. An NEC group Γ is a discrete subgroup of the group of isometries of the hyperbolic plane $\mathcal{H}$ with compact quotient $\mathcal{H}/\Gamma$.

For the convenience of the reader we give a minimum of preliminaries about NEC groups and Klein surfaces (for details see [4]).

An NEC group Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathcal{H}$, including orientation reversing elements, with compact quotient $X = \mathcal{H}/\Gamma$. Every NEC group Γ has associated the following symbol called *signature*:

$$\sigma(\Gamma) = (g, \pm, [m_1, \dots, m_r], \{(n_{i,1}, \dots, n_{i,s_i}),\ i = 1, \dots, k\}), \tag{1.1}$$

where the numbers g, r, k and s_i are non-negative integers, $m_i, n_{i,j}$ are integers such that $m_i, n_{i,j} \geq 2$. The number g is the topological genus of X, and the sign determines the orientability of X.

The numbers m_i are the *proper periods* corresponding to cone points in X. The brackets $(n_{i,1}, \dots, n_{i,s_i})$ are the *period-cycles.* The number k of period-cycles is equal to the number of boundary components of X. Numbers $n_{i,j}$ are the periods of the period-cycle $(n_{i,1}, \dots, n_{i,s_i})$ also called *link-periods*, corresponding to corner points in the boundary of X. The number $p = \alpha g + k - 1$, where $\alpha = 2$ or 1 according to the sign be "+" or "−", respectively, is called the *algebraic genus* of X.

An NEC group with the above signature is generated by x_i, $(i = 1, \dots, r)$; e_i, $(i = 1, \dots, k)$; $c_{i,j}$, $(i = 1, \dots, k;\ j = 0, \dots, s_i)$; and a_i, b_i $(i = 1, \dots, g)$ if σ has sign "+" or d_i $(i = 1, \dots, g)$ if σ has sign "−", and relations

$$\begin{array}{ll}
x_i^{m_i} = 1; & i = 1, \dots, r; \\
c_{i,j-1}^2 = c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{i,j}} = 1; & i = 1, \dots, k;\ j = 1, \dots, s_i; \\
e_i^{-1} c_{i,0} e_i c_{i,s_i} = 1; & i = 1, \dots, k; \\
\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g (a_i b_i a_i^{-1} b_i^{-1}) = 1; & (\text{if } \sigma \text{ has sign "} + \text{"}); \\
\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g d_i^2 = 1; & (\text{if } \sigma \text{ has sign "} - \text{"}).
\end{array}$$

The isometries x_i are elliptic, e_i, a_i, b_i are hyperbolic, c_i are reflections and d_i are glide reflections. They are called *canonical generators.*

Every NEC group Γ with signature (1.1) has associated a fundamental region whose area $\mu(\Gamma)$, called the *area of the group*, is

$$\mu(\Gamma) = 2\pi\left(\alpha g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{i,j}}\right)\right),$$

with $\alpha = 2$ or 1 according to the sign being "+" or "−". The group given by the presentation above can be represented as an NEC group with signature (1.1) if and only if its area is greater than 0. We denote by $|\Gamma|$ the expression $\mu(\Gamma)/2\pi$ and call it the *reduced area* of Γ.

If Γ is a subgroup of an NEC group Γ' of finite index N, then Γ is also an NEC group and the following Riemann-Hurwitz formula holds:

$$\mu(\Gamma) = N\mu(\Gamma').$$

If the group has neither proper periods nor link-periods, it is called a *surface group* and has the following signature

$$\sigma(\Gamma) = (g, \pm, [-], \{(-), \overset{k}{\ldots}, (-)\}),$$

For a Klein surface X with $p \geq 2$, there exists a NEC surface group Γ such that $X = \mathcal{H}/\Gamma$. A finite group G of order N is an automorphism group of $X = \mathcal{H}/\Gamma$ if and only if there exists an NEC group Λ such that Γ is a normal subgroup of Λ with index N and $G = \Lambda/\Gamma$. Since Γ is a surface group, it does not contain elements of finite order other than reflections. Therefore, there must be an epimorphism $\theta : \Lambda \to G$ with kernel Γ, such that the relations defining Λ are preserved by θ.

Given a finite group G there exist bordered Klein surfaces X such that G acts as an automorphism group of X, and also unbordered non-orientable surfaces S, such that G acts on S. The minimum algebraic genus of the surfaces X is called the *real genus* of G, $\rho(G)$, and the minimal topological genus of the surfaces S is the *symmetric crosscap number* of G, $\tilde{\sigma}(G)$. In order to obtain these parameters we need to study NEC groups Λ with minimal area such that $G = \Lambda/\Gamma$.

An extensive study has been made on both parameters $\rho(G)$ and $\tilde{\sigma}(G)$. The numbers which are $\rho(G)$ for some G form the *real genus spectrum*, whilst those which are $\tilde{\sigma}(G)$ form the *symmetric crosscap spectrum*. None of these spectra is still completely known, and the relationship between both parameters is a tool for that study. When an integer does not belong to either spectrum, it is called a gap of that spectrum.

Regarding the real genus, there is no group with real genus 2, 12 or 24 [14]. No other gap was currently known to exist, but in the very recent paper [6], it is proved that 72 is also a gap. Therefore, the first number for which it is not known whether it belongs to the spectrum is 84.

For the symmetric crosscap spectrum, the present knowledge is based on [1]. May proved that there does not exist any group G such that $\tilde{\sigma}(G) = 3$. For $N > 3$, if N is a gap of the symmetric crosscap spectrum then N lies in four congruence classes mod 120, namely 3, 51, 75 and 99, and it satisfies additional conditions. The present result will be given below in Theorem 3.4. However, many numbers satisfying those necessary conditions actually belong to the spectrum. In fact, no gap apart from 3 is currently known.

2 Results on real genus and symmetric crosscap number

The goal of the present work is to compare both parameters $\rho(G)$ and $\tilde{\sigma}(G)$. It is worth noting that very often

$$\tilde{\sigma}(G) = \rho(G) + 1. \tag{2.1}$$

This property holds for important classes of groups, but it is not true in general. When it holds for a group G, we say that G satisfies Property (2.1).

2.1 Groups of odd order

First, the authors proved in [1] that the Property (2.1) holds for all groups of odd order.

Theorem 2.1 ([1, Corollary 1])**.** *If G has odd order, then $\tilde{\sigma}(G) = \rho(G) + 1$.*

2.2 Abelian groups

Property (2.1) is also true for Abelian groups. In [18] J. Rodríguez mentions in Remark 6.2 that "the crosscap number of an Abelian group relates with its real genus straightforwardly: $\tilde{\sigma}(G) = \rho(G) + 1$". However, as far as we know this result has not appeared anywhere, and we are now providing its proof, taking into account that both parameters are already known in the case of Abelian groups, obtained by McCullough and Gromadzki in [16] and [11] respectively.

First, we quote the result on real genus.

Theorem 2.2 ([16])**.** *Let G be a non-cyclic Abelian group of order N, $G \neq C_2 \times C_2 \times C_2$, $C_2 \times C_{2k}$ $(k \geq 1)$. Write*

$$G = C_{e_1} \times \cdots \times C_{e_m} \times C_{d_1} \times \cdots \times C_{d_l} \times C_2^n,$$

e_i multiple of 4, d_j odd, $e_{i+1}|e_i$, $d_1|e_m$, $d_{j+1}|d_j$. Then $\rho(G)$ is

A) $1 + N\left(n + \sum_{i=1}^m \left(1 - \frac{1}{e_i}\right) + \sum_{j=1}^l \left(1 - \frac{1}{d_j}\right) - 1\right)$, *$n < m$.*

B) $1+N\left(m+t+\left(1-\frac{1}{2d_t}\right)+\sum_{j=t+1}^{l}\left(1-\frac{1}{d_j}\right)-2\right)$, *if* $m<n\leq m+2l-1$, $n-m=2t-1$.

C) $1+N\left(m+t+\sum_{j=t+1}^{l}\left(1-\frac{1}{d_j}\right)-1\right)$, *if* $m\leq n\leq m+2l$, $n-m=2t$.

D) $1+\frac{N(3m+2l+n-3)}{4}$, *if* $n\geq m+2l+1$.

On the other hand, for the symmetric crosscap number the result is the following

Theorem 2.3 ([11])**.** *Let G be a non-cyclic Abelian group of order N, $G\neq C_2\times C_2\times C_2$, $C_2\times C_{2k}$ $(k\geq 1)$. If G has non-cyclic 2-Sylow subgroup, write $G=C_{m_1}\times\cdots\times C_{m_k}\times C_2^s$, where $m_1,\ldots,m_l$ are odd, $m_{l+1}\ldots,m_k$ are even, $m_i|m_{i+1}$, and s is as large as possible. Then $\tilde{\sigma}(G)$ is*

i) $2+N\left(k-1-\sum_{i=1}^{k-s}\frac{1}{m_i}\right)$, *if* $s-(k-l)\leq 0$.

ii) $2+N(k-1)$, *if* $s-(k-l)=2l$.

iii) $2+N\left(k-1+\frac{s-k-l+1}{4}\right)$, *if* $s-(k-l)>2l$.

iv) $2+N\left(k-1-\sum_{i=1}^{(k+l-s)/2}\frac{1}{m_i}\right)$, *if* $0<s-(k-l)<2l$, $s-(k-l)$ *even.*

v) $2+N\left(k-1-\frac{1}{2m_{(k+l-s+1)/2}}-\sum_{i=1}^{(k+l-s-1)/2}\frac{1}{m_i}\right)$, *if* $0<s-(k-l)<2l$, $s-(k-l)$ *odd.*

And if N is odd, or G has cyclic 2-Sylow subgroup write $G=C_{m_1}\times\cdots\times C_{m_r}$, $m_i|m_{i+1}$ and then $\tilde{\sigma}(G)$ is

vi) $2+N\left(-1+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right)$.

Since in both Theorems the group G has been described in a different way, it is not too easy to compare $\rho(G)$ and $\tilde{\sigma}(G)$. We shall do it now, by proving

Theorem 2.4. *Let G be a non-cyclic Abelian group $G\neq C_2\times C_2\times C_2$, $C_2\times C_{2k}$ $(k\geq 1)$. Then $\tilde{\sigma}(G)=\rho(G)+1$.*

Proof. We start with each of the four possibilities for $\rho(G)$, namely A, B, C and D.

The translation of the parameters between both Theorems is as follows. In [16], m is the number of factors that are multiples of 4, l is the number of odd factors and n is the number of factors 2. Instead, in [11], $k-l$ is the number of factors multiple of 4, l is the number of odd factors and s is the number of factors 2.

We start with case A. Then $n<m$ in [16] is equivalent to $s<k-l$, what implies that $s-(k-l)<0$, and we are in case i) in [11]. Hence

$$\rho(G)=1+N\left(n+\sum_{i=1}^{l}\left(1-\frac{1}{d_i}\right)+\sum_{j=n+1}^{m}\left(1-\frac{1}{e_j}\right)-1\right)$$

translates to

$$\rho\left(G\right)=1+N\left(s+l+\left(k-l-s\right)-1-\sum_{i=1}^{l}\frac{1}{d_i}-\sum_{j=l+s+1}^{m+l}\frac{1}{e_j}\right)$$
$$=1+N\left(k-1-\sum_{i=1}^{l}\frac{1}{d_i}-\sum_{j=l+s+1}^{m+l}\frac{1}{e_j}\right)=1+N\left(k-1-\sum_{i=1}^{k-s}\frac{1}{m_i}\right)=\tilde{\sigma}\left(G\right)-1.$$

Now, we consider the case B. Then $m<n\leq m+2l-1$, $n-m=2t-1$ odd. This implies $k-l<s\leq k-l+2l-1=k+l-1$, $s-(k-l)$ odd, and so $0<s-(k-l)\leq 2l-1$ with $s-(k-l)$ odd. We are in case $v)$ in [11]. Then

$$\rho\left(G\right)=1+N\left(m+t+\left(1-\frac{1}{2d_t}\right)+\sum_{i=t+1}^{l}\left(1-\frac{1}{d_i}\right)-2\right)$$

translates to

$$\rho\left(G\right)=1+N\left(k-l+\frac{s-k+l+1}{2}+\left(1-\frac{1}{2d_{(s-k+l+1)/2}}\right)+\sum_{i=(s-k+l+3)/2}^{l}\left(1-\frac{1}{d_i}\right)-2\right)$$
$$=1+N\left(k-l+\frac{s-k+l+1}{2}+1+l-\frac{s-k+l+3}{2}+1-2-\frac{1}{2d_{(s-k+l+1)/2}}-\sum_{i=(s-k+l+3)/2}^{l}\frac{1}{d_i}\right)$$
$$=1+N\left(k-l-\frac{1}{2m_{(l-s+k+1)/2}}-\sum_{i=1}^{(l-s+k-1)/2}\frac{1}{m_i}\right)=\tilde{\sigma}\left(G\right)-1.$$

We move to case D, where $n\geq m+2l+1$. This implies $s\geq(k-l)+2l+1$, and so $s-(k-l)\geq 2l+1$. Hence $s-(k-l)>2l$, and this corresponds to the case $iii)$. In this case

$$\rho(G)=\frac{1+N(3m+2l+n-3)}{4}$$

corresponds to

$$\rho(G)=1+N\frac{3k-3l+2l+s-3}{4}=1+N\frac{3k-l+s-3}{4}$$
$$=1+N\left(k-1+\frac{s-k-l+1}{4}\right)=\tilde{\sigma}(G)-1.$$

Finally, we must deal with the case C, where $m\leq n\leq m+2l$, $n-m=2t$ is even. This means that $k-l\leq s\leq k+l$, with $s-(k-l)=2t$. This possibility splits into three subcases.

If $s-(k-l)=0$, we are in case $i)$, and

$$\rho(G)=1+N\left(m+t+\sum_{i=t+1}^{l}\left(1-\frac{1}{d_i}\right)\right)$$

means that

$$\rho(G) = 1 + N\left(k - l + l - \sum_{i=t+1}^{l} \frac{1}{d_i} - 1\right) = 1 + N\left(k - 1 - \sum_{i=1}^{k-s} \frac{1}{m_i}\right) = \tilde{\sigma}(G) - 1.$$

Now, if $s-(k-l) = 2l$, we are in case ii) and $s-(k-l) = 2l$ implies $t = (n-m)/2 = (s-(k-l))/2 = l$, and so

$$\rho(G) = 1 + N\left(m + t + \sum_{i=t+1}^{l}\left(1 - \frac{1}{d_i}\right)\right) = 1 + N(k - l + l - 1) = 1 + N(k-1) = \tilde{\sigma}(G) - 1.$$

For the remaining values of $s - (k - l)$ we go to case iv). Then

$$\begin{aligned}\rho(G) &= 1 + N\left(m + t - \sum_{i=t+1}^{l}\left(1 - \frac{1}{d_i}\right) - 1\right)\\ &= 1 + N\left(m + t + (l - t) - 1 - \sum_{i=t+1}^{l} \frac{1}{d_i}\right)\\ &= 1 + N\left(m + l - 1 - \sum_{i=t+1}^{l} \frac{1}{d_i}\right).\end{aligned}$$

Since $l - t = l - (s - (k - l))/2 = (k - s + l)/2$, we have

$$\rho(G) = 1 + N\left(k - 1 - \sum_{i=1}^{(k-s+l)/2} \frac{1}{m_i}\right) = \tilde{\sigma}(G) - 1. \qquad \square$$

Remark 2.5. *Theorem 2.4 enables a comparison of the results from both papers [2] and [15]. Call* $\mathcal{S}^c_{\text{ab}}$ *the set of numbers in the symmetric crosscap spectrum which are* $\tilde{\sigma}(A)$ *for some Abelian group* A*, and* $\mathcal{S}^r_{\text{ab}}$ *the set of numbers in the real genus spectrum which are* $\rho(A)$ *for some Abelian group* A*. The set* $\mathcal{S}^c_{\text{ab}}$ *was studied in [2], and the set* $\mathcal{S}^r_{\text{ab}}$ *in [15]. Since we have proved that* $\tilde{\sigma}(A) = \rho(A) + 1$ *for each Abelian group* A*, the results in both papers imply each other. For instance, if* n *is even, then* $n \in \mathcal{S}^c_{\text{ab}}$ *if and only if* $n \equiv 2 \pmod 4$ *(Theorem 2 of [2]), and if* n *is odd, then* $n \in \mathcal{S}^r_{\text{ab}}$ *if and only if* $n \equiv 1 \pmod 4$ *(Theorem 1 in [15]). In the same way, all partial results on the structure of each of both sets obtained in those two papers can be translated in terms of the other, by using the fundamental equality* $\tilde{\sigma}(A) = \rho(A) + 1$.

2.3 Groups $C_n \times DC_3$ and $C_n \times A_4$

Theorems 2.1 and 2.4 suggest that Property (2.1) holds often. Also other families of groups satisfy it. Consider the groups of order $12n$, $C_n \times DC_3$ and $C_n \times A_4$. The real genus and symmetric crosscap number of these groups were obtained in [5] and [9], respectively, and they are presented below

Table 1

n	$\rho(C_n \times DC_3)$	$\tilde{\sigma}(C_n \times DC_3)$
2	13	14
3	16	17
6	43	44
odd, $(n, 6) = 1$	$8n - 2$	$8n - 1$
odd, $3 \mid n$, $9 \nmid n$	$8n - 8$	$8n - 7$
odd, $9 \mid n$	$8n - 2$	$8n - 1$
even, $4 \nmid n$	$9n - 11$	$9n - 10$
even, $4 \mid n$	$8n + 1$	$8n + 2$

Hence, for all n, $\tilde{\sigma}(C_n \times DC_3) = \rho(C_n \times DC_3) + 1$.

For the groups $C_n \times A_4$ with n divisible by 3 we have $\rho(C_n \times A_4) = 8n-11$ and $\tilde{\sigma}(C_n \times A_4) = 8n-10$. So there exist families of non-Abelian groups of even order satisfying Property (2.1).

