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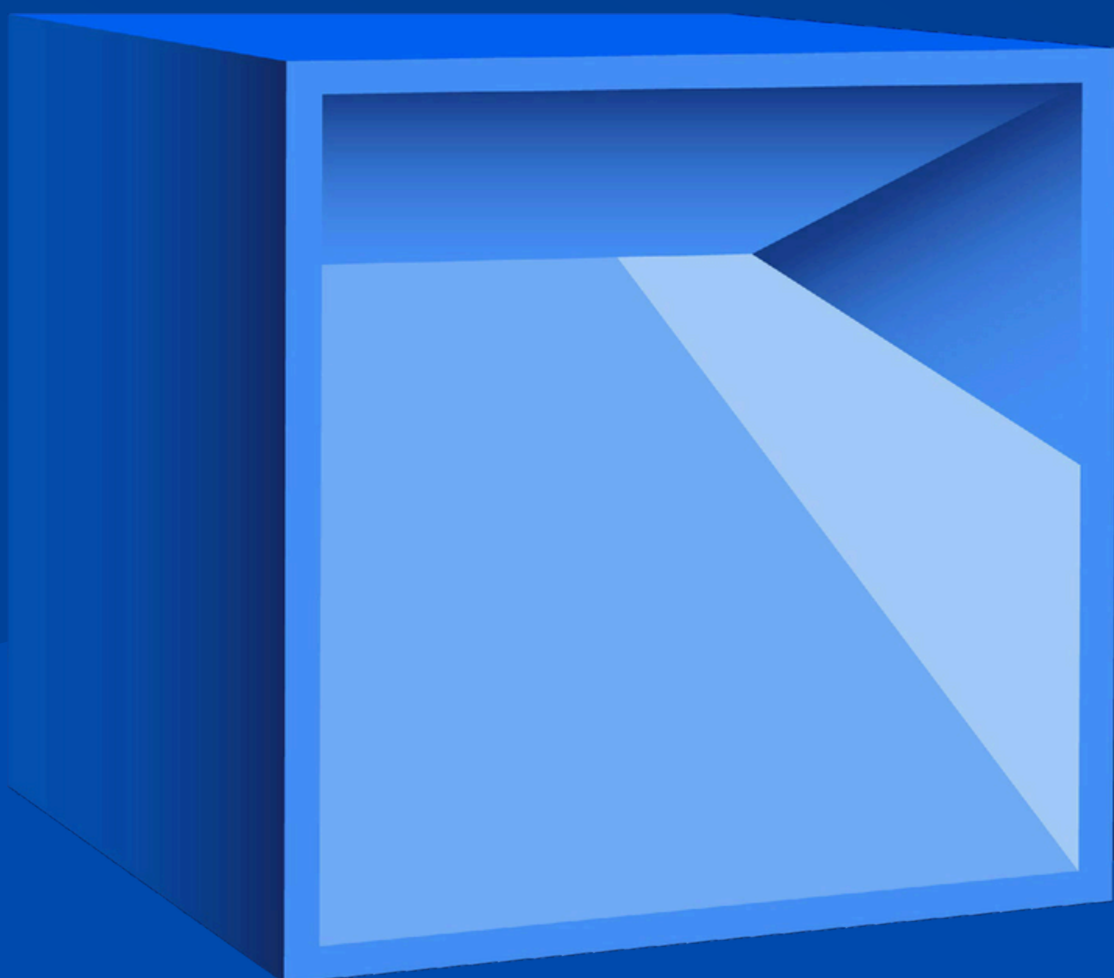

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SUMMARY

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Weak solutions of a discrete Robin problem involving the anisotropic $\vec{p}$-mean curvature operator

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ABSTRACT

This work investigates the existence and uniqueness of a solution to a discrete Robin boundary value problem involving the anisotropic $\vec{p}$-mean curvature operator. The existence result is established through variational methods, specifically by applying the Mountain Pass Theorem of Ambrosetti and Rabinowitz in combination with Ekeland's Variational Principle. Uniqueness is obtained under the assumption of Lipschitz continuity on the nonlinear term.

RESUMEN

Este trabajo investiga la existencia y unicidad de una solución a un problema discreto de valores en la frontera de Robin que involucra el operador de $\vec{p}$-curvatura media anisotrópico. El resultado de existencia se establece a través de métodos variacionales, específicamente aplicando el Teorema del Paso de la Montaña de Ambrosetti y Rabinowitz en combinación con el Principio Variacional de Ekeland. La unicidad se obtiene bajo la hipótesis de continuidad Lipschitz del término no-lineal.

Keywords and Phrases: Discrete Robin problem, boundary value problems, anisotropic $\vec{p}$-mean curvature operator, critical point, nontrivial solution, mountain pass theorem, Ekeland variational principle.

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1 Introduction

In this article, we study the following nonlinear discrete Robin problem.

$$\begin{cases} -\triangle\left(\left(1+\phi_{p(k-1)}\left(\triangle u(k-1)\right)\right)|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right) = \lambda f(k,u(k)), & k\in\mathbb{Z}\left[1,T\right], \\ \triangle u(0) = u(T+1) = 0, \end{cases} \tag{1.1}$$

where $T \geq 2$ is a positive integer.

For fixed integers a, b such that $a < b$, we denote by $\mathbb{Z}[a,b]$ the discrete interval $\{a, a+1, \ldots, b-1, b\}$. The parameter λ is positive. The forward difference operator is given by $\triangle u(k-1) = u(k)-u(k-1)$. The function $\phi_{p(k)} : \mathbb{R} \to \mathbb{R}$ is defined by $\phi_{p(k)}(s) = \dfrac{|s|^{p(k)}}{\sqrt{1+|s|^{2p(k)}}}$, for every $s \in \mathbb{R}$. The functions p and f will be defined precisely in the subsequent sections.

In problem (1.1), we consider two boundary conditions: a Neumann boundary condition ($\triangle u(0) = 0$) and a Dirichlet boundary condition ($u(T+1) = 0$). In the literature, these are referred to as mixed boundary conditions (see [25]).

Difference equations arise in many research fields as the discrete counterpart of partial differential equations and are often studied via numerical analysis. In this context, the operator in problem (1.1),

$$\triangle\left(\left(1+\frac{|\triangle u(k-1)|^{p(k-1)}}{\sqrt{1+|\triangle u(k-1)|^{2p(k-1)}}}\right)|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right)$$

represents the discrete counterpart of the following $\vec{p}$-anisotropic operator

$$\left(\left(1+\frac{|u'(t)|^{p(t)}}{\sqrt{1+|u'(t)|^{2p(t)}}}\right)|u'(t)|^{p(t)-2}u'(t)\right)'.$$

In recent years, equations involving the anisotropic $\vec{p}$-mean curvature operator, under various boundary conditions, have become a significant and captivating research topic. Problem (1.1) has been specifically analyzed in [4], where Dirichlet-type boundary conditions were applied through the use of variational methods and critical point theory. In this framework, problem (1.1) also serves as a discrete analogue of the following problem.

$$\begin{cases} -\left(\left(1+\dfrac{|u'(t)|^{p(t)}}{\sqrt{1+|u'(t)|^{2p(t)}}}\right)|u'(t)|^{p(t)-2}u'(t)\right)' = \lambda f(t,u(t)), & t\in(0,1), \\ u'(0) = u(1) = 0. \end{cases} \tag{1.2}$$

Problem (1.2) and its multi-dimensional variants arise in various applications, including elasticity mechanics [38,41], electrorheological fluids [14,20,37,38], and image restoration [11]. In [11], Chen *et al.* studied a functional with a variable exponent $1 \leq p(t) \leq 2$, which serves as a model for image denoising, enhancement and restoration.

The existence of a solution to a nonlinear difference equation can be proved using fixed point theory and the method of upper and lower solution techniques, as seen in [12,21] and the references therein. It is well known that critical point theory is a crucial tool for addressing problems involving differential equations.

Variational methods for difference equations were introduced by Guo and Yu [18]. The variational methods have been employed to study various equations, yielding different results. We refer to recent results involving anisotropic discrete boundary value problems [15–17, 23, 25, 26, 29, 39] and references therein. Discrete problems involving anisotropic exponents were firstly discussed in [24, 32].

In [32], by using the mountain pass theorem and Ekeland variational principle, the authors proved the existence of a continuous spectrum of eigenvalues for the following problem

$$\begin{cases} -\triangle\left(\phi_{p(k-1)}\left(\triangle u(k-1)\right)\right) = \lambda|u(k)|^{q(k)-2}, \quad k \in \mathbb{Z}\,[1,T]\,, \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.3}$$

where $\phi_{p(\cdot)}(s) = |s|^{p(\cdot)-2}s$, $p : \mathbb{Z}\,[0,T] \to [2,\infty)$, $q : \mathbb{Z}\,[1,T] \to [2,\infty)$ and λ is a positive constant.

In [24], Koné and Ouaro showed, by using the minimization method, the existence and uniqueness of weak solutions to the following problem

$$\begin{cases} -\triangle\left(a(k-1, \triangle u(k-1))\right) = f(k), \quad k \in \mathbb{Z}\,[1,T]\,, \\ u(0) = u(T+1) = 0. \end{cases} \tag{1.4}$$

We note that problem (1.4) is a generalization of (1.3). Indeed, in the particular case where $a(k,\xi) = |\xi|^{p(k)-2}\xi$ for all $k \in \mathbb{Z}\,[0,T]$ and $\xi \in \mathbb{R}$, the operator in (1.4) reduces to the $p(k)$-Laplacian, *i.e.*,

$$\triangle_{p(k-1)}u(k-1) := \phi_{p(k-1)}\left(\triangle u(k-1)\right) = |\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1).$$

In [22], the authors studied the following Robin problem

$$\begin{cases} \triangle^2 u(k-1) = f(k, u(k)), \quad k \in \mathbb{Z}\,[1,T]\,, \\ u(0) = \triangle u(T) = 0. \end{cases} \tag{1.5}$$

Using the strongly monotone operator principle and critical point theory, the authors proved the existence of nontrivial solutions for (1.5).

In [10], Chen *et al.* considered the following Robin problem

$$\begin{cases} \nabla\left(\dfrac{\triangle u_k}{\sqrt{1-(\triangle u_k)^2}}\right)+\lambda\mu_k(u_k)^q=0, \quad k\in\mathbb{Z}\left[1,T\right], \\ \triangle u_0=u_{T+1}=0. \end{cases} \tag{1.6}$$

By combining the method of upper and lower solutions with Brouwer degree theory and Szulkin's critical point theory for convex, lower semicontinuous perturbations of C^1 functions, the authors determined the ranges of the parameter λ for which problem (1.6) admits zero, one, or two positive solutions. In [28], by using critical point theory, the authors considered the existence of infinitely many positive solutions of the following discrete Robin problem with ϕ-Laplacian

$$\begin{cases} -\triangle(\varphi_p(\triangle u_{k-1}))+q_k\varphi_p(u_k)=\lambda f(k,u_k), \quad k\in\mathbb{Z}\left[1,T\right], \\ \triangle u_0=u_{T+1}=0, \end{cases} \tag{1.7}$$

where φ_p is a special ϕ-Laplacian operator (see [31]) defined by $\varphi_p(s)=\dfrac{p|s|^{p-2}s}{2\sqrt{1+|s|^p}}$ with $p\geq 2$.

In [19], by using variational methods, Hadjian and Bagheri established the existence of at least one nontrivial solution for the following problem

$$\begin{cases} -\triangle(\phi_c(\triangle u_{k-1}))=\lambda f(k,u_k), \quad k\in\mathbb{Z}\left[1,T\right], \\ u_0=u_{T+1}=0, \end{cases} \tag{1.8}$$

where ϕ_c is a special ϕ-Laplacian operator (see [31]) defined by $\phi_c(s)=\dfrac{s}{\sqrt{1+s^2}}$.

For the study of the following Robin problem involving a second-order nonlinear difference equation

$$\begin{cases} \nabla\left(\dfrac{\triangle u_k}{\sqrt{1-(\triangle u_k)^2}}\right)+\lambda f(k,u_k)=0, \quad k\in\mathbb{Z}\left[1,T\right], \\ \triangle u_0=\alpha u_1=0, \quad u_{T+1}=0, \end{cases} \tag{1.9}$$

we refer to [36]. In the particular case where $f(k,t)=\mu_k t^q$ and $\alpha=1$, we obtain the problem studied by Chen *et al.* [10]. The authors used different methods to obtain the existence and multiplicity of solutions for a discrete boundary value problem in [1, 2, 5, 7, 9, 34, 40].

In this article, we use the Ambrosetti-Rabinowitz mountain pass theorem (see [3]), Ekeland's variational principle and a Lipschitz continuity condition on the nonlinear term. Using these tools, we establish the existence and uniqueness of a nontrivial solution to a discrete Robin problem involving equations with the anisotropic $\vec{p}$-mean curvature operator.

The remainder of this article is organized as follows. In Section 2, we present some auxiliary results related to problem (1.1) and recall the abstract critical point theorem. Section 3 develops the variational framework associated with problem (1.1) and introduces our main results. Finally, we identify conditions under which problem (1.1) admits a unique nontrivial solution.

2 Preliminaries

Throughout this article, we denote

$$p^+ = \max_{k\in\mathbb{Z}[0,T]} p(k), \quad p^- = \min_{k\in\mathbb{Z}[0,T]} p(k), \quad r^+ = \max_{k\in\mathbb{Z}\in[1,T]} r(k) \quad \text{and} \quad r^- = \min_{k\in\mathbb{Z}[1,T]} r(k).$$

We consider the T-dimensional Banach space

$$H = \{u : \mathbb{Z}\,[0, T+1] \to \mathbb{R} \quad \text{such that} \quad \triangle u(0) = u(T+1) = 0\},$$

equipped with the norm

$$\|u\| = \left(\sum_{k=1}^{T} |\triangle u(k)|^{p^-}\right)^{1/p^-}. \tag{2.1}$$

However, we will use the following norm in H at times

$$\|u\|_\infty = \max_{k\in\mathbb{Z}[0,T+1]} |u(k)|, \quad \text{for all} \quad u \in H.$$

The space H will also be equipped with the following Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\left\{\mu > 0 : \sum_{k=1}^{T} \frac{1}{p(k)} \left|\frac{\triangle u(k)}{\mu}\right|^{p(k)} \leq 1\right\}.$$

Since on H, all norms are equivalent, then there exist two constants $0 < K_1 < K_2$ such that

$$K_1\|u\|_{p(\cdot)} \leq \|u\| \leq K_2\|u\|_{p(\cdot)}. \tag{2.2}$$

Next, let $\rho_{p(\cdot)} : H \to \mathbb{R}$ be given by

$$\rho_{p(\cdot)}(u) = \sum_{k=1}^{T} \frac{1}{p(k)} |\triangle u(k)|^{p(k)}.$$

Remark 2.1. *If $u \in H$, then the following properties hold.*

$$\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}, \tag{2.3}$$

$$\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}. \tag{2.4}$$

To establish our main result, we introduce the following quotient

$$\Lambda_1 = \inf_{u\in H\setminus\{0\}} \frac{\displaystyle\sum_{k=1}^{T} \frac{1}{p(k)}\left(|\triangle u(k)|^{p(k)} + \sqrt{1+|\triangle u(k)|^{2p(k)}} - 1\right)}{\displaystyle\sum_{k=1}^{T}\frac{1}{p(k)}|u(k)|^{p(k)}}. \tag{2.5}$$

We say that λ is an eigenvalue of problem (1.1) whenever the problem admits a nontrivial solution.

It should be emphasized that Λ_1 represents the first eigenvalue of problem (1.1) in the particular case where

$$f(k,u(k)) = |u(k)|^{p(k)-2}u(k).$$

In addition, Λ_1 serves as a critical threshold parameter governing the existence of nontrivial solutions to problem (1.1), thus guaranteeing the consistency of the analysis.

Let us also define the function

$$F(k,\xi) = \int_0^{\xi} f(k,s)\,ds, \quad \text{for all} \quad (k,\xi)\in\mathbb{Z}\,[1,T]\times\mathbb{R}.$$

We also make the following assumptions for the study of problem (1.1).

(H_1) For each $k\in\mathbb{Z}[1,T]$, the mapping $f(k,\cdot):\mathbb{R}\to\mathbb{R}$ is continuous.

(H_2) There exist a constant $C_1>0$ and a function

$$r(\cdot):\mathbb{Z}[1,T]\to[2,\infty)$$

such that:

(*i*) $|f(k,\xi)|\le C_1\big(1+|\xi|^{r(k)-1}\big)$, $\forall k\in\mathbb{Z}[1,T]$, $\forall\xi\in\mathbb{R}$.

(*ii*) $\liminf\limits_{|\xi|\to 0}\dfrac{F(k,\xi)}{|\xi|^{r(k)}}\ge 0$, for all $k\in\mathbb{Z}[1,T]$.

In particular, assumption $(H_2)(i)$ implies that there exists a constant $C_2>0$ such that

$$|F(k,\xi)|\le C_2\big(1+|\xi|^{r(k)}\big), \quad \forall k\in\mathbb{Z}[1,T], \quad \forall\xi\in\mathbb{R}.$$

(H_3) $\liminf\limits_{|\xi|\to\infty}\dfrac{F(k,\xi)}{|\xi|^{r^-}}\ge 0$, for all $k\in\mathbb{Z}[1,T]$.

(H_4) For every $\lambda\in(0,\Lambda_1)$,

$$\limsup_{|\xi|\to 0}\frac{\lambda f(k,\xi)}{|\xi|^{p(k)-2}\xi} < \Lambda_1, \quad \text{for all} \quad k\in\mathbb{Z}[1,T].$$

(H_5) $p(\cdot):\mathbb{Z}[0,T]\to(2,\infty)$.

Example 2.2. *The function*

$$f(x,t) := \begin{cases} |t|^{p(x)-2}t, & \text{if } |t| < 1 \\ |t|^{r(x)-2}t, & \text{if } |t| \geq 1, \end{cases}$$

with $r^- > p^+$, satisfies assumptions (H_1), (H_2), (H_3) and (H_4).

This example provides a concrete instance of the broader class of functions considered in problem (1.1).

In the sequel, we will use the following auxiliary results.

Lemma 2.3 ([16, 35])**.** *(a) For all $u \in H$ with $\|u\| > 1$,*

$$\sum_{k=1}^{T} \frac{1}{p(k)} |\triangle u(k)|^{p(k)} \geq \frac{1}{p^+} \left(\|u\|^{p^-} - T \right).$$

(b) For all $u \in H$ with $\|u\| < 1$,

$$\sum_{k=1}^{T} \frac{1}{p(k)} |\triangle u(k)|^{p(k)} \geq \frac{1}{p^+ T^{(p^+ - p^-)/p^-}} \|u\|^{p^+}.$$

(c) For all $u \in H$ and for any $m \geq 2$,

$$\sum_{k=1}^{T} |u(k)|^m \leq \left(T^{(p^- - 1)/p^-} \right)^m T \|u\|^m.$$

(d) For all $u \in H$ and all $p^+ \geq 2$,

$$\sum_{k=1}^{T} |\triangle u(k)|^{p^+} \leq 2^{p^+} \left(T^{(p^- - 1)/p^-} \right)^{p^+} T \|u\|^{p^+}.$$

(e) For all $u \in H$ and all $p^+ \geq 2$,

$$\sum_{k=1}^{T} \frac{1}{p(k)} |\triangle u(k)|^{p(k)} \leq \frac{T}{p^-} \left[2^{p^+} \left(T^{(p^- - 1)/p^-} \right)^{p^+} \|u\|^{p^+} + 1 \right].$$

The energy functional associated with problem (1.1) is defined by $J_\lambda : H \to \mathbb{R}$ as follows

$$J_\lambda(u) = \sum_{k=1}^{T} \left[\frac{1}{p(k)} \left(|\triangle u(k)|^{p(k)} + \sqrt{1 + |\triangle u(k)|^{2p(k)}} - 1 \right) - \lambda F(k, u(k)) \right]. \tag{2.6}$$

Definition 2.4. *We say that $u \in H$ is a weak solution of the problem* (1.1) *if*

$$\sum_{k=1}^{T}\Big[\big(1+\phi_{p(k)}(\triangle u(k))\big)\,|\triangle u(k)|^{p(k)-2}\triangle u(k)\triangle v(k)-\lambda f(k,u(k))v(k)\Big]=0, \tag{2.7}$$

for any $v \in H$ and

$$\sum_{k=1}^{T}\Big[\big(1+\phi_{p(k)}(\triangle u(k))\big)\,|\triangle u(k)|^{p(k)}-\lambda f(k,u(k))u(k)\Big]=0. \tag{2.8}$$

We define the functionals $\Phi, \Psi : H \to \mathbb{R}$ by

$$\Phi(u)=\sum_{k=1}^{T}\frac{1}{p(k)}\left(|\triangle u(k)|^{p(k)}+\sqrt{1+|\triangle u(k)|^{2p(k)}}-1\right)$$

and

$$\Psi(u)=\sum_{k=1}^{T}F(k,u(k)).$$

The functional is now written as: $J_\lambda(u) = \Phi(u) - \lambda\Psi(u)$.

Proposition 2.5. *The functional J_λ is well-defined on H and is of class $C^1(H,\mathbb{R})$ with the derivative given by*

$$\langle J_\lambda'(u),v\rangle=\sum_{k=1}^{T}\Big[\big(1+\phi_{p(k)}(\triangle u(k))\big)\,|\triangle u(k)|^{p(k)-2}\triangle u(k)\triangle v(k)-\lambda f(k,u(k))v(k)\Big], \tag{2.9}$$

for all $u, v \in H$.

The proof of Proposition 2.5 is a consequence of the proof of the following lemma.

Lemma 2.6. *The functionals Φ and Ψ are well-defined on H, and both belong to the class $C^1(H,\mathbb{R})$. Moreover, their derivatives are given by*

$$\langle\Phi'(u),v\rangle=\sum_{k=1}^{T}\big(1+\phi_{p(k)}(\triangle u(k))\big)\,|\triangle u(k)|^{p(k)-2}\triangle u(k)\triangle v(k),\quad \langle\Psi'(u),v\rangle=\sum_{k=1}^{T}f(k,u(k))v(k),$$

for all $u, v \in H$.

Furthermore, the critical points of the functional J_λ in H coincide with the weak solutions of problem (1.1).

Since the proof of Lemma 2.6 is very similar to that of Lemma 3.4 in [17] and Lemma 2.3 in [23], it is omitted.

Owing to the finite-dimensional setting, every weak solution of problem (1.1) is a strong (*i.e.*, classical) solution. Consequently, solving problem (1.1) amounts to finding the critical points of the functional J.

We now introduce the following results, which will be useful in the subsequent analysis.

Proposition 2.7 ([33]). *Assume that the condition* (H_5) *holds. Then,* $\Lambda_1 > 0$.

Definition 2.8. *Let* E *be a real Banach space and let* $J : E \to \mathbb{R}$ *be a functional. We say that* J *satisfies the* Palais-Smale condition *(abbreviated as* (PS) *condition) if every sequence* $\{u_n\} \subset E$ *such that* $\{J(u_n)\}$ *is bounded and* $J'(u_n) \to 0$ *as* $n \to \infty$*, admits a convergent subsequence in* E.

Moreover, a sequence $\{u_n\} \subset E$ *is said to satisfy the* Palais-Smale condition at level $c \in \mathbb{R}$*, denoted by* $(PS)_c$*, if*

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Lemma 2.9 ([39]). *Let* E *be a finite-dimensional Banach space and let* $J \in C^1(E, \mathbb{R})$ *be an anti-coercive functional. Then,* J *satisfies the (PS) condition.*

Lemma 2.10 ([30, Mountain pass lemma]). *Let* E *be a real Banach space. Assume that* $J \in C^1(E, \mathbb{R})$ *satisfies the (PS) condition. Suppose also that:*

(i) $J(0) = 0$;

(ii) there exist $\rho > 0$ *and* $\alpha > 0$ *such that* $J(u) \geq \alpha$ *for all* $u \in E$ *with* $\|u\| = \rho$;

(iii) there exists u_1 *in* E *with* $\|u_1\| \geq \rho$ *such that* $J(u_1) < 0$.

Then, J *has a critical value* $c \geq \alpha$ *which can be characterized by*

$$c = \inf_{h \in \Gamma} \max_{s \in [0,1]} J\left(h(s)\right),$$

where $\Gamma = \{h \in C\left([0,1], E\right) : h(0) = 0,\ h(1) = u_1\}$.

Theorem 2.11 ([30]). *Let* E *be a real Banach space and* $J : E \to \mathbb{R}$. *If* J *is weakly lower semicontinuous and coercive, i.e.* $\lim_{\|x\| \to \infty} J(x) = \infty$*, then there exists* $x_0 \in E$ *such that* $\inf_{x \in E} J(x) = J(x_0)$.

Moreover if $J \in C^1(E, \mathbb{R})$*, then* x_0 *is a critical point of* J*, i.e.* $J'(x_0) = 0$.

Theorem 2.12 ([13, Ekeland's variational principle]). *Let* X *be a complete metric space and* $\Phi : X \to \mathbb{R}$ *a lower semicontinuous function bounded from below. Then, for every* $\epsilon > 0$ *and* $\overline{u} \in X$ *be given such that*

$$\Phi(\overline{u}) \leq \inf_{u \in X} \Phi(u) + \epsilon,$$

there exists $u_\epsilon \in X$ *such that*

(i) $\Phi(u_\epsilon) \leq \Phi(\overline{u})$,

(ii) $d(u_\epsilon, \overline{u}) < \epsilon$,

(iii) $\Phi(u_\epsilon) < \Phi(u) + \epsilon d(u, u_\epsilon)$ *for each* $u \neq u_\epsilon$.

Corollary 2.13 ([13]). *Let X be a complete metric space and $\Phi : X \to \mathbb{R}$ be a lower semicontinuous function bounded below. Assume that $\Phi \in C^1(X, \mathbb{R})$. Then, for every $\varepsilon > 0$, there exists $u_\varepsilon \in X$ such that*

(i) $\Phi(u_\varepsilon) \leq \inf_{u \in X} \Phi(u) + \varepsilon$,

(ii) $\|\Phi'(u_\varepsilon)\| \leq \varepsilon$.

3 Existence and uniqueness of weak nontrivial solutions

This section focuses on the existence and uniqueness of nontrivial weak solutions to problem (1.1). We have the following result.

Theorem 3.1. *Assume that the hypotheses (H_1)-(H_5) hold. If $(r^- > p^+)$ or $(r^+ < p^-)$ or $(r^- < p^-)$, then there exist $\lambda^*, \rho, \Lambda^* > 0$ such that for any $\lambda > \lambda^*$ and $\Lambda_1 - \rho \in (\lambda, \Lambda^*)$, the problem (1.1) has at least one weak nontrivial solution.*

Proof. We can distinguish the following three cases:

Case 1: $r^- > p^+$

In this instance, we will demonstrate that J_λ possesses a "mountain pass geometry."

Lemma 3.2. *Assume that the hypotheses of Theorem 3.1 are satisfied, then.*

(i) *There exist $a, \varrho > 0$ and $\rho, \Lambda^* > 0$ such that for any $\lambda > 0$ and $\Lambda_1 - \rho \in (\lambda, \Lambda^*)$, one has*

$$J_\lambda(u) \geq a > 0 \quad \textit{for all} \quad u \in H \quad \textit{with} \quad \|u\| = \varrho.$$

(ii) *There exists $e \in H$ with $\|e\| > \varrho$ such that*

$$J_\lambda(e) < 0.$$

Proof. (i) Using hypothesis (H_4), for any $\lambda \in (0, \Lambda_1)$, we can find $\rho, \beta > 0$ such that $\lambda \leq \Lambda_1 - \rho$ and

$$\lambda f(k, \xi) \leq (\Lambda_1 - \rho) |\xi|^{p(k)-2}\xi, \quad \text{for all} \quad (k, \xi) \in \mathbb{Z}[1, T] \times \mathbb{R} \quad \text{and} \quad |\xi| \leq \beta.$$

In particular, if f is as in Example 2.2, then $\beta = 1$.

We deduce for $\xi \in (0, \beta]$, that

$$\lambda F(k,\xi) \leq (\Lambda_1 - \rho) \int_0^{\xi} |s|^{p(k)-2} s\, ds = (\Lambda_1 - \rho) \int_0^{\xi} s^{p(k)-2} s\, ds = \frac{1}{p(k)} (\Lambda_1 - \rho) |\xi|^{p(k)}$$

and for $\xi \in [-\beta, 0)$, we infer that

$$\lambda F(k,\xi) \leq (\Lambda_1 - \rho) \int_{\xi}^{0} |s|^{p(k)-2} s\, ds = (\Lambda_1 - \rho) \int_{\xi}^{0} (-s)^{p(k)-2} s\, ds = \frac{1}{p(k)} (\Lambda_1 - \rho) |\xi|^{p(k)}.$$

Then, it follows that

$$\lambda F(k,\xi) \leq \frac{1}{p(k)} (\Lambda_1 - \rho) |\xi|^{p(k)}, \quad \text{for all} \quad k \in \mathbb{Z}[1,T] \quad \text{and} \quad |\xi| \leq \beta. \tag{3.1}$$

Let $u \in H$ be such that $|u(k)| \leq \beta$ for all $k \in \mathbb{Z}[1,T]$. Then, by relation (2.1), we have

$$\|u\| \leq 2\beta T^{1/p^-}.$$

Now, let $u \in H$ be fixed such that $\|u\| \leq 1$. Define

$$\kappa = \min\left\{2\beta T^{1/p^-},\, 1\right\}.$$

Then, for any $u \in H$ satisfying $\|u\| \leq \kappa$, it follows from relations (2.5), (3.1), and assertions (b) and (c) of Lemma 2.3 that

$$\begin{aligned}
J_\lambda(u) &\geq \Phi(u) - (\Lambda_1 - \rho) \sum_{\substack{k=1 \\ |u(k)|>1}}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} - (\Lambda_1 - \rho) \sum_{\substack{k=1 \\ |u(k)|>1}}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} \\
&\geq \Phi(u) - (\Lambda_1 - \rho) \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{r(k)} - (\Lambda_1 - \rho) \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} \\
&\geq \frac{\Lambda_1 - (\Lambda_1 - \rho)}{\Lambda_1} \Phi(u) - \frac{(\Lambda_1 - \rho)}{p^+} \sum_{k=1}^{T} |u(k)|^{r^-} \\
&\geq \frac{\rho}{\Lambda_1 p^+} T^{(p^- - p^+)/p^-} \|u\|^{p^+} - \frac{(\Lambda_1 - \rho)}{p^+} \left(T^{(p^- - 1)/p^-}\right)^{r^-} T \|u\|^{r^-} \\
&= \left(c_1 \varrho^{p^+ - r^-} - (\Lambda_1 - \rho) c_2\right) \varrho^{r^-},
\end{aligned}$$

where c_1 and c_2 are positive constants.

Hence, choosing $\Lambda^* = \dfrac{c_1 \varrho^{p^+ - r^-}}{2c_2}$, then, for any $\Lambda_1 - \rho \in (\lambda, \Lambda^*)$, there exist some positive numbers $0 < \varrho < \kappa$ and $a = \dfrac{c_1 \varrho^{p^+}}{2} > 0$ such that $J_\lambda(u) \geq a > 0$ for all $u \in H$ with $\|u\| = \varrho$.

(ii) Fix $\lambda > 0$. By (H_3), for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$F(k,\xi) \geq \varepsilon|\xi|^{r^-}, \quad \text{for all} \quad k \in \mathbb{Z}[1,T] \quad \text{and all} \quad \xi \in \mathbb{R}, \quad \text{with} \quad |\xi| > \eta.$$

Since $\xi \to F(k,\xi) - \varepsilon|\xi|^{r^-}$ is continuous on $[-\eta,\eta]$, there is a constant $C_\eta > 0$ such that

$$F(k,\xi) - \varepsilon|\xi|^{r^-} \geq -C_\eta, \quad \text{for all} \quad k \in \mathbb{Z}[1,T] \quad \text{and all} \quad \xi \in [-\eta,\eta].$$

Hence, we get

$$F(k,\xi) \geq \varepsilon|\xi|^{r^-} - C_\eta, \quad \text{for all } (k,\xi) \in \mathbb{Z}[1,T] \times \mathbb{R}. \tag{3.2}$$

So, from (3.2) and Lemma 2.3 (e), we obtain

$$\begin{aligned} J_\lambda(u) &= \sum_{k=1}^{T} \frac{1}{p(k)} \left(|\triangle u(k)|^{p(k)} + \sqrt{1 + |\triangle u(k)|^{2p(k)}} - 1 \right) - \lambda \sum_{k=1}^{T} F(k,u(k)) \\ &\leq \frac{2}{p^-} \sum_{k=1}^{T} |\triangle u(k)|^{p(k)} - \lambda \sum_{k=1}^{T} \left(\varepsilon |u(k)|^{r^-} - C_\eta \right) \\ &\leq \frac{2T}{p^-} \left[2^{p^+} \left(T^{\frac{p^- - 1}{p^-}} \right)^{p^+} \|u\|^{p^+} + 1 \right] - \lambda\varepsilon \sum_{k=1}^{T} |u(k)|^{r^-} + \lambda T C_\eta. \end{aligned} \tag{3.3}$$

As

$$\|u\|^{p^-} \leq 2^{p^- - 1} \sum_{k=1}^{T} \left(|u(k+1)|^{p^-} + |u(k)|^{p^-} \right) \leq 2^{p^-} \sum_{k=1}^{T} |u(k)|^{p^-} \leq 2^{p^-} T^{\frac{r^- - p^-}{r^-}} \left(\sum_{k=1}^{T} |u(k)|^{r^-} \right)^{\frac{p^-}{r^-}},$$

which means that

$$\sum_{k=1}^{T} |u(k)|^{r^-} \geq 2^{-r^-} T^{\frac{p^- - r^-}{p^-}} \|u\|^{r^-}. \tag{3.4}$$

Then, it follows from (3.3) and (3.4) that

$$J_\lambda(u) \leq \frac{T}{p^-} 2^{p^+ + 1} \left(T^{\frac{p^- - 1}{p^-}} \right)^{p^+} \|u\|^{p^+} - \lambda\varepsilon 2^{-r^-} T^{\frac{p^- - r^-}{p^-}} \|u\|^{r^-} + \frac{2T}{p^-} + \lambda T C_\eta. \tag{3.5}$$

Since $r^- > p^+$, $J_\lambda(u) \to -\infty$ as $\|u\| \to \infty$. Thus, J_λ is anti-coercive. Consequently, there exists $e \in H$ with $\|e\| > \varrho$ such that $J_\lambda(e) < 0$. $\square$

Lemma 3.3. *Assume that the hypotheses of Theorem 3.1 hold. Then, for any $\lambda > 0$, the functional J_λ satisfies Palais-Smale condition.*

Proof. By Lemma 3.2 (ii), the functional J_λ is anti-coercive. Therefore, by Lemma 2.3, the functional J_λ satisfies the Palais-Smale condition for any $\lambda > 0$. Thus, our problem (1.1) has at least one nontrivial solution. □

Case 2: $r^+ < p^-$

In the second case, we apply a direct variational approach. We verify that the functional J_λ has a critical point. Let $\lambda > 0$ be fixed, since H is a finite-dimensional space and J_λ is of class $C^1(H, \mathbb{R})$, it is sufficient to prove that J_λ is coercive.

Let $\|u\| > 1$. Then, by (2.5), (2.6), (a) and (c) of Lemma 2.3, one has

$$\begin{aligned} J_\lambda(u) &\geq \Phi(u) - (\Lambda_1 - \rho) \sum_{\substack{k=1 \\ |u(k)|>1}}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} - (\Lambda_1 - \rho) \sum_{\substack{k=1 \\ |u(k)|>1}}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} \\ &\geq \Phi(u) - (\Lambda_1 - \rho) \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{r(k)} - (\Lambda_1 - \rho) \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} \\ &\geq \frac{\Lambda_1 - (\Lambda_1 - \rho)}{\Lambda_1} \Phi(u) - \frac{(\Lambda_1 - \rho)}{p^+} \sum_{k=1}^{T} |u(k)|^{r^+} \\ &\geq \frac{\rho}{\Lambda_1 p^+} \|u\|^{p^-} - \frac{(\Lambda_1 - \rho)}{p^+} \left(T^{(p^- - 1)/p^-}\right)^{r^+} T \|u\|^{r^+} - K(T), \end{aligned}$$

where $K(T)$ is a positive constant. Therefore, choosing $\Lambda^* = \dfrac{\rho}{\Lambda_1 \left(T^{(p^- - 1)/p^-}\right)^{r^+} T}$, since $r^+ < p^-$, one deduces that J_λ is coercive.

Now, let $u_* \in H$ be a global minimum of J_λ, which is a critical point of J_λ and, in turn, a weak solution of the problem (1.1).

We now show that u_* is nontrivial for λ large enough.

Let $d \in (0, 1)$ be a fixed real and $k_0 \in \mathbb{Z}[1, T]$, we define a function $w \in H$ by

$$w(k) = \begin{cases} d & \text{if} \quad k = k_0, \\ 0 & \text{if} \quad k \in \mathbb{Z}[1, T] - \{k_0\}. \end{cases}$$

Then, we deduce by $(H_2)(ii)$ that

$$\begin{aligned} J_\lambda(w) &= \frac{1}{p(k_0)}\left(|d|^{p(k_0)} + \sqrt{1+|d|^{2p(k_0)}} - 1\right) - \lambda F(k_0, w(k_0)) \\ &\leq \frac{2}{p(k_0)}|d|^{p(k_0)} - \lambda F(k_0, d) \leq \frac{2}{p^-}d^{p^-} - \lambda C d^{r^+}. \end{aligned}$$

Thus, if we choose λ^* as

$$\lambda^* = \frac{2}{p^- C}d^{p^- - r^+},$$

then for any $\lambda > \lambda^*$ and $r^+ < p^-$, $J_\lambda(w) < 0$. Since u_* is a global minimum of J_λ, it follows that $J_\lambda(u_*) < 0$ for any $\lambda > \lambda^*$; therefore u_* is a weak nontrivial solution of problem (1.1).

Case 3: $r^- < p^-$

In this case, we apply the Ekeland's variational principle.

Lemma 3.4. *Assume that $(H_2)(ii)$ holds and $r^- < p^-$. Then, there is $\overline{v} \in H$ such that $J_\lambda(\overline{v}) < 0$.*

Proof. Take $d \in (0, \kappa)$, where κ is as in the proof of Lemma 3.2 (ii), such that $d < \left(\frac{p^-\lambda C}{2}\right)^{\frac{1}{p^- - r^-}}$. Let $k_0 \in \mathbb{Z}[1,T]$ with $r(k_0) = r^-$. Consider any fixed $\overline{v} \in H$ such that $\overline{v}(k_0) = d$ and $\overline{v}(k) = 0$ for any $k \in \mathbb{Z}[1,T] \setminus \{k_0\}$. Using the condition $(H_2)(ii)$, we have

$$J_\lambda(\overline{v}) \leq \frac{2}{p(k_0)}d^{p(k_0)} - \lambda C d^{r(k_0)} \leq \frac{2}{p^-}d^{p^-} - \lambda C d^{r^-}.$$

Then,

$$J_\lambda(\overline{v}) < 0,$$

for all $d < \left(\frac{p^-\lambda C}{2}\right)^{\frac{1}{p^- - r^-}}$. The proof is thus complete. □

Relation (i) of Lemma 3.2 implies that

$$\inf_{u \in \partial B_\kappa} J_\lambda(u) > 0,$$

where $B_\kappa = \{u \in H \text{ such that } \|u\| \leq \kappa\}$. On the other hand, observe that Lemma 3.3 implies that there exists $\overline{v} \in H$ such that $J_\lambda(\overline{v}) < 0$, for every $d < \left(\frac{p^-\lambda C}{2}\right)^{\frac{1}{p^- - r^-}}$. Recall that $\overline{v} \in int\, B_\kappa$. Thus,

$$\inf_{u \in int\, B_\kappa} J_\lambda(u) < 0.$$

So, it follows

$$\inf_{u\in int\, B_\kappa} J_\lambda(u) < \inf_{u\in\partial B_\kappa} J_\lambda(u).$$

Let $\epsilon > 0$ be fixed, such that

$$0 < \epsilon < \inf_{u\in\partial B_\kappa} J_\lambda(u) - \inf_{u\in int\, B_\kappa} J_\lambda(u).$$

Applying Ekeland's variational principle to the functional $J_\lambda : B_\kappa \to \mathbb{R}$, there exists $u_\epsilon \in B_\kappa$ such that

$$J_\lambda(u_\epsilon) < \inf_{u\in B_\kappa} J_\lambda(u) + \epsilon \quad \text{and} \quad J_\lambda(u_\epsilon) < J_\lambda(u) + \epsilon\|u - u_\epsilon\| \quad \text{for all} \quad u \neq u_\epsilon.$$

Moreover,

$$J_\lambda(u_\epsilon) < \inf_{u\in B_\kappa} J_\lambda(u) + \epsilon \leq \inf_{u\in int\, B_\kappa} J_\lambda(u) + \epsilon < \inf_{u\in\partial B_\kappa} J_\lambda(u),$$

then, we infer that $u_\epsilon \in int\, B_\kappa$. Next, we introduce the function $\psi_\lambda : B_\kappa \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = J_\lambda(u) + \epsilon\|u - u_\epsilon\| \quad \text{for all} \quad u \neq u_\epsilon.$$

So, it follows that u_ϵ is a minimum point of ψ_λ and thus

$$\frac{\psi_\lambda(u_\epsilon + \theta v) - \psi_\lambda(u_\epsilon)}{\theta} \geq 0, \tag{3.6}$$

for all $v \in B_\kappa$ and all $\theta > 0$ small enough. Therefore, using relation (3.6), we deduce that

$$\frac{J_\lambda(u_\epsilon + \theta v) - J_\lambda(u_\epsilon)}{\theta} + \epsilon\|v\| \geq 0.$$

Letting $\theta \to 0^+$, we obtain

$$J'_\lambda(u_\epsilon, v) + \epsilon\|v\| \geq 0 \quad \text{for all} \quad u \in H, \tag{3.7}$$

where $J'_\lambda(u_\epsilon, v)$ is the directional derivative of the function J_λ at u_ϵ in the direction of v. Since

$$J'_\lambda(u_\epsilon, v) = \langle J'_\lambda(u_\epsilon), v\rangle = J'_\lambda(u_\epsilon)v,$$

we obtain from (3.7),

$$\|J'_\lambda(u_\epsilon)\| \leq \epsilon.$$

Thus, we deduce that there exists a sequence $\{u_n\} \subset int\, B_\kappa$ such that

$$J_\lambda(u_n) \to c = \inf_{u\in B_\kappa} J_\lambda(u) \quad \text{and} \quad J'_\lambda(u_n) \to 0 \quad \text{as} \quad n \to \infty.$$

As the sequence $\{u_n\}$ is bounded in H, then there exists $u_0 \in H$ such that, up to a subsequence, $\{u_n\}$ converges to u_0 in H. Hence, the problem (1.1) has a nontrivial solution. □

Lemma 3.5. *Let $\lambda > 0$. Suppose that conditions (H_1)-(H_5) are satisfied. If $u \in H$ is a solution of problem* (1.1)*, then there exist two positive constants κ_1 and κ_2 such that $\kappa_1 \leq \|u\| \leq \kappa_2$.*

Proof. The proof of this lemma is organized into two steps, as outlined below.

Step 1. Assume that $u \in H$ is a solution of (1.1) with $\|u\|_{p(\cdot)} \leq 1$. Set $\zeta = \dfrac{p^-}{\lambda\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}}$.

Since f satisfies (H_4), for any $\lambda \in (0, \Lambda_1)$, there exist $\rho, \beta > 0$ such that $\lambda \leq \Lambda_1 - \rho < \zeta$ and

$$\lambda f(k,\xi) \leq (\Lambda_1 - \rho)\, |\xi|^{p(k)-2}\xi \quad \text{for all} \quad k \in \mathbb{Z}\,[1,T] \quad \text{and} \quad \xi \in \mathbb{R} \quad \text{with} \quad |\xi| \leq \beta.$$

On the other hand, by $(H_2)(i)$, there exists a positive constant L such that

$$\lambda |f(k,\xi)| \leq L|\xi|^{r(k)-1}, \quad \text{for all} \quad (k,\xi) \in \mathbb{Z}\,[1,T] \times \mathbb{R} \quad \text{and} \quad |\xi| > \beta,$$

where $L = \lambda\left(\frac{1}{\beta^{r(k)-1}} + 1\right)$. Consequently, we get that

$$\lambda |f(k,\xi)| \leq (\Lambda_1 - \rho)\, |\xi|^{p(k)-1} + L|\xi|^{r(k)-1} \quad \text{for all} \quad k \in \mathbb{Z}\,[1,T] \quad \text{and} \quad \xi \in \mathbb{R}.$$

Using the above inequality, (2.2), (2.4), (2.8) and Lemma 2.3 (c), we obtain

$$\begin{aligned}
\|u\|_{p(\cdot)}^{p^+} &\leq \rho_{p(\cdot)}(u) = \sum_{k=1}^{T} \frac{1}{p(k)}\, |\triangle u(k)|^{p(k)} \leq \frac{1}{p^-} \sum_{k=1}^{T} |\triangle u(k)|^{p(k)} \\
&\leq \frac{1}{p^-} \sum_{k=1}^{T} \left(1 + \phi_{p(k)}(\triangle u(k))\right) |\triangle u(k)|^{p(k)} = \frac{\lambda}{p^-} \sum_{k=1}^{T} f(k, u(k))u(k) \\
&\leq \frac{\lambda}{p^-}(\Lambda_1 - \rho) \sum_{k=1}^{T} |u(k)|^{p(k)} + \frac{\lambda L}{p^-} \sum_{k=1}^{T} |u(k)|^{r(k)} \\
&\leq \frac{\lambda}{p^-}(\Lambda_1 - \rho) \sum_{k=1}^{T} |u(k)|^{p^+} + \frac{\lambda L}{p^-} \sum_{k=1}^{T} |u(k)|^{r^+} \\
&\leq \frac{\lambda}{p^-}(\Lambda_1 - \rho) \left(T^{\frac{p^--1}{p^-}}\right)^{p^+} T\|u\|^{p^+} + \frac{\lambda L}{p^-} \left(T^{\frac{p^--1}{p^-}}\right)^{r^+} T\|u\|^{r^+} \\
&\leq \frac{\lambda}{p^-}(\Lambda_1 - \rho) \left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}\, \|u\|_{p(\cdot)}^{p^+} + \frac{\lambda L}{p^-} \left(T^{\frac{p^--1}{p^-}}\right)^{r^+} TK_2^{r^+}\, \|u\|_{p(\cdot)}^{r^+}.
\end{aligned}$$

Therefore,

$$\|u\|_{p(\cdot)} \geq \left[\frac{1 - \frac{\lambda}{p^-} (\Lambda_1 - \rho) \left(T^{\frac{p^- - 1}{p^-}} \right)^{p^+} TK_2^{p^+}}{\frac{\lambda L}{p^-} \left(T^{\frac{p^- - 1}{p^-}} \right)^{r^+} TK_2^{r^+}} \right]^{\frac{1}{r^+ - p^+}}.$$

Set

$$\kappa_1^* = \left[\frac{1 - \frac{\lambda}{p^-} (\Lambda_1 - \rho) \left(T^{\frac{p^- - 1}{p^-}} \right)^{p^+} TK_2^{p^+}}{\frac{\lambda L}{p^-} \left(T^{\frac{p^- - 1}{p^-}} \right)^{r^+} TK_2^{r^+}} \right]^{\frac{1}{r^+ - p^+}}$$

and note that

$$0 < \kappa_1^* < 1.$$

Indeed, since

$$\lambda \leq \Lambda_1 - \rho < \frac{p^-}{\lambda \left(T^{\frac{p^- - 1}{p^-}} \right)^{p^+} TK_2^{p^+}},$$

it follows that

$$0 < 1 - \frac{\lambda}{p^-} (\Lambda_1 - \rho) \left(T^{\frac{p^- - 1}{p^-}} \right)^{p^+} TK_2^{p^+} < 1.$$

Clearly, $\lambda L \left(T^{\frac{p^- - 1}{p^-}} \right)^{r^+} TK_2^{r^+} > p^-$. Hence, $0 < \kappa_1^* < 1$.

Step 2. Suppose that $u \in H$ is a solution of (1.1) such that $\|u\|_{p(\cdot)} \geq 1$. Then, there exists a constant $\kappa_2^* > 1$ such that $\|u\|_{p(\cdot)} \leq \kappa_2^*$.

According to (2.6) and (2.8), one has

$$\begin{aligned}
&r^- \left(J_\lambda(u) + \lambda \sum_{k=1}^{T} F(k, u(k)) \right) - \lambda \sum_{k=1}^{T} f(k, u(k)) u(k) \\
&= r^- \sum_{k=1}^{T} \frac{1}{p(k)} \left(|\triangle u(k)|^{p(k)} + \sqrt{1 + |\triangle u(k)|^{2p(k)}} - 1 \right) - \sum_{k=1}^{T} \left(1 + \phi_{p(k)}(\triangle u(k))\right) |\triangle u(k)|^{p(k)} \\
&\geq \frac{r^-}{p^+} \sum_{k=1}^{T} |\triangle u(k)|^{p(k)} - \sum_{k=1}^{T} |\triangle u(k)|^{p(k)} = \left(\frac{r^-}{p^+} - 1 \right) \sum_{k=1}^{T} |\triangle u(k)|^{p(k)}.
\end{aligned}$$

Recall the Ambrosetti-Rabinowitz condition:

$$\frac{r^-}{\xi} \leq \frac{f(k,\xi)}{F(k,\xi)}, \quad \text{for all} \quad (k,\xi) \in \mathbb{Z}[1,T] \times \mathbb{R} \quad \text{and for some} \quad r^- > p^+. \tag{3.8}$$

Integrating, we obtain that (3.2) holds (see [6, Remark 5.2] or [8, Remark 3.7]).

Combining the above inequalities with (2.2), (3.5), (3.8) and Lemma 2.3, it follows that

$$
\begin{aligned}
& r^- \left(J_\lambda(u) + \lambda \sum_{k=1}^{T} F(k,u(k)) \right) - \lambda \sum_{k=1}^{T} f(k,u(k))u(k) \\
& \leq r^- J_\lambda(u) + r^- \lambda \sum_{k=1}^{T} \frac{1}{r^-} f(k,u(k))u(k) - \lambda \sum_{k=1}^{T} f(k,u(k))u(k) \\
& = r^- J_\lambda(u) = r^- \inf_{h\in\Gamma} \max_{s\in[0,1]} J_\lambda(h(s)) \leq r^- \max_{s\in[0,1]} J_\lambda(se) \leq r^- \max_{s\geq 0} J_\lambda\left(s\frac{e}{\|e\|_{p(\cdot)}} \right) \\
& \leq r^- \max_{s\geq 0} \left(\frac{T}{p^-} 2^{p^+ +1} \left(T^{\frac{p^- -1}{p^-}} \right)^{p^+} s^{p^+} \frac{\|e\|^{p^+}}{\|e\|_{p(\cdot)}^{p^+}} - \lambda\varepsilon 2^{-r^-} T^{\frac{p^- - r^-}{p^-}} s^{r^-} \frac{\|e\|^{r^-}}{\|e\|_{p(\cdot)}^{r^-}} + \frac{2T}{p^-} + \lambda T C_\eta \right) \\
& \leq r^- \max_{s\geq 0} \left(\frac{T}{p^-} 2^{p^+ +1} \left(T^{\frac{p^- -1}{p^-}} \right)^{p^+} s^{p^+} K_2^{p^+} - \lambda\varepsilon 2^{-r^-} T^{\frac{p^- - r^-}{p^-}} s^{r^-} K_1^{r^-} + \frac{2T}{p^-} + \lambda T C_\eta \right),
\end{aligned}
$$

where $e \in H$ is given by Lemma 3.2 (ii). Hence from (2.3), we infer that

$$
\begin{aligned}
& \left(\frac{r^-}{p^+} - 1 \right) \|u\|_{p(\cdot)}^{p^-} \leq \left(\frac{r^-}{p^+} - 1 \right) \rho_{p(\cdot)}(u) = \left(\frac{r^-}{p^+} - 1 \right) \sum_{k=1}^{T} \frac{1}{p(k)} |\triangle u(k)|^{p(k)} \\
& \leq \frac{r^-}{p^-} \max_{s\geq 0} \left(\frac{T}{p^-} 2^{p^+ +1} \left(T^{\frac{p^- -1}{p^-}} \right)^{p^+} s^{p^+} K_2^{p^+} - \lambda\varepsilon 2^{-r^-} T^{\frac{p^- - r^-}{p^-}} s^{r^-} K_1^{r^-} + \frac{2T}{p^-} + \lambda T C_\eta \right).
\end{aligned}
$$

Let

$$
\sigma(s) = \frac{T}{p^-} 2^{p^+ +1} \left(T^{\frac{p^- -1}{p^-}} \right)^{p^+} s^{p^+} K_2^{p^+} - \lambda\varepsilon 2^{-r^-} T^{\frac{p^- - r^-}{p^-}} s^{r^-} K_1^{r^-} + \frac{2T}{p^-} + \lambda T C_\eta
$$

and $\frac{d\sigma}{ds}(s) = 0$. Since $r^- > p^+$, then $\sigma(s) \to -\infty$ as $s \to \infty$.

Therefore,

$$
\frac{d\sigma}{ds}(s) = \frac{T}{p^-} 2^{p^+ +1} \left(T^{\frac{p^- -1}{p^-}} \right)^{p^+} p^+ K_2^{p^+} s^{p^+ -1} - \lambda\varepsilon 2^{-r^-} T^{\frac{p^- - r^-}{p^-}} r^- K_1^{r^-} s^{r^- -1},
$$

which implies that

$$
s^{r^- - p^+} = \frac{\frac{T}{p^-} 2^{p^+ +1} \left(T^{\frac{p^- -1}{p^-}} \right)^{p^+} p^+ K_2^{p^+}}{\lambda\varepsilon 2^{-r^-} T^{\frac{p^- - r^-}{p^-}} r^- K_1^{r^-}}.
$$

So,

$$
s = \left[\frac{\frac{T}{p^-} 2^{p^+ +1} \left(T^{\frac{p^- -1}{p^-}} \right)^{p^+} p^+ K_2^{p^+}}{\lambda\varepsilon 2^{-r^-} T^{\frac{p^- - r^-}{p^-}} r^- K_1^{r^-}} \right]^{\frac{1}{r^- - p^+}}.
$$

Let

$$\kappa_2^* = \left[\frac{\frac{r^-}{p^-}\left(\frac{T}{p^-}2^{p^++1}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} s^{p^+}K_2^{p^+} - \lambda\varepsilon 2^{-r^-}T^{\frac{p^--r^-}{p^-}}s^{r^-}K_1^{r^-} + \frac{2T}{p^-} + \lambda T C_\eta\right)}{\frac{r^-}{p^+} - 1} \right]^{1/p^-}.$$

Thus, by the definition of σ, one has

$$\sigma_{max}(s) \geq \frac{2T}{p^-} + \lambda T C_\eta,$$

which is equivalent to saying

$$r^-\sigma_{max}(s) \geq \frac{r^-}{p^-}2T + r^-\lambda T C_\eta > \frac{r^-}{p^-}2T > \frac{r^-}{p^-} \geq \frac{r^-}{p^+} > \frac{r^-}{p^+} - 1.$$

Since $r^- > p^+$ and $2 < p^- \leq p(\cdot) < p^+ < \infty$, we infer that $\kappa_2^* > 1$ and by (2.2), there exist some constants $\kappa_1 = K_1\kappa_1^*$, $\kappa_2 = K_2\kappa_2^*$ such that $\kappa_1 \leq \|u\| \leq \kappa_2$.

The proof of Lemma 3.5 is then complete. □

Next, we examine conditions under which our problem (1.1) has a unique non trivial solution.

Lemma 3.6. *There exists a constant $c > 0$ such that for all $k \in \mathbb{Z}[1,T]$ and $s > 0$,*

$$\min\left\{ \left(1+\phi_{p(k)}(s)\right)s^{p(k)-2}, s^{p(k)-1}\frac{\partial \phi_{p(k)}}{\partial s}(s) + (p(k)-1)\left(1+\phi_{p(k)}(s)\right)s^{p(k)-2} \right\} \geq cs^{p(k)-2},$$

where $c = \min\{1, p^- - 1\}$.

Proof. For all $s > 0$, we observe that

$$\left(1+\phi_{p(k)}(s)\right)s^{p(k)-2} \geq s^{p(k)-2} = 1 \times s^{p(k)-2}.$$

One also has

$$\frac{\partial \phi_{p(k)}}{\partial s}(s) = \frac{p(k)s^{p(k)-1}}{\left(1+s^{2p(k)}\right)^{3/2}}.$$

At more, one has

$$\begin{aligned} &s^{p(k)-1}\frac{\partial \phi_{p(k)}}{\partial s}(s) + (p(k)-1)\left(1+\phi_{p(k)}(s)\right)s^{p(k)-2} \\ &\quad = (p(k)-1)\,s^{p(k)-2} + \frac{(2p(k)-1)\,s^{2p(k)-2} + (p(k)-1)\,s^{4p(k)-2}}{\left(1+s^{2p(k)}\right)^{3/2}} \geq \left(p^- - 1\right)s^{p(k)-2}. \end{aligned}$$

Hence, for all $s > 0$,

$$\min\{\left(1+\phi_{p(k)}(s)\right) s^{p(k)-2}, s^{p(k)-1}\frac{\partial \phi_{p(k)}}{\partial s}(s) + (p(k)-1)\left(1+\phi_{p(k)}(s)\right) s^{p(k)-2}\} \geq \min\{1, p^- - 1\} s^{p(k)-2}. \quad \square$$

As in [27], one has the following result.

Lemma 3.7. *There exists a positive constant c such that*

$$\left(\left(1+\phi_{p(k)}(\xi)\right)|\xi|^{p(k)-2}\xi - \left(1+\phi_{p(k)}(\eta)\right)|\eta|^{p(k)-2}\eta\right)(\xi-\eta) \geq c4^{2-p(k)}|\xi-\eta|^{p(k)},$$

for all $\xi, \eta \in \mathbb{R}$ with $(\xi,\eta) \neq (0,0)$.

Let us now introduce the following hypothesis.

(H_6) There exist a constant $0 < \delta < \dfrac{p^- c 4^{2-p^+}}{\lambda\left(T^{\frac{p^- -1}{p^-}}\right)^{p^+} T K_2^{p^+}(2\kappa_2^*)^{p^+-p^-}}$ such that

$$|f(k,\xi) - f(k,\eta)| \leq \delta|\xi-\eta|^{p^+-1} \quad \text{for all} \quad k \in \mathbb{Z}[1,T] \quad \text{and} \quad \xi,\eta \in \mathbb{R} \quad \text{with} \quad \xi \neq \eta.$$

One has the following result.

Theorem 3.8. *Under assumptions (H_1)-(H_5) and (H_6), there exists a unique nontrivial solution of problem* (1.1).

Proof. Let u and v be two non-trivial solutions to problem (1.1). Then, by (2.7), we have

$$\sum_{k=1}^{T}\left(1+\phi_{p(k)}(\triangle u(k))\right)|\triangle u(k)|^{p(k)-2}\triangle u(k)\triangle(u-v)(k) = \lambda\sum_{k=1}^{T} f(k,u(k))(u-v)(k) \tag{3.9}$$

and

$$\sum_{k=1}^{T}\left(1+\phi_{p(k)}(\triangle v(k))\right)|\triangle v(k)|^{p(k)-2}\triangle v(k)\triangle(u-v)(k) = \lambda\sum_{k=1}^{T} f(k,v(k))(u-v)(k). \tag{3.10}$$

Subtracting (3.9) and (3.10), we obtain

$$\sum_{k=1}^{T}\Big[\left(1+\phi_{p(k)}(\triangle u(k))\right)|\triangle u(k)|^{p(k)-2}\triangle u(k) - \left(1+\phi_{p(k)}(\triangle v(k))\right)|\triangle v(k)|^{p(k)-2}\triangle v(k)\Big]\triangle(u-v)(k) = \lambda\sum_{k=1}^{T}\left[f(k,u(k)) - f(k,v(k))\right](u-v)(k). \tag{3.11}$$

If $\|u-v\|_{p(\cdot)} \leq 1$, then using (2.4), Lemma 3.6, (3.11), (H_6) and Lemma 2.3 (c), we deduce from (2.2) that

$$\begin{aligned}
c4^{2-p^+}\|u-v\|_{p(\cdot)}^{p^+} &\leq c4^{2-p^+}\rho_{p(\cdot)}(u-v) \leq \frac{1}{p^-}\sum_{k=1}^{T} c4^{2-p(k)}|\triangle u(k)-\triangle v(k)|^{p(k)} \\
&\leq \frac{1}{p^-}\sum_{k=1}^{T}\Bigg(\left(1+\phi_{p(k)}(\triangle u(k))\right)|\triangle u(k)|^{p(k)-2}\triangle u(k) \\
&- \left(1+\phi_{p(k)}(\triangle v(k))\right)|\triangle v(k)|^{p(k)-2}\triangle v(k)\Bigg)(\triangle u(k)-\triangle v(k)) \\
&= \frac{\lambda}{p^-}\sum_{k=1}^{T}\left[f(k,u(k))-f(k,v(k))\right](u-v)(k) \\
&\leq \frac{\lambda\delta}{p^-}\sum_{k=1}^{T}|u(k)-v(k)|^{p^+} \leq \frac{\lambda\delta}{p^-}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} T\|u-v\|^{p^+} \\
&\leq \frac{\lambda\delta}{p^-}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}\|u-v\|_{p(\cdot)}^{p^+}.
\end{aligned}$$

Therefore,

$$\left[c4^{2-p^+} - \frac{\lambda\delta}{p^-}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}\right]\|u-v\|_{p(\cdot)}^{p^+} \leq 0.$$

Recall that the constant δ is such that $\delta < \dfrac{p^-c4^{2-p^+}}{\lambda\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}}$.

Hence, $\|u-v\|_{p(\cdot)}^{p^+} = 0$, which implies that $u=v$.

Now, let $\|u-v\|_{p(\cdot)} \geq 1$. Similarly, we can deduce that

$$c4^{2-p^+}\|u-v\|_{p(\cdot)}^{p^-} \leq c4^{2-p^+}\rho_{p(\cdot)}(u-v) \leq \frac{\lambda\delta}{p^-}\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}\|u-v\|_{p(\cdot)}^{p^+}.$$

Consequently,

$$\|u-v\|_{p(\cdot)}^{p^+-p^-} \geq \frac{p^-c4^{2-p^+}}{\lambda\delta\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}}.$$

Which is equivalent to say

$$\|u-v\|_{p(\cdot)} \geq \left[\frac{p^-c4^{2-p^+}}{\lambda\delta\left(T^{\frac{p^--1}{p^-}}\right)^{p^+} TK_2^{p^+}}\right]^{\frac{1}{p^+-p^-}}.$$

It is then clear that if u, v are solutions to problem (1.1) and $\delta < \dfrac{p^- c 4^{2-p^+}}{\lambda \left(T^{\frac{p^- -1}{p^-}}\right)^{p^+} T K_2^{p^+} (2\kappa_2^*)^{p^+ - p^-}}$,

then

$$2\kappa_2^* < \|u-v\|_{p(\cdot)} \leq \|u\|_{p(\cdot)} + \|v\|_{p(\cdot)} \leq 2\kappa_2^*.$$

This contradicts the assumption that $\|u-v\|_{p(\cdot)} \geq 1$. Consequently, it follows that $u = v$. □

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Further results on the metric dimension and spectrum of graphs

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ABSTRACT

The concept of metric dimension in graphs has the aim of finding a set of vertices in a graph with the smallest size that can be used as a reference to identify all vertices in the graph uniquely. Formally, let G be a connected graph, and let $S = \{s_1, \ldots, s_k\} \subseteq V(G)$ be an ordered set. For every $v \in V(G)$, we define $r(v|S) = (d(v, s_1), \ldots, d(v, s_k))$ where d is the distance function of G. We call S a *resolving set* if $r(u|S) \neq r(v|S)$ for every $u, v \in V(G)$, $u \neq v$. The *metric dimension* of G, denoted by $\dim(G)$, is the smallest integer k such that G has a resolving set of size k. Recently, the authors have initiated research on the relation between the metric dimension of a graph and its nullity (that is, the multiplicity of 0 in its adjacency spectrum), and we have obtained several results. In this paper, we present some new relationships between the metric dimension and the spectrum of graphs. In detail, we present an inequality involving the metric dimension and nullity of any bipartite or singular graph. Then, we give an infinite class of graphs having equal metric dimension and nullity using the rooted product of graphs. Finally, for any connected graph G other than a path, we show that a submatrix of the distance matrix of G, associated with a minimal resolving set of G, has the full-rank property.

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RESUMEN

El concepto de dimensión métrica en grafos tiene como propósito encontrar un conjunto de vértices en un grafo con el menor tamaño que puede usarse como referencia para identificar únicamente todos los vértices del grafo. Formalmente, sea G un grafo conexo, y sea $S = \{s_1, \dots, s_k\} \subseteq V(G)$ un conjunto ordenado. Para todo $v \in V(G)$, definimos $r(v|S) = (d(v, s_1), \dots, d(v, s_k))$ donde d es la función de distancia de G. Llamamos a S un *conjunto resolvente* si $r(u|S) \neq r(v|S)$ para todo $u, v \in V(G)$, $u \neq v$. La *dimensión métrica* de G, denotada por $\dim(G)$, es el entero más pequeño k tal que G tiene un conjunto resolvente de tamaño k. Recientemente, los autores han comenzado a investigar sobre la relación entre la dimensión métrica de un grafo y su nulidad (es decir, la multiplicidad de 0 en su espectro de adyacencia), y hemos obtenido diversos resultados. En este artículo, presentamos algunas relaciones nuevas entre la dimensión métrica y el espectro de grafos. En detalle, presentamos una desigualdad que involucra la dimensión métrica y la nulidad de cualquier grafo bipartito o singular. Luego, entregamos una clase infinita de grafos con igual dimensión métrica y nulidad usando el producto enraizado de grafos. Finalmente, para todo grafo conexo G distinto de un camino, mostramos que una submatriz de la matriz de distancia de G, asociada a un conjunto resolvente mínimo de G, tiene la propiedad de rango completo.

Keywords and Phrases: Metric dimension, spectrum, nullity, distance matrix.

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1 Introduction

In the 1960s, Slater [14] and Harary and Melter [10] independently introduced the concept of metric dimension of graphs. They introduced the term *locating set* or *resolving set* which refers to a set of vertices used to identify each vertex in a graph uniquely. A resolving set with the smallest size is called a *basis*, and its cardinality is referred to as the *metric dimension* of the graph. Since the metric dimension of graphs and its variations have direct applicability to several real-world issues like robot navigation [12] and chemistry [3], research on them has grown rapidly in the recent few decades. See, for example, [15] and [13] for surveys on this topic. On the other hand, in 1972, Cvetković, Gutman, and Trinajstić [5], and then Cvetković and Gutman [4], introduced the nullity of a graph as a new invariant; it is the multiplicity of 0 as an eigenvalue of the graph's adjacency matrix. They further investigated the connection between graph nullity and chemical structures. Excellent overviews of graph nullity can be found in [1] and [9].