2.4 Groups $C_m \times D_n$

Now, we consider the groups $C_m \times D_n$. Their real genus and symmetric crosscap number were obtained respectively in [10] and [7]. However, it is necessary to correct a mistake in [7]. In Proposition 2.3 of that paper, it was stated that $\tilde{\sigma}(C_m \times D_n) = 2 + n(m-2)$ if m is a multiple of 4 and n is odd. The proof included the claim that it is not possible to obtain a suitable epimorphism $\theta : \Lambda \to C_m \times D_n$ for a group Λ with signature $(0, +, [-], \{(\alpha), (-)\})$ for an $\alpha \geq 2$. As we will see this is wrong, and the genus of a surface on which $C_m \times D_n$ acts can be lowered for those values of m and n if $2n < m$, as follows.

Proposition 2.6. *Let m be a multiple of 4, n odd, and $2n < m$. Then $\tilde{\sigma}(C_m \times D_n) = 2 + m(n-1)$.*

Proof. Let X be a generator of C_m, A and B generators of D_n of order 2, and Λ be an NEC group with signature $(0, +, [-], \{(n), (-)\})$. We define a homomorphism θ from Λ to $C_m \times D_n$ by

$$\theta(e_1) = XAB, \quad \theta(e_2) = X^{-1}BA, \quad \theta(c_{1,0}) = A, \quad \theta(c_{1,1}) = BAB, \quad \theta(c_{2,0}) = X^{m/2}.$$

Then, $\theta(c_{1,1}c_{1,0}) = (BA)^2$, and so $\theta((c_{1,1}c_{1,0})^{(n+1)/2}) = BA$. Now, $\theta(e_1(c_{1,1}c_{1,0})^{(n+1)/2}) = X$; and so, $\theta(c_{2,0}(e_1(c_{1,1}c_{1,0})^{(n+1)/2})^{m/2}c_{1,0}) = X^{m/2}X^{m/2}A = A$. Finally, since BA and A are images of orientation-preserving elements of Λ, so is B.

The reduced area of Λ is $\frac{1}{2}(1 - \frac{1}{n}) = \frac{n-1}{2n}$, and so $\tilde{\sigma}(C_m \times D_n) \leq 2 + \frac{n-1}{2n}2mn = 2 + m(n-1)$. We are going to see that this bound cannot be lowered. All possible signatures for the group Λ were already studied in the proof of Proposition 2.3 of [7], excepting those of the form $(0, +, [-], \{(\alpha), (-)\})$.

We complete the work now, considering these signatures. Therefore, suppose that there exists an epimorphism θ from an NEC group Λ with signature $(0, +, [-], \{(\alpha), (-)\})$ for an $\alpha \geq 2$ onto $C_m \times D_n$, and call ψ the composition of θ with the projection of $C_m \times D_n$ onto D_n. Since $c_{1,0}$ has order 2 and n is odd, necessarily $\psi(c_{1,0}) = (AB)^t A$ for a certain t. Then, $\psi(e_1)$ can have the form $(AB)^r$ or $(AB)^r A$. In any case those two images must generate D_n. If $\psi(e_1) = (AB)^r$, then $\psi(e_1 c_{1,0}) = (AB)^{r+t} A$ has order 2. So, in order to generate D_n, $(AB)^r$ must have order n. Besides, $\psi(c_{1,1}) = (BA)^r (AB)^t A (AB)^r = (AB)^{t-2r} A$, and so, $\psi(c_{1,0} c_{1,1}) = (AB)^t A (AB)^{t-2r} A = (AB)^{2r}$ has also order n. Thus, $\alpha = n$. On the other hand, if $\psi(e_1) = (AB)^r A$, then $\psi(e_1 c_{1,0}) = (AB)^{r-t}$, which must have order n. Since $\psi(c_{1,1}) = (AB)^r A (AB)^t A (AB)^r A = (AB)^{2t-t} A$, then $\psi(c_{1,0} c_{1,1}) = (AB)^t A (AB)^{2r-t} A = (AB)^{2t-2r}$. Now, both $\psi(e_1)$ and $\psi(c_{1,0})$ have order 2, and so $\psi(e_1 c_{1,0}) = (AB)^{r-t}$ must have order n. But then also $\psi(c_{1,0} c_{1,1})$ has order n, and again $\alpha = n$. We have finished, and the inequality $2 + m(n-1) < 2 + n(m-2)$ holds if and only if $2n < m$. $\square$

By results in [10] and [7], and Proposition 2.6, we have the following Theorem where for an abuse of notation we write ρ and $\tilde{\sigma}$ for $\rho(C_m \times D_n)$ and $\tilde{\sigma}(C_m \times D_n)$.

Theorem 2.7. *The real genus and the symmetric crosscap number of the groups $C_m \times D_n$ are the following*

m odd, n even, $n < 2m$	$\rho = 1 + m(n-2)$	$\tilde{\sigma} = 2 + m(n-2)$
m odd, n even, $n \geq 2m$	$\rho = 1 + n(m-1)$	$\tilde{\sigma} = 2 + n(m-1)$
m, n odd, $m > n$	$\rho = 1 + m(n-1)$	$\tilde{\sigma} = 2 + mn - m - n$
m, n odd, $m < n$	$\rho = 1 + n(m-1)$	$\tilde{\sigma} = 2 + mn - m - n$
$m = n$ odd	$\rho = 1 + m(m-2)$	$\tilde{\sigma} = 2 + m(m-2)$
m, n even	$\rho = 1 + mn$	$\tilde{\sigma} = 2 + mn$
m a multiple of 4, n odd, $m < 2n$	$\rho = 1 + n(m-2)$	$\tilde{\sigma} = 2 + n(m-2)$
m a multiple of 4, n odd, $m > 2n$	$\rho = 1 + m(n-1)$	$\tilde{\sigma} = 2 + m(n-1)$

Corollary 2.8. *Observe that $\tilde{\sigma}(C_m \times D_n) = \rho(C_m \times D_n) + 1$, except when m and n are different odd numbers. In such a case, for $m > n$, $\tilde{\sigma}(C_m \times D_n) = \rho(C_m \times D_n) + 1 - n$; and if $n > m$, $\tilde{\sigma}(C_m \times D_n) = \rho(C_m \times D_n) + 1 - m$. Both results provide all even negative numbers for the difference $\tilde{\sigma}(G) - \rho(G)$.*

2.5 Groups $D_m \times D_n$

Now we shall consider the groups $D_m \times D_n$. Their symmetric crosscap number was obtained in [7], and the real genus in [5]. Observe that the real genus for m and n odd was calculated in Proposition 2(a) of [5], and included with a misprint in Theorem 3 there. The result should be read as follows: If m and n are odd, $n < m$, then $\rho(D_m \times D_n) = 1 + m(n-1)$.

In turn, the mistake stated above on $\tilde{\sigma}(C_m \times D_n)$ produced a couple of wrong results on $\tilde{\sigma}(D_m \times D_n)$ which we must correct here. For m odd and n even, Proposition 8 in [7] states that $\tilde{\sigma}(D_m \times D_n) = m(n-2)+2$. This is correct for $n \leq 2m$, but if $2m < n$, then the symmetric crosscap number of $D_m \times D_n$ is in fact smaller, as given by the forthcoming two results.

Proposition 2.9. *Let m be an odd number, n an even number with $n/2$ odd and $2m < n$. Then, $\tilde{\sigma}(D_m \times D_n) = 2 + (m-1)n$.*

Proof. Let A and B be generators of D_m of order 2, and C and D generators of D_n of order 2. Take Λ to be an NEC group with signature $(0, +, [-], \{(2m, 2, 2, 2)\})$, and define a homomorphism θ from Λ to $D_m \times D_n$ by

$$\theta(c_{1,0}) = A, \quad \theta(c_{1,1}) = BD, \quad \theta(c_{1,2}) = B(CD)^{n/2}, \quad \theta(c_{1,3}) = (CD)^{n/2}C, \quad \theta(c_{1,4}) = A.$$

Then, $\theta(c_{1,0}c_{1,1}) = ABD$, and so, $\theta((c_{1,0}c_{1,1})^m) = D$, $\theta((c_{1,0}c_{1,1})^{m+1}) = AB$. Now, $\theta(c_{1,1}c_{1,3}) = B(DC)^{n/2+1}$. Since $(DC)^{n/2+1}$ has order $n/2$ which is odd, $\theta((c_{1,1}c_{1,3})^{n/2}) = B$. And so,

$$\theta((c_{1,0}c_{1,1})^{m+1}(c_{1,1}c_{1,3})^{n/2}) = A.$$

Finally, $\theta(c_{1,2}c_{1,3}) = BC$, and so, $\theta((c_{1,1}c_{1,3})^{n/2}c_{1,2}c_{1,3}) = C$. So $D_m \times D_n$ is generated by the images of orientation-preserving elements of Λ.

The reduced area of Λ is $\frac{1}{4} - \frac{1}{4m}$, and so $\tilde{\sigma}(D_m \times D_n) \leq 2 + 4mn\left(\frac{1}{4} - \frac{1}{4m}\right) = 2 + (m-1)n$.

We now prove that this is in fact $\tilde{\sigma}(D_m \times D_n)$ by comparing with $\tilde{\sigma}(C_m \times D_n)$ as obtained in [7]. By Proposition 2.2.i) of that paper, for m odd, n even, with $2m < n$, $\tilde{\sigma}(C_m \times D_n) = 2 + n(m-1)$. Since $\tilde{\sigma}(D_m \times D_n) \geq \tilde{\sigma}(C_m \times D_n)$, we have finished. □

Proposition 2.10. *Let m be an odd number, n a multiple of 4, and $2m < n$. Then, $\tilde{\sigma}(D_m \times D_n) = 2 + (m-1)n$.*

Proof. Let A and B generators of D_m, and C and D generators of D_n, all of them of order 2. Take Λ an NEC group of signature $(0, +, [-], \{(2m, 2, 2, 2)\})$, and define a homomorphism θ from Λ to $D_m \times D_n$ by

$$\theta(c_{1,0}) = A, \quad \theta(c_{1,1}) = BD, \quad \theta(c_{1,2}) = (CD)^{n/2}, \quad \theta(c_{1,3}) = C, \quad \theta(c_{1,4}) = A.$$

Then, $\theta(c_{1,0}c_{1,1}) = ABD$. Since m is odd, $\theta((c_{1,0}c_{1,1})^m) = D$, and $\theta((c_{1,0}c_{1,1})^{m+1}) = AB$. Now, $\theta(c_{1,3}(c_{1,0}c_{1,1})^m) = CD$, and so, $\theta((c_{1,3}(c_{1,0}c_{1,1})^m)^{n/2}) = (CD)^{n/2}$. So,

$$\theta(c_{1,0}c_{1,2}(c_{1,3}(c_{1,0}c_{1,1})^m)^{n/2}) = A \quad \text{and} \quad \theta(c_{1,3}c_{1,2}(c_{1,3}(c_{1,0}c_{1,1})^m)^{n/2}) = C.$$

Finally, $\theta(c_{1,0}c_{1,2}(c_{1,3}(c_{1,0}c_{1,1})^m)^{n/2}(c_{1,0}c_{1,1})^{m+1}) = B$. So, $D_m \times D_n$ is generated by the images of orientation-preserving elements of Λ.

The reduced area of Λ is $\frac{1}{4} - \frac{1}{4m}$, and so $\tilde{\sigma}(D_m \times D_n) \leq 2 + 4mn\left(\frac{1}{4} - \frac{1}{4m}\right) = 2 + (m-1)n = \tilde{\sigma}(C_m \times D_n)$. The proof is finished. □

Hence, from [10] and [8] along with Propositions 2.8 and 2.9, we have the following Theorem.

Theorem 2.11. *The real genus and symmetric crosscap number of the groups $D_m \times D_n$ are the following*

m *odd*, n *even*, $n < 2m$	$\rho = 1 + m(n-2)$	$\tilde{\sigma} = 2 + m(n-2)$
m *odd*, n *even*, $n \geq 2m$	$\rho = 1 + n(m-1)$	$\tilde{\sigma} = 2 + n(m-1)$
m, n *odd*, $m > n$	$\rho = 1 + m(n-1)$	$\tilde{\sigma} = 1 + (m-1)(n-1)$
$m = n$ *odd*	$\rho = 1 + m(m-2)$	$\tilde{\sigma} = 2 + m(m-2)$
m, n *even*	$\rho = 1 + mn$	$\tilde{\sigma} = 2 + mn$

Remark 2.12. *Thus, the groups $D_m \times D_n$ satisfy Property* (2.1)*, except when m and n are different odd numbers. In that case, $\tilde{\sigma}(D_m \times D_n) - \rho(D_m \times D_n) = 1 - n$, and so this difference provides again, as in Corollary 2.8, all even negative numbers.*

3 Gaps in the symmetric crosscap spectrum

Our next results are inspired by [14, Theorem 6]. In that result, C. L. May studied the groups $C_n \times G_{pq}$.

Let $p < q$ be two odd primes such that $p \mid q-1$. Then there exists a non-Abelian group of order pq, denoted by G_{pq}. This group admits a presentation given by generators S and T, and relations $S^q = T^p = 1$, $T^{-1}ST = S^r$, where $r^p \equiv 1 (\mathrm{mod}\ q)$, $r \not\equiv 1 (\mathrm{mod}\ q)$. Then ST has order p, and so $X = T$, $Y = ST$, are two generators of G_{pq} of order p. It follows that $\rho(G_{pq}) = q(p-2)+1$, [13, Theorem 4], and, applying Theorem 2.1, we have:

Theorem 3.1. *Let $p < q$ be two odd primes such that $p \mid q-1$. Then $\tilde{\sigma}(G_{pq}) = q(p-2)+2$.*

Now consider the groups $G = C_n \times G_{pq}$. We are going to study the real genus and the symmetric crosscap number of G. In the case when n is coprime with pq, the real genus of G is given by the following theorem of May:

Theorem 3.2 ([14], Theorem 6)**.** *Let $p < q$ be two odd primes such that $p \mid q-1$, and n an integer coprime with pq. Then $\rho(C_n \times G_{pq}) = 1 + q(pn - n - 1)$.*

Now we turn to the symmetric crosscap number of these groups.

Theorem 3.3. *Let $p < q$ be two odd primes such that $p \mid q-1$, and n an integer coprime with pq. Then $\tilde{\sigma}(C_n \times G_{pq}) = 2 + q(pn - n - 1)$.*

Proof. If n is odd, then $C_n \times G_{pq}$ has odd order, and we apply Theorem 2.1 and Theorem 3.2.

Now, we show that these groups satisfy Property (2.1) also in the case when n is even. Let us take X and Y to be the generators of G_{pq} of order p as above, and denote by A the generator of C_n. Consider an NEC group Γ with signature $(0, +, [p, np], \{(-)\})$, and define the epimorphism θ from Γ onto $C_n \times G_{pq}$ by $\theta(x_1) = X$, $\theta(x_2) = AY$, $\theta(e_1) = (AXY)^{-1}$, $\theta(c_{1,0}) = A^{n/2}$. Since n and p are coprime, there exist integers α, β, such that $\alpha n + \beta p = 1$. Then, $\theta(x_2^{\alpha n}) = (AY)^{\alpha n} = Y^{\alpha n} = Y^{1-\beta p} = Y$, $\theta(x_2^{\beta p}) = (AY)^{\beta p} = A^{\beta p} = A^{1-\alpha n} = A$. Besides, $\theta\left(x_2^{\beta pn/2} c_{1,0}\right) = 1$, and so the kernel contains an orientation reversing element. So, $\tilde{\sigma}(C_n \times G_{pq}) \le 2 + q(pn - n - 1)$.

Now we need to see that the area of Γ is minimal. The only possibility to reduce the area is to substitute n with one of its factors, say k, and take signature

$$(0, +, [p, kp], \{(-)\}) \quad \text{or} \quad (1, -, [p, kp], \{-\}).$$

Then the image of x_2 must be $A^{n/k}Y$, and either the image of $c_{1,0}$ is $A^{n/2}$ or the image of d_1 is $A^{(n-n/k)/2}(XY)^{(p-1)/2}$.

In the first case it is not possible to generate A as an image of an orientation preserving element, because the image of any word with an even number of copies of $c_{1,0}$ will have, as projection onto C_n, a power of $A^{n/k}$. In the second case, the exponent n/k must be even, in order to get that the image of $d_1^2 x_1 x_2$ be 1. But then also the orientation preserving elements contain an even number of copies of d_1, and so only powers of A with even exponent can be obtained. Therefore, also in this case the element A is not the image of an orientation preserving element.

Thus the area of Γ is minimal, and we have that $\tilde{\sigma}(C_n \times G_{pq}) = 2 + q(pn - n - 1)$, and these groups satisfy Property (2.1). □

We are now going to use the above results to eliminate many possible gaps in the symmetric crosscap spectrum. This problem was studied in [1], and the main result was the following:

Theorem 3.4 ([1], Theorem 2)**.** *Let $N > 3$ be a gap of the symmetric crosscap spectrum. Then $N \equiv 3$, 51, 75 or 99 (mod 120), $N \not\equiv 651$ (mod 660), $N - 2$ is not a square, and $N - 2$ has some prime factor $p \equiv 5$ (mod 6).*

These conditions, necessary for a number to be a gap, are not sufficient. For $N < 10000$, they left sixty-seven numbers which were possible gaps. Three of them are in fact the symmetric crosscap number of a group, thanks to Theorems 2.3 and 3.1. We show them in the Table 2, where we indicate N, its class (mod 120), the prime factors of $N - 2$, and the group G such that $\tilde{\sigma}(G) = N$.

Table 2

N	$N \equiv \pmod{120}$	$N-2$	$G,\ \tilde{\sigma}(G)=N$
1443	3	$1441 = 11 \cdot 131$	$G_{13\cdot131}$
4875	15	$4873 = 11 \cdot 443$	$G_{13\cdot443}$
6051	51	$6049 = 23 \cdot 263$	$C_{23} \times C_{276}$

This leaves sixty-four numbers which are candidates for being a gap, but forty of them are actually $\tilde{\sigma}(C_n \times G_{pq})$ for some n, p, q as obtained in Theorem 3.3. We display the respective data in Table 3.