Despite the growth of interest in the metric dimension of graphs, its connection to the graph's spectrum has not been studied further. Recently, the authors [7] have initiated research on the relation between the metric dimension of a graph and its spectrum, and we have obtained several results. This research was motivated by the observation that the equality $\dim(G) = \eta(G)$, where $\dim(G)$ and $\eta(G)$ respectively denote the metric dimension and nullity of the graph G, holds for complete bipartite graphs $K_{r,s}$ where $r \neq s$, paths P_n where n is odd, and cycles C_n where $n \equiv 0 \pmod 4$. This paper aims to provide further connections between the two concepts. In detail, we first give an inequality involving $\dim(G)$ and $\eta(G)$ for any bipartite or singular graphs G, generalizing our previous result for trees. Then, we give an infinite class of graphs G where $\dim(G) = \eta(G)$ using the rooted product of graphs. Finally, we give another relation between the metric dimension of a graph and its distance matrix. We show that for any connected graph G, a submatrix of its distance matrix, associated with a minimal resolving set of G, has the full-rank property.

All the graphs considered in this study are finite, simple, and undirected. We refer to Diestel [6] for the basic definitions related to graphs. An *empty graph* $\emptyset$ is the graph without any vertices and edges. Let $G = (V(G), E(G))$ be a graph. We simply write $V = V(G)$ and $E = E(G)$ if the graph is clear from context. Two vertices $u, v \in V$ are said to be *adjacent* if $uv \in E$. The *open neighborhood* of a vertex $u \in V$ is the set $N_G(u) := \{v \in V : uv \in E\}$, and the *closed neighborhood* of u is $N_G[u] := \{u\} \cup N_G(u)$. The *degree* of a vertex $u \in V$, denoted by $\deg(u)$, is the size of $N_G(u)$. A vertex is called *pendant* if it has degree one, and let $p(G)$ denote the number of pendant vertices of G. For two distinct vertices u, v in a graph G, the *distance* $d(u, v)$ of u and v is the length of a shortest path connecting u and v. We denote by P_n, C_n, $K_{m,n}$, and K_n for paths, cycles, complete bipartite, and complete graphs. For two integers $a \leq b$, we define $[a, b] := \{x \in \mathbb{Z} : a \leq x \leq b\}$.

Let $u, v \in V$, $u \neq v$. We say that a vertex $s \in V$ *resolves* u and v if $d(u, s) \neq d(v, s)$. Let

$S = \{s_1, s_2, \ldots, s_k\} \subseteq V$ be an ordered subset of V. The *representation* of $v \in V$ with respect to S, denoted by $r(v|S)$, is the vector $r(v|S) = (d(v,s_1), d(v,s_2), d(v,s_3), \ldots, d(v,s_k))$. We call S a *resolving set* of G if $r(u|S) \neq r(v|S)$ for every distinct pair $u, v \in V$, that is, if each vertex of G has a unique representation with respect to S. In other words, S is a resolving set if and only if every pair of distinct vertices $u, v \in V$ is resolved by an element of S. A resolving set of G with minimum size is called a *basis* of G. The cardinality of a basis of G is called the *metric dimension* of G which is denoted by $\dim(G)$. A resolving set of G is called minimal if for every $S_0 \subset S$, S_0 is not a resolving set of G, that is, S does not contain a smaller resolving set of G.

Let $G = (V, E)$ be a graph of order n with $V = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of G is the $n \times n$ matrix $\mathbf{A} = \mathbf{A}(G) = (a_{ij})$ whose entry a_{ij} is equal to 1 if v_i and v_j are adjacent, and 0 otherwise. The *distance matrix* of G is the matrix $\mathbf{D} = \mathbf{D}(G) = (d_{ij})$, where $d_{ij} = d(v_i, v_j)$. For $\mathbf{M} \in \{\mathbf{A}, \mathbf{D}\}$, the $\mathbf{M}$-*spectrum* of G, denoted by $\mathrm{spec}_{\mathbf{M}}(G)$, is the set of eigenvalues of $\mathbf{M}(G)$ together with their multiplicities. If the distinct eigenvalues of $\mathbf{M}(G)$ are $\lambda_1 > \lambda_2 > \cdots > \lambda_s$, and their multiplicities are $m_1, m_2, \ldots, m_s$, respectively, then we write $\mathrm{spec}_{\mathbf{M}}(G) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_s^{m_s}\}$. For an eigenvalue λ, we may write $m_{\mathbf{M}}(\lambda)$ to denote the multiplicity of λ in $\mathrm{spec}_{\mathbf{M}}(G)$. The *nullity* of G, denoted by $\eta(G)$, is the multiplicity of eigenvalue 0 in $\mathrm{spec}_{\mathbf{A}}(G)$, that is, $\eta(G) = m_{\mathbf{A}}(0)$. We call a graph G *singular* if $\eta(G) > 0$. For the trivial case, we define $\eta(\emptyset) = 0$.

2 Preliminary Results

In this section, we provide some known results that are useful in our discussions.

Theorem 2.1 ([3,12])**.** *A graph G has $\dim(G) = 1$ if and only if G is a path.*

Theorem 2.2 ([15])**.** *For every integer $n \geq 3$, $\dim(C_n) = 2$.*

Let G and H be two graphs. The union $G \cup H$ is the graph where $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The join $G \vee H$ is the graph obtained by taking the two graphs and connecting, by an edge, each vertex in G to each vertex in H. Furthermore, the complement $\overline{G}$ of G has $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : uv \notin E(G), u, v \in V(G)\}$.

Theorem 2.3 ([3])**.** *Let G be a graph of order $n \geq 4$. Then, $\dim(G) = n - 2$ if and only if $G = K_{r,s}$ $(r, s \geq 1)$, $G = K_s \vee \overline{K_t}$ $(s \geq 1, t \geq 2)$, or $G = K_s \vee (K_1 \cup K_t)$ $(s, t \geq 1)$.*

For the case of trees, we need the following definitions. A vertex of degree at least 3 in a graph G is called a *major vertex* of G. A pendant vertex u of G is called a *terminal vertex of a major vertex* v of G if $d(u, v) < d(u, w)$ for every other major vertex w of G. In other words, a pendant vertex u is a terminal vertex of v if v is the closest major vertex from u. The *terminal degree* $\mathrm{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is called an *exterior major vertex* of G if $\text{ter}(v) > 0$. Let $\sigma(G)$ denote the sum of the terminal degrees of all major vertices of G, and let $\text{ex}(G)$ denote the number of exterior major vertices of G. With these definitions, we may calculate the metric dimension of trees other than a path by the following formula.

Theorem 2.4 ([3, 12, 14])**.** *If T is a tree other than a path, then*

$$\dim(T) = \sigma(T) - \text{ex}(T) = \sum_{\substack{v \in V \\ \text{ter}(v) > 1}} (\text{ter}(v) - 1).$$

The proof of Theorem 2.4 utilizes the following general bound for any connected graphs.

Lemma 2.5 ([3])**.** *If G is a connected graph, then* $\dim(G) \geq \sigma(G) - \text{ex}(G)$.

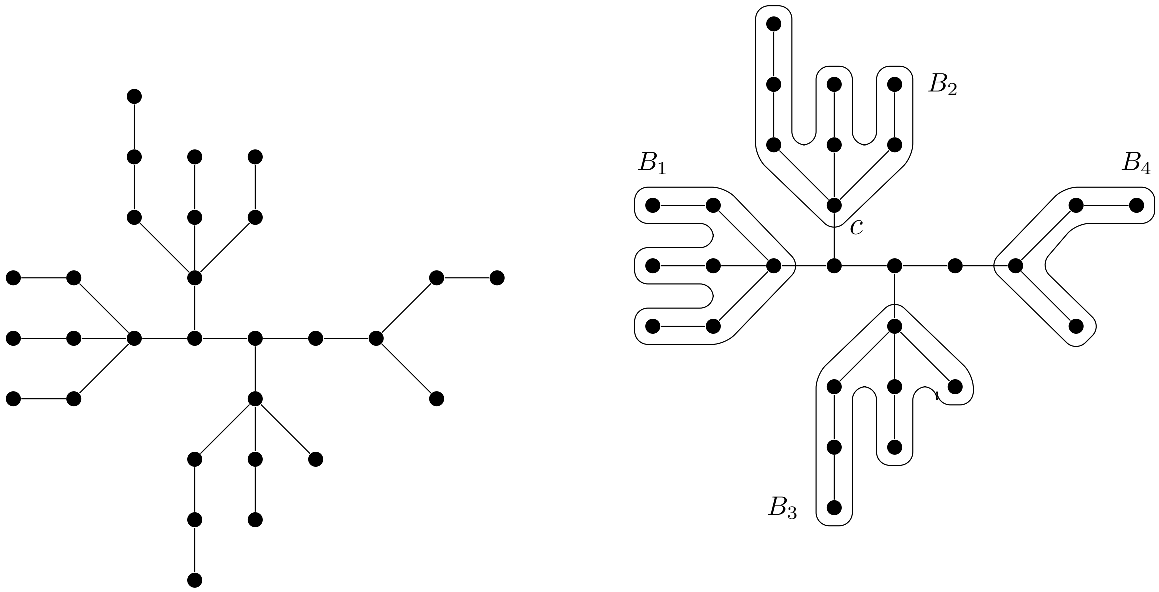
For an exterior major vertex v in G, a *tail of* v is a path connecting v to one of its terminal vertex, excluding v. Thus, an exterior major vertex v has $\text{ter}(v)$ tails. We call a tail *odd* or *even* if it has an odd or even number of vertices, respectively. A *branch* B is a subgraph of G induced by an exterior major vertex v in G and all its tails. In this case, we call v the *stem vertex* of B. Thus, a branch with n tails is a subdivision of the star graph $K_{1,n}$. We say a branch B is of Type I if it has at least one odd tail and Type II otherwise. In Figure 1b, the branches of T in Figure 1a are the blocked subgraphs B_1, B_2, B_3, and B_4. The vertex c is the stem of B_2. The branches B_2, B_3, and B_4 are of Type I, while the branch B_1 is of Type II. With these additional definitions, observe that the second equality in Theorem 2.4 indicates that the metric dimension of a tree depends only on the structure of its branches.

We now discuss the rooted and corona product of graphs. Let G be a graph where $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $\mathcal{H}$ be a set of n graphs $H_1, H_2, \ldots, H_n$ where a vertex in H_i is chosen as the *root* of H_i, $i \in [1, n]$. The *rooted product* of G by $\mathcal{H}$, denoted by $G(\mathcal{H})$, is the graph obtained by identifying the root of H_i and v_i for every $i \in [1, n]$ [8]. A special case of rooted product of graphs is the caterpillar graph. A caterpillar is a tree such that the removal of its pendants produces a path. For positive integers k and $n_1, n_2, \ldots, n_k$, a caterpillar $CP(n_1, n_2, \ldots, n_k)$ is the graph $P_k(\{K_{1,n_1}, \ldots, K_{1,n_k}\})$ by taking the center vertex of each K_{1,n_i} as its root.

Let G and H be two graphs with $|G| = n$. The *corona product* $G \odot H$ is defined as the graph obtained by taking one copy of G and n copies of H, and we connect (by an edge) every vertex in the ith copy of H with the ith vertex of G [16]. For the case where $H = \overline{K_m}$ for some positive integer m, we have $G \odot \overline{K_m} = G(\mathcal{H})$ where $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$, $H_i = K_{1,m}$ for every $i \in [1, n]$.

Theorem 2.6 ([11])**.** *If G is a connected graph of order n, and $t \in \mathbb{N}$, $t \geq 2$, then* $\dim(G \odot \overline{K_t}) = n(t-1)$

Theorem 2.7 ([11])**.** *If G is a connected graph of order n, and $\mathcal{H} = \{K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_n}\}$ where $m_i \geq 2$ for every $i \in [1, n]$, then* $\dim(G(\mathcal{H})) = \sum_{i=1}^{n} (m_i - 1)$.



(a) The tree T (b) The branches of T

Figure 1: A tree and its branches

We now discuss the results related to the spectrum and nullity of graphs.

Theorem 2.8 ([2])**.**

(1) *For every positive integers* r, s, $\text{spec}_{\mathbf{A}}(K_{r,s}) = \{\pm\sqrt{rs}, 0^{r+s-2}\}$.

(2) *For every integer* $n \geq 2$, $\text{spec}_{\mathbf{A}}(C_n) = \{2\cos(2\pi k/n) : k \in [1,n]\}$.

(3) *For every integer* $n \geq 1$, $\text{spec}_{\mathbf{A}}(P_n) = \{2\cos(\pi k/(n+1)) : k \in [1,n]\}$.

We can see from Theorem 2.8 that $\eta(K_{r,s}) = r + s - 2$; $\eta(C_n) = 2$ if $n \equiv 0 \pmod 4$, and 0 if otherwise; and $\eta(P_n) = 1$ if n is odd, and 0 if n is even. The following observation is immediate from Theorems 2.8, 2.1, 2.2, and 2.3.

Observation 2.9. *The condition* $\dim(G) = \eta(G)$ *holds if* G *is one of the following graphs:*

(1) $K_{r,s}$ *where* $r \neq s$, *or*

(2) C_n *where* $n \equiv 0 \pmod 4$, *or*

(3) P_n *where* n *is odd.*

Lemma 2.10 ([9])**.** *Let* G *be a graph order* n. *Then,* $\eta(G) = n$ *if and only if* $G = \overline{K_n}$.

The following lemmas are very useful in many parts of our discussion.

Lemma 2.11 ([4]). *Let G be a bipartite graph containing a pendant vertex, say v, and H be the graph obtained from G by deleting v and its neighbor. Then, $\eta(G) = \eta(H)$.*

Lemma 2.12 ([9]). *Let $G = \bigcup_{i=1}^{t} G_i$, where $G_1, \ldots, G_t$ are connected components of G. Then, $\eta(G) = \sum_{i=1}^{t} \eta(G_i)$.*

We now mention our previous result.

Theorem 2.13 ([7]). *Let T be a tree other than a path. Let $\mathcal{B}_{\mathrm{I}}$ and $\mathcal{B}_{\mathrm{II}}$ be the sets of Type I and Type II branches in T, respectively. Let e_2 be the number of even tails in T. If T has an odd tail, then*

$$\dim(T) = \eta(T) - \eta(T - \mathcal{B}_{\mathrm{I}}) - |\mathcal{B}_{\mathrm{II}}| + e_2,$$

where $T - \mathcal{B}_{\mathrm{I}}$ is the graph obtained from T by deleting all Type I branches in T.

3 Main results

3.1 The metric dimension and nullity of bipartite or singular graphs

We first present an inequality involving $\dim(G)$ and $\eta(G)$ for any connected bipartite/singular graph having an odd tail. The proof of this theorem is similar to the proof of Theorem 2.13. However, for completeness, we present the proof.

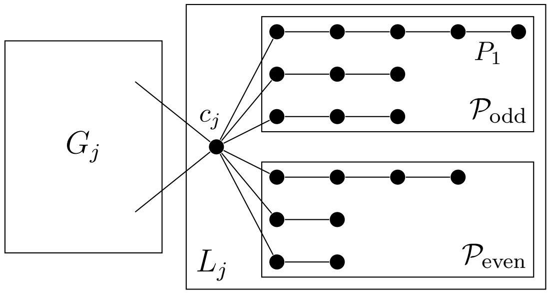
Theorem 3.1. *Let G be a connected bipartite or singular graph other than a path. Let $\mathcal{B}_{\mathrm{I}}$ and $\mathcal{B}_{\mathrm{II}}$ be the sets of Type I and Type II branches in G, respectively. Let e_2 be the number of even tails in G. If G has an odd tail, then*

$$\dim(G) \geq \eta(G) - \eta(G - \mathcal{B}_{\mathrm{I}}) - |\mathcal{B}_{\mathrm{II}}| + e_2$$

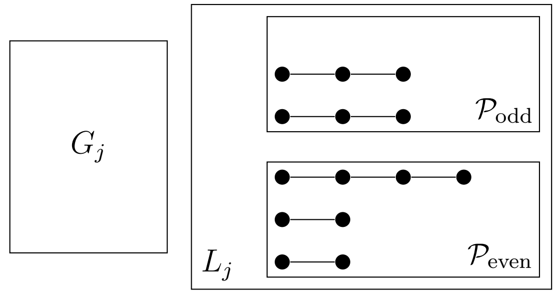
where $G - \mathcal{B}_{\mathrm{I}}$ is the graph obtained from G by deleting all Type I branches in G.

Proof. Let $B_1, \ldots, B_k$ be the branches in G. Since G has at least one odd tail, there exists a Type I branch in G. Suppose that $|\mathcal{B}_{\mathrm{I}}| = p \geq 1$. Without loss of generality, let $\mathcal{B}_{\mathrm{I}} = \{B_1, B_2, \ldots, B_p\}$ and $\mathcal{B}_{\mathrm{II}} = \{B_{p+1}, B_{p+2}, \ldots, B_k\}$. Observe that we may construct a sequence of graphs $G_0, G_1, \ldots, G_p$ where $G_0 := G$, $G_p = G - \mathcal{B}_{\mathrm{I}}$, and $G_j = G_{j-1} - B_j = G - \bigcup_{i=1}^{j} B_i$ for $j \in [1, p]$. So, the graph G_j is obtained from G by deleting the branches $B_1, B_2, \ldots, B_j$ of G.

For an arbitrary $j \in [1, p]$, consider the graph G_{j-1} and Type I branch B_j with stem vertex c_j. Suppose that B_j has $e^{(j)}$ tails, $e_1^{(j)}$ odd tails, and $e_2^{(j)}$ even tails, hence $e^{(j)} = e_1^{(j)} + e_2^{(j)}$ and $e_2 = \sum_{i=1}^{k} e_2^{(i)}$. Let $\mathcal{P}_{\mathrm{odd}}$ be the set of all odd tails of B_j, and let $\mathcal{P}_{\mathrm{even}}$ be the set of all even tails of B_j. Pick an arbitrary odd tail, say P_1, and then delete P_1 and c_j from G_{j-1}. Since P_1 is an odd



(a) G_{j-1}



(b) $G_{j-1} - P_1 - c_j$

Figure 2: The grouping of the vertices in G_{j-1} and $G_{j-1} - P_1 - c_j$

tail, we have $\eta(G_{j-1}) = \eta(G_{j-1} - P_1 - c_j)$ by Lemma 2.11. Observe that the graph $G_{j-1} - P_1 - c_j$ has several connected components (see Figure 2): G_j, odd tails of B_j except P_1, and even tails of B_j. By Lemma 2.11, we have

$$\eta(P) = \begin{cases} 1, & \text{if } P \in \mathcal{P}_{\text{odd}}, \\ 0, & \text{if } P \in \mathcal{P}_{\text{even}}, \end{cases}$$

since successively deleting a pendant vertex and its neighbor of a path yields a single vertex if it has an odd order, and an empty graph if it has an even order.

Consequently, by Lemma 2.12, we have

$$\eta(G_{j-1}) = \eta(G_{j-1} - P_1 - c_j) = \eta(G_j) + \sum_{P \in \mathcal{P}_{\text{odd}}} \eta(P) + \sum_{P \in \mathcal{P}_{\text{even}}} \eta(P) = \eta(G_j) + \left(e_1^{(j)} - 1\right).$$

Therefore, we have the relation $\eta(G_j) = \eta(G_{j-1}) - \left(e_1^{(j)} - 1\right)$ for $j \in [1, p]$. By applying this relation successively, we obtain

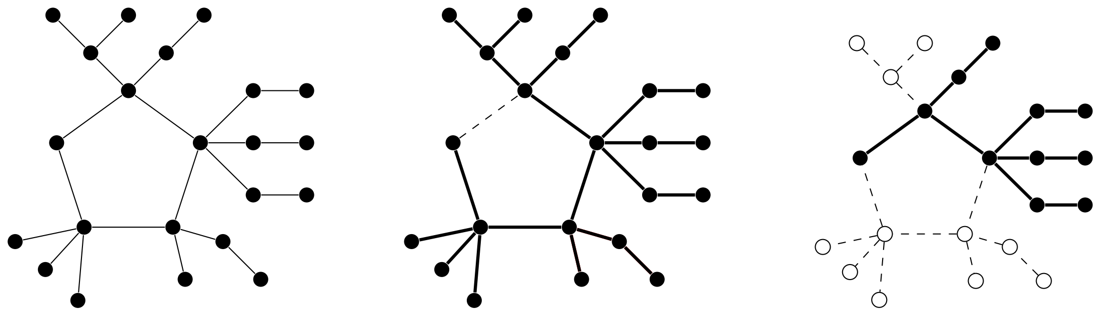
$$\eta(G - \mathcal{B}_{\text{I}}) = \eta(G_p) = \eta(G_0) - \sum_{i=1}^{p} \left(e_1^{(i)} - 1\right) = \eta(G) - \sum_{i=1}^{p} \left(e_1^{(i)} - 1\right).$$

Finally, since $\dim(G) \geq \sum_{i=1}^{k} (e^{(i)} - 1)$ by Lemma 2.5, we have

$$\begin{aligned} \eta(G - \mathcal{B}_{\text{I}}) &= \eta(G) - \sum_{i=1}^{k} \left(e_1^{(i)} - 1\right) + \sum_{i=p+1}^{k} \left(e_1^{(i)} - 1\right) \\ &= \eta(G) - \sum_{i=1}^{k} \left(e^{(i)} - 1 - e_2^{(i)}\right) + \sum_{i=p+1}^{k} (0 - 1) \\ &= \eta(G) - \sum_{i=1}^{k} \left(e^{(i)} - 1\right) + \sum_{i=1}^{k} e_2^{(i)} - (k - p) \\ &\geq \eta(G) - \dim(G) + e_2 - |\mathcal{B}_{\text{II}}|. \end{aligned}$$

□

Example 3.2. *Let G be the graph shown in Figure 3a. The graph $G - \mathcal{B}_{\mathrm{I}}$ is the bold subgraph shown in Figure 3c. With some calculations, we obtain $\eta(G) = 4$ (so G is singular), $\eta(G-\mathcal{B}_{\mathrm{I}}) = 1$, $|\mathcal{B}_{\mathrm{II}}| = 2$, and $e_2 = 5$. Thus, by Theorem 3.1, we obtain* $\dim(G) \geq \eta(G) - \eta(G - \mathcal{B}_{\mathrm{I}}) - |\mathcal{B}_{\mathrm{II}}| + e_2 = 4 - 1 - 2 + 5 = 6$.



(a) G (b) T (c) $G - \mathcal{B}_{\mathrm{I}}$

Figure 3: The graph G, spanning tree T of G, and $G - \mathcal{B}_{\mathrm{I}}$

3.2 The metric dimension and nullity of the rooted product of some graphs

Next, we discuss some relationships between the metric dimension and nullity of the rooted product of some graphs. For certain conditions, this product will establish an infinite class of graphs whose metric dimension and nullity are equal. For that, we need a useful class of graph called *branch graph* which is simply a subdivision of $K_{1,n}$ for some positive integer n. The number of subdivision processes in each "leg" of $K_{1,n}$ is arbitrary. The following proposition gives the metric dimension of $G(\mathcal{H})$ for *any* set of branch graphs $\mathcal{H}$ (see Figure 4). Observe that this proposition generalizes Theorems 2.6 and 2.7.

Proposition 3.3. *Let $\mathcal{H} = \{B_1, B_2, \ldots, B_n\}$ be a set of $n \geq 1$ branch graphs. For every $i \in [1, n]$, the graph B_i has $e_i \geq 2$ tails, and the center of B_i is chosen as the root of B_i. For every connected graph G of order n,* $\dim(G(\mathcal{H})) = \sum_{i=1}^{n}(e_i - 1) = p(G(\mathcal{H})) - n$.

Proof. Let G be a connected graph of order n. First, we show that $\dim(G(\mathcal{H})) \geq \sum_{i=1}^{n}(e_i - 1)$. Let $V(G) = \{v_1, \ldots, v_n\}$. The graph $G(\mathcal{H})$ is obtained by identifying v_i with the center of B_i. Consequently, the pendant vertices of all B_i's become the pendant vertices in $G(\mathcal{H})$, so $p(G(\mathcal{H})) = \sum_{i=1}^{n} e_i$. Moreover, all vertices in G become the exterior major vertices in $G(\mathcal{H})$, so $\mathrm{ex}(G(\mathcal{H})) = n$. Thus, by Lemma 2.5, we have

$$\dim(G(\mathcal{H})) \geq p(G(\mathcal{H})) - \mathrm{ex}(G(\mathcal{H})) = \sum_{i=1}^{n} e_i - n = \sum_{i=1}^{n}(e_i - 1).$$

Next, we show that $\dim(G(\mathcal{H})) \leq \sum_{i=1}^{n}(e_i - 1)$. For every $v_i \in V(G) \subset V(G(\mathcal{H}))$, let $T_i := \{v_i^1, v_i^2, \ldots, v_i^{e_i}\}$ be the set of all terminal vertices of v_i, where v_i^j is the terminal vertex of v_i in the jth tail, $j \in [1, e_i]$. Let $S = \bigcup_{i=1}^{n}(T_i \setminus \{v_i^{e_i}\})$. We will show that S is a resolving set of $G(\mathcal{H})$. Let $x, y \in V(G(\mathcal{H}))$ be two distinct vertices. There are some cases for x and y.

(1) Let $x, y \in V(B_i)$, $i \in [1, n]$, that is, x and y are in the same branch.

  (a) If x and y are in the same tail, say the jth tail, $j \in [1, e_i]$, then $d(x, v_i^1) \neq d(y, v_i^1)$.

  (b) Suppose that x and y are in different tails, say j_1th and j_2th tails, respectively. Observe that at least one of $v_i^{j_1}$ and $v_i^{j_2}$ must be in S; say $v_i^{j_1} \in S$ without loss of generality. Consequently, $d(y, v_i^{j_1}) = d(y, v_i) + d(v_i, x) + d(x, v_i^{j_1}) > d(x, v_i^{j_1})$ since $d(y, v_i) > 0$.

  (c) Suppose that $x = v_i$ and y is in the jth tail. If $j \in [1, e_i - 1]$, then $d(y, v_i^j) < d(x, v_i^j)$. If $j = e_i$, then $d(y, v_i^1) = d(y, x) + d(x, v_i^1) > d(x, v_i^1)$ since $d(y, x) > 0$.

(2) Let $x \in V(B_s)$ and $y \in V(B_t)$, $s \neq t \in [1, n]$, that is, x and y are in different branches. Consequently, $d(v_s, v_t) > 0$.

  (a) If x is in the jth tail, $j \in [1, e_s - 1]$, then wherever y may be in B_t, we have $d(y, v_s^j) = d(y, v_t) + d(v_t, v_s) + d(v_s, x) + d(x, v_s^j) > d(x, v_s^j)$. Similar argument also applies if y is in the jth tail, $j \in [1, e_t - 1]$, that is, $d(x, v_t^j) > d(y, v_t^j)$ wherever x may be in B_s.

  (b) If $x = v_s$ and $y = v_t$, then $d(y, v_s^1) = d(y, v_t) + d(v_t, v_s) + d(v_s, v_s^1) > d(v_s, v_s^1) = d(x, v_s^1)$.

  (c) For the last case, suppose that x and y are in the e_sth and e_tth tails, respectively. If $d(x, v_s^1) \neq d(y, v_s^1)$, then we are done. Now, let us assume that $d(x, v_s^1) = d(y, v_s^1)$. Observe that since $d(v_t, v_s) > 0$, we have

$$
\begin{aligned}
d(x, v_t^1) &= d(x, v_s) + d(v_s, v_t) + d(v_t, v_t^1) \\
&= (d(x, v_s) + d(v_s, v_s^1)) + d(v_s, v_t) + d(v_t, v_t^1) - d(v_s, v_s^1) \\
&= d(x, v_s^1) + d(v_s, v_t) + d(v_t, v_t^1) - d(v_s, v_s^1) \\
&= d(y, v_s^1) + d(v_s, v_t) + d(v_t, v_t^1) - d(v_s, v_s^1) \\
&= (d(y, v_t) + d(v_t, v_s) + d(v_s, v_s^1)) + d(v_s, v_t) + d(v_t, v_t^1) - d(v_s, v_s^1) \\
&= (d(y, v_t) + d(v_t, v_t^1)) + 2d(v_t, v_s) \\
&= d(y, v_t^1) + 2d(v_t, v_s) \\
&> d(y, v_t^1).
\end{aligned}
$$

Thus, for every case of x and y, there is an element of S resolving them. Consequently, S is a resolving set of $G(\mathcal{H})$, and since $|S| = \sum_{i=1}^{n}(e_i - 1)$, we have $\dim(G(\mathcal{H})) \leq \sum_{i=1}^{n}(e_i - 1)$. Therefore, $\dim(G(\mathcal{H})) = \sum_{i=1}^{n}(e_i - 1)$. □

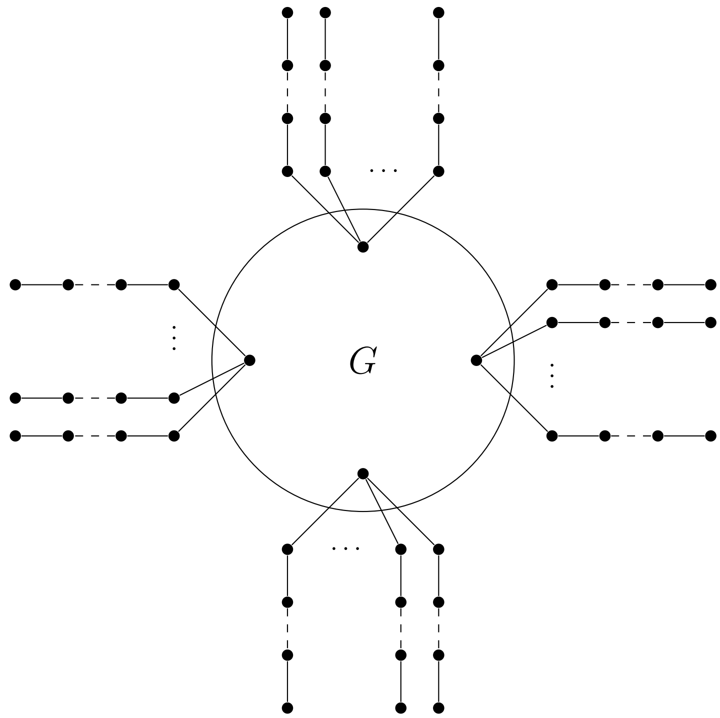


Figure 4: The graph $G(\mathcal{H})$

Theorem 3.4. *Let $\mathcal{H} = \{B_1, B_2, \ldots, B_n\}$ be a set of $n \geq 1$ branch graphs whose tails are all odd tails. For every $i \in [1, n]$, the graph B_i has $e_i \geq 2$ tails, and the center of B_i is chosen as the root of B_i. For every connected bipartite graph G of order n,*

$$\dim(G(\mathcal{H})) = \eta(G(\mathcal{H})) = \sum_{i=1}^{n}(e_i - 1).$$

Proof. From Proposition 3.3, $\dim(G(\mathcal{H})) = \sum_{i=1}^{n}(e_i - 1)$. We only need to show that $\eta(G(\mathcal{H})) = \sum_{i=1}^{n}(e_i - 1)$. Observe that G is bipartite implies $G(\mathcal{H})$ is also bipartite. Consider an arbitrary branch B_i in $G(\mathcal{H})$. By applying Lemma 2.11 consecutively, we may delete one tail from B_i together with the vertex v_i without changing the nullity, that is, the nullity of the resulting graph is the same as of $G(\mathcal{H})$. Moreover, this deletion leaves only $e_i - 1$ tails of B_i. From Lemma 2.11 again, these $e_i - 1$ tails leave $e_i - 1$ isolated vertices (since every tail in B_i is an odd tail) without changing the nullity. Thus, the deletion process on the branch B_i leaves the graph $G(\mathcal{H}) - B_i$ and $e_i - 1$ isolated vertices with the same nullity as $G(\mathcal{H})$. By applying the same process to the other branches, we get a graph consisting of $\sum_{i=1}^{n}(e_i - 1)$ isolated vertices whose nullity equals the nullity of $G(\mathcal{H})$. Thus, $\eta(G(\mathcal{H})) = \sum_{i=1}^{n}(e_i - 1)$. Therefore, $\dim(G(\mathcal{H})) = \eta(G(\mathcal{H}))$. □

The following corollary is a consequence of Theorem 3.4 by observing that corona product of graphs and caterpillar graphs are special cases of rooted product of graphs.

Corollary 3.5. *The condition* $\dim(G) = \eta(G)$ *holds if* G *is one of the following graphs:*

(1) $H \odot \overline{K_p}$ *for every connected bipartite graph* H *and positive integer* $p \geq 2$*, or*

(2) $CP(n_1, n_2, \ldots, n_k)$ *for every positive integers* k *and* $n_i \geq 2$*,* $i \in [1, k]$*.*

In contrast to Theorem 3.4, if all branch graphs in $\mathcal{H}$ have only even tails, then the metric dimension of $G(\mathcal{H})$ is strictly greater than its nullity as we show in the following theorem.

Theorem 3.6. *Let* $\mathcal{H}$ *be a set of* $n \geq 2$ *branch graphs with at least 2 tails whose tails are all even tails, and for every* $B \in \mathcal{H}$*, the center of* B *is chosen as the root of* B*. For every connected bipartite graph* G *of order* n*,* $\dim(G(\mathcal{H})) > \eta(G(\mathcal{H}))$*.*

Proof. Let $\mathcal{H} = \{B_1, \ldots, B_n\}$, where every $B_i \in \mathcal{H}$ has $e_i \geq 2$ tails. Assume to the contrary that there exists a connected bipartite graph G of order n satisfying $\dim(G(\mathcal{H})) \leq \eta(G(\mathcal{H}))$. Since G is connected and has an order $n \geq 2$, we have $G \neq \overline{K_n}$, so $\eta(G) \leq n - 1$ from Lemma 2.10. From Proposition 3.3, we have $\dim(G(\mathcal{H})) = \sum_{i=1}^{n} e_i - n$, and by applying Lemma 2.11 on $G(\mathcal{H})$ consecutively, we obtain $\eta(G(\mathcal{H})) = \eta(G)$. Therefore,

$$n = 2n - n \leq \sum_{i=1}^{n} e_i - n = \dim(G(\mathcal{H})) \leq \eta(G(\mathcal{H})) = \eta(G) \leq n - 1,$$

a contradiction. □

3.3 The metric dimension and distance matrix of graphs

Finally, we discuss a relationship between the metric dimension of a graph and its distance matrix. For that, we need the following notations. For a connected graph G and $\emptyset \neq S \subseteq V(G)$, the distance matrix $\mathbf{D}$ of G can be partitioned into

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}[S] & \mathbf{D}[V \backslash S] \end{bmatrix}$$

where $\mathbf{D}[S] \in \mathbb{R}^{|G| \times |S|}$ and $\mathbf{D}[V \backslash S] \in \mathbb{R}^{|G| \times |V \backslash S|}$ are the submatrices obtained from $\mathbf{D}$ by taking all the columns corresponding to the elements of S and $V \backslash S$, respectively. Observe that the vth row of $\mathbf{D}[S]$ is $r(v|S)^\top$. Observation 3.7 is a direct consequence of this definition. Recall that a resolving set S of G is called *minimal* if S does not contain a smaller resolving set of G. A basis is a minimal resolving set, but the converse is not necessarily true.

Observation 3.7. *Let* G *be a connected graph with distance matrix* $\mathbf{D}$ *and* $\emptyset \neq S \subseteq V(G)$*.*

(1) S *is a resolving set of* G *if and only if* $\mathbf{D}[S]$ *has no two identical rows.*

(2) *S is a minimal resolving set of G if and only if (1) $\mathbf{D}[S]$ has no two identical rows, and (2) for every $s \in S$, $\mathbf{D}[S\backslash\{s\}]$ has two identical rows.*

Theorem 3.8. *Let G be a connected graph other than a path with distance matrix $\mathbf{D}$. If S is a minimal resolving set of G, then $\operatorname{rank}(\mathbf{D}[S]) = |S|$. Consequently, $\dim(G) \leq \operatorname{rank}(\mathbf{D})$.*

Proof. Let S be a minimal resolving set of G with $|S| = k$. Since G is not a path, we have $k \geq 2$ from Theorem 2.1. Let $i \in [1, k]$ be arbitrary. According to Observation 3.7, there are two rows $\mathbf{d}_u = (d_{u1}, \ldots, d_{uk})^\top$ and $\mathbf{d}_v = (d_{v1}, \ldots, d_{vk})^\top$ $(u \neq v)$ of $\mathbf{D}[S]$ such that $d_{us} = d_{vs}$ for every $s \in [1, k]\backslash\{i\}$, but $d_{ui} > d_{vi}$, without loss of generality. Define $c_i := d_{ui} - d_{vi} > 0$. Observe that $\frac{1}{c_i}(\mathbf{d}_u - \mathbf{d}_v) = \mathbf{e}_i$ where $\mathbf{e}_i := (0, \ldots, 0, 1, 0, \ldots, 0)^\top$ with entry 1 is in the ith column. This means that $\mathbf{e}_i$ is in the row space of $\mathbf{D}[S]$. Since $i \in [1, k]$ is arbitrary, the linearly independent set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$ is contained in the row space of $\mathbf{D}[S]$, hence $\operatorname{rank}(\mathbf{D}[S]) \geq |S|$. By the property of rank, we obtain $\operatorname{rank}(\mathbf{D}[S]) \leq \min\{|G|, |S|\} = |S|$. Therefore, $\operatorname{rank}(\mathbf{D}[S]) = |S|$. Consequently, $\dim(G) \leq |S| = \operatorname{rank}(\mathbf{D}[S]) \leq \operatorname{rank}(\mathbf{D})$. □

The contrapositive of Theorem 3.8 and the fact that $\operatorname{rank}(\mathbf{D}[S]) \leq |S|$ produce the following corollary.

Corollary 3.9. *Let G be a connected graph other than a path with distance matrix $\mathbf{D}$. If S is a resolving set of G and $\operatorname{rank}(\mathbf{D}[S]) < |S|$, then S contains a smaller resolving set of G.*

4 Conclusion and open problems

In this paper, we gave a lower bound of the metric dimension $\dim(G)$ of any connected bipartite/singular graph G in terms of its nullity $\eta(G)$. Then, we gave infinite examples of graphs having equal metric dimension and nullity using the rooted product of graphs. We found that $\dim(G(\mathcal{H})) = \eta(G(\mathcal{H}))$ if $\mathcal{H}$ is the set of branch graphs having only odd tails and having at least two tails. It is still an open problem to characterize or list other graphs having equal metric dimension and nullity.

Problem 4.1. *Give other examples of graphs G with $\dim(G) = \eta(G)$.*

Another interesting problem is to investigate $\dim(G(\mathcal{H}))$ when $\mathcal{H}$ is the set of complete graphs of order at least 3. As a preliminary observation, it is known that for every integer $n \geq 2$, $\dim(K_n) = n - 1$. On the other hand, we also have $m_{\mathbf{A}(K_n)}(-1) = n - 1$, thus $\dim(K_n) = m_{\mathbf{A}(K_n)}(-1)$. We conjectured that there is a relationship between the metric dimension of a graph with the multiplicity of eigenvalue -1 through the existence of a clique.

Problem 4.2. *Investigate the relationships between the metric dimension of a graph having cliques and the multiplicity of -1 in their spectrum. In particular, if $F = G(\mathcal{H})$ where G is any connected* *bipartite graph and* $\mathcal{H}$ *is the set of complete graphs of order at least 3, then compare* $\dim(F)$ *and* $m_{\mathbf{A}(F)}(-1)$.

Lastly, we gave a relationship between the metric dimension of a graph and its distance matrix. We showed that if S is a minimal resolving set of G having distance matrix $\mathbf{D}$, then $\mathbf{D}[S]$ is full-rank. Since the metric dimension of a graph is closely related to the graph distance, there may be more relationships between the metric dimension and the distance matrix of a graph.

Problem 4.3. *Find other relationships between the metric dimension of a graph and its distance matrix.*

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Modified convergences to the Euler-Mascheroni constant

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ABSTRACT

We introduce in this paper some new sequences that converge to the Euler-Mascheroni constant. These sequences have a higher convergence rate than the classical one. Further properties are given.

RESUMEN

En este artículo introducimos nuevas sucesiones que convergen a la constante de Euler-Mascheroni. Estas sucesiones tienen una tasa de convergencia mayor que la clásica. También entregamos propiedades adicionales de las mismas.

Keywords and Phrases: Euler-Mascheroni constant, Euler's gamma function, digamma function, approximations, speed of convergence.

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1 Introduction and motivation

The Euler-Mascheroni constant, represented by the symbol gamma γ, is a key mathematical constant that appears in numerous areas of number theory and analysis. Introduced by the Swiss mathematician Leonhard Euler in 1734, this constant is defined as the limit of the difference between the harmonic series and the natural logarithm. Mathematically, it is defined as the limit of the sequence:

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} + \ln \frac{1}{n}.$$

The approximate value of γ is $0.57721\ldots$, although its precise nature –whether it is rational or irrational– remains unresolved in the field of mathematics.

Throughout history, the Euler-Mascheroni constant has been extensively studied and computed. Euler initially determined its value to six decimal places, and later mathematicians, including the Italian mathematician Lorenzo Mascheroni, have worked to refine this calculation.

Despite its long-standing history, many aspects of γ continue to captivate mathematicians, making it a subject of ongoing research and investigation.

In particular, many researchers are focused on developing new, rapidly converging sequences to approximate γ.

This interest stems from the hypothesis that the unresolved question of whether γ is rational or irrational may be attributed to the slow convergence rate of the classical sequence $(\gamma_n)_{n\geq 1}$.

Recent studies have introduced various sequences with faster convergence rates (but a sacrifice of simplicity), aiming to shed light on the true nature of this enigmatic number. The methods used range from modifying some terms from the harmonic series to changing the argument of the logarithm to polynomial or rational functions. See, *e.g.*, [2–5].

This paper aims to introduce some new faster convergences to γ, keeping a simple form.

2 The results

Along with the classical sequence $(\gamma_n)_{n\geq 1}$ (that converges to γ decreasingly), the following sequence

$$\gamma_n' = \sum_{k=1}^{n} \frac{1}{k} + \ln \frac{1}{n+1}$$

converges increasingly to γ.

Both sequences $(\gamma_n)_{n\geq 1}$ and $(\gamma_n')_{n\geq 1}$ converge to γ like n^{-1}, since

$$\lim_{n\to\infty} n(\gamma_n - \gamma) = \frac{1}{2} \quad \text{and} \quad \lim_{n\to\infty} n(\gamma_n' - \gamma) = -\frac{1}{2}.$$

We introduce in this paper new sequences by modifying the argument of the logarithm to $\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)$, then to $\frac{1}{n^2}+\frac{1}{(n+1)^2}$.

For the sake of simplicity, we propose the sequence

$$\mu_n=\sum_{k=1}^{n}\frac{1}{k}+\ln\left(\frac{1}{n}+\frac{1}{n+1}\right)$$

that converges (to $\gamma+\ln 2$) at a higher rate of convergence, as we can see from the following:

Theorem 2.1. *a) The sequence $(\mu_n)_{n\geq 1}$ converges decreasingly to $\gamma+\ln 2$, at a rate of convergence n^{-2}. More precisely,*

$$\lim_{n\to\infty} n^2\left(\mu_n-(\gamma+\ln 2)\right)=\frac{7}{24}.$$

b) The following inequalities hold true, for every integer $n\geq 1$:

$$\frac{7}{24\,(n+1)\,(n+2)}\leq \mu_n-(\gamma+\ln 2)\leq \frac{7}{24n\,(n+1)}.$$

Keeping in mind that the number $\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)$, which appears in the expression of the sequence $(\mu_n)_{n\geq 1}$, is the arithmetic mean of $\frac{1}{n}$ and $\frac{1}{n+1}$, we introduce the following sequence involving the quadratic mean of $\frac{1}{n}$ and $\frac{1}{n+1}$:

$$\eta_n=\sum_{k=1}^{n}\frac{1}{k}+\frac{1}{2}\ln\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}\right).$$

The sequence $(\eta_n)_{n\geq 1}$ converges $\left(\text{to } \gamma+\frac{1}{2}\ln 2\right)$ with a rate of convergence n^{-2}, as we can see from the following:

Theorem 2.2. *a) The sequence $(\eta_n)_{n\geq 1}$ converges decreasingly to $\gamma+\frac{1}{2}\ln 2$, at a rate of convergence n^{-2}. More precisely,*

$$\lim_{n\to\infty} n^2\left(\eta_n-\left(\gamma+\frac{1}{2}\ln 2\right)\right)=\frac{5}{12}.$$

b) The following inequalities hold, for every integer $n\geq 1$:

$$\frac{5}{12\,(n+1)\,(n+2)}\leq \eta_n-\left(\gamma+\frac{1}{2}\ln 2\right)\leq \frac{5}{12n\,(n+1)}.$$

3 The proofs

A main tool for computing the speed of convergence is the following lemma, first stated in [6].

Lemma 3.1. *If* $(x_n)_{n\geq 1}$ *is convergent to zero and*

$$\lim_{n\to\infty} n^k \left(x_n - x_{n+1}\right) = l \in (-\infty, \infty),$$

for some $k > 1$ *and* $l \neq 0$, *then*

$$\lim_{n\to\infty} n^{k-1} x_n = \frac{l}{k-1}.$$

This lemma is useful especially when the sequence $(x_n)_{n\geq 1}$ is defined as a sum and consequently, the difference $x_n - x_{n+1}$ becomes of a simpler form.

Proof of Theorem 1. *a*) We have $\mu_n - \mu_{n+1} = f(n)$, where

$$f(x) = -\frac{1}{x+1} + \ln\left(\frac{1}{x} + \frac{1}{x+1}\right) - \ln\left(\frac{1}{x+1} + \frac{1}{x+2}\right).$$

This function f is decreasing on $(0, \infty)$, since

$$f'(x) = -\frac{14x + 7x^2 + 6}{x(2x+3)(2x+1)(x+2)(x+1)^2} < 0.$$

As $\lim_{x\to\infty} f(x) = 0$, it follows that $f > 0$ on $(0, \infty)$ and consequently, the sequence $(\mu_n)_{n\geq 1}$ is decreasing.

By standard calculations (or faster, using the Maple software) we get:

$$\lim_{n\to\infty} n^3 \left(\mu_n - \mu_{n+1}\right) = \frac{7}{12}.$$

According to Lemma 3.1, we obtain:

$$\lim_{n\to\infty} n^2 \left(\mu_n - (\gamma + \ln 2)\right) = \frac{7}{24}.$$

b) First we prove the following inequalities, for every integer $n \geq 1$:

$$\frac{7}{12n(n+1)(n+2)} - \frac{7}{4n(n+1)(n+2)(n+3)} < \mu_n - \mu_{n+1} < \frac{7}{12n(n+1)(n+2)}, \tag{3.1}$$

namely $u(x) < 0$ and $v(x) > 0$, for all $x \in (0, \infty)$, where

$$u(x) = f(x) - \frac{7}{12x(x+1)(x+2)}$$

and

$$v(x) = f(x) - \left(\frac{7}{12x(x+1)(x+2)} - \frac{7}{4x(x+1)(x+2)(x+3)} \right).$$

The function u is increasing, while the function v is decreasing, as

$$u'(x) = \frac{94x + 47x^2 + 42}{12x^2(2x+3)(2x+1)(x+2)^2(x+1)^2} > 0, \quad x > 0,$$

and

$$v'(x) = -\frac{4305x + 4748x^2 + 2137x^3 + 336x^4 + 1296}{12x(2x+3)(2x+1)(x+3)^2(x+2)^2(x+1)^2} < 0, \quad x > 0.$$

But $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = 0$, thus $u(x) < 0$ and $v(x) > 0$, for all $x \in (0,\infty)$, as we have announced before. The inequality (3.1) is true.

Now we plan to sum the inequalities (3.1) from n to $n+k-1$, where k is any positive number:

$$\frac{7}{12}\sum_{i=n}^{n+k-1} \frac{1}{i(i+1)(i+2)} - \frac{7}{4}\sum_{i=n}^{n+k-1} \frac{1}{i(i+1)(i+2)(i+3)} < \mu_n - \mu_{n+k} < \frac{7}{12}\sum_{i=n}^{n+k-1} \frac{1}{i(i+1)(i+2)}. \tag{3.2}$$

These are telescopic sums, as

$$\frac{1}{i(i+1)(i+2)} = \frac{1}{2}\left(\frac{1}{i(i+1)} - \frac{1}{(i+1)(i+2)} \right) \tag{3.3}$$

and

$$\frac{1}{i(i+1)(i+2)(i+3)} = \frac{1}{3}\left(\frac{1}{i(i+1)(i+2)} - \frac{1}{(i+1)(i+2)(i+3)} \right). \tag{3.4}$$

The inequality (3.2) becomes:

$$\frac{7}{24}\left(\frac{1}{n(n+1)} - \frac{1}{(n+k)(n+k+1)} \right) - \frac{7}{12}\left(\frac{1}{n(n+1)(n+2)} - \frac{1}{(n+k)(n+k+1)(n+k+2)} \right) < \mu_n - \mu_{n+k} < \frac{7}{24}\left(\frac{1}{n(n+1)} - \frac{1}{(n+k)(n+k+1)} \right).$$

By taking the limit as $k \to \infty$, we obtain:

$$\frac{7}{24}\frac{1}{n(n+1)} - \frac{7}{12}\frac{1}{n(n+1)(n+2)} \le \mu_n - (\gamma + \ln 2) \le \frac{7}{24}\frac{1}{n(n+1)},$$

which is the conclusion. □

Proof of Theorem 2. *a*) We have $\eta_n - \eta_{n+1} = g(n)$, where

$$g(x) = -\frac{1}{x+1} + \frac{1}{2}\ln\left(\frac{1}{x^2} + \frac{1}{(x+1)^2}\right) - \frac{1}{2}\ln\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2}\right)$$

This function g is decreasing on $(0,\infty)$, since

$$g'(x) = -\frac{38x + 59x^2 + 40x^3 + 10x^4 + 10}{x(x+2)(2x+2x^2+1)(6x+2x^2+5)(x+1)^2} < 0.$$

As $\lim_{x\to\infty} g(x) = 0$, it follows that $g > 0$ on $(0,\infty)$ and consequently, the sequence $(\eta_n)_{n\geq 1}$ is decreasing.

By standard calculations (or faster, using the Maple software) we get:

$$\lim_{n\to\infty} n^3(\eta_n - \eta_{n+1}) = \frac{5}{6}.$$

According to the Lemma 3.1, we obtain:

$$\lim_{n\to\infty} n^2\left(\eta_n - \left(\gamma + \frac{1}{2}\ln 2\right)\right) = \frac{5}{12}.$$

b) First we prove the following inequalities, for every integer $n \geq 1$:

$$\frac{5}{6n(n+1)(n+2)} - \frac{5}{2n(n+1)(n+2)(n+3)} < \eta_n - \eta_{n+1} < \frac{5}{6n(n+1)(n+2)}, \quad (3.5)$$

namely $s(x) < 0$ and $t(x) > 0$, for all $x \in (0,\infty)$, where

$$s(x) = g(x) - \frac{5}{6x(x+1)(x+2)}$$

and

$$t(x) = g(x) - \left(\frac{5}{6x(x+1)(x+2)} - \frac{5}{2x(x+1)(x+2)(x+3)}\right).$$

The function s is increasing, while the function t is decreasing, as

$$s'(x) = \frac{190x + 279x^2 + 184x^3 + 46x^4 + 50}{6x^2(2x+2x^2+1)(6x+2x^2+5)(x+2)^2(x+1)^2} > 0$$

and

$$t'(x) = -\frac{5089x + 10\,460x^2 + 11\,283x^3 + 6620x^4 + 1994x^5 + 240x^6 + 1080}{6x(2x+2x^2+1)(6x+2x^2+5)(x+3)^2(x+2)^2(x+1)^2} < 0.$$

But $\lim_{x\to\infty} s(x) = \lim_{x\to\infty} t(x) = 0$, thus $s(x) < 0$ and $t(x) > 0$, for all $x \in (0,\infty)$, as we have announced before. The inequality (3.5) is true.

Now we plan to sum the inequalities (3.5) from n to $n+k-1$, where k is any positive number:

$$\frac{5}{6}\sum_{i=n}^{n+k-1}\frac{1}{i\left(i+1\right)\left(i+2\right)}-\frac{5}{2}\sum_{i=n}^{n+k-1}\frac{1}{i\left(i+1\right)\left(i+2\right)\left(i+3\right)}$$
$$<\eta_n-\eta_{n+k}<\frac{5}{6}\sum_{i=n}^{n+k-1}\frac{1}{i\left(i+1\right)\left(i+2\right)}. \quad (3.6)$$

These are telescopic sums, as we can see from (3.3)-(3.4). The inequality (3.6) becomes:

$$\frac{5}{12}\left(\frac{1}{n\left(n+1\right)}-\frac{1}{\left(n+k\right)\left(n+k+1\right)}\right)$$
$$-\frac{5}{6}\left(\frac{1}{n\left(n+1\right)\left(n+2\right)}-\frac{1}{\left(n+k\right)\left(n+k+1\right)\left(n+k+2\right)}\right)$$
$$<\eta_n-\eta_{n+k}<\frac{5}{12}\left(\frac{1}{n\left(n+1\right)}-\frac{1}{\left(n+k\right)\left(n+k+1\right)}\right).$$

By taking the limit as $k\to\infty$, we obtain:

$$\frac{5}{12\left(n+1\right)\left(n+2\right)}\leq\eta_n-\left(\gamma+\frac{1}{2}\ln 2\right)\leq\frac{5}{12n\left(n+1\right)}.$$

□

4 Further remarks

We believe that the ideas in this paper could be of interest to other researchers to obtain new generalizations, or results.

To be more precisely, recall that the harmonic sum is closely related to the digamma function ψ, *i.e.* the logaritmic derivative of the Euler-gamma function:

$$\psi\left(x\right)=\frac{d}{dx}\left(\ln\Gamma\left(x\right)\right).$$

Here,

$$\Gamma\left(x\right)=\int_0^\infty t^{x-1}e^{-t}dt,\quad x>0.$$

We have $\psi\left(1\right)=-\gamma$ and for every integer $n\geq 2$,

$$\psi\left(n\right)=-\gamma+\sum_{k=1}^{n-1}\frac{1}{k}.$$

Furthermore,

$$\psi\left(x+1\right)=\psi\left(x\right)+\frac{1}{x}.$$

For proofs and other properties, please see [1, p. 258].

Under these circumstances, the sequences we deal with in the above sections admit continuous forms on $(1,\infty)$, as follows:

$$\gamma(x) = \gamma + \psi(x) + \ln\frac{1}{x-1} \tag{4.1}$$

$$\mu(x) = \gamma + \psi(x) + \ln\left(\frac{1}{x-1} + \frac{1}{x}\right) \tag{4.2}$$

$$\eta(x) = \gamma + \psi(x) + \frac{1}{2}\ln\left(\frac{1}{(x-1)^2} + \frac{1}{x^2}\right), \tag{4.3}$$

for $x > 1$. We have: $\gamma_n = \gamma(n+1)$, $\mu_n = \mu(n+1)$, $\eta_n = \eta(n+1)$, for all integers $n \geq 1$.

Bounds for the functions γ, μ, η given in (4.1)-(4.3) and consequently for the sequences $(\gamma_n)_{n\geq 1}$, $(\mu_n)_{n\geq 1}$, $(\eta_n)_{n\geq 1}$ can be obtained by using the asymptotic series of the digamma function:

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{i=1}^{\infty}\frac{B_{2i}}{2ix^{2i}} = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^5} + \ldots, \text{ as } x \to \infty. \tag{4.4}$$

Here, B_j are the Bernoulli numbers given by the generating function:

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}.$$

We have $B_1 = -1/2$ and $B_{2j+1} = 0$, for all positive integers j, while the first few Bernoulli numbers are $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42\ldots$ For detalis, see, *e.g.*, [1, p. 804].

The above announced bounds can be obtained by truncation of the (4.4) series. More precisely, under and upper approximations are given by alternatively truncate the (4.4) series:

$$\ln x - \frac{1}{2x} - \sum_{i=1}^{2m-1}\frac{B_{2i}}{2ix^{2i}} < \psi(x) < \ln x - \frac{1}{2x} - \sum_{i=1}^{2n}\frac{B_{2i}}{2ix^{2i}}.$$

In this way, along bounds, other monotonicity, even complete monotonicity properties can be discovered.

Declarations

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Normalized solutions for coupled Kirchhoff equations with critical and subcritical nonlinearities

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ABSTRACT

In this paper, we study Kirchhoff equations with constraint conditions

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u_1|^2\,dx\right)\Delta u_1=\lambda_1 u_1 \\ \qquad +\mu_1|u_1|^{p_1-2}u_1+\beta r_1|u_1|^{r_1-2}u_1|u_2|^{r_2} \quad \text{in } \mathbb{R}^3, \\ -\left(a+b\int_{\mathbb{R}^3}|\nabla u_2|^2\,dx\right)\Delta u_2=\lambda_2 u_2 \\ \qquad +\mu_2|u_2|^{p_2-2}u_2+\beta r_2|u_1|^{r_1}|u_2|^{r_2-2}u_2 \quad \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3}|u_1|^2\,dx=c_1, \quad \int_{\mathbb{R}^3}|u_2|^2\,dx=c_2, \\ u_1\in H^1\left(\mathbb{R}^3\right), \quad u_2\in H^1\left(\mathbb{R}^3\right). \end{cases} \tag{P}$$

where a, b, β, μ_i, $c_i>0$, $r_i>1$, $2<p_i<\frac{14}{3}<r:=r_1+r_2\leq 2^*$ for $i=1,2$, and λ_1, $\lambda_2\in\mathbb{R}$ appear as Lagrange multipliers. The existence of normalized solutions for p_1 and p_2 within a specific range of $(2,\frac{14}{3})$ has been considered both the Sobolev subcritical case ($r<2^*$) and the critical case ($r=2^*$) by the Minimax principle and variational methods. This paper provides a refinement and extension of the results for the normalized solutions to Kirchhoff equations.

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RESUMEN

En este artículo, estudiamos ecuaciones de Kirchhoff con condiciones de restricción

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u_1|^2\,dx\right)\Delta u_1=\lambda_1 u_1 \\ \qquad +\mu_1|u_1|^{p_1-2}u_1+\beta r_1|u_1|^{r_1-2}u_1|u_2|^{r_2} \quad \text{en } \mathbb{R}^3, \\ -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u_2|^2\,dx\right)\Delta u_2=\lambda_2 u_2 \\ \qquad +\mu_2|u_2|^{p_2-2}u_2+\beta r_2|u_1|^{r_1}|u_2|^{r_2-2}u_2 \quad \text{en } \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^3}|u_1|^2\,dx=c_1, \quad \int_{\mathbb{R}^3}|u_2|^2\,dx=c_2, \\ u_1\in H^1\left(\mathbb{R}^3\right), \quad u_2\in H^1\left(\mathbb{R}^3\right). \end{cases} \tag{P}$$

donde a, b, β, μ_i, $c_i > 0$, $r_i > 1$, $2 < p_i < \frac{14}{3} < r := r_1+r_2 \leq 2^*$ para $i = 1, 2$, y λ_1, $\lambda_2 \in \mathbb{R}$ aparecen como multiplicadores de Lagrange. La existencia de soluciones normalizadas para p_1 y p_2 en un rango específico de $(2, \frac{14}{3})$ ha sido considerado tanto el caso Sobolev subcrítico ($r < 2^*$) y el caso crítico ($r = 2^*$) a través del principio Minimax y métodos variacionales. Este artículo entrega un refinamiento y una extensión de los resultados para soluciones normalizadas de ecuaciones de Kirchhoff.

Keywords and Phrases: Normalized solution, Kirchhoff equations, variational methods.

2020 AMS Mathematics Subject Classification: 35J60, 47J30, 35J20.

1 Introduction and main results

In this paper, we are concerned with the existence of normalized solutions to following Kirchhoff equations in $H^1\left(\mathbb{R}^3\right) \times H^1\left(\mathbb{R}^3\right)$,

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}\left|\nabla u_1\right|^2 dx\right)\Delta u_1=\lambda_1 u_1+\mu_1\left|u_1\right|^{p_1-2}u_1+\beta r_1\left|u_1\right|^{r_1-2}u_1\left|u_2\right|^{r_2}, \\ -\left(a+b\displaystyle\int_{\mathbb{R}^3}\left|\nabla u_2\right|^2 dx\right)\Delta u_2=\lambda_2 u_2+\mu_2\left|u_2\right|^{p_2-2}u_2+\beta r_2\left|u_1\right|^{r_1}\left|u_2\right|^{r_2-2}u_2, \end{cases} \tag{1.1}$$

under mass constraints,

$$\int_{\mathbb{R}^3}\left|u_1\right|^2 dx=c_1, \quad \int_{\mathbb{R}^3}\left|u_2\right|^2 dx=c_2, \tag{1.2}$$

where c_1, c_2 are prescribed positive constants.

The Kirchhoff-type problems, initially proposed by Kirchhoff in 1883 [18], extend the classical d'Alembert wave equations. Following the foundational work by Lions [22], Kirchhoff-type equations have attracted significant interest, leading to extensive exploration of their steady-state models. Early classical studies on Kirchhoff equations can be found in [1,12,13,19,23] and the references therein.

Currently, physicists are particularly interested in solutions that satisfy normalized conditions: $\int_{\mathbb{R}^3}|u_i|^2dx=c_i$, for $i=1,2$, due to their clear physical significance, particularly regarding mass. For example, from a physical perspective, the normalized condition can represent the number of particles in each component of Bose-Einstein condensates or the power supply in nonlinear optics. In this context, λ_i appears as an unknown quantity in the Kirchhoff equations (1.1). It is therefore natural to prescribe the value of the mass so that λ_i can be interpreted as Lagrange multipliers. From this perspective, problem (P) can be addressed by studying certain constrained variational problems, obtaining normalized solutions by identifying critical points of the energy functional $J : H^1\left(\mathbb{R}^3\right) \times H^1\left(\mathbb{R}^3\right) \to \mathbb{R}$ defined by

$$J\left(u_1,u_2\right)=\frac{a}{2}\sum_{i=1}^{2}\|\nabla u_i\|_2^2+\frac{b}{4}\sum_{i=1}^{2}\|\nabla u_i\|_2^4-\sum_{i=1}^{2}\frac{\mu_i}{p_i}\|u_i\|_{p_i}^{p_i}-\beta\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx,$$

constrained on $\mathcal{S} := S\left(c_1\right)\times S\left(c_2\right)$, where $\|\cdot\|_p$ denotes the standard norm in $L^p\left(\mathbb{R}^3\right)$ for $p \in [1,+\infty)$ and $S(c) := \left\{u \in H^1\left(\mathbb{R}^3\right) : \|u\|_2^2 = c\right\}$ for any $c > 0$.

When $b = 0$, the Kirchhoff equations (1.1) reduce to a nonlinear Schrödinger equations. In this case, we note that the mass critical exponent $\frac{10}{3}$. If the problem (P) is purely mass subcritical, *i.e.*, $2 < p_1,\ p_2,\ r < \frac{10}{3}$, Gou and Jeanjean [10] searched for a critical point of J as a global minimizer of J on $\mathcal{S}$. In the purely mass supercritical case, *i.e.*, $\frac{10}{3} < p,\ q,\ r < 2^*$, Bartsch *et al.* [3] first considered the case of $p = q = r = 4$. They obtained the existence of positive solutions to problem (P) provided $0 < \beta < \beta_1(c_1,c_2)$ or $\beta > \beta_2(c_1,c_2)$. Bartsch and Jeanjean [2] extended these results of [3] to $\frac{10}{3} < p_1,\, p_2,\, r < 2^*$. Recently, Jeanjean *et al.* [17] focused on the coupled purely mass supercritical case and proved the existence of solutions for all c_1, c_2, and without restrictions on β. For the mixed cases such as $2 < p_1,\, p_2 < \frac{10}{3} < r < 2^*$ or $2 < r < \frac{10}{3} < p_1,\, p_2 < 2^*$, Gou and Jeanjean [11] explored the multiplicity of solutions to problem (P). Later, Bartsch and Jeanjean [2] used the mountain pass lemma and a compactness argument to show that problem (P) has a positive solution for suitable $c_1,\, c_2 > 0$ when $2 < p_1 < \frac{10}{3} < p_2$ and $r < 2^*$. In the Sobolev critical case, Li and Zou [21] investigated the condition that $2 < p_1,\, r < 2^*,\, p_2 \le 2^*$. Bartsch *et al.* [4] also considered the Sobolev critical case with $2 < r < 2^* = p_1 = p_2$. When $\frac{10}{3} < p_1,\, p_2 < r = 2^*$, Liu and Fang [24] demonstrated that problem (P) has a mountain pass solution. Zhang and Han [34] obtained a positive ground state solution of problem (P) with $2 < p_1,\, p_2 < \frac{10}{3}$ and $r = 2^*$.

When $b > 0$, there are several results in the literature dealing with normalized solutions to problem (P). Ye [32, 33] considered this constrained problem for a single Kirchhoff equation

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u = \lambda u + \mu |u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^3} |u|^2 dx = c. \end{cases} \tag{1.3}$$

Ye proved that $p = \frac{14}{3}$ is a mass critical exponent for Kirchhoff equation. To be more precise, the functional corresponding to problem (1.3) is

$$I_\mu(u) := \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\mu}{p}\|u\|_p^p,$$

which is bounded from below on manifold $S(c)$ when $2 < p < \frac{14}{3}$. However, when $\frac{14}{3} < p < 6$, the functional is not bounded from below on $S(c)$. By Ekeland's variational principle and the strict monotonicity of a energy function, Cao *et al.* [5] considered the existence of positive solutions to problem (P) with the purely mass subcritical case $2 < p_1,\, p_2,\, r < \frac{14}{3}$. Recently, Yang [31] showed the existence of positive solutions to problem (P) in the purely mass supercritical case $\frac{14}{3} < p_1$, $p_2,\, r < 2^*$ and in the mixed case $2 < r < \frac{14}{3} < p_1,\, p_2 < 2^*$. Hu and Mao [15] further obtained the existence of two solution (local minimizer and Mountain-Pass type) for the mixed cases $2 < p_1$, $p_2 \le \frac{10}{3}$ and $\frac{14}{3} < r < 2^*$. More results about the normalized solutions, we refer the readers to [8, 14, 29, 30].

To provide clarity in the discussion, we summarize some of the results on normalized solutions to problem (P) in Table 1.

Motivated by the aforementioned works, we study normalized solutions to problem (P) in three distinct cases: (H_1): $\frac{10}{3} < p_1,\, p_2 < \frac{14}{3} < r < 2^*$; (H_2): $2 < p_1 < \frac{14}{3} < p_2,\, r < 2^*$ and (H_3): $2 < p_1,\, p_2 < \frac{10}{3},\, r = 2^*$. To address compactness issues, we work within the radial space $\mathcal{S}_r := S_r(c_1) \times S_r(c_2)$, where $S_r(c) := \left\{u \in H_r^1(\mathbb{R}^3) : \|u\|_2^2 = c\right\}$, and $H_r^1(\mathbb{R}^3)$ denotes the space of

Table 1

b	p_1, p_2, r	Types of solutions	References
$b=0$	$2<p_1,p_2,r<\frac{10}{3}$	a global minimizer	[2,10]
$b=0$	$\frac{10}{3}<p_1,p_2,r<6$	Mountain Pass solution	[2,3]
$b=0$	$2<p_1<\frac{10}{3}<p_2,r<6$	Mountain Pass solution	[2]
$b=0$	$2<r<\frac{10}{3}<p_1,p_2<6$	Mountain Pass solution, a local minimizer	[11]
$b=0$	$r=6$ or $p_1,p_2=6$	Mountain Pass solution, ground state solution	[4,21,24,34]
$b>0$	$2<p_1,p_2,r<\frac{14}{3}$	a global minimizer	[5]
$b>0$	$\frac{14}{3}<p_1,p_2,r<6$; $2<r<\frac{14}{3}<p_1,p_2<6$	Mountain Pass solution, a local minimizer	[31]
$b>0$	$2<p_1,p_2\leq\frac{10}{3}, \frac{14}{3}<r<6$	Mountain Pass solution, a local minimizer	[15]
$b>0$	$\frac{10}{3}<p_1,p_2<\frac{14}{3}, \frac{14}{3}<r<6$	open problem	
$b>0$	$2<p_1<\frac{14}{3}<p_2,r<6$	open problem	
$b>0$	$2<p_1,p_2<\frac{14}{3}, r=6$	open problem	

radial functions on $\mathbb{R}^3$. By the principle of symmetric criticality, the critical points of J constrained on $\mathcal{S}_r$ are also critical points of J constrained on $\mathcal{S}$.