Table 3

N	$N \equiv \pmod{120}$	$N-2$	$G,\ \tilde{\sigma}(G)=N$
915	75	$913 = 11 \cdot 83$	$C_{21} \times G_{5\cdot11}$
1179	99	$1177 = 11 \cdot 107$	$C_{27} \times G_{5\cdot11}$
1539	99	$1537 = 29 \cdot 53$	$C_{9} \times G_{7\cdot29}$
1635	75	$1633 = 23 \cdot 71$	$C_{6} \times G_{5\cdot71}$
1923	3	$1921 = 17 \cdot 113$	$C_{3} \times G_{7\cdot113}$
2235	75	$2233 = 7 \cdot 11 \cdot 29$	$C_{13} \times G_{7\cdot29}$
2499	99	$2497 = 11 \cdot 227$	$C_{57} \times G_{5\cdot11}$
2739	99	$2737 = 7 \cdot 17 \cdot 23$	$C_{12} \times G_{11\cdot23}$
2763	3	$2761 = 11 \cdot 251$	$C_{3} \times G_{5\cdot251}$
3339	99	$3337 = 47 \cdot 71$	$C_{8} \times G_{7\cdot71}$
3555	75	$3553 = 11 \cdot 17 \cdot 19$	$C_{81} \times G_{5\cdot11}$
3819	99	$3817 = 11 \cdot 347$	$C_{87} \times G_{5\cdot11}$
4083	3	$4081 = 7 \cdot 11 \cdot 53$	$C_{93} \times G_{5\cdot11}$
4323	3	$4321 = 29 \cdot 149$	$C_{25} \times G_{7\cdot29}$
4395	75	$4393 = 23 \cdot 191$	$C_{6} \times G_{5\cdot191}$
4899	99	$4897 = 59 \cdot 83$	$C_{3} \times G_{29\cdot59}$
5139	99	$5137 = 11 \cdot 467$	$C_{117} \times G_{5\cdot11}$
5403	3	$5401 = 11 \cdot 491$	$C_{3} \times G_{5\cdot491}$
5499	99	$5497 = 23 \cdot 239$	$C_{4} \times G_{7\cdot239}$
5595	75	$5593 = 7 \cdot 17 \cdot 47$	$C_{400} \times G_{3\cdot7}$

N	$N \equiv \pmod{120}$	$N-2$	$G,\ \tilde{\sigma}(G) = N$
5715	75	$5713 = 29 \cdot 197$	$C_5 \times G_{7\cdot 197}$
6195	75	$6193 = 11 \cdot 563$	$C_{141} \times G_{5\cdot 11}$
6411	51	$6409 = 13 \cdot 17 \cdot 29$	$C_{37} \times G_{7\cdot 29}$
6459	99	$6457 = 11 \cdot 587$	$C_{147} \times G_{5\cdot 11}$
6723	3	$6721 = 11 \cdot 13 \cdot 47$	$C_{259} \times G_{3\cdot 13}$
7155	75	$7153 = 23 \cdot 311$	$C_6 \times G_{5\cdot 311}$
7515	75	$7513 = 11 \cdot 683$	$C_{171} \times G_{5\cdot 11}$
7635	75	$7633 = 17 \cdot 449$	$C_3 \times G_{7\cdot 449}$
7731	51	$7729 = 59 \cdot 131$	$C_5 \times G_{13\cdot 131}$
7779	99	$7777 = 7 \cdot 11 \cdot 101$	$C_{177} \times G_{5\cdot 11}$
7803	3	$7801 = 29 \cdot 269$	$C_{45} \times G_{7\cdot 29}$
8043	3	$8041 = 11 \cdot 17 \cdot 43$	$C_{94} \times G_{3\cdot 43}$
8259	99	$8257 = 23 \cdot 359$	$C_{36} \times G_{11\cdot 23}$
8451	51	$8449 = 7 \cdot 17 \cdot 71$	$C_{20} \times G_{7\cdot 71}$
8835	75	$8833 = 11^2 \cdot 73$	$C_{61} \times G_{3\cdot 73}$
8979	99	$8977 = 47 \cdot 191$	$C_{12} \times G_{5\cdot 191}$
9099	99	$9097 = 11 \cdot 827$	$C_{207} \times G_{5\cdot 11}$
9195	75	$9193 = 29 \cdot 317$	$C_{53} \times G_{7\cdot 29}$
9363	3	$9361 = 11 \cdot 23 \cdot 37$	$C_{127} \times G_{3\cdot 37}$
9915	75	$9913 = 23 \cdot 431$	$C_6 \times G_{5\cdot 431}$

According to above results only twenty-four numbers N remain as potential gaps in the symmetric crosscap spectrum, with $3 < N < 10000$. They are shown in Table 4.

These results reinforce the conjecture that there is no other gap besides 3 in the spectrum of the symmetric crosscap number.

Now, we are going to study the particular case $N = 699$, the smallest number for which it is unknown whether it represents a gap in the spectrum. This will demonstrate how to use the relationship between the real genus and symmetric crosscap number, and how Property (2.1) is useful when it holds. Unfortunately, this is not the case for this value of N and the group G already known to satisfy $\rho(G) = N - 1$.

Table 4: Table 4

N	$N \equiv \pmod{120}$	$N-2$
699	99	$697 = 17 \cdot 41$
1083	3	$1081 = 23 \cdot 47$
1515	75	$1513 = 17 \cdot 89$
2331	51	$2329 = 17 \cdot 137$
3651	51	$3649 = 41 \cdot 89$
3843	3	$3841 = 23 \cdot 167$
3963	3	$3961 = 17 \cdot 233$
4371	51	$4369 = 17 \cdot 257$
4635	75	$4633 = 41 \cdot 113$
5019	99	$5017 = 29 \cdot 173$
5355	75	$5353 = 53 \cdot 101$
5619	99	$5617 = 41 \cdot 137$
6003	3	$6001 = 17 \cdot 353$
6315	75	$6313 = 59 \cdot 107$
6819	99	$6817 = 17 \cdot 401$
7851	51	$7849 = 47 \cdot 167$
7899	99	$7897 = 53 \cdot 149$
8499	99	$8497 = 29 \cdot 293$
8811	51	$8809 = 23 \cdot 383$
8859	99	$8857 = 17 \cdot 521$
8883	3	$8881 = 83 \cdot 107$
9171	51	$9169 = 53 \cdot 173$
9555	75	$9553 = 41 \cdot 233$
9675	75	$9673 = 17 \cdot 569$

Since $41 \equiv 1 \pmod 4$, there exists a semidirect product $C_4 \rtimes C_{41}$, with presentation

$$\langle X, Y \mid Y^4 = X^{41} = 1, XY = YX^9 \rangle.$$

Now call $G = C_9 \times (C_4 \rtimes C_{41})$, and Z a generator of C_9. This group G has real genus 698, see Corollary 6 of [14]. So, if it satisfies Property (2.1), we have a group with symmetric crosscap number 699. Let us study G. Its elements of order 2 lie in $C_4 \rtimes C_{41}$, and they have the form X^kY^2. For, $(X^kY^2)^2 = X^kY^2X^kY^2 = YX^{9k}YX^kY^2 = Y^2X^{81k}X^kY^2 = Y^2X^{82k}Y^2 = 1$, and it is clear that no other element has order 2.

Now, consider an NEC group Λ with signature $(0, +, [2, 36], \{(-)\})$ and an epimorphism $\theta : \Lambda \to G$ defined by

$$\theta(x_1) = XY^2, \quad \theta(x_2) = YZ, \quad \theta(e_1) = YX^{-1}Z^{-1}, \quad \theta(c_{1,0}) = X^{10}Y^2.$$

The kernel of this epimorphism is a non-orientable unbordered surface group, because $o(XY^2) = 2$, $o(YZ) = 36$, and

$$\begin{aligned}\theta(x_1x_2e_1) &= XY^2YZYX^{-1}Z^{-1} = XY^4X^{-1} = 1,\\ \theta(e_1^{-1}c_{1,0}e_1c_{1,0}) &= XY^3ZX^{10}Y^2YX^{-1}Z^{-1}X^{10}Y^2 = XY^3X^{10}Y^3X^9Y^2\\ &= Y^3X^{729}X^{10}Y^3X^9Y^2 = Y^3X^{739}Y^3X^9Y^2 = Y^6X^{739\cdot 729+9}Y^2 = Y^8 = 1.\end{aligned}$$

Besides, $\theta(\Lambda^+) = G$, because

$$\theta(x_2^9) = (YZ)^9 = Y \quad \theta(x_2^{28}) = (YZ)^{28} = Z \quad \theta(x_1x_2^{18}) = (XY^2)Y^2 = X$$

The genus of the corresponding surface is

$$(9\cdot 4\cdot 41)\left(1-\frac{1}{2}+1-\frac{1}{36}-1\right)+2 = 9\cdot 4\cdot 41\cdot\frac{17}{36}+2 = 17\cdot 41+2 = 699.$$

It only remains to prove that this is the minimum genus of a non-orientable unbordered surface on which G acts. But this is not the case. Consider an NEC group Γ with signature $(0, +, [36], \{(41)\})$ and an epimorphism $\theta : \Gamma \to G$ defined by

$$\theta(x_1) = YZ, \quad \theta(e_1) = Y^{-1}Z^{-1}, \quad \theta(c_{1,0}) = XY^2, \quad \theta(c_{1,1}) = X^{32}Y^2.$$

Then,

$$\theta(e_1^{-1}c_{1,0}e_1c_{1,1}) = YZXY^2Y^{-1}Z^{-1}X^{32}Y^2 = YXYX^{32}Y^2 = YYX^9X^{32}Y^2 = 1.$$

Besides, $\theta(\Gamma^+) = G$, because

$$\begin{aligned}\theta(x_1^{28}) &= Z,\\ \theta(x_1^9) &= Y,\\ \theta(c_{1,0}c_{1,1}) &= XY^2X^{32}Y^2 = Y^2X^{81}X^{32}Y^2 = Y^2X^{31}Y^2 = Y^4X^{31\cdot 81} = X^{10}.\end{aligned}$$

So that, $\theta((c_{1,0}c_{1,1})^{37}) = X^{370} = X$. Now, we compute the genus, and it is

$$(9\cdot 4\cdot 41)\left(\left(1-\frac{1}{36}\right)+\frac{1}{2}\left(1-\frac{1}{41}\right)-1\right)+2 = 9\cdot 4\cdot 41\cdot\left(\frac{20}{41}-\frac{1}{36}\right)+2 = 20\cdot 9\cdot 4-41+2 = 681.$$

Hence $\tilde{\sigma}(G) \leq 681$, in fact it equals 681, and so the group G does not satisfy Property (2.1), and no group with symmetric crosscap number 699 is known yet.

4 Gaps in the real genus spectrum

All odd numbers belong to the real genus spectrum, since C. L. May proved in [12] that the dicyclic group DC_n of order $4n$ has real genus $2n+1$. So the problem of determining the spectrum of the real genus restricts to even numbers. It is known that 2, 12, 24 and 72 are not the real genus of any group. In his paper [14], C. L. May obtained families of groups whose real genera cover most of the even numbers. For instance, for $N < 10000$, his results leave 328 numbers for which it is unknown whether they belong to the real genus spectrum. M. Pires has calculated explicitly those numbers in [17]. Most of them are multiple of 12, but there are also numbers $N \equiv 2, 6, 8 \pmod{12}$.

Unfortunately, the groups G for which we know that $\tilde{\sigma}(G) \equiv 1, 7, 9 \pmod{12}$ do not satisfy Property (2.1) and cannot be used to eliminate gaps in the real genus spectrum. The situation is very different for $N \equiv 2 \pmod{12}$. According to [17], the numbers $N \equiv 2 \pmod{12}$ with $N < 10000$, which are not yet known to belong to the real genus spectrum are 1082, 3842, 6266, 7850, 8810 and 8882. Let us pay attention to $6266 \equiv 26 \pmod{60}$. In [1] it was proved that for each $k \geq 0$, a semidirect product $G_k = C_5 \rtimes C_{8+16k}$ satisfies $\tilde{\sigma}(G_k) = 60k + 27$. We are going to show that these groups satisfy Property (2.1), and so $\rho(G_k) = 60k + 26$.

Proposition 4.1. *Let $k \geq 0$, and $G_k = C_5 \rtimes C_{8+16k}$, with presentation $\langle A, B \mid B^5 = A^{8+16k} = 1, BA = AB^2 \rangle$. Then, $\rho(G_k) = 60k + 26$.*

Proof. One can see in [1] or [17] that the element BA^{2+4k} has order 4, and A^{4+8k} is the unique element of G_k of order 2. Since BA^{2+4k} and A generate G_k, take an NEC group Λ with signature $(0, +, [4, 8+16k], \{(-)\})$, and define $\theta : \Lambda \to G_k$ by

$$\theta(x_1) = BA^{2+4k}, \quad \theta(x_2) = A, \quad \theta(e_1) = A^{5+12k}B^{-1}, \quad \theta(c_{1,0}) = 1.$$

Then, θ is an epimorphism, the reduced area of Λ is $|\Lambda| = \frac{5+12k}{8+16k}$, and $\rho(G_k) \leq 1 + o(G_k)|\Lambda| = 1 + (40+80k)\frac{5+12k}{8+16k} = 60k+26$. In order to see that this is in fact $\rho(G_k)$, recall that the signature of the suitable group Λ must have a period-cycle with two consecutive link-periods equal to 2, or an empty period-cycle, see [3]. Since G_k has a unique element of order 2, the first possibility does not hold. So, Λ must have an empty period-cycle, and for getting a smaller reduced area, its signature must have the form $(0, +, [m_1, m_2], \{(-)\})$. Then, by using the same arguments as in Proposition 7 of [1], it follows that the minimal area is indeed attained for the signature $(0, +, [4, 8+16k], \{(-)\})$. Thus, $\rho(G_k) = 60k + 26$. Observe that in particular $\rho(G_{104}) = 6266$. □

On the contrary, for the five other values of N, namely 1082, 3842, 7850, 8810 and 8882, it is not known whether $N+1$ belongs to the symmetric crosscap spectrum, see Table 4. Hence, these pairs $(N, N+1)$ seem to be a convenient target for identifying possible gaps in both spectra.

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

All necessary data are stated and quoted in the paper.

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References

[1] A. Bacelo, J. J. Etayo, and E. Martínez, "Filling gaps of the symmetric crosscap spectrum," *Mosc. Math. J.*, vol. 17, no. 3, pp. 357–369, 2017, doi: 10.17323/1609-4514-2017-17-3-357-369.

[2] A. Bacelo, J. J. Etayo, and E. Martínez, "The symmetric crosscap spectrum of Abelian groups," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, vol. 112, no. 3, pp. 633–640, 2018, doi: 10.1007/s13398-017-0434-3.

[3] E. Bujalance and E. Martinez, "A remark on NEC groups representing surfaces with boundary," *Bull. London Math. Soc.*, vol. 21, no. 3, pp. 263–266, 1989.

[4] E. Bujalance, J. J. Etayo, J. M. Gamboa, and G. Gromadzki, *Automorphism groups of compact bordered Klein surfaces*, ser. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1990, vol. 1439, doi: 10.1007/BFb0084977.

[5] C. Cano, "Sobre el género real de grupos finitos," Ph.D. dissertation, Universidad Nacional de Educación a Distancia (UNED), 2011.

[6] J. J. Etayo, E. Martínez, and P. Rodríguez Calderón, "There is no group of real genus 72," *Anales de la Real Academia de Doctores de España*, vol. 10, no. 4, pp. 663–680, 2025.

[7] J. J. Etayo Gordejuela and E. Martínez, "The symmetric cross-cap number of the groups $C_m \times D_n$," *Proc. Roy. Soc. Edinburgh Sect. A*, vol. 138, no. 6, pp. 1197–1213, 2008, doi: 10.1017/S0308210507000169.

[8] J. J. Etayo Gordejuela and E. Martínez, "The action of the groups $D_m \times D_n$ on unbordered Klein surfaces," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, vol. 105, no. 1, pp. 97–108, 2011, doi: 10.1007/s13398-011-0007-9.

[9] J. J. Etayo Gordejuela and E. Martínez, "The symmetric crosscap number of the families of groups $DC_3 \times C_n$ and $A_4 \times C_n$," *Houston J. Math.*, vol. 38, no. 2, pp. 345–358, 2012.

[10] J. J. Etayo Gordejuela and E. Martínez, "The real genus of cyclic by dihedral and dihedral by dihedral groups," *J. Algebra*, vol. 296, no. 1, pp. 145–156, 2006, doi: 10.1016/j.jalgebra.2005.03.038.

[11] G. Gromadzki, "Abelian groups of automorphisms of compact nonorientable Klein surfaces without boundary," *Comment. Math. Prace Mat.*, vol. 28, no. 2, pp. 197–217, 1989.

[12] C. L. May, "Finite groups acting on bordered surfaces and the real genus of a group," *Rocky Mountain J. Math.*, vol. 23, no. 2, pp. 707–724, 1993, doi: 10.1216/rmjm/1181072586.

[13] C. L. May, "Finite metacyclic groups acting on bordered surfaces," *Glasgow Math. J.*, vol. 36, no. 2, pp. 233–240, 1994, doi: 10.1017/S0017089500030779.

[14] C. L. May, "Groups of even real genus," *J. Algebra Appl.*, vol. 6, no. 6, pp. 973–989, 2007, doi: 10.1142/S0219498807002612.

[15] C. L. May and J. Zimmerman, "The real genus spectrum of abelian groups," *J. Algebra Appl.*, vol. 18, no. 8, 2019, Art. ID 1950158.