It is known that critical points of $J|_{\mathcal{S}_r}$ stay in

$$\mathcal{P}:=\left\{(u_1,u_2)\in\mathcal{S}_r : P\left(u_1,u_2\right)=0\right\},$$

as a consequence of Pohozaev identity, where

$$P\left(u_1,u_2\right):=a\sum_{i=1}^{2}\|\nabla u_i\|_2^2+b\sum_{i=1}^{2}\|\nabla u_i\|_2^4-\sum_{i=1}^{2}\mu_i\gamma_{p_i}\|u_i\|_{p_i}^{p_i}-\beta r\gamma_r\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx.$$

Moreover, we define for $u\in S(c)$ the map

$$(s\star u)(x):=e^{\frac{3s}{2}}u\left(e^s x\right),\quad s\in\mathbb{R},$$

which preserves the L^2 norm and plays a special role in the study of structures of $J(u_1,u_2)$ and $P(u_1,u_2)$ on the constraint $\mathcal{S}_r$. We introduce the fiber mapping for $J(u_1,u_2)$,

$$\begin{aligned}\Phi_{u_1,u_2}(s):&=J(s\star u_1,s\star u_2)\\&=\frac{ae^{2s}}{2}\sum_{i=1}^{2}\|\nabla u_i\|_2^2+\frac{be^{4s}}{4}\sum_{i=1}^{2}\|\nabla u_i\|_2^4-\sum_{i=1}^{2}\frac{\mu_i e^{p_i\gamma_{p_i}s}}{p_i}\|u_i\|_{p_i}^{p_i}-\beta e^{r\gamma_r s}\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx,\end{aligned}\tag{1.4}$$

for any $(u_1,u_2)\in\mathcal{S}_r$. It is easy to verify that $(s\star u,s\star v)\in\mathcal{P}$ if and only if s is a critical point of $\Phi_{u_1,u_2}(s)$. In particular, $(u,v)\in\mathcal{P}$ if only if $s=0$ is a critical point of $\Phi_{u_1,u_2}(s)$.

We will require some preliminary results regarding problem (1.3). Let $m(c,\mu)$ denote the ground state level, defined as

$$m(c,\mu):=\inf\left\{I_\mu(u):u\in S(c)\text{ such that }\left(I_\mu\big|_{S(c)}\right)'(u)=0\right\},$$

and introduce the Pohozaev set for the single Kirchhoff equation:

$$V(c) := \left\{u \in S(c) : 0 = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \mu\gamma_p\|u\|_p^p\right\}.$$

Now, we state the first result about the mass sub-critical case as follows.

Theorem 1.1. *Assume the following assumptions (H_1) holds,*

$$(H_1): \ \frac{10}{3} < p_1,\ p_2 < \frac{14}{3} < r < 2^*.$$

There exists $\beta_0 := \beta_0(c_1, c_2) > 0$, such that for $0 < \beta \leq \beta_0$ and $c_1, c_2 < c^*$, problem (P) *has a positive normalized solution.*

Inspired by [2], Bartsch and Jeanjean constructed a minimax level and proved the existence of a positive normalized solution for Schrödinger equations with $2 < p_1 < \frac{10}{3} < p_2$, $r < 2^*$. Our second result deals with the case

$$(H_2): \ 2 < p_1 < \frac{14}{3} < p_2,\ r < 2^*;\ 2 < r_2 < \frac{10}{3}.$$

which we call it mix mass sup-critical case.

Theorem 1.2. *Assume that (H_2) holds. For*

(p_1) *$2 < p_1 \leq \frac{10}{3}$ and $c_1 > 0$, or $\frac{10}{3} < p_1 < \frac{14}{3}$ and $c_1 > c_*$, where c_* is positive constant only depend on a, b, μ_1,*

if $m(c_1, \mu_1) + m(c_2, \mu_2) < 0$, problem (P) *has a positive normalized solution.*

As a corollary of Theorem 1.2, we obtain the following results.

Corollary 1.3. *Assume that (H_2) holds.*

(i) For any $c_2 > 0$, there exists $\bar{c}_1$, such that for $c_1 \geq \bar{c}_1$, problem (P) *has a positive normalized solution.*

(ii) For any $c_1 > c_*$, there exists $\bar{c}_2$, such that for $c_2 \geq \bar{c}_2$, problem (P) *has a positive normalized solution.*

Last, we consider the mass sub-critical and Sobolev critical case,

$$(H_3): \ 2 < p_1,\ p_2 \leq \frac{10}{3},\ r = 2^*.$$

Theorem 1.4. *Assume that* (H_3) *holds. There exist* $\beta_* := \beta_*(c_1, c_2)$ *and* μ_**, such that for* $0 < \beta < \beta_*$ *and* μ_1*,* $\mu_2 < \mu_*$*, problem* (P) *has a ground state solution.*

Remark 1.5. *(i) Theorem 1.1 serves as a complement to the work of Hu and Mao [15], specifically addressing the case of problem* (P) *with* $2 < p_1, p_2 \leq \frac{10}{3}$ *and* $\frac{14}{3} < r < 2^*$*. Compared with a single equation, the main difficulty for systems is how to exclude the semi-trivial solutions. In [15], the authors heavily rely on* $p < \frac{10}{3}$ *since that* $m(c,\mu) < 0$ *to excluding semi-trivial solutions. However, we partially extend to the case that* $\frac{10}{3} < p_1, p_2 < \frac{14}{3}$ *with the mass constrained suitable small to overcome this difficulty.*

(ii) Theorems 1.2 and 1.4 complement the results of Zhang and Han [34] and Bartsch and Jeanjean [2], which extended the study from Schrödinger equations to Kirchhoff equations.

(iii) Compared Kirchhoff equations with single Kirchhoff equation, the existence and types of solutions to problem (P) *are similar to the result of single equation,*

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = c, \end{cases} \tag{1.5}$$

where a*,* b*,* c *are positive constants and* $2 < q < p \leq 2^*$*. Feng et al. in [7] have proven that under condition* $2 < q < \frac{10}{3} < p = 2^*$*, problem* (1.5) *has a second solution. It is an interesting question whether problem* (P) *also has a second solution under condition* (H_3)*?*

The rest of this paper is organized as follows. In Section 2, we present some preliminary results. Sections 3-5 are devoted to the proofs of Theorems 1.1-1.4.

Notation: In this paper, we denote $H := H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ and $H_r := H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3)$. $\to$ and $\rightharpoonup$ denote the strong and weak convergence in the related function space, respectively. $H^{-1}(\mathbb{R}^3)$ is the dual space of $H^1(\mathbb{R}^3)$. C, $C(\cdot), \ldots$ denote positive constants. $o_n(1)$ represents a real sequence with $o_n(1) \to 0$ as $n \to +\infty$. $D^{1,2}(\mathbb{R}^3)$ denotes the closure of the function space $C_c^\infty(\mathbb{R}^3)$ with the norm $\|u\|_{D^{1,2}(\mathbb{R}^3)} = \|\nabla u\|_2$. The best Sobolev constant S is given by $S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}$.

2 Preliminary results

Before we proceed further, let us first revisit the Gagliardo-Nirenberg inequality in [27, 28]. For $2 \leq p \leq 2^*$, there exists a constant $C_p > 0$ such that for any $u \in H^1(\mathbb{R}^3)$,

$$\|u\|_p \leq C_p \|\nabla u\|_2^{\gamma_p} \|u\|_2^{1-\gamma_p},$$

where $\gamma_p = \frac{3(p-2)}{2p}$. For $2 \le r_1 + r_2 \le 2^*$, there exists $q > 1$ such that

$$\max\left\{\frac{2}{r_1}, \frac{2^*}{2^* - r_2}\right\} \le q \le \min\left\{\frac{2^*}{r_1}, \frac{2}{(2-r_2)^+}\right\}. \tag{2.1}$$

Set $q' := \frac{q}{q-1}$, $2 \le r_1 q$, $r_2 q' \le 2^*$, by the Hölder inequality, we have

$$\int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2}\, dx \le \|u_1\|_{r_1 q}^{r_1} \|u_2\|_{r_2 q'}^{r_2} < \infty,$$

which implies that the functional J is well defined. For $\frac{14}{3} < r = r_1 + r_2 < 2^*$, by the Hölder inequality and the Gagliardo-Nirenberg inequality, we know

$$\begin{aligned}\int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2}\, dx &\le \|u_1\|_r^{r_1} \|u_2\|_r^{r_2} \le C \|\nabla u_1\|_2^{r_1\gamma_r} \|\nabla u_2\|_2^{r_2\gamma_r} \\ &\le C \left(\sum_{i=1}^{2} \|\nabla u_i\|_2^2\right)^{\frac{r_1\gamma_r}{2}} \left(\sum_{i=1}^{2} \|\nabla u_i\|_2^2\right)^{\frac{r_2\gamma_r}{2}} \le C \left(\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2\right)^{\frac{r\gamma_r}{2}}. \end{aligned} \tag{2.2}$$

Specifically, for $r = 2^*$, $r\gamma_r = 2^*$, then $C = S^{-\frac{2^*}{2}}$. Next, we need a splitting lemma similar to Brézis-Lieb Lemma as follows.

Lemma 2.1 ([11, Lemma 2.4], [6, Lemma 2.3])**.** *Assume that r_1, $r_2 > 1$, $2 < r_1 + r_2 \le 2^*$. If*

$$(u_1^n, u_2^n) \rightharpoonup (u_1, u_2) \ \text{ in } H,$$

then up to a subsequence

$$\int_{\mathbb{R}^3} |u_1^n|^{r_1} |u_2^n|^{r_2}\, dx = \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2}\, dx + \int_{\mathbb{R}^3} |u_1^n - u_1|^{r_1} |u_2^n - u_2|^{r_2}\, dx + o_n(1).$$

Moreover, a description of the PPS sequence is also needed as follows.

Lemma 2.2 ([15, Lemma 2.5, 2.6])**.** *Assume that $2 < p_1, p_2 < 2^*$, $2 < r < 2^*$. If $\{(u_1^n, u_2^n)\}$ is a bounded Palais-Smale sequence for J on $\mathcal{S}_r$, there exist $(u_1, u_2) \in H_r$ and a sequence $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R}^2$, such that up to a subsequence*

(i) $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ *in* H_r, $(u_1^n, u_2^n) \to (u_1, u_2)$ *in* $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ *for* $p \in (2, 2^*)$.

(ii) $(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$ *in* $\mathbb{R}^2$.

(iii) $J'(u_1^n, u_2^n) - \lambda_1^n (u_1^n, 0) - \lambda_2^n (0, u_2^n) \to 0$ *in* $H_r^{-1}(\mathbb{R}^3) \times H_r^{-1}(\mathbb{R}^3)$.

(iv) (u_1, u_2) *is a solution of equations* (1.1) *for* λ_1, $\lambda_2 \le 0$ *if* $P(u_1^n, u_2^n) \to 0$. *In addition,* $u_1^n \to u_1$ *in* $H_r^1(\mathbb{R}^3)$ *if* $\lambda_1 < 0$. *Similarly,* $u_2^n \to u_2$ *in* $H_r^1(\mathbb{R}^3)$ *if* $\lambda_2 < 0$.

Lemma 2.3 ([16]). *Let $p \in (1,3]$. If $u \in L^p(\mathbb{R}^3) \cap C^2(\mathbb{R}^3)$ is non-negative and satisfies $-\Delta u \geq 0$ in $\mathbb{R}^3$, then $u = 0$.*

Lemma 2.4. *Let $p_i \in (2, 2^*)$, $i = 1, 2$. If $(u_1, u_2) \in H_r$ is a solution of Kirchhoff equations (1.1) with $u_1 \geq 0$, $u_1 \neq 0$, and $u_2 \geq 0$, then $\lambda_1 < 0$. Similarly, if $u_1 \geq 0$, $u_2 \geq 0$, and $u_2 \neq 0$, then $\lambda_2 < 0$.*

Proof. Similar proofs can be referenced in [5, Lemma 2.4]. □

The following existing results concerning the single Kirchhoff equation is rather significant to the main proof of Theorems.

Proposition 2.5. *Assume that $p \in (2, 2^*)$ and $\mu > 0$. Then*

(i) *[5, Lemma 2.2], [26, Theorem 1.1, 1.4]: Assume that $2 < p < \frac{10}{3}$, the problem (1.3) has a unique positive ground state solution for any $c > 0$. If $p = \frac{10}{3}$, there exists c' such that the problem (1.3) has a unique positive ground state solution for $c > c'$. Moreover, $m(c, \mu) < 0$, $m(c, \mu) \to -\infty$ as $c \to \infty$.*

(ii) *[5, Lemma 2.2], [26, Theorem 1.1], [25, Theorem 1.1]: Assume that $p \in (\frac{10}{3}, \frac{14}{3})$, there exists $0 < c^* < c_*$, such that the problem (1.3) admits exactly two positive normalized solutions w_1, w_2 if $c > c^*$ and no solution if $c < c^*$. If $c \geq c_*$, one of the above positive solutions is the unique normalized ground state solution. Without loss of generality, let w_1 be the normalized ground state and w_2 be the high-energy, then there holds that $I_\mu(w_1) = m(c, \mu) \leq 0 < I_\mu(w_2)$, and $m(c, \mu) \to -\infty$ as $c \to \infty$.*

(iii) *[33], [31, Lemma 3.1]: If $p \in (\frac{14}{3}, 2^*)$ and problem (1.3) admits a unique solution u_c for any $c > 0$, $m(c, \mu) = I_\mu(u_c) = \max_{s \in \mathbb{R}} \Phi_{u_c}(s) = \min_{u \in V(c)} I_\mu(u) > 0$, where*

$$\Phi_u(s) := I_\mu(s \star u) = \frac{ae^{2s}}{2} \|\nabla u\|_2^2 + \frac{be^{4s}}{4} \|\nabla u\|_2^4 - \frac{\mu e^{p\gamma_p s}}{p} \|u\|_p^p.$$

Moreover, $m(c, \mu)$ is strictly decreasing with respect to c.

3 The proof of Theorem 1.1

We shall investigate the mountain pass geometry of $J(u_1, u_2)$ on $\mathcal{S}_r$.

Lemma 3.1. *Assume that (H_1) holds.*

(i) *There exist $\rho_0 = \rho_0(c_1, c_2)$ and $\beta_0 = \beta_0(c_1, c_2) > 0$, such that for $0 < \beta \leq \beta_0$,*

$$\inf_{A(2\rho_0) \setminus A(\rho_0)} J(u_1, u_2) > 0,$$

where $A(\rho_0) := \left\{(u_1, u_2) \in \mathcal{S}_r : \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2 < \rho_0\right\}$ *for* $\rho_0 > 0$.

(*ii*) *There exists* $(u_1, u_2) \in \mathcal{S}_r \backslash A(2\rho_0)$, *such that* $J(u_1, u_2) < 0$.

Proof. (*i*) Let $\rho := \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2$. By (2.2) and the Gagliardo-Nirenberg inequality, for $(u_1, u_2) \in \mathcal{S}_r$, we have:

$$\begin{aligned} J(u_1, u_2) &= \frac{a}{2}\rho + \frac{b}{4}\sum_{i=1}^{2}\|\nabla u_i\|_2^4 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}\|u_i\|_{p_i}^{p_i} - \beta\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}\,dx \\ &\geq \frac{b}{8}\rho^2 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}C_i\|\nabla u_i\|_2^{p_i\gamma_{p_i}} - \beta C_3\rho^{\frac{r\gamma_r}{2}} \\ &\geq \frac{b}{8}\rho^2 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}C_i\rho^{\frac{p_i\gamma_{p_i}}{2}} - \beta C_3\rho^{\frac{r\gamma_r}{2}}, \end{aligned}$$

where $C_i := C(c_1, c_2)$ for $(i = 1, 2, 3)$. If (H_1) holds, then $2 < p_i\gamma_{p_i} < 4$ and $4 < r\gamma_r < 2^*$. Let $\rho_0 > 0$ be large enough such that

$$\sum_{i=1}^{2}\frac{\mu_i}{p_i}C_i(\rho_0)^{\frac{p_i\gamma_{p_i}-4}{2}} \leq \frac{b}{32}, \tag{3.1}$$

and then choose $\beta_0 > 0$ small enough such that

$$\beta_0 C_3(2\rho_0)^{\frac{r\gamma_r-4}{2}} \leq \frac{b}{32}.$$

Hence, for any $0 < \beta \leq \beta_0$ and $(u_1, u_2) \in A(2\rho_0)\backslash A(\rho_0)$, *i.e.*, $\rho_0 \leq \rho < 2\rho_0$, we have

$$\begin{aligned} J(u_1, u_2) &\geq \frac{b}{8}\rho^2 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}C_i\rho^{\frac{p_i\gamma_{p_i}}{2}} - \beta C_3\rho^{\frac{r\gamma_r}{2}} = \rho^2\left(\frac{b}{8} - \sum_{i=1}^{2}C_i\rho^{\frac{p_i\gamma_{p_i}-4}{2}} - \beta C_3\rho^{\frac{r\gamma_r-4}{2}}\right) \\ &\geq b\rho_0^2\left(\frac{1}{8} - \frac{1}{32} - \frac{1}{32}\right) = \frac{b}{16}\rho_0^2. \end{aligned}$$

(*ii*) Let $u^t(x) := t^{\frac{3}{2}}u(tx)$. Then,

$$\|u^t\|_2^2 = \|u\|_2^2, \quad \|\nabla u^t\|_2^2 = t^2\|\nabla u\|_2^2, \quad \|u^t\|_p^p = t^{p\gamma_p}\|u\|_p^p, \quad \text{for all } p \in (2, 2^*).$$

Fix $(u_1, u_2) \in \mathcal{S}_r$, $(u_1^t, u_2^t) \in \mathcal{S}_r\backslash A(2\rho_0)$ when t is sufficiently large. Since

$$J(u_1^t, u_2^t) = \frac{a}{2}t^2\sum_{i=1}^{2}\|\nabla u_i\|_2^2 + \frac{b}{4}t^4\sum_{i=1}^{2}\|\nabla u_i\|_2^4 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}t^{p_i\gamma_{p_i}}\|u_i\|_{p_i}^{p_i} - \beta t^{r\gamma_r}\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx,$$

it is straightforward to check that $\psi_{(u_1,u_2)}(t) := J(u_1^t, u_2^t) < 0$ for t large enough. □

Thanks to Lemma 3.1, we introduce a minimax structure of the mountain pass type. Specifically, there exists,

$$\gamma\left(c_1, c_2\right) := \inf_{g \in \Gamma} \max_{t \in [0,1]} J(g(t)),$$

where $\Gamma := \left\{ g \in C\left([0,1], \mathcal{S}_r\right) : g(0) \in \partial A(\rho_0),\ g(1) \notin \overline{A\left(2\rho_0\right)},\ J(g(1)) < 0 \right\}$. This framework allows us to search for a critical point of the mountain pass type at the level $\gamma\left(c_1, c_2\right)$. It is clear that $\gamma\left(c_1, c_2\right) \geq \inf_{u \in \partial A(\rho_0)} J\left(u_1, u_2\right) > 0$.

Lemma 3.2. *Assume that (H_1) holds. There exists a Palais-Smale sequence $\{(u_1^n, u_2^n)\}$ for $J|_{\mathcal{S}_r}$ at the level $\gamma\left(c_1, c_2\right)$, which satisfies $\{u_1^n\}^- \to 0$, $\{u_2^n\}^- \to 0$, and $P\left(u_1^n, u_2^n\right) \to 0$.*

Proof. The proof of the theorem is standard, and we omit the detailed steps here. For a comprehensive explanation, refer to [15, Lemma 3.1], [2, Lemma 5.5], and [9, Theorem 4.1]. □

Lemma 3.3. *Assume that (H_1) holds. There exists a pair of positive solution (u_1, u_2) to equations (1.1) for some (λ_1, λ_2), and $J\left(u_1, u_2\right) = \gamma\left(c_1, c_2\right) > 0$.*

Proof. By Lemma 3.2, there exists a Palais-Smale sequence $\{(u_1^n, u_2^n)\}$ for $J|_{\mathcal{S}_r}$ at the level $\gamma\left(c_1, c_2\right)$. We first prove that $\{(u_1^n, u_2^n)\}$ is bounded in H_r. Since $P\left(u_1^n, u_2^n\right) \to 0$, we have

$$a \sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + b \sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 = \sum_{i=1}^{2} \mu_i \gamma_{p_i} \|u_i^n\|_{p_i}^{p_i} + \beta r \gamma_r \int_{\mathbb{R}^3} |u_1^n|^{r_1} \, |u_2^n|^{r_2} \, dx + o_n(1). \tag{3.2}$$

Thus,

$$\begin{aligned}
\gamma(c_1, c_2) + o_n(1) &= \frac{a}{2} \sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + \frac{b}{4} \sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 - \sum_{i=1}^{2} \frac{\mu_i}{p_i} \|u_i^n\|_{p_i}^{p_i} - \beta \int_{\mathbb{R}^3} |u_1^n|^{r_1} \, |u_2^n|^{r_2} \, dx \\
&= a \left(\frac{1}{2} - \frac{1}{r\gamma_r} \right) \sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + b \left(\frac{1}{4} - \frac{1}{r\gamma_r} \right) \sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 \\
&- \sum_{i=1}^{2} \mu_i \gamma_{p_i} \left(\frac{1}{p_i \gamma_{p_i}} - \frac{1}{r\gamma_r} \right) \|u_i^n\|_{p_i}^{p_i} \\
&\geq a \left(\frac{1}{2} - \frac{1}{r\gamma_r} \right) \rho + \frac{b}{2} \left(\frac{1}{4} - \frac{1}{r\gamma_r} \right) \rho^2 - \sum_{i=1}^{2} C_i \mu_i \gamma_{p_i} \left(\frac{1}{p_i \gamma_{p_i}} - \frac{1}{r\gamma_r} \right) \rho^{\frac{p_i \gamma_{p_i}}{2}},
\end{aligned}$$

where $\rho = \|\nabla u_1^n\|_2^2 + \|\nabla u_2^n\|_2^2$, $4 < r\gamma_r < 2^*$, $2 < p_i \gamma_{p_i} < 4$. Hence, $\{(u_1^n, u_2^n)\}$ is bounded in H_r. Then, for p, $q \in (2, 2^*)$, we may assume that

$$(u_1^n, u_2^n) \rightharpoonup (u_1, u_2) \text{ in } H_r, \quad (u_1^n, u_2^n) \to (u_1, u_2) \text{ in } L^p(\mathbb{R}^3) \times L^q(\mathbb{R}^3). \tag{3.3}$$

By Lemmas 2.2, 3.2, there exists a sequence $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R}^2$, such that $(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$, $\lambda_1, \lambda_2 \leq 0$. Consequently, (u_1, u_2) is a solution to equations (1.1) and satisfies $P\left(u_1, u_2\right) = 0$.

Since $(u_1^n)^- \to 0$, $(u_2^n)^- \to 0$, it follows that $u_1,\, u_2 \geq 0$.

Now, we prove $J(u_1, u_2) = \gamma(c_1, c_2)$. By (3.3) and Lemma 2.1, the right hand side of (3.2) converges to

$$\sum_{i=1}^{2} \mu_i \gamma_{p_i} \|u_i\|_{p_i}^{p_i} + \beta r \gamma_r \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx.$$

Combining this with $P(u_1, u_2) = 0$, we have

$$\lim_{n\to+\infty} a \sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + b \sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 = a \sum_{i=1}^{2} \|\nabla u_i\|_2^2 + b \sum_{i=1}^{2} \|\nabla u_i\|_2^4.$$

Therefore, $J(u_1^n, u_2^n) \to J(u_1, u_2)$, and hence, $J(u_1, u_2) = \gamma(c_1, c_2)$. □

Proof of Theorem 1.1. As known from Lemma 3.3, it is sufficient to prove that $(u_1, u_2) \in \mathcal{S}_r$. Using the fact that (u_1, u_2) is a solution to equations (1.1), we deduce that

$$\lambda_1 \|u_1\|_2^2 + \lambda_2 \|u_2\|_2^2 = a \sum_{i=1}^{2} \|\nabla u_i\|_2^2 + b \sum_{i=1}^{2} \|\nabla u_i\|_2^4 - \sum_{i=1}^{2} \mu_i \|u_i\|_{p_i}^{p_i} - \beta r \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} \, dx.$$

Combining Pohozaev identity and the fact that $\gamma_{p_i}, \gamma_r < 1$, we get

$$\lambda_1 \|u_1\|_2^2 + \lambda_2 \|u_2\|_2^2 = \sum_{i=1}^{2} \mu_i \left(\gamma_{p_i} - 1\right) \|u_i\|_{p_i}^{p_i} + \beta r \left(\gamma_r - 1\right) \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} \, dx < 0.$$

Hence, at least one of λ_1 and λ_2 is negative. Without loss of generality, we may assume $\lambda_1 < 0$. By Lemma 2.2, we have $u_1^n \to u_1$ in $H_r^1(\mathbb{R}^3)$, and then $u_1 \in S_r(c_1)$. For the sake of contradiction, suppose that $\lambda_2 \geq 0$, then

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u_2|^2 \, dx\right) \Delta u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2 - 2} u_2 + \beta r_2 |u_1|^{r_1} |u_2|^{r_2 - 2} u_2 \geq 0.$$

It follows from Lemma 2.3 that $u_2 = 0$. Thus, $J(u_1, u_2) = J(u_1, 0)$, and $u_1 \in S_r(c_1)$ satisfies the equation

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right) \Delta u = \lambda_1 u + \mu_1 |u|^{p_1 - 2} u. \tag{3.4}$$

However, this equation contradicts Proposition 2.5 (ii) that equation (3.4) admits no solution if $c < c^*$. Therefore, $\lambda_2 < 0$, and then, $u_2 \in S_r(c_2)$. Finally, by the maximum principle, we deduce that $u_1, u_2 > 0$ in $\mathbb{R}^3$. □

4 The proof of Theorem 1.2

Inspired by [2], let p_1 and p_2 be in different ranges *i.e.*, (H_2). For any $K>0$, set

$$T_K := \left\{u_2 \in S(c_2) : \|\nabla u_2\|_2^2 \le K\right\} \quad \text{and} \quad B_K := \left\{u_2 \in S(c_2) : \|\nabla u_2\|_2^2 = 2K\right\}.$$

Rewriting that $J_{u_1}(u_2) := J(u_1,u_2)$ for $u_1 \in S(c_1)$ and

$$J_{u_1}(u_2) = J_{u_1}(0) + \frac{a}{2}\|\nabla u_2\|_2^2 + \frac{b}{4}\|\nabla u_2\|_2^4 - \frac{\mu_2}{p_2}\|u_2\|_{p_2}^{p_2} - \beta\int_{\mathbb{R}^3} |u_1|^{r_1}|u_2|^{r_2}\,dx.$$

Lemma 4.1. *Assume that (H_2) holds. There exists a continuous function K from $S(c_1)$ to $\mathbb{R}$, $u_1 \mapsto K(u_1)$, such that*

$$\sup_{T_{K(u_1)}} J_{u_1}(u_2) < \inf_{B_{K(u_1)}} J_{u_1}(u_2), \quad \textit{for all } u_1 \in S(c_1).$$

The function K is bounded, and it is bounded away from 0 on bounded subsets of $S(c_1)$.

Proof. Fixing $u_1 \in S(c_1)$, for $u_2 \in T_K$, we have that,

$$J_{u_1}(u_2) \le J_{u_1}(0) + \frac{a}{2}\|\nabla u_2\|_2^2 + \frac{b}{4}\|\nabla u_2\|_2^4 \le J_{u_1}(0) + \frac{aK(u_1)}{2} + \frac{bK(u_1)^2}{4}.$$

For $u_2 \in B_K$, $\gamma' := \frac{3(r_2q'-2)}{2q'}$, where q' is defined in (2.1). Using the Gagliardo-Nirenberg inequality and (2.2), we obtain,

$$\begin{aligned} J_{u_1}(u_2) &\ge J_{u_1}(0) + aK(u_1) + bK(u_1)^2 - \frac{\mu_2}{p_2}C\|\nabla u_2\|_2^{p_2\gamma_{p_2}}\|u_2\|_2^{p_2(1-\gamma_{p_2})} - C\beta\|u_1\|_{r_1q}^{r_1}\|u_2\|_{r_2q'}^{r_2} \\ &\ge J_{u_1}(0) + aK(u_1) + bK(u_1)^2 - C_1K(u_1)^{\frac{p_2\gamma_{p_2}}{2}} - C_2\|u_1\|_{r_1q}^{r_1}K(u_1)^{\frac{\gamma'}{2}}. \end{aligned}$$

Observe that $C_1K(u_1)^{\frac{p_2\gamma_{p_2}}{2}} \le \frac{a}{8}K(u_1)$ if $K(u_1)>0$ is sufficiently small for $\frac{p_2\gamma_{p_2}}{2}>1$. Similarly, $C_2\|u_1\|_{r_1q}^{r_1}K(u_1)^{\frac{\gamma'}{2}} \le \frac{a}{8}K(u_1)$ if $K(u_1)>0$ is sufficiently small for $\frac{\gamma'}{2}>1$, provided that $q < \frac{6}{10-3r_2}$. We can choose q satisfying this inequality and (2.1) because

$$\frac{6}{10-3r_2} > \max\left\{\frac{2}{r_1}, \frac{2^*}{2^*-r_2}\right\},$$

which is a consequence of $r_1+r_2>\frac{14}{3}$ and $2<r_2<\frac{10}{3}$. More precisely, let $K : S(a_1) \to \mathbb{R}^+$ satisfy

$$K(u_1) \le \min\left\{\left(\frac{a}{8C_1}\right)^{\frac{2}{p_2\gamma_{p_2}-2}}, \left(\frac{a}{8C_2\|u_1\|_{r_1q}^{r_1}}\right)^{\frac{2}{\gamma'-2}}\right\}. \tag{4.1}$$

For $u_2 \in B_{K(u_1)}$, we have

$$\begin{aligned} J_{u_1}\left(u_2\right) &\geq J_{u_1}(0)+aK\left(u_1\right)+bK^2\left(u_1\right)-\frac{a}{8}K\left(u_1\right)-\frac{a}{8}K\left(u_1\right) \\ &> J_{u_1}(0)+\frac{a}{2}K\left(u_1\right)+\frac{b}{4}K^2\left(u_1\right) \geq \sup_{T_{K(u_1)}} J_{u_1}(u_2). \end{aligned} \tag{4.2}$$

Clearly, we define a continuous function $K : S\left(c_1\right) \to \mathbb{R}^+$ that satisfies (4.1) and is bounded away from 0 on bounded subsets of $S\left(c_1\right)$. In fact, the right-hand side of (4.1) can serve as a definition. By (4.1), K is also bounded from above. □

Now, we denote

$$T\left(u_1\right) := T_{K(u_1)}, \quad B\left(u_1\right) := B_{K(u_1)},$$

and

$$B := \left\{\left(u_1, u_2\right) : u_1 \in S\left(c_1\right), u_2 \in B\left(u_1\right)\right\}.$$

It follows from the assumption (p_1) in Theorem 1.2 and Proposition 2.5 that there exists a ground state solution $\underline{u} \in S\left(c_1\right)$ for problem (1.3) satisfying

$$J(\underline{u}, 0) = m(c_1, \mu_1) = I_{\mu_1}(\underline{u}) = \min_{u \in S(c_1)} J(u, 0) < 0.$$

Lemma 4.2. *Assume that (H_2) holds. There exist $\bar{v} \in T(\underline{u})$ and $\bar{w} \in S\left(c_2\right) \backslash T_{2K(\underline{u})}$ such that*

$$\max\{J(\underline{u}, \bar{v}), J(\underline{u}, \bar{w})\} < \inf_{(u_1, u_2) \in B} J\left(u_1, u_2\right).$$

Proof. Since $J\left(\underline{u}, u_2\right) \to J(\underline{u}, 0)$ as $\|\nabla u_2\|_2 \to 0$, to obtain $\bar{v} \in T(\underline{u})$, we claim that $J(\underline{u}, 0) < \inf_B J$. The functional $J(\cdot, 0) : S(c_1) \to \mathbb{R}$ is coercive because $2 < p_1 < \frac{14}{3}$. Choose $R > 0$ such that $J(u_1, 0) \geq J(\underline{u}, 0) + 1$ if $\|\nabla u_1\|_2 \geq R$. It follows from (4.2) and $(u_1, u_2) \in B$ with $\|\nabla u_1\|_2 \geq R$ that

$$J(u_1, u_2) \geq J(u_1, 0) + \frac{3}{4}K(u_1) > J(\underline{u}, 0) + 1. \tag{4.3}$$

For $(u_1, u_2) \in B$ with $\|\nabla u_1\|_2 \leq R$, there holds,

$$J(u_1, u_2) \geq J(u_1, 0) + \frac{3}{4}K(u_1) \geq J(\underline{u}, 0) + \frac{3}{4}\varepsilon, \tag{4.4}$$

where $\varepsilon := \inf_{\|\nabla u_1\|_2 \leq R} K(u_1) > 0$ from Lemma 4.1. By (4.3) and (4.4), the claim holds.

To find $\bar{w} \in S(c_2) \backslash T_{2K(\underline{u})}$ as required, consider any $u \in S(c_2)$. Clearly, $t \star u \in S(c_2)$ for every $t > 0$, and $\|\nabla(t \star u)\|_2 \to \infty$ as $t \to \infty$. Since $p_2 > \frac{14}{3}$, fixing an arbitrary $u \in S(c_2)$, we see that $J(\underline{u}, (t \star u)) \to -\infty$ as $t \to \infty$. □

As a result of Lemma 4.2, the set

$$\Gamma_1 := \Big\{ g' \in \mathcal{C}([0,1], \mathcal{S}_r) : g'(0) = (v_1, v_2)\,,\, g'(1) = (w_1, w_2)\,,\, v_2 \in T\,(v_1)\,,\, w_2 \notin T_{2K(w_1)}, \\ \max\left\{J\,(v_1, v_2)\,, J\,(w_1, w_2)\right\} < \inf_B J \Big\},$$

is nonempty.

Lemma 4.3. $\bar{\gamma}\,(c_1, c_2) := \inf_{g' \in \Gamma_1} \max_{t \in [0,1]} J(g'(t)) \geq \inf_B J$.

Proof. It is sufficient to show that for each $g'(t) := (g_1'(t), g_2'(t)) \in \Gamma_1$, there exists $t \in [0,1]$ such that $g'(t) \in B$. Consider the map $\alpha : [0,1] \to \mathbb{R}$ defined by $t \to \|\nabla g_2'(t)\|_2^2 - 2K\,(g_1'(t))$. This map satisfies

$$\alpha(0) = \|\nabla v_2\|_2^2 - 2K\,(v_1) \leq K\,(v_1) - 2K\,(v_1) < 0,$$

and $\alpha(1) = \|\nabla w_2\|_2^2 - 2K\,(w_1) > 0$. Thus, there exists $t \in [0,1]$ such that $\alpha(t) = 0$, which implies that $g'(t) \in B$. □

Lemma 4.4. *Assume that the conditions of Theorem 1.2 hold. Then, we have*

$$\bar{\gamma}\,(c_1, c_2) \leq m(c_1, \mu_1) + m(c_2, \mu_2).$$

Proof. By Proposition 2.5 (iii), there exists $\bar{u} \in V\,(c_2)$ such that

$$\min_{u \in V(c_2)} I_{\mu_2}(u) = \max_{t \in \mathbb{R}} I_{\mu_2}(t \star \bar{u}) = m(c_2, \mu_2) = I_{\mu_2}(0 \star \bar{u}) = I_{\mu_2}(\bar{u}) = J(0, \bar{u}). \tag{4.5}$$

Next, we consider the path $h : [0,1] \to \mathcal{S}_r$ defined by $h(t) = (\underline{u}, h_s(t))$, where

$$h_s(t)(x) = \mathrm{e}^{\frac{s(2t-1)3}{2}}\,\bar{u}\left(\mathrm{e}^{s(2t-1)}x\right).$$

Here, $s > 0$ is chosen sufficiently large so that

$$h_s(0)(\cdot) = \mathrm{e}^{\frac{-3s}{2}}\,\bar{u}\left(\mathrm{e}^{-s}\cdot\right) \in T(\underline{u}), \quad h_s(1)(\cdot) = \mathrm{e}^{\frac{3s}{2}}\,\bar{u}\left(\mathrm{e}^{s}\cdot\right) \notin T_{2K(\underline{u})},$$

and

$$\max\left\{J\,(\underline{u}, h_s(0))\,, J\,(\underline{u}, h_s(1))\right\} < \inf_B J.$$

Therefore, h belongs to Γ_1. Utilizing (4.5) and $\beta \geq 0$, we get

$$\max_{t \in [0,1]} J(h(t)) \leq J(\underline{u}, 0) + \max_{t \in [0,1]} J\left(0, h_s(t)\right) = m(c_1, \mu_1) + m(c_2, \mu_2).$$

This completes the proof. □

Lemma 4.5. *Assume that* (H_2) *holds. There exists a Palais-Smale sequence* $\{(u_1^n, u_2^n)\} \subset \mathcal{S}_r$ *for* J *at the level* $\bar{\gamma}(c_1, c_2)$ *that satisfies* $\{u_1^n\}^- \to 0$, $\{u_2^n\}^- \to 0$ *in* H_r, *and the additional property that* $P(u_1^n, u_2^n) \to 0$. *Moreover, the sequence* $\{(u_1^n, u_2^n)\}$ *is bounded.*

Proof. The existence of the sequence $\{(u_1^n, u_2^n)\}$ can be referenced in Lemma 3.2. Here, we only provide the proof of boundedness. Given that $P(u_1^n, u_2^n) = 0$, for any $\varepsilon > 0$, we have:

$$\begin{aligned} J(u_1^n, u_2^n) &= \frac{a}{2}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^2 + \frac{b}{4}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^4 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}\|u_i^n\|_{p_i}^{p_i} - \beta\int_{\mathbb{R}^3}|u_1^n|^{r_1}|u_2^n|^{r_2}\,dx \\ &= \frac{(1+\epsilon)a}{4}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^2 + \frac{\epsilon b}{4}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^4 + \delta_1(\epsilon)\|u_1^n\|_{p_1}^{p_1} + \delta_2(\epsilon)\|u_2^n\|_{p_2}^{p_2} \\ &\quad + \beta\delta_3(\epsilon)\int_{\mathbb{R}^3}|u_1^n|^{r_1}|u_2^n|^{r_2}\,dx + \frac{(1-\epsilon)}{4}P(u_1^n, u_2^n), \end{aligned}$$

where

$$\delta_1(\epsilon) = \frac{(1-\epsilon)\mu_1\gamma_{p_1}}{4} - \frac{\mu_1}{p_1}, \quad \delta_2(\epsilon) = \frac{(1-\epsilon)\mu_2\gamma_{p_2}}{4} - \frac{\mu_2}{p_2}, \quad \delta_3(\epsilon) = \frac{(1-\epsilon)r\gamma_r}{4} - 1.$$

Note that the coefficients satisfy $\delta_1(\epsilon) < 0$ and $\delta_2(\epsilon), \delta_3(\epsilon) > 0$ for sufficiently small $\varepsilon > 0$. Although $\delta_1(\epsilon) < 0$, the term $\|u_1^n\|_{p_1}^{p_1}$ is controlled by $\sum_{i=1}^{2}\|\nabla u_i^n\|_2^4$ because $p_1 < \frac{14}{3}$. Hence, we conclude that J is coercive. Consequently, the sequence $\{(u_1^n, u_2^n)\} \subset \mathcal{S}_r$ is bounded. ☐

Proof of Theorem 1.2. By Lemmas 2.2 and 4.5, we can assume that $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ in H_r, where $u_1 \geq 0$ and $u_2 \geq 0$. As shown in Lemma 3.3, we have $J(u_1, u_2) = \bar{\gamma}(c_1, c_2)$. To establish strong convergence, it suffices to show, according to Lemmas 2.4 and 2.2 (iv), that $u_1 \neq 0$ and $u_2 \neq 0$.

We first claim that: if $\bar{\gamma}(c_1, c_2) < 0$, *then* $u_1 \neq 0$ *and* $u_2 \neq 0$.

For contradiction, that at least one of u_1 or u_2 is zero. Then, by Lemma 2.1,

$$(u_1^n, u_2^n) \to (u_1, u_2) \text{ in } L^p(\mathbb{R}^3) \times L^q(\mathbb{R}^3) \text{ for } p, q \in (2, 2^*) \text{ and } \beta\int_{\mathbb{R}^3}|u_1^n|^{r_1}|u_2^n|^{r_2}\,dx \to 0.$$

For the sequence $\{(u_1^n, u_2^n)\}$ satisfying $P(u_1^n, u_2^n) \to 0$, we deduce that

$$a\sum_{i=1}^{2}\|\nabla u_i^n\|_2^2 + b\sum_{i=1}^{2}\|\nabla u_i^n\|_2^4 - \sum_{i=1}^{2}\mu_i\gamma_{p_i}\|u_i^n\|_{p_i}^{p_i} = o_n(1).$$

By the weak lower semi-continuity, we have

$$
\begin{aligned}
J\left(u_1^n, u_2^n\right) &= \frac{a}{2}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^2 + \frac{b}{4}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^4 - \sum_{i=1}^{2}\frac{\mu_i}{p_i}\|u_i^n\|_{p_i}^{p_i} - \beta\int_{\mathbb{R}^3}|u_1^n|^{r_1}|u_2^n|^{r_2}dx \\
&= \frac{a}{4}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^2 - \sum_{i=1}^{2}\mu_i\gamma_{p_i}\left(\frac{1}{p_i\gamma_{p_i}} - \frac{1}{4}\right)\|u_i^n\|_{p_i}^{p_i} + o_n(1) \qquad (4.6)\\
&\geq \frac{a}{4}\sum_{i=1}^{2}\|\nabla u_i\|_2^2 - C_1\|u_1\|_{p_1}^{p_1} + C_2\|u_2\|_{p_2}^{p_2},
\end{aligned}
$$

where $C_1 > 0$ and $C_2 > 0$. We now distinguish three cases.

Case 1. $(u_1 = u_2 = 0)$**:** From (4.6), we obtain $J\left(u_1^n, u_2^n\right) \geq 0$. Since we have assumed that $\gamma\left(c_1, c_2\right) < 0$, this case cannot occur.

Case 2. $(u_1 = 0$ **and** $u_2 \neq 0)$**:** By Lemmas 2.2, 2.4, we have $\lambda_2 < 0$, and hence $u_2^n \to u_2 \in S_r\left(c_2\right)$. From (4.6), we get

$$
0 > \bar{\gamma}(c_1, c_2) = J\left(u_1^n, u_2^n\right) \geq \frac{a}{4}\|\nabla u_2\|_2^2 + C_2\|u_2\|_{p_2}^{p_2} > 0, \quad \text{as} \quad n \to \infty. \qquad (4.7)
$$

This results in a contradiction.

Case 3. $(u_1 \neq 0$ **and** $u_2 = 0)$**:** Since $u_2 = 0$ and $J(u_1, u_2) = \bar{\gamma}\left(c_1, c_2\right)$, we have

$$
\bar{\gamma}(c_1, c_2) = J\left(u_1, u_2\right) = J\left(u_1, 0\right) = I_{\mu_1}\left(u_1\right).
$$

We know u_1 satisfies

$$
-\left(a + b\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)\Delta u = \lambda_1 u + \mu_1|u|^{p_1-2}u.
$$

For $2 < p_1 \leq \frac{10}{3}$, u_1 is a positive ground state solution by Propsition 2.5 (i). Then $m(c_1, \mu_1) = I_{\mu_1}(u_1)$. From Lemmas 4.1, 4.3 and the definitions of B, Γ_1, we know that

$$
\bar{\gamma}\left(c_1, c_2\right) \geq \inf_B J > J(u_1, 0) = I_{\mu_1}(u_1) = m(c_1, \mu_1), \qquad (4.8)
$$

which contradicts $\bar{\gamma}\left(c_1, c_2\right) = m(c_1, \mu_1)$. When $\frac{10}{3} < p_1 < \frac{14}{3}$, u_1 can be characterized as either a high energy solution or a ground state solution. If u_1 is ground state solution, we can get a contradiction similar to (4.8). If u_1 is high energy solution, we have a contradiction as $0 < I_\mu(u_1) = \bar{\gamma}(c_1, c_2) < 0$. Thus, the claim holds.

In view of Lemmas 2.2, 4.4 and 4.5, to establish the theorem, it is enough to prove that $m(c_1, \mu_1) + m(c_2, \mu_2) < 0$. Note also that $u_1 > 0$ and $u_2 > 0$ follow directly from the strong maximum principle. □

Proof of Corollary 1.3. The Corollary is a straightforward consequence of Theorem 1.2 and Proposition 2.5. □

5 The proof of Theorem 1.4

In this section, we first consider the case that (H_3). Recalling Proposition 2.5 (i), for $2 < p_1$, $p_2 \leq \frac{10}{3}$, there exist u^1 and u^2 such that

$$m(c_1, \mu_1) = I_{\mu_1}(u^1) \quad \text{and} \quad m(c_2, \mu_2) = I_{\mu_2}(u^2).$$

Lemma 5.1. *Assume that (H_3) holds. There exist $\beta_1 := \beta_1(c_1, c_2)$ and $\rho_* := \rho_*(c_1, c_2) > \|\nabla u^1\|_2^2 + \|\nabla u^2\|_2^2$ such that*

$$J(u_1, u_2) > 0 \text{ on } A(2\rho_*) \setminus A(\rho_*) \quad \text{for } 0 < \beta < \beta_1,$$

where $A(\rho_*) = \left\{(u_1, u_2) \in \mathcal{S}_r : \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2 < \rho_*\right\}$ for $\rho_* > 0$.

Proof. Recalling the proof of Lemma 3.1, we can take a sufficiently large ρ_* such that

$$\rho_* > \|\nabla u^1\|_2^2 + \|\nabla u^2\|_2^2,$$

and

$$\sum_{i=1}^{2} \frac{\mu_i}{p_i} C_i \left(\rho_*\right)^{\frac{p_i \gamma_{p_i} - 4}{2}} \leq \frac{b}{32}. \tag{5.1}$$

Next, we choose $\beta_1 > 0$ to be sufficiently small, such that

$$\beta_1 C_3 \left(2\rho_*\right)^{\frac{2^* - 4}{2}} \leq \frac{b}{32}. \tag{5.2}$$

The lemma follows directly from (5.1) and (5.2). □

Now we can set

$$\gamma'(c_1, c_2) := \inf_{A(2\rho_*)} J(u_1, u_2).$$

The following lemma plays a crucial role in overcoming compactness.

Lemma 5.2. *Assume that (H_3) holds. Then, for any $0 < \beta < \beta_1$, the following statements are true:*

- (i) $\gamma'(c_1, c_2) < m(c_1, \mu_1) + m(c_2, \mu_2) < 0$.
- (ii) $\gamma'(c_1, c_2) \leq \gamma'(c_1', c_2')$, *for all* $0 < c_1' \leq c_1$, $0 < c_2' \leq c_2$.

Proof. (*i*) From Lemma 5.1, we know that $\left(u^1, u^2\right) \in A(\rho_*)$. Furthermore, using Proposition 2.5 (*i*) and the fact that $\beta > 0$, we deduce that

$$\gamma'(c_1, c_2) \leq J(u^1, u^2) = I_{\mu_1}(u^1) + I_{\mu_2}(u^2) - \beta \int_{\mathbb{R}^3} \left|u^1\right|^{r_1} \left|u^2\right|^{r_2} dx < m(c_1, \mu_1) + m(c_2, \mu_2) < 0.$$

(*ii*) To prove this, we need to show that for any $\varepsilon > 0$, $\gamma'(c_1, c_2) \leq \gamma'(c_1', c_2') + \varepsilon$, for all $0 < c_1' \leq c_1$ and $0 < c_2' \leq c_2$. Let $\varphi(x) \in C_c^\infty\left(\mathbb{R}^N\right)$ be a cut-off function such that

$$0 \leq \phi(x) \leq 1 \quad \text{and} \quad \phi(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

By the definition of $\gamma'(c_1', c_2')$ and Lemma (5.1), there exists $(u_1, u_2) \in A\left(\rho_*\right)$ such that

$$J(u_1, u_2) \leq \gamma'(c_1', c_2') + \frac{\varepsilon}{2}. \tag{5.3}$$

For any $\delta > 0$, we define $(u_{\delta_1}(x), u_{\delta_2}(x)) := (u_1\phi(\delta x), u_2\phi(\delta x))$. Since $(u_{\delta_1}, u_{\delta_2}) \to (u_1, u_2)$ in H_r as $\delta \to 0^+$, there exists a sufficiently small δ such that

$$J\left(u_{\delta_1}, u_{\delta_2}\right) \leq J(u_1, u_2) + \frac{\varepsilon}{4} \quad \text{and} \quad \|\nabla u_{\delta_1}\|_2^2 + \|\nabla u_{\delta_2}\|_2^2 \leq \frac{3}{2}\rho_*. \tag{5.4}$$

Let $\varphi(x) \in C_c^\infty\left(\mathbb{R}^3\right)$ such that $\operatorname{supp}(\varphi) \subset \left\{x \in \mathbb{R}^3 : \frac{4}{\delta} \leq |x| \leq 1 + \frac{4}{\delta}\right\}$ and set

$$(\tilde{u}_1, \tilde{u}_2) = \left(\frac{\sqrt{c_1 - \|u_{\delta_1}\|_2^2}}{\|\varphi\|_2}\varphi, \frac{\sqrt{c_2 - \|u_{\delta_2}\|_2^2}}{\|\varphi\|_2}\varphi\right).$$

Noting that, for any $s \leq 0$,

$$\operatorname{supp}\left(u_{\delta_1}\right) \cap \operatorname{supp}(s \star \tilde{u}_1) = \emptyset \quad \text{and} \quad \operatorname{supp}\left(u_{\delta_2}\right) \cap \operatorname{supp}(s \star \tilde{u}_2) = \emptyset.$$

As $s \to -\infty$, we have

$$J(s \star \tilde{u}_1, s \star \tilde{u}_2) \to 0 \quad \text{and} \quad \|\nabla s \star \tilde{u}_1\|_2^2 + \|\nabla s \star \tilde{u}_2\|_2^2 \leq \frac{\varepsilon}{12\rho_* b}. \tag{5.5}$$

It follows that

$$\left(u_{\delta_1} + s \star \tilde{u}_1, u_{\delta_2} + s \star \tilde{u}_2\right) \in A\left(2\rho_*\right),$$

and by (5.3)-(5.5), for $s<0$ large enough, we have

$$\begin{aligned}\gamma'(c_1,c_2) &\leq J\left(u_{\delta_1}+s\star\tilde{u}_1, u_{\delta_2}+s\star\tilde{u}_2\right)\\ &= J\left(u_{\delta_1},u_{\delta_2}\right)+J(s\star\tilde{u}_1,s\star\tilde{u}_2)+\frac{b}{2}\sum_{i=1}^{2}\|\nabla u_{\delta_i}\|_2^2\|\nabla s\star\tilde{u}_i\|_2^2\\ &\leq J(u_1,u_2)+\frac{\varepsilon}{4}+\frac{\varepsilon}{8}+\frac{\varepsilon}{8}\leq \gamma'(c_1',c_2')+\varepsilon.\end{aligned}$$

The proof is completed. □

Lemma 5.3. *Assume that* (H_3) *holds. For any* $0<\beta<\beta_1$*, there exists*

$$\mu_* := \mu_*(a,b,c_1,c_2,p_1,p_2,\beta,\rho)$$

such that for $\mu_1,\mu_2<\mu_*$ *and* $(u_1,u_2)\in\mathcal{S}_r$*, the function* $\Phi_{u_1,u_2}(s)$*, defined in* (1.4) *has two critical points* $t_{u_1,u_2}<\tau_{u_1,u_2}$ *and two zeros* $c_{u_1,u_2}<d_{u_1,u_2}$ *with* $t_{u_1,u_2}<c_{u_1,u_2}<\tau_{u_1,u_2}<d_{u_1,u_2}$*. Moreover, for* $s\in\mathbb{R}$*,*

(*i*) *If* $(s\star u_1, s\star u_2)\in\mathcal{P}$*, then either* $s=t_{u_1,u_2}$ *or* $s=\tau_{u_1,u_2}$*.*

(*ii*) $\|\nabla s\star u_1\|_2^2+\|\nabla s\star u_2\|_2^2\leq\rho_*$ *for every* $s\leq c_{u_1,u_2}$ *and*

$$J(t_{u_1,u_2}\star u_1, t_{u_1,u_2}\star u_2)=\min\left\{J(s\star u_1,s\star u_2): \|\nabla s\star u_1\|_2^2+\|\nabla s\star u_2\|_2^2\leq\rho_*\right\}<0.$$

(*iii*) *We have* $J\left(\tau_{u_1,u_2}\star u_1,\tau_{u_1,u_2}\star u_2\right)=\max\{J(s\star u_1,s\star u_2): s\in\mathbb{R}\}$*.*

Proof. (*i*) Since $p_i\gamma_{p_1}<2$ for $i=1,\ 2$, and $r=2^*$, it is evident that $\Phi_{u_1,u_2}(-\infty)=0^-$ and $\Phi_{u_1,u_2}(+\infty)=-\infty$. By Lemma 5.1, we know that $\Phi_{u_1,u_2}(s)$ has at least two critical points $t_{u_1,u_2}<\tau_{u_1,u_2}$, where t_{u_1,u_2} is a local minimum point of $\Phi_{u_1,u_2}(s)$ at negative level and τ_{u_1,u_2} is a global maximum point at positive level. On the other hand, it is standard to prove that $\Phi_{u_1,u_2}(s)$ has at most two critical points as in [20, Lemma 4.5]. The (*ii*) and (*iii*) follow from Lemma 5.1 and (*i*). □

Proof of Theorem 1.4. Consider a minimizing sequence $\{(u_1^n,u_2^n)\}\subset\mathcal{S}_r$ for $J|_{A(2\rho_*)}$. By Lemma 5.3, we have $\|\nabla t_{u_1^n,u_2^n}\star u_1^n\|_2^2+\|\nabla t_{u_1^n,u_2^n}\star u_2^n\|_2^2\leq\rho_*$, and the sequence $\left\{t_{u_1^n,u_2^n}\star u_1^n, t_{u_n,u_2^n}\star u_2^n\right\}$ remains a minimizing sequence for $J|_{A(2\rho_*)}$. According to [9, Theorem 4.1], there exists a new minimizing sequence, still denoted by $\{(u_1^n,u_2^n)\}\subset A\left(2\rho_*\right)$, such that

$$J\left(u_1^n,u_2^n\right)\to\gamma'(c_1,c_2),\quad P\left(u_1^n,u_2^n\right)\to 0,\quad J'|_{\mathcal{S}_r}\left(u_1^n,u_2^n\right)\to 0,\qquad \text{as } n\to\infty. \tag{5.6}$$

Given that $J'|_{\mathcal{S}_r}(u_1^n,u_2^n)\to 0$, there exist sequences $\{\lambda_1^n\}\subset\mathbb{R}$ and $\{\lambda_2^n\}\subset\mathbb{R}$ such that

$$\begin{aligned} a\sum_{i=1}^{2}\int_{\mathbb{R}^3}\nabla u_i^n\nabla\varphi_i dx+b\sum_{i=1}^{2}\|\nabla u_i^n\|_2^2\int_{\mathbb{R}^3}\nabla u_i^n\nabla\varphi_i dx-\sum_{i=1}^{2}\mu_i\int_{\mathbb{R}^3}|u_i^n|^{p_i-2}u_i^n\varphi_i dx\\ -\beta r_1\int_{\mathbb{R}^3}|u_1^n|^{r_1-2}|u_2^n|^{r_2}u_1^n\varphi_1 dx-\beta r_2\int_{\mathbb{R}^3}|u_1^n|^{r_1}|u_2^n|^{r_2-2}u_2^n\varphi_2 dx\\ =\int_{\mathbb{R}^3}(\lambda_1^n u_1^n\varphi_1+\lambda_2^n u_2^n\varphi_2)dx+o_n(1), \end{aligned} \tag{5.7}$$

for any $(\varphi_1,\varphi_2)\in H_r$. Taking $(u_1^n,0)$ and $(0,u_2^n)$ as test functions, we have

$$\begin{cases} \lambda_1^n c_1+o_n(1)=a\|\nabla u_1^n\|_2^2+b\|\nabla u_1^n\|_2^4-\mu_1\|u_1^n\|_{p_1}^{p_1},\\ \lambda_2^n c_2+o_n(1)=a\|\nabla u_2^n\|_2^2+b\|\nabla u_2^n\|_2^4-\mu_2\|u_2^n\|_{p_2}^{p_2}. \end{cases}$$

Since the sequence $\{u_1^n,u_2^n\}\subset A(2\rho_*)$ is bounded, we suppose that $\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_i^n|^2dx=A_i\geq 0$. Without loss of generality, let us assume that, up to a subsequence, $(\lambda_1^n,\lambda_2^n)\to(\lambda_1,\lambda_2)\in\mathbb{R}^2$, $(u_1^n,u_2^n)\rightharpoonup(u_1,u_2)\in H_r$ and $(u_1^n,u_2^n)\to(u_1,u_2)$ in $L^p(\mathbb{R}^3)\times L^q(\mathbb{R}^3)$ for any $p,\,q\in(2,2^*)$. Then, we know that,

$$\begin{cases} -(a+bA_1)\Delta u_1=\lambda_1u_1+\mu_1|u_1|^{p_1-2}u_1+\beta r_1|u_1|^{r_1-2}u_1|u_2|^{r_2},\\ -(a+bA_2)\Delta u_2=\lambda_2u_2+\mu_2|u_2|^{p_2-2}u_2+\beta r_2|u_1|^{r_1}|u_2|^{r_2-2}u_2. \end{cases} \tag{5.8}$$

From (5.8), we have

$$0=P_A(u_1,u_2):=a\sum_{i=1}^{2}\|\nabla u_i\|_2^2+b\sum_{i=1}^{2}A_i\|\nabla u_i\|_2^2-\sum_{i=1}^{2}\mu_i\gamma_{p_i}\|u_i\|_{p_i}^{p_i}-\beta 2^*\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx.$$

Let $(\bar{u}_1^n,\bar{u}_2^n):=(u_1^n-u_1,u_2^n-u_2)$. Then $\bar{u}_1^n\to 0$ in $L^{p_1}(\mathbb{R}^3)$, $\bar{u}_2^n\to 0$ in $L^{p_2}(\mathbb{R}^3)$ and we have

$$\begin{aligned} P(u_1^n,u_2^n)&=P_A(u_1,u_2)+a\sum_{i=1}^{2}\|\nabla\bar{u}_i^n\|_2^2+b\sum_{i=1}^{2}A_i\|\nabla\bar{u}_i^n\|_2^2-\beta 2^*\int_{\mathbb{R}^3}|\bar{u}_1^n|^{r_1}|\bar{u}_2^n|^{r_2}dx\\ &=a\sum_{i=1}^{2}\|\nabla\bar{u}_i^n\|_2^2+b\sum_{i=1}^{2}A_i\|\nabla\bar{u}_i^n\|_2^2-\beta 2^*\int_{\mathbb{R}^3}|\bar{u}_1^n|^{r_1}|\bar{u}_2^n|^{r_2}dx+o_n(1). \end{aligned} \tag{5.9}$$

From (2.2), (5.9) and Lemma 2.1, we obtain

$$\begin{aligned} a\sum_{i=1}^{2}\|\nabla\bar{u}_i^n\|_2^2\leq a\sum_{i=1}^{2}\|\nabla\bar{u}_i^n\|_2^2+b\sum_{i=1}^{2}A_i\|\nabla\bar{u}_i^n\|_2^2=\beta 2^*\int_{\mathbb{R}^3}|\bar{u}_1^n|^{r_1}\,|\bar{u}_2^n|^{r_2}\,dx+o_n(1)\\ \leq\beta 2^*S^{-\frac{2^*}{2}}\left(\sum_{i=1}^{2}\|\nabla\bar{u}_i^n\|_2^2\right)^{\frac{2^*}{2}}+o_n(1). \end{aligned} \tag{5.10}$$

Up to a subsequence, we assume that $\sum_{i=1}^{2}\|\nabla \bar{u}_i^n\|_2^2 \to l \geq 0$. According to (5.10), we have $l=0$ or $l \geq \left(\frac{a}{\beta 2^*}\right)^{\frac{1}{2}} S^{\frac{3}{2}}$. If $l \geq \left(\frac{a}{\beta 2^*}\right)^{\frac{1}{2}} S^{\frac{3}{2}}$, then from (5.6), (5.10), and Lemma 2.1, we conclude

$$\begin{aligned}
\gamma'(c_1,c_2) &= \lim_{n\to\infty} J\left(u_1^n,u_2^n\right) = J(u_1,u_2) + \lim_{n\to\infty} J\left(\bar{u}_1^n,\bar{u}_2^n\right) + \frac{b}{2}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^2\|\nabla u_i\|_2^2 \\
&\geq J(u_1,u_2) + \lim_{n\to\infty} J\left(\bar{u}_1^n,\bar{u}_2^n\right) \\
&\geq \gamma'\left(\|u_1\|_2^2,\|u_2\|_2^2\right) + \lim_{n\to\infty}\left(\frac{a}{2}\sum_{i=1}^{2}\|\nabla \bar{u}_i^n\|_2^2 + \frac{b}{4}\sum_{i=1}^{2}\|\nabla \bar{u}_i^n\|_2^4 - \beta\int_{\mathbb{R}^3}|\bar{u}_1^n|^{r_1}|\bar{u}_2^n|^{r_2}\,dx\right) \\
&\geq \gamma'\left(\|u_1\|_2^2,\|u_2\|_2^2\right) + \lim_{n\to\infty}\left(\frac{a}{2}\sum_{i=1}^{2}\|\nabla \bar{u}_i^n\|_2^2 - \beta\int_{\mathbb{R}^3}|\bar{u}_1^n|^{r_1}|\bar{u}_2^n|^{r_2}\,dx\right) \\
&\geq \gamma'\left(\|u_1\|_2^2,\|u_2\|_2^2\right) + \lim_{n\to\infty}\left[\frac{a}{2}\sum_{i=1}^{2}\|\nabla \bar{u}_i^n\|_2^2\left(1-\frac{2}{a}\beta S^{-\frac{2^*}{2}}\left(\sum_{i=1}^{2}\|\nabla \bar{u}_i^n\|_2^2\right)^2\right)\right].
\end{aligned}$$

By $\sum_{i=1}^{2}\|\nabla \bar{u}_i^n\|_2^2 \leq \rho_*$, there exists $\beta_* < \beta_1$ such that $\left(\frac{a}{2\beta_*}S^{\frac{2^*}{2}}\right)^{\frac{1}{2}} \geq \rho_*$. Then

$$\left(1-\frac{2}{a}\beta S^{-\frac{2^*}{2}}\left(\sum_{i=1}^{2}\|\nabla \bar{u}_i\|_2^2\right)^2\right) \geq 0,$$

when $\beta < \beta_*$, which contradicts with (ii) of Lemma 5.2. Thus, $\sum_{i=1}^{2}\|\nabla \bar{u}_i^n\|_2^2 \to 0$, as $n \to \infty$. Then $(u_1^n,u_2^n) \to (u_1,u_2)$ in $D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ and (u_1,u_2) is a solution to equations (1.1).

Finally, we will prove that $(u_1^n,u_2^n) \to (u_1,u_2)$ in H_r. Taking (u_1^n,u_2^n) as the test function in (5.7), we obtain

$$\langle J'\left(u_1^n,u_2^n\right),\left(u_1^n,u_2^n\right)\rangle = \lambda_1^n c_1 + \lambda_2^n c_2 + o_n(1).$$

Given that $P\left(u_1^n,u_2^n\right) \to 0$, $(\lambda_1^n,\lambda_2^n) \to (\lambda_1,\lambda_2)$, we have

$$\lambda_1 c_1 + \lambda_2 c_2 = \lambda_1^n c_1 + \lambda_2^n c_2 + o_n(1) = \sum_{i=1}^{2}\mu_i\left(\gamma_{p_i}-1\right)\|u_i^n\|_{p_i}^{p_i} < 0.$$

Since $\lambda_1 c_1 + \lambda_2 c_2 < 0$, at least one of λ_1 and λ_2 is negative. Next, we consider three possible conditions.

Case 1. ($\lambda_1 < 0$ and $\lambda_2 < 0$): Using the fact that

$$\langle J'\left(u_1^n,u_2^n\right) - \lambda_1^n\left(u_1^n,0\right),\left(u_1^n,0\right)\rangle \to \langle J'\left(u_1,u_2\right) - \lambda_1\left(u_1,0\right),\left(u_1,0\right)\rangle = 0,$$

we have

$$\begin{cases}
\lambda_1\|u_1^n\|_2^2 + o_n(1) = a\|\nabla u_1^n\|_2^2 + b\|\nabla u_1^n\|_2^4 - \mu_1\|u_1^n\|_{p_1}^{p_1}, \\
\lambda_1\|u_1\|_2^2 = a\|\nabla u_1\|_2^2 + b\|\nabla u_1\|_2^4 - \mu_1\|u_1\|_{p_1}^{p_1}.
\end{cases}$$

Since Lemma 2.1, $\lambda_1^n \to \lambda_1 < 0$, $\|u_1^n\|_{p_1}^{p_1} \to \|u_1\|_{p_1}^{p_1}$, and $u_1^n \to u_1$ in $D^{1,2}(\mathbb{R}^3)$, we get $\|u_1^n\|_2^2 \to \|u_1\|_2^2$, leading to strong convergence. The case where $\lambda_2 < 0$ is treated similarly.

Case 2. ($\lambda_1 < 0$ and $\lambda_2 \geq 0$): Using the method of Case 1, it can be concluded that $u_1^n \to u_1$ in $H_r^1\left(\mathbb{R}^3\right)$ and $u_1 \in S_r\left(c_1\right)$. Assume, by contradiction, that $\lambda_2 \geq 0$, then

$$-\left(a+b\int_{\mathbb{R}^3}\left|\nabla u_2\right|^2 dx\right)\Delta u_2 = \lambda_2 u_2 + \mu_2\left|u_2\right|^{p_2-2}u_2 + \beta r_2\left|u_1\right|^{r_1}\left|u_2\right|^{r_2-2}u_2 \geq 0.$$

By Lemma 2.3, we deduce that $u_2 = 0$. Thus, $J\left(u_1, u_2\right) = J\left(u_1, 0\right)$, $u_1^n \to u_1$, and $u_1 \in S_r\left(c_1\right)$ satisfies the equation

$$-\left(a+b\int_{\mathbb{R}^3}\left|\nabla u\right|^2 dx\right)\Delta u = \lambda_1 u + \mu_1\left|u\right|^{p_1-2}u.$$

Therefore, $I_{\mu_1}(u_1) \geq m(c_1, \mu_1)$. On the other hand, by Hölder inequality,

$$0 \leq \int_{\mathbb{R}^3}\left|u_1^n\right|^{r_1}\left|u_2^n\right|^{r_2} dx \leq \left\|u_1^n\right\|_{2^*}^{r_1}\left\|u_2^n\right\|_{2^*}^{r_2}.$$

Using the fact $u_2^n \to 0$ in $D^{1,2}\left(\mathbb{R}^3\right)$, we have

$$\gamma'(c_1, c_2) = \lim_{n\to\infty} J\left(u_1^n, u_2^n\right) = I_{\mu_1}(u_1) + \lim_{n\to\infty} I_{\mu_2}\left(u_2^n\right) - \beta\lim_{n\to\infty}\int_{\mathbb{R}^3}\left|u_1^n\right|^{r_1}\left|u_2^n\right|^{r_2} dx \geq m(c_1, \mu_1),$$

which contradicts Lemma 5.2 (i)

Case 3. ($\lambda_2 < 0$ and $\lambda_1 \geq 0$): By similar arguments as in Case 2, we obtain a contradiction $\gamma'(c_1, c_2) \geq m(c_2, \mu_2)$. Therefore, we conclude that $(u_1^n, u_2^n) \to (u_1, u_2)$ in H_r.

By Lemma 5.3 and $\gamma'(c_1, c_2) < 0$, we have

$$\gamma'(c_1, c_2) = J(u_1, u_2) = \inf_{\mathcal{P}} J = \inf_{A(\rho_*)} J < 0.$$

This implies that (u_1, u_2) is a ground state solution. The proof of Theorem 1.4 is completed. □

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Blow-up and global existence of solutions for a higher-order reaction diffusion equation with singular potential

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ABSTRACT

In this work, we consider the higher-order reaction-diffusion parabolic problem with time dependent coefficient. We prove the blow-up of solutions and obtain a lower and an upper bound for the blow-up time. Finally, we investigate the existence of a global weak solution to the problem.

RESUMEN

En este trabajo, consideramos un problema parabólico de reacción-difusión de alto orden con coeficiente dependiente del tiempo. Demostramos la explosión de soluciones y obtenemos cotas inferior y superior para el tiempo de explosión. Finalmente, investigamos la existencia de una solución débil global del problema.

Keywords and Phrases: Blow-up, higher-order, singular potential, global existence, reaction-diffusion.

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1 Introduction

In this work, we investigate the following reaction-diffusion parabolic problem with singular potential:

$$\begin{cases} \dfrac{z_t}{|x|^{2m}} + \mathcal{A}z = \alpha(t)|z|^{r-1}z, & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial^i z(x,t)}{\partial \nu^i} = 0, \quad i = 0,1,\ldots,m-1, & (x,t) \in \partial\Omega \times (0,T), \\ z(x,0) = z_0(x) \in H_0^m(\Omega) \cap L^{r+1}(\Omega), \quad x \in \Omega, \end{cases} \tag{1.1}$$

here $n \geq 1$ and $\Omega \subset R^n$ is open and bounded with Lipschitz boundary, where $T > 0$, $r > 1$, $\mathcal{A} = (-\Delta)^m$, $m > 1$ is an integer constant and a unit outer normal ν, $x = (x_1, x_2, \ldots, x_n)$, $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. The coefficient $\alpha(t)$ is chosen such that

$$\alpha \in C^1[0,\infty), \ \alpha(0) > 0 \text{ and } \ \alpha'(t) \geq 0 \quad \text{for all } t \in [0,\infty). \tag{1.2}$$

Explosive phenomena commonly arise in solutions to reaction-diffusion partial differential equations of various types (see *e.g.* [4, 6, 15] and references therein). Understanding the conditions under which such phenomena occur is of practical interest. However, accurately computing the precise blow-up time is often challenging. Despite this challenge, it is still possible to estimate the blow-up time using various methods. Notable approaches for investigation include the first eigenvalue method proposed by Kaplan in 1963, the potential well method developed by Levine and Payne in 1970, the comparison method, and other techniques involving integration. A recent comprehensive overview of these methods can be found in the monograph by Hu [11], Al'shin *et al.* [2] and Pişkin [17]. Additionally, readers may refer to the survey articles by Galaktionov [8] and Levine [13] for insights into the blow-up properties of more general evolution problems. Specifically, sufficient conditions for blow-up estimates are discussed in works of Philippin [16] and Han [9] provided insights for the equation of the form:

$$z_t + \Delta^2 z = k(t) f(z).$$

In another study, Han [10] investigated the equation of the form

$$\frac{z_t}{|x|^2} - \Delta z = k(t)|z|^{p-1}z,$$

in which the author derived the lower and upper bounds on the blow-up time of weak solutions.

In [23], Thanh *et al.* considered the reaction-diffusion parabolic problem with time dependent coefficients

$$\frac{z_t}{|x|^4} + \Delta^2 z = k(t)|z|^{p-1}z.$$

They proved an upper and lower bound for blow-up time. Do *et al.* [5] investigated the existence of a global weak solution to the problem together with the decaying and blow-up properties using the potential well method.

Recently, Thanh *et al.* [24] proved the higher-order version $\Delta\left(|\Delta|^{m-2}\Delta\right)$ of the $p-$Laplacian and the function $k(t)$ non-Newtonian filtration equation and obtained the blow-up result with lower and upper bounds. The reader is directed to [19–21] for a detailed discussion of higher-order hyperbolic equations.

In our research, we employed various types of Dirichlet-Neumann boundary conditions in conjunction with a general nonlinear term. Additionally, we derived the primary outcomes of this paper using a methodology distinct from those discussed in prior works. While some of the literature has addressed blow-up solutions for higher-order parabolic equation, to the best of our knowledge, there is currently no article available that specifically explores the finite-time blow-up solutions for a higher-order parabolic equation with a variable coefficient term $\alpha(t)$. Consequently, we endeavored to investigate and present new and noteworthy findings in this regard. For a more in-depth exploration of blow-up phenomena in higher-order parabolic equation, readers are encouraged to consult the book by Galaktionov [7].

Motivated by above-mentioned papers, in this paper, we investigate to prove the upper and lower bounds for the blow-up time of solutions for problem (1.1), which was not previously studied, where we study higher-order parabolic equation with time dependent coefficient source terms $\alpha(t)|z|^{r-1}z$.

The rest of the work is as follows: In Section 2, we give some assumptions needed in this work. In Section 3, under suitable conditions, we obtain an upper bound for the blow-up time. In Section 4, we obtain a lower bound for the blow-up time. In Section 5, under suitable conditions, we investigate the existence of a global weak solution to the problem.

2 Preliminaries

In this part, we present certain lemmas and assumptions required for the formulation and proof of our results. Let $\|.\|$, $\|.\|_r$ and $\|.\|_{W^{m,r}(\Omega)}$ indicate the typical $L^2(\Omega)$, $L^r(\Omega)$ and $W^{m,r}(\Omega)$ norms (see [1,18]).

Now, we consider some energy estimates: Let $n \geq 1$ and $\Omega \subset R^n$ be open bounded with Lipschitz boundary. For each $z \in H_0^m(\Omega) \cap L^{r+1}(\Omega)$ and $t \in [0,\infty)$ define the following functionals of the problem (1.1):

- Energy functional is as follows:

$$J(z,t) = \frac{1}{2}\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 - \frac{\alpha(t)}{r+1}\|z\|^{r+1},$$

- and Nehari functional is as follows:

$$I(z,t) = \left\|\mathcal{A}^{\frac{1}{2}} z\right\|^{2} - \alpha(t)\|z\|^{r+1}.$$

We strive to establish both upper and lower bounds for the blow-up time of a weak solution to equation (1.1), the precise definitions of which are provided in the following.

Definition 2.1. *A function z is termed a weak solution to equation* (1.1) *if* $z \in L^2\left(0,T; H_0^m(\Omega) \cap L^{r+1}(\Omega)\right)$ *and* $\frac{z_t}{|x|^{2m}} \in L^2\left(0,T; L^2(\Omega)\right)$ *where z satisfies the following equation:*

$$\left(\frac{z_t}{|x|^{2m}}, \varphi\right) + \left(\mathcal{A}^{\frac{1}{2}} z, \mathcal{A}^{\frac{1}{2}} \varphi\right) = \alpha(t)\left(|z|^{r-1} z, \varphi\right), \tag{2.1}$$

for all $\varphi \in H_0^m(\Omega) \cap L^{r+1}(\Omega)$ *and* $t \in [0,\infty)$.

When $\Omega \subset R^n$ is an open and bounded set with a Lipschitz boundary, the existence of a local weak solution can be established using standard Ordinary Differential Equation (ODE) theory, coupled with the Faedo-Galerkin approximation technique, as is well-known in the literature.

Definition 2.2. *Assume that $z(t)$ is a weak solution to* (1.1)*. If $z(t)$ exists for all t in the interval $[0,T^*)$, and the limit as to blow up at a finite time T^* if $z(t)$ exists for all $t \in [0,T^*)$ and*

$$\lim_{t\to T^*}\left\|\frac{z_t}{|x|^m}\right\|^2 = \infty. \tag{2.2}$$

Such a T^* is called the maximal existence time as well as the blow up time for $z(t)$. If (2.2) *does not happen for any finite time T^*, then $z(t)$ is called a global solution and the maximal existence time of $z(t)$ is ∞.*

We are able to define the stable and unstable sets as follows for each $t \geq 0$:

- Stable set:

$$\Sigma_1(t) = \left\{z \in H_0^m(\Omega) : J(z,t) < n_\infty \text{ and } I(z,t) > 0\right\}.$$

- Unstable set:

$$\Sigma_2(t) = \left\{z \in H_0^m(\Omega) : J(z,t) < n_\infty \text{ and } I(z,t) < 0\right\}.$$

$\Sigma_1(t)$ and $\Sigma_2(t)$ are crucial to our paper. Where

$$n_\infty = \lim_{t\to\infty} n(t).$$

Note that J, I, C_0, n, Σ_1 and Σ_2 are all time-dependent, as indicated by the presence of $\alpha(t)$ in

(1.1). The introduction of this time-dependent factor introduces additional technical complexity into our analysis.

Because of the presence of the inverse coefficient $1/|x|^{2m}$, it is important to highlight the distinction between the two cases when $0 \in \Omega$ and $0 \notin \Omega$. If $0 \in \Omega$ then $1/|x|^{2m}$ develops a singularity. This requires the application of Rellich's inequality, which is valid for $n \geq 2m+1$, in the proofs of our main results. However, if $0 \notin \Omega$ then there is no singularity and (1.1) can be considered as a slight extension of the model in [10]. In this case our results are valid for all $n \geq 1$. To deal with these two cases at the same time, we use the notation

$$n_\Omega = \begin{cases} 2m+1, & \text{if } 0 \in \Omega \\ 1, & \text{if } 0 \notin \Omega \end{cases} \quad \text{and} \quad 2^* = \begin{cases} \infty, & \text{if } n \leq 2m, \\ \frac{2n}{n-2m} = 2 + \frac{2m}{n-2m}, & \text{if } n \geq 2m+1. \end{cases}$$

Let us start with the following Rellich inequality Lemma.

Lemma 2.3. *Assume that $n \geq 2m+1$ and $\Omega \subset R^n$ be open bounded. Let $z \in H_0^m(\Omega)$. Then $\frac{z}{|x|^{2m}} \in L^2(\Omega)$ and*

$$\int_\Omega \frac{|z|^2}{|x|^{2m}} dx \leq \left(\frac{m^2}{n(m-1)(n-2m)} \right)^2 \int_\Omega \left| \mathcal{A}^{\frac{1}{2}} z \right|^2 dx = \mathcal{C} \int_\Omega \left| \mathcal{A}^{\frac{1}{2}} z \right|^2 dx.$$

Proof. Let $z \in H_0^m(\Omega)$ and $\check{z}$ be its zero extension to R^n. Then $\check{z} \in H^m(R^n)$ by [1, Lemma 3.27], and

$$\begin{aligned} \int_\Omega \frac{|z|^2}{|x|^{2m}} dx &\leq \int_{R^n} \frac{|\check{z}|^2}{|x|^{2m}} dx \leq \left(\frac{m^2}{n(m-1)(n-2m)} \right)^2 \int_{R^n} \left| \mathcal{A}^{\frac{1}{2}} \check{z} \right|^2 dx \\ &\leq \left(\frac{m^2}{n(m-1)(n-2m)} \right)^2 \int_{R^n} \left| \mathcal{A}^{\frac{1}{2}} z \right|^2 dx, \end{aligned} \tag{2.3}$$

here we used [3, Corollary 6.3.5], in the second step of the argument. This provides the justification for the claim. □

The next result below is the Gagliardo-Nirenberg inequality.

Lemma 2.4. *Let $n \geq 2m+1$ and Ω be open and bounded subset of R^n, $1 < r < 1 + \frac{4m}{n-2m}$. Then there exists $C_0 = C_0(\Omega, n, r) > 0$ so that*

$$\|z\|_{L^{r+1}(\Omega)}^{r+1} \leq C_0 \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{\beta(r+1)} \|z\|^{(1-\beta)(r+1)}, \quad \forall z \in H_0^m(\Omega),$$

where

$$\beta = \frac{n(r-1)}{4(r+1)} \in (0,1). \tag{2.4}$$

Proof. Let $z \in H_0^m(\Omega)$. It follows from Gagliardo-Nirenberg inequality that

$$\|z\|_{L^{r+1}(\Omega)}^{r+1} \leq C(\Omega, n, r) \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^{\beta(r+1)} \|z\|^{(1-\beta)(r+1)},$$

where used

$$\left\| \nabla^2 z \right\| \leq C(\Omega, n) \left\| \mathcal{A}^{\frac{1}{2}} z \right\|,$$

by [22, Chapter 3, Proposition 3]. □

Lemma 2.5. *Assume that $n \geq 1$ and $\Omega \subset R^n$ be open and bounded with Lipschitz boundary. Suppose α is defined by (1.2). Let z be a weak solution to equation (1.1) with $T > 0$. Then the following identities hold:*

(H1)

$$J(z(h), h) + \int_0^h \left(\left\| \frac{z_t(s)}{|x|^m} \right\|^2 - \frac{\alpha'(s)}{r+1} \|z\|_{L^{r+1}(\Omega)}^{r+1} \right) ds = J(z_0, 0),$$

and

(H2)

$$\frac{d}{dt}\left(\frac{1}{2} \left\| \frac{z(h)}{|x|^m} \right\|^2 \right) = \left(\frac{z(h)}{|x|^{2m}}, z_t(h) \right) = -I(z(h), h),$$

for a.e. $h \in [0, T)$.

Proof. For **(H1)**, first assume that $z_t \in L^2\left(0, T; H_0^m(\Omega) \cap L^{r+1}(\Omega)\right)$. Then, utilizing z_t as a test function in (2.1) we have

$$\left\| \frac{z_t}{|x|^m} \right\|^2 + \left(\mathcal{A}^{\frac{1}{2}} z, \mathcal{A}^{\frac{1}{2}} z_t \right) = \alpha(t) \left(|z|^{r-1} z, z_t \right).$$

Moreover, direct calculations provide

$$\frac{d}{dt} J(z(t), t) = \left(\mathcal{A}^{\frac{1}{2}} z, \mathcal{A}^{\frac{1}{2}} z_t \right) - \alpha(t) \left(|z|^{r-1} z, z_t \right) - \frac{\alpha'(t)}{r+1} \|z\|_{L^{r+1}(\Omega)}^{r+1},$$

as a function of t in the interval $[0, T)$. Combining these two identities together results in

$$\frac{d}{dt} J(z(t), t) = -\left\| \frac{z_t}{|x|^m} \right\|^2 - \frac{\alpha'(t)}{r+1} \|z\|_{L^{r+1}(\Omega)}^{r+1}, \tag{2.5}$$

as a function of t in the interval $[0, T)$.

Now **(H1)** follows by integrating both sides of (2.5) with respect to t over $(0, h)$, where $h \in (0, T)$.

To conclude, with an approximation argument we examine that (2.5) holds without the assumption that $z_t \in L^2\left(0, T; H_0^m(\Omega) \cap L^{r+1}(\Omega)\right)$.

The proof of **(H2)** is the same way and is omitted. □

The result we give below is obtained directly from Lemma 2.4 and the Friedrichs inequality (*cf.* [14, Theorem 1.10]).

Lemma 2.6. *Let $n \geq 1$, $z \in H_0^m(\Omega)$ and $2 < r+1 < 2^*$. Then there exists a constant $S_r = S_r(n,r) > 0$ so that*

$$\|z\|_{L^{r+1}(\Omega)} \leq S_r \|\Delta z\|.$$

In addition, we note that the constant S_r may be made explicit and sharp when $n \geq 2m+1$.

Our next result is known as the concavity argument, which is widely used in the literature and is used for the sufficient condition of blow-up.

Lemma 2.7 ([12,13])**.** *Suppose that a positive, twice-differentiable on $(0,\infty)$ function $\psi(t)$ satisfies the inequality*

$$\psi''(t)\psi(t) - (1+\theta)(\psi'(t))^2 \geq 0,$$

where $\theta > 0$. If $\psi(0) > 0$ and $\psi'(0) > 0$ for all $t \in (0,\infty)$. Then there exists $T > 0$ such that

$$\lim_{t\to T^-} \psi(t) = \infty, \quad \textit{and} \quad T \leq \frac{\psi(0)}{\theta\psi'(0)}.$$

3 Upper bound for blow-up time

In this part, we are going to obtain the upper bounds for the finite time blow-up results. For simplicity, we shall write

$$\mathcal{L}(t) = \frac{1}{2}\left\|\frac{z(t)}{|x|^m}\right\|^2,$$

for each $t \in [0,T)$.

We start with the proof of Theorem 3.1. This is related to the upper limit on the explosion time of the weak solution when the initial energy functional is negative (1.1) .

Theorem 3.1. *Assume that $n \geq 2m+1$ and $\Omega \subset R^n$ be open and bounded with Lipschitz boundary. Let $r > 1$ and α be given by (1.2). Such that $0 \neq z_0 \in H_0^m(\Omega) \cap L^{r+1}(\Omega)$ and*

$$J(z_0,0) = \frac{1}{2}\left\|\mathcal{A}^{\frac{1}{2}} z_0\right\|^2 - \frac{\alpha(0)}{r+1}\|z_0\|_{L^{r+1}(\Omega)}^{r+1} < 0.$$

Suppose that $z(t)$ is a weak solution to (1.1) with $T > 0$. Then z blows up at a finite time T^* which satisfies

$$T^* \leq \frac{\left\|\frac{z_0}{|x|^m}\right\|^2}{(1-r^2)J(z_0,0)}.$$

Proof. Here we set $T^* < \infty$, where $T^* \geq 0$ is the maximum existence time of z, and then we aim to provide an upper bound for T^*.

Set for this purpose

$$\mathcal{K}(t) = -J(z(t), t),$$

for every $t \in [0, T^*)$. According to the hypothesis $\mathcal{L}(0) > 0$ and $\mathcal{K}(0) > 0$.

We can also write from Lemma 2.5:

$$\mathcal{K}'(t) = -\frac{d}{dt}J(z(t), t) = \left\| \frac{z_t}{|x|^m} \right\|^2 + \frac{\alpha'(s)}{r+1} \|z\|_{L^{r+1}(\Omega)}^{r+1} \geq 0, \tag{3.1}$$

for each $t \in [0, T^*)$, so $\mathcal{K}$ increases over $[0, T^*)$. Consequently, $\mathcal{K}(t) \geq \mathcal{K}(0) > 0$ for all $t \in [0, T^*)$.

Assume that $t \in [0, T^*)$. Same way,

$$\mathcal{L}'(t) = \left(\frac{z}{|x|^{2m}}, z_t \right) = -I(z(t), t) = \frac{r-1}{2} \left\| \mathcal{A}^{\frac{1}{2}} z \right\|^2 - (r+1) J(z(t), t) \geq (r+1) \mathcal{K}(t). \tag{3.2}$$

Thus,

$$\mathcal{L}(t) \mathcal{K}'(t) \geq \frac{1}{2} \left\| \frac{z}{|x|^m} \right\|^2 \left\| \frac{z_t}{|x|^m} \right\|^2 \geq \frac{1}{2} \left(\frac{z}{|x|^{2m}}, z_t \right)^2 = \frac{1}{2} (\mathcal{L}'(t))^2 \geq \frac{r+1}{2} \mathcal{L}'(t) \mathcal{K}(t). \tag{3.3}$$

From (3.1), (3.2) and (3.3), we get

$$\left(\mathcal{K}(t) \mathcal{L}^{-(r+1)/2}(t) \right)' = \mathcal{L}^{-(r+3)/2}(t) \left(\mathcal{K}'(t) \mathcal{L}(t) - \frac{r+1}{2} \mathcal{K}(t) \mathcal{L}'(t) \right) \geq 0.$$

This means that $\mathcal{K}\mathcal{L}^{-(r+1)/2}$ strictly increases over $[0, T^*)$, which follows:

$$\begin{aligned} 0 < \xi_0 = \mathcal{K}(0) \mathcal{L}^{-(r+1)/2}(0) &< \mathcal{K}(t) \mathcal{L}^{-(r+1)/2}(t) \\ &\leq \frac{1}{r+1} \mathcal{L}'(t) \mathcal{L}^{-(r+1)/2}(t) = \frac{2}{1-r^2} \left(\mathcal{L}^{(1-r)/2}(t) \right)', \end{aligned}$$

here we used (3.2).

Integrating this last representation with respect to t over $(0, \tau)$, where $\tau \in (0, T^*)$, we obtain:

$$\xi_0 \tau \leq \frac{2}{1-r^2} \left(\mathcal{L}^{(1-r)/2}(\tau) - \mathcal{L}^{(1-r)/2}(0) \right).$$

Since this inequality only holds for a finite period of time, we deduce $T^* < \infty$. Moreover,

$$0 \leq \mathcal{L}^{(1-r)/2}(\tau) \leq \mathcal{L}^{(1-r)/2}(0) - \frac{\left(r^2 - 1\right) \xi_0}{2} \tau,$$

for all $\tau \in [0, T^*)$. This reveals that

$$T^* \leq \frac{2}{(r^2-1)\,\xi_0}\mathcal{L}^{(1-r)/2}(0) = \frac{2\mathcal{L}(0)}{(1-r^2)\,J(z_0,0)}.$$

The proof is complete. □

Next we state and prove Theorem 3.2. Here it provides an upper bound on the explosion time for a weak solution to (1.1) when the initial energy functional is positive.

Theorem 3.2. *Suppose that $n \geq 2m+1$ and $\Omega \subset R^n$ be open and bounded with Lipschitz boundary. Let $r > 1$ and α be given by (1.2). Assume that $0 \neq z_0 \in H_0^m(\Omega) \cap L^{r+1}(\Omega)$ and*

$$0 \leq C_1 J(z_0, 0) < \frac{1}{2}\left\|\frac{z_0}{|x|^m}\right\|^2 = \mathcal{L}(0),$$

where

$$C_1 = \frac{(r+1)\,\mathcal{C}}{r-1} \quad \text{and} \quad \mathcal{C} = \left(\frac{m^2}{n\,(m-1)\,(n-2m)}\right)^2.$$

Suppose that $z(t)$ be a weak solution to (1.1) with $T > 0$. Then z blows up at a finite time T^* which satisfies

$$T^* \leq \frac{4rC_1\mathcal{L}(0)}{(r-1)^2\,(r+1)\,(\mathcal{L}(0) - C_1 J(z_0,0))}.$$

Proof. Here we set $T^* < \infty$, where $T^* \geq 0$ is the maximum existence time of z, and then we aim to provide an upper bound for T^*.

From (3.2)

$$\begin{aligned}\mathcal{L}'(t) &\geq \frac{r-1}{2}\left\|\mathcal{A}^{\frac{1}{2}}z\right\|^2 - (r+1)\,J(z(t),t) \geq \frac{r-1}{2\mathcal{C}}\left\|\frac{z(t)}{|x|^m}\right\|^2 - (r+1)\,J(z(t),t) \\ &= \frac{r-1}{\mathcal{C}}\left[\mathcal{L}(t) - C_1 J(z(t),t)\right] = \frac{r-1}{\mathcal{C}}\mathcal{M}(t),\end{aligned}$$

for each $t \in (0, T^*)$, where in the second step we used Lemma 2.3.

Observe from the inequality above:

$$\mathcal{M}'(t) = \mathcal{L}'(t) - C_1\frac{d}{dt}J(z(t),t) \geq \mathcal{L}'(t) \geq \frac{r-1}{\mathcal{C}}\mathcal{M}(t),$$

for each $t \in (0, T^*)$, here we used (3.1) in the second step.

Moreover,

$$\mathcal{M}(0) = \mathcal{L}(0) - C_1 J(z_0, 0) > 0,$$

by assumption. Consequently, an application of Gronwall's inequality gives

$$\mathcal{M}(t) \geq \mathcal{M}(0) \exp\left(\frac{r-1}{\mathcal{C}} t\right) > 0.$$

This means that $\mathcal{L}'(t) > 0$ for every $t \in (0, T^*)$. That is, $\mathcal{L}$ increases strictly over $[0, T^*)$ and hence

$$\mathcal{L}(t) > \mathcal{L}(0), \tag{3.4}$$

for every $t \in [0, T^*)$.

And by C_1 and $\mathcal{C}$ given in the statement of this theorem. Fix $\tau \in [0, T^*)$ and

$$\beta \in \left(0, \frac{r+1}{rC_1}\right) \mathcal{M}(0) \quad \text{and} \quad \sigma \in \left(\frac{\mathcal{L}(0)}{(r-1)\beta}, \infty\right). \tag{3.5}$$

The choices of β and σ are justified below with (3.8) and (3.9) respectively. Define non-negative functional

$$\Psi(h) = \int_0^h \mathcal{L}(s)\, ds + (\tau - h)\mathcal{L}(0) + \beta(h+\sigma)^2,$$

where $h \in [0, \tau]$. Then

$$\Psi'(h) = \mathcal{L}(h) - \mathcal{L}(0) + 2\beta(h+\sigma) = 2\int_0^h \left(\frac{z(s)}{|x|^m}, z_t(s)\right) ds + 2\beta(h+\sigma),$$

and

$$\begin{aligned}
\Psi''(h) &= 2\left(\frac{z(h)}{|x|^m}, z_t(h)\right) + 2\beta = -2I(z(h), h) + 2\beta \\
&\geq -2(r+1) J(z(h), h) + (r-1)\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 + 2\beta \\
&\geq -2(r+1)\left[J(z_0, 0) - \int_0^h \left(\left\|\frac{z_t(s)}{|x|^m}\right\|^2 + \frac{\alpha'(s)}{r+1}\|z\|_{L^{r+1}(\Omega)}^{r+1}\right) ds\right] + (r-1)\left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 + 2\beta \\
&\geq -2(r+1)\left[J(z_0, 0) - \int_0^h \left(\left\|\frac{z_t(s)}{|x|^m}\right\|^2 + \frac{\alpha'(s)}{r+1}\|z\|_{L^{r+1}(\Omega)}^{r+1}\right) ds\right] + \frac{2(r-1)}{\mathcal{C}}\mathcal{L}(h) + 2\beta,
\end{aligned} \tag{3.6}$$

for each $h \in [0, \tau)$, where we used Lemmas 2.5 and 2.3 in the third and fourth lines, respectively.

In what follows it is convenient to denote

$$\begin{aligned}
\theta(h) &= \left(2\int_0^h \mathcal{L}(s)\, ds + \beta(h+\sigma)^2\right)\left(\int_0^h \left\|\frac{z_t(s)}{|x|^m}\right\|_{L^2(\Omega)}^2 ds + \beta\right) \\
&\quad - \left(\int_0^h \left(\frac{z(s)}{|x|^m}, z_t(s)\right) ds + \beta(h+\sigma)\right)^2 \geq 0,
\end{aligned} \tag{3.7}$$

for every $h \in [0,\tau]$, where in the last step of (3.7) we used the Cauchy-Schwarz inequality.

From Lemma 2.7, (3.6) and (3.4), we obtain

$$\begin{aligned}
\Psi(h)\Psi''(h) - \frac{r+1}{2}\left(\Psi'(h)\right)^2 &= \Psi(h)\Psi''(h) - 2(r+1)\left[\int_0^h \left(\frac{z(s)}{|x|^m}, z_t(s)\right) ds + \beta(h+\sigma)\right]^2 \\
&= \Psi(h)\Psi''(h) + 2(r+1)\left[\theta(h) - \left(\Psi(h) - (\tau-h)\mathcal{L}(0)\right)\left(\int_0^h \left\|\frac{z_t(s)}{|x|^m}\right\|^2 ds + \beta\right)\right] \\
&\geq \Psi(h)\Psi''(h) - 2(r+1)\Psi(h)\left(\int_0^h \left\|\frac{z_t(s)}{|x|^m}\right\|^2 ds + \beta\right) \\
&\geq \Psi(h)\left[\Psi''(h) - 2(r+1)\left(\int_0^h \left\|\frac{z_t(s)}{|x|^m}\right\|^2 ds + \beta\right)\right] \\
&\geq \Psi(h)\left[-2(r+1)J(z_0,0) + \frac{2(r-1)}{\mathcal{C}}\mathcal{L}(h) - 2r\beta\right] \\
&\geq \Psi(h)\left[-2(r+1)J(z_0,0) + \frac{2(r-1)}{\mathcal{C}}\mathcal{L}(0) - 2r\beta\right] \\
&= 2(r+1)\Psi(h)\left[-J(z_0,0) + \frac{1}{C_1}\mathcal{L}(0) - \frac{r\beta}{r+1}\right] \geq 0,
\end{aligned} \tag{3.8}$$

for all $h \in [0,\tau]$.

Then observe this

$$\Psi(0) = \tau\mathcal{L}(0) + \beta\sigma^2 > 0, \quad \text{and} \quad \Psi'(0) = 2\beta\sigma > 0.$$

Consequently, from Lemma 2.7:

$$\tau \leq \frac{2\Psi(0)}{(r-1)\Psi'(0)} = \frac{2\left(\tau\mathcal{L}(0) + \beta\sigma^2\right)}{2(r-1)\beta\sigma} = \frac{\mathcal{L}(0)}{(r-1)\beta\sigma}\tau + \frac{\sigma}{r-1}.$$

This is as a result

$$\tau\left(1 - \frac{\mathcal{L}(0)}{(r-1)\beta\sigma}\right) \leq \frac{\sigma}{r-1},$$

or equivalently, we can write

$$\tau \leq \frac{\sigma}{r-1}\left(1 - \frac{\mathcal{L}(0)}{(r-1)\beta\sigma}\right)^{-1} = \frac{\beta\sigma^2}{(r-1)\beta\sigma - \mathcal{L}(0)}. \tag{3.9}$$

Reducing the expression mentioned in (3.5) across the range of σ results in

$$\tau \leq \frac{4\mathcal{L}(0)}{(r-1)^2\beta}. \tag{3.10}$$

Next, we aim to minimize the expression referenced by (3.10) within the specified range of β as outlined in (3.5). This leads to the following inequality:

$$\tau \leq \frac{4rC_1\mathcal{L}(0)}{(r-1)^2(r+1)\mathcal{M}(0)}. \tag{3.11}$$

Finally, the inequality stated in reference (3.11) remains valid for all $\tau \in (0, T^*)$. From this, we can conclude that

$$T^* \leq \frac{4rC_1\mathcal{L}(0)}{(r-1)^2(r+1)\mathcal{M}(0)},$$

as needed. □

4 Lower bound for blow-up time

In this section we consider with the lower bound for the finite time blow-up results. This is the content of Theorem 4.1. For simplicity, we shall write

$$\mathcal{L}(t) = \frac{1}{2}\left\|\frac{z(t)}{|x|^m}\right\|^2,$$

for each $t \in [0, T)$.

We start with the proof of Theorem 4.1. This is related to the lower limit on the explosion time of the weak solution when the initial energy functional is negative (1.1) .

Theorem 4.1. *Assume that $n \geq 2m+1$ and $\Omega \subset R^n$ be open bounded with Lipschitz boundary. Let α is given by (1.2) which enjoys a further property that*

$$\alpha_\infty = \lim_{t\to\infty} \alpha(t) < \infty.$$

Suppose that $1 < r < 1 + \frac{4m}{n}$. Let $z(t)$ be a weak solution to (1.1) with $T > 0$ and $0 \neq z_0 \in H_0^m(\Omega)$. Assume that $z(t)$ blows up at T^*. Then

$$T^* \geq \frac{\mathcal{L}^{1-\gamma}(0)}{C^*(\gamma-1)},$$

where

$$\beta = \frac{n(r-1)}{4(r+1)} \in (0,1), \quad \gamma = \frac{(1-\beta)(r+1)}{2-\beta(r+1)} > 1,$$

and

$$C^* = \frac{2-\beta(r+1)}{2}\left(\frac{2}{\alpha_\infty C_0 \beta(r+1)}\right)^{-\beta(r+1)/(2-\beta(r+1))}\left(\sup_{x\in\Omega}|x|\right)^{4\gamma},$$

with $C_0 = C_0(\Omega, n, r) > 0$.

Proof. By assumption $1 < r < 1 + \frac{4m}{n}$ this leads to

$$0 < \beta(r+1) = \frac{(r-1)n}{4} < m.$$

This allows us to apply Young's inequality below.

Based on the constants defined in the expression of this theorem and utilizing the Lemma 2.4 and Young's inequality. We get

$$\begin{aligned}
\mathcal{L}'(h) &= \left(\frac{z(h)}{|x|^m}, z_t(h)\right) = -I(z(h), h) = \alpha(h)\|z\|_{L^{r+1}(\Omega)}^{r+1} - \left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 \\
&\leq C_0 \alpha_\infty \left\|\mathcal{A}^{\frac{1}{2}} z\right\|^{\beta(r+1)} \|z\|^{(1-\beta)(r+1)} - \left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 \\
&\leq \left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 + \frac{2-\beta(r+1)}{2}\left(\frac{2}{\alpha_\infty \mathcal{C}_0 \beta(r+1)}\right)^{-\beta(r+1)/(2-\beta(r+1))} \|z\|^{2\gamma} - \left\|\mathcal{A}^{\frac{1}{2}} z\right\|^2 \\
&= \frac{2-\beta(r+1)}{2}\left(\frac{2}{\alpha_\infty \mathcal{C}_0 \beta(r+1)}\right)^{-\beta(r+1)/(2-\beta(r+1))} \|z\|^{2\gamma} \\
&\leq \frac{2-\beta(r+1)}{2}\left(\frac{2}{\alpha_\infty \mathcal{C}_0 \beta(r+1)}\right)^{-\beta(r+1)/(2-\beta(r+1))} \left(\sup_{x\in\Omega}|z|\right)^{4\gamma} \mathcal{L}(t)^\gamma \\
&= C^* \mathcal{L}(t)^\gamma,
\end{aligned}$$

for all $h \in (0, T^*)$. Equivalency one has

$$\frac{\mathcal{L}'(t)}{\mathcal{L}(t)^\gamma} \leq C^*,$$

where do we get it

$$\frac{1}{1-\gamma}\left(\mathcal{L}^{1-\gamma}(t) - \mathcal{L}^{1-\gamma}(0)\right) \leq C^* t.$$

Lastly, since $\gamma > 1$ and $\lim_{t\to T^*} \mathcal{L}(t) = \infty$, allowing $t \to T^*$ in the above inequality, we have

$$T^* \geq \frac{\mathcal{L}^{1-\gamma}(0)}{C^*(\gamma-1)},$$

as required. □

5 Global existence

In this Section, we establish the existence of a global weak solution to the equation referenced as (1.1), which corresponds to Theorem 5.2. While the proof follows the conventional arguments of Faedo-Galerkin approximation, the presence of the fourth-order operator in (1.1) requires a thorough justification, particularly when the initial datum z_0 belongs to the stable set Σ_1. For the sake of simplicity in notation, we utilize the dot notation in this part

$$z_k' = (z_k)_t = \frac{\partial}{\partial t} z_k.$$

Hereafter

$$a \wedge b = \min\{a, b\} \quad \text{and} \quad a \vee b = \max\{a, b\}.$$

Remember we set

$$n_\Omega = \begin{cases} 2m+1, & \text{if } 0 \in \Omega \\ 1, & \text{if } 0 \notin \Omega \end{cases} \quad \text{and} \quad 2^* = \begin{cases} \infty, & \text{if } n \leq 2m, \\ \frac{2n}{n-2m}, & \text{if } n \geq 2m+1, \end{cases}$$

with

$$\Sigma_1(t) = \{z \in H_0^m(\Omega) : J(z,t) < n_\infty \ \text{ and } \ I(z,t) > 0\},$$

and

$$\Sigma_2(t) = \{z \in H_0^m(\Omega) : J(z,t) < n_\infty \ \text{ and } \ I(z,t) > 0\},$$

for every $t \geq 0$.

We begin with a problem of approximation.

Lemma 5.1 ([5])**.** *Assume that $n \geq n_\Omega$ and $2 < r+1 < 2^*$. Suppose that $k \in N$, $T > 0$ and $z_{k0} \in C_c^\infty(\Omega)$. Then the problem*

$$\begin{cases} \rho_k(x) z_k' + \mathcal{A} z_k = \beta_k(z_k), & (x,t) \in \Omega \times (0,T], \\ \dfrac{\partial^i z_k(x,t)}{\partial \nu^i} = 0, \ i = 0, 1, \ldots, m-1, & (x,t) \in \Omega \times (0,T], \\ z_k(x,0) = z_{k0}, & x \in \Omega, \end{cases} \tag{5.1}$$

accepts a global solution $z_k \in C([0,T]; H_0^m(\Omega))$ so that $z_k' \in L^2(0,T; H_0^m(\Omega))$, where

$$\rho_k(x) = |x|^{-2m} \wedge n \quad \text{and} \quad \beta_k(z_k) = \alpha(t)\left[(-k) \vee \left(|z_k|^{r-1} z_k\right) \wedge k\right].$$

Finally, we present the existence of a global weak solution to (1.1) when the initial datum z_0 belongs to the stable set Σ_1.

Theorem 5.2. *Suppose that $n \geq n_\Omega$ and $\Omega \subset R^n$ be open bounded with Lipschitz boundary. Assume that $2 < r+1 < 2^*$. Let $z_0 \in \sum_1(0)$. Suppose $\alpha \in C^1[0,\infty)$ satisfies $\alpha(0) > 0$ and $\alpha'(t) \geq 0$ for all $t \in [0,\infty)$. Morever suppose that $\lim_{t\to\infty} \alpha(t) = 1$. Then there exists a global weak solution to* (1.1).

Proof. Since $z_0 \in \Sigma_1(0)$, there exists a constant $\epsilon_0 > 0$ so that

$$J(u_0, 0) + \epsilon_0 < n_\infty.$$

From Lemma 5.1 for every $k \in N$ there exists a weak solution $z_k \in C([0,T]; H_0^m(\Omega))$ with $z_k' \in L^2(0,T; H_0^m(\Omega))$ to the problem (5.1), here $z_{k0} \in C_c^\infty(\Omega)$ is so that

$$\lim_{k\to\infty} z_{k0} = z_0 \quad \text{in } H_0^m(\Omega).$$

By choosing a sufficiently large $k \in N$, we can also assume that

$$J(z_{k0}, 0) \le J(z_0, 0) + \epsilon_0 < n_\infty. \tag{5.2}$$

Using z_k' as a test function in (5.1), we get

$$\int_0^t \int_\Omega \rho_k^2 z_k'(s)^2 \, dx\, ds + \int_0^t \int_\Omega \mathcal{A} z_k(s) z_k'(s)\, dx\, ds$$
$$= \int_0^t \int_\Omega \beta_k(z_k) z_k'(s)\, dx\, ds \le \int_0^t \int_\Omega |z_k(s)|^{r-1} z_k(s) z_k'(s)\, dx\, ds.$$

When you realize this

$$\int_\Omega \mathcal{A} z_k z_k' dx = \frac{d}{dt}\left(\frac{1}{2}\int_\Omega \left\|\mathcal{A}^{1/2} z_k\right\|^2 dx\right),$$

and

$$\int_\Omega |z_k|^{r-1} z_k z_k' dx = \frac{d}{dt}\left(\frac{1}{r+1}\int_\Omega \|z_k\|_{L^{r+1}(\Omega)}^{r+1} dx\right).$$

We can rewrite the above inequality as follows:

$$\int_0^t \int_\Omega \rho_k z_k'(s)^2 \, dx\, ds + J(z_k(t), t) \le J(z_{k0}, 0) < n_\infty, \tag{5.3}$$

here we used (5.2) in the last step. This implies $z_k(t) \in \Sigma_1$ for every $t \in [0,T]$. Indeed, let us express the opposite statement by way of contradiction. Let t^* denote the minimal time at which $z_k(t^*) \notin \Sigma_1$. Utilizing the fact that $z_k \in C([0,T]; H_0^m(\Omega))$ we deduce that $z_k(t^*) \in \partial\Sigma_1$. In other words, either $J(z_k(t^*), t^*) = n_\infty$ or $I(z_k(t^*), t^*) = 0$. The former is impossible due to (5.3).

As a result, it is necessary to satisfy $I(z_k(t^*), t^*) = 0$ or equivalently,

$$\left\|\mathcal{A}^{\frac{1}{2}} z_k(t^*)\right\|^2 = \alpha(t^*) \|z_k(t^*)\|_{L^{r+1}(\Omega)}^{r+1},$$

which implies

$$J(z_k(t^*), t^*) = \frac{r-1}{2(r+1)}\left\|\mathcal{A}^{\frac{1}{2}} z_k(t^*)\right\|^2 \ge \frac{r-1}{2(r+1)} S_r^{-2} \|z_k(t^*)\|_{L^{r+1}(\Omega)}^2$$
$$= \frac{r-1}{2(r+1)} S_r^{-2} \left(\frac{\alpha(t^*)^{-1/2}\left\|\mathcal{A}^{\frac{1}{2}} z_k(t^*)\right\|}{\|z\|_{L^{r+1}(\Omega)}}\right)^{\frac{2}{r+1}\left(\frac{1}{2}-\frac{1}{r+1}\right)^{-1}}$$

$$\geq \frac{r-1}{2(r+1)} \alpha(t^*)^{2/(1-r)} S_r^{-2(r+1)/(r-1)} = n(t^*) \geq n_\infty.$$

This statement contradicts the information provided in inequality (5.3). Therefore, $z_k(t)$ belongs to the set Σ_1 for each t in the interval $[0,T]$, as asserted.

For $t \in [0,T]$, if $z_k(t) \in \Sigma_1$, it implies

$$\left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 = \alpha(t) \|z_k(t)\|_{L^{r+1}(\Omega)}^{r+1}.$$

By utilizing equation (5.3) we can derive the following inequality:

$$\int_0^t \int_\Omega \rho_k z_k'(s)^2 \, dx \, ds + \left(\frac{1}{2} - \frac{\alpha(t)}{r+1} \right) \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 < J(z_{k0}, 0) < n_\infty. \tag{5.4}$$

There is one in particular

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{r+1} \right) \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 &= \left(\frac{1}{2} - \frac{\alpha_\infty}{r+1} \right) \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 \\ &< \left(\frac{1}{2} - \frac{\alpha(t)}{r+1} \right) \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 < J(z_{k0}, 0), \end{aligned} \tag{5.5}$$

here $\alpha_\infty = \lim_{t\to\infty} \alpha(t) = 1$ by hypothesis. Utilizing the Lemma 2.6, (5.5) and (5.2), we get

$$\begin{aligned} \int_\Omega |z_k(t)|^{r+1} dx &< S_r^{r+1} \left(\left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 \right)^{(r+1)/2} = S_r^{r+1} \left(\left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 \right)^{(r+1)/2-1} \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 \\ &< S_r^{r+1} \left[\left(\frac{1}{2} - \frac{1}{r+1} \right)^{-1} J(z_{k0}, 0) \right]^{(r+1)/2-1} \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 \\ &< S_r^{r+1} \left[\left(\frac{1}{2} - \frac{1}{r+1} \right)^{-1} (J(z_0, 0) + \epsilon_0) \right]^{(r+1)/2-1} \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2 \\ &= \delta \left\| \mathcal{A}^{\frac{1}{2}} z_k(t) \right\|^2. \end{aligned} \tag{5.6}$$

Note that

$$0 < \delta < S_r^{r+1} \left[\left(\frac{1}{2} - \frac{1}{r+1} \right)^{-1} d_\infty \right]^{(r+1)/2-1} = \left[\left(\frac{1}{2} - \frac{1}{r+1} \right)^{-1} \frac{r-1}{2(r+1)} \right]^{(r-1)/2} = 1.$$

Next, we employ z_k as a test function in (5.1) to obtain

$$\begin{aligned} \frac{1}{2} \int_\Omega \rho_k z_k^2 \, dx + \int_0^t \int_\Omega \left| \mathcal{A}^{\frac{1}{2}} z_k(s) \right|^2 dx \, ds &\leq \int_0^t \int_\Omega |z_k(s)|^{r+1} dx \, ds + \frac{1}{2} \int_\Omega \rho_k z_{k0}^2 \, dx \\ &< \delta \int_0^t \int_\Omega \left| \mathcal{A}^{\frac{1}{2}} z_k(s) \right|^2 dx \, ds + \frac{1}{2} \int_\Omega \rho_k z_{k0}^2 \, dx, \end{aligned}$$

where we utilized reference (5.6) in the second step.

It can be deduced that

$$\frac{1}{2}\int_{\Omega}\rho_k z_k^2\,dx + (1-\delta)\int_0^t\int_{\Omega}\left|\mathcal{A}^{\frac{1}{2}} z_k\,(s)\right|^2 dx\,ds < \frac{1}{2}\int_{\Omega}\rho_k z_{k0}^2\,dx < C, \tag{5.7}$$

here $C > 0$ is independent of k and T. As a result, the sequence $\{z_k\}_{k\in N}$ is uniformly bounded in $L^2\,(0, T; H_0^m\,(\Omega))$.

By (5.4) and (5.7), the following properties are satisfied:

$$\begin{cases} z_k \to z & \text{a.e. in } (0,T)\times\Omega, \\ \rho_k^{1/2} z_k \overset{\omega}{\to} \frac{z_t}{|x|^m} & \text{in } L^2\left(0,T;L^2\,(\Omega)\right), \\ \mathcal{A}^{\frac{1}{2}} z_k \overset{\omega}{\to} \mathcal{A}^{\frac{1}{2}} z & \text{in } L^2\left(0,T;L^2\,(\Omega)\right), \\ z_k \overset{\omega}{\to} z & \text{in } L^2\left(0,T;L^{r+1}\,(\Omega)\right), \\ z_k \overset{\omega}{\to} z & \text{in } L^2\left(0,T;L^{r+1}\,(\Omega)\right), \end{cases}$$

for all $T > 0$. The theorem now follows by taking limits as $k \to \infty$ in (5.1). Since $T > 0$ is arbitrary, the solution is global. □

Conflict of interest and data availability

The authors state that there is no conflict of interest and data availability not applicable to this article.

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An asymptotic estimate of Aoki's function

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ABSTRACT

The Aoki's function $A(x) := \left(1+\frac{1}{x}\right)^x + \left(1-\frac{1}{x}\right)^{-x}$ is sharply estimated for $x \gg 1$. For example, we have the zero approximation given as

$$2e\left(1+\frac{1}{4x^2-1}\right) < A(x) < 2e\left(1+\frac{3}{4x^2-1}\right), \quad x \geq \frac{29}{14}.$$

RESUMEN

Estimamos ajustadamente la función de Aoki $A(x) := \left(1+\frac{1}{x}\right)^x + \left(1-\frac{1}{x}\right)^{-x}$ para $x \gg 1$. Por ejemplo, tenemos la aproximación cero dada por

$$2e\left(1+\frac{1}{4x^2-1}\right) < A(x) < 2e\left(1+\frac{3}{4x^2-1}\right), \quad x \geq \frac{29}{14}.$$

Keywords and Phrases: Approximation, estimate, expansion, exponential function, inequality.

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1 Introduction

The Aoki's function $A(x)$,

$$A(x) := \left(1+\frac{1}{x}\right)^{x} + \left(1-\frac{1}{x}\right)^{-x}, \tag{1.1}$$

the sum of two strictly monotonic functions, increasing and decreasing respectively, has been estimated in [1, Theorem 1] as

$$\frac{e(2x^e-1)}{x^e-1} =: A_1(x) < A(x) < A_2(x) := \frac{e(2x^2-1)}{x^2-1} \quad (x>1). \tag{1.2}$$

Figure 1 (left), showing[1] the graphs of the functions $A_1(x)$, $A(x)$ and $A_2(x)$, discloses that the double inequality (1.2) is relatively rough. This fact has encouraged us to give more accurate approximations, which are illustrated in Figure 1 (right), where there are plotted the graphs of the functions $A_1^*(x)$, $A(x)$ and $A_2^*(x)$ from Example 3.5.

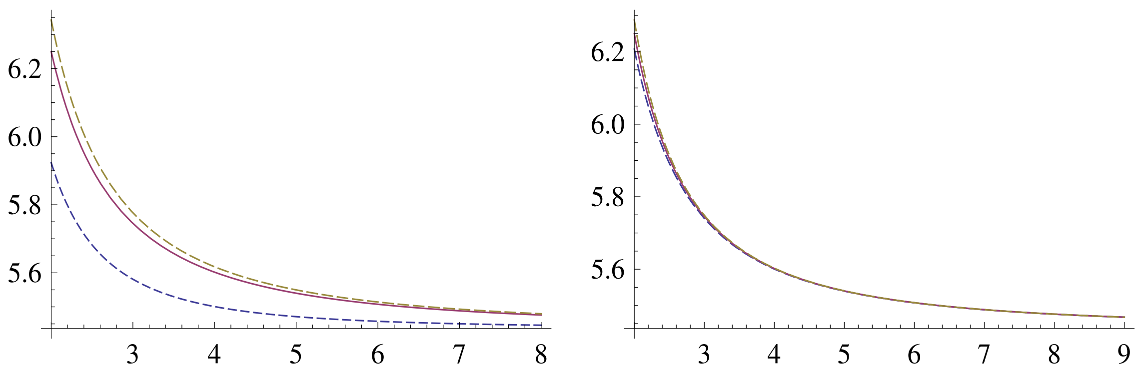

Figure 1: Left there are the graphs of the functions $A_1(x)$, $A(x)$ and $A_2(x)$. Right are illustrated the inequalities (3.1)–(3.2) in Example 3.5.

The main purpose of this article is to provide a sharp estimate of the function $A(x)$. The emphasis is on its brevity, a simple approach and its concrete sharpness (double inequalities), which is also important in some numerical treatments.

[1] All graphics in this paper are made using Mathematica [4].

2 Background – an expansion of the function $(1+y)^{1/y}$

According to [3, (20) and Theorem and Corollaries 1–2 on p. 105] there holds the following lemma.

Lemma 2.1. *For every real $y > -1$, we have the expansion*

$$(1+y)^{1/y} = \frac{2e}{y+2} \sum_{i=0}^{\infty} B_{2i} \cdot \left(\frac{y}{y+2}\right)^{2i}, \tag{2.1}$$

where the sequence B_{2n} is strictly monotonically decreasing, bounded as

$$B_2 = B_3 = \frac{5}{6} \quad \text{and} \quad \frac{7}{10} < \lim_{n\to\infty} B_n < B_n < \frac{8}{10}, \quad \text{for } n \geq 4, \tag{2.2}$$

and is given recursively as

$$B_0 = B_1 = 1, \quad B_{2m+1} = B_{2m} = \frac{1}{m} \sum_{j=1}^{m} \frac{4j+1}{4j+2} B_{2m-2j}, \quad \text{for } m \geq 1. \tag{2.3}$$

Lemma 2.1 implies the next lemma.

Lemma 2.2. *The equation* (2.1) *holds for any real y such that $|y| < 1$.*

Remark 2.3. *Instead of Lemma 2.1, we could also use the results of the paper [2], which provides the expansion $(1+x)^{1/x} = e\sum_{j=0}^{\infty}(-1)^j b_j x^j$ ($b_j \in \mathbb{R}^+$, $-1 < x \leq 1$). However, in this expansion, the convergence of the series is slower than the convergence of the series in the expansion $(1+x)^{1/x} = e \cdot \sum_{j=0}^{\infty}(-1)^j B_j \cdot \left(\frac{x}{x+2}\right)^j$ ($B_j \in \mathbb{R}^+$, $-1 < x \neq 0$), given in the paper [3].*

3 Expansion of the Aoki's function

Using $y = \pm\frac{1}{x}$ in Lemma 2.2, we get the following theorem.

Theorem 3.1. *The expansion*

$$A(x) = 2e\,x \sum_{i=0}^{\infty} B_{2i} \cdot \left(\frac{1}{(2x+1)^{2i+1}} + \frac{1}{(2x-1)^{2i+1}}\right)$$

holds true for any $x > 1$.

Proof. For $x > 1$, we have $\left|\pm\frac{1}{x}\right| < 1$. Consequently, using Lemma 2.2, the equation (2.1) holds for $y = \frac{1}{x}$ and also for $y = -\frac{1}{x}$. Therefore we obtain

$$\left(1+\frac{1}{x}\right)^x = \frac{2ex}{1+2x} \sum_{i=0}^{\infty} B_{2i} \cdot \left(\frac{1}{1+2x}\right)^{2i}$$

and

$$\left(1-\frac{1}{x}\right)^{-x} = \frac{2ex}{2x-1}\sum_{i=0}^{\infty} B_{2i}\cdot\left(\frac{1}{2x-1}\right)^{2i}. \qquad \square$$

Corollary 3.2. *For any integer $m \ge 0$ and every real $x > 1$, we have*

$$A(x) = A_m^*(x) + \delta_m(x),$$

where

$$A_m^*(x) := 2e\,x\sum_{i=0}^{m} B_{2i}\cdot\left(\frac{1}{(2x+1)^{2i+1}}+\frac{1}{(2x-1)^{2i+1}}\right)$$

and

$$0<\delta_m(x)<\delta_m^*(x) := \frac{e\,B_{2m+2}}{(x-1)(2x-1)^{2m+1}} < \frac{e}{(x-1)(2x-1)^{2m+1}}.$$

Proof. Referring to Theorem 3.1 and (2.2) in Lemma 2.1, we have

$$0<\delta_m(x) = 2e\,x\cdot\sum_{i=m+1}^{\infty} B_{2i}\cdot\frac{2}{(2x-1)^{2i+1}} < 4e\,x\cdot B_{2m+2}\cdot(2x-1)^{-(2m+3)}\sum_{i=0}^{\infty}(2x-1)^{-2i}$$
$$= 4e\,x\cdot B_{2m+2}\cdot(2x-1)^{-(2m+3)}\cdot\frac{1}{1-(2x-1)^{-2}} = \frac{e\,B_{2m+2}}{(x-1)(2x-1)^{2m+1}}.$$

Hence, referring to the estimates (2.2), we prove Corollary 3.2. $\square$

Remark 3.3. *In Corollary 3.2, m is a parameter that affect the error term $\delta_m(x)$.*

Example 3.4 (zero approximation)**.** *Setting $m = 0$ in Corollary 3.2 and using* (2.2)*, we estimate*

$$2e\left(1+\frac{1}{4x^2-1}\right) < A(x) < 2e\left(1+\frac{1}{4x^2-1}\right)+\frac{5e}{6(x-1)(2x-1)}, \quad x>1$$
$$\le 2e\left(1+\frac{3}{4x^2-1}\right), \quad x\ge\frac{29}{14}.$$

Example 3.5. *Putting $m = 1$ in Corollary 3.2 and considering the equality $B_4 = \frac{287}{360}$, given by* (2.3)*, we obtain the following inequalities*

$$A(x) > 2e\left(1+\frac{1}{4x^2-1}+\frac{10x^2(4x^2+3)}{3(4x^2-1)^3}\right) \tag{3.1}$$

$$A(x) < 2e\left(1+\frac{1}{4x^2-1}+\frac{10x^2(4x^2+3)}{3(4x^2-1)^3}\right)+\frac{287e}{360(x-1)(2x-1)^3}. \tag{3.2}$$

Corollary 3.6. *For an integer $m \ge 0$ and a real $x > 1$, the relative error*

$$\rho_m(x) := \frac{A(x)-A_m^*(x)}{A(x)}$$

of the approximation $A(x) \approx A_m^*(x)$ satisfies the double inequality

$$0<\rho_m(x)<\rho_m^*(x) := \frac{B_{2m+2}}{2(x-1)(2x-1)^{2m+1}} < \frac{1}{2(x-1)(2x-1)^{2m+1}}.$$

Proof. According to Example 3.4, we have $A(x) > 2e$. Therefore, using Corollary 3.2, we get

$$\rho_m(x) = \frac{\big(A_m^*(x) + \delta_m(x)\big) - A_m^*(x)}{A(x)} < \frac{\delta_m^*(x)}{2e} = \frac{B_{2m+2}}{2(x-1)(2x-1)^{2m+1}}. \qquad \square$$

Example 3.7. *Thanks to Lemma 2.1 and Corollary 3.6, we have*

$$\rho_0^*(x) = \frac{5}{12(x-1)(2x-1)} \quad \textit{and} \quad \rho_1^*(x) = \frac{287}{720(x-1)(2x-1)^3}, \quad x > 1.$$

Figure 2 shows the graphs of the errors $\rho_1(x)$ and $\rho_1^*(x)$ on the left and the graphs of the quotient $\rho_1^*(x)/\rho_1(x)$ on the right respectively.

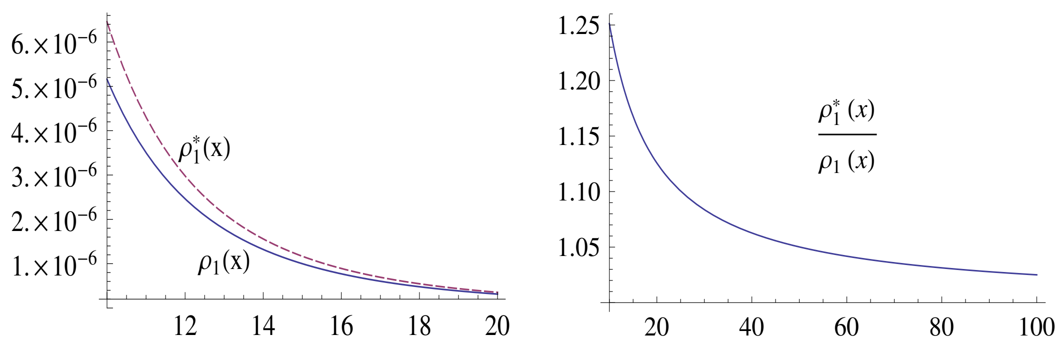

Figure 2: On the left are the graphs of the errors $\rho_1(x)$ and $\rho_1^*(x)$; on the right is the graph of the quotient $\rho_1^*(x)/\rho_1(x)$.

Remark 3.8. *A reviewer of this article suggested that the author rewrite the article following reviewer's suggestions, which, in his opinion, also include a better and much simpler approach to the problem at hand. The result of reviewer's intervention is his expansion*

$$A(x) = \sum_{n=0}^{\infty} \frac{a_{-2n}}{x^{2n}} = \sum_{n=0}^{m} \frac{a_{-2n}}{x^{2n}} + E_m(x),$$

where $a_{-2n} := \frac{2e}{(2n)!} D_{2n}$ *with* D_n *defined recursively as*

$$D_0 := 1, \quad D_m := \sum_{j=0}^{m-1} (-1)^{m-j} \frac{(m-1)!}{j!} \cdot \frac{m-j}{m+1-j} D_j, \quad m \geq 1$$

and estimated as

$$\big|D_m\big| < \frac{m!}{2}, \quad \big|E_m(x)\big| < \frac{2e|D_{2m+2}|}{(2m+2)! \cdot (x^{2m+2} - x^{2m+1})} < \frac{e}{(x-1)x^{2m+1}},$$

for $m \geq 1$*. However, the sequence* $(D_n)_{n\geq 0}$ *is not simple. Additionally, the crucial fact is that the series* $\sum_{n=0}^{\infty} \frac{a_{-2n}}{x^{2n}}$ *converges more slowly than the series* $\sum_{i=0}^{\infty} B_{2i} \cdot \left(\frac{1}{(2x+1)^{2i+1}} + \frac{1}{(2x-1)^{2i+1}}\right)$*, see Corollary 3.2.*

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Generalized translation and convolution operators in the realm of linear canonical deformed Hankel transform with applications

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ABSTRACT

Among the class of generalized Fourier transformations, the linear canonical transform is of pivotal importance mainly due to its higher degrees of freedom in lieu of the conventional Fourier and fractional Fourier transforms. This article is a continuation of our recent work *"Linear canonical deformed Hankel transform and the associated uncertainty principles, J. Pseudo-Differ. Oper. Appl.(2023), 14:29"*. Building upon this, we formulate the generalized translation and convolution operators associated with this newly proposed transformation. Besides, the obtained results are invoked to examine and obtain an analytical solution of the generalized heat equation. Finally, we study the heat semigroup pertaining to the generalized heat equation.

RESUMEN

Entre la clase de transformaciones de Fourier generalizadas, la transformada lineal canónica es de importancia central, mayormente debido a sus grados de libertad más altos en lugar de las transformadas convencionales de Fourier y de Fourier fraccionaria. Este artículo es una continuación de nuestro trabajo reciente *"Linear canonical deformed Hankel transform and the associated uncertainty principles, J. Pseudo-Differ. Oper. Appl.(2023), 14:29"*. Construyendo a partir de esto, formulamos los operadores de traslación y convolución generalizados asociados a esta nueva transformación propuesta. Además, los resultados obtenidos se utilizan para examinar y obtener una solución analítica de la ecuación de calor generalizada. Finalmente, estudiamos el semigrupo de calor pertinente a la ecuación de calor generalizada.

Keywords and Phrases: Deformed Hankel transform, linear canonical deformed Hankel transform, generalized translation, generalized convolution, heat semigroup, heat equation.

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1 Introduction

The Fourier transform is regarded as one of the remarkable discoveries in mathematical sciences as it profoundly influenced diverse branches of science and engineering. In the realm of harmonic analysis, the Fourier transform plays a pivotal role in analyzing signals wherein the characteristics are statistically invariant over time [6]. In the higher-dimensional scenario, there are several ways to arrive at the definition of the Fourier transform. The most basic formulation in $\mathbb{R}^d$ is given by the integral transform

$$\mathcal{F}(f)(\lambda) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)\, e^{-i\langle \lambda, x\rangle}\, dx. \tag{1.1}$$

Alternatively, one can rewrite the transform as

$$\mathcal{F}(f)(\lambda) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)\, \mathcal{K}(\lambda, x)\, dx, \tag{1.2}$$

where $\mathcal{K}(\lambda, x)$ is the unique solution to the system of partial differential equations

$$\begin{cases} \partial_{x_j} \mathcal{K}(\lambda, x) = -i\lambda_j \mathcal{K}(\lambda, x), & j = 1, \ldots, d, \\ \mathcal{K}(\lambda, 0) = 1, & \lambda \in \mathbb{R}^d \end{cases}$$

Yet another mathematical description of the higher-dimensional Fourier transform was proposed by Howe [44] via the Laplace operator $\triangle$ on $\mathbb{R}^d$ as follows:

$$\mathcal{F} = \exp\left(\frac{i\pi d}{4}\right) \exp\left(\frac{i\pi}{4}\left(\triangle - \|x\|^2\right)\right). \tag{1.3}$$

It is pertinent to mention that each of the above alternative representations has its specific use cases, and a detailed description regarding different ramifications of the Fourier transform can be found in [10]. Many generalizations of the Fourier transform can be attributed to a deeper understanding of the fundamental operators in Harmonic analysis. In the d-dimensional Euclidean space, the three elementary operators are the Laplace operator $\triangle$, norm $\|\cdot\|$, and the Euler operator $\mathbb{E}$, respectively defined as follows:

$$\triangle := \sum_{j=1}^{d} \partial_{x_j}^2, \quad \|x\|^2 := \sum_{j=1}^{d} x_j^2, \quad \mathbb{E} := \sum_{j=1}^{d} x_j \partial_{x_j},$$

As observed in [44], the operators

$$E = \frac{\|x\|^2}{2}, \quad F = -\frac{\triangle}{2}, \quad \text{and} \quad H = E + \frac{d}{2}$$

are invariant under $O(d)$ and generate the Lie algebra $\mathfrak{sl}_2$:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

Recently, there has been a lot of interest in other differential or difference operator realizations of $\mathfrak{sl}_2$ or other Lie (super) algebras. The focus is in particular on the generalized Fourier transforms that subsequently arise from these operator theoretic notions including the Dunkl transform [13], various discrete Fourier transforms in $\mathbb{R}^d$ [23], Fourier transforms in Clifford algebras [11] and many more. However, the hard problem in this context is to find explicit closed formulas for the integral kernel of the associated Fourier transforms. For further useful details regarding the generalized Fourier transforms and their implications, we refer the interested reader to [10].

Very recently, Ben Said *et al.* [3] have given a foundation for the deformation theory of the classical case, by constructing a generalization $\mathcal{F}_{k,a}$ of the Fourier transform, and the holomorphic semigroup $\mathcal{I}_{k,a}$ with infinitesimal generator

$$\mathcal{L}_{k,a,d} := \|x\|^{2-a} \triangle_k - \|x\|^a, \quad a > 0, \tag{1.4}$$

acting on a concrete Hilbert space deforming $L^2(\mathbb{R}^d)$, where $\triangle_k$ is the Dunkl Laplace operator. The authors have analyzed $\mathcal{F}_{k,a}$ and $\mathcal{I}_{k,a}(z)$ in the context of integral operators as well as representation theory. The deformation parameters consist of a real parameter a coming from the interpolation of the minimal unitary representations of two different reductive groups by keeping smaller symmetries, and a parameter k coming from Dunkl's theory of differential-difference operators associated with a finite Coxeter group (see [3]). In case $a = \frac{2}{n}$, $n \in \mathbb{N}$ and $d = 1$, we call the generalized Fourier transform $\mathcal{F}_{k,\frac{2}{n}}$, the deformed *Hankel transform* and will be denoted by $\mathcal{F}_{k,n}$.

As of now, the deformed Hankel transform $\mathcal{F}_{k,n}$ has witnessed an ample amount of research in the realm of harmonic analysis, which includes the study of kernel of the deformed Hankel transform [9], the generalized translation operator [2, 5, 30], the generalized maximal function [2], the Flett potentials [4], the deformed wavelet packets [19], uncertainty principles [25], the (k, n)-generalized wavelet multipliers [26], the (k, n)-generalized wavelet transform [27,29], the localization operators [34], the (k, n)-generalized Gabor transform [28], the (k, n)-generalized Stockwell transform [30], the (k, n)-generalized Wigner transform [32] and many more.

This paper is a continuation of the recent work carried out in the article *Linear canonical deformed Hankel transform and the associated uncertainty principles* [33]. Nonetheless, in [33], we have introduced and studied the linear canonical transform in the deformed Hankel frame (*i.e.* special case $a = \frac{2}{n}$, $n \in \mathbb{N}$ and $d = 1$). Recall that the classical linear canonical transform (LCT) was independently introduced by Collins [8] in paraxial optics, and Moshinsky, and Quesne [35] in quantum mechanics, to study the conservation of information and uncertainty under linear maps of phase space. The LCT is an integral transformation associated with a general homogeneous lossless linear mapping in phase space endowed with a total of three free parameters. The involved parameters constitute a 2×2 uni-modular matrix mapping the position x and the wave number y

into

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $ad - bc = 1$. The transformation maps any convex body into another convex body while preserving the area of the body. Such transformations constitute the homogeneous special group $SL(2, \mathbb{R})$. The linear canonical transform of any signal f with respect to a real matrix $M = (a, b; c, d) \in SL(2, \mathbb{R})$ with $b \neq 0$ is defined by

$$\mathcal{F}^M\big[f(x)\big](y) = \frac{1}{\sqrt{ib}} \int_{\mathbb{R}} f(x)\, \mathcal{K}^M(x, y)\, dy, \tag{1.5}$$

where

$$\mathcal{K}^M(x, y) = \exp\left\{ \frac{i}{2} \left(\frac{dx^2 + ay^2 - ixy}{b} \right) \right\}. \tag{1.6}$$

It is important to emphasize that the LCT provides a unified treatment of many generalized Fourier transforms in the sense that it is an embodiment of several well-known integral transforms including the Fourier transform [6,42], the fractional Fourier transform [1], the Fresnel transform [24], scaling operations and so on [7,21]. Due to the extra degrees of freedom and simple geometrical manifestation, the LCT is more flexible than other transforms and is as such suitable as well as a powerful tool for investigating deep problems in optics, quantum physics and signal processing [7,21]. Indeed, over a couple of decades, the application areas for LCT have been growing at an exponential rate and is as such befitting for investigating deep problems in signal analysis, filter design, phase retrieval problems, pattern recognition, radar analysis, holographic three-dimensional television, quantum physics, and many more. Apart from applications, the theoretical framework of LCT has likewise been extensively studied and investigated which has led to the formulation of convolution theorems [40], sampling theorems [22], Poisson summation formulae [45] and uncertainty principles [41]. For more about LCT and their applications, we allude to [7,21,37–39].