[16] D. McCullough, "Minimal genus of abelian actions on Klein surfaces with boundary," *Math. Z.*, vol. 205, no. 3, pp. 421–436, 1990, doi: 10.1007/BF02571253.

[17] M. Pires, "Los grupos de género real par," *Anales de la Real Academia de Doctores de España*, vol. 9, no. 4, pp. 863–898, 2024.

[18] J. Rodríguez, "Abelian actions on compact nonorientable Riemann surfaces," *Glasg. Math. J.*, vol. 64, no. 3, pp. 634–648, 2022, doi: 10.1017/S0017089521000410.

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Some inequalities associated with a partial differential operator

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ABSTRACT

We study uncertainty principles for a generalized Fourier transform $\mathcal{F}_\alpha$, associated with the pair of partial differential operators (D, D_α) originally introduced by Flensted-Jensen and later extended by Trimèche. This transform, is defined via the Jacobi kernel and an appropriate weighted measure. We establish an $\mathrm{L}^p - \mathrm{L}^q$ version of Miyachi's theorem, from which we deduce Cowling-Price-type results. Additionally, we establish a local uncertainty principle in the sense of Faris and provide related numerical estimates.

RESUMEN

Estudiamos principios de incertidumbre para una transformada de Fourier generalizada $\mathcal{F}_\alpha$, asociada al par de operadores diferenciales parciales (D, D_α) originalmente introducidos por Flensted-Jensen y luego extendidos por Trimèche. Esta transformada está definida a través del núcleo de Jacobi y una medida pesada apropiada. Establecemos una versión $\mathrm{L}^p - \mathrm{L}^q$ del teorema de Miyachi, a partir del cual deducimos resultados de tipo Cowling-Price. Adicionalmente, establecemos un principio de incertidumbre local en el sentido de Faris y entregamos estimaciones numéricas relacionadas.

Keywords and Phrases: Partial differential operators, generalized Fourier transform, Jacobi kernel, Miyachi theorem, Cowling-Price theorem, uncertainty principle.

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1 Introduction

In the context of harmonic analysis on symmetric spaces, Flensted-Jensen [7] introduced a pair of partial differential operators fundamental to the study of spherical functions on simply connected semisimple Lie groups:

$$D = \frac{\partial}{\partial\theta} \quad \text{and} \quad D_n = \frac{\partial^2}{\partial y^2} + [(2n-1)\coth y + \tanh y]\,\frac{\partial}{\partial y} - \frac{1}{\cosh^2 y}\frac{\partial^2}{\partial\theta^2},$$

where n is a positive integer. Trimèche [16] extended these operators by generalizing the integer parameter $n-1$ to a positive real parameter $\alpha > 0$, thereby developing an associated harmonic analysis framework centered around a generalized Fourier transform $\mathcal{F}_\alpha$. For suitable functions, this transform is given by

$$\mathcal{F}_\alpha f(\lambda,\mu) = \iint_{\mathbb{R}_+\times\mathbb{R}} f(y,\theta)\varphi_{-\lambda,\mu}(y,\theta)\,dm_\alpha(y,\theta),$$

where $\varphi_{-\lambda,\mu}$ is constructed from the classical Jacobi kernel $\varphi_\mu^{\alpha,\lambda}$ via the formula:

$$\varphi_{\lambda,\mu}(y,\theta) = e^{i\lambda\theta}(\cosh y)^\lambda \varphi_\mu^{\alpha,\lambda}(y)$$

and the measure

$$dm_\alpha(y,\theta) = 2^{2(\alpha+1)}(\sinh y)^{2\alpha+1}\cosh y\,dy\,d\theta$$

reflects the intrinsic non-Euclidean geometry of the underlying space. Unlike classical Jacobi transforms, where λ is fixed, $\mathcal{F}_\alpha$ treats λ as a spectral variable. This key innovation makes $\mathcal{F}_\alpha$ a natural and powerful tool for analyzing radial functions on the universal covering group of $\mathbf{U}(n,1)$. Although significant work has been done to explore various aspects of this transform [7, 9, 12, 16], its potential within the framework of uncertainty principles remains largely unexplored. This paper aims to address this gap by establishing several uncertainty principles for $\mathcal{F}_\alpha(f)$. We begin by recalling that classical examples of such principles include decay-based results like Hardy's theorem [8], which states that if

$$|f(x)| \le ce^{-ax^2} \quad \text{and} \quad |\widehat{f}(y)| \le ce^{-by^2},$$

then $f = 0$ when $ab > \frac{1}{4}$, and f is Gaussian otherwise. Cowling-Price [2] extended this to $\mathrm{L}^p - \mathrm{L}^q$ integrability conditions, while Miyachi [13] introduced logarithmic integrability conditions, requiring

$$e^{ax^2} f \in \mathrm{L}^1(\mathbb{R}) + \mathrm{L}^\infty(\mathbb{R}), \qquad \int_{\mathbb{R}} \log^+\left(\frac{|\widehat{f}(y)|e^{by^2}}{\beta}\right) dy < \infty,$$

where $\log^+ x = \max(\log x, 0)$. Miyachi extended Hardy's theorem by replacing pointwise decay with logarithmic integrability conditions, thereby enlarging the class of admissible functions.

This work establishes an analogue of Miyachi's theorem for the generalized Fourier transform $\mathcal{F}_\alpha$, associated with the operator pair (D, D_α). Our approach, which leverages sharp estimates of the generalized Jacobi kernel, distinguishes itself from previous techniques. These include methods reliant on Bessel operators and the Dunkl setting [1,10], Laguerre polynomials for Riemann-Liouville operators [10], or Abel transforms and heat kernels in Jacobi analysis [3]. This builds upon several related studies on uncertainty principles found in [4, 9, 12, 14].

Alongside these decay-based principles, a distinct, support-based perspective was developed by Faris and Price. This approach quantifies uncertainty not through rates of decay, but through the spatial concentration of a function and the frequency dispersion of its transform. The Faris-Price [5, 15] expresses this idea via measurable sets: for $f \in \mathrm{L}^2(\mathbb{R}^n)$ and a measurable set $E \subset \mathbb{R}^n$, one has

$$\int_E |\widehat{f}(\xi)|^2 d\xi \leq K_\alpha |E|^{\frac{2\alpha}{n}} \||x|^\alpha f\|_2^2, \quad 0 < \alpha < \frac{n}{2}.$$

Such support-based principles provide explicit constants that govern the trade-off between spatial localization and spectral dispersion.

A second main contribution is the establishment of a local uncertainty principle of Faris-type for $\mathcal{F}_\alpha$. The theoretical result guarantees the existence of an optimal constant $K_{\alpha,a,q}(\gamma_\alpha(F))$ but does not provide its explicit form. To bridge this gap, we employ numerical optimization techniques to compute this constant, quantifying the precise trade-off between spatial and spectral localization.

The paper is organized as follows. Section 2 develops the harmonic analysis framework for $\mathcal{F}_\alpha$ and provides the necessary kernel bounds. Section 3 proves Miyachi- and Cowling-Price-type theorems. Section 4 establishes the Faris-type principle and conducts a numerical investigation to compute the associated optimal constants.

2 Mathematical framework

2.1 Generalized Jacobi Kernel

Let α be a positive real number and let $\mathbb{K} = [0, +\infty[\times\mathbb{R}$. Following [16], we consider the differential operators:

$$\begin{cases} D = \dfrac{\partial}{\partial\theta}, \\ D_\alpha = \dfrac{\partial^2}{\partial y^2} + [(2\alpha+1)\coth y + \tanh y]\dfrac{\partial}{\partial y} - \dfrac{1}{\cosh^2 y}\dfrac{\partial^2}{\partial\theta^2} + (\alpha+1)^2. \end{cases} \tag{2.1}$$

For complex parameters $\lambda, \mu \in \mathbb{C}$, the system

$$\begin{cases} Du = i\lambda u, \\ D_\alpha u = -\mu^2 u, \\ u(0,0) = 1, \quad \frac{\partial u}{\partial y}(0,\theta) = 0 \text{ for } \theta \in \mathbb{R} \end{cases} \tag{2.2}$$

has a unique solution given by the generalized Jacobi kernel:

$$\varphi_{\lambda,\mu}(y,\theta) = e^{i\lambda\theta}(\cosh y)^\lambda \varphi_\mu^{\alpha,\lambda}(y), \tag{2.3}$$

where $\varphi_\mu^{\alpha,\lambda}$ is the Jacobi kernel [6]:

$$\varphi_\mu^{\alpha,\lambda}(y) = {}_2F_1\left(\frac{\alpha+\lambda+1+i\mu}{2}, \frac{\alpha+\lambda+1-i\mu}{2}; \alpha+1; -\sinh^2 y\right), \tag{2.4}$$

expressed in terms of the Gaussian hypergeometric function ${}_2F_1$.

For $y > 0$ and $\theta \in \mathbb{R}$, the kernel admits the integral representation [16]:

$$\varphi_{\lambda,\mu}(y,\theta) = \frac{2^\alpha \alpha}{\pi}(\sinh y)^{-2\alpha} \int_0^y \int_{-\omega}^{\omega} (\cosh y \cos\psi - \cosh s)^{\alpha-1} \cos(\mu s) e^{i\lambda(\theta+\psi)} d\psi\, ds, \tag{2.5}$$

where $\omega = \omega(s,y) = \arccos(\cosh s/\cosh y)$. When $y = 0$, the kernel simplifies to $\varphi_{\lambda,\mu}(0,\theta) = e^{i\lambda\theta}$.

The spectral space $\widehat{\mathbb{K}} = \mathbb{L} \cup \Omega$ consists of:

$$\mathbb{L} = \mathbb{R} \times [0, +\infty[, \quad \Omega = \bigcup_{m\in\mathbb{N}} (D_m^+ \cup D_m^-),$$

where:

$$D_m^+ = \{(\alpha+2m+1+\eta, i\eta) \mid \eta > 0\} \quad \text{and} \quad D_m^- = \{(-\alpha-2m-1-\eta, i\eta) \mid \eta > 0\}.$$

The kernel satisfies the uniform bound [16]:

$$\forall(\lambda,\mu) \in \widehat{\mathbb{K}}, \quad \sup_{(y,\theta)\in\mathbb{K}} |\varphi_{\lambda,\mu}(y,\theta)| = 1. \tag{2.6}$$

The kernel relates to the generalized Riemann-Liouville transform $\mathfrak{X}_\alpha$ through:

$$\varphi_{\lambda,\mu}(y,\theta) = \mathfrak{X}_\alpha\left(\cos(\mu\cdot)e^{i\lambda\cdot}\right)(y,\theta),$$

where

$$\mathfrak{X}_\alpha f(y,\theta) = \int_{\mathbb{K}} f(x,t)\mathrm{K}(x,t,y,\theta)dx\,dt$$

with kernel

$$\mathrm{K}(x,t,y,\theta) = \frac{2^\alpha \alpha}{\pi}\chi_{[0,y]}(x)\chi_{[-\omega,\omega]}(t-\theta)(\cosh y\cos(t-\theta) - \cosh x)^{\alpha-1}(\sinh y)^{-2\alpha}.$$

For the constant function $\mathbf{1}$, we have the bound:

$$\mathfrak{X}_\alpha(\mathbf{1})(y,\theta) = \int_{\mathbb{K}} \mathrm{K}(x,t,y,\theta)dx\,dt \leq 1. \tag{2.7}$$

2.2 Generalized Fourier transform

For $p \in [1,+\infty]$, we define the weighted Lebesgue spaces as follows:

- For $1 \leq p < \infty$, the space $\mathrm{L}^p_\alpha(\mathbb{K})$ consists of measurable functions $f : \mathbb{K} \to \mathbb{C}$ satisfying

$$\|f\|_{p,m_\alpha} = \left(\int_{\mathbb{K}} |f(y,\theta)|^p dm_\alpha(y,\theta)\right)^{1/p} < \infty,$$

  where the measure is given by

$$dm_\alpha(y,\theta) = 2^{2(\alpha+1)}(\sinh y)^{2\alpha+1}\cosh y\,dy\,d\theta. \tag{2.8}$$

- For $p = \infty$, the space $\mathrm{L}^\infty_\alpha(\mathbb{K})$ consists of measurable functions with finite essential supremum norm

$$\|f\|_{\infty,m_\alpha} = \operatorname*{ess\,sup}_{(y,\theta)\in\mathbb{K}} |f(y,\theta)|.$$

The generalized Fourier transform $\mathcal{F}_\alpha$ on $\mathrm{L}^1_\alpha(\mathbb{K})$ is defined by:

$$\mathcal{F}_\alpha f(\lambda,\mu) = \int_{\mathbb{K}} f(y,\theta)\varphi_{-\lambda,\mu}(y,\theta)dm_\alpha(y,\theta),$$

satisfying the following inequality:

$$\forall(\lambda,\mu) \in \widehat{\mathbb{K}}, \quad |\mathcal{F}_\alpha f(\lambda,\mu)| \leq \|f\|_{1,m_\alpha}. \tag{2.9}$$

The Plancherel measure $d\gamma_\alpha$ combines continuous and discrete parts:

$$\begin{aligned}\int_{\widehat{\mathbb{K}}} g(\lambda,\mu)d\gamma_\alpha(\lambda,\mu) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}\times[0,+\infty[} g(\lambda,\mu)\frac{d\lambda\,d\mu}{|C_1(\lambda,\mu)|^2} \\ &+ \frac{1}{(2\pi)^2}\sum_{m=0}^{\infty}\left\{\int_0^\infty g(\kappa+\eta,i\eta)C_2(\kappa+\eta,i\eta)d\eta + \int_0^\infty g(-\kappa-\eta,i\eta)C_2(-\kappa-\eta,i\eta)d\eta\right\},\end{aligned}$$

where $\kappa = \alpha + 2m + 1$ and:

$$C_1(\lambda,\mu) = \frac{2^{\alpha+1-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma\left(\frac{\alpha+\lambda+1+i\mu}{2}\right)\Gamma\left(\frac{\alpha-\lambda+1+i\mu}{2}\right)}, \quad C_2(\lambda,\mu) = -2i\pi \operatorname*{Res}_{z=\mu}[C_1(\lambda,z)C_1(\lambda,-z)]^{-1}.$$

The weight functions satisfy [16]:

$$K_1|\mu|^2 \leq |C_1(\lambda,\mu)|^{-2} \leq K_2(1+|\lambda|^2+|\mu|^2)^{2[\alpha+\frac{1}{2}]+1}. \tag{2.10}$$

$$|C_2(\lambda,\mu)| \leq K_3(1+|\lambda|^2+|\mu|^2)^{2[\alpha+\frac{1}{2}]+1}. \tag{2.11}$$

The transform $\mathcal{F}_\alpha$ satisfies the Plancherel identity

$$\|\mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha} = \|f\|_{2,m_\alpha}.$$

For $1 \leq p \leq 2$, the Hausdorff-Young inequality holds:

$$\|\mathcal{F}_\alpha(f)\|_{q,\gamma_\alpha} \leq \|f\|_{p,m_\alpha}, \tag{2.12}$$

where q is the conjugate of p. The inversion formula is given by:

$$f(y,\theta) = \int_{\widehat{\mathbb{K}}} \mathcal{F}_\alpha(f)(\lambda,\mu)\varphi_{\lambda,\mu}(y,\theta)d\gamma_\alpha(\lambda,\mu) \tag{2.13}$$

The heat kernel relates to Gaussians via:

$$\mathrm{E}_a^\alpha(y,\theta) = \int_{\widehat{\mathbb{K}}} e^{-a(\lambda^2+\mu^2+(\alpha+1)^2)}\varphi_{\lambda,\mu}(y,\theta)d\gamma_\alpha(\lambda,\mu), \tag{2.14}$$

with more general heat functions:

$$W_{k,j}^\alpha(a,(y,\theta)) = i^k \int_{\widehat{\mathbb{K}}} \lambda^k(-\mu)^{2j} e^{-a(\lambda^2+\mu^2+(\alpha+1)^2)}\varphi_{\lambda,\mu}(y,\theta)d\gamma_\alpha(\lambda,\mu). \tag{2.15}$$

3 Miyachi-type theorem for the generalized Fourier transform

To establish our main result, we first derive kernel estimates on $\mathbb{C}^2$.

Proposition 3.1. *For all $(\lambda,\mu) \in \mathbb{C}^2$ and $(y,\theta) \in \mathbb{K}$,*

$$|\varphi_{\lambda,\mu}(y,\theta)| \leq C(1+y)e^{(|\Im\mu|-(\alpha+1))y}e^{|\Im\lambda|(|\theta|+\pi)}, \tag{3.1}$$

where $C > 0$. Moreover, since $y \geq 0$,

$$|\varphi_{\lambda,\mu}(y,\theta)| \leq Ce^{|\Im\mu|y+|\Im\lambda|(|\theta|+\pi)}. \tag{3.2}$$

Proof. By [11, Lemma 2.3], for $\lambda = \mu = 0$,

$$\varphi_{0,0}(y,\theta) \leq C(1+y)e^{-(\alpha+1)y}.$$

Using $|\cos(\mu s)| \leq e^{|\Im\mu|s}$ and the integral representation (2.5),

$$|\varphi_{\lambda,\mu}(y,\theta)| \leq Ce^{|\Im\mu|y+|\Im\lambda|(|\theta|+\pi)}\varphi_{0,0}(y,\theta),$$

since $\omega \in [-\pi,\pi]$. This proves (3.1). Inequality (3.2) follows by analyzing the decay of $f(y) = (1+y)e^{-(\alpha+1)y}$ on $[0,+\infty[$. □

We now state a Phragmén-Lindelöf-type lemma sufficient for our needs:

Lemma 3.2 ([10])**.** *Let h be entire on $\mathbb{C}^2$. Suppose there exist constants $C, B > 0$ such that*

$$|h(z_1,z_2)| \leq Ce^{B((\Re z_1)^2+(\Re z_2)^2)} \quad \textit{and} \quad \int_{\mathbb{R}^2} \log^+|h(x,y)|dx\,dy < \infty.$$

Then h is constant.

Lemma 3.3. *Let $p,q \in [1,+\infty]$ and f be measurable on $\mathbb{K}$ satisfying*

$$e^{a(y^2+(|\theta|+\pi)^2)+2(\alpha+1)y}f \in \mathrm{L}^p_\alpha(\mathbb{K}) + \mathrm{L}^q_\alpha(\mathbb{K}), \quad a > 0. \tag{3.3}$$

Then $\mathcal{F}_\alpha(f)$ is well-defined and entire on $\mathbb{C}^2$. Moreover, for all $(\lambda,\mu) \in \mathbb{C}^2$,

$$|\mathcal{F}_\alpha(f)(\lambda,\mu)| \leq Ce^{\frac{|\Im\lambda|^2+|\Im\mu|^2}{4a}}. \tag{3.4}$$

Proof. The function $(\lambda,\mu) \mapsto \varphi_{-\lambda,\mu}(y,\theta)$ is entire by (2.3) and (2.4). Using Proposition 3.1,

$$|f(y,\theta)\varphi_{-\lambda,\mu}(y,\theta)m_\alpha(y,\theta)| \leq Ce^{|\Im\lambda|(|\theta|+\pi)+|\Im\mu|y}|f(y,\theta)|m_\alpha(y,\theta).$$

By (3.3), there exist $f_1 \in \mathrm{L}^p_\alpha(\mathbb{K})$ and $f_2 \in \mathrm{L}^q_\alpha(\mathbb{K})$ such that

$$|f\varphi_{-\lambda,\mu}m_\alpha| \leq \sum_{k=1}^{2} g_k(\lambda,\mu,y,\theta),$$

where

$$g_k(\lambda,\mu,y,\theta) = Ce^{|\Im\lambda|(|\theta|+\pi)+|\Im\mu|y}e^{-a(y^2+(|\theta|+\pi)^2)-2(\alpha+1)y}|f_k(y,\theta)|m_\alpha(y,\theta).$$

Observe that

$$|\Im\lambda|(|\theta|+\pi) + |\Im\mu|y - a(y^2+(|\theta|+\pi)^2) = -\Delta_{\lambda,\mu}(y,\theta) + \frac{|\Im\lambda|^2+|\Im\mu|^2}{4a},$$

where

$$\Delta_{\lambda,\mu}(y,\theta) = \left(\sqrt{a}y - \frac{|\Im\mu|}{2\sqrt{a}}\right)^2 + \left(\sqrt{a}(|\theta|+\pi) - \frac{|\Im\lambda|}{2\sqrt{a}}\right)^2 \geq 0.$$

Thus,

$$g_k(\lambda,\mu,y,\theta) \leq Ce^{\frac{|\Im\lambda|^2+|\Im\mu|^2}{4a}} e^{-\Delta_{\lambda,\mu}(y,\theta)}|f_k(y,\theta)|e^{-2(\alpha+1)y}m_\alpha(y,\theta).$$

For a compact $K \subset \mathbb{C}^2$, there exists $(\lambda_0,\mu_0) \in K$ such that

$$\min_{(\lambda,\mu)\in K} \Delta_{\lambda,\mu}(y,\theta) = \Delta_{\lambda_0,\mu_0}(y,\theta).$$

Since $e^{\frac{|\Im\lambda|^2+|\Im\mu|^2}{4a}}$ is bounded on K,

$$g_k(\lambda,\mu,y,\theta) \leq G_k(y,\theta) = Ce^{-\Delta_{\lambda_0,\mu_0}(y,\theta)}|f_k(y,\theta)|e^{-2(\alpha+1)y}m_\alpha(y,\theta).$$

To show $\mathcal{F}_\alpha f$ is entire, it suffices to prove $G_k \in \mathrm{L}^1_\alpha(\mathbb{K})$. By Hölder's inequality,

$$\int_{\mathbb{K}} |G_1(y,\theta)|dy\,d\theta \leq C \left\| f_1 e^{-\frac{2(\alpha+1)y}{p}} \right\|_{p,m_\alpha} \left(\int_{\mathbb{K}} e^{-\Delta_{\lambda_0,\mu_0}(y,\theta)p'} e^{-2(\alpha+1)y} m_\alpha(y,\theta) dy\,d\theta \right)^{\frac{1}{p'}}.$$

Using (2.8), $e^{-2(\alpha+1)y}m_\alpha(y,\theta) \leq C$, so

$$\int_{\mathbb{K}} |G_1(y,\theta)|dy\,d\theta \leq C \|f_1\|_{p,m_\alpha} \left(\int_{\mathbb{K}} e^{-\Delta_{\lambda_0,\mu_0}(y,\theta)p'} dy\,d\theta \right)^{\frac{1}{p'}} < \infty.$$

Similarly, for q' conjugate to q,

$$\int_{\mathbb{K}} |G_2(y,\theta)|dy\,d\theta < \infty.$$

Thus $\mathcal{F}_\alpha f$ is entire.

To prove (3.4), apply Hölder's inequality to g_1 and g_2:

$$|\mathcal{F}_\alpha f(\lambda,\mu)| \leq Ce^{\frac{|\Im\lambda|^2+|\Im\mu|^2}{4a}} \left(\|f_1\|_{p,m_\alpha} \left(\int_{\mathbb{K}} e^{-\Delta_{\lambda,\mu}(y,\theta)p'} dy\,d\theta \right)^{\frac{1}{p'}} \right.$$
$$\left. + \|f_2\|_{q,m_\alpha} \left(\int_{\mathbb{K}} e^{-\Delta_{\lambda,\mu}(y,\theta)q'} dy\,d\theta \right)^{\frac{1}{q'}} \right) \leq Ce^{\frac{|\Im\lambda|^2+|\Im\mu|^2}{4a}} \left(\|f_1\|_{p,m_\alpha} + \|f_2\|_{q,m_\alpha} \right). \quad \square$$

Remark 3.4. *Condition* (3.3) *implies* $f \in \mathrm{L}^1_\alpha(\mathbb{K})$. *Indeed, by* (2.8) *and Hölder's inequality,*

$$\int_{\mathbb{K}} |f(y,\theta)|dm_\alpha(y,\theta) \leq \left\| f_1 e^{-\frac{2(\alpha+1)y}{p}} \right\|_{p,m_\alpha} \left(\int_{\mathbb{K}} e^{-ap'(y^2+(|\theta|+\pi)^2)} e^{-2(\alpha+1)y} dm_\alpha \right)^{\frac{1}{p'}}$$
$$+ \left\| f_2 e^{-\frac{2(\alpha+1)y}{q}} \right\|_{q,m_\alpha} \left(\int_{\mathbb{K}} e^{-aq'(y^2+(|\theta|+\pi)^2)} e^{-2(\alpha+1)y} dm_\alpha \right)^{\frac{1}{q'}}$$
$$\lesssim \|f_1\|_{p,m_\alpha} + \|f_2\|_{q,m_\alpha} < \infty.$$

Theorem 3.5. *Let $a, b, \beta > 0$, $p, q \in [1, \infty]$, and f be measurable on $\mathbb{R}^2$, even in the first variable, satisfying*

$$e^{a(y^2+(|\theta|+\pi)^2)+2(\alpha+1)y} f \in \mathrm{L}^p_\alpha(\mathbb{K}) + \mathrm{L}^q_\alpha(\mathbb{K})$$

and

$$\int_{\mathbb{R}^2} \log^+ \frac{|\mathcal{F}_\alpha f(\lambda,\mu)| e^{b(\mu^2+\lambda^2)}}{\beta} d\lambda\, d\mu < \infty. \tag{3.5}$$

Then:

- *If $ab > \frac{1}{4}$, then $f = 0$ a.e.*
- *If $ab = \frac{1}{4}$, then $f = C\mathrm{E}^\alpha_{\frac{1}{4a}}$ with $|C| \le \beta$, where $\mathrm{E}^\alpha_{\frac{1}{4a}}$ is the heat kernel* (2.14)*.*

Proof. Define $h(\lambda,\mu) = e^{\frac{\lambda^2+\mu^2}{4a}} \mathcal{F}_\alpha f(\lambda,\mu)$. By Lemma 3.3, h is entire and satisfies

$$|h(\lambda,\mu)| \le C e^{\frac{(\Re\lambda)^2+(\Re\mu)^2}{4a}}.$$

Now consider

$$\int_{\mathbb{R}^2} \log^+ |h(\lambda,\mu)| d\lambda\, d\mu = \int_{\mathbb{R}^2} \log^+ \left(|\mathcal{F}_\alpha f(\lambda,\mu)| e^{b(\mu^2+\lambda^2)} e^{(\frac{1}{4a}-b)(\lambda^2+\mu^2)} \right) d\lambda\, d\mu.$$

- **Case $ab > \frac{1}{4}$:** Since $e^{(\frac{1}{4a}-b)(\lambda^2+\mu^2)} \le 1$ and $\int_{\mathbb{R}^2} e^{(\frac{1}{4a}-b)(\lambda^2+\mu^2)} d\lambda\, d\mu < \infty$,

  $$\int_{\mathbb{R}^2} \log^+ |h(\lambda,\mu)| d\lambda\, d\mu < \infty.$$

  Lemma 3.2 implies h is constant, so $\mathcal{F}_\alpha f = Ce^{-\frac{\lambda^2+\mu^2}{4a}}$. Condition (3.5) forces $C = 0$ when $ab > \frac{1}{4}$, so $f = 0$ by injectivity of $\mathcal{F}_\alpha$.

- **Case $ab = \frac{1}{4}$:** Then

  $$\int_{\mathbb{R}^2} \log^+ |h(\lambda,\mu)| d\lambda\, d\mu \le \int_{\mathbb{R}^2} \log^+ \frac{|\mathcal{F}_\alpha f(\lambda,\mu)| e^{b(\mu^2+\lambda^2)}}{\beta} d\lambda\, d\mu < \infty.$$

  Lemma 3.2 gives $\mathcal{F}_\alpha f = Ce^{-\frac{\lambda^2+\mu^2}{4a}}$, and (3.5) implies $|C| \le \beta$. Inverting $\mathcal{F}_\alpha$ yields $f = C\mathrm{E}^\alpha_{\frac{1}{4a}}$. □

Corollary 3.6. *Let $a, b > 0$, $p, q \in [1, \infty]$, $1 \le r < \infty$, and f measurable on $\mathbb{R}^2$, even in the first variable, satisfying*

$$e^{a(y^2+(|\theta|+\pi)^2)+2(\alpha+1)y} f \in \mathrm{L}^p_\alpha(\mathbb{K}) + \mathrm{L}^q_\alpha(\mathbb{K})$$

and

$$\int_{\mathbb{R}^2} e^{br(\mu^2+\lambda^2)} |\mathcal{F}_\alpha f(\lambda,\mu)|^r d\lambda\, d\mu < \infty. \tag{3.6}$$

If $ab \ge \frac{1}{4}$, then $f = 0$ a.e.

Proof. Since $\log^+ x \leq x$ for $x > 0$,

$$\log^+ \frac{|\mathcal{F}_\alpha f(\lambda,\mu)|e^{b(\mu^2+\lambda^2)}}{\beta} \leq \left(\frac{|\mathcal{F}_\alpha f(\lambda,\mu)|e^{b(\mu^2+\lambda^2)}}{\beta}\right)^r.$$

Choosing $\beta = 1$, (3.6) implies

$$\int_{\mathbb{R}^2} \log^+|\mathcal{F}_\alpha f(\lambda,\mu)|e^{b(\mu^2+\lambda^2)} d\lambda\, d\mu < \infty.$$

By Theorem 3.5, $f = 0$ if $ab > \frac{1}{4}$. If $ab = \frac{1}{4}$, $f = C\mathrm{E}^\alpha_{\frac{1}{4a}}$ with $|C| \leq 1$, but (3.6) holds only if $C = 0$. □

Theorem 3.7 (Cowling-Price Type)**.** *Let f be measurable on $\mathbb{R}^2$, even in the first variable, with $a, b > 0$, $1 \leq p, q < \infty$, satisfying*

$$e^{a(y^2+(|\theta|+\pi)^2)+2(\alpha+1)y} f \in \mathrm{L}^p_\alpha(\mathbb{K})$$

and

$$e^{b(\mu^2+\lambda^2)}|\mathcal{F}_\alpha f(\lambda,\mu)| \in \mathrm{L}^q_\alpha(\widehat{\mathbb{K}}). \tag{3.7}$$

If $ab \geq \frac{1}{4}$, then $f = 0$ a.e.

Proof. Since $\mathrm{L}^p(\mathbb{K}) \subset \mathrm{L}^p(\mathbb{K}) + \mathrm{L}^q(\mathbb{K})$, (3.3) holds. From (3.7) and (2.10),

$$\int_{\mathbb{L}} e^{bq(\mu^2+\lambda^2)}|\mathcal{F}_\alpha f(\lambda,\mu)|^q |C_1(\lambda,\mu)|^{-2} d\lambda\, d\mu < \infty$$

implies

$$\int_{\mathbb{R}^2} e^{bq(\mu^2+\lambda^2)}|\mathcal{F}_\alpha f(\lambda,\mu)|^q d\lambda\, d\mu < \infty$$

by the evenness of $\mathcal{F}_\alpha f$ in μ. Corollary 3.6 with $r = q$ completes the proof. □

Remark 3.8. *This work establishes a Cowling-Price-type uncertainty principle (Theorem 3.7) within the Miyachi framework. It is instructive to compare this result with those derived from the Beurling-Hörmander framework, such as the one found in [12]. The two approaches are distinct in their hypotheses and their conclusions, particularly at the critical exponent $ab = 1/4$.*

(1) ***Comparison of hypotheses:***

- *In the Miyachi framework requires strict exponential decay without polynomial weights:*

$$e^{a(y^2+(|\theta|+\pi)^2)+2(\alpha+1)y} f \in \mathrm{L}^p_\alpha(\mathbb{K}), \quad e^{b(\lambda^2+\mu^2)}|\mathcal{F}_\alpha f| \in \mathrm{L}^q_\alpha(\widehat{\mathbb{K}}).$$

- *In the Beurling-Hörmander framework [12] permits a tempered decay, allowing polynomial weights:*

$$\int_{\mathbb{K}} \frac{|f|e^{a|y+\pi,\theta|^2}}{(1+|y,\theta|)^N}\, dm_\alpha < \infty, \quad \int_{\widehat{\mathbb{K}}} \frac{|\mathcal{F}_\alpha f|e^{b|\lambda,\mu|^2}}{(1+|\lambda,\mu|)^N}\, d\gamma_\alpha < \infty.$$

(2) ***Comparison of conclusions at*** $ab = 1/4$***:***

- *Under the Miyachi hypotheses, the conclusion is a sharp uniqueness result:* $f = 0$ *is the only function that satisfies the conditions.*
- *Under the Beurling-Hörmander hypotheses, the conclusion is a characterization result: the function* f *must be a finite linear combination of heat kernel modes:*

$$f(y,\theta) = \sum_{k+j<N-1} a_{k,j}\mathcal{W}^\alpha_{k,j}(y,\theta),$$

where $\mathcal{W}^\alpha_{k,j}$ *are defined by relation* (2.15)*.*

4 Local uncertainty principle and numerical study

In this section, we provide a local uncertainty principle of Faris-type for the generalized Fourier transform $\mathcal{F}_\alpha$. This result quantifies the impossibility of a function f and its transform $\mathcal{F}_\alpha(f)$ being simultaneously concentrated on sets of finite measure. We derive an inequality bounding the concentration of $\mathcal{F}_\alpha(f)$ on a set F by the spatial dispersion of f. We then compute the optimal constant numerically, quantifying the precise trade-off between spatial and spectral localization.