The main goal of this article is twofold. First, by employing the fundamental tools associated with the linear canonical deformed Hankel transform (LCDHT) [33], we introduce and investigate a generalized translation operator corresponding to the LCDHT. This operator is then utilized to define a convolution product, and several of its essential properties are examined. Subsequently, we establish the main theorems pertaining to the harmonic analysis in the framework of the LCDHT. Recognizing that the LCDHT represents a recent addition to the class of integral transforms, offering several additional degrees of freedom, we are further motivated to apply it to the heat equation. Therefore, the second objective of this paper is to study the generalized heat equation and the corresponding heat semigroup within the LCDHT setting. Thus, we can conclude that the principal contribution of this work lies in developing the harmonic analysis and exploring the generalized heat equation associated with a family of integral transforms such as the Dunkl, Bessel, and linear canonical Bessel (LCB) transforms [12,15–17]. Besides, our analysis extends to other integral transforms that have not yet been studied in this context, including the Dunkl fractional transform, the Dunkl Fresnel transform, and the LCD transform.

The remainder of this paper is organized as follows. Section 2 recalls the main results of the harmonic analysis associated with the deformed Hankel transform and the linear canonical deformed Hankel transform (LCDHT). Section 3 introduces and investigates the generalized translation operator corresponding to the LCDHT, along with an examination of its fundamental properties, including symmetry, commutativity, and continuity on certain functional spaces. Section 4 is devoted to the development and analysis of the generalized convolution product. In Section 5, we consider the generalized heat equation and the associated heat semigroup operator within the LCDHT framework. Finally, Section 6 presents the concluding remarks, summarizing the principal findings and outlining possible directions for future research.

2 Deformed Hankel transforms, translation and convolutions

In this section, we shall present the prerequisites concerning the deformed Hankel transform which shall be frequently used in formulating the main results. More precisely, we shall briefly review the conventional translation operators, deformed Hankel transform and the corresponding generalized translation and convolutions. For a detailed perspective, we refer to the articles [3, 5, 30] and the references therein.

2.1 Deformed Hankel transform

Let $L^p_{k,n}(\mathbb{R}), 1 \leq p \leq \infty$, be the space of measurable functions on $\mathbb{R}$ such that

$$\|f\|_{L^p_{k,n}(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p d\gamma_{k,n}(x)\right)^{1/p} < \infty, \quad \text{if} \quad 1 \leq p < \infty,$$
$$\|f\|_{L^\infty_{k,n}(\mathbb{R})} = \text{ess} \sup_{x\in\mathbb{R}} |f(x)| < \infty,$$

where

$$d\gamma_{k,n}(x) := M_{k,n}|x|^{\frac{(2k-2)n+2}{n}} dx, \quad M_{k,n} = \frac{n^{\frac{n(2k-1)}{2}}}{2^{\frac{n(2k-1)+2}{2}}\Gamma\left(\frac{n(2k-1)+2}{2}\right)}, \quad k \geq \frac{n-1}{n}, \quad n \in \mathbb{N}.$$

For $p = 2$, the space is equipped with the scalar product:

$$\langle f, g\rangle_{L^2_{k,n}(\mathbb{R})} := \int_{\mathbb{R}} f(x)\overline{g(x)}\, d\gamma_{k,n}(x).$$

To facilitate our narrative, we set some notations as under:

- $C_b(\mathbb{R})$ the space of bounded continuous functions on $\mathbb{R}$.
- $C_{b,e}(\mathbb{R})$ the space of even bounded continuous functions on $\mathbb{R}$.
- $C_0(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$ and vanishing at infinity. We provide $C_0(\mathbb{R})$ with the topology of uniform convergence.
- $C_c(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$ and with compact support.
- $C^p(\mathbb{R})$ the space of functions of class C^p on $\mathbb{R}$.
- $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.
- $\mathfrak{S}_{k,n}(\mathbb{R})$ the space of all functions $f \in C^\infty(\mathbb{R}^*)$ such that

$$\sup_{x\in\mathbb{R}^*} |(|x|^{\frac{2}{n}})^j (|x|^{2-\frac{2}{n}} \triangle_k)^s (x^m f^{(m)}(x))| < \infty, \quad \text{for all } j, s, m \in \mathbb{N}_0.$$

- $SL(2,\mathbb{R})$ the group of 2×2 real matrices with determinant one.

We are now in a position to recall the notion of Dunkl operator. In this direction, we have the following definition:

For any $f \in C^1(\mathbb{R})$, the Dunkl operator T_k on $\mathbb{R}$ is defined by

$$T_k f(x) := f'(x) + 2k\,\frac{f(x) - f(-x)}{x}, \tag{2.1}$$

where as the corresponding Dunkl-Laplace operator $\triangle_k$, for any $f \in C^2(\mathbb{R})$, is given by

$$\triangle_k f(x) := T_k^2 f(x) = f''(x) + 2k\left(\frac{f'(x)}{x} - \frac{f(x) - f(-x)}{2x^2}\right). \tag{2.2}$$

Consider the operator

$$\Delta_{k,n} := \big|x\big|^{2-\frac{2}{n}} \triangle_k - \big|x\big|^{\frac{2}{n}}. \tag{2.3}$$

In the following, we recall some spectral properties of the differential-difference operator $\Delta_{k,n}$.

- $\Delta_{k,n}$ is an essentially self-adjoint operator on $L^2_{k,n}(\mathbb{R})$.
- There is no continuous spectrum of $\Delta_{k,n}$.
- The discrete spectrum of $-\Delta_{k,n}$ is $\left\{\frac{4m}{n} + 2k + \frac{2}{n} \pm 1 : m \in \mathbb{N}\right\}$.

Definition 2.1. *For any* $f \in L^1_{k,n}(\mathbb{R})$ *and* $k \geq \dfrac{n-1}{n}$, $n \in \mathbb{N}$, *the deformed Hankel transform is denoted by* $\mathcal{F}_{k,n}(f)$ *and is given as*

$$\mathcal{F}_{k,n}(f)(\xi) = \int_{\mathbb{R}} f(x)\, B_{k,n}(\lambda, x)\, d\gamma_{k,n}(x), \quad \textit{for all } \lambda \in \mathbb{R}, \tag{2.4}$$

where $B_{k,n}(\lambda, x)$ *is the deformed Hankel kernel given by*

$$B_{k,n}(\lambda, x) = \jmath_{nk-\frac{n}{2}}\left(n|\lambda x|^{\frac{1}{n}}\right) + (-i)^n \left(\frac{n}{2}\right)^n \frac{\Gamma\left(nk - \frac{n}{2} + 1\right)}{\Gamma\left(nk + \frac{n}{2} + 1\right)} \lambda x \jmath_{nk+\frac{n}{2}}\left(n|\lambda x|^{\frac{1}{n}}\right). \tag{2.5}$$

Observe that

$$\jmath_\alpha(u) := \Gamma(\alpha+1)\left(\frac{u}{2}\right)^{-\alpha} J_\alpha(u) = \Gamma(\alpha+1)\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\alpha+m+1)}\left(\frac{u}{2}\right)^{2m} \tag{2.6}$$

denotes the normalized Bessel function of index α.

Example 2.2. *The function* α_t, $t > 0$, *defined on* $\mathbb{R}$ *by*

$$\alpha_t(x) = \frac{1}{(2t)^{\frac{2nk+2-n}{2}}} e^{-\frac{n|x|^{\frac{2}{n}}}{4t}},$$

satisfies

$$\mathcal{F}_{k,n}(\alpha_t)(\xi) = e^{-nt|\xi|^{\frac{2}{n}}}, \quad \forall\, \xi \in \mathbb{R}.$$

Here, we list some important properties of the deformed Hankel kernel and transform:

(i) $B_{k,n}(z,t) = B_{k,n}(t,z), \quad B_{k,n}(z,0) = 1, \ \overline{B_{k,n}(z,t)} = B_{k,n}((-1)^n z, t),$

$B_{k,n}(\lambda z, t) = B_{k,n}(z, \lambda t), \ \forall z, t, \lambda \in \mathbb{R}.$

(ii) $B_{k,n}(.,.)$ solves the following differential-difference equations on $\mathbb{R} \times \mathbb{R}$

$$\begin{cases} |\lambda|^{2-\frac{2}{n}} \triangle_k^\lambda B_{k,n}(\lambda, x) = -|x|^{\frac{2}{n}} B_{k,n}(\lambda, x), \\ |x|^{2-\frac{2}{n}} \triangle_k^x B_{k,n}(\lambda, x) = -|\lambda|^{\frac{2}{n}} B_{k,n}(\lambda, x). \end{cases}$$

where the superscript in $\triangle_k^x$ denotes the relevant variable.

(iii) For $k \geq 1/2$, $B_{k,n}(.,.)$ satisfies the following inequality

$$\left|B_{k,n}(x,y)\right| \leq 1, \quad \forall\, x, y \in \mathbb{R} \tag{2.7}$$

(iv) $B_{k,n}(.,.)$ is bounded if and only if
$$k \geq \frac{n-1}{2n}. \tag{2.8}$$

(v) Under the bounded condition (2.8), there always exists a finite positive constant C depending on n and k such that
$$\big|B_{k,n}(x,y)\big| \leq C, \quad \forall\, x,y \in \mathbb{R}. \tag{2.9}$$

(vi) ([31]). For $x,y \in \mathbb{R}$ and $\delta \in \mathbb{C}$ with $\operatorname{Re}\delta > 0$, we have
$$\int_{\mathbb{R}} e^{-\delta|\xi|^{2/n}}\, B_{k,n}(x,\xi)\, B_{k,n}(y,\xi)\, d\gamma_{k,n}(\xi) = \frac{e^{-(n^2/4\delta)(|x|^{2/n}+|y|^{2/n})}}{\left(\frac{2\delta}{n}\right)^{\frac{(2k-1)n+2}{2}}}\, B_{k,n}\left(\frac{x}{(\frac{2\delta}{n})^n}, (-i)^n y\right). \tag{2.10}$$

(vii) Under the bounded condition (2.8), the deformed Hankel transform $\mathcal{F}_{k,n}$ is bounded on $L^1_{k,n}(\mathbb{R})$. In particular, if $k \geq 1/2$,
$$\big\|\mathcal{F}_{k,n}(f)\big\|_{L^\infty_{k,n}(\mathbb{R})} \leq \big\|f\big\|_{L^1_{k,n}(\mathbb{R})}. \tag{2.11}$$

(viii) The deformed Hankel transform $\mathcal{F}_{k,n}$ provides a natural generalization of the conventional Hankel transform. For instance, if we set
$$B^{even}_{k,n}(x,y) = \frac{1}{2}\left(B_{k,n}(x,y) + B_{k,n}(x,-y)\right) = j_{nk-\frac{n}{2}}\left(n|xy|^{\frac{1}{n}}\right). \tag{2.12}$$

Then, $\mathcal{F}_{k,n}$ of an even function f on $\mathbb{R}$ specializes to a Hankel type transform on $\mathbb{R}_+$. In fact, when $f(x) = F(|x|)$ is an even function on $\mathbb{R}$ and belongs to $L^1_{k,n}(\mathbb{R})$, then
$$\mathcal{F}_{k,n}(f)(\xi) = \frac{(\frac{n}{2})^{(\frac{2nk-n}{2})}}{\Gamma\left(\frac{2nk+2-n}{2}\right)} \int_0^\infty F(r)\, j_{\frac{2nk-n}{2}}\left(n(r|\xi|)^{\frac{1}{n}}\right) r^{\frac{(2k-2)n+2}{n}}\, dr, \quad \forall\, \xi \in \mathbb{R}. \tag{2.13}$$

(ix) The deformed Hankel transform $f \mapsto \mathcal{F}_{k,n}(f)$ is an isometric isomorphism on $L^2_{k,n}(\mathbb{R})$ and satisfies [3]
$$\int_{\mathbb{R}} \big|\mathcal{F}_{k,n}(f)(\lambda)\big|^2 d\gamma_{k,n}(\lambda) = \int_{\mathbb{R}} \big|f(x)\big|^2 d\gamma_{k,n}(x). \tag{2.14}$$

(x) For all $f,g \in L^2_{k,n}(\mathbb{R})$, we have
$$\int_{\mathbb{R}} \mathcal{F}_{k,n}(f)(\lambda)\overline{\mathcal{F}_{k,n}(g)(\lambda)}\, d\gamma_{k,n}(\lambda) = \int_{\mathbb{R}} f(x)\overline{g(x)}\, d\gamma_{k,n}(x). \tag{2.15}$$

(xi) The deformed Hankel transform $\mathcal{F}_{k,n}$ is an involutive unitary operator on $L^1_{k,n}(\mathbb{R})$, that is;
$$\mathcal{F}^{-1}_{k,n}(f)(x) = \mathcal{F}_{k,n}(f)((-1)^n x), \quad x \in \mathbb{R}. \tag{2.16}$$

(xii) For any $f \in L^p_{k,n}(\mathbb{R}), 1 \leq p \leq 2$, the deformed Hankel transform $\mathcal{F}_{k,n}(f)$ belongs to $L^{p'}_{k,n}(\mathbb{R})$ and satisfies the following inequality:

$$\|\mathcal{F}_{k,n}(f)\|_{L^{p'}_{k,n}(\mathbb{R})} \leq \|f\|_{L^p_{k,n}(\mathbb{R})}, \tag{2.17}$$

where p' denotes the conjugate exponent of p.

(xiii) $\mathcal{F}_{k,n}(\mathcal{S}(\mathbb{R})) \subset C^\infty(\mathbb{R})$ if and only if $n = 1$.

(xiv) $\mathcal{F}_{k,n}(\mathcal{S}(\mathbb{R})) = \mathcal{S}(\mathbb{R})$ if and only if $n = 1$.

(xv) For any $f \in \mathcal{S}(\mathbb{R})$, we have

$$\mathcal{F}_{k,n}(f)(y) = F_1\left(|y|^{\frac{1}{n}}\right) + yF_2\left(|y|^{\frac{1}{n}}\right), \tag{2.18}$$

where the even functions $F_1, F_2 \in \mathcal{S}(\mathbb{R})$.

(xvi) The space $\mathfrak{S}_{k,n}(\mathbb{R})$ satisfies the following properties: (see [14]).

- $\mathcal{F}_{k,n}(\mathfrak{S}_{k,n}(\mathbb{R})) = \mathfrak{S}_{k,n}(\mathbb{R})$.
- The embedding $\mathfrak{S}_{k,n}(\mathbb{R}) \hookrightarrow L^p_{k,n}(\mathbb{R})$, $1 \leq p < \infty$, is continuous.
- $\mathfrak{S}_{k,n}(\mathbb{R})$ is a dense subset of $L^p_{k,n}(\mathbb{R})$, $1 \leq p < \infty$.

(xvii) The unitary operator $\mathcal{F}_{k,n}$ satisfies the following intertwining relations on a dense subspace of $L^2_{k,n}(\mathbb{R})$:

$$\mathcal{F}_{k,n} \circ |x|^{\frac{2}{n}} = -|x|^{2-\frac{2}{n}}\Delta_k \circ \mathcal{F}_{k,n}, \quad \mathcal{F}_{k,n} \circ |x|^{2-\frac{2}{n}}\Delta_k = -|x|^{\frac{2}{n}} \circ \mathcal{F}_{k,n}. \tag{2.19}$$

2.2 Generalized translation and convolution operators

Definition 2.3 ([27]). *The generalized translation operator $f \mapsto \tau^{k,n}_x f$ on $L^2_{k,n}(\mathbb{R})$ is defined by*

$$\mathcal{F}_{k,n}(\tau^{k,n}_x f) = \overline{B_{k,n}(.,x)}\mathcal{F}_{k,n}(f). \tag{2.20}$$

It is fruitful to have a class of functions in which (2.20) holds pointwise. One such class is the generalized Wigner space $\mathcal{W}_{k,n}(\mathbb{R})$ given by

$$\mathcal{W}_{k,n}(\mathbb{R}) := \left\{f \in L^1_{k,n}(\mathbb{R}) : \mathcal{F}_{k,n}(f) \in L^1_{k,n}(\mathbb{R})\right\}.$$

Following, we give several properties of the generalized translation operator [27].

(i) For any $f \in L^2_{k,n}(\mathbb{R})$, we have

$$\left\|\tau_x^{k,n} f\right\|_{L^2_{k,n}(\mathbb{R})} \leq \left\|f\right\|_{L^2_{k,n}(\mathbb{R})}, \quad \forall\, x \in \mathbb{R}. \tag{2.21}$$

(ii) For any $f \in \mathcal{W}_{k,n}(\mathbb{R})$, we have

$$\tau_x^{k,n} f(y) = \int_{\mathbb{R}} B_{k,n}((-1)^n x, \xi) B_{k,n}((-1)^n y, \xi) \mathcal{F}_{k,n}(f)(\xi)\, d\gamma_{k,n}(\xi), \quad \forall x, y \in \mathbb{R}. \tag{2.22}$$

(iii) For any $f \in \mathcal{W}_{k,n}(\mathbb{R})$, we have

$$\tau_x^{k,n} f(y) = \tau_y^{k,n}(f)(x), \quad \forall x, y \in \mathbb{R}. \tag{2.23}$$

(iv) For all f in $\mathcal{W}_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \tau_x^{k,n} f(y) g(y)\, d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y) \tau_{(-1)^n x}^{k,n} g(y)\, d\gamma_{k,n}(y), \quad \forall x \in \mathbb{R}. \tag{2.24}$$

(v) ([31]). For every $\delta > 0$, the (k,n)-generalized translation of the generalized Gaussian function is given by

$$\tau_x^{k,n}\left(e^{-\frac{n^2 |s|^{\frac{2}{n}}}{4\delta}}\right)(y) = e^{-n^2 \frac{|x|^{\frac{2}{n}} + |y|^{\frac{2}{n}}}{4\delta}} B_{k,n}\left(\frac{x}{(\frac{2\delta}{n})^n}, (i)^n y\right). \tag{2.25}$$

Recently, an explicit formula for the generalized translation operator $\tau_x^{k,n}$ has been reported in [5]:

Theorem 2.4. *For any $f \in C_b(\mathbb{R})$ and $k \geq \dfrac{n-1}{n}$, the generalized translation operator $\tau_x^{k,n}$ is given by*

$$\tau_x^{k,n} f(y) = \int_{\mathbb{R}} f(z)\, d\zeta_{x,y}^{k,n}(z), \tag{2.26}$$

where

$$d\zeta_{x,y}^{k,n}(z) = \begin{cases} \mathcal{K}_{k,n}(x,y,z) d\gamma_{k,n}(z), & \text{if } xy \neq 0, \\ d\delta_x(z), & \text{if } y = 0, \\ d\delta_y(z), & \text{if } x = 0, \end{cases} \tag{2.27}$$

$$\mathcal{K}_{k,n}(x,y,z) = K_{\mathrm{B}}^{nk-\frac{n}{2}}(|x|^{\frac{1}{n}}, |y|^{\frac{1}{n}}, |z|^{\frac{1}{n}})\, \nabla_{k,n}(x,y,z), \tag{2.28}$$

having support on the set $\left\{z \in \mathbb{R} : \big|\, |x|^{\frac{1}{n}} - |y|^{\frac{1}{n}} \big| < |z|^{\frac{1}{n}} < |x|^{\frac{1}{n}} + |y|^{\frac{1}{n}}\right\}$,

$$\nabla_{k,n}(x,y,z) = \frac{M_{k,n}}{2n}\left\{1 + (-1)^n \frac{n!\,\mathrm{sgn}(xy)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}\left(\Delta(|x|^{\frac{2}{n}}, |y|^{\frac{2}{n}}, |z|^{\frac{2}{n}})\right)\right.$$
$$\left. + \frac{n!\,\mathrm{sgn}(xz)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}\left(\Delta(|z|^{\frac{2}{n}}, |x|^{\frac{2}{n}}, |y|^{\frac{2}{n}})\right) + \frac{n!\,\mathrm{sgn}(yz)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}\left(\Delta(|z|^{\frac{2}{n}}, |y|^{\frac{2}{n}}, |x|^{\frac{2}{n}})\right)\right\}, \tag{2.29}$$

$$\Delta(u,v,w) = \frac{1}{2\sqrt{uv}}(u+v-w), \quad u,v,w \in \mathbb{R}_+^*, \tag{2.30}$$

$C_n^{nk-\frac{n}{2}}$ *the Gegenbauer polynomials and* $K_{\mathrm{B}}^{nk-\frac{n}{2}}$ *is the positive kernel given by*

$$K_{\mathrm{B}}^{nk-\frac{n}{2}}(u,v,w) =$$
$$\begin{cases} \frac{\Gamma(nk-\frac{n}{2}+1)}{2^{2nk-n-1}\Gamma(nk-\frac{n-1}{2})\Gamma(\frac{1}{2})} \frac{\left\{\left[(u+v)^2-w^2\right]\left[w^2-(u-v)^2\right]\right\}^{nk-\frac{n+1}{2}}}{(uvw)^{2nk-n}} & \text{if } |u-v| < w < u+v, \\ 0 & \text{elsewhere.} \end{cases}$$

Remark 2.5. *(i) For all $x, y, \lambda \in \mathbb{R}$, we have the following product formula:*

$$\tau_x^{k,n} B_{k,n}(\lambda, y) = B_{k,n}(\lambda, x) B_{k,n}(\lambda, y). \tag{2.31}$$

(ii) For all $x, y \in \mathbb{R}^*$, we have

$$\int_{\mathbb{R}} \mathcal{K}_{k,n}(x,y,z)\, d\gamma_{k,n}(z) = 1. \tag{2.32}$$

(iii) For all $x, y, z \in \mathbb{R}^*$, we have

$$\mathcal{K}_{k,n}(x,y,z) = \mathcal{K}_{k,n}(y,x,z). \tag{2.33}$$

(iv) For all $x, y, z \in \mathbb{R}^*$, we have

$$\mathcal{K}_{k,n}(x,y,z) = \mathcal{K}_{k,n}((-1)^n x, z, y). \tag{2.34}$$

(v) For all $x, y, z \in \mathbb{R}^*$, we have

$$\mathcal{K}_{k,n}(x,(-1)^n y,z) = \mathcal{K}_{k,n}(x,(-1)^n z, y). \tag{2.35}$$

(vi) For any $x, y \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} |\mathcal{K}_{k,n}(x,y,z)|\, d\gamma_{k,n}(z) \le 4. \tag{2.36}$$

On what follows we will recall the "trigonometric" form of the generalized translation operator proved in [30].

Theorem 2.6. *(i) For $f \in C_b(\mathbb{R})$ write $f = f_{\mathrm{e}} + f_{\mathrm{o}}$ as a sum of even and odd functions. Then*

$$\begin{aligned}\tau_x^{k,n} f(y) = \frac{M_{k,n}}{2n} \Bigg[&\int_0^\pi f_{\mathrm{e}}\big(\langle\!\langle x, y\rangle\!\rangle_{\phi,n}\big) \left\{1 + (-1)^n \frac{n!\,\mathrm{sgn}(xy)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}(\cos\phi)\right\} (\sin\phi)^{2nk-n} d\phi \\ &+ \int_0^\pi f_{\mathrm{o}}\left(\langle\!\langle x, y\rangle\!\rangle_{\phi,n}\right) \left\{ \frac{n!\,\mathrm{sgn}(x)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}} \left(\frac{|x|^{\frac{1}{n}} - |y|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x, y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}\right) \right. \\ &+ \left. \frac{n!\,\mathrm{sgn}(y)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}} \left(\frac{|y|^{\frac{1}{n}} - |x|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x, y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}\right) \right\} (\sin\phi)^{2nk-n} d\phi \Bigg], \qquad (2.37)\end{aligned}$$

where

$$\langle\!\langle x, y\rangle\!\rangle_{\phi,n} := \left(|x|^{\frac{2}{n}} + |y|^{\frac{2}{n}} - 2|xy|^{\frac{1}{n}}\cos\phi\right)^{\frac{n}{2}}. \qquad (2.38)$$

(ii) For every $f \in C_{b,\mathrm{e}}(\mathbb{R})$, we have

$$\tau_x^{k,n} f(y) = \frac{M_{k,n}}{2n} \int_0^\pi f\big(\langle\!\langle x, y\rangle\!\rangle_{\phi,n}\big) \left\{1 + (-1)^n \frac{n!\,\mathrm{sgn}(xy)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}(\cos\phi)\right\} (\sin\phi)^{2nk-n} d\phi. \qquad (2.39)$$

(iii) For every $\lambda > 0$, we have

$$\tau_x^{k,n}\left(e^{-\lambda|.|^{\frac{2}{n}}}\right)(y) = \frac{M_{k,n}}{2n} e^{-\lambda\left(|x|^{\frac{2}{n}} + |y|^{\frac{2}{n}}\right)} V_{k,n}(\lambda; x, y), \qquad (2.40)$$

where

$$V_{k,n}(\lambda; x, y) := \int_0^\pi e^{2\lambda|xy|^{\frac{1}{n}}\cos\phi} \left\{1 + (-1)^n \frac{n!\,\mathrm{sgn}(xy)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}(\cos\phi)\right\} (\sin\phi)^{2nk-n} d\phi.$$

(iv) ([30]). Using (2.40), properties of the Gegenbauer polynomials and by simple calculations, we obtain

$$\left|\tau_x^{k,n}\left(e^{-\lambda|.|^{\frac{2}{n}}}\right)(y)\right| \le \frac{M_{k,n}}{2n} e^{-\lambda\left(|x|^{\frac{1}{n}} - |y|^{\frac{1}{n}}\right)^2}. \qquad (2.41)$$

Theorem 2.7 ([5])**.** *Let $\tau_x^{k,n}$ be the generalized translation operation as defined in* (2.19)*. Then,*

(i) For any $f \in L^1_{\mathrm{loc}}(d\gamma_{k,n})$ and $k \geq \dfrac{n-1}{n}$, we have

$$\tau_x^{k,n} f(y) = \tau_y^{k,n} f(x), \quad \tau_0^{k,n} f = f.$$

(ii) For any $f \in L^p_{k,n}(\mathbb{R}), 1 \leq p \leq \infty$, we have

$$\big\|\tau_x^{k,n} f\big\|_{L^p_{k,n}(\mathbb{R})} \leq 4\big\|f\big\|_{L^p_{k,n}(\mathbb{R})}. \tag{2.42}$$

(iii) For every $f \in L^1_{k,n}(\mathbb{R})$, we have

$$\mathcal{F}_{k,n}(\tau_x^{k,n} f)(\lambda) = B_{k,n}((-1)^n\lambda, x)\mathcal{F}_{k,n} f(\lambda), \quad \lambda \in \mathbb{R}.$$

(iv) For any $f \in L^p_{k,n}(\mathbb{R}), 1 \leq p \leq 2$, we have

$$\mathcal{F}_{k,n}(\tau_x^{k,n} f)(\lambda) = B_k((-1)^n\lambda, x)\mathcal{F}_k(f)(\lambda), \quad a.e.\ \lambda \in \mathbb{R}. \tag{2.43}$$

(v) For all $f \in C_b(\mathbb{R})$ or belongs in $L^p_{k,n}(\mathbb{R}), 1 \leq p \leq \infty$, we have

$$\tau_x^{k,n}\tau_y^{k,n}(f) = \tau_y^{k,n}\tau_x^{k,n}(f). \tag{2.44}$$

Proposition 2.8. *If $f \in C_0(\mathbb{R})$, then we have*

$$\lim_{|x|\to\infty} \tau_x^{k,n}(f)(y) = 0.$$

Proof. For $f \in C_0(\mathbb{R})$, $y \in \mathbb{R}$ and $\phi \in [0,\pi]$, we have

$$\lim_{|x|\to\infty} f_{\mathrm{e}}\big(\langle\!\langle x, y\rangle\!\rangle_{\phi,n}\big) = \lim_{|x|\to\infty} f_{\mathrm{o}}\big(\langle\!\langle x, y\rangle\!\rangle_{\phi,n}\big) = 0.$$

Using Theorem 2.6 *(i)*, the properties of the Gegenbauer polynomials, an application of dominated convergence theorem give the desired result. □

Theorem 2.9 ([30]). *Let $L^p_{k,n,e}(\mathbb{R})$ be the space of even functions in $L^p_{k,n}(\mathbb{R})$. Then,*

(i) For every bounded and non-negative function $f \in L^1_{k,n,e}(\mathbb{R})$, we have $\tau^{k,n}_x f \geq 0$, $\tau^{k,n}_x f \in L^1_{k,n}(\mathbb{R})$, $\forall\, x \in \mathbb{R}$, and

$$\int_{\mathbb{R}} \tau^{k,n}_x f(y)\, d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y)\, d\gamma_{k,n}(y). \tag{2.45}$$

(ii) For any $f \in L^p_{k,n,e}(\mathbb{R})$, we have

$$\big\|\tau^{k,n}_x f\big\|_{L^p_{k,n,e}(\mathbb{R})} \leq \big\|f\big\|_{L^p_{k,n,e}(\mathbb{R})}. \tag{2.46}$$

(iii) For every $f \in L^1_{k,n}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \tau^{k,n}_x f(y)\, d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y)\, d\gamma_{k,n}(y). \tag{2.47}$$

(iv) If f_1 and f_2 are two suitable functions, we have

$$\int_{\mathbb{R}} \tau^{k,n}_y f_1((-1)^n t) f_2(t) d\gamma_{k,n}(t) = \int_{\mathbb{R}} \tau^{k,n}_y f_2((-1)^n t) f_1(t) d\gamma_{k,n}(t), \quad y \in \mathbb{R}. \tag{2.48}$$

Definition 2.10. *The generalized convolution product of two suitable functions $f, g \in L^2_{k,n}(\mathbb{R})$ is defined by*

$$f *_{k,n} g(x) = \int_{\mathbb{R}} \tau^{k,n}_x f((-1)^n y)\, g(y)\, d\gamma_{k,n}(y). \tag{2.49}$$

It is pertinent to mention that the convolution product (2.49) is both commutative and associative. We culminate this subsection by giving the following important results.

Proposition 2.11 ([5]). *Let $f *_{k,n} g(x)$ be the generalized convolution as defined in (2.49). Then,*

(i) For any $f \in L^2_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R})$, we have

$$f *_{k,n} g(x) = \int_{\mathbb{R}} \tau^{k,n}_x f((-1)^n y) g(y)\, d\gamma_{k,n}(y). \tag{2.50}$$

(ii) For every $f \in L^p_{k,n}(\mathbb{R})$ and $g \in L^q_{k,n}(\mathbb{R})$ with $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$, the convolution product $f *_{k,n} g$ belongs to $L^r_{k,n}(\mathbb{R})$ and satisfies the inequality:

$$\big\|f *_{k,n} g\big\|_{L^r_{k,n}(\mathbb{R})} \leq 4\|f\|_{L^p_{k,n}(\mathbb{R})}\|g\|_{L^q_{k,n}(\mathbb{R})}. \tag{2.51}$$

(iii) For every $f \in L^2_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R})$, we have

$$\mathcal{F}_{k,n}(f *_{k,n} g) = \mathcal{F}_{k,n}(f)\mathcal{F}_{k,n}(g). \tag{2.52}$$

(iv) *For $f, g \in L^2_{k,n}(\mathbb{R})$, the convolution $f *_{k,n} g \in L^2_{k,n}(\mathbb{R})$ if and only if $\mathcal{F}_{k,n}(f)\mathcal{F}_{k,n}(g) \in L^2_{k,n}(\mathbb{R})$ and satisfies [27]*

$$\mathcal{F}_{k,n}(f *_{k,n} g) = \mathcal{F}_{k,n}(f)\, \mathcal{F}_{k,n}(g). \tag{2.53}$$

(v) *For every $f, g \in L^2_{k,n}(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} \left| f *_{k,n} g(x) \right|^2 d\gamma_{k,n}(x) = \int_{\mathbb{R}} \left| \mathcal{F}_{k,n}(f)(\xi) \right|^2 |\mathcal{F}_{k,n}(g)(\xi)|^2 d\gamma_{k,n}(\xi). \tag{2.54}$$

2.3 Deformed Hankel transform in linear canonical domain

In this section, we recall some results proved in [33].

Definition 2.12. *The deformed linear canonical Hankel transform of any function $f \in L^1_{k,n}(\mathbb{R})$, with respect to the uni-modular matrix $M = (a, b; c, d) \in SL(2, \mathbb{R})$ is defined by*

$$\mathcal{F}^M_{k,n}(f)(x) = \frac{1}{(ib)^{\frac{(2k-1)n+2}{2n}}} \int_{\mathbb{R}} K^M_{k,n}(x, y) f(y)\, d\gamma_{k,n}(y), \tag{2.55}$$

where

$$K^M_{k,n}(x, y) = e^{\frac{i}{2}\left(\frac{d}{b}x^2 + \frac{a}{b}y^2\right)} B_{k,n}\left(\frac{x}{b}, y\right). \tag{2.56}$$

Definition 2.12 allows us to make the followings comments:

(i) For $M = (1, b, 0, 1)$, the deformed linear canonical Hankel transform (2.55) coincides with the Fresnel transform associated with the deformed Hankel transform:

$$\mathcal{W}^b_{k,n} f(x) = \begin{cases} \dfrac{1}{(ib)^{\frac{(2k-1)n+2}{2n}}} \displaystyle\int_{\mathbb{R}} E^b_{k,n}(x, y) f(y)\, d\gamma_{k,n}(y), & b \neq 0, \\ f(x), & b = 0, \end{cases}$$

where $E^b_{k,n}(x, y) = e^{\frac{i}{2b}(x^2+y^2)} B_{k,n}\left(\frac{x}{b}, y\right)$.

(ii) For $M = (\cosh(b), \sinh(b); \sinh(b), \cosh(b))$, $b \in \mathbb{R}$, the deformed linear canonical Hankel transform (2.55) boils down to the following integral transform

$$\mathcal{V}^b_{k,n} f(x) = \begin{cases} \dfrac{1}{(i \sinh(b))^{\frac{(2k-1)n+2}{2n}}} \displaystyle\int_{\mathbb{R}} R^b_{k,n}(x, y) f(y)\, d\gamma_{k,n}(y), & b \neq 0, \\ f(x), & b = 0, \end{cases}$$

where $R^b_{k,n}(x, y) = e^{\frac{i}{2}\coth(b)(x^2+y^2)} B_{k,n}\left(\frac{x}{\sinh(b)}, y\right)$.

(iii) For $M=(\cos\alpha,\sin\alpha;-\sin\alpha,\cos\alpha)$, $\alpha\in\mathbb{R}$, the deformed linear canonical Hankel transform (2.55) coincides with the fractional deformed Hankel transform $\mathcal{F}^{\alpha}_{k,n}$:

$$\mathcal{F}^{\alpha}_{k,n}f(x)=\begin{cases}\dfrac{e^{i\left(\frac{(2k-1)n+2}{2n}\right)(\alpha-2n\pi)-\hat{\alpha}\pi/2)}}{|\sin(\alpha)|^{\frac{(2k-1)n+2}{2n}}}\displaystyle\int_{\mathbb{R}}\mathcal{K}^{\alpha}_{k,n}(x,y)f(y)\,d\gamma_{k,n}(y), & (2j-1)\pi<\alpha<(2j+1)\pi,\\ f(x), & \alpha=2j\pi,\\ f(-x), & \alpha=(2j+1)\pi,\end{cases}$$

where $\hat{\alpha}=\operatorname{sgn}(\sin(\alpha))$, $\mathcal{K}^{\alpha}_{k,n}(x,y)=e^{-\frac{i}{2}\cot(\alpha)(x^2+y^2)}B_{k,n}\left(\frac{x}{\sin(\alpha)},y\right)$.

Definition 2.13. *For any uni-modular matrix $M\in SL(2,\mathbb{R})$, the differential-difference operator $\triangle^M_{k,n}$ is defined by*

$$\triangle^M_{k,n}:=|x|^{2(1-\frac{1}{n})}\left\{\frac{d^2}{dx^2}+\left(\frac{2k}{x}-2i\frac{d}{b}x\right)\frac{d}{dx}-\left(\frac{d^2}{b^2}x^2+(2k+1)i\frac{d}{b}+\frac{k}{x^2}(1-s)\right)\right\},\tag{2.57}$$

where $s(u(x)):=u(-x)$.

Definition 2.13 allows us to make the following comments:

(i) For $M=(0,1;-1,0)$, $\triangle^M_{k,n}$ boils down to the deformed Laplace operator $\triangle_{k,n}$ whereas $\mathcal{F}^M_{k,n}$ coincides with the deformed Hankel transform $\mathcal{F}_{k,n}$ (except for a constant unimodular factor $(e^{i\frac{\pi}{2}})^{\frac{(2k-1)n+2}{2n}}$).

(ii) $\triangle^M_{k,n}$ is related to the deformed Laplace operator $\triangle_{k,n}$ via

$$e^{-\frac{i}{2}\frac{d}{b}x^2}\circ\triangle^M_{k,n}\circ e^{\frac{i}{2}\frac{d}{b}x^2}=\triangle_{k,n}+|x|^{\frac{2}{n}}.\tag{2.58}$$

(iii) For any $f,g\in\mathcal{S}(\mathbb{R})$, we have

$$\int_{\mathbb{R}}\triangle^M_{k,n}f(x)\overline{g(x)}\,d\gamma_{k,n}(x)=\int_{\mathbb{R}}f(x)\overline{\triangle^M_{k,n}g(x)}\,d\gamma_{k,n}(x).\tag{2.59}$$

(iv) For each $y\in\mathbb{R}$, the kernel $K^M_{k,n}(.,y)$ of the linear canonical deformed Hankel transform $\mathcal{F}^M_{k,n}$ satisfy the following:

$$\begin{cases}\triangle^M_{k,n}K^M_{k,n}(.,y)=-|\frac{y}{b}|^{\frac{2}{n}}K^M_{k,n}(.,y),\\ K^M_{k,n}(0,y)=e^{\frac{i}{2}\frac{a}{b}y^2}.\end{cases}\tag{2.60}$$

(v) For each $x,y\in\mathbb{R}$, we have

$$|K^M_{k,n}(x,y)|\leq 1.\tag{2.61}$$

Theorem 2.14. *Let $M = (a, b; c, d) \in SL(2, \mathbb{R})$. Then,*

(i) For any $f \in L^1_{k,n}(\mathbb{R})$, $\mathcal{F}^M_{k,n}(f)$ belongs to $C_0(\mathbb{R})$ and satisfies the following inequality:

$$\left\|\mathcal{F}^M_{k,n}(f)\right\|_{L^\infty_{k,n}(\mathbb{R})} \leq |b|^{-\frac{(2k-1)n+2}{2n}} \left\|f\right\|_{L^1_{k,n}(\mathbb{R})}. \tag{2.62}$$

(ii) For every $f \in L^1_{k,n}(\mathbb{R})$ with $\mathcal{F}^M_{k,n}(f) \in L^1_{k,n}(\mathbb{R})$, we have

$$\left(\mathcal{F}^M_{k,n} \circ \mathcal{F}^{M^{-1}}_{k,n}\right)(f) = \left(\mathcal{F}^{M^{-1}}_{k,n} \circ \mathcal{F}^M_{k,n}\right)(f) = s_{n+1}(f) \quad a.e., \tag{2.63}$$

where $s_j(f)(x) := f((-1)^j x)$, $\forall\, x \in \mathbb{R}$, $j \in \mathbb{N}$.

(iii) $\mathcal{F}^M_{k,n}$ is a topological isomorphism from $L^2_{k,n}(\mathbb{R})$ into itself.

(iv) $\mathcal{F}^M_{k,n}$ is a topological isomorphism from $\mathfrak{S}_{k,n}(\mathbb{R})$ into itself.

(v) For any $f, g \in L^1_{k,n}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \mathcal{F}^M_{k,n}(f)(x)\overline{g(x)}\, d\gamma_{k,n}(x) = \int_{\mathbb{R}} f(x)\overline{\mathcal{F}^{M^{-1}}_{k,n}(g)(x)}\, d\gamma_{k,n}(x).$$

(vi) If $f \in L^1_{k,n}(\mathbb{R}) \cap L^2_{k,n}(\mathbb{R})$, then $\mathcal{F}^M_{k,n}(f) \in L^2_{k,n}(\mathbb{R})$ and

$$\left\|\mathcal{F}^M_{k,n}(f)\right\|_{L^2_{k,n}(\mathbb{R})} = \left\|f\right\|_{L^2_{k,n}(\mathbb{R})}. \tag{2.64}$$

(vii) For any $f, g \in L^2_{k,n}(\mathbb{R})$, we have

$$\left\langle \mathcal{F}^M_{k,n}(f), g\right\rangle_{L^2_{k,n}(\mathbb{R})} = \left\langle f, \mathcal{F}^{M^{-1}}_{k,n} g\right\rangle_{L^2_{k,n}(\mathbb{R})}. \tag{2.65}$$

(viii) (Operational formulas). Let $M \in SL(2, \mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$. Then we have

$$\mathcal{F}^M_{k,n}\left[|y|^{\frac{2}{n}} f(y)\right] = -|b|^{\frac{2}{n}} \triangle^M_{k,n}\left[\mathcal{F}^M_{k,n}(f)\right], \tag{2.66}$$

and

$$|x|^{\frac{2}{n}} \mathcal{F}^M_{k,n}(f) = -|b|^{\frac{2}{n}} \mathcal{F}^M_{k,n}\left[\triangle^{M^{-1}}_{k,n}(f)\right]. \tag{2.67}$$

Definition 2.15. *The deformed linear canonical Hankel transform of any function* $f \in L^p_{k,n}(\mathbb{R})$, $1 \leq p \leq 2$ *with respect to the uni-modular matrix* $M = (a, b; c, d) \in SL(2, \mathbb{R})$ *is defined by*

$$\mathcal{F}^M_{k,n}(f) = e^{-i\left(\frac{(2k-1)n+2}{2n}\right)\frac{\pi}{2}\operatorname{sgn}(b)}\left(\mathbf{L}_{\frac{d}{b}} \circ \Delta_b \circ \mathcal{F}_{k,n} \circ \mathbf{L}_{\frac{a}{b}}\right)(f), \tag{2.68}$$

where $\mathcal{F}_{k,n} : L^p_{k,n}(\mathbb{R}) \to L^{p'}_{k,n}(\mathbb{R})$ *is the deformed Hankel transformation on* $L^p_{k,n}(\mathbb{R})$, $\mathbf{L}_{d/b}$ *and* Δ_b *are the chirp multiplication and dilation operators, defined respectively, by*

$$\mathbf{L}_s f(x) = e^{\frac{is}{2}x^2} f(x), \ s \in \mathbb{R} \quad \text{and} \quad \Delta_s f(x) = \frac{1}{|s|^{\frac{(2k-1)n+2}{2n}}} f\left(\frac{x}{s}\right), \ s \in \mathbb{R}^*. \tag{2.69}$$

Theorem 2.16 (Young's inequality)**.** *For any uni-modular matrix* $M = (a, b; c, d) \in SL(2, \mathbb{R})$ *and* $1 \leq p \leq 2$, $\mathcal{F}^M_{k,n}$ *satisfies the following inequality:*

$$\left\|\mathcal{F}^M_{k,n}(f)\right\|_{L^{p'}_{k,n}(\mathbb{R})} \leq |b|^{\left(\frac{(2k-1)n+2}{2n}\right)\left(\frac{2}{p'}-1\right)} \left\|f\right\|_{L^p_{k,n}(\mathbb{R})}. \tag{2.70}$$

3 Generalized translations associated with LCDHT

Definition 3.1. *Let* $M = (a, b; c, d) \in SL(2, \mathbb{R})$, $b \neq 0$, *a given uni-modular matrix. For suitable function* f, *we define the generalized translation operator associated with the operator* $\triangle^M_{k,n}$ *by*

$$T^{M,k,n}_x f(y) = e^{\frac{i}{2}\frac{d}{b}(x^2+y^2)} \tau^{k,n}_x \left[e^{-\frac{i}{2}\frac{d}{b}s^2} f(s)\right](y), \tag{3.1}$$

where $\tau^{k,n}_x$ *is the* (k, n)*-generalized translation operator associated with* $\Delta_{k,n}$.

We will rely on this definition for each function on the following spaces:

- $L^p_{k,n}(\mathbb{R}), 1 \leq p \leq \infty$.
- $C_b(\mathbb{R})$.

Some important properties of the generalized translation operator $T^{M,k,n}_x$ are assembled in the following theorem.

Theorem 3.2. *Let* $M = (a, b; c, d) \in SL(2, \mathbb{R})$, $b \neq 0$, *then the generalized translation operator* $T^{M,k,n}_x$ *as defined in* (3.1) *satisfies the following properties:*

(i) Linearity: $T^{M,k,n}_x [\alpha f + \beta g](y) = \alpha T^{M,k,n}_x f(y) + \beta T^{M,k,n}_x g(y)$, $\alpha, \beta \in \mathbb{R}$.

(ii) Symmetry: $T^{M,k,n}_0 = Id$, $T^{M,k,n}_x f(y) = T^{M,k,n}_y f(x)$, $\forall\, x, y \in \mathbb{R}$.

(iii) Product Formula: For every $x, , y, z \in \mathbb{R}$, we have

$$T_x^{M,k,n}\left[K_{k,n}^M(.,y)\right](z) = e^{-\frac{i}{2}\frac{a}{b}y^2}\, K_{k,n}^M(x,y)K_{k,n}^M(z,y). \tag{3.2}$$

(iv) Commutative: We have

$$T_x^{M,k,n} \circ T_y^{M,k,n} = T_y^{M,k,n} \circ T_x^{M,k,n} \quad \text{and} \quad \triangle_{k,n}^M \circ T_x^{M,k,n} = T_x^{M,k,n} \circ \triangle_{k,n}^M. \tag{3.3}$$

(v) Let $f \in \mathfrak{S}_{k,n}(\mathbb{R})$. The function $u(x,y) = T_x^{M,k,n} f(y)$ is a solution of the problem

$$\begin{cases} \Delta_{x,k,n}^M \; u(x,y) = \Delta_{y,k,n}^M \; u(x,y) \\ u(x,0) = f(x). \end{cases} \tag{3.4}$$

(vi) For all $x, y \in \mathbb{R}$, we have

$$T_x^{M,k,n} f(y) = \int_{\mathbb{R}} e^{-i\frac{d}{b}z^2} f(z)\; \mathcal{W}_{k,n}^M(x,y,z)\, d\gamma_{k,n}(z), \tag{3.5}$$

where

$$\mathcal{W}_{k,n}^M(x,y,z) = e^{\frac{i}{2}\frac{d}{b}(x^2+y^2+z^2)}\, \mathcal{K}_{k,n}(x,y,z). \tag{3.6}$$

(vii) The generalized translation operator $T_x^{M,k,n}$ is continuous from $C_b(\mathbb{R})$ into itself. Moreover, the operator is also continuous from $L_{k,n}^p(\mathbb{R})$, $1 \le p \le \infty$, into itself and satisfies the following inequality:

$$\left\| T_x^{M,k,n} f \right\|_{L_{k,n}^p(\mathbb{R})} \le 4 \left\| f \right\|_{L_{k,n}^p(\mathbb{R})}. \tag{3.7}$$

(viii) For any $f \in L_{k,n}^1(\mathbb{R})$ and $g \in C_b(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \left[T_x^{M,k,n} f((-1)^n y)\right] \; \left[e^{-i\frac{d}{b}y^2} g(y)\right] d\gamma_{k,n}(y) = \int_{\mathbb{R}} \left[e^{-i\frac{d}{b}y^2} f(y)\right] \left[T_x^{M,k,n} g((-1)^n y)\right] d\gamma_{k,n}(y). \tag{3.8}$$

(ix) For any $f \in L_{k,n}^1(\mathbb{R})$, we have

$$\mathcal{F}_{k,n}^M\left[T_x^{M^{-1},k,n} f\right](\lambda) = e^{\frac{i}{2}\frac{d}{b}\lambda^2}\; \overline{K_{k,n}^M(\lambda,x)}\, \mathcal{F}_{k,n}^M(f)(\lambda), \quad \lambda \in \mathbb{R}. \tag{3.9}$$

(x) For every $f \in L_{k,n}^p(\mathbb{R})$, $1 < p \le 2$, we have

$$\mathcal{F}_{k,n}^M\left[T_x^{M^{-1},k,n} f\right](\lambda) = e^{\frac{i}{2}\frac{d}{b}\lambda^2}\; \overline{K_{k,n}^M(\lambda,x)}\, \mathcal{F}_{k,n}^M(f)(\lambda), \quad a.e. \tag{3.10}$$

(xi) If $f \in C_0(\mathbb{R})$, then we have

$$\lim_{|x|\to\infty} T_x^{M^{-1},k,n} f(y) = 0, \quad y \in \mathbb{R}. \tag{3.11}$$

Proof. Using (3.1), we establish the proof of (i) and (ii).

(iii) Invoking Definition 3.1 and (2.31), we observe that

$$\begin{aligned} T_x^{M,k,n}\left[K_{k,n}^M(\cdot,y)\right](z) &= e^{\frac{i}{2}\frac{d}{b}(x^2+z^2)}\tau_x^{k,n}\left[s\mapsto e^{\frac{i}{2}\frac{a}{b}y^2}B_{k,n}\left(\frac{s}{b},y\right)\right](z)\\ &= e^{\frac{i}{2}\frac{d}{b}(x^2+z^2)}e^{\frac{i}{2}\frac{a}{b}y^2}\tau_x^{k,n}\left[s\mapsto B_{k,n}\left(\frac{s}{b},y\right)\right](z)\\ &= e^{\frac{i}{2}\frac{d}{b}(x^2+z^2)}e^{\frac{i}{2}\frac{a}{b}y^2}B_{k,n}\left(\frac{x}{b},y\right)B_{k,n}\left(\frac{z}{b},y\right)\\ &= e^{-\frac{i}{2}\frac{a}{b}y^2}K_{k,n}^M(x,y)K_{k,n}^M(z,y). \end{aligned}$$

(iv) For any $f \in L^p_{k,n}(\mathbb{R})$, $1 \le p \le \infty$ (or $f \in C_b(\mathbb{R})$), (3.1) and Theorem 2.7 imply that

$$\begin{aligned} \left[T_x^{M,k,n}\circ T_y^{M,k,n}\right]f(z) &= e^{\frac{i}{2}\frac{d}{b}(x^2+y^2+z^2)}\left[\tau_x^{k,n}\circ\tau_y^{k,n}\right]\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](z)\\ &= e^{\frac{i}{2}\frac{d}{b}(x^2+y^2+z^2)}\left[\tau_y^{k,n}\circ\tau_x^{k,n}\right]\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](z)\\ &= \left[T_y^{M,k,n}\circ T_x^{M,k,n}\right]f(z). \end{aligned}$$

Moreover, for any $f \in \mathfrak{S}_{k,n}(\mathbb{R})$, identities (2.58) and (2.19) imply that

$$\begin{aligned} \left[\triangle_{k,n}^M\circ T_x^{M,k,n}\right]f(y) &= e^{\frac{i}{2}\frac{d}{b}(x^2+y^2)}\left[|x|^{2-\frac{2}{n}}\triangle_k\circ\tau_x^{k,n}\right]\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](y)\\ &= e^{\frac{i}{2}\frac{d}{b}(x^2+y^2)}\left[\tau_x^{k,n}\circ|x|^{2-\frac{2}{n}}\triangle_k\right]\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](y)\\ &= \left[T_x^{M,k,n}\circ\triangle_{k,n}^M\right]f(y). \end{aligned}$$

(v) Since system (3.4) is equivalent to

$$\begin{cases} |x|^{2-\frac{2}{n}}\triangle_{k,x}\tilde{u}(x,y) = |y|^{2-\frac{2}{n}}\triangle_{k,y}\tilde{u}(x,y),\\ \tilde{u}(x,0) = e^{-\frac{i}{2}\frac{d}{b}x^2}f(x), \end{cases}$$

where $\tilde{u}(x,y) = e^{-\frac{i}{2}\frac{d}{b}(x^2+y^2)}u(x,y)$. Therefore, by invoking the transmutation property

$$e^{-\frac{i}{2}\frac{d}{b}x^2}\circ\triangle_{k,n}^M\circ e^{\frac{i}{2}\frac{d}{b}x^2} = |x|^{2-\frac{2}{n}}\triangle_k,$$

together with the identity (2.19) and $\tau_x^{k,n}\triangle_k = \triangle_k\tau_x^{k,n}$, we obtain that the function

$$\tilde{u}(x,y) = \tau_x^{k,n}\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](y)$$

is a solution of the previous system. Consequently, we get

$$u(x,y) = e^{\frac{i}{2}\frac{d}{b}(x^2+y^2)}\tau_x^{k,n}\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](y) = T_x^{M,k,n}(f)(y)$$

is a solution of (3.4).

(vi) This is a direct consequence of (3.1) and (2.26).

(vii) The continuous property of $T_x^{M,k,n}$ follows directly from the fact that

$$T_x^{M,k,n}f = \left[\mathbf{L}_{\frac{d}{b},x}\circ\mathbf{L}_{\frac{d}{b},y}\circ\tau_x^{k,n}\circ\mathbf{L}_{-\frac{d}{b}}\right]f$$

where $\mathbf{L}_{\frac{d}{b}}$, $\mathbf{L}_{-\frac{d}{b}}, \tau_x^{k,n}$ are continuous from $C_b(\mathbb{R})$ into itself and $L^p_{k,n}(\mathbb{R})$ into itself, respectively. Moreover, for any $f \in L^p_{k,n}(\mathbb{R})$, the operator $T_x^{M,k,n}f$ belongs to $L^p_{k,n}(\mathbb{R})$ and satisfies

$$\left\|T_x^{M,k,n}f\right\|_{L^p_{k,n}(\mathbb{R})} = \left\|\tau_x^{k,n}\left[\mathbf{L}_{-\frac{d}{b}}f\right]\right\|_{L^p_{k,n}(\mathbb{R})} \leq 4\left\|\mathbf{L}_{-\frac{d}{b}}f\right\|_{L^p_{k,n}(\mathbb{R})} = 4\left\|f\right\|_{L^p_{k,n}(\mathbb{R})}.$$

(viii) For any $f \in L^1_{k,n}(\mathbb{R})$ and $g \in C_b(\mathbb{R})$, (3.1) and (2.49) yield

$$\begin{aligned}
&\int_{\mathbb{R}}\left[T_x^{M,k,n}f((-1)^n y)\right]\left[e^{-i\frac{d}{b}y^2}g(y)\right]d\gamma_{k,n}(y)\\
&\qquad = e^{\frac{i}{2}\frac{d}{b}x^2}\int_{\mathbb{R}}\tau_x^{k,n}\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right]((-1)^n y)\left[e^{-\frac{i}{2}\frac{d}{b}y^2}g(y)\right]d\gamma_{k,n}(y)\\
&\qquad = e^{\frac{i}{2}\frac{d}{b}x^2}\int_{\mathbb{R}}\left[e^{-\frac{i}{2}\frac{d}{b}y^2}f(y)\right]\tau_x^{k,n}\left[e^{-\frac{i}{2}\frac{d}{b}s^2}g(s)\right]((-1)^n y)\,d\gamma_{k,n}(y)\\
&\qquad = \int_{\mathbb{R}}\left[e^{-i\frac{d}{b}y^2}f(y)\right]\left[T_x^{M,k,n}g((-1)^n y)\right]d\gamma_{k,n}(y).
\end{aligned}$$

(ix) For any $f \in L^1_{k,n}(\mathbb{R})$, (2.55), (2.56), (3.1) and Theorem 2.7 imply that

$$\begin{aligned}
&\left[(ib)^{\frac{(2k-1)n+2}{2n}}\right]\mathcal{F}^M_{k,n}\left[T_x^{M^{-1},k,n}f\right](\lambda)\\
&\qquad = e^{\frac{i}{2}(\frac{d}{b}\lambda^2-\frac{a}{b}x^2)}\int_{\mathbb{R}}\tau_x^{k,n}\left[e^{\frac{i}{2}\frac{a}{b}s^2}f(s)\right](y)B_{k,n}\left(\frac{\lambda}{b},y\right)d\gamma_{k,n}(y)\\
&\qquad = e^{\frac{i}{2}(\frac{d}{b}\lambda^2-\frac{a}{b}x^2)}\int_{\mathbb{R}}e^{\frac{i}{2}\frac{a}{b}y^2}f(y)\tau_x^{k,n}\left[s\mapsto B_{k,n}\left(\frac{\lambda}{b},s\right)\right]((-1)^n y)\,d\gamma_{k,n}(y)\\
&\qquad = e^{\frac{i}{2}(\frac{d}{b}\lambda^2-\frac{a}{b}x^2)}\overline{B_{k,n}\left(\frac{\lambda}{b},x\right)}\int_{\mathbb{R}}e^{\frac{i}{2}\frac{a}{b}y^2}f(y)B_{k,n}\left(\frac{\lambda}{b},y\right)d\gamma_{k,n}(y)\\
&\qquad = \left[(ib)^{\frac{(2k-1)n+2}{2n}}\right]e^{\frac{i}{2}\frac{d}{b}\lambda^2}\,\overline{K^M_{k,n}(\lambda,x)}\,\mathcal{F}^M_{k,n}(f)(\lambda).
\end{aligned}$$

(x) For any $f \in L^1_{k,n}(\mathbb{R}) \cap L^p_{k,n}(\mathbb{R})$, the result follows directly by virtue of property (ix) while as Young inequality (2.70) and relation (3.7) show that the mappings $f \mapsto \mathcal{F}^M_{k,n}\left[T_x^{M^{-1},k,n}f\right]$ and $f \mapsto \mathcal{F}^M_{k,n}(f)$ are continuous from $L^p_{k,n}(\mathbb{R})$ into $L^{p'}_{k,n}(\mathbb{R})$. As such, the result follows immediately by the density of $L^1_{k,n}(\mathbb{R}) \cap L^p_{k,n}(\mathbb{R})$ in $L^p_{k,n}(\mathbb{R})$.

(xi) Using the relation (3.1) and Proposition 2.8, we derive the result. □

Corollary 3.3. *For any $f \in \mathcal{S}(\mathbb{R})$, we have*

$$T_x^{M^{-1},k,n} f(y) = \frac{1}{(-ib)^{\frac{(2k-1)n+2}{2n}}} e^{-\frac{i}{2}\frac{a}{b}y^2} \int_{\mathbb{R}} B_{k,n}\left((-1)^n \frac{\lambda}{b}, y\right) \overline{K^M_{k,n}(\lambda,x)}\, \mathcal{F}^M_{k,n}(f)(\lambda)\, d\gamma_{k,n}(\lambda). \tag{3.12}$$

Proof. For any $f \in \mathcal{S}(\mathbb{R})$, inequality (3.7) implies that $y \mapsto [T_x^{M^{-1},k,n} f](y)$ is continuous function of $L^1_{k,n}(\mathbb{R})$. Therefore, as a consequence of (3.9) and the inversion formula of the deformed linear canonical Hankel transform, the result follows immediately. □

We conclude this section with the following important result.

Theorem 3.4. *Let $T_y^{M,k,n}$ be the generalized translation operator associated with the uni-modular matrix $M = (a, b; c, d)$, $b \neq 0$. Then,*

(i) For all $f \in C_0(\mathbb{R})$, we have

$$\lim_{y \to 0} \left\| T_y^{M,k,n} f - f \right\|_\infty = 0. \tag{3.13}$$

(ii) For any $f \in L^p_{k,n}(\mathbb{R})$, $1 \leq p < \infty$, we have

$$\lim_{y \to 0} \left\| T_y^{M,k,n} f - f \right\|_{L^p_{k,n}(\mathbb{R})} = 0. \tag{3.14}$$

Proof. (i) **First step:** We shall prove the result for any $f \in C_c(\mathbb{R})$. Using the fact that

$$\frac{M_{k,n}}{2n} \int_0^\pi (\sin\phi)^{2nk-n} d\phi = 1 \quad \text{and} \quad \int_0^\pi C_n^{nk-\frac{n}{2}}(\cos\phi)(\sin\phi)^{2nk-n} d\phi = 0,$$

the generalized translation operator $T_y^{M,k,n}$ we can be expressed

$$T_y^{M,k,n} f(x) - f(x) = a_y(x) + b_y(x) + c_y(x) + d_y(x), \tag{3.15}$$

where

$$a_y(x) = \frac{M_{k,n}}{2n} f_e(x) \int_0^\pi \left[e^{i\frac{d}{2b}(x^2+y^2-\langle\langle x,y\rangle\rangle^2_{\phi,n})} - 1 \right] \left\{ 1 + (-1)^n \frac{n!\,\mathrm{sgn}(xy)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}}(\cos\phi) \right\} (\sin\phi)^{2nk-n} d\phi,$$

$$b_y(x) = \frac{M_{k,n}}{2n} \int_0^\pi e^{i\frac{d}{2b}\left(x^2+y^2-\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}-2|xy|^{\frac{1}{n}}\cos\phi\right)^n\right)} \left[f_e\left(\langle\langle x,y\rangle\rangle_{\phi,n}\right) - f_e(x) \right] (\sin\phi)^{2nk-n} d\phi$$

$$c_y(x) = \frac{M_{k,n}}{2n} f_o(x) \int_0^\pi \left[e^{i\frac{d}{2b}(x^2+y^2-\langle\langle x,y\rangle\rangle^2_{\phi,n})} - 1 \right] R_{k,n}(x,y,\phi)(\sin\phi)^{2nk-n} d\phi$$

$$d_y(x) = \frac{M_{k,n}}{2n} \int_0^\pi e^{i\frac{d}{2b}\left(x^2+y^2-\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}-2|xy|^{\frac{1}{n}}\cos\phi\right)^n\right)} \left[f_o\left(\langle\langle x,y\rangle\rangle_{\phi,n}\right) - f_o(x) \right] R_{k,n}(x,y,\phi)(\sin\phi)^{2nk-n} d\phi,$$

and

$$R_{k,n}(x,y,\phi)=\frac{n!\,\mathrm{sgn}(x)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\left(\frac{|x|^{\frac{1}{n}}-|y|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}\right)+\frac{n!\,\mathrm{sgn}(y)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\left(\frac{|y|^{\frac{1}{n}}-|x|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}\right).$$

Invoking the properties of the Gegenbauer polynomials, we observe that there exists a positive constant $\mathfrak{C}(k,n)$ such that

$$\left\|a_y\right\|_\infty\le\mathfrak{C}(k,n)\|f\|_\infty\int_0^\pi\left|e^{i\frac{d}{b}(x^2+y^2-\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^2)}-1\right|(\sin\phi)^{2nk-n}d\phi.$$

Therefore, we have

$$\lim_{y\to0}e^{i\frac{d}{b}(x^2+y^2-\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^2)}-1=0,\quad |e^{i\frac{d}{b}(x^2+y^2-\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^2)}-1|\le2,$$

and

$$\int_0^\pi(\sin\phi)^{2nk-n}d\phi=\frac{2n}{M_{k,n}}<\infty.$$

Then, an application of dominated convergence theorem implies that

$$\lim_{y\to0}\int_0^\pi\left|e^{i\frac{d}{b}(x^2+y^2-\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^2)}-1\right|(\sin\phi)^{2nk-n}d\phi=0.$$

So, we derive that

$$\lim_{y\to0}\left\|a_y\right\|_\infty=0.$$

As $\lim_{y\to0}f_e\left(\langle\!\langle x,y\rangle\!\rangle_{\phi,n}\right)=f_e(x)$, we derive from the uniform continuity of f, that for given $\epsilon>0$, there exists $\delta>0$ such that $|y|<\delta$ and

$$\left|b_y(x)\right|\le\frac{M_{k,n}}{2n}\int_0^\pi\left|f_e\left(\langle\!\langle x,y\rangle\!\rangle_{\phi,n}\right)-f_e(x)\right|(\sin\phi)^{2nk-n}d\phi\le\epsilon.$$

Hence

$$\lim_{y\to0}\|b_y\|_\infty=0.$$

Similarly, one can prove that

$$\lim_{y\to0}\|c_y\|_\infty=\lim_{y\to0}\|d_y\|_\infty=0.$$

Thus, we conclude that for any $f\in C_c(\mathbb{R})$, we have

$$\lim_{y\to0}\left\|T_y^{M,k,n}f-f\right\|_\infty=0.$$

Second step: Assume that $f \in C_0(\mathbb{R})$. Using the fact that $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, there exists a function $g \in C_c(\mathbb{R})$ such that $\|f-g\|_\infty \leq \frac{\epsilon}{10}$ so that

$$\begin{aligned}\left\|T_y^{M,k,n}f - f\right\|_\infty &\leq \left\|T_y^{M,k,n}(f-g)\right\|_\infty + \left\|T_y^{M,k,n}g - g\right\|_\infty + \|f-g\|_\infty \\ &\leq 5\,\|f-g\|_\infty + \left\|T_y^{M,k,n}g - g\right\|_\infty \leq \frac{\epsilon}{2} + \left\|T_y^{M,k,n}g - g\right\|_\infty.\end{aligned}$$

From the first step, for sufficiently small values of y, the quantity $\left\|T_y^{M,k,n}g - g\right\|_\infty$ can be made less than $\epsilon/2$. As such, we shall get the desired result.