4.1 Faris-type local uncertainty principle

Faris local uncertainty theorem for the generalized Fourier $\mathcal{F}_\alpha$ states

Theorem 4.1. *If* $1 < p \leq 2$, $q = \frac{p}{p-1}$ *and* $0 < a < \frac{2}{q}$ *then for all* $f \in \mathrm{L}^p_\alpha(\mathbb{K})$ *and all measurable subset* $F \subset \widehat{\mathbb{K}}$ *satisfying* $0 < \gamma_\alpha(F) < +\infty$*,*

$$\left(\int_F |\mathcal{F}_\alpha f(\lambda,\mu)|^q \, d\gamma_\alpha(\lambda,\mu)\right)^{\frac{1}{q}} \leq K_{\alpha,a,q}\left(\gamma_\alpha(F)\right)\left(\int_{\mathbb{K}} |(y,\theta)|^p \, |f(y,\theta)|^p \, dm_\alpha(y,\theta)\right)^{\frac{1}{p}}, \tag{4.1}$$

where $K_{\alpha,a,q}$ *is a constant which depend on the measure of the subset* F*,* $\gamma_\alpha(F)$*.*

Proof. Let F be a measurable subset of $\widehat{\mathbb{K}}$. Let us denote B the Euclidean ball of radius $r > 0$.

$$B = \left\{(y,\theta) \in \mathbb{K}, \quad |(y,\theta)| = \sqrt{y^2+\theta^2} < r\right\}.$$

We get

$$\|\mathcal{F}_\alpha(f)\ \chi_F\|_{q,\gamma_\alpha} \leq \ \|\mathcal{F}_\alpha(f\chi_B)\ \chi_F\|_{q,\gamma_\alpha} + \|\mathcal{F}_\alpha(f\chi_{B^c})\ \chi_F\|_{q,\gamma_\alpha}.$$

On the other hand

$$\begin{aligned}\|\mathcal{F}_\alpha(f\chi_B)\ \chi_F\|^q_{q,\gamma_\alpha} &= \int_{\widehat{\mathbb{K}}} |\mathcal{F}_\alpha(f\chi_B)(\lambda,\mu)\ \chi_F(\lambda,\mu)|^q\ \ d\gamma_\alpha(\lambda,\mu)\\ &\leq \|\mathcal{F}_\alpha(f\chi_B)\|^q_{\infty,\gamma_\alpha}\int_{\widehat{\mathbb{K}}}\chi_F(\lambda,\mu)\ d\gamma_\alpha(\lambda,\mu).\end{aligned}$$

Then

$$\|\mathcal{F}_\alpha(f\chi_B)\ \chi_F\|_{q,\gamma_\alpha} \leq (\gamma_\alpha(F))^{\frac{1}{q}}\ \ \|\mathcal{F}_\alpha(f\chi_B)\|_{\infty,\gamma_\alpha}. \tag{4.2}$$

Moreover

$$\|\mathcal{F}_\alpha(f\chi_{B^c})\ \chi_F\|_{q,\gamma_\alpha} \leq \|\mathcal{F}_\alpha(f\chi_{B^c})\|_{q,\gamma_\alpha}. \tag{4.3}$$

According to relations (4.2) and (4.3), we obtain

$$\|\mathcal{F}_\alpha(f)\ \chi_F\|_{q,\gamma_\alpha} \leq (\gamma_\alpha(F))^{\frac{1}{q}}\ \|\mathcal{F}_\alpha(f\chi_B)\|_{\infty,\gamma_\alpha} + \|\mathcal{F}_\alpha(f\chi_{B^c})\|_{q,\gamma_\alpha}.$$

Therefore (2.9) and (2.12) yield to

$$\|\mathcal{F}_\alpha f\chi_F\|_{q,\gamma_\alpha} \leq (\gamma_\alpha(F))^{\frac{1}{q}}\ \|f\chi_B\|_{1,m_\alpha} + \|f\chi_{B^c}\|_{p,m_\alpha}. \tag{4.4}$$

Using Hölder inequality, we get

$$\|f\chi_B\|_{1,m_\alpha} \leq \left(\int_{\mathbb{K}} |f(y,\theta)|^p\ \ |(y,\theta)|^{ap}\,dm_\alpha(y,\theta)\right)^{\frac{1}{p}}\left(\int_{\mathbb{K}} |(y,\theta)|^{-aq}\,\chi_B(y,\theta)dm_\alpha(y,\theta)\right)^{\frac{1}{q}}.$$

Applying polar coordinates we get

$$\int_{\mathbb{K}} \frac{\chi_B(y,\theta)}{\|(y,\theta)\|^{aq}}dy\,d\theta = \frac{\pi}{2-qa}r^{2-qa}.$$

Since

$$\int_{\mathbb{K}} |(y,\theta)|^{-aq}\,\chi_B(y,\theta)dm_\alpha(y,\theta) \leq 2^{2(\alpha+1)}e^{2(\alpha+1)r}\int_{\mathbb{K}}\frac{\chi_B(y,\theta)}{|(y,\theta)|^{aq}}dy\,d\theta$$

then we deduce that

$$\|f\chi_B\|_{1,m_\alpha} \leq C_{\alpha,a,q}e^{\frac{2}{q}(\alpha+1)r}\ r^{\frac{2}{q}-a}\ \|\ \ |(y,\theta)|^a\,f\|_{p,m_\alpha}, \tag{4.5}$$

where

$$C_{\alpha,a,q} = \left(\frac{\pi\,2^{2(\alpha+1)}}{2-qa}\right)^{\frac{1}{q}}. \tag{4.6}$$

According to relations (4.5) and (4.4) and the fact that

$$\|f\chi_{B^c}\|_{p,m_\alpha}^p \leq \| \, |(y,\theta)|^a \, f\|_{p,m_\alpha}^p \| \, |(y,\theta)|^{-ap} \, \chi_B^c \|_{\infty,m_\alpha} \leq r^{-ap} \| \, |(y,\theta)|^a \, f\|_{p,m_\alpha}^p$$

we conclude that

$$\|\mathcal{F}_\alpha f \chi_F\|_{q,\gamma_\alpha} \leq g(r) \| \, |(y,\theta)|^a \, f\|_{p,m_\alpha}, \tag{4.7}$$

where g is a function from $]0,+\infty[$ into $\mathbb{R}$, given by

$$g(r) = Ae^{br} r^c + r^{-a}, \tag{4.8}$$

where

$$A = C_{\alpha,a,q}\big(\gamma_\alpha(F)\big)^{\frac{1}{q}} > 0, \quad b = \frac{2}{q}(\alpha+1) > 0, \quad c = \frac{2}{q} - a > 0. \tag{4.9}$$

The function g is continuous and coercive on $]0,+\infty[$ since

$$\lim_{r\to 0^+} g(r) = +\infty \quad \text{and} \quad \lim_{r\to +\infty} g(r) = +\infty.$$

Thus, g attains a minimum. Differentiating, we get

$$g'(r) = Ae^{br} r^{c-1}(br + c) - ar^{-a-1}. \tag{4.10}$$

Setting $g'(r) = 0$ is equivalent to solving

$$h(r) := Ae^{br} r^{c+a}(br + c) - a.$$

Since $c + a = \frac{2}{q} > 0$, the function h is continuous and strictly increasing on $]0,+\infty[$, with

$$\lim_{r\to 0^+} h(r) = -a < 0, \quad \lim_{r\to +\infty} h(r) = +\infty.$$

Therefore, there exists a unique $r^* > 0$ such that $h(r^*) = a$, so $g'(r^*) = 0$. Since g is coercive, this critical point is the unique global minimum of g. Let us denote this unique minimum of g by

$$K_{\alpha,a,q}\big(\gamma_\alpha(F)\big) := \min_{r>0} g(r). \tag{4.11}$$

Finally, relation (4.7) yields (4.1), completing the proof. □

4.2 Numerical study of the optimal constant

This section presents a comprehensive numerical investigation of the function $g(r)$ defined in Equation (4.7), which determines the optimal constant $K_{\alpha,a,q}(\gamma_\alpha(F))$ in Theorem 4.1. We recall that

$$g(r) = Ae^{br}r^c + r^{-a},$$

where the parameters are defined in relation (4.9).

To find the global minimizer $r^* > 0$ of $g(r)$, we implement the Newton-Raphson method to solve the equation $g'(r) = 0$. The first and second derivatives of $g(r)$ are:

$$\begin{aligned} g'(r) &= Ae^{br}r^{c-1}(br + c) - ar^{-a-1}, \\ g''(r) &= Ae^{br}r^{c-2}\left[(br + c)^2 + (c-1)(br + c) - c\right] + a(a+1)r^{-a-2}. \end{aligned}$$

The Newton-Raphson iteration scheme is given by:

$$r_{n+1} = r_n - \frac{g'(r_n)}{g''(r_n)}.$$

We initialize the algorithm with $r_0 = 0.1$ and use a convergence criterion of

$$|r_{n+1} - r_n| < 10^{-6}.$$

- **Numerical computation.** We choose specific parameter values:

$$\begin{cases} p = 1.5 & \rightarrow \text{ so } q = 3, \\ \alpha = 0.5, & \\ a = 0.5 & \rightarrow \text{ satisfies } a < \frac{2}{q}, \\ \gamma_\alpha(F) = 1 & \rightarrow \text{ for simplicity.} \end{cases}$$

  Now compute the constants:

$$\begin{cases} A = C_{\alpha,a,q} \cdot \left(\gamma_\alpha(F)\right)^{1/3} = \left(\frac{\pi \cdot 2^{2(0.5+1)}}{2-3\cdot 0.5}\right)^{1/3} = \left(\frac{\pi\cdot 2^3}{0.5}\right)^{1/3} \approx (50.265)^{1/3} \approx 3.691, \\ b = \frac{2}{3}(0.5+1) = 1, \\ c = \frac{2}{3} - 0.5 \approx 0.1667. \end{cases}$$

  Thus, the function simplifies to

$$g(r) \approx 3.691 \cdot e^r \cdot r^{0.1667} + r^{-0.5}.$$

The Newton-Raphson method converges rapidly to the solution, as demonstrated in Table 1.

Table 1: Newton-Raphson iterations.

Iteration (n)	r_n	$g'(r_n)$
0	0.100000	-12.456
1	0.157832	-2.891
2	0.180214	-0.327
3	0.183105	-0.006
4	0.183127	-0.000012
5	0.183127	≈ 0

The algorithm converges in 5 iterations to $r^* \approx 0.1831$, yielding the minimum value $g(r^*) \approx 5.677$. The following Figure 1 illustrates the behavior of $g(r)$, confirming the existence of a unique minimum where the term r^{-a} dominates as $r \to 0^+$ and the term $Ae^{br}r^c$ dominates as $r \to +\infty$.

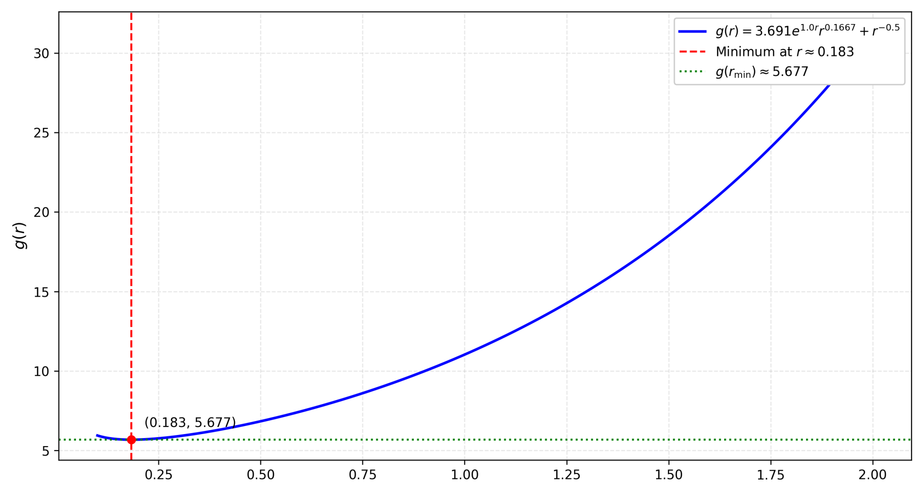


Figure 1: Behavior of $g(r)$ for $p = 1.5$, $\alpha = 0.5$, $a = 0.5$.

4.3 Asymptotic behavior of $K_{\alpha,a,q}(\gamma_\alpha(F))$

In the previous numerical study, the measure of the frequency set was fixed at $\gamma_\alpha(F) = 1$ to compute a specific value for the optimal constant. We now analyze the behavior of $K_{\alpha,a,q}(\gamma_\alpha(F))$ over the full range of its domain, particularly in the asymptotic regimes where $\gamma_\alpha(F) \to 0^+$ or $\gamma_\alpha(F) \to +\infty$. This analysis reveals the intrinsic scaling properties of the uncertainty principle and provides practical insight into the trade-off between spatial and frequency localization governed by the parameters α, a, p.

- **Behavior as $\gamma_\alpha(F) \to 0^+$**

  When $\gamma_\alpha(F) \to 0^+$, by relation (4.9) we have $A \to 0^+$. From (4.8), the dominant term in $g(r)$ becomes r^{-a}, so we expect the minimizing r^* to grow. We have

$$g'(r) = 0 \Longleftrightarrow Ae^{br}r^{c-1}(br+c) - ar^{-a-1} = 0 \Longleftrightarrow Ae^{br}r^{c-1}(br+c) = ar^{-a-1}.$$

  Applying logarithms, we get

$$\ln A + br + (c+a)\ln r + \ln(br+c) = \ln a.$$

  For small A, the term br dominates, so we approximate:

$$br^* \approx \ln\left(\frac{a}{A}\right) \implies r^* \approx \frac{1}{b}\ln\left(\frac{a}{A}\right).$$

  By substituting into $g(r^*)$, we obtain

$$g(r^*) \approx Ae^{br^*}(r^*)^c + (r^*)^{-a} \approx a(r^*)^{-a} \approx a\left(\frac{b}{\ln(a/A)}\right)^a.$$

  Since A is proportional to $\left(\gamma_\alpha(F)\right)^{\frac{1}{q}}$, we derive

$$K_{\alpha,a,q}(\gamma_\alpha(F)) \sim a\left(\frac{b}{\ln\left(\frac{a}{C_{\alpha,a,q}\gamma_\alpha(F)^{1/q}}\right)}\right)^a \quad \text{as} \quad \gamma_\alpha(F) \to 0^+,$$

  where $C_{\alpha,a,q}$ is given by (4.6).

- **Behavior as $\gamma_\alpha(F) \to +\infty$**

  Since $\gamma_\alpha(F) \to +\infty$, then $A \to +\infty$. On the other hand, the dominant term in $g(r)$ is $Ae^{br}r^c$, so we expect the minimizing r^* to shrink. The equation $g'(r) = 0$ gives us

$$Ae^{br}r^{c-1}(br+c) = ar^{-a-1}.$$

  For large A, the left hand side dominates, so we balance terms by taking $r^* \to 0^+$. Assume r^* is small and expand $e^{br} \approx 1 + br$. Then

$$A(1+br^*)(r^*)^{c-1}(br^*+c) \approx a(r^*)^{-a-1}.$$

  Yields to

$$Ac(r^*)^{c-1} \approx a(r^*)^{-a-1} \implies (r^*)^{c+a} \approx \frac{a}{Ac}.$$

Thus:

$$r^* \approx \left(\frac{a}{Ac}\right)^{\frac{1}{c+a}} = \left(\frac{a}{C_{\alpha,a,q} c \gamma_\alpha(F)^{1/q}}\right)^{\frac{1}{c+a}}.$$

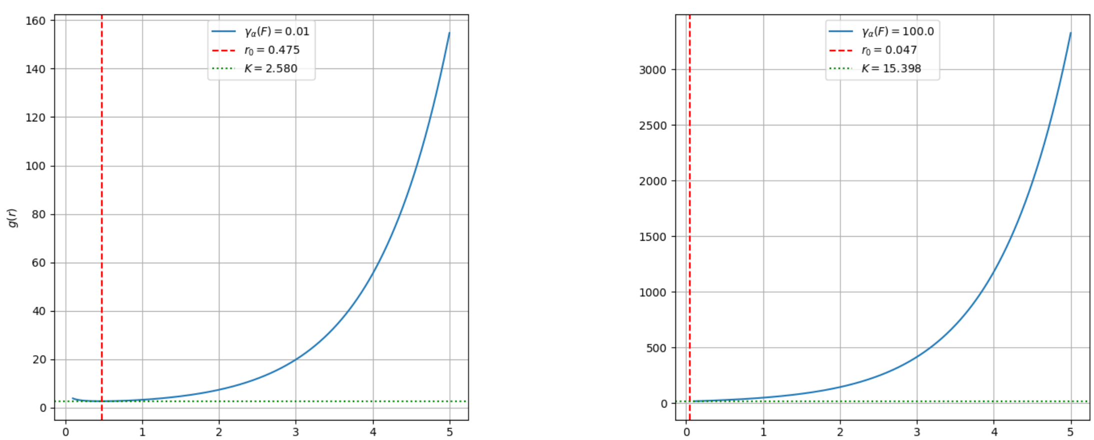
Substituting into $g(r^*)$:

$$g(r^*) \approx A e^{br^*} (r^*)^c + (r^*)^{-a} \approx A(r^*)^c + (r^*)^{-a}.$$

Since $r^* \to 0^+$, the second term dominates:

$$K_{\alpha,a,q}(\gamma_\alpha(F)) \approx (r^*)^{-a} \approx \left(\frac{C_{\alpha,a,q} c \gamma_\alpha(F)^{1/q}}{a}\right)^{\frac{a}{c+a}}.$$

This contrasting behavior is illustrated in Figure 2, which shows the function $g(r)$ for extreme values of $\gamma_\alpha(F)$. The left panel shows the slow logarithmic decay for $\gamma_\alpha(F) \to 0^+$, while the right panel demonstrates the power-law growth for $\gamma_\alpha(F) \to +\infty$. The vertical dashed lines indicate the minimizing radius r^* in each case.



(a) Behavior of $g(r)$ for small $\gamma_\alpha(F)$.

(b) Behavior of $g(r)$ for large $\gamma_\alpha(F)$.

Figure 2

- **Numerical computation** The following table presents numerical values of the minimizing radius r_0 and the optimal constant $K_{\alpha,a,q}(\gamma_\alpha(F))$ for different values of $\gamma_\alpha(F)$, using the parameters:

$$p = 1.5, \quad \alpha = 0.5, \quad a = 0.5.$$

Table 2: Numerical values of the optimal radius r^* and constant $K_{\alpha,a,q}$.

$\gamma_\alpha(F)$	A	r^*	$K_{\alpha,a,q}$
10^{-6}	0.037	13.12	0.276
10^{-5}	0.079	11.72	0.295
10^{-4}	0.171	10.32	0.316
10^{-3}	0.369	8.92	0.341
10^{-2}	0.795	7.52	0.372
10^{-1}	1.713	6.12	0.404
1	3.691	0.183	5.677
10	7.937	0.089	12.309
10^2	17.088	0.042	24.891
10^3	36.913	0.020	48.712
10^4	79.370	0.009	94.868

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References

[1] F. Chouchene, R. Daher, T. Kawazoe, and H. Mejjaoli, "Miyachi's theorem for the Dunkl transform," *Integral Transforms Spec. Funct.*, vol. 22, no. 3, pp. 167–173, 2011, doi: 10.1080/10652469.2010.505029.

[2] M. Cowling and J. F. Price, "Generalisations of Heisenberg's inequality," in *Harmonic analysis (Cortona, 1982)*, ser. Lecture Notes in Math. Springer, Berlin, 1983, vol. 992, pp. 443–449, doi: 10.1007/BFb0069174.

[3] R. Daher and T. Kawazoe, "Generalized Hardy's theorem for the Jacobi transform," *Hiroshima Math. J.*, vol. 36, no. 3, pp. 331–337, 2006.

[4] R. Daher, A. Khadari, and S. Omri, "Uncertainty principle for the spherical mean operator," *J. Math. Inequal.*, vol. 8, no. 3, pp. 475–487, 2014, doi: 10.7153/jmi-08-35.