(ii) Let $f \in C_c(\mathbb{R})$ such that $\operatorname{supp} f \subset [-R, R]$ and $y \in [-1, 1]$. Involving Theorem 3.2 of [4], we derive that the functions $T_y^{M,k,n}f$ are also supported in a common compact set $[-(R^{\frac{1}{n}} + |y|^{\frac{1}{n}})^n, (R^{\frac{1}{n}} + |y|^{\frac{1}{n}})^n] \subset [-2^n(R+1), 2^n(R+1)]$. Consequently, we have

$$\left\|T_y^{M,k,n}f - f\right\|^p_{L^p_{k,n}(\mathbb{R})} \leq \left(\int_{-2^n(R+1)}^{2^n(R+1)} d\gamma_{k,n}(x)\right)\left\|T_y^{M,k,n}f - f\right\|_\infty \to 0, \quad \text{as } y \to 0.$$

Therefore, the general case follows immediately by the density of $C_c(\mathbb{R})$ in $L^p_{k,n}(\mathbb{R})$. This completes the proof of the theorem. □

4 Generalized convolutions product associated with LCDHT

Definition 4.1. *For a given uni-modular matrix $M = (a, b; c, d) \in SL(2, \mathbb{R})$, $b \neq 0$, the generalized convolution product, associated with $\mathcal{F}^M_{k,n}$, for two suitable functions f and g is defined by*

$$f \underset{M,k,n}{\odot} g(x) = \int_{\mathbb{R}} \left[T_x^{M,k,n}f\right]((-1)^n y)\ \left[e^{-i\frac{d}{b}y^2}g(y)\right] d\gamma_{k,n}(y). \tag{4.1}$$

Some elementary properties of convolution (4.1) are summarized below:

(i) An application of Fubini's theorem together with (2.35), (3.5) and (3.6), we have

$$\begin{aligned}f \underset{M,k,n}{\odot} g &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}} e^{-i\frac{d}{b}z^2} f(z)\mathcal{W}^M_{k,n}(x, (-1)^n y, z)\, d\gamma_{k,n}(z)\right]\left[e^{-i\frac{d}{b}y^2}g(y)\right] d\gamma_{k,n}(y) \\ &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}} e^{-i\frac{d}{b}y^2} g(y)\mathcal{W}^M_{k,n}(x, (-1)^n y, z)\, d\gamma_{k,n}(y)\right]\left[e^{-i\frac{d}{b}z^2}f(z)\right] d\gamma_{k,n}(z) = g \underset{M,k,n}{\odot} f.\end{aligned}$$

(ii) Using Fubini's theorem, we have

$$\begin{aligned}T_x^{M,k,n}\left(f \underset{M,k,n}{\odot} g\right)(y) &= \int_{\mathbb{R}} e^{-i\frac{d}{b}z^2}\left(f \underset{M,k,n}{\odot} g\right)(z)\mathcal{W}^M_{k,n}(x, y, z)\, d\gamma_{k,n}(z) \\ &= \int_{\mathbb{R}} e^{-i\frac{d}{b}z^2}\left[\int_{\mathbb{R}}\left[T_z^{M,k,n}f((-1)^n s)\right]\left[e^{-i\frac{d}{b}s^2}g(s)\right] d\gamma_{k,n}(s)\right]\mathcal{W}^M_{k,n}(x, y, z)\, d\gamma_{k,n}(z)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\frac{d}{b}z^2} \left[T^{M,k,n}_{(-1)^n s} f(z) \right] \mathcal{W}^M_{k,n}(x,y,z)\, d\gamma_{k,n}(z) \right] \left[e^{-i\frac{d}{b}s^2} g(s) \right] d\gamma_{k,n}(s) \\
&= \int_{\mathbb{R}} T^{M,k,n}_x \left[T^{M,k,n}_{(-1)^n s} f \right](y) \left[e^{-i\frac{d}{b}s^2} g(s) \right] d\gamma_{k,n}(s) \\
&= \int_{\mathbb{R}} T^{M,k,n}_y \left[T^{M,k,n}_x f \right]((-1)^n s) \left[e^{-i\frac{d}{b}s^2} g(s) \right] d\gamma_{k,n}(s) \\
&= \left(\left[T^{M,k,n}_x f \right] \underset{M,k,n}{*} g \right)(y).
\end{aligned}
$$

The following proposition contain the basic facts about convolutions of $L^p_{k,n}(\mathbb{R})$, $1 \le p \le \infty$.

Proposition 4.2 (Young's Inequality). *Let $1 \le p,q,r \le \infty$ with $p^{-1}+q^{-1} = r^{-1}+1$. If $f \in L^p_{k,n}(\mathbb{R})$ and $g \in L^q_{k,n}(\mathbb{R})$, then $f \underset{M,k,n}{\odot} g \in L^r_{k,n}(\mathbb{R})$ and satisfies the following inequality:*

$$
\left\| f \underset{M,k,n}{\odot} g \right\|_{L^r_{k,n}(\mathbb{R})} \le 4 \left\| f \right\|_{L^p_{k,n}(\mathbb{R})} \left\| g \right\|_{L^q_{k,n}(\mathbb{R})}. \tag{4.2}
$$

Proof. Using Hölder's inequality, we obtain

$$
\begin{aligned}
&\left| T^{M,k,n}_x f((-1)^n y) e^{-i\frac{d}{b}y^2} g(y) \right| \\
&\qquad = \left(\left| T^{M,k,n}_x f((-1)^n y) \right|^p |g(y)|^q \right)^{1/r} \left(\left| T^{M,k,n}_x f((-1)^n y) \right|^p \right)^{1/p-1/r} \left(|g(y)|^q \right)^{1/q-1/r}.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\int_{\mathbb{R}} \left| T^{M,k,n}_x f((-1)^n y) e^{-i\frac{d}{b}y^2} g(y) \right| d\gamma_{k,n}(y) &\le \left(\int_{\mathbb{R}} \left| T^{M,k,n}_x f((-1)^n y) \right|^p |g(y)|^q \, d\gamma_{k,n}(y) \right)^{1/r} \\
&\quad \left(\int_{\mathbb{R}} \left| T^{M,k,n}_x f((-1)^n y) \right|^p d\gamma_{k,n}(y) \right)^{\frac{r-p}{rp}} \left(\int_{\mathbb{R}} |g(y)|^q \, d\gamma_{k,n}(y) \right)^{\frac{r-q}{rq}},
\end{aligned}
$$

which leads us to

$$
\begin{aligned}
\left| \left(f \underset{M,k,n}{\odot} g \right)(x) \right|^r &\le \left(\int_{\mathbb{R}} \left| T^{M,k,n}_x f((-1)^n y) \right|^p d\gamma_{k,n}(y) \right)^{\frac{r-p}{p}} \|g\|^{r-q}_{L^q_{k,n}(\mathbb{R})} \\
&\qquad \int_{\mathbb{R}} \left| T^{M,k,n}_x f((-1)^n y) \right|^p |g(y)|^q \, d\gamma_{k,n}(y).
\end{aligned}
$$

By invoking (3.7), we observe that

$$
\left| \big(f \underset{M,k,n}{\odot} g \big)(x) \right|^r \le 4^{r-p} \|f\|^{r-p}_{L^p_{k,n}(\mathbb{R})} \|g\|^{r-q}_{L^q_{k,n}(\mathbb{R})} \int_{\mathbb{R}} \left| T^{M,k,n}_x f((-1)^n y) \right|^p |g(y)|^q \, d\gamma_{k,n}(y).
$$

After multiply both sides by $d\gamma_{k,n}(x)$ and integrating over $\mathbb{R}$, we get

$$\begin{aligned}\left\| f \underset{M,k,n}{\odot} g \right\|^r_{L^r_{k,n}(\mathbb{R})} &\leq 4^{r-p} \|f\|^{r-p}_{L^p_{k,n}(\mathbb{R})} \|g\|^{r-q}_{L^q_{k,n}(\mathbb{R})} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left|T_x^{M,k,n} f((-1)^n y)\right|^p |g(y)|^q \, d\gamma_{k,n}(y)\right] d\gamma_{k,n}(x)\\ &= 4^{r-p} \|f\|^{r-p}_{L^p_{k,n}(\mathbb{R})} \|g\|^{r-q}_{L^q_{k,n}(\mathbb{R})} \int_{\mathbb{R}} |g(y)|^q \left[\int_{\mathbb{R}} \left|T^{M,k,n}_{(-1)^n y} f(x)\right|^p d\gamma_{k,n}(x)\right] d\gamma_{k,n}(y)\\ &\leq 4^r \, \|f\|^r_{L^p_{k,n}(\mathbb{R})} \|g\|^r_{L^q_{k,n}(\mathbb{R})}.\end{aligned}$$

Or equivalently,

$$\left\| f \underset{M,k,n}{\odot} g \right\|_{L^r_{k,n}(\mathbb{R})} \leq 4 \left\|f\right\|_{L^p_{k,n}(\mathbb{R})} \left\|g\right\|_{L^q_{k,n}(\mathbb{R})}.$$

□

Theorem 4.3. *Let* $\underset{M,k,n}{\odot}$ *be the generalized convolution as defined by (4.1) associated with unimodular matrix* $M = (a, b; c, d) \in SL(2, \mathbb{R})$, $b \neq 0$. *Then,*

(i) For any $f, g \in L^1_{k,n}(\mathbb{R})$, *we have*

$$\mathcal{F}^M_{k,n}\left(f \underset{M^{-1},k,n}{\odot} g\right)(x) = \left((ib)^{\frac{(2k-1)n+2}{2n}}\right) e^{-\frac{i}{2}\frac{d}{b}x^2} \mathcal{F}^M_{k,n}(f)(x) \mathcal{F}^M_{k,n}(g)(x), \quad \text{for all } x \in \mathbb{R}. \tag{4.3}$$

(ii) For any $f \in L^1_{k,n}(\mathbb{R})$ *and* $g \in L^p_{k,n}(\mathbb{R})$, $1 < p \leq 2$, *we have*

$$\mathcal{F}^M_{k,n}\left(f \underset{M^{-1},k,n}{\odot} g\right)(x) = \left((ib)^{\frac{(2k-1)n+2}{2n}}\right) e^{-\frac{i}{2}\frac{d}{b}x^2} \mathcal{F}^M_{k,n}(f)(x) \mathcal{F}^M_{k,n}(g)(x), \quad a.e.\ x \in \mathbb{R}. \tag{4.4}$$

(iii) For $f, g, h \in L^1_{k,n}(\mathbb{R})$, *we have*

$$\left(f \underset{M,k,n}{\odot} g\right) \underset{M,k,n}{\odot} h = f \underset{M,k,n}{\odot} \left(g \underset{M,k,n}{\odot} h\right). \tag{4.5}$$

Proof. (i) Using the definition of $\mathcal{F}^M_{k,n}$ along with (3.9), it follows that

$$\begin{aligned}&\mathcal{F}^M_{k,n}\left(f \underset{M^{-1},k,n}{\odot} g\right)(x)\\ &= \frac{1}{(ib)^{\frac{(2k-1)n+2}{2n}}} \int_{\mathbb{R}} K^M_{k,n}(x,y) \left[\int_{\mathbb{R}} T_y^{M^{-1},k,n} f((-1)^n z) \left[e^{i\frac{a}{b}z^2} g(z)\right] d\gamma_{k,n}(z)\right] d\gamma_{k,n}(y)\\ &= \frac{1}{(ib)^{\frac{(2k-1)n+2}{2n}}} \int_{\mathbb{R}} \left[e^{i\frac{a}{b}z^2} g(z)\right] \left[\int_{\mathbb{R}} K^M_{k,n}(x,y) T^{M^{-1},k,n}_{(-1)^n z} f(y) \, d\gamma_{k,n}(y)\right] d\gamma_{k,n}(z)\\ &= \int_{\mathbb{R}} \left[e^{i\frac{a}{b}z^2} g(z)\right] \left[\mathcal{F}^M_{k,n}\left(T^{M^{-1},k,n}_{(-1)^n z} f\right)(x)\right] d\gamma_{k,n}(z)\\ &= \left((ib)^{\frac{(2k-1)n+2}{2n}}\right) \; e^{-\frac{i}{2}\frac{d}{b}x^2} \mathcal{F}^M_{k,n}(f)(x) \, \mathcal{F}^M_{k,n}(g)(x).\end{aligned}$$

It is pertinent to mention that Fubini theorem has been used in the second line as

$$\int_{\mathbb{R}^2} \left| K^M_{k,n}(x,y) T_y^{M^{-1},k,n} f((-1)^n z) e^{i\frac{a}{b}z^2} g(z) \right| d\gamma_{k,n}(y)\, d\gamma_{k,n}(z)$$

$$\leq C \int_{\mathbb{R}^2} \left| T_y^{M^{-1},k,n} f((-1)^n z) \right| |g(z)|\, d\gamma_{k,n}(y)\, d\gamma_{k,n}(z) \leq 4C \|f\|_{L^1_{k,n}(\mathbb{R})} \|g\|_{L^1_{k,n}(\mathbb{R})} < \infty.$$

(ii) The result is true for $g \in L^1_{k,n}(\mathbb{R}) \cap L^p_{k,n}(\mathbb{R})$ by virtue of (i). On the other hand, the Young's inequality (2.70) for the deformed linear canonical Hankel transform and Proposition 4.2 show that the mappings $g \mapsto \mathcal{F}^M_{k,n}\left(f \underset{M^{-1},k,n}{*} g\right)$ and $g \mapsto \mathcal{F}^M_{k,n}(f)\ \mathcal{F}^M_{k,n}(g)$ are continuous from $L^p_{k,n}(\mathbb{R})$ into $L^{p'}_{k,n}(\mathbb{R})$. Finally, the result follows directly from density of $L^1_{k,n}(\mathbb{R}) \cap L^p_{k,n}(\mathbb{R})$ in $L^p_{k,n}(\mathbb{R})$.

(iii) The result follows immediately by an application of result (i). □

5 Generalized heat equation and the associated operators

In this section, we shall illustrate our proposed theory developed in previous sections to the following generalized heat equation associated with the operator $\triangle^{M^{-1}}_{k,n}$:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \sigma \triangle^{M^{-1}}_{k,n} u(t,x), \ (t,x) \in (0,\infty) \times \mathbb{R} \\ u(0,x) = f(x), \end{cases} \tag{5.1}$$

where f is defined on the Banach space $\mathfrak{B}$ which could be either $L^p_{k,n}(\mathbb{R})$, $1 \leq p \leq \infty$, $(C_b(\mathbb{R}), \|.\|_\infty)$ or $(C_0(\mathbb{R}), \|.\|_\infty)$, $\sigma > 0$ is the coefficient of heat conductivity and the initial data $u(0,x) = f(x)$ means that $u(t,x) \to f(x)$ as $t \to 0$ in the norm of $\mathfrak{B}$.

5.1 Generalized heat kernel associated with $\sigma \triangle^{M^{-1}}_{k,n}$

Given a uni-modular matrix $M = (a,b;c,d) \in SL(2,\mathbb{R})$, $b \neq 0$ and $\sigma, t > 0$, we define

$$\mathcal{P}^{M^{-1}}_t(y) := \frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} \exp\left\{ -\frac{iay^2}{2b} - \frac{ny^2}{2\sigma t} \right\}, \quad y \in \mathbb{R}. \tag{5.2}$$

Using the relations (2.55), (2.56), (5.2) and Example 2.2, we obtain

$$\mathcal{F}^M_{k,n}\left(\mathcal{P}^{M^{-1}}_t\right)(x) = \exp\left\{ \frac{idx^2}{2b} - t\sigma \left(\frac{x}{|b|}\right)^2 \right\}, \quad \forall\, t > 0, \quad x \in \mathbb{R}. \tag{5.3}$$

Definition 5.1. *Given a uni-modular matrix $M = (a, b; c, d) \in SL(2, \mathbb{R})$, $b \neq 0$, the generalized heat kernel associated with $\triangle_{k,n}^{M^{-1}}$ is denoted as $G_t^{M^{-1}}$ and defined by*

$$G_t^{M^{-1}}(x, y) = T_x^{M^{-1},k,n}\left[\mathcal{P}_t^{M^{-1}}\right](y), \quad x, y \in \mathbb{R}, \quad t > 0. \tag{5.4}$$

We collect some basic properties of the generalized heat kernel $G_t^{M^{-1}}$ in the following proposition.

Proposition 5.2. *The generalized heat kernel $G_t^{M^{-1}}$ as defined in (5.4) satisfies the following properties:*

(i) For $t > 0$, we have

$$G_t^{M^{-1}}(x, y) = \frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} \exp\left\{\frac{-ia(x^2 + y^2)}{2b} - \frac{n(|x|^{\frac{2}{n}} + |y|^{\frac{2}{n}})}{2\sigma t}\right\} B_{k,n}\left(\frac{x}{(\sigma t)^n}, (-i)^n y\right). \tag{5.5}$$

(ii) For $t > 0$, there exists a positive constant $\mathcal{C}(k, n)$ such that

$$\left|G_t^{M^{-1}}(x, y)\right| \leq \mathcal{C}(k, n) \frac{e^{-n\frac{\left(|x|^{\frac{1}{n}} - |y|^{\frac{1}{n}}\right)^2}{2\sigma t}}}{(\sigma t)^{\frac{(2k-1)n+2}{2}}}. \tag{5.6}$$

(iii) For $t > 0$, we have

$$\int_{\mathbb{R}} e^{\frac{i}{2}\frac{a}{b}(x^2+y^2)} G_t^{M^{-1}}(x, y)\, d\gamma_{k,n}(y) = 1. \tag{5.7}$$

(iv) For $s, t > 0$, we have

$$G_{t+s}^{M^{-1}}(x, y) = \int_{\mathbb{R}} G_t^{M^{-1}}(x, z)\, G_s^{M^{-1}}(y, z)\; e^{i\frac{a}{b}z^2} d\gamma_{k,n}(z). \tag{5.8}$$

(v) For fixed $t > 0$ and $y \in \mathbb{R}$, we have

$$\mathcal{F}_{k,n}^{M}\left(G_t^{M^{-1}}(., y)\right)(\xi) = e^{i\frac{d}{b}\xi^2}\overline{K_{k,n}^{M}(\xi, y)} \exp\left\{-t\sigma\left|\frac{\xi}{b}\right|^{\frac{2}{n}}\right\}. \tag{5.9}$$

(vi) For a fixed $y \in \mathbb{R}$, $u(t, x) = G_t^{M^{-1}}(x, y)$ is the solution of the generalized heat equation (5.1).

Proof. (i) Using the Definition 3.1, we observe that

$$G_t^{M^{-1}}(x, y) = \frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} e^{-\frac{i}{2}\frac{a}{b}(x^2+y^2)} \tau_x^{k,n}\left[e^{-\frac{n|s|^{\frac{2}{n}}}{2\sigma t}}\right](y). \tag{5.10}$$

Therefore, by simple application of (2.25), we derive the desired assertion.

(ii) The assertion follows directly from the relation (5.10) and inequality (2.41).

(iii) An application of (5.10) leads us to

$$\int_{\mathbb{R}} e^{\frac{i}{2}\frac{a}{b}(x^2+y^2)} G_t^{M^{-1}}(x,y)\, d\gamma_{k,n}(y) = \frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} \int_{\mathbb{R}} \tau_x^{k,n}\left(e^{-\frac{n|s|^{\frac{2}{n}}}{2\sigma t}}\right)(y)\, d\gamma_{k,n}(y).$$

Thus, we obtain the desired result by applying (2.47) and simple calculations.

(iv) Using the identity (5.5), we obtain

$$\int_{\mathbb{R}} G_t^{M^{-1}}(x,z)\, G_s^{M^{-1}}(y,z)\ e^{i\frac{a}{b}z^2} d\gamma_{k,n}(z) = \frac{1}{((\sigma)^2 ts)^{\frac{(2k-1)n+2}{2}}} e^{\frac{-i}{2}\frac{a}{b}(x^2+y^2) - \left[n\frac{|x|^{\frac{2}{n}}}{2\sigma t} + n\frac{|y|^{\frac{2}{n}}}{2\sigma s}\right]}$$
$$\int_{\mathbb{R}} e^{-n\left[\frac{|z|^{\frac{2}{n}}}{2\sigma t} + \frac{|z|^{\frac{2}{n}}}{2\sigma s}\right]} B_{k,n}\left(\frac{x}{(\sigma t)^n}, (-i)^n z\right) B_{k,n}\left(\frac{y}{(\sigma s)^n}, (-i)^n z\right) d\gamma_{k,n}(z).$$

From the relation (2.10), we deduce that

$$\int_{\mathbb{R}} e^{-n\left[\frac{|z|^{\frac{2}{n}}}{2\sigma t} + \frac{|z|^{\frac{2}{n}}}{2\sigma s}\right]} B_{k,n}\left(\frac{x}{(\sigma t)^n}, (i)^n z\right) B_{k,n}\left(\frac{y}{(\sigma s)^n}, (i)^n z\right) d\gamma_{k,n}(z)$$
$$= \left(\frac{\sigma ts}{t+s}\right)^{\frac{(2k-1)n+2}{2}} e^{n\left[\frac{s|x|^{\frac{2}{n}}}{2\sigma t(t+s)} + \frac{t|y|^{\frac{2}{n}}}{2\sigma s(t+s)}\right]} B_{k,n}\left(\frac{x}{(\sigma(s+t))^n}, (i)^n z\right),$$

which leads to the given desired result.

(v) Involving the relations (5.4), (3.9) and (5.3), we get the desired result.

(vi) For fixed $y \in \mathbb{R}$ and $t > 0$, we put $v(x,t) := G_t^{M^{-1}}(x,y)$. Using (5.4) and Corollary 3.3, we deduce that

$$G_t^{M^{-1}}(x,y) = \frac{1}{(-ib)^{\frac{(2k-1)n+2}{2n}}} \int_{\mathbb{R}} e^{i\frac{d}{b}\lambda^2} B_{k,n}\left((-1)^n \frac{\lambda}{b}, y\right) \overline{K_{k,n}^M(\lambda,x)} \exp\left\{-t\sigma\left|\frac{\lambda}{b}\right|^{\frac{2}{n}}\right\} d\gamma_{k,n}(\lambda). \tag{5.11}$$

By taking differentiations under integral, the identities (2.66), (2.60) and by standard analysis, we see that

$$\left[\frac{\partial}{\partial t} - \sigma\triangle_{k,n}^{M^{-1}}\right] G_t^{M^{-1}}(x,y) = 0.$$

This completes the proof of the Proposition 5.2. □

Theorem 5.3. *Assume that* $M = (a, b; c, d) \in SL(2, \mathbb{R})$ *such that* $b \neq 0$. *Let* $\mathfrak{B}$ *be one of the Banach spaces* $L^p_{k,n}(\mathbb{R})$ $(1 \leq p \leq \infty)$, $(C_b(\mathbb{R}), \|.\|_\infty)$ *or* $(C_0(\mathbb{R}), \|.\|_\infty)$. *Then:*

(i) For each $f \in X$, *the function* $u(t,x) = \left(\mathcal{P}_t^{M^{-1}} \underset{M^{-1},k,n}{\odot} f\right)(x)$ *satisfies the generalized heat equation*

$$\frac{\partial u(t,x)}{\partial t} = \sigma \triangle_{k,n}^{M^{-1}} u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \tag{5.12}$$

and

$$\left\|u(t,\cdot)\right\|_{L^r_{k,n}(\mathbb{R})} \leq \frac{4\left(2\Gamma\left(\frac{(2k-1)n+2}{nq}\right) M_{k,n}\right)^{1/q}}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} \left\|f\right\|_{L^p_{k,n}(\mathbb{R})}, \tag{5.13}$$

where $p, q, r \in [1, \infty]$ *satisfying* $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.

(ii) Let $f(x) = e^{-\frac{i}{2}\frac{a}{b}x^2} p\left(|x|^{\frac{2}{n}}\right)$ *with* $p(s) = \sum_{j=0}^{m} c_j s^j$.

We define the function u as $u(t,x) = \left(\mathcal{P}_t^{M^{-1}} \underset{M^{-1},k,n}{\odot} f\right)(x)$. *We have*

$$u(t,x) = e^{-\frac{i}{2}\frac{a}{b}x^2} \sum_{j=0}^{n} j!\, c_j \left(\frac{2\sigma t}{n}\right)^j L_j^{\left(\frac{(2k-1)n}{2}\right)} \left(-\frac{n|x|^{\frac{2}{n}}}{2\sigma t}\right), \tag{5.14}$$

where $L_j^{\left(\frac{(2k-1)n}{2}\right)}$ *denote the Laguerre functions of degree j [43]. Moreover,*

$$\frac{\partial u(t,x)}{\partial t} = \sigma \triangle_{k,n}^{M^{-1}} u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \quad \textit{with } u(0,x) = f(x).$$

Proof. (i) In view of (5.11) and Fubini's theorem, the function $u(t,x)$ can be expressed as

$$u(t,x) = \frac{1}{(-ib)^{\frac{(2k-1)n+2}{2n}}} \int_{\mathbb{R}} e^{i\frac{d}{b}\lambda^2} \overline{K^M_{k,n}(\lambda,x)} \exp\left\{-t\sigma\left|\frac{\lambda}{b}\right|^{\frac{2}{n}}\right\} \mathcal{F}_{k,n}(f)(\lambda)\, d\gamma_{k,n}(\lambda). \tag{5.15}$$

Moreover, as above take again differentiation under the integral in (5.15) and (2.66), we derive the result.

Furthermore, the Young's inequality (4.2) implies that

$$\left\|u(t,.)\right\|_{L^r_{k,n}(\mathbb{R})} = \left\|\mathcal{P}_t^{M^{-1}} \underset{M^{-1},k,n}{\odot} f\right\|_{L^r_{k,n}(\mathbb{R})} \leq 4\left\|\mathcal{P}_t^{M^{-1}}\right\|_{L^q_{k,n}(\mathbb{R})} \left\|f\right\|_{L^p_{k,n}(\mathbb{R})}. \tag{5.16}$$

Using (5.16) and the fact that

$$\left\|\mathcal{P}_t^{M^{-1}}\right\|_{L^q_{k,n}(\mathbb{R})} = \frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} \left(\int_{\mathbb{R}} e^{-\frac{nq}{2\sigma t}|y|^{\frac{2}{n}}} d\gamma_{k,n}(y)\right)^{1/q} = \frac{\left(2\Gamma\left(\frac{(2k-1)n+2}{nq}\right) M_{k,n}\right)^{1/q}}{(\sigma t)^{\frac{(2k-1)n+2}{2}}},$$

we obtain the desired inequality (5.12).

(ii) Firstly, it is easy to see that

$$u(t,x) = \int_{\mathbb{R}} G_t^{M^{-1}}(x,(-1)^n y)\, e^{\frac{i}{2}\frac{a}{b}y^2} p\left(|y|^{\frac{2}{n}}\right) d\gamma_{k,n}(y). \tag{5.17}$$

Now, if we write $p\left(|y|^{\frac{2}{n}}\right) = \sum_{j=1}^{m} c_j |y|^{\frac{2j}{n}}$, then using (5.5) and by the change of variables $u = \frac{y}{(\sigma t)^{\frac{n}{2}}}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} G_t^{M^{-1}}(x,(-1)^n y)\, e^{\frac{i}{2}\frac{a}{b}y^2} p\left(|y|^{\frac{2}{n}}\right) d\gamma_{k,n}(y) \\
&\quad = \frac{M_{k,n} e^{-\frac{i}{2}\frac{a}{b}x^2} e^{-\frac{n|x|^{\frac{2}{n}}}{2\sigma t}}}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} \sum_{j=1}^{m} c_j \int_{\mathbb{R}} e^{n\frac{-|y|^{\frac{2}{n}}}{2\sigma t}} B_{k,n}\left(\frac{x}{(\sigma t)^n}, (i)^n y\right) |y|^{\frac{(2k-2)n+2+2j}{n}}\, dy \\
&\quad = \sum_{j=1}^{m} c_j \left(\frac{2\sigma t}{n}\right)^j e^{-\frac{i}{2}\frac{a}{b}x^2} e^{-n\frac{|x|^{\frac{2}{n}}}{2\sigma t}}\, I_t(x),
\end{aligned} \tag{5.18}$$

where

$$I_t(x) = \frac{2}{\Gamma\left(\frac{(2k-1)n+2}{2}\right)} \int_{\mathbb{R}} e^{-u^2} j_{\frac{(2k-1)n}{2}}\left(\frac{2i|x|^{\frac{1}{n}}u}{\sqrt{\frac{2\sigma t}{n}}}\right) u^{(2k-1)n+1+2j}\, du.$$

Using the identity (6.631(10) in [20]), we get

$$\int_0^{\infty} e^{-u^2} j_\alpha(uz) u^{2j+2\alpha+1} du = \frac{\Gamma(\alpha+1)}{2} j!\, e^{-\frac{z^2}{4}} L_j^{\alpha}\left(\frac{z^2}{4}\right), \quad z \geq 0.$$

Further, by simple calculations, we see that

$$I_t(x) = j!\, e^{-\frac{i}{2}\frac{a}{b}x^2} e^{n\frac{|x|^{\frac{2}{n}}}{2\sigma t}} L_j^{\left(\frac{(2k-1)n}{2}\right)}\left(-\frac{n|x|^{2/n}}{2\sigma t}\right), \quad x \in \mathbb{R}. \tag{5.19}$$

Substituting (5.19) in (5.18), we get the desired identity:

$$u(t,x) = e^{-\frac{i}{2}\frac{a}{b}x^2} \sum_{j=0}^{n} j!\, c_j \left(\frac{2\sigma t}{n}\right)^j L_j^{\left(\frac{(2k-1)n}{2}\right)}\left(-\frac{n|x|^{\frac{2}{n}}}{2\sigma t}\right).$$

Finally, using (i) we observe that the function u solves (5.12). Moreover using the identity, (*cf.* [43]),

$$\left(\frac{2\sigma t}{n}\right)^j L_j^{\frac{(2k-1)n}{2}}\left(-\frac{n|x|^{2/n}}{2\sigma t}\right)\Big|_{t=0} = \frac{|x|^{\frac{2j}{n}}}{j!}$$

we derive that $u(0,x) = f(x)$. This completes the proof of the Theorem 5.3. □

5.2 Heat semi-groups associated with $\sigma\triangle_{k,n}^{M^{-1}}$

We begin this subsection by recalling the necessary tools on semigroups.

Definition 5.4 ([36]). *Let X be a Banach space. A one-parameter family $S = \{S(t);\ t \geq 0\}$ of bounded linear operators on X is called a strongly continuous semigroup if it satisfies:*

(i) $S(0) = Id_X$.

(ii) $S(t+s) = S(t)S(s)$ *for all* $t, s \geq 0$.

(iii) The mapping $t \mapsto S(t)u$ is continuous on $[0,\infty)$ for all $u \in X$. A strongly continuous semigroup is called a contraction semigroup, if $\|S(t)\| \leq 1$ for all $t \geq 0$.

Let $S = (S(t))_{t\geq 0}$ be a strongly continuous semigroup. The generator $\mathfrak{O}$ of S is defined by the formula

$$\mathfrak{O}u = \lim_{t\to 0}\frac{S(t)u - u}{t} = \frac{d}{dt}S(t)u\Big|_{t=0},$$

the domain $\mathcal{D}(\mathfrak{O})$ of $\mathfrak{O}$ being the set of all $u \in X$ for which the limit defined above exists.

In this subsection, we shall denote $\mathfrak{B}$ as one of the Banach spaces $L^p_{k,n}(\mathbb{R})$ $(1 \leq p < \infty)$ or $(C_0(\mathbb{R}), \|.\|_\infty)$.

Definition 5.5. *Let $M = (a, b; c, d) \in SL(2,\mathbb{R})$ be a uni-modular matrix such that $b \neq 0$. Then, for each $t \geq 0$ and $f \in X$, we define a family of operators*

$$S_{k,n}^{M^{-1}}(t)f = \begin{cases} \frac{1}{4}\left[\mathcal{P}_t^{M^{-1}} \underset{M^{-1},k,n}{\odot} f\right] & \text{if} \quad t > 0, \\ f & \text{if} \quad t = 0. \end{cases} \tag{5.20}$$

The family of operators (5.20) is often called the heat semigroup associated with $\sigma\triangle_{k,n}^{M^{-1}}$.

Theorem 5.6. *The family of operators* $\left\{S_{k,n}^{M^{-1}}(t) : t \geq 0\right\}$ *is strongly continuous contraction on* $\mathfrak{B}$.

Proof. We shall divide the proof of the theorem into two steps.

First step: (i) Assume that $\mathfrak{B} = C_0(\mathbb{R})$. Then, the result is trivial when $t = 0$. For any $f \in C_0(\mathbb{R})$ and $t > 0$, (3.8) and (5.2), implies that

$$\begin{aligned}\left(S_{k,n}^{M^{-1}}(t)f\right)(x) &= \frac{1}{4}\int_{\mathbb{R}} \left[T_x^{M^{-1},k,n}\mathcal{P}_t^{M^{-1}}\right]((-1)^n y)\left[e^{i\frac{a}{b}y^2}f(y)\right] d\gamma_{k,n}(y)\\ &= \frac{1}{4}\int_{\mathbb{R}} e^{i\frac{a}{b}y^2}\mathcal{P}_t^{M^{-1}}(y)\left[T_x^{M^{-1},k,n}f\right]((-1)^n y)\, d\gamma_{k,n}(y)\\ &= \frac{1}{4}\frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}}\int_{\mathbb{R}} e^{\frac{i}{2}\frac{a}{b}y^2} e^{\frac{-n|y|^{\frac{2}{n}}}{2\sigma t}}\left[T_x^{M^{-1},k,n}f\right]((-1)^n y)\, d\gamma_{k,n}(y)\\ &= \frac{1}{4}\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}\int_{\mathbb{R}} e^{i\frac{a}{2b}\left(\frac{2\sigma t}{n}\right)^n} e^{-|v|^{\frac{2}{n}}}\left[T_x^{M^{-1},k,n}f\right]\left((-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}} v\right) d\gamma_{k,n}(v).\end{aligned} \tag{5.21}$$

Clearly the mapping $(x,v) \longmapsto \left[T_x^{M^{-1},k,n}f\right]\left((-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}} v\right)$ is continuous on $\mathbb{R}^2$. Moreover, using (3.11) and (3.7), we have

$$\lim_{|x|\to\infty}\left[T_x^{M^{-1},k,n}f\right]\left((-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}} v\right) = 0$$

and

$$\left|e^{i\frac{a}{2b}\left(\frac{2\sigma t}{n}\right)^n} e^{-|v|^{\frac{2}{n}}}\left[T_x^{M^{-1},k,n}f\right]\left((-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}} v\right)\right| \leq 4\left\|f\right\|_\infty e^{-|v|^{\frac{2}{n}}} \in L_{k,n}^1(\mathbb{R}).$$

Therefore, it follows by the dominated convergence theorem that $S_{k,n}^{M^{-1}}(t)f \in C_0(\mathbb{R})$ and satisfies the inequality:

$$\left\|S_{k,n}^{M^{-1}}(t)f\right\|_\infty \leq \left\{\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}\int_{\mathbb{R}} e^{-|y|^{\frac{2}{n}}}\, d\gamma_{k,n}(y)\right\}\left\|f\right\|_\infty = \left\|f\right\|_\infty.$$

By taking supremum over all $f \in C_0(\mathbb{R})$ and noting that $\|f\|_\infty \leq 1$, we obtain $\left\|S_{k,n}^{M^{-1}}(t)\right\| \leq 1$.

(ii) For all $t, s > 0$ and $f \in C_0(\mathbb{R})$, from (5.8) we have

$$\begin{aligned}
&S_{k,n}^{M^{-1}}(s)\left(S_{k,n}^{M^{-1}}(t)f\right)(x)\\
&\quad= \frac{1}{4}\int_{\mathbb{R}} G_s^{M^{-1}}(x,z)e^{i\frac{a}{b}z^2}\left(\int_{\mathbb{R}} G_t^{M^{-1}}(y,z)e^{i\frac{a}{b}y^2}f(y)\,d\gamma_{k,n}(y)\right)d\gamma_{k,n}(z)\\
&\quad= \frac{1}{4}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} G_s^{M^{-1}}(x,z)G_t^{M^{-1}}(y,z)\; e^{i\frac{a}{b}z^2}d\gamma_{k,n}(z)\right)e^{i\frac{a}{b}y^2}f(y)\,d\gamma_{k,n}(y)\\
&\quad= \frac{1}{4}\int_{\mathbb{R}} G_{s+t}^{M^{-1}}(x,y)e^{i\frac{a}{b}y^2}f(y)\,d\gamma_{k,n}(y)\\
&\quad= S_{k,n}^{M^{-1}}(s+t)f(x).
\end{aligned}$$

(iii) Using the fact

$$\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}\int_{\mathbb{R}} e^{-|y|^{\frac{2}{n}}}d\gamma_{k,n}(y) = 1,$$

identity (5.21) gives the freedom to write

$$\left(S_{k,n}^{M^{-1}}(t)f\right)(x) - f(x) = a_t(x) + b_t(x) \tag{5.22}$$

where

$$a_t(x) = \frac{\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}}{4}\int_{\mathbb{R}} e^{-|v|^{\frac{2}{n}}}\left(e^{i\frac{a}{2b}\left(\frac{2\sigma t}{n}\right)^n|v|^2}-1\right)f(x)\,d\gamma_{k,n}(v), \tag{5.23}$$

$$b_t(x) = \frac{\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}}{4}\int_{\mathbb{R}} e^{i\frac{a}{2b}\left(\frac{2\sigma t}{n}\right)^n|v|^2}e^{-|v|^{\frac{2}{n}}}\left(\left[T_x^{M^{-1},k,n}f\right]\left((-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v\right)-f(x)\right)d\gamma_{k,n}(v). \tag{5.24}$$

Using the fact that

$$\left\|T^{M^{-1},k,n}_{(-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v}f - f\right\|_\infty \le 5\|f\|_\infty \quad\text{and}\quad \lim_{t\to 0}\left\|T^{M^{-1},k,n}_{(-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v}f - f\right\|_\infty = 0,$$

together as above with an application of the dominated convergence theorem, we get the desired result as

$$\|a_t\|_\infty \le \frac{\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}}{4}\left[\int_{\mathbb{R}} e^{-|v|^{\frac{2}{n}}}\left|e^{i\frac{\alpha}{2b}\left(\frac{2\sigma t}{n}\right)^n|v|^2}-1\right|d\gamma_{k,n}(v)\right]\|f\|_\infty \longrightarrow 0, \quad \text{as } t\to 0,$$

$$\|b_t\|_\infty \le \frac{\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}}{4}\int_{\mathbb{R}} e^{-|v|^{\frac{2}{n}}}\left\|T^{M^{-1},k,n}_{(-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v}f - f\right\|_\infty d\gamma_{k,n}(v) \longrightarrow 0, \quad \text{as } t\to 0.$$

Second step: (i) Assume that $X = L^p_{k,n}(\mathbb{R})$, $1 \leq p < \infty$. For any $f \in L^p_{k,n}(\mathbb{R})$, Young's inequality (4.2) yields

$$\left\| S^{M^{-1}}_{k,n}(t)f \right\|_{L^p_{k,n}(\mathbb{R})} = \frac{1}{4} \left\| \mathcal{P}^{M^{-1}}_t \underset{M^{-1},k,n}{\odot} f \right\|_{L^r_{k,n}(\mathbb{R})} \leq \left\| \mathcal{P}^{M^{-1}}_t \right\|_{L^1_{k,n}(\mathbb{R})} \|f\|_{L^p_{k,n}(\mathbb{R})}.$$

Since

$$\left\| \mathcal{P}^{M^{-1}}_t \right\|_{L^1_{k,n}(\mathbb{R})} = \frac{1}{(\sigma t)^{\frac{(2k-1)n+2}{2}}} \int_{\mathbb{R}} e^{-\frac{n|y|^{\frac{2}{n}}}{2\sigma t}} d\gamma_{k,n}(y) = 1.$$

Thus, we obtain

$$\left\| S^{M^{-1}}_{k,n}(t)f \right\|_{L^p_{k,n}(\mathbb{R})} \leq \|f\|_{L^p_{k,n}(\mathbb{R})}.$$

By taking supremum over all $f \in L^p_{k,n}(\mathbb{R})$ and noting that $\|f\|_{L^p_{k,n}(\mathbb{R})} \leq 1$, we obtain for each $t \geq 0$, $S^{M^{-1}}_{k,n}(t)$ is a bounded linear operator on $L^p_{k,n}(\mathbb{R})$ and $\|S^{M^{-1}}_{k,n}(t)\| \leq 1$.

(ii) Since $\mathcal{S}(\mathbb{R}) \subset C_0(\mathbb{R})$, we derive that

$$S^{M^{-1}}_{k,n}(s+t) = S^{M^{-1}}_{k,n}(s)\, S^{M^{-1}}_{k,n}(t) \quad \text{on } \mathcal{S}(\mathbb{R}).$$

On the other hand, $S^{M^{-1}}_{k,n}(s)$, $S^{M^{-1}}_{k,n}(t)$ and $S^{M^{-1}}_{k,n}(s+t)$ are continuous from $L^p_{k,n}(\mathbb{R})$ into itself. Therefore, the result follows immediately by the density of $\mathcal{S}(\mathbb{R})$ in $L^p_{k,n}(\mathbb{R})$.

(iii) Firstly, we show that if $f \in C_c(\mathbb{R})$, then

$$\lim_{t \to 0} \left\| S^{M^{-1}}_{k,n}(t)f - f \right\|_{L^p_{k,n}(\mathbb{R})} = 0. \tag{5.25}$$

By virtue of the relation (5.22), it follows that

$$\left\| S^{M^{-1}}_{k,n}(t)f - f \right\|_{L^p_{k,n}(\mathbb{R})} \leq \|a_t\|_{L^p_{k,n}(\mathbb{R})} + \|b_t\|_{L^p_{k,n}(\mathbb{R})},$$

with

$$\|a_t\|_{L^p_{k,n}(\mathbb{R})} \leq \frac{\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}}{4} \left[\int_{\mathbb{R}} e^{-|v|^{\frac{2}{n}}} \left| e^{i\frac{a}{2b}\left(\frac{2\sigma t}{n}\right)^n |v|^2} - 1 \right| d\gamma_{k,n}(v) \right] \|f\|_{L^p_{k,n}(\mathbb{R})} \longrightarrow 0, \quad \text{as } t \to 0,$$

whereas the Minkowski's inequality yields that

$$\|b_t\|_{L^p_{k,n}(\mathbb{R})} \leq \frac{\left(\frac{2}{n}\right)^{\frac{(2k-1)n+2}{2}}}{4} \int_{\mathbb{R}} e^{-|v|^{\frac{2}{n}}} \left\| T^{M^{-1},k,n}_{(-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}} v} f - f \right\|_{L^p_{k,n}(\mathbb{R})} d\gamma_{k,n}(v) \longrightarrow 0, \quad \text{as } t \to 0.$$

Implementation of the dominated convergence theorem implies that

$$\left\|T^{M^{-1},k,n}_{(-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v}f-f\right\|_{L^p_{k,n}(\mathbb{R})}\leq 5\,\|f\|_{L^p_{k,n}(\mathbb{R})}\quad(\text{by }(3.7)),$$

$$\lim_{t\to 0}\left\|T^{M^{-1},k,n}_{(-1)^n\left(\frac{2\sigma t}{n}\right)^{\frac{n}{2}}v}f-f\right\|_{L^p_{k,n}(\mathbb{R})}=0,\quad(\text{see Theorem 3.4}),$$

and $v\mapsto e^{-|v|^{\frac{2}{n}}}\in L^1_{k,n}(\mathbb{R})$.

Since $C_c(\mathbb{R})$ is dense in $L^p_{k,n}(\mathbb{R})$, therefore, for any $f\in L^p_{k,n}(\mathbb{R})$, there exists $g\in C_c(\mathbb{R})$ such that

$$\|f-g\|_{L^p_{k,n}(\mathbb{R})}\leq\frac{\epsilon}{3},$$

and

$$\begin{aligned}\left\|S^{M^{-1}}_{k,n}(t)f-f\right\|_{L^p_{k,n}(\mathbb{R})}&\leq\left\|S^{M^{-1}}_{k,n}(t)(f-g)\right\|_{L^p_{k,n}(\mathbb{R})}+\left\|S^{M^{-1}}_{k,n}(t)g-g\right\|_{L^p_{k,n}(\mathbb{R})}\\&\quad+\|f-g\|_{L^p_{k,n}(\mathbb{R})}\\&\leq 2\|f-g\|_{L^p_{k,n}(\mathbb{R})}+\left\|S^{M^{-1}}_{k,n}(t)g-g\right\|_{L^p_{k,n}(\mathbb{R})}\\&\leq\frac{2\epsilon}{3}+\left\|S^{M^{-1}}_{k,n}(t)g-g\right\|_{L^p_{k,n}(\mathbb{R})}.\end{aligned}$$

Further the relation (5.25) implies that, for sufficiently small values of t, we have

$$\left\|S^{M^{-1}}_{k,n}(t)g-g\right\|_{L^p_{k,n}(\mathbb{R})}\leq\frac{\epsilon}{3}.$$

Subsequently, we obtain

$$\lim_{t\to 0}\left\|S^{M^{-1}}_{k,n}(t)f-f\right\|_{L^p_{k,n}(\mathbb{R})}=0.$$

This completes the proof of Theorem 5.6. □

We close this section by the following statement for the semigroup $\left(S^{M^{-1}}_{k,n}(t),t\geq 0\right)$ acting on the Banach spaces $\mathfrak{B}=L^p_{k,n}(\mathbb{R})$ $(1\leq p<\infty)$ or $(C_0(\mathbb{R}),\|.\|_\infty)$.

Proposition 5.7. *The operator $\triangle^{M^{-1}}_{k,n}$ is closable and its closure generates the semigroup $\left(S^{M^{-1}}_{k,n}(t),t\geq 0\right)$ acting on the Banach spaces $\mathfrak{B}$.*

Proof. Let $f\in\mathfrak{S}_{k,n}(\mathbb{R})$. Involving the relations (5.20) and (5.15), we observe that

$$\mathcal{F}^M_{k,n}\left(\frac{S^{M^{-1}}_{k,n}(t)-Id}{t}f\right)(\lambda)=\frac{e^{-t\sigma\left|\frac{\lambda}{b}\right|^{\frac{2}{n}}}-1}{t}\mathcal{F}^M_{k,n}(f)(\lambda).$$

Thus, we derive that

$$\lim_{t\to 0} \mathcal{F}_{k,n}^{M} \left(\frac{S_{k,n}^{M^{-1}}(t) - Id}{t} f \right)(\lambda) = -\sigma \left| \frac{\lambda}{b} \right|^{\frac{2}{n}} \mathcal{F}_{k,n}^{M}(f)(\lambda) = \mathcal{F}_{k,n}^{M} \left(\sigma \triangle_{k,n}^{M^{-1}} f \right)(\lambda).$$

Using the injectivity of $\mathcal{F}_{k,n}^{M}$ on $\mathfrak{S}_{k,n}(\mathbb{R})$, we infer that the generator of the semigroup $\left(S_{k,n}^{M^{-1}}(t), t \geq 0\right)$, denoted by $\mathfrak{D}_{k,n}$, satisfies

$$\mathfrak{D}_{k,n} f = \lim_{t\to 0} \frac{S_{k,n}^{M^{-1}}(t) - Id}{t} f = \sigma \triangle_{k,n}^{M^{-1}} f.$$

As $\mathfrak{S}_{k,n}(\mathbb{R})$ is invariant under $\mathcal{F}_{k,n}$, we derive that $\mathfrak{S}_{k,n}(\mathbb{R})$ is invariant under $\left(S_{k,n}^{M^{-1}}(t), t \geq 0\right)$ which is a strongly continuous semigroup of contractions on $\mathfrak{B}$. So, we observe that $\mathfrak{S}_{k,n}(\mathbb{R})$ is subset of $\mathfrak{D}_{k,n}$. Moreover since $\mathfrak{S}_{k,n}(\mathbb{R})$ is dense in $\mathfrak{B}$, Then by [36, Corollary 1.2.2], it follows that $\mathfrak{S}_{k,n}(\mathbb{R})$ is a core for the generator $\mathfrak{D}_{k,n}$ and the desired result is proved. □

6 Potential applications and simulation perspectives

The theoretical framework developed in this article admits several potential applications in diverse areas of harmonic analysis, signal processing, and mathematical physics. Owing to the additional degrees of freedom offered by the parameters of the linear canonical deformed Hankel transform (LCDHT), the corresponding generalized translation and convolution operators introduced here extend the analytical and practical scope of existing transform methods.

6.1 Uncertainty principles

The LCDHT provides a natural platform for establishing new variants of classical uncertainty relations, including the Heisenberg, Donoho–Stark, and Hardy-type inequalities. By incorporating linear canonical and deformed Hankel parameters, the LCDHT allows sharper localization bounds in both the time and transform domains. Such results are expected to find applications in quantum mechanics, optical tomography, and time–frequency localization theory, where precise phase–space characterizations are essential.

6.2 Signal reconstruction

The generalized translation and convolution structures developed in this work constitute the foundation for signal reconstruction and sampling theorems in the LCDHT domain. These results facilitate the recovery of signals that are bandlimited with respect to the LCDHT rather than the classical Fourier transform, offering significant advantages in nonuniform sampling, filter design, and inverse problems. Potential applications include optical field recovery, radar and sonar imaging, seismic data interpretation, and medical image reconstruction, where signals often exhibit Hankel-type or radial symmetries.

6.3 Simulation and error analysis perspectives

Although the present work is primarily theoretical, the proposed framework can be extended toward numerical validation and simulation studies. A theoretical error analysis may focus on the stability and convergence of the generalized translation and convolution operators under discretization or kernel truncation. Synthetic test signals, such as Gaussian–Bessel or chirp-type functions, may be used to verify reconstruction accuracy and energy preservation. Quantitative measures like mean square error (MSE) and signal-to-noise ratio (SNR) would help assess computational fidelity. Such experiments would not only corroborate the analytical findings but also demonstrate the robustness and applicability of the LCDHT in signal reconstruction and time–frequency localization problems.

7 Conclusion and future work

In this paper, we have investigated the generalized translation and convolution operators within the framework of the linear canonical deformed Hankel transform (LCDHT). Although the results presented here are primarily theoretical, they have been effectively applied to the analysis of the generalized heat equation and the associated heat semigroup. It is pertinent to mention that the proposed transform not only unifies several existing integral transforms such as the classical and fractional Fourier transforms, as well as the linear canonical transform in the Dunkl and Hankel settings but also leads to the formulation of new integral transforms, including the fractional (k, n)-generalized Fourier transform and the generalized Fresnel transform. Furthermore, building upon the harmonic analysis developed in the earlier sections, we have explored the Gabor, wavelet, Wigner, and wavelet multiplier transforms in the context of the LCDHT framework [18]. For future research, we plan to extend this work by investigating additional applications in time-frequency analysis and by developing the reproducing kernel theory associated with the LCDHT. These directions are expected to further enrich the theoretical foundations and broaden the applicability of this new class of integral transforms.

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Data Availability

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Conflict of Interest

The authors declare that they have no conflicts of interest.

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Inertial viscosity Mann-type subgradient extragradient algorithms for solving variational inequality and fixed point problems in real Hilbert spaces

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ABSTRACT

This paper presents two inertial viscosity Mann-type extrapolated algorithms for finding a common solution to the variational inequality problem involving a monotone and Lipschitz continuous operator and the fixed-point problem for a demicontractive mapping in real Hilbert spaces. The proposed algorithms feature an adaptive step size strategy, computed iteratively, which circumvents the need for prior knowledge of the operator's Lipschitz constant. Under appropriate assumptions, we establish two strong convergence theorems guaranteeing the robustness of the methods. Furthermore, we provide a comparative performance analysis of the proposed algorithms against some existing strongly convergent schemes, supported by numerical experiments with MATLAB-based graphical illustrations.

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RESUMEN

Este artículo presenta dos algoritmos extrapolados de tipo Mann con viscosidad inercial para encontrar una solución común al problema de desigualdad variacional que involucra un operador continuo, monótono y Lipschitz y al problema de punto fijo para una aplicación semicontractiva en espacios de Hilbert reales. Los algoritmos propuestos presentan una estrategia de tamaño de paso adaptativo, calculado iterativamente, que evita la necesidad del conocimiento previo de la constante de Lipschitz del operador. Bajo hipótesis apropiadas, establecemos dos teoremas de convergencia fuertes que garantizan la robustez de los métodos. Más aún, entregamos un análisis comparativo del desempeño de los algoritmos propuestos contra algunos esquemas existentes fuertemente convergentes, sobre la base de experimentos numéricos con ilustraciones gráficas basadas en MATLAB.

Keywords and Phrases: Subgradient extragradient method, extragradient method, Mann-like method, inertial method, viscosity method.

2020 AMS Mathematics Subject Classification: 47H05, 47H09, 49J15, 47J20, 65K15.

1 Introduction

Consider a real Hilbert space $\mathcal{D}$ equipped with the inner product $\langle ., . \rangle$, and the corresponding norm $\|.\|$, and $\varnothing \neq E$ be a closed, convex subset of $\mathcal{D}$. This study is devoted to the pursuit of a common solution to problems involving variational inequalities and fixed point theory within the framework of real Hilbert spaces. The impetus for this investigation arises from the significant role these problems play in numerous mathematical models, where constraints are naturally formulated as variational inequalities and/or fixed point conditions. This situation occurs especially in practical problems, such as signal processing, composite minimization problems, optimal control problems, and image restoration. The relevance and applicability of this framework have been well-established in prior works [3, 17, 20, 23, 32]. Let us recall the involved problems.

The variational inequality problem associated with the operator $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ over the set E seeks to determine a point $v \in E$ such that the following condition is satisfied:

$$\langle \mathcal{F}v, s - v \rangle \geq 0, \quad \forall s \in E. \tag{VIP}$$

The solution set of the (VIP) is denoted by $VI(E, \mathcal{F})$. Variational inequality problems provide a useful and indispensable tool for investigating various interesting issues emerging in many areas, such as social, physics, engineering, economics, network analysis, medical imaging, inverse problems, transportation and much more; see, *e.g.*, [4, 12, 23]. Variational inequality theory has been proven to provide a simple, universal, and consistent structure to deal with possible problems. In the past few decades, researchers have shown tremendous interest in exploring different extensions of variational inequality problems. Recent advancements, as evidenced in works such as [1, 10, 24, 28, 29] underscore a growing emphasis on the development of efficient and practically implementable numerical algorithms for addressing variational inequalities. Under fairly general conditions, two prominent strategies have emerged for solving monotone variational inequalities: projection-type methods and regularization-based approaches. In this study, we concentrate on projection-type methods, with particular attention to the projection gradient method, arguably the most straightforward among them for solving (VIP) given as:

$$s_{n+1} = \mathcal{P}_E(s_n - \eta \mathcal{F} s_n),$$

where $\mathcal{P}_E$, denotes the metric projection onto the set E and $\eta > 0$ is an appropriately chosen step size.

It is worth emphasizing that the projected gradient method necessitates only a single projection onto the feasible set per iteration, making it computationally appealing. However, its convergence typically hinges on relatively strong assumptions, most notably, that the underlying operator is either strongly monotone or inverse strongly monotone. To relax these stringent conditions, Korpelevich [15] proposed the extragradient method, originally designed to solve saddle point problems in Euclidean spaces. The method introduces an additional intermediate step to enhance convergence properties under weaker assumptions. The iterative scheme of the extragradient method is given by:

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta \mathcal{F} s_n), \\ s_{n+1} = \mathcal{P}_E(s_n - \eta \mathcal{F} t_n), \end{cases} \tag{1.1}$$

where operator $\mathcal{F}$ is assumed to be monotone and $\mathcal{L}$-Lipschitz continuous, $\mathcal{P}_E$ represents the metric projection from $\mathcal{D}$ onto E, and $\eta \in (0, 1/\mathcal{L})$. It is established that the sequence $\{s_n\}$ produced by the process (1.1) converges to an element in $VI(E, \mathcal{F})$.

It is essential to recognize that solving the shortest distance problem is equivalent to computing the metric projection onto a closed convex set E. As previously noted, the extragradient method involves two projections onto E in each iteration. While effective, this requirement can pose significant computational challenges, particularly when E is a general closed and convex set with a complex structure. To mitigate this issue, Censor *et al.* [9] introduced the subgradient extragradient method as a refinement of the original extragradient algorithm. The key innovation in this approach lies in replacing the second projection onto E with a projection onto a carefully constructed half-space. This modification is advantageous because projecting onto a half-space is computationally explicit and significantly simpler. The modified algorithm is formulated as follows:

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta \mathcal{F} s_n), \\ T_n = \{s \in \mathcal{D} | \langle s_n - \eta \mathcal{F} s_n - t_n, s - t_n \rangle \leq 0\}, \\ s_{n+1} = \mathcal{P}_{T_n}(s_n - \eta \mathcal{F} t_n), \end{cases} \tag{1.2}$$

The sequence $\{s_n\}$ produced by (1.2) converges weakly to a solution of the variational inequality in this case where $VI(E, \mathcal{F}) \neq \varnothing$.

On the other hand, the fixed point problem plays a pivotal role in the theory and solution of variational inequalities. Let $\mathcal{S} : E \to E$ be a nonlinear mapping. A point $s \in \mathcal{D}$ is called a fixed point of the mapping $\mathcal{S}$ if it satisfies the condition $\mathcal{S}s = s$. The set of all fixed points of $\mathcal{S}$ is denoted as $Fix(\mathcal{S})$. The fixed point problem is formulated as follows:

$$\text{find } v \in E \text{ such that } \mathcal{S}v = v. \tag{FPP}$$

The principal objective of this paper is to determine a common solution to both the (VIP) and the (FPP). Specifically, the goal is to find a point v such that

$$v \in VI(E, \mathcal{F}) \cap Fix(\mathcal{S}). \tag{VIFPP}$$

A wide range of numerical algorithms have been developed to tackle the combined variational inequality and fixed point problem (VIFPP) in infinite-dimensional spaces as documented in [6, 7, 11, 36], and the references therein. Notably, Takahashi and Toyoda [26] proposed an iterative scheme for approximating a solution to the (VIFPP) which is described as follows:

$$s_{n+1} = (1-\zeta_n)s_n + \zeta_n \mathcal{S}\mathcal{P}_E(s_n - \eta_n \mathcal{F}s_n), \tag{1.3}$$

where $\mathcal{F} : E \to \mathcal{D}$ is μ-inverse strongly monotone, $\mathcal{S} : E \to E$ is nonexpansive, $\zeta_n \in (0,1)$ is a control sequence, $\eta_n > 0$ is a stepsize parameter, $\mathcal{P}_E$ denotes the metric projection onto the convex set E. They proved $\{s_n\}$ generated by (1.3) converges weakly to a solution of (VIFPP) under certain conditions. More recently, Censor *et al.* [8] established the following iterative scheme and proved its weak convergence to the solution of the (VIFPP),

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta\mathcal{F}s_n), \\ T_n = \{s \in \mathcal{D} | \langle s_n - \eta\mathcal{F}s_n - t_n, s - t_n\rangle \leq 0\}, \\ s_{n+1} = \zeta_n s_n + (1-\zeta_n)\mathcal{S}\mathcal{P}_{T_n}(s_n - \eta\mathcal{F}t_n). \end{cases} \tag{1.4}$$

In the context of infinite-dimensional Hilbert spaces, strong (norm) convergence is generally more desirable than weak convergence, particularly for practical applications. To ensure strong convergence when solving the combined (VIFPP), Kraikaew and Saejung [16] introduced the Halpern Subgradient Extragradient Method (HSEGM). This method integrates the Halpern iteration scheme with the subgradient extragradient framework, providing a robust approach for approximating common solutions to variational inequality and fixed point problems, which is described as:

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta\mathcal{F}s_n), \\ T_n = \{s \in \mathcal{D} | \langle s_n - \eta\mathcal{F}s_n - t_n, s - t_n\rangle \leq 0\}, \\ u_n = \zeta_n s_0 + (1-\zeta_n)\mathcal{P}_{T_n}(s_n - \eta\mathcal{F}t_n), \\ s_{n+1} = \tau_n s_n + (1-\tau_n)\mathcal{S}u_n, \end{cases} \tag{HSEGM}$$

They proved that the sequence $\{s_n\}$ generated by the (HSEGM) converges strongly to $\mathcal{P}_{VI\cap Fix(\mathcal{S})}(s_0)$, the metric projection of the initial point s_0 onto the set of common solutions of the variational inequality and fixed point problems.

Recently, Thong and Hieu [34] proposed the Modified Subgradient Extragradient Method (MSEGM) by integrating the subgradient extragradient technique with the Mann-type iteration scheme. The primary objective of this algorithm is to identify common solution elements belonging to both the solution set of the variational inequality problem (VIP) and the fixed point set of a demicontractive mapping. The algorithm is formally outlined as follows:

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta \mathcal{F} s_n), \\ T_n = \{s \in \mathcal{D} | \langle s_n - \eta \mathcal{F} s_n - t_n, s - t_n \rangle \leq 0\}, \\ u_n = \mathcal{P}_{T_n}(s_n - \eta \mathcal{F} t_n), \\ s_{n+1} = (1 - \zeta_n - \tau_n) u_n + \tau_n \mathcal{S} u_n, \end{cases} \tag{MSEGM}$$

They proved its strong convergence to an element $v \in VI(E, \mathcal{F}) \cap Fix(\mathcal{S})$, where $\|v\| = min\{\|u\| : u \in VI(E, \mathcal{F}) \cap Fix(\mathcal{S})\}$.

A notable limitation of both the (HSEGM) and (MSEGM) algorithms is their reliance on prior knowledge of the Lipschitz constant of the mapping $\mathcal{F}$. However, in many practical situations, this information is either unavailable or difficult to estimate accurately. To address this issue, Thong and Hieu [35] proposed two extragradient-viscosity algorithms, designed to solve the combined (VIFPP) without requiring the Lipschitz constant. Their approach incorporates an adaptive step-size rule, allowing automatic updates at each iteration. The algorithms are formulated as follows:

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta_n \mathcal{F} s_n), \\ T_n = \{s \in \mathcal{D} | \langle s_n - \eta_n \mathcal{F} s_n - t_n, s - t_n \rangle \leq 0\}, \\ u_n = \mathcal{P}_{T_n}(s_n - \eta_n \mathcal{F} t_n), \\ s_{n+1} = \zeta_n \mathcal{J}(s_n) + (1 - \zeta_n)[(1 - \tau_n) u_n + \tau_n \mathcal{S} u_n], \end{cases} \tag{VSEGM}$$

and

$$\begin{cases} t_n = \mathcal{P}_E(s_n - \eta_n \mathcal{F} s_n), \\ u_n = t_n - \eta_n(\mathcal{F} t_n - \mathcal{F} s_n), \\ s_{n+1} = \zeta_n \mathcal{J}(s_n) + (1 - \zeta_n)[(1 - \tau_n) u_n + \tau_n \mathcal{S} u_n], \end{cases} \tag{VTEGM}$$

where algorithms (VSEGM) and (VTEGM) update the step size $\{\eta_n\}$ by the following rule:

$$\eta_{n+1} = \begin{cases} \min\left\{\dfrac{\nu \|s_n - t_n\|}{\|\mathcal{F} s_n - \mathcal{F} t_n\|}, \eta_n\right\}, & \text{if } \mathcal{F} s_n - \mathcal{F} t_n \neq 0 \\ \eta_n, & \text{otherwise,} \end{cases}$$

The sequences produced by (VTEGM) and (VTEGM) converges strongly under mild assumptions to $q \in Fix(\mathcal{S}) \cap VI(E, \mathcal{F})$, where $q = \mathcal{P}_{Fix(\mathcal{S}) \cap VI(E, \mathcal{F})}(\mathcal{J}(q))$.

In recent years, fast iterative algorithms have attracted considerable interest, especially those employing inertial techniques inspired by discrete analogues of second-order dissipative dynamical systems [2, 19]. These inertial methods accelerate convergence by incorporating momentum-like terms into the iterative process. Leveraging this framework, Tan *et al.* [33] proposed the following inertial algorithm for solving the combined variational inequality and fixed point problem (VIFPP)

$$\begin{cases} w_n = s_n + \mathcal{K}_n(s_n - s_{n-1}), \\ t_n = \mathcal{P}_E(w_n - \eta_n \mathcal{F} w_n), \\ T_n = \{s \in \mathcal{D} | \langle w_n - \eta_n \mathcal{F} w_n - t_n, s - t_n \rangle \leq 0\}, \\ u_n = \mathcal{P}_{T_n}(w_n - \eta_n \mathcal{F} t_n), \\ s_{n+1} = \zeta_n \mathcal{J}(s_n) + (1 - \zeta_n)[(1 - \tau_n) u_n + \tau_n \mathcal{S} u_n], \end{cases} \tag{IVSEGM}$$

where the step size $\{\mathcal{K}_n\}$ and $\{\eta_n\}$ are updated by the following rules:

$$\mathcal{K}_n = \begin{cases} \min\left\{ \dfrac{\delta_n}{\|s_n - s_{n-1}\|}, \mathcal{K} \right\}, & \text{if } s_n \neq s_{n-1}, \\ \mathcal{K}, & \text{otherwise,} \end{cases}$$

and

$$\eta_{n+1} = \begin{cases} \min\left\{ \dfrac{\nu \|s_n - t_n\|}{\|\mathcal{F} s_n - \mathcal{F} t_n\|}, \eta_n \right\}, & \text{if } \mathcal{F} s_n - \mathcal{F} t_n \neq 0, \\ \eta_n, & \text{otherwise.} \end{cases}$$

Recently, Mewomo *et al.* [18] integrated the inertial, viscosity, and Tseng's approaches and introduced two Generalized Viscosity Inertial Tseng Methods (GVITMs) for solving pseudomonotone variational inequalities with fixed point constraints, formulated as follows:

$$\begin{cases} w_n = s_n + \delta_n(s_n - s_{n-1}), \\ t_n = P_C(w_n - \gamma_n \mathcal{F} w_n), \\ z_n = t_n - \gamma_n(\mathcal{F} t_n - \mathcal{F} w_n), \\ u_n = \beta_{n,0} z_n + \sum_{i=1}^m \beta_{n,i} v_{n,i}, \quad v_{n,i} \in S_i z_n, \\ s_{n+1} = \alpha_n \gamma \mathcal{J}(w_n) + (I - \alpha_n G) u_n, \end{cases} \tag{GVITM$_\mathrm{I}$}$$

where δ_n and γ_n are updated by (1.5) and (1.6), respectively.

$$\delta_n = \begin{cases} \min\left\{ \dfrac{\epsilon_n}{\|s_n - s_{n-1}\|}, \delta \right\}, & \text{if } s_n \neq s_{n-1}, \\ \delta, & \text{otherwise,} \end{cases} \tag{1.5}$$

$$\gamma_{n+1} = \begin{cases} \min\left\{ \dfrac{\phi \|w_n - t_n\|}{\|\mathcal{F} w_n - \mathcal{F} t_n\|}, \gamma_n + \phi_n \right\}, & \text{if } \mathcal{F} w_n - \mathcal{F} t_n \neq 0, \\ \gamma_n + \phi_n, & \text{otherwise,} \end{cases} \tag{1.6}$$

and

$$\begin{cases} w_n = s_n + \delta_n(s_n - s_{n-1}), \\ t_n = P_C(w_n - \gamma_n \nabla\psi w_n), \\ z_n = t_n - \gamma_n(\nabla\psi t_n - \nabla\psi w_n), \\ u_n = \beta_{n,0} z_n + \sum_{i=1}^{m} \beta_{n,i} v_{n,i}, \quad v_{n,i} \in S_i z_n, \\ s_{n+1} = \alpha_n \gamma \mathcal{J}(w_n) + (I - \alpha_n G) u_n, \end{cases} \tag{GVITM$_{\rm II}$}$$

where δ_n and γ_n are updated by (1.5) and (1.7), respectively.

$$\gamma_{n+1} = \begin{cases} \min\left\{ \dfrac{\phi\|w_n - t_n\|}{\|\nabla\psi w_n - \nabla\psi t_n\|}, \gamma_n + \phi_n \right\}, & \text{if } \nabla\psi w_n - \nabla\psi y_n \neq 0, \\ \gamma_n + \phi_n, & \text{otherwise.} \end{cases} \tag{1.7}$$

where $\delta > 0$, $\gamma_1 > 0$, ϕ_n is a nonnegative sequence such that $\sum_{n=1}^{\infty} \phi_n < +\infty$, and $\phi \in (0,1)$. The authors established strong convergence results for the sequences generated by (GVITM$_{\rm I}$) and (GVITM$_{\rm II}$) without imposing the sequential weak continuity of the pseudomonotone operator and without requiring prior knowledge of the Lipschitz constants.

Recently, Kesornprom *et al.* [14] proposed a new variant of the proximal gradient algorithm incorporating double inertial extrapolation for solving constrained convex minimization problems in Hilbert spaces, formulated as follows:

$$\begin{aligned} z^n &= s^n + \theta_n(s^n - s^{n-1}) + \eta_n(s^{n-1} - s^{n-2}), \quad n \geq 1, \\ s^{n+1} &= P_E(prox_{\alpha_n g}(z^n - \alpha_n \nabla f(z^n))), \end{aligned}$$

where

$$\alpha_{n+1} = \begin{cases} \min\left\{ \dfrac{\delta\|z^n - prox_{\alpha_n g}(z^n - \alpha_n \nabla f(z^n))\|}{\|\nabla f(z^n) - \nabla f(prox_{\alpha_n g}(z^n - \alpha_n \nabla f(z^n)))\|}, \alpha_n \right\}, \\ \qquad\qquad\qquad\qquad \text{if } \nabla f(z^n) - \nabla f(prox_{\alpha_n g}(z^n - \alpha_n \nabla f(z^n))) \neq 0, \\ \alpha_n, \qquad\qquad\qquad\quad \text{otherwise.} \end{cases}$$

where $\theta_n \geq 0$, $\eta_n \geq 0$, $\alpha_1 > 0$ and $\delta \in (0, \frac{1}{2})$. They established the weak convergence of the proposed method to a point in $\operatorname{argmin}(f+g) \cap E$. For an extensive discussion on fast iterative algorithms and their recent advancements, the reader may consult [21,25,31,38,39] and the references therein.

Motivated and inspired by existing studies in this area, the purpose of this paper is to develop two inertial extragradient algorithms that combine the Mann iteration, viscosity approximation, and subgradient extragradient methods with a new step size for discovering a common solution of a monotone and Lipschitz variational inequality problem and of the fixed point problem involving a demicontractive mapping in real Hilbert spaces. The suggested algorithms need to calculate the projection on the feasible set only once per iteration, which makes them faster. Strong convergence theorems of the algorithms are established without the prior information of the Lipschitz constant of the operator. Lastly, some computational tests appearing in finite and infinite dimensions are proposed to support the theoretical results.

The organizational structure of our paper is built up as follows. In Section 2, we recall some preliminary results and lemmas that need to be used in the next section. In Section 3, we propose the algorithms and analyse their convergence. Some numerical experiments to verify our theoretical results are presented in Section 4. At last, the paper ends with a brief summary in Section 5, the final section.

2 Preliminaries

Consider $\varnothing \neq E$ (closed, convex) subset of a real Hilbert space $\mathcal{D}$. The weak convergence and strong convergence of the sequence $\{s_n\}$ to s are denoted as $s_n \rightharpoonup s$ and $s_n \to s$, respectively. For any $s, t \in \mathcal{D}$ and $\zeta \in R$ the following statements hold:

(i) $\|s+t\|^2 = \|s\|^2 + 2\langle s, t\rangle + \|t\|^2$.

(ii) $\|s+t\|^2 \leq \|s\|^2 + 2\langle t, s+t\rangle$.

(iii) $\|\zeta s + (1-\zeta)t\|^2 = \zeta\|s\|^2 + (1-\zeta)\|t\|^2 - \zeta(1-\zeta)\|s-t\|^2$.

For any point $s \in \mathcal{D}$, there exists a distinct nearest point in the closed and convex subset E identified as $\mathcal{P}_E(s)$ satisfying $\mathcal{P}_E(s) = \operatorname{argmin}\{\|s-t\|, t \in E\}$. $\mathcal{P}_E$ is termed as the metric projection of $\mathcal{D}$ onto E. It is established that $\mathcal{P}_E$ is a nonexpansive mapping and it possesses the following fundamental properties:

(i) $\langle s - \mathcal{P}_E(s), t - \mathcal{P}_E(s)\rangle \leq 0$, $\forall t \in E$.

(ii) $\|\mathcal{P}_E(s) - \mathcal{P}_E(t)\|^2 \leq \langle \mathcal{P}_E(s) - \mathcal{P}_E(t), s - t\rangle$, $\forall t \in \mathcal{D}$.

Definition 2.1 ([27]). *A mapping $\mathcal{A} : \mathcal{D} \to \mathcal{D}$ is said to be:*

(i) $\mathcal{L}$-Lipschitz continuous with $\mathcal{L} > 0$ if

$$\|\mathcal{A}s - \mathcal{A}t\| \leq \mathcal{L}\|s-t\|, \quad \forall s, t \in \mathcal{D}.$$

(ii) ζ-strongly monotone if there exists $\zeta > 0$ such that

$$\langle \mathcal{A}s - \mathcal{A}t, s-t\rangle \geq \zeta\|s-t\|^2, \quad \forall s, t \in \mathcal{D}.$$

(iii) *ζ-inverse strongly monotone if there exists $\zeta > 0$ such that*

$$\langle \mathcal{A}s - \mathcal{A}t, s - t\rangle \geq \zeta\|\mathcal{A}s - \mathcal{A}t\|^2, \quad \forall s, t \in \mathcal{D}.$$

Remark 2.2 ([5]). *if $\mathcal{A} : \mathcal{D} \to \mathcal{D}$ be an injective operator so that $\mathcal{A}^{-1}$ is well defined, then:*

(a) If $\mathcal{A}$ is ζ-strongly monotone, then its inverse $\mathcal{A}^{-1}$ is ζ-inverse strongly monotone.

(b) If $\mathcal{A}$ is ζ-inverse strongly monotone, then its inverse $\mathcal{A}^{-1}$ is ζ-strongly monotone.

(iv) *monotone if*

$$\langle \mathcal{A}s - \mathcal{A}t, s - t\rangle \geq 0, \quad \forall s, t \in \mathcal{D}.$$

(v) *quasi-nonexpansive if*

$$\|\mathcal{A}s - u\| \leq \|s - t\|, \quad \forall u \in Fix(\mathcal{A}), \quad s \in \mathcal{D}.$$

(vi) *μ-strictly pseudocontractive with $0 \leq \mu < 1$ if*

$$\|\mathcal{A}s - \mathcal{A}t\|^2 \leq \|s - t\|^2 + \mu\|(I - \mathcal{A})s - (I - \mathcal{A})t\|^2, \quad \forall s, t \in \mathcal{D}.$$

(vii) *τ-demicontractive with $0 \leq \tau < 1$ if*

$$\|\mathcal{A}s - u\|^2 \leq \|s - u\|^2 + \tau\|(I - \mathcal{A})s\|^2, \quad \forall u \in Fix(\mathcal{A}), \quad s \in \mathcal{D}. \tag{2.1}$$

or equivalently

$$\langle \mathcal{A}s - s, s - u\rangle \leq \frac{\tau - 1}{2}\|s - \mathcal{A}s\|^2, \quad \forall u \in Fix(\mathcal{A}), \quad s \in \mathcal{D}. \tag{2.2}$$

Definition 2.3 ([37]). *If $\mathcal{A} : \mathcal{D} \to \mathcal{D}$ is a nonlinear operator with $Fix(\mathcal{A}) \neq \varnothing$. Then, $I - \mathcal{A}$ is said to be demiclosed at zero if for any $\{s_n\}$ in $\mathcal{D}$, the following implications holds:*

$$s_n \rightharpoonup s \text{ and } (I - \mathcal{A})s_n \to 0 \implies s \in Fix(\mathcal{A}).$$

Lemma 2.4 ([33]). *Consider $\mathcal{S} : \mathcal{D} \to \mathcal{D}$ as a τ-demicontractive operator with $Fix(\mathcal{S}) \neq \varnothing$. Let $\mathcal{S}_\mu = (1 - \mu)I + \mu\mathcal{S}$, where $\mu \in (0, 1 - \tau)$. Then:*

(i) *$Fix(\mathcal{S}) = Fix(\mathcal{S}_\mu)$.*

(ii) *$\|\mathcal{S}_\mu s - u\|^2 \leq \|s - u\|^2 - \mu(1 - \tau - \mu)\|(I - \mathcal{S})s\|^2$, $\forall s \in \mathcal{D}$, $u \in Fix(\mathcal{S})$.*

(iii) *$Fix(\mathcal{S})$ is a closed convex subset of $\mathcal{D}$.*

Lemma 2.5 ([16]). *Consider $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ as a monotone and $\mathcal{L}$-Lipschitz continuous mapping on E. Let $S = \mathcal{P}_E(I - \nu\mathcal{F})$, where $\nu > 0$. If s_n is a sequence in $\mathcal{D}$ such that $s_n \rightharpoonup q$ and $s_n - Ss_n \to 0$, then it follows that $q \in VI(E, \mathcal{F}) = Fix(S)$.*

Lemma 2.6 ([22]). *Consider a positive sequence $\{r_n\}$, a sequence of real numbers $\{b_n\}$ and a sequence $\{a_n\}$ in the interval $(0,1)$ such that $\sum_{n=1}^{\infty} a_n = \infty$. Assuming*

$$r_{n+1} \leq a_n b_n + (1 - a_n) r_n, \quad \forall n \geq 1$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{r_{n_k}\}$ of $\{r_n\}$ satisfying $\liminf_{k\to\infty}(r_{n_k+1} - r_{n_k}) \geq 0$, then $\lim_{n\to\infty} r_n = 0$.

3 Main result

This section presents two inertial extragradient algorithms that are specifically designed to solve (VIFPP), and provides a convergence analysis of them. We first assume that the following conditions are met by the suggested algorithms.

(A1) $Fix(\mathcal{S}) \cap VI(E, \mathcal{F}) \neq \varnothing$.

(A2) $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is monotone and $\mathcal{L}$-Lipschitz continuous.

(A3) $\mathcal{S} : \mathcal{D} \to \mathcal{D}$ is μ-demicontractive such that $(I - \mathcal{S})$ is demiclosed at zero.

(A4) $\mathcal{J} : \mathcal{D} \to \mathcal{D}$ is $\mathcal{Q}$-contraction with constant $\mathcal{Q} \in [0,1)$.

3.1 Algorithm-I

Algorithm 3.1 Algorithm-I

Initialization: Choose $\mathcal{K} > 0$, $\eta_1 > 0$, and $\nu \in (0,1)$.
Select arbitrary s_0 and s_1 from $\mathcal{D}$.
Iterative step:
Step 1. Given the iterates s_{n-1} and s_n (for $n \geq 1$), set

$$w_n = s_n + \mathcal{K}_n(s_n - s_{n-1}),$$

where

$$\mathcal{K}_n = \begin{cases} \min\left\{\dfrac{\delta_n}{\|s_n - s_{n-1}\|}, \mathcal{K}\right\}, & \text{if } s_n \neq s_{n-1}; \\ \mathcal{K}, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute

$$t_n = \mathcal{P}_E(w_n - \eta_n \mathcal{F} w_n).$$

Algorithm 3.1 Algorithm-I

Step 3. Compute

$$u_n = \mathcal{P}_{T_n}(w_n - \eta_n \mathcal{F} t_n),$$

where the half-space T_n is defined by

$$T_n := \{s \in \mathcal{D} \mid \langle w_n - \eta_n \mathcal{F} w_n - t_n, s - t_n\rangle \leq 0\}.$$

Step 4. Compute

$$s_{n+1} = \zeta_n \mathcal{J}(s_n) + (1 - \zeta_n)\big[(1 - \gamma_n - \tau_n)u_n + \tau_n \mathcal{S} u_n\big]$$

and update

$$\eta_{n+1} = \begin{cases} \min\left\{\dfrac{\nu\|w_n - t_n\|}{\|\mathcal{F}w_n - \mathcal{F}t_n\|}, \eta_n\right\}, & \text{if } \mathcal{F}w_n - \mathcal{F}t_n \neq 0; \\ \eta_n, & \text{otherwise.} \end{cases} \tag{3.2}$$

Set $n := n + 1$ and go to **Step 1**.

The subsequent lemmas prove to be valuable for analyzing the convergence of the algorithm.

Lemma 3.1 ([33]). *The sequence $\{\eta_n\}$ produced by (3.2) is a nonincreasing sequence and*

$$\lim_{n\to\infty} \eta_n = \eta \geq \min\left\{\eta_1, \frac{\nu}{\mathcal{L}}\right\}.$$

Lemma 3.2 ([30]). *Assume that condition* **(A2)** *holds. Let $\{u_n\}$ be a sequence produced by Algorithm 3.1, then*

$$\|u_n - v\|^2 \leq \|w_n - v\|^2 - \left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|t_n - w_n\|^2 - \left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|u_n - t_n\|^2 \tag{3.3}$$

for all $v \in VI(E, \mathcal{F})$.

Theorem 3.3. *Under the fulfillment of Conditions* **(A1)-(A4)**, *$\{\delta_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\delta_n}{\zeta_n} = 0$, where $\zeta_n \subset (0,1)$ satisfies $\sum_{n=1}^{\infty} \zeta_n = \infty$ and $\lim_{n\to\infty} \zeta_n = 0$. Furthermore, for some $a > 0$, $b > 0$, $\gamma_n \in (0,1)$, $\lim_{n\to\infty} \gamma_n = 0$, and $\sum_{n=0}^{\infty} \gamma_n = \infty$, let $\tau_n \in (a, b) \subset (0, (1-\mu)(1-\gamma_n))$, then the sequence $\{s_n\}$ produced by Algorithm 3.1 converges in norm to $v \in Fix(\mathcal{S}) \cap VI(E, \mathcal{F})$, where $v = \mathcal{P}_{Fix(\mathcal{S})\cap VI(E,\mathcal{F})}(\mathcal{J}(v))$.*

Proof. Since $VI(E, \mathcal{F})$ is a closed convex subset, and by Lemma 2.4, $Fix(\mathcal{S})$ is also a closed convex subset. Therefore, the mapping $\mathcal{P}_{Fix(\mathcal{S})\cap VI(E,\mathcal{F})}(\mathcal{J}) : \mathcal{D} \to \mathcal{D}$ forms a contraction. By applying the Banach contraction principle, there exists a unique point $v \in \mathcal{D}$ such that $v = \mathcal{P}_{Fix(\mathcal{S})\cap VI(E,\mathcal{F})}(\mathcal{J})$. Specifically, $v \in Fix(\mathcal{S}) \cap VI(E, \mathcal{F})$, and

$$\langle \mathcal{J}(v) - v, u - v\rangle \leq 0, \quad \forall u \in Fix(\mathcal{S}) \cap VI(E, \mathcal{F}).$$

The proof is split up into four sections.

Claim 1. $\{s_n\}$ is a bounded sequence. Put, $t_n = (1-\gamma_n-\tau_n)u_n + \tau_n \mathcal{S}u_n$, we have

$$\begin{aligned}\|t_n - v\| &= \|(1-\gamma_n-\tau_n)u_n + \tau_n \mathcal{S}u_n - v\| \\ &= \|(1-\gamma_n-\tau_n)(u_n - v) + \tau_n(\mathcal{S}u_n - v) - \gamma_n v\| \\ &= \|(1-\gamma_n-\tau_n)(u_n - v) + \tau_n(\mathcal{S}u_n - v)\| + \|\gamma_n v\|. \end{aligned} \tag{3.4}$$

Additionally, it can be deduced from (2.1), (2.2), and Lemma 3.2 that

$$\begin{aligned}\|(1-\gamma_n-\tau_n)(u_n - v) + \tau_n(\mathcal{S}u_n - v)\|^2 &= (1-\gamma_n-\tau_n)^2\|(u_n - v)\|^2 \\ &\quad + 2(1-\gamma_n-\tau_n)\tau_n\langle \mathcal{S}u_n - v, u_n - v\rangle + \tau_n^2\|\mathcal{S}u_n - v\|^2 \\ &\leq (1-\gamma_n-\tau_n)^2\|(u_n - v)\|^2 \\ &\quad + 2(1-\gamma_n-\tau_n)\tau_n\left[\|u_n - v\|^2 - \frac{1-\mu}{2}\|u_n - \mathcal{S}u_n\|^2\right] \\ &\quad + \tau_n^2\left[\|u_n - v\|^2 + \mu\|u_n - \mathcal{S}u_n\|^2\right] \\ &= (1-\gamma_n)^2\|u_n - v\|^2 + \tau_n\left(\tau_n - (1-\gamma_n)(1-\mu)\right)\|u_n - \mathcal{S}u_n\|^2 \\ &\leq (1-\gamma_n)^2\|u_n - v\|^2 \leq (1-\gamma_n)^2\|w_n - v\|^2\end{aligned}$$

signifying that

$$\|(1-\gamma_n-\tau_n)(u_n - v) + \tau_n(\mathcal{S}u_n - v)\| \leq (1-\gamma_n)\|w_n - v\|. \tag{3.5}$$

By the definition of w_n, we obtain

$$\|w_n - v\| = \|s_n + \mathcal{K}_n(s_n - s_{n-1}) - v\| \leq \|s_n - v\| + \zeta_n\frac{\mathcal{K}_n}{\zeta_n}\|s_n - s_{n-1}\|.$$

From (3.1), it can be deduced that

$$\lim_{n\to\infty}\frac{\mathcal{K}_n}{\zeta_n}\|s_n - s_{n-1}\| = 0.$$

This result holds true, since $\mathcal{K}_n\|s_n - s_{n-1}\| \leq \delta_n$ for all $n \geq 1$. Moreover, considering the limit $\lim_{n\to\infty}\frac{\delta_n}{\zeta_n} = 0$, it follows that

$$\lim_{n\to\infty}\|s_n - s_{n-1}\| \leq \lim_{n\to\infty}\frac{\delta_n}{\zeta_n} = 0.$$

Therefore, there exists a constant $\mathcal{M}_* > 0$ such that

$$\frac{\mathcal{K}_n}{\zeta_n}\|s_n - s_{n-1}\| \leq \mathcal{M}_*, \quad \forall n \geq 1. \tag{3.6}$$

Thus utilizing above, we get

$$\|w_n - v\| \leq \|s_n - v\| + \zeta_n \mathcal{M}_*. \tag{3.7}$$

which in turn implies

$$\|(1-\gamma_n-\tau_n)(u_n - v) + \tau_n(\mathcal{S}u_n - v)\| \leq \|s_n - v\| + \zeta_n \mathcal{M}_*.$$

Referring to (3.4), we obtain

$$\|t_n - v\| \leq \|s_n - v\| + \zeta_n \left[\mathcal{M}_* + \frac{\gamma_n}{\zeta_n}\|v\|\right] \leq \|s_n - v\| + \zeta_n \mathcal{M}, \tag{3.8}$$

where $\left[\mathcal{M}_* + \frac{\gamma_n}{\zeta_n}\|v\|\right] \leq \mathcal{M}$ for some $\mathcal{M} > 0$. Now,

$$\begin{aligned}
\|s_{n+1} - v\| &\leq \|\zeta_n \mathcal{J}(s_n) + (1-\zeta_n)\eta_n - v\| \\
&\leq \zeta_n \|\mathcal{J}(s_n) - \mathcal{J}(v)\| + \zeta_n \|\mathcal{J}(v) - v\| + (1-\zeta_n)\|t_n - v\| \\
&\leq \zeta_n \mathcal{Q}\|s_n - v\| + \zeta_n \|\mathcal{J}(v) - v\| + (1-\zeta_n)[\|s_n - v\| + \zeta_n \mathcal{M}] \\
&= (1-\zeta_n(1-\mathcal{Q}))\|s_n - v\| + \zeta_n(\|\mathcal{J}(v) - v\| + \mathcal{M}) \\
&\leq \max\left\{\|s_n - v\|, \frac{\|\mathcal{J}(v) - v\| + \mathcal{M}}{1-\mathcal{Q}}\right\} \leq \cdots \leq \max\left\{\|s_0 - v\|, \frac{\|\mathcal{J}(v) - v\| + \mathcal{M}}{1-\mathcal{Q}}\right\}.
\end{aligned}$$

This implies that the sequence $\{s_n\}$ is bounded. Consequently, the sequences $\{w_n\}$, $\mathcal{J}(s_n)$, $\{t_n\}$, and $\{u_n\}$ are also bounded.

Claim 2.

$$\begin{aligned}
(1-\zeta_n)&\left(1-\nu\frac{\eta_n}{\eta_{n+1}}\right)\|t_n - w_n\|^2 + (1-\zeta_n)\left(1-\nu\frac{\eta_n}{\eta_{n+1}}\right)\|u_n - t_n\|^2 \\
&+ (1-\zeta_n)\tau_n[(1-\mu) - \tau_n]\|u_n - \mathcal{S}u_n\| \\
&\leq \|s_n - v\|^2 - \|s_{n+1} - v\|^2 + \zeta_n\|\mathcal{J}(s_n) - v\|^2 + \zeta_n \mathcal{M}_{**} + (1-\zeta_n)\gamma_n \mathcal{M}_{***}.
\end{aligned}$$

Since from (3.7),

$$\|w_n - v\|^2 \leq (\|s_n - v\| + \zeta_n \mathcal{M}_*)^2 = \|s_n - v\|^2 + \zeta_n\left(2\mathcal{M}_*\|s_n - v\| + \zeta_n \mathcal{M}_*^2\right) \leq \|s_n - v\|^2 + \zeta_n \mathcal{M}_{**}, \tag{3.9}$$

for some $\mathcal{M}_{**} > 0$.

$$\|s_{n+1} - v\|^2 = \|\zeta_n(\mathcal{J}(s_n) - v) + (1-\zeta_n)(t_n - v)\|^2 \leq \zeta_n\|\mathcal{J}(s_n) - v\|^2 + (1-\zeta_n)\|t_n - v\|^2. \tag{3.10}$$

Now,

$$\begin{aligned}\|t_n - v\|^2 &= \|(1-\gamma_n-\tau_n)u_n + \tau_n \mathcal{S}u_n - v\|^2 = \|(u_n - v) + \tau_n(\mathcal{S}u_n - u_n) - \gamma_n u_n\|^2 \\ &\leq \|(u_n - v) + \tau_n(\mathcal{S}u_n - u_n)\|^2 - 2\gamma_n\langle u_n, \eta_n - v\rangle \\ &= \|u_n - v\|^2 + \tau_n^2\|\mathcal{S}u_n - u_n\|^2 + 2\tau_n\langle \mathcal{S}u_n - u_n, u_n - v\rangle + 2\gamma_n\langle u_n, v - \eta_n\rangle.\end{aligned}$$

It follows from Lemma(2.4),

$$\begin{aligned}\|t_n - v\|^2 &\leq \|u_n - v\|^2 + \tau_n^2\|\mathcal{S}u_n - u_n\|^2 - \tau_n(1-\mu)\|u_n - \mathcal{S}u_n\|^2 + 2\gamma_n\langle u_n, v - \eta_n\rangle \\ &\leq \|w_n - v\|^2 + \tau_n[\tau_n - (1-\mu)]\|u_n - \mathcal{S}u_n\|^2 + \gamma_n\mathcal{M}_{***}. \end{aligned} \tag{3.11}$$

for some $\mathcal{M}_{***} > 0$, from (3.10)

$$\begin{aligned}\|s_{n+1} - v\|^2 &\leq \zeta_n\|\mathcal{J}(s_n) - v\|^2 \\ &\quad + (1-\zeta_n)\left[\|w_n - v\|^2 + \tau_n[\tau_n - (1-\mu)]\|u_n - \mathcal{S}u_n\|^2 + \gamma_n\mathcal{M}_{***}\right] \\ &\leq \zeta_n\|\mathcal{J}(s_n) - v\|^2 + \|s_n - v\|^2 + \zeta_n\mathcal{M}_{**} \\ &\quad - (1-\zeta_n)\left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|t_n - w_n\|^2 - (1-\zeta_n)\left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|u_n - t_n\|^2 \\ &\quad - (1-\zeta_n)\tau_n[(1-\mu) - \tau_n]\|u_n - \mathcal{S}u_n\| + (1-\zeta_n)\gamma_n\mathcal{M}_{***}.\end{aligned}$$

By a straightforward manipulation, we attain the desired result.

Claim 3.

$$\|s_{n+1} - v\|^2 = (1-(1-\mathcal{Q})\zeta_n)\|s_n - v\|^2 + \zeta_n(1-\mathcal{Q})\left[\frac{(1-\zeta_n)\zeta_n\mathcal{N} + 2\langle\mathcal{J}(v) - v, s_{n+1} - v\rangle}{1-\mathcal{Q}}\right].$$

Since by (3.8),

$$\|t_n - v\|^2 \leq [\|s_n - v\| + \zeta_n\mathcal{M}]^2 = \|s_n - v\|^2 + \zeta_n^2\mathcal{M}^2 + 2\zeta_n\langle\mathcal{M}, s_n - v\rangle \leq \|s_n - v\|^2 + \zeta_n^2\mathcal{N},$$

where $\mathcal{M}^2 + \frac{2}{\zeta_n}\langle\mathcal{M}, s_n - v\rangle \leq \mathcal{N}$ for some $\mathcal{N} > 0$.

$$\begin{aligned}\|s_{n+1} - v\|^2 &= \|\zeta_n\mathcal{J}(s_n) + (1-\zeta_n)t_n - v\|^2 \\ &= \|\zeta_n(\mathcal{J}(s_n) - \mathcal{J}(v)) + (1-\zeta_n)(t_n - v) + \zeta_n(\mathcal{J}(v) - v)\|^2 \\ &\leq \|\zeta_n(\mathcal{J}(s_n) - \mathcal{J}(v)) + (1-\zeta_n)(t_n - v)\|^2 + 2\zeta_n\langle\mathcal{J}(v) - v, s_{n+1} - v\rangle \\ &\leq \zeta_n\mathcal{Q}\|s_n - v\|^2 + (1-\zeta_n)\|t_n - v\|^2 + 2\zeta_n\langle\mathcal{J}(v) - v, s_{n+1} - v\rangle \\ &\leq \zeta_n\mathcal{Q}\|s_n - v\|^2 + (1-\zeta_n)\left[\|s_n - v\|^2 + \zeta_n^2\mathcal{N}\right] + 2\zeta_n\langle\mathcal{J}(v) - v, s_{n+1} - v\rangle \\ &= (1-(1-\mathcal{Q})\zeta_n)\|s_n - v\|^2 + \zeta_n(1-\mathcal{Q})\left[\frac{(1-\zeta_n)\zeta_n\mathcal{N} + 2\langle\mathcal{J}(v) - v, s_{n+1} - v\rangle}{1-\mathcal{Q}}\right].\end{aligned}$$

Claim 4. The sequence $\|s_n - v\|^2$ converges to zero. In fact, using Lemma 2.6, it is sufficient to show that for each subsequence $\|s_{n_k} - v\|$ of $\|s_n - v\|$ satisfying $\limsup_{k\to\infty}\langle \mathcal{J}(v)-v, s_{n+1}-v\rangle \leq 0$ with

$$\liminf_{k\to\infty}(\|s_{n_k+1} - v\| - \|s_{n_k} - v\|) \geq 0. \tag{3.12}$$

We assume that $\|s_{n_k} - v\|$ is a subsequence of $\|s_n - v\|$, such that (3.12) holds, for the purposes of this analysis. Next,

$$\begin{aligned}&\liminf_{k\to\infty}\left(\|s_{n_k+1} - v\|^2 - \|s_{n_k} - v\|^2\right)\\ &\qquad = \liminf_{k\to\infty}\left[(\|s_{n_k+1} - v\| - \|s_{n_k} - v\|)(\|s_{n_k+1} - v\| + \|s_{n_k} - v\|)\right] \geq 0.\end{aligned}$$

Based on Claim 2, we have,

$$\begin{aligned}&\limsup_{k\to\infty}(1-\zeta_{n_k})\left(1-\nu\frac{\eta_{n_k}}{\eta_{n_k+1}}\right)\|t_{n_k} - w_{n_k}\|^2 + (1-\zeta_{n_k})\left(1-\nu\frac{\eta_{n_k}}{\eta_{n_k+1}}\right)\|u_{n_k} - t_{n_k}\|^2\\ &\quad + (1-\zeta_{n_k})\tau_{n_k}[(1-\mu)-\tau_{n_k}]\|u_{n_k} - \mathcal{S}u_{n_k}\|\\ &\quad \leq \limsup_{k\to\infty}\left[\|s_{n_k} - v\|^2 - \|s_{n_k+1} - v\|^2 + \zeta_{n_k}\|\mathcal{J}(s_{n_k}) - v\|^2\right.\\ &\quad \left. + \zeta_{n_k}\mathcal{M}_{**} + (1-\zeta_{n_k})\gamma_{n_k}\mathcal{M}_{***}\right]\\ &\quad = -\liminf_{k\to\infty}\left[\|s_{n_k+1} - v\|^2 - \|s_{n_k} - v\|^2\right]\end{aligned}$$

signifying that

$$\lim_{k\to\infty}\|w_{n_k} - t_{n_k}\| = 0, \lim_{k\to\infty}\|u_{n_k} - \mathcal{S}u_{n_k}\| = 0, \|u_{n_k} - t_{n_k}\| = 0. \tag{3.13}$$

Therefore, we can infer that $\lim_{k\to\infty}\|u_{n_k} - w_{n_k}\| = 0$. Referring to the definition of w_n, we have

$$\|s_{n_k} - w_{n_k}\| = \mathcal{K}_{n_k}\|s_{n_k} - s_{n_{k-1}}\| = \zeta_{n_k}\frac{\mathcal{K}_{n_k}}{\zeta_{n_k}}\|s_{n_k} - s_{n_{k-1}}\| \to 0 \text{ as } k\to\infty. \tag{3.14}$$

This in conjunction with $\lim_{k\to\infty}\|u_{n_k} - w_{n_k}\| = 0$, implies that

$$\lim_{k\to\infty}\|u_{n_k} - s_{n_k}\| = 0. \tag{3.15}$$

Considering $t_{n_k} = (1-\gamma_{n_k} - \tau_{n_k})u_{n_k} + \tau_{n_k}\mathcal{S}u_{n_k}$, it is evident that

$$\|t_{n_k} - u_{n_k}\| \leq \tau_n\|(\mathcal{S}u_n - u_{n_k})\| + \gamma_n\|u_{n_k}\|.$$

Hence, we obtain

$$\|t_{n_k} - u_{n_k}\| = 0. \tag{3.16}$$

By using (3.15) and (3.16), we can deduce that

$$\begin{aligned}\|s_{n_{k+1}} - s_{n_k}\| &\leq \|\zeta_{n_k}\mathcal{J}(s_{n_k}) + (1-\zeta_{n_k})t_{n_k} - s_{n_k}\| \\ &\leq \zeta_{n_k}\|\mathcal{J}(s_{n_k}) - s_{n_k}\| + (1-\zeta_{n_k})\|t_{n_k} - s_{n_k}\| \\ &\leq \zeta_{n_k}\|\mathcal{J}(s_{n_k}) - s_{n_k}\| + \|t_{n_k} - u_{n_k}\| + \|u_{n_k} - s_{n_k}\| \to 0 \text{ as } k \to \infty. \quad (3.17)\end{aligned}$$

Given that the sequence $\{s_{n_k}\}$ is bounded, it can be inferred that there exists a subsequence $\{s_{n_{k_j}}\}$ of $\{s_{n_k}\}$ such that $s_{n_{k_j}} \rightharpoonup u$. This further implies that

$$\limsup_{k\to\infty}\langle \mathcal{J}(v) - v, s_{n_k} - v\rangle = \lim_{j\to\infty}\langle \mathcal{J}(v) - v, s_{n_{k_j}} - v\rangle = \langle \mathcal{J}(v) - v, u - v\rangle. \quad (3.18)$$

From (3.14), it follows that $w_{n_k} \rightharpoonup u$. Combining (3.13), $\lim_{n\to\infty}\eta_n = \eta$ and Lemma 2.5, one can conclude that $u \in VI(E,\mathcal{F})$. Utilizing (3.15), we have $u_{n_k} \rightharpoonup u$. By the demiclosedness of $(I - \mathcal{S})$, we obtain $u \in Fix(\mathcal{S})$. Consequently, $u \in Fix(\mathcal{S}) \cap VI(E,\mathcal{F})$. Combining (3.18), the definition of v and $u \in Fix(\mathcal{S}) \cap VI(E,\mathcal{F})$, we obtain

$$\limsup_{k\to\infty}\langle \mathcal{J}(v) - v, s_{n_k} - v\rangle = \langle \mathcal{J}(v) - v, u - v\rangle \leq 0, \quad (3.19)$$

which in conjunction with (3.19) and (3.18), implies that

$$\begin{aligned}\limsup_{k\to\infty}\langle \mathcal{J}(v) - v, s_{n_k+1} - v\rangle &\leq \limsup_{k\to\infty}\langle \mathcal{J}(v) - v, s_{n_k+1} - s_{n_k}\rangle + \limsup_{k\to\infty}\langle \mathcal{J}(v) - v, s_{n_k} - v\rangle \\ &= \langle \mathcal{J}(v) - v, u - v\rangle \leq 0 \quad (3.20)\end{aligned}$$

Therefore (3.20) and Claim 3 in the light of Lemma 2.6 indicates that $s_n \to v$ as $n \to \infty$. Thus, completes the proof. □

Specifically, we may design a new algorithm for (VIP) if $\mathcal{S} = I$ (identity operator) in Algorithm 3.1. To be more exact, we have the corollary that follows:

Corollary 3.4. *If $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is Lipschitz continuous, monotone and $\mathcal{J} : \mathcal{D} \to \mathcal{D}$ is a $\mathcal{Q}$-contraction with $\mathcal{Q} \in [0,1)$. If the sequences γ_n, ζ_n, and τ_n be same as in Theorem 3.3 and if $VI(E,\mathcal{F}) \neq \varnothing$, let $s_0, s_1 \in \mathcal{D}$ and let the sequence $\{s_n\}$ be generated by*

$$\begin{cases} w_n = s_n + \mathcal{K}_n(s_n - s_{n-1}), \\ t_n = \mathcal{P}_E(w_n - \eta_n\mathcal{F}w_n), \\ u_n = \mathcal{P}_{T_n}(w_n - \eta_n\mathcal{F}t_n), \ \textit{where the half-space } T_n \textit{ is defined by} \\ T_n := \{s \in \mathcal{D} | \langle w_n - \eta_n\mathcal{F}w_n - t_n, s - t_n\rangle \leq 0\}, \\ s_{n+1} = \zeta_n\mathcal{J}(s_n) + (1-\zeta_n)((1-\gamma_n)u_n), \end{cases} \quad (3.21)$$

where $\mathcal{K}_n$ and η_n are defined by (3.1) *and* (3.2)*, respectively. Then the iterative sequence* $\{s_n\}$ *generated by* (3.21) *converges to* $v \in VI(E, \mathcal{F})$ *in norm, where* $v = \mathcal{P}_{VI(E,\mathcal{F})}(\mathcal{J}(v))$.

3.2 Algorithm-II

Algorithm 3.2 Algorithm-II

Initialization: Choose $\mathcal{K} > 0, \eta_1 > 0, \nu \in (0, 1)$. Let $s_0, s_1 \in \mathcal{D}$ be arbitrary.
Iterative step: Calculate s_{n+1} as follows:
Step 1. Given the iterates s_{n-1} and $s_n (n \geq 1)$. Set $w_n = s_n + \mathcal{K}_n(s_n - s_{n-1})$, where $\mathcal{K}_n$ is defined by (3.1).
Step 2. Compute $t_n = \mathcal{P}_E(w_n - \eta_n \mathcal{F} w_n)$.
Step 3. Compute $u_n = \mathcal{P}_{T_n}(w_n - \eta_n \mathcal{F} t_n)$, where the half-space T_n is defined by

$$T_n := \{s \in \mathcal{D} \mid \langle w_n - \eta_n \mathcal{F} w_n - t_n, s - t_n \rangle \leq 0\}.$$

Step 4. Compute $s_{n+1} = \zeta_n \mathcal{J}(s_n) + (1 - \zeta_n)[(1 - \tau_n)(\gamma_n u_n) + \tau_n \mathcal{S} u_n]$, and update η_{n+1} by (3.2).
Set $n := n + 1$ and go to **Step 1.**

Theorem 3.5. *Let conditions* **(A1)-(A4)** *holds and* $\{\delta_n\}$ *be a positive sequence with* $\lim_{n\to\infty} \frac{\delta_n}{\zeta_n} = 0$*, where* $\zeta_n \subset (0, 1)$ *satisfies* $\sum_{n=1}^{\infty} \zeta_n = \infty$ *and* $\lim_{n\to\infty} \zeta_n = 0$*. Furthermore, for some* $a > 0$, $\gamma_n \in (0, 1)$, $\lim_{n\to\infty} \gamma_n = 1$*, and* $\sum_{n=0}^{\infty}(1 - \gamma_n) = \infty$*, let* $\tau_n \in \left(a, \frac{(1-\mu)\gamma_n}{2+\mu+\gamma_n}\right) \subset (a, 1 - \mu)$*, then the sequence* $\{s_n\}$ *produced by Algorithm 3.2 converges in norm to* $v \in Fix(\mathcal{S}) \cap VI(E, \mathcal{F})$*, where* $v = \mathcal{P}_{Fix(\mathcal{S})\cap VI(E,\mathcal{F})}\mathcal{J}(v)$.

Proof. **Claim 1.** The sequence s_n is bounded. Define $t_n = (1 - \tau_n)(\gamma_n u_n) + \tau_n \mathcal{S} u_n$.

$$\begin{aligned} \|t_n - v\| &= \|(1 - \tau_n)(\gamma_n u_n) + \tau_n \mathcal{S} u_n - v\| \\ &\leq \|(1 - \tau_n)\gamma_n(u_n - v) + \tau_n(\mathcal{S} u_n - v)\| + (1 - \tau_n)(1 - \gamma_n)\|v\|. \end{aligned} \tag{3.22}$$

On the other hand,

$$\begin{aligned} \|(1 - \tau_n)\gamma_n(u_n - v) + \tau_n(\mathcal{S} u_n - v)\|^2 &= ((1 - \tau_n)\gamma_n)^2\|u_n - v\|^2 + \tau_n^2\|\mathcal{S} u_n - v\|^2 \\ &+ 2(1 - \tau_n)\gamma_n\tau_n\langle \mathcal{S} u_n - v, u_n - v\rangle \\ &\leq ((1 - \tau_n)\gamma_n + \tau_n)^2 \|u_n - v\|^2 \\ &+ \tau_n (\mu\tau_n - (1 - \mu)(1 - \tau_n)\gamma_n) \|\mathcal{S} u_n - u_n\|^2 \\ &\leq ((1 - \tau_n)\gamma_n + \tau_n)^2 \|u_n - v\|^2. \end{aligned} \tag{3.23}$$

we obtained the above inequality because $\tau_n < \dfrac{(1 - \mu)\gamma_n}{2 + \mu + \gamma_n}$.

Thus it is implied from (3.23) that

$$\begin{aligned}\|(1-\tau_n)\gamma_n(u_n-v)+\tau_n(\mathcal{S}u_n-v)\| &\leq ((1-\tau_n)\gamma_n+\tau_n)\,\|u_n-v\|\\ &\leq (1-(1-\tau_n)(1-\gamma_n))\,\|u_n-v\|\\ &\leq (1-(1-\tau_n)(1-\gamma_n))\,\|w_n-v\|\\ &\leq (1-(1-\tau_n)(1-\gamma_n))\,[\|s_n-v\|+\zeta_n\mathcal{M}_*]\,. \end{aligned} \tag{3.24}$$

From (3.22), we have

$$\begin{aligned}\|t_n-v\| &\leq (1-(1-\tau_n)(1-\gamma_n))\,[\|s_n-v\|+\zeta_n\mathcal{M}_*]+(1-\tau_n)(1-\gamma_n)\|v\|\\ &\leq (1-(1-\tau_n)(1-\gamma_n))\,\|s_n-v\|+\zeta_n\mathcal{M}_*+(1-\tau_n)(1-\gamma_n)\|v\|\\ &= (1-(1-\tau_n)(1-\gamma_n))\,\|s_n-v\|\\ &+(1-\tau_n)(1-\gamma_n)\left[\frac{\zeta_n\mathcal{M}_*}{(1-\tau_n)(1-\gamma_n)}+\|v\|\right]\\ &\leq \max\left\{\|s_n-v\|,\frac{\zeta_n\mathcal{M}_*}{(1-\tau_n)(1-\gamma_n)}+\|v\|\right\}:=M^*\end{aligned}$$

for some $M^*>0$, hence

$$\begin{aligned}\|s_{n+1}-v\| &= \|\zeta_n\mathcal{J}(s_n)+(1-\zeta_n)t_n-v\|\\ &\leq \zeta_n\|\mathcal{J}(s_n)-\mathcal{J}(v)\|+\zeta_n\|\mathcal{J}(v)-v\|+(1-\zeta_n)\|t_n-v\|\\ &\leq \zeta_n\mathcal{Q}\|s_n-v\|+\zeta_n\|\mathcal{J}(v)-v\|+(1-\zeta_n)M^*\\ &= \zeta_n\mathcal{Q}\|s_n-v\|+(1-\zeta_n)\left[M^*+\frac{\zeta_n}{1-\zeta_n}\|\mathcal{J}(v)-v\|\right]\\ &\leq \max\left\{M^*+\frac{\zeta_n}{1-\zeta_n}\|\mathcal{J}(v)-v\|,\mathcal{Q}\|s_n-v\|\right\}\\ &\leq \cdots \leq \max\{M^*,\mathcal{Q}\|s_0-v\|\}\,.\end{aligned}$$

Which ensures the boundedness of $\{s_n\}$, so the sequences $\{w_n\}$, $\{\mathcal{J}(s_n)\}$, $\{t_n\}$, and $\{u_n\}$ are also bounded.

Claim 2.

$$\begin{aligned}&(1-\zeta_n)\left(1-\nu\frac{\eta_n}{\eta_{n+1}}\right)\|t_n-w_n\|+(1-\zeta_n)\left(1-\nu\frac{\eta_n}{\eta_{n+1}}\right)\|u_n-t_n\|\\ &+(1-\zeta_n)\tau_n(1-\mu-\tau_n)\|\mathcal{S}u_n-u_n\|^2\\ &\leq \|s_n-v\|^2-\|s_{n+1}-v\|^2+\zeta_n\|\mathcal{J}(s_n)-v\|^2+(1-\gamma_n)M^{**}+\zeta_nM^{***}.\end{aligned}$$

$$\begin{aligned}\|t_n-v\|^2 &= \|(1-\tau_n)(\gamma_nu_n)+\tau_n\mathcal{S}u_n-v\|^2\\ &= \|(u_n-v)+\tau_n(\mathcal{S}u_n-u_n)-(1-\tau_n)(1-\gamma_n)u_n\|^2\\ &\leq \|(u_n-v)+\tau_n(\mathcal{S}u_n-u_n)\|^2-2(1-\tau_n)(1-\gamma_n)\langle u_n,\eta_n-v\rangle\end{aligned}$$

$$= \|u_n - v\|^2 + \tau_n^2\|\mathcal{S}u_n - u_n\|^2 + 2\tau_n\langle \mathcal{S}u_n - u_n, u_n - v\rangle -$$
$$2(1-\tau_n)(1-\gamma_n)\langle u_n, \eta_n - v\rangle$$
$$\leq \|u_n - v\|^2 + \tau_n^2\|\mathcal{S}u_n - u_n\|^2 - \tau_n(1-\mu)\|\mathcal{S}u_n - u_n\|^2$$
$$- 2(1-\tau_n)(1-\gamma_n)\langle u_n, \eta_n - v\rangle$$
$$= \|u_n - v\|^2 - \tau_n(1-\mu-\tau_n)\|\mathcal{S}u_n - u_n\|^2$$
$$- 2(1-\tau_n)(1-\gamma_n)\langle u_n, \eta_n - v\rangle$$
$$\leq \|u_n - v\|^2 - \tau_n(1-\mu-\tau_n)\|\mathcal{S}u_n - u_n\|^2 + (1-\gamma_n)M^{**}$$

for some $M^{**} > 0$. Now,

$$\begin{aligned}
\|s_{n+1} - v\|^2 &= \|\zeta_n(\mathcal{J}(s_n) - v) + (1-\zeta_n)(t_n - v)\|^2 \\
&\leq \zeta_n\|\mathcal{J}(s_n) - v\|^2 + (1-\zeta_n)\|t_n - v\|^2 \\
&\leq \zeta_n\|\mathcal{J}(s_n) - v\|^2 + \|s_n - v\|^2 + \zeta_n M^{***} \\
&- (1-\zeta_n)\left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|t_n - w_n\| - (1-\zeta_n)\left(1 - \nu\frac{\eta_n}{\eta_{n+1}}\right)\|u_n - t_n\| \\
&- (1-\zeta_n)\tau_n(1-\mu-\tau_n)\|\mathcal{S}u_n - u_n\|^2 + (1-\gamma_n)M^{**}.
\end{aligned}$$

Hence, by simple deformation, we obtain the desired result.

Claim 3.

$$\begin{aligned}
\|s_{n+1} - v\|^2 &= (1 - (1-\mathcal{Q})\zeta_n)\|s_n - v\|^2 \\
&\quad + \zeta_n(1-\mathcal{Q})\left[\frac{(1-\zeta_n)\zeta_n\mathcal{M}_* + 2\langle \mathcal{J}(v) - v, s_{n+1} - v\rangle}{1-\mathcal{Q}}\right].
\end{aligned}$$

By using the identical reasons as in Claim 3 of Theorem 3.3, the required result can be produced.

Claim 4. Sequence $\{\|s_n - v\|^2\}$ converges to zero. We do not include the proof here because it is comparable to Claim 4 of Theorem 3.3. $\square$

The following Corollary will be obtained if we put $\mathcal{S} = I$ in Algorithm 3.2.

Corollary 3.6. *Consider $\mathcal{F}, \mathcal{J}$ as in Corollary 3.4 and let ζ_n, γ_n, τ_n be same as in Theorem 3.5. Then the sequence $\{s_n\}$ with $s_0, s_1 \in \mathcal{D}$ generated by* (3.25)

$$\begin{cases}
w_n = s_n + \mathcal{K}_n(s_n - s_{n-1}), \\
t_n = \mathcal{P}_E(w_n - \eta_n\mathcal{F}w_n), \\
u_n = \mathcal{P}_{T_n}(w_n - \eta_n\mathcal{F}t_n), \textit{where the half-space } T_n \textit{ is defined by} \\
T_n := \{s \in \mathcal{D} | \langle w_n - \eta_n\mathcal{F}w_n - t_n, s - t_n\rangle \leq 0\}, \\
s_{n+1} = \zeta_n\mathcal{J}(s_n) + (1-\zeta_n)(\gamma_n u_n + \tau_n(1-\gamma_n)u_n),
\end{cases} \tag{3.25}$$

converges to $v \in VI(E,\mathcal{F})$ *in norm, where* $v = \mathcal{P}_{VI(E,\mathcal{F})}(\mathcal{J}(v))$. *where* $\mathcal{K}_n$ *and* η_n *are defined by* (3.1) *and* (3.2)*, respectively.*

4 Numerical example

In this section, we provide a numerical example to illustrate the behavior of the proposed algorithms and compare them with some existing strongly convergent algorithms. The parameters are set as follows: $\zeta_n = \frac{1}{n+1}$, $\tau_n = \frac{n}{2n+1}$, $\gamma_n = \frac{n}{30n+1}$, $\eta_1 = 1$, $\nu = 0.5$, $\mathcal{J}(s) = 0.5s$, $\mathcal{K} = 0.3$, $\delta_n = \frac{100}{(n+1)^2}$. The solution s^* is known, so we use $D_n = \|s_n - s^*\|$ to measure the n-th iteration error and convergence of D_n to 0 indicates that $\{s_n\}$ converges to the problem's solution.

Example 4.1. *We take the nonlinear operator* $\mathcal{F} : R^2 \to R^2$ *defined by* $\mathcal{F}(s,t) = (s+t+\sin s, -s+t+\sin s)$*, feasible set* $E = [-1,1] \times [-1,1]$*. Clearly* $\mathcal{F}$ *is monotone and Lipschitz continuous with constant* $\mathcal{L} = 3$ *and let the matrix* $F = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$*. We consider the mapping* $\mathcal{S} : R^2 \to R^2$ *by* $\mathcal{S}u = \|F\|^{-1}Fu$*, where* $u = (s,t)^T$*. It is obvious to see that* $\mathcal{S}$ *is* 0*-demicontractive and thus* $\tau = 0$*. The solution of the problem is* $s^* = (0,0)^T$*. The initial values* $s_0 = s_1$ *are randomly generated by* $k * rand(2,1)$ *in MATLAB. The numerical results of all the algorithms with different initial values are described in Figures (Figure 1, Figure 2, Figure 3, Figure 4).*

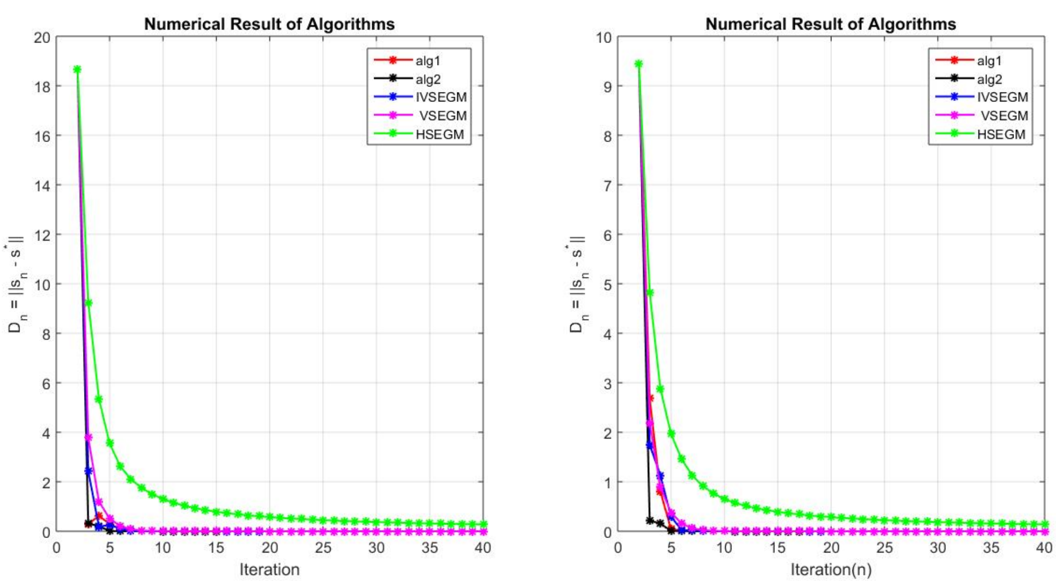

Figure 1: The convergence graphs of $\{D_n = \|s_n - s^*\|\}$ vs iteration ($n = 40$).

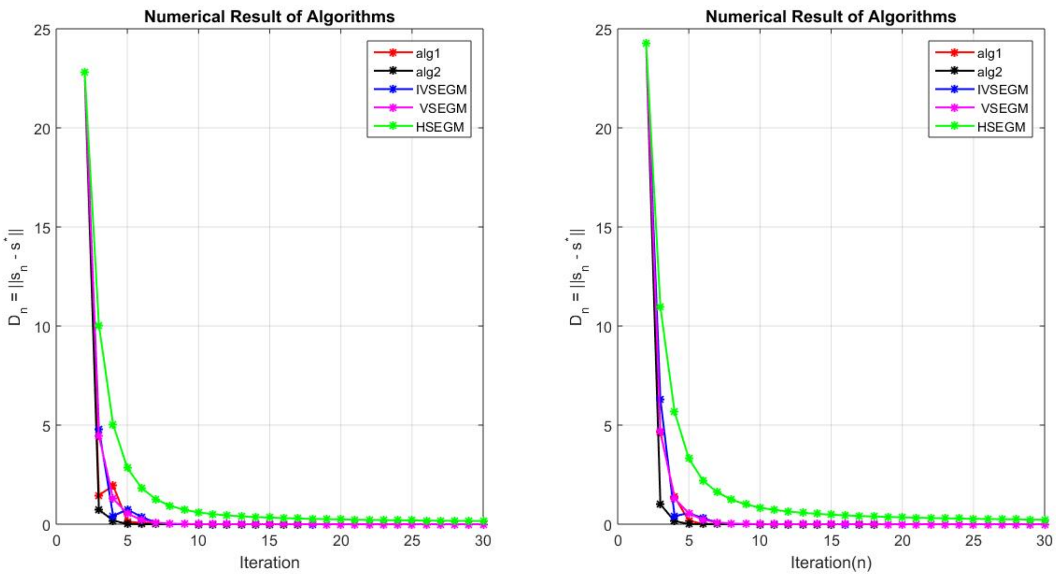

Figure 2: The convergence graphs of $\{D_n = \|s_n - s^*\|\}$ vs iteration ($n = 30$).

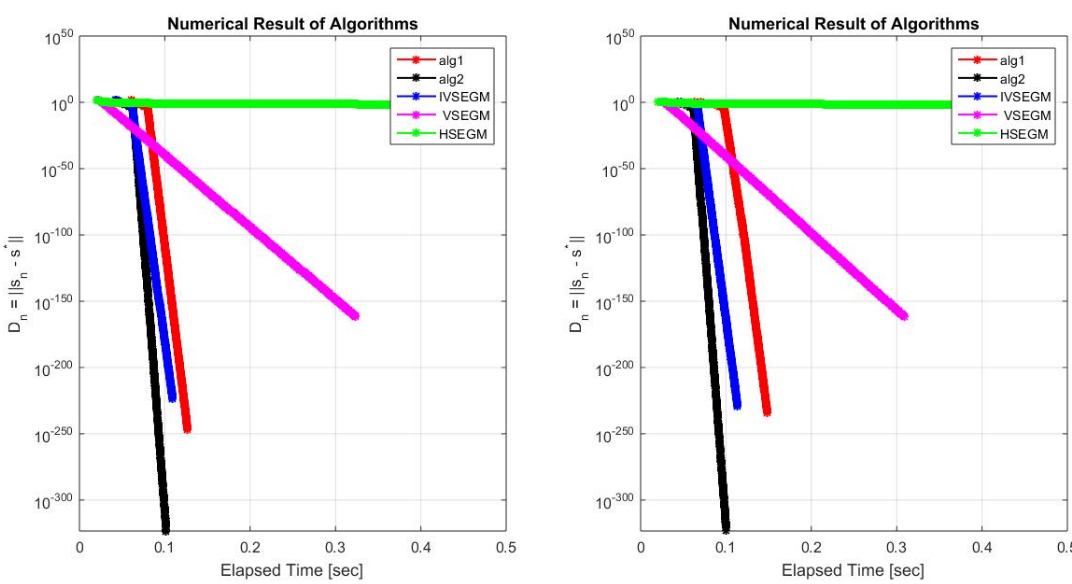

Figure 3: The Elapsed time graph of the sequence $\{D_n = \|s_n - s^*\|\}$ with initial values $s_0 = s_1 = 30rand(2, 1)$ and $n = 300$

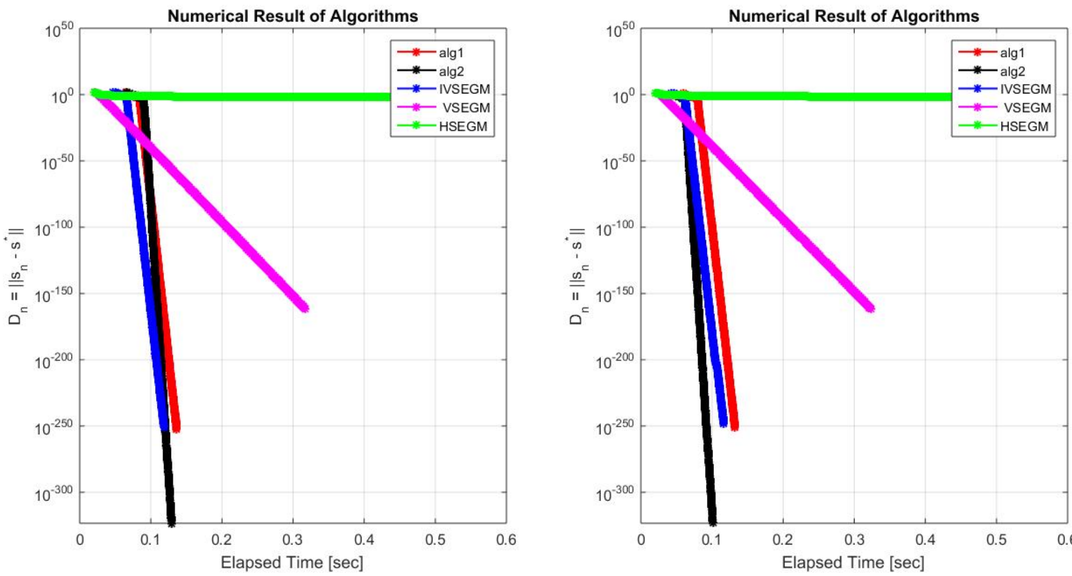

Figure 4: The Elapsed time graph of the sequence $\{D_n = \|s_n - s^*\|\}$ with initial values $s_0 = s_1 = 40rand(2,1)$ and $n = 250$.

Example 4.2. *Consider the linear operator $\mathcal{F} : R^m \to R^m$ $(m = 50, 100, 150, 200)$ in the form $\mathcal{F}(s) = Ms + q$, where $q \in R^m$ and $M = NN^T + Q + D$, N is a $m \times m$ matrix, Q is a $m \times m$ skew-symmetric matrix, and D is a $m \times m$ diagonal matrix with its diagonal entries being nonnegative (hence M is positive symmetric definite). The feasible set E is given by $E = \{s \in R^m : -2 \leq s_i \leq 5, i = 1, \ldots, m\}$. It is clear that $\mathcal{F}$ is monotone and Lipschitz continuous with constant $L = \|M\|$. In this experiment, all entries of N, D are generated randomly in $[0, 2]$, Q is generated randomly in $[-2, 2]$ and $q = 0$. Let $\mathcal{S} : \mathcal{D} \to \mathcal{D}$ be given by $\mathcal{S}s = 0.5s$. It is easy to see that the solution of the problem in this case is $s^* = \{0\}$. The initial values $s_0 = s_1$ are randomly generated by $k * rand(2,1)$ in MATLAB. Figure 5 shows the numerical behavior of all the algorithms in different dimensions ($m = 50$, $m = 100$, $m = 150$, $m = 200$).*

Example 4.3. *Finally, we consider our problem in the infinite-dimensional Hilbert space $\mathcal{D} = L^2([0,1])$ with inner product $\langle s, y \rangle = \int_0^1 s(t)y(t)dt$ and norm $\|s\| = \left(\int_0^1 |s(t)|^2 dt\right)^{\frac{1}{2}}$, $\forall s, y \in \mathcal{D}$. Let the feasible set be the unit ball $E = \{s \in \mathcal{D} : \|s\| \leq 1\}$. Define an operator $\mathcal{F} : E \to \mathcal{D}$ by*

$$(\mathcal{F}s)(t) = \int_0^1 (s(t) - G(t,u)g(s(u)))du + h(t)), \quad t \in [0,1], \quad s \in E,$$

where,

$$G(t,u) = \frac{2tue^{t+u}}{e\sqrt{e^2 - 1}}, \quad g(s) = \cos(s), \quad h(t) = \frac{2te^t}{e\sqrt{e^2 - 1}}.$$

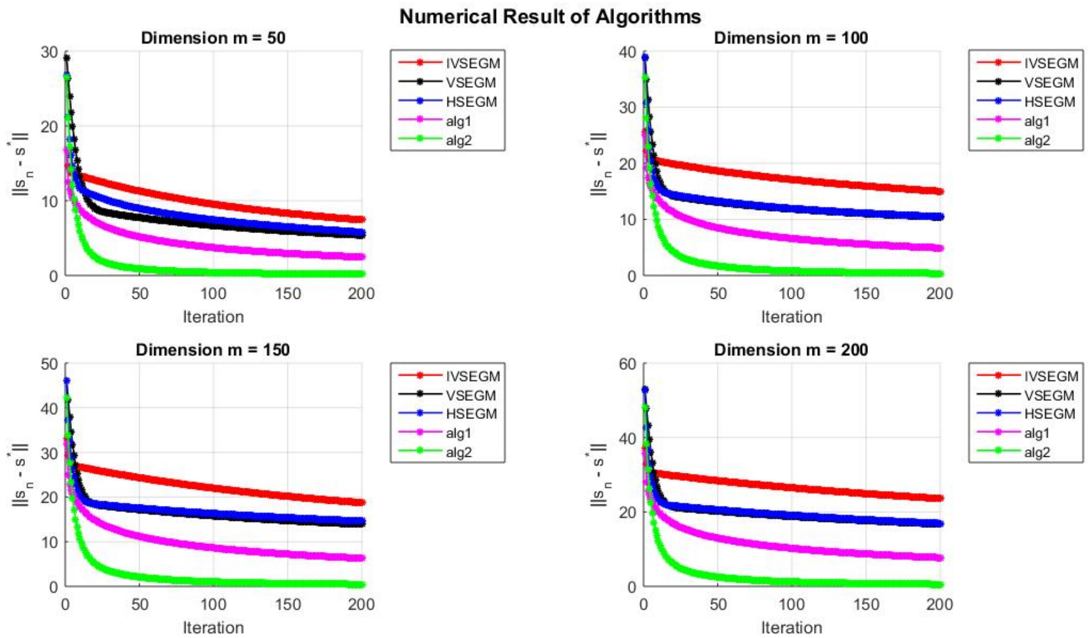

Figure 5: The convergence graphs of $\{D_n = \|s_n - s^*\|\}$ vs iteration(n = 200).

It is known that $\mathcal{F}$ is monotone and L-Lipschitz continuous with $L = 2$ ([13]). The projection on E is inherently explicit, that is,

$$P_E(s) = \begin{cases} \dfrac{s}{\|s\|}, & \text{if } \|s\| > 1; \\ s, & \text{if } \|s\| \leq 1. \end{cases}$$

The mapping $\mathcal{S} : L^2([0,1]) \to L^2([0,1])$ is of the form

$$(\mathcal{S}s)(t) = \int_0^1 ts(u)\,du, \quad t \in [0,1].$$

A straightforward computation implies that $\mathcal{S}$ is 0-demicontractive. The solution of the problem is $s^*(t) = 0$. The maximum number of iterations 50 is used as a common stopping criterion for all algorithms. Figure 6 shows the behaviors of $D_n = \|s_n(t) - s^*(t)\|$ generated by all the algorithms with four starting points.

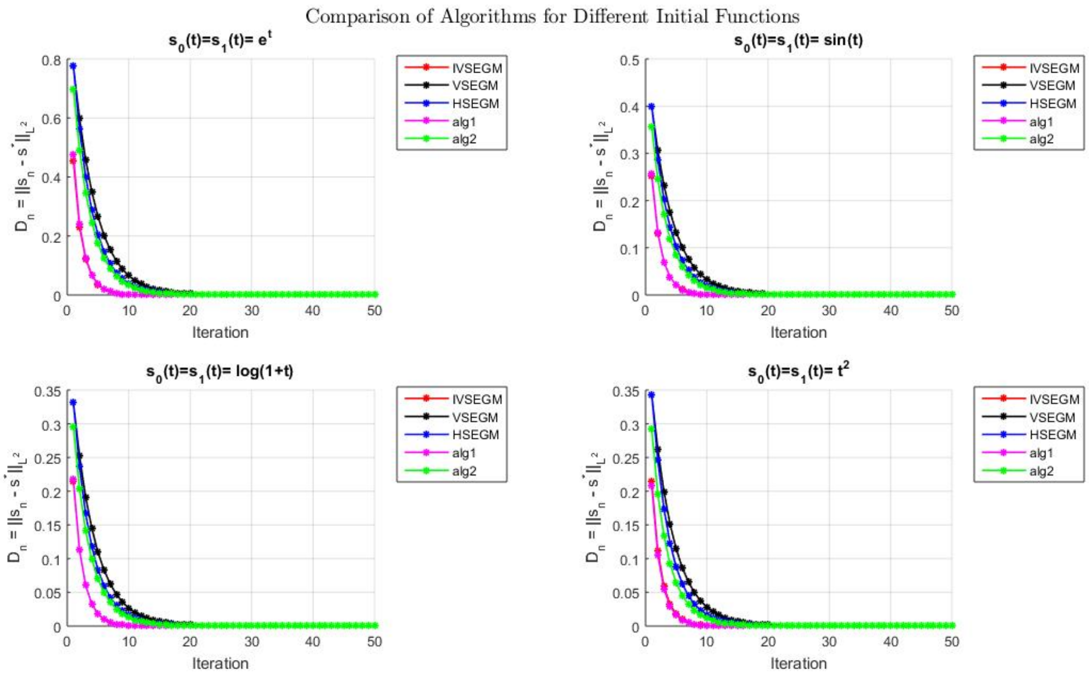

Figure 6: The convergence graphs of $\{D_n = \|s_n - s^*\|\}$ vs iteration ($n = 50$).

5 Conclusion

In this study, we investigated two self-adaptive iterative schemes for seeking a common solution to the variational inequality problem involving a monotone and Lipschitz continuous mapping and the fixed point problem with a demicontractive mapping. We proposed two new inertial extragradient methods with a new step size to compute the approximate solutions of problems in a real Hilbert space. The strong convergence of the suggested methods is established under standard and suitable conditions. Finally, some computational tests are given to explain our convergent results. The algorithms obtained in this paper improved and summarized some of the recent results in the literature.

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Hausdorff operators associated with the linear canonical Sturm-Liouville transform

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ABSTRACT

In the present paper, we introduce the canonical Sturm-Liouville operator $L^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b}\right)$, where A is a nonnegative function satisfying certain conditions. We prove the boundedness of the canonical Sturm-Liouville Hausdorff operators on the space $L^p(\mathbb{R}_+, A(x)\,\mathrm{d}x)$, $p \in [1, \infty)$. We investigate canonical Sturm-Liouville wavelet transform, and obtain some useful results. The relation between the canonical Sturm-Liouville wavelet transform and canonical Sturm-Liouville Hausdorff operator is also established. The properties of the adjoint canonical Sturm-Liouville Hausdorff operators are further discussed. The harmonic analysis associated with the operator L^M plays an important role in establishing the results of this paper.

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RESUMEN

En el presente arículo, introducimos el operador de Sturm-Liouville canónico $L^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b}\right)$, donde A es una función no-negativa que satisface ciertas condiciones. Demostramos el acotamiento de los operadores Hausdorff de Sturm-Liouville canónicos en el espacio $L^p(\mathbb{R}_+, A(x)\,\mathrm{d}x)$, $p \in [1,\infty)$. Investigamos la transformada de ondeletas de Sturm-Liouville canónica y obtenemos algunos resultados útiles. También se establece la relación entre la transformada de ondeletas de Sturm-Liouville canónica y el operador Hausdorff de Sturm-Liouville canónico. Se discuten las propiedades de los adjuntos a operadores Hausdorff de Sturm-Liouville canónicos. El análisis armónico asociado al operador L^M juega un rol importante para establecer los resultados de este artículo.

Keywords and Phrases: Canonical Sturm-Liouville transform, canonical Sturm-Liouville convolution, canonical Sturm-Liouville Hausdorff operators, canonical Sturm-Liouville wavelet transform.

2020 AMS Mathematics Subject Classification: 44A05, 44A20, 47G10.

1 Introduction

The study of Hausdorff operators, which originated from some classical summation methods, has a long history in real and complex analysis. In the one-dimensional setting, Hausdorff operators on the real line were introduced in [10] and studied on the Hardy space in [18]. The natural generalization in several dimensions was introduced and studied in [3,5,16]. Particularly, Hausdorff operators are interesting operators in harmonic analysis [19]. It contains some important operators, such as Hardy operator, adjoint Hardy operator [6,15], and the Cesàro operator [14] in one dimension. The Hardy-Littlewood-Pólya operator and the Riemann-Liouville fractional integral operator can also be derived from the Hausdorff operator [1, 25]. The modern study of general Hausdorff operators on $L^1(\mathbb{R})$ and the real Hardy space $H^1(\mathbb{R})$ over the real line was pioneered by Liflyand and Móricz in [18]. Many research papers have addressed the boundedness of the Hausdorff operator on Hardy spaces. For instance, Liflyand and his collaborators in [16,17] proved, by more effective ways, that the Hausdorff operator has the same behavior on the Hardy space $H^1(\mathbb{R})$ as that in the Lebesgue space $L^1(\mathbb{R})$. Recently, Daher and Saadi in [7, 8] investigated the Dunkl Hausdorff operator on the Lebesgue space $L^1_\alpha(\mathbb{R})$ and on the Hardy space $H^1_\alpha(\mathbb{R})$. Subsequently, Mondal and Poria [22] studied Hausdorff operators associated with the Opdam-Cherednik operator. Furthermore, Tyr [35] studied the boundedness of q-Hausdorff operators on q-Hardy spaces. Another fundamental tool in harmonic analysis is the canonical Sturm-Liouville Hausdorff operators, which is the main object of study in this paper.