[5] W. G. Faris, "Inequalities and uncertainty principles," *J. Mathematical Phys.*, vol. 19, no. 2, pp. 461–466, 1978, doi: 10.1063/1.523667.

[6] M. Flensted-Jensen, "Paley-Wiener type theorems for a differential operator connected with symmetric spaces," *Ark. Mat.*, vol. 10, pp. 143–162, 1972, doi: 10.1007/BF02384806.

[7] M. Flensted-Jensen, "Spherical functions on a simply connected semisimple Lie group. II. The Paley-Wiener theorem for the rank one case," *Math. Ann.*, vol. 228, no. 1, pp. 65–92, 1977, doi: 10.1007/BF01360773.

[8] G. H. Hardy, "A Theorem Concerning Fourier Transforms," *J. London Math. Soc.*, vol. 8, no. 3, pp. 227–231, 1933, doi: 10.1112/jlms/s1-8.3.227.

[9] L. Kamoun and R. Laffi, "Benedicks and Donoho-Stark type theorems," *Turkish J. Math.*, vol. 44, no. 5, pp. 1724–1735, 2020, doi: 10.3906/mat-2005-57.

[10] H. Khaled and O. Slim, "An L^p-L^q version of Miyachi's theorem for the Riemann-Liouville operator," *Indian J. Pure Appl. Math.*, vol. 46, no. 2, pp. 121–138, 2015, doi: 10.1007/s13226-015-0125-8.

[11] T. Koornwinder, "A new proof of a Paley-Wiener type theorem for the Jacobi transform," *Ark. Mat.*, vol. 13, pp. 145–159, 1975, doi: 10.1007/BF02386203.

[12] R. Laffi and S. Negzaoui, "Uncertainty principle related to Flensted-Jensen partial differential operators," *Asian-Eur. J. Math.*, vol. 14, no. 1, 2021, Art. ID 2150004.

[13] A. Miyachi, "A generalization of theorem of hardy," in *Harmonic Analysis Seminar held at Izunagaoka*, Shizuoka-Ken, Japan, 1997, pp. 44–51.

[14] S. Negzaoui, "Beurling-Hörmander's theorem related to Bessel-Struve transform," *Integral Transforms Spec. Funct.*, vol. 27, no. 9, pp. 685–697, 2016, doi: 10.1080/10652469.2016.1188814.

[15] J. F. Price, "Inequalities and local uncertainty principles," *J. Math. Phys.*, vol. 24, no. 7, pp. 1711–1714, 1983, doi: 10.1063/1.525916.

[16] K. Trimèche, "Opérateurs de permutation et analyse harmonique associés à des opérateurs aux dérivées partielles," *J. Math. Pures Appl. (9)*, vol. 70, no. 1, pp. 1–73, 1991.

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Absolutely continuous spectrum preservation: A new proof for unitary operators under finite-rank multiplicative perturbations

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ABSTRACT

We will provide a new proof of the Birman-Krein theorem for unitary operators multiplicatively perturbed by finite-rank operators, which is nothing more than the Kato-Rosenblum theorem, but instead of self-adjoint operators. In other words, U is a unitary operator and X is a unitary operator given by a finite rank perturbation of the identity, *i.e.*, $X = \mathbf{1} + W$ with W finite rank. We show that U and its perturbed version UX (or XU) are unitarily equivalent on their absolutely continuous subspaces.

RESUMEN

Entregamos una nueva demostración del teorema de Birman-Krein para operadores unitarios perturbados multiplicativamente por operadores de rango finito, que no es más que el teorema de Kato-Rosenblum, pero en lugar de operadores autoadjuntos. En otras palabras, U es un operador unitario y X es un operador unitario dado por una perturbación de rango finito de la identidad, *i.e.*, $X = \mathbf{1} + W$ con W de rango finito. Mostramos que U y su versión perturbada UX (o XU) son unitariamente equivalentes en sus subspacios absolutamente continuos.

Keywords and Phrases: Absolutely continuous measure, finite rank perturbations, multiplicative perturbation, unitary operators.

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1 Introduction

One of the great theorems in spectral theory is the famous Kato-Rosenblum theorem [4]:

Theorem 1.1. *If A and T are self-adjoint operators, and A is trace class, then the absolutely continuous parts of T and $T + A$ are unitarily equivalent.*

This theorem tells us that T and $T + A$ have the same absolutely continuous spectrum.

Our motivation is to provide an alternative proof of the Birman-Krein theorem [1], which serves as the unitary counterpart to the Kato-Rosenblum theorem, for the case of unitary operators under multiplicative finite-rank perturbations (hence trace-class operators). Specifically, we are interested in the preservation of absolutely continuous spectrum under transformations of the form $U \mapsto UX$ (or XU), where U and X unitary operator. It is worth mentioning here that X is a unitary operator, but not of finite rank, however, it can be expressed as $X = \mathbf{1} + W$, where $W = X - \mathbf{1}$ is indeed a finite-rank operator. This ensures that X differs from the identity only on a finite-dimensional subspace. While Birman and Krein mention how the proof would proceed if X were of rank 1 or finite rank, they do not provide a detailed demonstration. Our work fills this gap by presenting a novel proof of this theorem.

In this proof, we avoid the use of scattering theory, which has been the traditional approach to this problem. Notably, L. de Branges and L. Shulman previously addressed similar in [5,6] and [2] where they employed scattering theory (wave operator limits). In contrast our approach is more restrictive than the general case, as it applies only when X is a unitary operator perturbed by a finite rank operator.

In Section 2, we introduce the general framework for multiplicatively perturbed unitary operators. In Section 3, to illustrate our general result, we examine the case where the perturbation is of rank 1. Finally, in Section 4, we present our main result.

2 Multiplicative perturbations

It is often convenient to express the unitary operator X as $X = e^{iY}$, where Y is a self-adjoint and bounded operator. If Y is of trace class, it can be written as

$$Y = \sum_{j=1}^{\infty} \omega_j P_{\varphi_j},$$

where $P_{\varphi_j} = \langle \varphi_j, \cdot \rangle \varphi_j$, $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal sequence, and $\sum_j |\omega_j| < \infty$.

Consider the perturbation

$$U \mapsto UX.$$

Then, $UX = U(\mathbf{1}+W) = U + UW$, where $W = \sum_{j=1}^{\infty} \frac{(iY)^j}{j!}$. Since $P_{\varphi_j}P_{\varphi_k} = 0$ for $j \neq k$, we have

$$e^{i(\omega_1 P_{\varphi_1} + \omega_2 P_{\varphi_2})} = e^{i\omega_1 P_{\varphi_1}} e^{i\omega_2 P_{\varphi_2}}.$$

Let $X_n = e^{i\sum_{j=1}^{n} \omega_j P_{\varphi_j}}$. Then, the commutator $[X_n, X_m] = 0$ for all $n, m \in \mathbb{N}$, meaning the operators commute. Thus,

$$X_{n+k} = e^{i\sum_{j=1}^{n+k} \omega_j P_{\varphi_j}} = e^{i\sum_{i=1}^{n} \omega_i P_{\varphi_i}} e^{i\sum_{j=n+1}^{n+k} \omega_j P_{\varphi_j}}.$$

Formally, the unitary operator X can be expressed as

$$e^{iY} = e^{i\omega_1 P_{\varphi_1}} \cdot e^{i\omega_2 P_{\varphi_2}} \cdots e^{i\omega_k P_{\varphi_k}} \cdots = \cdots e^{i\omega_k P_{\varphi_k}} \cdots e^{i\omega_2 P_{\varphi_2}} \cdot e^{i\omega_1 P_{\varphi_1}}. \tag{2.1}$$

Remark 2.1. (1) *If Y is of rank 1 and $\mathbf{1}$ is the identity operator, then*

$$e^{i\omega P_\varphi} = \sum_{j\geq 0} \frac{(i\omega P_\varphi)^j}{j!} = \mathbf{1} + \sum_{j\geq 1} \frac{(i\omega P_\varphi)^j}{j!} = \mathbf{1} + \sum_{j\geq 1} \frac{(i\omega)^j P_\varphi}{j!} = \mathbf{1} + (e^{i\omega} - 1)P_\varphi.$$

(2) *Let $\beta_j = (e^{i\omega_j} - 1) \in \mathbb{C}$. Since P_{φ_j} is a projection operator for all $j \in \mathbb{N}$, we have*

$$(\mathbf{1} + \beta_j P_{\varphi_j})(\mathbf{1} + \beta_k P_{\varphi_k}) = \mathbf{1} + \beta_j P_{\varphi_j} + \beta_k P_{\varphi_k}, \quad \textit{for } j \neq k.$$

(3) *For $\omega_j \in \mathbb{R}$,*

$$|\beta_j| = |e^{i\omega_j} - 1| \leq |\omega_j|,$$

for all $j \in \mathbb{N}$.

To justify equality (2.1), we present the following lemma.

Lemma 2.2. *Let $X_n = e^{i\sum_{j=1}^{n} \omega_j P_{\varphi_j}}$, then $\{X_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $B(\mathcal{H})$.*

Proof. Using Remark 2.1, we have:

$$\begin{aligned}
||X_{n+k} - X_n|| &= \left|\left| X_n e^{i\sum_{j=n+1}^{n+k} \omega_j P_{\varphi_j}} - X_n \right|\right| \\
&= \left|\left| X_n \left[e^{i\omega_{n+1} P_{\varphi_{n+1}}} \cdot e^{i\omega_{n+2} P_{\varphi_{n+2}}} \cdots e^{i\omega_{n+k} P_{\varphi_{n+k}}} - \mathbf{1} \right] \right|\right| \\
&= \left|\left| X_n \left[\left(\mathbf{1} + \left(e^{i\omega_{n+1}} - 1\right) P_{\varphi_{n+1}}\right) \cdots \left(\mathbf{1} + \left(e^{i\omega_{n+k}} - 1\right) P_{\varphi_{n+k}}\right) - \mathbf{1} \right] \right|\right| \\
&= \left|\left| X_n \left[\left(\mathbf{1} + \beta_{n+1} P_{\varphi_{n+1}}\right) \cdot \left(\mathbf{1} + \beta_{n+2} P_{\varphi_{n+2}}\right) \cdots \left(\mathbf{1} + \beta_{n+k} P_{\varphi_{n+k}}\right) - \mathbf{1} \right] \right|\right| \\
&\leq ||X_n|| \cdot \left|\left| \left(\mathbf{1} + \beta_{n+1} P_{\varphi_{n+1}}\right) \cdot \left(\mathbf{1} + \beta_{n+2} P_{\varphi_{n+2}}\right) \cdots \left(\mathbf{1} + \beta_{n+k} P_{\varphi_{n+k}}\right) - \mathbf{1} \right|\right| \\
&= \left|\left| \left(\mathbf{1} + \beta_{n+1} P_{\varphi_{n+1}}\right) \cdot \left(\mathbf{1} + \beta_{n+2} P_{\varphi_{n+2}}\right) \cdots \left(\mathbf{1} + \beta_{n+k} P_{\varphi_{n+k}}\right) - \mathbf{1} \right|\right| \\
&= \left|\left| \left(\mathbf{1} + \beta_{n+1} P_{\varphi_{n+1}} + \beta_{n+2} P_{\varphi_{n+2}}\right) \cdots \left(\mathbf{1} + \beta_{n+k-1} P_{\varphi_{n+k-1}} + \beta_{n+k} P_{\varphi_{n+k}}\right) - \mathbf{1} \right|\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq |\beta_{n+1}| \cdot \left|\left|P_{\varphi_{n+1}}\right|\right| + |\beta_{n+2}| \cdot \left|\left|P_{\varphi_{n+2}}\right|\right| + \cdots + |\beta_{n+k}| \cdot \left|\left|P_{\varphi_{n+k}}\right|\right| \\
&= \left|\left|\mathbf{1} + \beta_{n+1}P_{\varphi_{n+1}} + \beta_{n+2}P_{\varphi_{n+2}} + \cdots + \beta_{n+k}P_{\varphi_{n+k}} - \mathbf{1}\right|\right| \\
&= \left|\left|\beta_{n+1}P_{\varphi_{n+1}} + \beta_{n+2}P_{\varphi_{n+2}} + \cdots + \beta_{n+k}P_{\varphi_{n+k}}\right|\right| \\
&= |\beta_{n+1}| + |\beta_{n+2}| + \cdots + |\beta_{n+k}| \\
&= \left|e^{i\omega_{n+1}} - 1\right| + \left|e^{i\omega_{n+2}} - 1\right| + \cdots + \left|e^{i\omega_{n+k}} - 1\right| \\
&\leq |\omega_{n+1}| + |\omega_{n+2}| + \cdots + |\omega_{n+k}| \\
&= \sum_{j=n+1}^{n+k} |\omega_j| \to 0,
\end{aligned}
$$

for $n \to \infty$, since it is of trace class, that is, $\{\omega_j\}_{j\in\mathbb{N}} \in l^1$. □

As an immediate result, we have:

Corollary 2.3. *Let U be another unitary operator, then $\{UX_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $B(\mathcal{H})$.*

Lemma 2.4. *$\|UX_n - UX\| \to 0$, when $n \to \infty$.*

Proof. Analogous to Lemma 2.2 and Corollary 2.3, if we have that $Y = \sum_{j=1}^{\infty} \omega_j P_{\varphi_j}$, then

$$
\begin{aligned}
||X_n - X|| = \left|\left|X_n - e^{iY}\right|\right| &= \left|\left|e^{i\sum_{j=1}^{n}\omega_j P_{\varphi_j}} - e^{i\sum_{j=1}^{n}\omega_j P_{\varphi_j}} \cdot e^{i\sum_{j>n}\omega_j P_{\varphi_j}}\right|\right| \\
&= \left|\left|e^{i\sum_{i=1}^{n}\omega_j P_{\varphi_j}}\left(\mathbf{1} - e^{i\sum_{j>n}\omega_j P_{\varphi_j}}\right)\right|\right| \leq \left|\left|\mathbf{1} - e^{i\sum_{j>n}\omega_j P_{\varphi_j}}\right|\right| \leq \sum_{j>n} |\omega_j| \to 0,
\end{aligned}
$$

when $n \to \infty$, and therefore $||UX_n - UX|| \to 0$. □

For unitary operators, the Cauchy and Borel transforms of a Borel measure μ on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ are given by

$$
F_\mu(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t), \quad |z| < 1, \quad R_\mu(z) = \int_0^{2\pi} \frac{d\mu(t)}{e^{it} - z}, \quad |z| < 1,
$$

respectively. Here F_μ and R_μ are related by:

$$
F_\mu(z) = 1 + 2zR_\mu(z). \tag{2.2}
$$

Here we are interested in the properties of F_μ; for this, we have the following theorem [7].

Theorem 2.5. *Let μ be a Borel measure on the unit circle $\mathbb{T}$, then*

(1) $\lim_{r\to 1} F_\mu\left(re^{it}\right)$ *exists for almost every t, and if*

$$d\mu(t) = f(t)\frac{dt}{2\pi} + d\mu_s(t)$$

defines $f(t)$, then $f(t) = \Re F_\mu\left(e^{it}\right)$.

(2) t_0 *is a pure point of μ if and only if* $\lim_{r\to 1}(1-r)\,\Re F_\mu\left(re^{it_0}\right) \neq 0$ *and in general*

$$\lim_{r\to 1}(1-r)\,\Re F_\mu\left(re^{it_0}\right) = \mu\left(\{t_0\}\right).$$

(3) $d\mu_s$ *is supported in* $\left\{t \mid \lim_{r\to 1} F_\mu\left(re^{it}\right) = \infty\right\}$.

Remark 2.6. *This last theorem relates nontangential limits of this transform to the singular $d\mu_s$ and absolutely continuous parts of $d\mu$.*

3 Rank 1 case

Now, let us consider the case of a rank 1 perturbation:

$$U_\omega = UX_\omega = U\left(\mathbf{1} + \left(e^{i\omega} - 1\right)P_\varphi\right).$$

Note that the intensity parameter ω exhibits periodicity, and it suffices to consider $0 \le \omega < 2\pi$. Here, φ is a normalized vector in the Hilbert space $\mathcal{H}$ that is cyclic for the unitary operator U, meaning that the closure

$$\overline{\mathrm{Lin}\left\{U^j\varphi \mid j \in \mathbb{Z}\right\}} = \mathcal{H},$$

with $U^0 = \mathbf{1}$. Since φ is cyclic for U, it is also cyclic for U_ω, for all $\omega \in \mathbb{R}$.

To simplify the notation, let μ^ω denote the spectral measure of the pair (U_ω, φ), $U_0 = U$, $\mu^0 = \mu$, $F_\omega = F_{\mu^\omega}$ and $R_\omega = R_{\mu^\omega}$. Clearly, $F_0(z) = F_\mu(z)$ and $R_0(z) = R_\mu(z)$, where the Cauchy and Borel transforms are respectively given by

$$F_\omega(z) = \left\langle \varphi, \left(U_\omega + z\mathbf{1}\right)\left(U_\omega - z\mathbf{1}\right)^{-1}\varphi \right\rangle, \quad R_\omega(z) = \left\langle \varphi, R_z\left(U_\omega\right)\varphi\right\rangle,$$

where $R_z(U_\omega) = \left(U_\omega - z\mathbf{1}\right)^{-1}$ is the resolvent operator.

Our goal is to prove that the measures μ^ω_{ac} and μ_{ac} are equivalent, which implies that their Radon-Nikodym derivatives are equal almost everywhere with respect to the Lebesgue measure (up to a non-vanishing factor, which in this case is 1 due to the specific form of the transformation). This equivalence of the measures implies the unitary equivalence of the absolutely continuous parts of U_ω and U.

Lemma 3.1. *$R_z(U)(U\varphi) = (\mathbf{1} + zR_z(U))\varphi$.*

Proof. In fact,

$$\begin{aligned} R_z(U)(U\varphi) - zR_z(U)(\varphi) &= R_z(U)\left[U(\varphi) - z\varphi\right] \\ &= R_z(U)\left[U - z\mathbf{1}\right](\varphi) = (U - z\mathbf{1})^{-1}(U - z\mathbf{1})\varphi = \varphi. \qquad \square \end{aligned}$$

Remark 3.2. *By the previous lemma, we then have that:*

$$\begin{aligned} \langle \varphi, R_z(U)(U\varphi)\rangle &= \langle \varphi, (\mathbf{1} + zR_z(U))\varphi\rangle \\ &= \langle \varphi, \mathbf{1}\varphi\rangle + z\langle \varphi, R_z(U)\varphi\rangle = \langle \varphi, \varphi\rangle + z\langle \varphi, R_z(U)\varphi\rangle = 1 + zR_\mu(z). \end{aligned}$$

Lemma 3.3. *For $|z| \neq 1$*

$$R_\omega(z) = \frac{R_0(z)}{e^{i\omega} + z(e^{i\omega} - 1)R_0(z)} \quad \text{and} \quad F_\omega(z) = \frac{(e^{i\omega} - 1) + (e^{i\omega} + 1)F_0(z)}{(e^{i\omega} + 1) + (e^{i\omega} - 1)F_0(z)}.$$

Proof. By the second resolvent identity, we have that

$$R_z(U) - R_z(U_\omega) = R_z(U)(U_\omega - U)R_z(U_\omega) = R_z(U)((e^{i\omega} - 1)UP_\varphi)R_z(U_\omega),$$

then

$$\begin{aligned} \langle \varphi, R_z(U)\varphi\rangle - \langle \varphi, R_z(U_\omega)\varphi\rangle &= \left\langle \varphi, R_z(U)((e^{i\omega} - 1)UP_\varphi)R_z(U_\omega)\varphi\right\rangle \\ &= (e^{i\omega} - 1)\left\langle \varphi, R_z(U)U\left(\langle \varphi, R_Z(U_\omega)\varphi\rangle \varphi\right)\right\rangle \\ &= (e^{i\omega} - 1)\langle \varphi, R_z(U_\omega)\varphi\rangle \langle \varphi, R_z(U)U\varphi\rangle \\ &= (e^{i\omega} - 1)\langle \varphi, R_z(U_\omega)\varphi\rangle \left[1 + z\langle \varphi, R_z(U)\varphi\rangle\right], \end{aligned}$$

that is, $R_0(z) - R_\omega(z) = (e^{i\omega} - 1)R_\omega(z)\left[1 + zR_0(z)\right]$, therefore,

$$R_\omega(z) = \frac{R_0(z)}{e^{i\omega} + z(e^{i\omega} - 1)R_0(z)}.$$

Now, by (2.2), we have

$$\begin{aligned} F_\omega(z) = 2zR_\omega(z) + 1 &= 2z\frac{R_0(z)}{e^{i\omega} + z(e^{i\omega} - 1)R_0(z)} + 1 \\ &= \frac{e^{i\omega} + z(e^{i\omega} - 1)R_0(z) + 2zR_0(z)}{e^{i\omega} + z(e^{i\omega} - 1)R_0(z)} = \frac{e^{i\omega} + z(e^{i\omega} + 1)R_0(z)}{e^{i\omega} + z(e^{i\omega} - 1)R_0(z)} \end{aligned}$$

$$= \frac{2e^{i\omega} + 2ze^{i\omega}R_0(z) + 2zR_0(z)}{2e^{i\omega} + 2ze^{i\omega}R_0(z) - 2zR_0(z)} = \frac{e^{i\omega} - 1 + e^{i\omega} + 2e^{i\omega}zR_0(z) + 1 + 2zR_0(z)}{e^{i\omega} + 1 + e^{i\omega} + 2e^{i\omega}zR_0(z) - 1 - 2zR_0(z)}$$
$$= \frac{(e^{i\omega} - 1) + (e^{i\omega} + 1)(1 + 2zR_0(z))}{(e^{i\omega} + 1) + (e^{i\omega} - 1)(1 + 2zR_0(z))} = \frac{(e^{i\omega} - 1) + (e^{i\omega} + 1)F_0(z)}{(e^{i\omega} + 1) + (e^{i\omega} - 1)F_0(z)}. \qquad \square$$

Remark 3.4. *From the previous lemma, we can express $F_0(z)$ in terms of $F_{\omega_1}(z)$ as follows:*

$$F_0(z) = \frac{(e^{i\omega_1} + 1)F_{\omega_1}(z) - (e^{i\omega_1} - 1)}{-(e^{i\omega_1} - 1)F_{\omega_1}(z) + (e^{i\omega_1} + 1)}, \tag{3.1}$$

for ω_1 in $[0, 2\pi)$. Now, if $\omega_2 \in [0, 2\pi)$ with $\omega_1 \neq \omega_2$, we can express $F_{\omega_2}(z)$ in terms of $F_0(z)$. Using (3.1)*, we can establish a relationship between F_{ω_2} and F_{ω_1} as follows:*

$$F_{\omega_2}(z) = \frac{e^{i\omega_2} - e^{i\omega_1} + \left(e^{i\omega_1} + e^{i\omega_2}\right) F_{\omega_1}(z)}{e^{i\omega_2} + e^{i\omega_1} + \left(e^{i\omega_2} - e^{i\omega_1}\right) F_{\omega_1}(z)} \quad \textit{and} \quad \Re F_{\omega_2}(z) = \frac{(1 + y^2)\Re F_{\omega_1}(z)}{\left|1 + iyF_{\omega_1}(z)\right|^2},$$

with $iy = \dfrac{e^{i\omega_2} - e^{i\omega_1}}{e^{i\omega_2} + e^{i\omega_1}}$.

Remark 3.5. *By Theorem 2.5, we know that the singular part of the measure μ_s is supported on*

$$S = \left\{ t \mid \lim_{r \to 1} F(re^{it}) = \infty \right\}.$$

Let us define the sets

$$S_1 = \left\{ t \mid \lim_{r \to 1} F_{\omega_1}(re^{it}) = \infty \right\} \quad \textit{and} \quad S_2 = \left\{ t \mid \lim_{r \to 1} F_{\omega_2}(re^{it}) = \infty \right\}.$$

These sets are mutually disjoint. Indeed, since $\omega_1 \neq \omega_2$, and using Remark 3.4, it follows that if $t \in S_1$, then

$$\lim_{r \to 1} F_{\omega_2}(re^{it}) = \frac{e^{i\omega_2} + e^{i\omega_1}}{e^{i\omega_2} - e^{i\omega_1}} \neq \infty.$$

Therefore, $t \notin S_2$, which implies $S_1 \cap S_2 = \emptyset$. Thus, the measures $\mu_s^{\omega_1}$ and $\mu_s^{\omega_2}$ are mutually singular. This last result could be considered the equivalent of Donoghue's Theorem, but for unitary operators [3].

Theorem 3.6. *For all $\omega_1 \neq \omega_2$, the absolutely continuous parts of U_{ω_1} and U_{ω_2} are unitarily equivalent.*

Proof. Since $U_{\omega_1} = U_{\omega_2} + \left(e^{i\omega_1} - e^{i\omega_2}\right) UP_\varphi$, let us define the sets

$$L_1 = \left\{ t \mid \lim_{r \to 1} F_{\omega_1}(re^{it}) = \infty \quad \text{or} \quad \nexists \lim_{r \to 1} F_{\omega_1}(re^{it}) \right\}, \quad L_2 = \left\{ t \mid \lim_{r \to 1} F_{\omega_1}(re^{it}) = \frac{e^{i\omega_2} + e^{i\omega_1}}{e^{i\omega_1} - e^{i\omega_2}} \right\},$$

moreover, by Theorem 2.5, the measures of these sets are zero. If we define $G = L_1 \cup L_2$, then the measure of G is also zero. From Remark 3.4, we obtain:

$$\left\{t \in \mathbb{T}\backslash G \mid \lim_{r\to 1} F_{\omega_1}(re^{it}) = 0\right\} = \left\{t \in \mathbb{T}\backslash G \mid \lim_{r\to 1} F_{\omega_2}(re^{it}) = 0\right\},$$

thus, for almost every $t \in \mathbb{T}\backslash G$, we have: $\lim_{r\to 1} \Re_{\omega_1}(re^{it}) \neq 0$ if and only if $\lim_{r\to 1} \Re_{\omega_2}(re^{it}) \neq 0$. By Theorem 2.5, the Radon-Nikodym derivative of the absolutely continuous part of the spectral measure is given by the real part of the boundary value of the Cauchy transform. Therefore, the above equivalence implies that the set of points t where the density of $(d\mu^{\omega_1})_{ac}$ is zero (or non-zero) coincides, up to a set of Lebesgue measure zero, with the set where the density of $(d\mu^{\omega_2})_{ac}$ is zero (or non-zero). This means that the measures $(d\mu^{\omega_1})_{ac}$ and $(d\mu^{\omega_2})_{ac}$ are mutually absolutely continuous with respect to each other (and with respect to Lebesgue measure), hence equivalent. The unitary equivalence of the absolutely continuous parts of U_{ω_1} and U_{ω_2} then follows from the spectral theorem. □

Remark 3.7. *From this theorem, under the specific choices $\omega_2 = \omega$ and $\omega_1 = 0$, establishes the equality $\mu^{\omega}_{ac} = \mu_{ac}$ of the absolutely continuous spectral measures. Consequently, the absolutely continuous parts of the operators U_ω and U_0 are unitarily equivalent, proving our original claim.*

4 Finite rank case

We consider the perturbation of the unitary operator U_0 by another unitary operator X, defined as:

$$U = U_0 X = U_0(\mathbf{1} + W) = U_0 + U_0 W,$$

where W is an operator given by:

$$W = \sum_{j=1}^{n} \beta_j P_{\varphi_j},$$

with $\beta_j = (e^{i\omega_j} - 1)$ and $\omega_j \in [0, 2\pi)$ for $j = 1, 2, \ldots, n$.

Using the second resolvent identity, we have:

$$R_z(U_0) - R_z(U) = R_z(U_0)(U - U_0)R_z(U),$$

substituting $U - U_0 = U_0 W$, we obtain:

$$R_z(U_0) - R_z(U) = R_z(U_0)(U_0 W)R_z(U).$$

Furthermore, we observe that:

$$R_z(U_0) - R_z(U) = W R_z(U) + z R_z(U_0) W R_z(U).$$

To simplify the notations, as in the rank 1 case, we will use that $R_{U_0} = R_0$, $R_0^{k,m}(z) = \langle \varphi_k, R_z(U_0)\varphi_m \rangle$ and $R_U^{k,m}(z) = \langle \varphi_k, R_z(U)\varphi_m \rangle$ for any $k, m \in \{1, 2, \ldots, n\}$, and viewing these as matrix elements, we have

$$R_0^{k,m}(z) - R_U^{k,m}(z) = \beta_k R_U^{k,m}(z) + z \sum_{j=1}^{n} R_0^{k,j}(z)\beta_j R_U^{j,k}(z),$$

which means

$$R_0(z) - R_U(z) = MR_U(z) + zR_0(z)MR_U(z),$$

where

$$M = \begin{bmatrix} \beta_1 & 0 & 0 & \cdots & 0 \\ 0 & \beta_2 & 0 & \cdots & 0 \\ 0 & 0 & \beta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_n \end{bmatrix}, \quad \Omega = \begin{bmatrix} e^{i\omega_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{i\omega_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{i\omega_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{i\omega_n} \end{bmatrix} = M + I,$$

with I the $n \times n$ identity matrix, then

$$R_U(z) = (M + I + zR_0(z)M)^{-1} R_0(z) = (\Omega + zR_0(z)(\Omega - I))^{-1} R_0(z).$$

And since $F_U(z) = I + 2zR_U(z)$, we have that

$$F_U(z) = (2I + M + F_0(z)M)^{-1}(M + F_0(z)(M + 2I)),$$

or

$$F_U(z) = [(\Omega + I) + F_0(z)(\Omega - I)]^{-1} ((\Omega - I) + F_0(z)(\Omega + I)).$$

and if we separate the matrix Ω in the following way

$$\Omega = \begin{bmatrix} \cos(\omega_1) + i\sin(\omega_1) & 0 & 0 & \cdots & 0 \\ 0 & \cos(\omega_2) + i\sin(\omega_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cos(\omega_n) + i\sin(\omega_n) \end{bmatrix} := C + iS,$$

then $\Omega + \Omega^* = 2C$ and $MM^* = 2(I - C)$, with M^* is the conjugate matrix of M, therefore

$$2\Re F_U(z) = F_U(z) + F_U(z) = 2((\Omega + I) + F_0(z)(\Omega - I))^{-1} \Re F_0(z)((\Omega + I) + (\Omega - I)F_0(z))^{-1},$$

$$\Re F_U(z) = ((\Omega + I) + F_0(z)(\Omega - I))^{-1} \Re F_0(z)((\Omega^* + I) + (\Omega^* - I)F_0^*(z))^{-1}, \tag{4.1}$$

or

$$\Re F_0(z) = ((\Omega + I) + F_0(z)(\Omega - I))\Re F_U(z)((\Omega^* + I) + (\Omega^* - I)F_0^*(z)), \tag{4.2}$$

for $|z| < 1$.

Remark 4.1. *By the second resolvent identity,*

$$R_z(U_0) = R_z(U) + R_z(U_0)(U_0W)R_z(U) = (I + R_z(U_0)(U_0W))R_z(U),$$

let

$$A = I + R_z(U_0)(U_0W) = R_z(U_0)(U - z) = I + W + zR_z(U_0)W,$$

and since $R_z(U_0) = \frac{1}{2z}(F_z(U_0) - I)$*, then* $A = I + \frac{W}{2} + \frac{1}{2}F_z(U_0)W$*, and given that* A *is bounded with a bounded inverse, we have* $A = R_z(U_0)(U - z)$ *and* $A^{-1} = R_z(U)(U_0 - z)$*. Then, if* $2A = T = 2I + W + F_z(U_0)W$ *and since* A *is invertible,* T *is invertible*

$$T^{-1} = \frac{1}{2}A^{-1} = \frac{1}{2}R_z(U)(U_0 - z) = \frac{1}{4z}(F_z(U) - I)(U_0 - z).$$

Therefore $(2I + M + F_0(z)M)^{-1}$ *is invertible.*

Let us consider the sets

$$I_{m,k}(U_0) := \left\{ t \in [0, 2\pi) \mid \left| \lim_{r\uparrow 1} F_0^{m,k}(re^{it}) \right| = \infty \ \text{ or } \ \nexists \lim_{r\uparrow 1} F_0^{m,k}(re^{it}) \right\}$$
$$N_m(U_0) := \left\{ t \in [0, 2\pi) \mid \lim_{r\uparrow 1} F_0^{m,n}(re^{it})(\omega_m - 1) = -\Omega - I \right\},$$

where $\lim_{r\uparrow 1} \omega_m F_0^{m,n}(re^{it})$ is an element of $\lim_{r\uparrow 1} \Omega F_0^{m,n}(re^{it})$. Then, the measures of $I_{m,k}(U_0)$ and $N_m(U_0)$ are zero, by Theorem 2.5, for all m, k and a.e. $t \in [0, 2\pi)$. Now, let us consider the union of these two sets, this is

$$G := \bigcup_{m,k=1}^{n} \Big(N_m(U_0) \cup N_m(U) \cup I_{m,k}(U_0) \cup I_{m,k}(U) \Big),$$

then the measure of G is also zero a.e. $t \in [0, 2\pi)$ and from the equations (4.1) and (4.2), we have

$$\left\{ t \in [0, 2\pi)\backslash G \mid \lim_{r\uparrow 1} \Re F_U(re^{it}) = 0 \right\} \subset \left\{ t \in [0, 2\pi)\backslash G \mid \lim_{r\uparrow 1} \Re F_0(re^{it}) = 0 \right\}$$

and

$$\left\{ t \in [0, 2\pi)\backslash G \mid \lim_{r\uparrow 1} \Re F_0(re^{it}) = 0 \right\} \subset \left\{ t \in [0, 2\pi)\backslash G \mid \lim_{r\uparrow 1} \Re F_U(re^{it}) = 0 \right\},$$

therefore, for almost every $t \in [0, 2\pi)\backslash G$, we have: $\lim_{r\to 1} \Re F_0(re^{it}) \neq 0$ if and only if $\lim_{r\to 1} \Re F_U(re^{it}) \neq 0$. Applying Theorem 2.5 again, we conclude that the absolutely continuous parts of the spectral measures for U_0 and U are mutually absolutely continuous. This equivalence of measures implies the unitary equivalence of the absolutely continuous parts of the operators U and U_0.

Remark 4.2. *In the same way, we can obtain this result for a perturbation* $U \mapsto XU$.

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References

[1] M. Š. Birman and M. G. Kreĭn, "On the theory of wave operators and scattering operators," *Dokl. Akad. Nauk SSSR*, vol. 144, pp. 475–478, 1962.

[2] L. de Branges and L. Shulman, "Perturbations of unitary transformations," *J. Math. Anal. Appl.*, vol. 23, pp. 294–326, 1968, doi: 10.1016/0022-247X(68)90069-3.

[3] J. S. Howland, "On a theorem of Aronszajn and Donoghue on singular spectra," *Duke Math. J.*, vol. 41, pp. 141–143, 1974.

[4] T. Kato, *Perturbation theory for linear operators*, ser. Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag New York, Inc., New York, 1966, vol. 132.

[5] L. Shulman, "Perturbations of unitary transformations," *J. Math. Anal. Appl.*, vol. 28, pp. 231–254, 1969, doi: 10.1016/0022-247X(69)90025-0.

[6] L. Shulman, "Perturbations of unitary transformations," *Amer. J. Math.*, vol. 91, pp. 267–288, 1969, doi: 10.2307/2373282.

[7] B. Simon, "Analogs of the m-function in the theory of orthogonal polynomials on the unit circle," *J. Comput. Appl. Math.*, vol. 171, no. 1-2, pp. 411–424, 2004, doi: 10.1016/j.cam.2004.01.022.

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