Here, we denote by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitary matrix in $SL(2,\mathbb{R})$ such that $b > 0$. We define the canonical Sturm-Liouville operator L^M on $\mathbb{R}_+^*$ by

$$L^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b}\right),$$

where A is a nonnegative function satisfying certain conditions.

Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the operator L^M is reduced to the Sturm-Liouville operator L:

$$L := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x}.$$

The classical Sturm-Liouville operator L plays an important role in analysis [2, 39]. In particular, the two references [4,33] investigate standard constructions of harmonic analysis, such as translation operators, convolution product, and Fourier transform, in connection with the operator L.

Using the Sturm-Liouville harmonic analysis [4, 33], for all $\lambda \in \mathbb{C}$, the system

$$\begin{cases} L^M u = -\dfrac{\lambda^2}{b^2} u, \\ u(0) = e^{\frac{id}{2b}\lambda^2}, \qquad u'(0) = 0, \end{cases}$$

admits a unique solution, denoted by φ_λ^M and given by

$$\varphi_\lambda^M(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)} \varphi_{\frac{\lambda}{b}}(x), \quad x \in \mathbb{R}_+,$$

where $\varphi_\lambda(x)$ is the Sturm-Liouville kernel [29, 30].

In this paper, we introduce the canonical Sturm-Liouville transform $\mathscr{F}^M$:

$$\mathscr{F}^M(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda^M(x) f(x) A(x)\, \mathrm{d}x, \quad \lambda \in \mathbb{R}_+.$$

The canonical Sturm-Liouville transform $\mathscr{F}^M$ can be regarded as a generalization of the Sturm-Liouville transform $\mathscr{F}$ (see [20, 27–32]):

$$\mathscr{F}(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda(x) f(x) A(x)\, \mathrm{d}x, \quad \lambda \in \mathbb{R}_+.$$

Let $\phi \in L^1(\mathbb{R}_+)$. We define the Hausdorff operator H_ϕ associated with the canonical Sturm-Liouville operator L^M for $f \in L^1(\mathbb{R}_+, A(x)\,\mathrm{d}x)$ by

$$H_\phi f(x) := \int_{\mathbb{R}_+} f_t(x) \phi(t)\, \mathrm{d}t,$$

where f_t is the dilation of f given by

$$f_t(x) := \frac{A\left(\frac{x}{t}\right)}{tA(x)} f\left(\frac{x}{t}\right), \quad x \in \mathbb{R}_+.$$

The main purpose of this paper is to extend some results of the classical Hausdorff operator given in [38] to the framework of canonical Sturm-Liouville theory, and to investigate the canonical Sturm-Liouville wavelet transform. We prove the boundedness of canonical Sturm-Liouville Hausdorff operator in space $L^p(\mathbb{R}_+, A(x)\,\mathrm{d}x)$, $p \in [1, \infty)$. The relation between the canonical Sturm-Liouville wavelet transform and the canonical Sturm-Liouville Hausdorff operator is also established. Next, we introduce the adjoint operator H_ϕ^* on $L^2(\mathbb{R}_+, A(x)\,\mathrm{d}x)$ by

$$H_\phi^* f(x) := \int_{\mathbb{R}_+} f(tx) \phi(t)\, \mathrm{d}t, \quad x \in \mathbb{R}_+.$$

We present the properties of the adjoint operator H_ϕ^*, including its boundedness on $L^p(\mathbb{R}_+, A(x)\,\mathrm{d}x)$,

$p \in [1, \infty)$. We also establish a relation between the canonical Sturm-Liouville wavelet transform and the adjoint operator H_ϕ^*.

Note that if $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$, the operator L^M is reduced to the canonical Bessel operator L_α^M:

$$L_\alpha^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{2\alpha+1}{x} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + 2i(\alpha+1)\frac{a}{b}\right).$$

In this case $\varphi_\lambda^M(x) = \varphi_{\lambda,\alpha}^M(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)} j_\alpha(\frac{\lambda x}{b})$, where j_α is the normalized Bessel function of the first kind and order α. The canonical transform $\mathscr{F}^M$ is the canonical Fourier-Bessel transform $\mathscr{F}_\alpha^M$:

$$\mathscr{F}_\alpha^M(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_{\lambda,\alpha}^M(x) f(x) x^{2\alpha+1} \mathrm{d}x, \quad \lambda \in \mathbb{R}_+.$$

Recently, the canonical Fourier-Bessel transform $\mathscr{F}_\alpha^M$ is the goal of many applications in the harmonic analysis (see [9, 11, 12, 21, 26]).

This paper is organized as follows. In Section 2, we recall some results about the Sturm-Liouville transform $\mathscr{F}$, the Sturm-Liouville translation τ_y and the Sturm-Liouville convolution $*$. In Section 3, we introduce the canonical Sturm-Liouville operator L^M, and we investigate the properties of the canonical Sturm-Liouville transform $\mathscr{F}^M$, the canonical Sturm-Liouville translation τ_y^M and the canonical Sturm-Liouville convolution $*^M$ associated with this operator. In Section 4, we introduce the canonical Sturm-Liouville Hausdorff operators H_ϕ and we establish their properties. In the last section, we investigate the canonical Sturm-Liouville wavelet transform and derive its relation with the operators H_ϕ and H_ϕ^*.

2 Sturm-Liouville harmonic analysis

In this section we recall some results about the harmonic analysis associated with the Sturm-Liouville operator (Sturm-Liouville transform, Sturm-Liouville translation and Sturm-Liouville convolution).

We consider the second-order differential operator L defined on $\mathbb{R}_+^*$ by

$$L := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x},$$

where

$$A(x) = x^{2\alpha+1}B(x), \quad \alpha > -1/2,$$

for B a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0) = 1$. Moreover we assume that A satisfies the following conditions:

(i) A is increasing and $\lim_{x\to\infty} A(x) = \infty$.

(ii) $\frac{A'}{A}$ is decreasing and $\lim_{x\to\infty} \frac{A'(x)}{A(x)} = 0$.

(iii) There exists a constant $\delta > 0$ such that

$$\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + e^{-\delta x} D(x), \tag{2.1}$$

where D is an infinitely differentiable function on $\mathbb{R}_+^*$, bounded and with bounded derivatives on all intervals $[x_0, \infty)$, for $x_0 > 0$.

This operator was studied in [4, 33], and the following results have been established:

(I) For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} Lu = -\lambda^2 u, \\ u(0) = 1, \quad u'(0) = 0, \end{cases}$$

admits a unique solution, denoted by φ_λ, with the following properties:

- for $x \in \mathbb{R}_+$, the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic on $\mathbb{C}$.
- For $\lambda \in \mathbb{C}$, the function $x \mapsto \varphi_\lambda(x)$ is even and infinitely differentiable on $\mathbb{R}$.

(II) For nonzero $\lambda \in \mathbb{C}$, the equation

$$Lu = -\lambda^2 u,$$

has a solution Φ_λ satisfying

$$\Phi_\lambda(x) = \frac{e^{i\lambda x}}{\sqrt{A(x)}} V(x, \lambda),$$

with

$$\lim_{x\to\infty} V(x, \lambda) = 1.$$

Consequently there exists a function (spectral function) $\lambda \mapsto c(\lambda)$, such that

$$\varphi_\lambda(x) = c(\lambda)\Phi_\lambda(x) + c(-\lambda)\Phi_{-\lambda}(x), \quad x \in \mathbb{R}_+,$$

for nonzero $\lambda \in \mathbb{C}$.

Moreover there exist positive constants k_1, k_2, k, such that

$$k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1},$$

for all λ such that $\operatorname{Im}\lambda \leq 0$ and $|\lambda| \geq k$.

(III) The Sturm–Liouville kernel $\varphi_\lambda(x)$ possesses the following integral representation of Mehler-type

$$\varphi_\lambda(x) = \int_0^x K(x,y)\cos(\lambda y)\,\mathrm{d}y, \quad x > 0, \tag{2.2}$$

where $K(x,.)$ is an even positive continuous function on $(-x,x)$ and supported in $[-x,x]$.

Using the Mehler integral representation formula (2.2), we obtain

$$-1 \leq \varphi_\lambda(x) \leq 1, \quad \lambda, x \in \mathbb{R}_+. \tag{2.3}$$

We denote by

- μ the measure defined on $\mathbb{R}_+$ by

$$\mathrm{d}\mu(x) := A(x)\,\mathrm{d}x,$$

and by $L^p(\mu)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbb{R}_+$, such that

$$\|f\|_{L^p(\mu)} := \left[\int_{\mathbb{R}_+} |f(x)|^p\,\mathrm{d}\mu(x)\right]^{1/p} < \infty, \quad p \in [1,\infty),$$
$$\|f\|_{L^\infty(\mu)} := \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} |f(x)| < \infty.$$

- ν the measure defined on $\mathbb{R}_+$ by

$$\mathrm{d}\nu(\lambda) := \frac{\mathrm{d}\lambda}{2\pi|c(\lambda)|^2},$$

and by $L^p(\nu)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbb{R}_+$, such that $\|f\|_{L^p(\nu)} < \infty$.

The Sturm-Liouville transform is the Fourier transform associated with the operator L and is defined for $f \in L^1(\mu)$ by

$$\mathscr{F}(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda(x) f(x)\mathrm{d}\mu(x), \quad \lambda \in \mathbb{R}_+. \tag{2.4}$$

Some of the properties of the Sturm-Liouville transform $\mathscr{F}$ are collected bellow.

Theorem 2.1 ([2, 4, 33, 39]). *(i)* ***Plancherel theorem.*** *The Sturm–Liouville transform $\mathscr{F}$ extends uniquely to an isometric isomorphism of $L^2(\mu)$ onto $L^2(\nu)$. In particular,*

$$\|f\|_{L^2(\mu)} = \|\mathscr{F}(f)\|_{L^2(\nu)}.$$

(ii) ***Inversion theorem.*** *Let $f \in L^1(\mu)$, such that $\mathscr{F}(f) \in L^1(\nu)$. Then*

$$f(x) = \int_{\mathbb{R}_+} \varphi_\lambda(x)\mathscr{F}(f)(\lambda)\,\mathrm{d}\nu(\lambda), \quad a.e.\ x \in \mathbb{R}_+.$$

The Sturm-Liouville kernel φ_λ satisfies the product formula [4, 33]

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_{\mathbb{R}_+} \varphi_\lambda(z)w(x,y,z)\,\mathrm{d}\mu(z) \quad \text{for} \quad x, y \in \mathbb{R}_+; \tag{2.5}$$

where $w(x, y, .)$ is a measurable positive function on $\mathbb{R}_+$, with support in $[|x-y|, x+y]$, satisfying

$$\int_{\mathbb{R}_+} w(x,y,z)\,\mathrm{d}\mu(z) = 1,$$

$$w(x,y,z) = w(y,x,z) \quad \text{for} \quad z \in \mathbb{R}_+, \tag{2.6}$$

$$w(x,y,z) = w(x,z,y) \quad \text{for} \quad z > 0. \tag{2.7}$$

We now define the generalized translation operator induced by (2.5). For $f \in L^1(\mu)$, the linear operator

$$\tau_y f(x) := \int_{\mathbb{R}_+} f(z)w(x,y,z)\,\mathrm{d}\mu(z), \quad x, y \in \mathbb{R}_+, \tag{2.8}$$

will be called Sturm-Liouville translation [4, 33].

As a first remark, we note that the relation (2.6) means that

$$\tau_y f(x) = \tau_x f(y), \quad x, y \in \mathbb{R}_+.$$

Theorem 2.2 ([23, 29, 30]). *(i) For all $y \geq 0$ and $f \in L^p(\mu)$, $p \in [1, \infty]$, we have*

$$\|\tau_y f\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)}.$$

(ii) For $f \in L^2(\mu)$ and $y \in \mathbb{R}_+$, we have

$$\mathscr{F}(\tau_y f)(\lambda) = \varphi_\lambda(y)\mathscr{F}(f)(\lambda), \quad \lambda \in \mathbb{R}_+.$$

Let $f, g \in L^2(\mu)$. The Sturm-Liouville convolution $f * g$ of f and g is defined by

$$f * g(x) := \int_{\mathbb{R}_+} \tau_x f(y)g(y)\,\mathrm{d}\mu(y), \quad x \in \mathbb{R}_+. \tag{2.9}$$

The convolution $*$ is commutative, associative and satisfies the Young inequality (see [23]). Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then for $f \in L^p(\mu)$ and $g \in L^q(\mu)$ we have

$$\|f * g\|_{L^r(\mu)} \leq \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Theorem 2.3 ([23, 34]). *(i) For $f, g \in L^2(\mu)$, the function $f * g$ belongs to $L^\infty(\mu)$, and*

$$f * g(x) = \int_{\mathbb{R}_+} \varphi_\lambda(x)\, \mathscr{F}(f)(\lambda) \mathscr{F}(g)(\lambda)\, \mathrm{d}\nu(\lambda), \quad x \in \mathbb{R}_+.$$

(ii) Let $f, g \in L^2(\mu)$. Then

$$\int_{\mathbb{R}_+} |f * g(x)|^2\, \mathrm{d}\mu(x) = \int_{\mathbb{R}_+} |\mathscr{F}^M(f)(\lambda)|^2\, |\mathscr{F}^M(g)(\lambda)|^2\, \mathrm{d}\nu(\lambda),$$

where both sides are finite or infinite.

Example 2.4 ([13, 24]). *Note that if $A(x) = x^{2\alpha+1}$, with $\alpha > -1/2$, the operator L is reduced to the Bessel operator L_α:*

$$L_\alpha := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\alpha + 1}{x} \frac{\mathrm{d}}{\mathrm{d}x}.$$

In this case $\varphi_\lambda(x) = j_\alpha(\lambda x)$, where j_α is the normalized Bessel function of the first kind and order α. We denote by μ_α the measure defined by $\mathrm{d}\mu_\alpha(x) := x^{2\alpha+1}\mathrm{d}x$.

The Fourier-Bessel transform $\mathscr{F}_\alpha$ is defined for $f \in L^1(\mu_\alpha)$ by

$$\mathscr{F}_\alpha(f)(\lambda) := \int_{\mathbb{R}_+} j_\alpha(\lambda x) f(x)\, \mathrm{d}\mu_\alpha(x), \quad \lambda \in \mathbb{R}_+.$$

The Fourier-Bessel translation operators are defined for $f \in L^1(\mu_\alpha)$ by

$$\tau_y^\alpha f(x) := \int_{\mathbb{R}_+} f(z) w_\alpha(x, y, z)\, \mathrm{d}\mu_\alpha(z), \quad x, y \in \mathbb{R}_+,$$

being $w_\alpha(x, y, .)$ the kernel given by

$$w_\alpha(x, y, z) = a_\alpha \frac{[(x+y)^2 - z^2]^{\alpha - \frac{1}{2}} [z^2 - (x-y)^2]^{\alpha - \frac{1}{2}}}{2^{2\alpha - 1}(xyz)^{2\alpha}}\, \chi_{(|x-y|, x+y)}(z), \tag{2.10}$$

where $a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$ and $\chi_{(|x-y|, x+y)}$ is the characteristic function of the interval $(|x-y|, x+y)$. Let $f, g \in L^2(\mu_\alpha)$. The Fourier-Bessel convolution $f *_\alpha g$ of f and g is defined by

$$f *_\alpha g(x) := \int_{\mathbb{R}_+} \tau_x^\alpha f(y) g(y)\, \mathrm{d}\mu_\alpha(y), \quad x \in \mathbb{R}_+.$$

3 Canonical Sturm-Liouville operator

Throughout this paper, we denote by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitary matrix in $SL(2,\mathbb{R})$ such that $b > 0$. We define the canonical Sturm-Liouville operator L^M on $\mathbb{R}_+^*$ by

$$L^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b}\right),$$

where A is the nonnegative function given in Section 2.

Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the operator L^M is reduced to the Sturm-Liouville operator L:

$$L := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x}.$$

For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} L^M u = -\dfrac{\lambda^2}{b^2}u, \\ u(0) = e^{\frac{id}{2b}\lambda^2}, \quad u'(0) = 0, \end{cases}$$

admits a unique solution, denoted by φ_λ^M and given by

$$\varphi_\lambda^M(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)}\varphi_{\frac{\lambda}{b}}(x), \quad x \in \mathbb{R}_+.$$

For $f \in L^1(\mu)$, we define the canonical Sturm-Liouville transform $\mathscr{F}^M(f)$ by

$$\mathscr{F}^M(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda^M(x) f(x)\,\mathrm{d}\mu(x), \quad \lambda \in \mathbb{R}_+.$$

This transform can be written as

$$\mathscr{F}^M(f)(\lambda) = e^{\frac{id}{2b}\lambda^2}\mathscr{F}\left(e^{\frac{ia}{2b}x^2}f\right)\left(\frac{\lambda}{b}\right), \quad f \in L^1(\mu), \tag{3.1}$$

where $\mathscr{F}$ is the Sturm-Liouville transform given by (2.4).

We denote by ν_b, $b > 0$, the measure defined on $\mathbb{R}_+$ by

$$\mathrm{d}\nu_b(\lambda) := \frac{\mathrm{d}\lambda}{2\pi b|c(\frac{\lambda}{b})|^2},$$

and by $L^p(\nu_b)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbb{R}_+$, such that $\|f\|_{L^p(\nu_b)} < \infty$.

Theorem 3.1. *(i) Let $f \in L^1(\mu)$, such that $\mathscr{F}^M(f) \in L^1(\nu_b)$. Then*

$$f(x) = \int_{\mathbb{R}_+} \varphi_\lambda^N(x)\mathscr{F}^M(f)(\lambda)\,\mathrm{d}\nu_b(\lambda), \quad a.e. \quad x \in \mathbb{R}_+,$$

where N is the matrix given by $N = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$.

(ii) For $f \in L^2(\mu)$ we have

$$\|\mathscr{F}^M(f)\|_{L^2(\nu_b)} = \|f\|_{L^2(\mu)}.$$

Proof. (i) follows from Theorem 2.1 (ii) and relation (3.1). (ii) follows from Theorem 2.1 (i) and relation (3.1). □

For $f \in L^1(\mu)$, we define the canonical Sturm-Liouville translation operators by

$$\tau_y^N f(x) := e^{-\frac{ia}{2b}(x^2+y^2)} \int_{\mathbb{R}_+} f(z)e^{\frac{ia}{2b}z^2} w(x,y,z)\,\mathrm{d}\mu(z), \quad x, y \in \mathbb{R}_+. \tag{3.2}$$

It is easy to prove the following results.

Theorem 3.2. *The operators τ_y^N, $y \in \mathbb{R}_+$, satisfy:*

(i) $\tau_y^N f(x) = \tau_x^N f(y)$, $x, y \in \mathbb{R}_+$.

(ii) $\tau_y^N f(x) = e^{-\frac{ia}{2b}(x^2+y^2)}\tau_y\left(f(z)e^{\frac{ia}{2b}z^2}\right)(x)$, *where τ_y is the Sturm-Liouville translation given by* (2.8).

(iii) $\tau_y^M \varphi_\lambda^M(x) = e^{-\frac{id}{2b}\lambda^2}\varphi_\lambda^M(x)\varphi_\lambda^M(y)$.

Theorem 3.3. *(i) For all $y \in \mathbb{R}_+$ and $f \in L^p(\mu)$, $p \in [1, \infty]$, we have*

$$\|\tau_y^N f\|_{L^p(\mu)} \le \|f\|_{L^p(\mu)}.$$

(ii) For $f \in L^2(\mu)$ and $y \in \mathbb{R}_+$, we have

$$\mathscr{F}^M(\tau_y^N f)(\lambda) = e^{\frac{id}{2b}\lambda^2}\varphi_\lambda^N(y)\mathscr{F}^M(f)(\lambda), \quad \lambda \in \mathbb{R}_+,$$

where $N = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$.

Proof. (i) follows from Theorem 2.2 (i) and Theorem 3.2 (ii).

(ii) Let $f \in L^1(\mu) \cap L^2(\mu)$. Then

$$\begin{aligned}\mathscr{F}^M(\tau_y^N f)(\lambda) &= \int_{\mathbb{R}_+} \tau_y^N f(x) \varphi_\lambda^M(x) \, \mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}_+} \left[e^{-\frac{ia}{2b}(x^2+y^2)} \int_{\mathbb{R}_+} f(z) e^{\frac{ia}{2b}z^2} w(x,y,z) \, \mathrm{d}\mu(z) \right] \varphi_\lambda^M(x) \, \mathrm{d}\mu(x).\end{aligned}$$

By using Fubini's theorem, (2.6) and (2.7) we obtain

$$\mathscr{F}^M(\tau_y^N f)(\lambda) = e^{-\frac{ia}{2b}y^2} \int_{\mathbb{R}_+} f(z) e^{\frac{ia}{2b}z^2} \left[\int_{\mathbb{R}_+} \varphi_\lambda^M(x) e^{-\frac{ia}{2b}x^2} w(z,y,x) \, \mathrm{d}\mu(x) \right] \mathrm{d}\mu(z).$$

And by Theorem 3.2 (iii) we deduce that

$$\mathscr{F}^M(\tau_y^N f)(\lambda) = e^{\frac{id}{2b}\lambda^2} \varphi_\lambda^N(y) \mathscr{F}^M(f)(\lambda), \quad \lambda \in \mathbb{R}_+. \tag{3.3}$$

Since $L^1(\mu) \cap L^2(\mu)$ is dense in $L^2(\mu)$, the formula (3.3) remains valid for $f \in L^2(\mu)$. □

Let $f, g \in L^2(\mu)$. The canonical Sturm-Liouville convolution $f *^N g$ of f and g is defined by

$$f *^N g(x) := \int_{\mathbb{R}_+} \tau_x^N f(y) \left[e^{\frac{ia}{b}y^2} g(y) \right] \mathrm{d}\mu(y), \quad x \in \mathbb{R}_+. \tag{3.4}$$

Then we can write

$$f *^N g(x) = e^{-\frac{ia}{2b}x^2} \left(e^{\frac{ia}{2b}z^2} f \right) * \left(e^{\frac{ia}{2b}z^2} g \right)(x), \quad x \in \mathbb{R}_+, \tag{3.5}$$

where $*$ is the Sturm-Liouville convolution given by (2.9).

The canonical Sturm-Liouville convolution $*^N$ is commutative, associative and satisfies the Young inequality. Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then for $f \in L^p(\mu)$ and $g \in L^q(\mu)$ we have

$$\|f *^N g\|_{L^r(\mu)} \leq \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Theorem 3.4. *(i) For $f, g \in L^2(\mu)$, the function $f *^N g$ belongs to $L^\infty(\mu)$, and*

$$f *^N g(x) = \int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \varphi_\lambda^N(x) \mathscr{F}^M(f)(\lambda) \mathscr{F}^M(g)(\lambda) \, \mathrm{d}\nu_b(\lambda), \quad x \in \mathbb{R}_+.$$

(ii) Let $f, g \in L^2(\mu)$. Then

$$\int_{\mathbb{R}_+} |f *^N g(x)|^2 \, \mathrm{d}\mu(x) = \int_{\mathbb{R}_+} |\mathscr{F}^M(f)(\lambda)|^2 |\mathscr{F}^M(g)(\lambda)|^2 \, \mathrm{d}\nu_b(\lambda),$$

where both sides are finite or infinite.

Proof. (i) follows from (3.5), Theorem 2.3 (i) and (3.1). (ii) follows from (3.5), Theorem 2.3 (ii) and (3.1). □

Example 3.5 ([9,11,12,21,26])**.** *Note that if $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$, the operator L^M is reduced to the canonical Bessel operator L^M_α:*

$$L^M_\alpha := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{2\alpha+1}{x} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + 2i(\alpha+1)\frac{a}{b}\right).$$

In this case $\varphi^M_\lambda(x) = \varphi^M_{\lambda,\alpha}(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)} j_\alpha(\frac{\lambda x}{b})$.

The canonical Fourier-Bessel transform $\mathscr{F}^M_\alpha$ is defined for $f \in L^1(\mu_\alpha)$ by

$$\mathscr{F}^M_\alpha(f)(\lambda) := \int_{\mathbb{R}_+} \varphi^M_{\lambda,\alpha}(x) f(x)\,\mathrm{d}\mu_\alpha(x), \quad \lambda \in \mathbb{R}_+.$$

Recently, the canonical Fourier-Bessel transform $\mathscr{F}^M_\alpha$ is the goal of many applications in the harmonic analysis.

The canonical Fourier-Bessel translation operators are defined for $f \in L^1(\mu_\alpha)$ by

$$\tau^{\alpha,N}_y f(x) := e^{-\frac{ia}{2b}(x^2+y^2)} \int_{\mathbb{R}_+} f(z) e^{\frac{ia}{2b}z^2} w_\alpha(x,y,z)\,\mathrm{d}\mu_\alpha(z), \quad x, y \in \mathbb{R}_+,$$

being $w_\alpha(x,y,.)$ the kernel given by (2.10).

Let $f, g \in L^2(\mu_\alpha)$. The canonical Fourier-Bessel convolution $f *^N_\alpha g$ of f and g is defined by

$$f *^N_\alpha g(x) := \int_{\mathbb{R}_+} \tau^{\alpha,N}_x f(y) \left[e^{\frac{ia}{b}y^2} g(y)\right] \mathrm{d}\mu_\alpha(y), \quad x \in \mathbb{R}_+.$$

4 Canonical Sturm-Liouville Hausdorff operator

In this section we define and study the Hausdorff operator associated with the canonical Sturm-Liouville operator L^M.

Let $f \in L^p(\mu)$, $p \in [1, \infty)$ and $t > 0$. We define the dilation function f_t by

$$f_t(x) := \frac{A\left(\frac{x}{t}\right)}{tA(x)} f\left(\frac{x}{t}\right), \tag{4.1}$$

and satisfies

$$\|f_t\|_{L^p(\mu)} \le \left(\frac{k(t)}{t}\right)^{1-\frac{1}{p}} \|f\|_{L^p(\mu)}, \tag{4.2}$$

where

$$k(t) = \sup_{x \in \mathbb{R}_+} \left(\frac{A(x)}{A(tx)}\right).$$

From (2.1), there exist two constants $C_1, C_2 > 0$, such that

$$C_1 x^{2\alpha+1} \leq A(x) \leq C_2 x^{2\alpha+1}, \quad x \in \mathbb{R}_+^*.$$

Therefore,

$$\frac{1}{Ct^{2\alpha+1}} \leq k(t) \leq \frac{C}{t^{2\alpha+1}}, \quad t > 0,$$

where $C = \frac{C_2}{C_1}$.

Let $\phi \in L^1(\mathbb{R}_+)$. We define the Hausdorff operator H_ϕ associated with the canonical Sturm-Liouville operator L^M for $f \in L^1(\mu)$ by

$$H_\phi f(x) := \int_{\mathbb{R}_+} f_t(x)\phi(t)\, \mathrm{d}t. \tag{4.3}$$

If we choose $\phi(t) = \beta(1-t)^{\beta-1}\chi_{(0,1)}(t)$, $\beta > 0$, we obtain the canonical Sturm-Liouville Cesàro operator of order β denoted by $\mathcal{C}_\beta$ and given by

$$\mathcal{C}_\beta f(x) := \beta \int_0^1 f_t(x)(1-t)^{\beta-1}\, \mathrm{d}t.$$

A brief history of the study of Cesàro operator can be found in [14].

If we choose $\phi(t) = \frac{1}{t}\chi_{(1,\infty)}(t)$, we obtain the canonical Sturm-Liouville Hardy operator denoted by $\mathcal{H}$ and given by

$$\mathcal{H}f(x) := \int_1^\infty f_t(x)\, \frac{\mathrm{d}t}{t}.$$

It is well known that Hardy operators are important operators in harmonic analysis, for instance, see [6, 15].

If we choose $\phi(t) = \frac{1}{\max(1,t)}$, we obtain the canonical Sturm-Liouville Hardy-Littlewood-Pólya operator denoted by $\mathcal{P}$ and given by

$$\mathcal{P}f(x) := \int_0^1 f_t(x)\mathrm{d}t + \int_1^\infty f_t(x)\, \frac{\mathrm{d}t}{t}.$$

The study of Hardy-Littlewood-Pólya operators can be found in [1].

If we choose $\phi(t) = \frac{1}{\Gamma(\eta)}\frac{(1-\frac{1}{t})^{\eta-1}}{t}\chi_{(1,\infty)}(t)$, $\eta > 0$ we obtain the canonical Sturm-Liouville Riemann-Liouville fractional integral operator denoted by $\mathcal{I}$ and given by

$$\mathcal{I}f(x) := \frac{1}{\Gamma(\eta)} \int_1^\infty f_t(x) \left(1 - \frac{1}{t}\right)^{\eta-1} \frac{\mathrm{d}t}{t}.$$

The study of Riemann-Liouville fractional integral operators can be found in [25].

Theorem 4.1. *Let $\phi \in L^1(\mathbb{R}_+)$. Then for $f \in L^1(\mu)$, we have*

$$\mathscr{F}^M(H_\phi f)(\lambda) = \int_{\mathbb{R}_+} \mathscr{F}^M(f_t)(\lambda)\phi(t)\,\mathrm{d}t, \quad \lambda \in \mathbb{R}_+.$$

Proof. Let $\phi \in L^1(\mathbb{R}_+)$, and let $f \in L^1(\mu)$. Then by (4.3) we have

$$\mathscr{F}^M(H_\phi f)(\lambda) = \int_{\mathbb{R}_+} H_\phi f(x)\varphi_\lambda^M(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}_+}\left[\int_{\mathbb{R}_+} f_t(x)\phi(t)\,\mathrm{d}t\right]\varphi_\lambda^M(x)\,\mathrm{d}\mu(x).$$

Since

$$\int_{\mathbb{R}_+^2} |f_t(x)||\phi(t)||\varphi_\lambda^M(x)|\,\mathrm{d}t\,\mathrm{d}\mu(x) \le \|\phi\|_{L^1(\mathbb{R}_+)}\|f\|_{L^1(\mu)} < \infty,$$

by Fubini's theorem we obtain

$$\mathscr{F}^M(H_\phi f)(\lambda) = \int_{\mathbb{R}_+}\left[\int_{\mathbb{R}_+} f_t(x)\varphi_\lambda^M(x)\,\mathrm{d}\mu(x)\right]\phi(t)\,\mathrm{d}t = \int_{\mathbb{R}_+} \mathscr{F}^M(f_t)(\lambda)\phi(t)\,\mathrm{d}t.$$

The theorem is proved. □

Theorem 4.2. *Let ϕ be a measurable function on $\mathbb{R}_+$ such that*

$$C_{\phi,p} := \int_{\mathbb{R}_+}\left(\frac{k(t)}{t}\right)^{1-\frac{1}{p}} |\phi(t)|\,\mathrm{d}t < \infty. \tag{4.4}$$

Then the Hausdorff operator H_ϕ is bounded on $L^p(\mu)$, $p \in [1,\infty)$ with

$$\|H_\phi f\|_{L^p(\mu)} \le C_{\phi,p}\|f\|_{L^p(\mu)}.$$

Proof. By using Minkowski's inequality for integrals, we have

$$\begin{aligned}\|H_\phi f\|_{L^p(\mu)} &= \left[\int_{\mathbb{R}_+}\left|\int_{\mathbb{R}_+} f_t(x)\phi(t)\,\mathrm{d}t\right|^p \mathrm{d}\mu(x)\right]^{1/p} \le \left[\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+} |f_t(x)||\phi(t)|\,\mathrm{d}t\right)^p \mathrm{d}\mu(x)\right]^{1/p}\\ &\le \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+} |f_t(x)|^p|\phi(t)|^p\,\mathrm{d}\mu(x)\right)^{1/p}\mathrm{d}t = \int_{\mathbb{R}_+} \|f_t\|_{L^p(\mu)}|\phi(t)|\,\mathrm{d}t.\end{aligned}$$

Then by (4.2) we obtain

$$\|H_\phi f\|_{L^p(\mu)} \le C_{\phi,p}\|f\|_{L^p(\mu)}.$$

Going back to the definition of

$$\left[\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+} |f_t(x)||\phi(t)|\,\mathrm{d}t\right)^p \mathrm{d}\mu(x)\right]^{1/p},$$

we deduce that the integral

$$H_\phi f(x) = \int_{\mathbb{R}_+} f_t(x)\phi(t)\,\mathrm{d}t,$$

is absolutely convergent for almost all $x \in \mathbb{R}_+$, and defines a function $H_\phi f \in L^p(\mathbb{R}_+)$. □

Let $f, g \in L^2(\mu)$, and let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition

$$C_{\phi,2} := \int_{\mathbb{R}_+} \left(\frac{k(t)}{t}\right)^{\frac{1}{2}} |\phi(t)|\,\mathrm{d}t < \infty. \tag{4.5}$$

We define the adjoint operator H_ϕ^* by the relation

$$\int_{\mathbb{R}_+} H_\phi^* f(x) g(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}_+} f(x) H_\phi g(x)\,\mathrm{d}\mu(x).$$

Theorem 4.3. *Let $f \in L^2(\mu)$, and let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition* (4.5). *Then*

$$H_\phi^* f(x) = \int_{\mathbb{R}_+} f(tx)\phi(t)\,\mathrm{d}t. \tag{4.6}$$

Proof. Let $f, g \in L^2(\mu)$, and let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition (4.5). From (4.3) and Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{R}_+} f(x) H_\phi g(x)\,\mathrm{d}\mu(x) &= \int_{\mathbb{R}_+} f(x)\left[\int_{\mathbb{R}_+} g_t(x)\phi(t)\,\mathrm{d}t\right]\mathrm{d}\mu(x)\\ &= \int_{\mathbb{R}_+}\left[\int_{\mathbb{R}_+} f(x) g_t(x)\,\mathrm{d}\mu(x)\right]\phi(t)\,\mathrm{d}t = \int_{\mathbb{R}_+}\left[\int_{\mathbb{R}_+} f(tx) g(x)\,\mathrm{d}\mu(x)\right]\phi(t)\,\mathrm{d}t.\end{aligned}$$

Using (4.2), this calculation is justified by the fact that

$$\int_{\mathbb{R}_+^2} |f(x)||g_t(x)|\mathrm{d}\mu(x)|\phi(t)|\,\mathrm{d}t \leq C_{\phi,2}\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)} < \infty.$$

Then according to Fubini's theorem we obtain

$$\int_{\mathbb{R}_+} f(x) H_\phi g(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}_+}\left[\int_{\mathbb{R}_+} f(tx)\phi(t)\,\mathrm{d}t\right] g(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}_+} H_\phi^* f(x) g(x)\,\mathrm{d}\mu(x),$$

where

$$H_\phi^* f(x) = \int_{\mathbb{R}_+} f(tx)\phi(t)\,\mathrm{d}t.$$

This calculation is justified by the fact that

$$\int_{\mathbb{R}_+^2} |f(tx)||g(x)|\,\mathrm{d}\mu(x)|\phi(t)|\,\mathrm{d}t \leq C_{\phi,2}\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)} < \infty.$$

This completes the proof of the theorem. □

Remark 4.4. *From Theorem 4.2, the operator* H^*_ϕ *is bounded on* $L^p(\mu)$, $p \in [1,\infty)$, *with*

$$\|H^*_\phi f\|_{L^p(\mu)} \leq C_{\phi,\frac{p}{p-1}} \|f\|_{L^p(\mu)},$$

where $C_{\phi,p}$ *is the constant given by* (4.4).

As in the same of Theorem 4.1, we obtain the following result.

Theorem 4.5. *Let* ϕ *be a measurable function on* $\mathbb{R}_+$ *satisfying the condition*

$$C_{\phi,\infty} := \int_{\mathbb{R}_+} \frac{k(t)}{t} |\phi(t)| \, \mathrm{d}t < \infty. \tag{4.7}$$

Then for $f \in L^1(\mu)$, *we have*

$$\mathscr{F}^M(H^*_\phi f)(\lambda) = \int_{\mathbb{R}_+} \mathscr{F}^M(f^*_t)(\lambda)\phi(t) \, \mathrm{d}t, \quad \lambda \in \mathbb{R}_+,$$

where $f^*_t(x) = f(tx)$.

Proof. Let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition (4.7), and let $f \in L^1(\mu)$. Then by (4.6) we have

$$\mathscr{F}^M(H^*_\phi f)(\lambda) = \int_{\mathbb{R}_+} H^*_\phi f(x) \varphi^M_\lambda(x) \, \mathrm{d}\mu(x) = \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} f(tx)\phi(t) \, \mathrm{d}t \right] \varphi^M_\lambda(x) \, \mathrm{d}\mu(x).$$

Since

$$\int_{\mathbb{R}^2_+} |f(tx)||\phi(t)||\varphi^M_\lambda(x)| \, \mathrm{d}t \, \mathrm{d}\mu(x) \leq C_{\phi,\infty} \|f\|_{L^1(\mu)} < \infty,$$

by Fubini's theorem we obtain

$$\mathscr{F}^M(H^*_\phi f)(\lambda) = \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} f(tx) \varphi^M_\lambda(x) \, \mathrm{d}\mu(x) \right] \phi(t) \, \mathrm{d}t = \int_{\mathbb{R}_+} \mathscr{F}^M(f^*_t)(\lambda)\phi(t) \, \mathrm{d}t.$$

The theorem is proved. □

Example 4.6. *Note that if* $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$, *we have*

$$f_t(x) = \frac{1}{t^{2\alpha+2}} f\left(\frac{x}{t}\right), \quad k(t) = \frac{1}{t^{2\alpha+1}}, \quad C_{\phi,p} = \int_{\mathbb{R}_+} \frac{|\phi(t)|}{t^{(2\alpha+2)(1-\frac{1}{p})}} \, \mathrm{d}t.$$

Therefore,

- *the canonical Bessel-Hausdorff operator is given by*

$$H_\phi f(x) = \int_{\mathbb{R}_+} f\left(\frac{x}{t}\right) \frac{\phi(t)}{t^{2\alpha+2}} \, \mathrm{d}t.$$

- *The canonical Bessel-Cesàro operator of order β is given by*

$$\mathcal{C}_\beta f(x) = \beta \int_0^1 f\left(\frac{x}{t}\right) \frac{(1-t)^{\beta-1}}{t^{2\alpha+2}}\, \mathrm{d}t.$$

- *The canonical Bessel-Hardy operator is given by*

$$\mathcal{H} f(x) = \int_1^\infty f\left(\frac{x}{t}\right) \frac{\mathrm{d}t}{t^{2\alpha+3}}.$$

- *The canonical Bessel-Hardy-Littlewood-Pólya operator is given by*

$$\mathcal{P} f(x) = \int_0^1 f\left(\frac{x}{t}\right) \frac{\mathrm{d}t}{t^{2\alpha+2}} + \int_1^\infty f\left(\frac{x}{t}\right) \frac{\mathrm{d}t}{t^{2\alpha+3}}.$$

- *The canonical Bessel-Riemann-Liouville fractional integral operator is given by*

$$\mathcal{I} f(x) = \frac{1}{\Gamma(\eta)} \int_1^\infty f\left(\frac{x}{t}\right) \left(1 - \frac{1}{t}\right)^{\eta-1} \frac{\mathrm{d}t}{t^{2\alpha+3}}.$$

5 Canonical Sturm-Liouville wavelet transform

In this section, we first recall some fundamental results on the canonical Sturm-Liouville wavelet transform. The classical Sturm-Liouville wavelet transform has been studied extensively in [23,34] where detailed definitions, illustrative examples, and comprehensive discussions of its properties can be found. In the following we establish a relation between the canonical Sturm-Liouville wavelet transform and the canonical Sturm-Liouville Hausdorff operator.

As in the same of [23,34] and by using Theorem 3.1 (ii), we prove following lemma.

Theorem 5.1. *Let $g \in L^2(\mu)$, and $t > 0$. Then there exists a function $g_r^\sharp$ in $L^2(\mu)$, such that*

$$\mathscr{F}^M(g_r^\sharp)(\lambda) = \mathscr{F}^M(g)(r\lambda), \quad \lambda \in \mathbb{R}_+, \tag{5.1}$$

and satisfies

$$\|g_r^\sharp\|_{L^2(\mu)} \le \frac{\ell_b(r)}{\sqrt{r}} \|g\|_{L^2(\mu)}, \tag{5.2}$$

where

$$\ell_b(r) = \sup_{\lambda > 0} \frac{|c(\frac{\lambda}{b})|}{|c(\frac{\lambda}{rb})|}.$$

We say that a function $g \in L^2(\mu)$ is a canonical Sturm-Liouville wavelet, if it satisfies the admissibility condition

$$0 < \omega_g := \int_{\mathbb{R}_+} |\mathscr{F}^M(g)(\lambda)|^2 \frac{\mathrm{d}\lambda}{\lambda} < \infty. \tag{5.3}$$

Example 5.2. *The function g given by*

$$g(x) := \int_{\mathbb{R}_+} \lambda^2 e^{-\lambda^2} \varphi_\lambda^N(x)\,\mathrm{d}\nu_b(\lambda), \quad x \in \mathbb{R}_+,$$

is a canonical Sturm-Liouville wavelet and $\omega_g = \frac{1}{8}$. Note that if $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$, we have

$$g(x) := -\frac{e^{-\frac{ia}{2b}x^2}}{2^\alpha \Gamma(\alpha+1)} \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{e^{-\frac{x^2}{2(ibd+2tb^2)}}}{(ibd+2tb^2)^{\alpha+1}} \right]_{t=0}, \quad x \in \mathbb{R}_+,$$

For a function $g \in L^2(\mu)$ and for $(r,s) \in \mathbb{R}_+^* \times \mathbb{R}_+$ we denote by $g_{r,s}$ the function defined on $\mathbb{R}_+$ by

$$g_{r,s}^\sharp(y) := \tau_s^N g_r^\sharp(y),$$

where τ_s^N are the generalized translation operators given by (3.2).

From Theorem 3.3 (i) and (5.2), the function $g_{r,s}^\sharp$ satisfies

$$\|g_{r,s}^\sharp\|_{L^2(\mu)} \leq \frac{\ell_b(r)}{\sqrt{r}} \|g\|_{L^2(\mu)}. \tag{5.4}$$

Let $g \in L^2(\mu)$ be a canonical Sturm-Liouville wavelet. We define for regular functions on $\mathbb{R}_+$, the canonical Sturm-Liouville wavelet transform by

$$\Phi_g^N(f)(r,s) := \int_{\mathbb{R}_+} e^{\frac{ia}{b}y^2} f(y) g_{r,s}^\sharp(y)\,\mathrm{d}\mu(y), \tag{5.5}$$

which can also be written in the form

$$\Phi_g^N(f)(r,s) = f *^N g_r^\sharp(s), \tag{5.6}$$

where $*^N$ is the generalized convolution product given by (3.4).

From (5.4) and (5.5) with Hölder's inequality, we have

$$\|\Phi_g^N(f)(r,.)\|_{L^\infty(\mu)} \leq \frac{\ell_b(r)}{\sqrt{r}} \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)}.$$

From (5.6), Theorem 3.4 (i) and (5.1), we have

$$\Phi_g^N(f)(r,s) = \int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \varphi_\lambda^N(s) \mathscr{F}^M(f)(\lambda) \mathscr{F}^M(g)(r\lambda)\,\mathrm{d}\nu_b(\lambda). \tag{5.7}$$

We denote by γ the measure defined on $\mathbb{R}_+^2$ by

$$\mathrm{d}\gamma(r,s) := \mathrm{d}\mu(s)\,\frac{\mathrm{d}r}{r},$$

and by $L^2(\gamma)$ the space of measurable functions f on $\mathbb{R}_+^2$, such that

$$\|f\|_{L^2(\gamma)} := \left[\int_{\mathbb{R}_+^2} |f(r,s)|^2\,\mathrm{d}\mu(s)\,\frac{\mathrm{d}r}{r}\right]^{1/2} < \infty.$$

Theorem 5.3. *Let $g \in L^2(\mu)$ be a canonical Sturm-Liouville wavelet.*

(i) Plancherel formula for Φ_g^N. For $f \in L^2(\mu)$ we have

$$\|f\|_{L^2(\mu)}^2 = \frac{1}{\omega_g}\|\Phi_g^N(f)\|_{L^2(\gamma)}^2.$$

(ii) Parseval formula for Φ_g^N. For $f, h \in L^2(\mu)$ we have

$$\langle f, h\rangle_{L^2(\mu)} = \frac{1}{\omega_g}\langle \Phi_g^N(f), \Phi_g^N(h)\rangle_{L^2(\gamma)}.$$

Proof. (i) Using Fubini's theorem, Theorem 3.4 (ii), and the relation (5.6), we obtain

$$\begin{aligned}\frac{1}{\omega_g}\|\Phi_g^N(f)\|_{L^2(\gamma)}^2 &= \frac{1}{\omega_g}\int_{\mathbb{R}_+^2} |f *^N g_r^\sharp(s)|^2\,\mathrm{d}\mu(s)\,\frac{\mathrm{d}r}{r}\\ &= \frac{1}{\omega_g}\int_{\mathbb{R}_+^2} |\mathscr{F}^M(f)(\lambda)|^2|\mathscr{F}^M(g_r^\sharp)(\lambda)|^2\,\mathrm{d}\nu_b(\lambda)\,\frac{\mathrm{d}r}{r}\\ &= \int_{\mathbb{R}_+} |\mathscr{F}^M(f)(\lambda)|^2\left(\frac{1}{\omega_g}\int_{\mathbb{R}_+} |\mathscr{F}^M(g)(r\lambda)|^2\,\frac{\mathrm{d}r}{r}\right)\mathrm{d}\nu_b(\lambda).\end{aligned}$$

By relation (5.3) we have

$$\frac{1}{\omega_g}\int_{\mathbb{R}_+} |\mathscr{F}^M(g)(r\lambda)|^2\,\frac{\mathrm{d}r}{r} = 1.$$

Then we deduce the desired result from Theorem 3.1 (ii).

(ii) The result is easily deduced from (i). □

We obtain a relation between the canonical Sturm-Liouville wavelet transform and the canonical Sturm-Liouville Hausdorff operator.

Theorem 5.4. *Let $g \in L^2(\mu)$ be a canonical Sturm-Liouville wavelet, and let $\phi \in L^1(\mathbb{R}_+)$ satisfying the condition* (4.5)*. Then for $f \in L^1(\mu) \cap L^2(\mu)$ we have*

$$\Phi_g^N(H_\phi f)(r,s) = \int_{\mathbb{R}_+} \Phi_g^N(f_t)(r,s)\phi(t)\,\mathrm{d}t,$$

where f_t is the dilation of f given by (4.1)*.*

Proof. Let $g \in L^2(\mu)$ be a canonical Sturm-Liouville wavelet, and let $f \in L^1(\mu) \cap L^2(\mu)$. From Theorem 4.2 we have $H_\phi f \in L^2(\mu)$. Then by (5.7) and Theorem 4.1, we get

$$\begin{aligned}
\Phi_g^N(H_\phi f)(r,s) &= \int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \mathscr{F}^M(H_\phi f)(\lambda)\mathscr{F}^M(g)(r\lambda)\varphi_\lambda^N(s)\,\mathrm{d}\nu_b(\lambda) \\
&= \int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \left[\int_{\mathbb{R}_+} \mathscr{F}^M(f_t)(\lambda)\phi(t)\,\mathrm{d}t\right] \mathscr{F}^M(g)(r\lambda)\varphi_\lambda^N(s)\,\mathrm{d}\nu_b(\lambda) \\
&= \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \mathscr{F}^M(f_t)(\lambda)\mathscr{F}^M(g)(r\lambda)\varphi_\lambda^N(s)\,\mathrm{d}\nu_b(\lambda)\right] \phi(t)\,\mathrm{d}t \\
&= \int_{\mathbb{R}_+} \Phi_g^N(f_t)(r,s)\phi(t)\,\mathrm{d}t.
\end{aligned}$$

Using (4.2), this calculation is justified by the fact that

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |\mathscr{F}^M(f_t)(\lambda)||\mathscr{F}^M(g_r^\sharp)(\lambda)|\,\mathrm{d}\nu_b(\lambda)|\phi(t)|\,\mathrm{d}t \le C_{\phi,2}\|f\|_{L^2(\mu)}\|g_r^\sharp\|_{L^2(\mu)} < \infty.$$

This ends the proof of the theorem. □

As in the same of Theorem 5.4, we obtain the following result.

Theorem 5.5. *Let $g \in L^2(\mu)$ be a canonical Sturm-Liouville wavelet, and Let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the conditions* (4.5) *and* (4.7)*. Then for $f \in L^1(\mu) \cap L^2(\mu)$ we have*

$$\Phi_g^N(H_\phi^* f)(r,s) = \int_{\mathbb{R}_+} \Phi_g^N(f_t^*)(r,s)\phi(t)\,\mathrm{d}t,$$

where $f_t^*(x) = f(tx)$.

Proof. Let $g \in L^2(\mu)$ be a canonical Sturm-Liouville wavelet, and let $f \in L^1(\mu) \cap L^2(\mu)$. From Remark 4.4 we have $H_\phi^* f \in L^2(\mu)$. Then by (5.7) and Theorem 4.5, we get

$$\begin{aligned}
\Phi_g^N(H_\phi^* f)(r,s) &= \int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \mathscr{F}^M(H_\phi^* f)(\lambda)\mathscr{F}^M(g)(r\lambda)\varphi_\lambda^N(s)\,\mathrm{d}\nu_b(\lambda) \\
&= \int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \left[\int_{\mathbb{R}_+} \mathscr{F}^M(f_t^*)(\lambda)\phi(t)\,\mathrm{d}t\right] \mathscr{F}^M(g)(r\lambda)\varphi_\lambda^N(s)\,\mathrm{d}\nu_b(\lambda)
\end{aligned}$$

$$= \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} e^{-\frac{id}{2b}\lambda^2} \mathscr{F}^M(f_t^*)(\lambda) \mathscr{F}^M(g)(r\lambda) \varphi_\lambda^N(s) \, d\nu_b(\lambda) \right] \phi(t) \, dt$$
$$= \int_{\mathbb{R}_+} \Phi_g^N(f_t^*)(r,s)\phi(t) \, dt.$$

This calculation is justified by the fact that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |\mathscr{F}^M(f_t^*)(\lambda)||\mathscr{F}^M(g_r^\sharp)(\lambda)| \, d\nu_b(\lambda)|\phi(t)| \, dt \leq C_{\phi,2} \|f\|_{L^2(\mu)} \|g_r^\sharp\|_{L^2(\mu)} < \infty.$$

This ends the proof of the theorem. □

Conclusion

In this work we have succeeded in generalizing the results of Móricz for the classical Hausdorff operator [38], Upadhyay *et al.* for the Hankel Hausdorff operator [36, 37] and Daher *et al.* for the Dunkl Hausdorff operator [7, 8] to the setting of canonical Sturm-Liouville theory. In this paper, we have studied the canonical Sturm-Liouville Hausdorff operator on the Lebesgue space $L^p(\mu)$, $p \in [1, \infty)$. Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we obtain the results of the classical Sturm-Liouville case.

Conflicts of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Data availability statement

There are no data used in this manuscript.

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Characterization of bc and strongly bc-polyharmonic functions

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ABSTRACT

We provide new characterizations of the bicomplex harmonic and strongly bc-harmonic functions in terms of bc-holomorphic functions. An extension to the bc-polyharmonic setting is investigated. We also derive similar bicomplex analog for strongly bc-polyharmonic functions of finite bi-order.

RESUMEN

Entregamos nuevas caracterizaciones de las funciones bicomplejas armónicas y fuertemente bc-armónicas en términos de funciones bc-holomorfas. Se investiga una extensión al marco bc-poliarmónico. También derivamos análogos bicomplejos similares para funciones fuertemente bc-poliarmónicas de bi-orden finito.

Keywords and Phrases: bc-polyholomorphic functions, bc-harmonic functions, strongly bc-harmonic functions, bc-polyharmonicity, Almansi's theorem.

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1 Introduction

Polyharmonic functions with respect to the familiar Laplace operator are a natural extension of harmonic functions [7]. The latter have been extensively studied in the literature [7, 11, 28] and have played a crucial role in different areas of mathematics and physics, including the theory of holomorphic functions, the study of elliptic partial differential equations, minimal surfaces, digital processing and electrical engineering. Recall that a $2m$ times continuously differentiable complex-valued function f in the n-dimensional Euclidean space $\mathbb{R}^n$ is said to be polyharmonic of order m in a domain $\Omega \subset \mathbb{R}^n$, if it satisfies $\Delta^m f(x) = 0$ for $x \in \Omega$, where Δ^m is the m-th iterate of the Laplace operator

$$\Delta = \frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right), \quad x = (x_1 x_2, \dots, x_n).$$

For $m = 2$, they are the so-called biharmonic functions, intervening in elasticity theory. We should point out that polyharmonic functions have been studied by the end of the nineteenth century by the classical paper [4] by Almansi. His main result states that for every polyharmonic function f of order m on a star domain Ω, there exist some harmonic functions h_k, $k = 0, \dots, m$, on Ω such that

$$f(x) = |x|^{2m} h_m(x) + |x|^{2(m-1)} h_{m-1}(x) + \cdots + h_0(x).$$

This extends in fact the Gauss decomposition of a polynomial [3, 26]. The development of their theory is due to Nicolesco [30] and Aronszajn [6] works. Recently, they have been the subject of many investigations in a variety of mathematical and engineering fields, including numerical analysis, approximation of functions, wavelet analysis, the construction of multivariate splines and image processing. For a broader overview of these matters and its various applications see, *e.g.* [5, 8, 22, 26, 29] and the references therein.

On the other hand, the analysis within the bicomplex numbers generalizing complex numbers is currently a fully developed field of study. Its introduction goes back to Segre [39]. Next, they have been elaborated by the Italian school in the early twentieth century [14,40]. Comprehensive studies were later carried out in [32, 34, 41]. In the last decades, they have been rediscovered, developed, and have attracted growing attention with some intriguing new advances with wide applications [2, 9, 12, 13, 18, 19, 21, 31, 37, 38, 42]. In fact, they have been used to discuss different aspects of the bicomplex neural networks [25,43], and furthermore serve as an appropriate model for representing color image encoding in image processing [3, 17]. Bicomplex analysis was also investigated in the finite element method with a significant improvement when compared to the real and complex cases [33]. Moreover, they are an ideal context to extend the classical results concerning signal processing and time-frequency analysis using tools from frame theory [15].

One of the well-developed axes in bicomplex analysis is the theory of holomorphic functions of a bicomplex variable. In fact, it was widely studied in [32] (see also [14,36,40]) with a close connection with functional calculus, theory of function spaces and integral transforms [15, 19, 21]. Contrary to this theory, harmonic and potential theories are new areas of research that emerge within the framework of bicomplex numbers. For some of their fundamentals, one refers for instance to [1,16]. Notice that different bicomplex analogs of the classical mean value theorems (MVT) have been obtained in [1] for bc-harmonic and strongly bc-harmonic functions, as well as their analytical and geometrical converses, including the bicomplex analog of Hansen and Nadirashvili's result [23]. While a complete characterization of hyperbolic-valued bc-harmonic functions, in terms of the bicomplex holomorphic functions, has been provided in [16]. It is proved in particular that a real-valued bicomplex harmonic function is not necessarily the hyperbolic real part of a bicomplex holomorphic function, but of a bicomplex polyholomorphic one. A result that was next extended to the bicomplex polyharmonic functions.

In the present paper, we intend to pursue such investigation of extending to bicomplex context the fecund theory of harmonic and polyharmonic functions of complex variable. In fact, we are concerned with the bicomplex versions of some known results satisfied by the classical harmonic functions on the complex plane $\mathbb{C}$. Namely, we establish a concrete characterization of the strongly bc-harmonic functions (Theorem 3.1), as well as different bicomplex analogs of the additive decomposition theorem for bc-harmonic and strongly bc-harmonic functions. The initial motivation for the second task is a classical fact in complex analysis asserting that harmonic functions are exactly those that can be rewritten as $H+\overline{G}$ for certain holomorphic functions H and G, which usually is proved using the characterization of holomorphic functions in terms of the Wirtinger operators. The proof of "only if" can also be handled starting from the fact that a real-valued harmonic function is the real part of a holomorphic function, which fails when dealing with bc-harmonic functions as pointed out in [16]. Accordingly, it seems to be natural and interesting to know whether bc-harmonic (or bc-polyharmonic in general) functions can still have a similar additive decomposition. This paper contains then an answer to this question. To this end, one makes use of the expected characterization of an hyperbolic-valued bc-harmonic function F being the hyperbolic real part of a bc-holomorphic function if and only if F belongs to the kernels of some bicomplex first order differential operators. We also show that a bicomplex-valued function F on $\mathbb{BC}$ in $\ker(\partial_{\widetilde{Z}}) \cap \ker(\partial_{Z^\dagger})$ is bc-harmonic if and only if there exist certain bicomplex holomorphic functions H and G such that $F = H + G^*$, where $*$ denotes the complex conjugation in $\mathbb{BC}$ with respect to the bicomplex ij. More generally, we derive an additional decomposition without assuming the condition of belonging to $\ker(\partial_{\widetilde{Z}}) \cap \ker(\partial_{Z^\dagger})$, see Theorem 3.7. Similar characterization for bc-polyharmonic functions of finite order in terms of special subclass of bc-polyholomorphic functions is also obtained in Theorem 3.3. The main tool in its proof relies on [16, Proposition 3.8]. However, for a formal proof, see Remark 3.4, where one makes use of Proposition 4.4 in

[16], giving a bicomplex analog of Almansi's theorem for the representation of bc-polyharmonic in terms of bc-harmonic functions. An explicit characterization of the so-called strongly bc-harmonic is also provided (Theorem 3.1). This result is then employed to give a precise description of the bc-harmonic functions arising as $H_0 + H_1^* + H_2^\dagger + \widetilde{H}_3$, for some bc-holomorphic functions H_ℓ, $\ell = 0, 1, 2, 3$. See Theorem 3.5 for an exact statement. The motivation for considering strongly bc-harmonic functions lies in the fact that an explicit and complete description of some spectral aspects of the bc-harmonic functions needs in general an additional harmonicity condition with respect to the $*$-conjugation, see for example [1, 2]. This phenomena will be confirmed in the present investigation.

We anticipate that the findings will be helpful for ulterior uses and applications. In fact, we claim that they can be employed to give the explicit formula for special bicomplex Bergman and Bargmann spaces of bc-harmonic functions as well as the integral representation for their elements by Bargmann type transform. We also anticipate extending the obtained results to the bicomplex analog of the so-called (α, β)-harmonic functions (see *e.g.* [10, 20, 24] and the references therein), which are defined as those that are twice continuously differentiable functions u solutions of the homogeneous equation $L^\varepsilon_{\alpha,\beta} u = 0$ on the complex plane ($\varepsilon = 0$) or the hyperbolic unit disc ($\varepsilon = +1$), where

$$L^\varepsilon_{\alpha,\beta} := (1 - \varepsilon|z|^2)\left\{(1 - \varepsilon|z|^2)\Delta + \alpha z \partial_z + \beta \overline{z} \partial_{\overline{z}} - \alpha\beta\right\}.$$

Notice that for $\alpha = -\beta$, it has been initiated and implicitly investigated in [2], by considering a pair of bicomplex magnetic Laplacians on $\mathbb{BC}$ and the disc.

The paper is outlined as follows. In Section 2, we fix the notations, including those announced above and related to the bicomplex numbers. We also define the bicomplex Laplace type operator and different notions of bc-harmonicity that we will work with. Section 3 deals with the proof of Theorem 3.1, giving a complete description of strongly bc-harmonic functions, as well as the additive decomposition theorems characterizing the bc-harmonic (Theorems 3.2 and 3.7) and bc-polyharmonic (Theorem 3.3) functions. The last section deals with some concluding remarks to answer the question how can the obtained conclusions be properly adapted to product-type domains.

2 Preliminaries

In this section, we briefly review some basic and needed notions from bicomplex analysis, we fix notations, and we introduce the different notions of harmonicity in the bicomplex setting that we will consider in this paper.

2.1 Bicomplex numbers

Bicomplex numbers are defined by complexifying the complex numbers $z = x + iy \in \mathbb{C}$ $(x, y \in \mathbb{R})$. Their 4-dimensional real algebra is then defined as $\mathbb{BC} = \{Z = z_1 + jz_2; z_1, z_2 \in \mathbb{C}\}$, where j is an imaginary unit, $j^2 = -1$, independent of i and satisfying $ij = ji =: k$. This turns k into what is known as hyperbolic unit, leading to the particular subset $\mathbb{D}$ of hyperbolic numbers, which is constituted of the bi-reals $x + ky$. The computation rules in $\mathbb{BC}$ extend, in a natural way, those in $\mathbb{C}$, giving rise to similar algebraic properties, except for division. More precisely, the null cone coincides with $\mathcal{NC} = \{\lambda(1 \pm ij);\ \lambda \in \mathbb{C}, \lambda \neq 0\}$. The particular elements

$$e_+ = \frac{(1+ij)}{2} \quad \text{and} \quad e_- = \frac{(1-ij)}{2}$$

are idempotent and satisfy $e_+e_- = 0$. Moreover, they yield the idempotent decomposition $\alpha e_+ + \beta e_- = Z$ of every $Z = z_1 + jz_2 \in \mathbb{BC}$, with unique complex components

$$\alpha = z_1 - iz_2 =: Proj^+(z_1 + jz_2) \quad \text{and} \quad \beta = z_1 + iz_2 =: Proj^-(z_1 + jz_2).$$

Thus, the map $P = (Proj^+, Proj^-)$,

$$P(z_1 + jz_2) := (z_1 - iz_2, z_1 + iz_2) = (\alpha, \beta), \tag{2.1}$$

realizes the algebra isomorphism $\mathbb{BC} \simeq \mathbb{C} \oplus \mathbb{C}$. Given such decomposition, the set $\mathbb{D}$ reads equivalently as the set of all $xe_+ + ye_-$ with $x, y \in \mathbb{R}$, leading to the partial order $\preceq$ $(xe_+ + ye_- \preceq x'e_+ + y'e_-$ if $x \leq x'$ and $y \leq y'$ in $\mathbb{R})$. A particular exception in the theory of bicomplex numbers is the attribution of three complex conjugates $Z^\dagger = z_1 - jz_2 = \beta e_+ + \alpha e_-$, $\widetilde{Z} = \overline{z_1} + j\overline{z_2} = \overline{\beta}e_+ + \overline{\alpha}e_-$, $Z^* = \overline{z_1} - j\overline{z_2} = \overline{\alpha}e_+ + \overline{\beta}e_-$, to each bicomplex number $Z = z_1 + jz_2$. By means of the above projection operators, one defines

$$\Omega^\pm := Proj^\pm(\Omega) = \{z_1 \mp iz_2 \in \mathbb{C},\ z_1 + jz_2 \in \Omega\}, \tag{2.2}$$

for given $\Omega \subset \mathbb{BC}$. We will write $\Omega = \Omega^+e_+ + \Omega^-e_-$, whenever Ω is a generic product-type set in $\mathbb{BC}$, *i.e.* those for which there exists a one-to-one correspondence from Ω onto $\Omega^+e_+ + \Omega^-e_-$. By Theorem 8.6 in [32, p. 37], such product-type sets are exactly those subsets in $\mathbb{BC}$ such that $P(\Omega) = \Omega^+ \times \Omega^-$, where P is as in (2.1). It should be pointed out that the openness of the components $\Omega^\pm$ in $\mathbb{C}$ follows from the openness of Ω in $\mathbb{BC}$, which is seen as the four-dimensional Euclidean space (see Riley's notes [34] or [32, Theorem 8.7]). For further details on the different topological considerations related to $\mathbb{BC}$, one refers to [32, 34].

2.2 Bicomplex holomorphy

Recall that a bicomplex-valued function

$$F(Z) = F_1(z_1, z_2) + jF_2(z_1, z_2),$$

on a given open set $\Omega \subset \mathbb{BC}$, is said in [32] to be bicomplex holomorphic (bc-holomorphic for short) in Ω, if for every $Z_0 \in \Omega$, the bicomplex limit

$$\lim_{\substack{H \to 0 \\ H \notin \mathcal{NC}}} \frac{F(Z_0 + H) - F(Z_0)}{H}$$

is finite. Another interesting characterization of the bc-holomorphicity is the Ringleb decomposition theorem [35] (see also [32, Theorem 15.5]), asserting that a bicomplex-valued function f is bc-holomorphic if and only if it is of the form

$$f(Z) = f(\alpha e_+ + \beta e_-) = \phi^+(\alpha)e_+ + \phi^-(\beta)e_-, \tag{2.3}$$

where $\phi^\pm : \mathbb{C} \longrightarrow \mathbb{C}$ are holomorphic $\mathbb{C}$-valued functions on $\mathbb{C}$. For a product-type domain this remains equivalent to F_1, F_2 be holomorphic in the complex variables $(z_1, z_2) \in \Omega^+ \times \Omega^-$ and satisfying in addition the complex Cauchy-Riemann equations [36, Theorem 1]

$$\frac{\partial F_1}{\partial z_1} = \frac{\partial F_2}{\partial z_2} \quad \text{and} \quad \frac{\partial F_2}{\partial z_1} = -\frac{\partial F_1}{\partial z_2}.$$

Analogously to the classical complex derivatives $\partial_z = \partial/\partial z$ and its complex conjugate $\partial_{\overline{z}} = \partial/\partial \overline{z}$, there are the first order differential operators with respect to the different bicomplex conjugates

$$\partial_Z = \frac{\partial}{\partial Z} := \frac{1}{2}\left(\frac{\partial}{\partial z_1} - j\frac{\partial}{\partial z_2}\right), \quad \partial_{Z^*} = \frac{\partial}{\partial Z^*} := \frac{1}{2}\left(\frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}}\right),$$
$$\partial_{Z^\dagger} = \frac{\partial}{\partial Z^\dagger} := \frac{1}{2}\left(\frac{\partial}{\partial z_1} + j\frac{\partial}{\partial z_2}\right), \quad \partial_{\widetilde{Z}} = \frac{\partial}{\partial \widetilde{Z}} := \frac{1}{2}\left(\frac{\partial}{\partial \overline{z_1}} - j\frac{\partial}{\partial \overline{z_2}}\right),$$

which can be used to provide a special realization of the so-called bicomplex holomorphic functions as solutions of a system of linear differential equations with constant coefficients. Namely, a real differentiable bicomplex-valued function F on an open set in $\mathbb{BC}$ is bc-holomorphic if and only if it is solution of (see [13, Theorem 2.7] or also [27, p. 159])

$$\frac{\partial F}{\partial Z^*} = \frac{\partial F}{\partial Z^\dagger} = \frac{\partial F}{\partial \widetilde{Z}} = 0. \tag{2.4}$$

The system provided in (2.4) is a central tool in the theory of bc-holomorphic functions, and can be used to extend the bc-holomorphy to polyanalytic setting, so that the discussed bc-holomorphic functions appear as the $(1,1,1)$-bc holomorphic functions in the definition below.

Definition 2.1 ([21])**.** *A bicomplex–valued function F having continuous partial derivatives on an open set $\Omega \subset \mathbb{BC}$, up to order $2\max(m,n,k)$, and satisfying the system*

$$\partial^m_{Z^*} F = \partial^n_{\widetilde{Z}} F = \partial^k_{Z^\dagger} F = 0 \tag{2.5}$$

is said to be (m,n,k)-bc-polyholomorphic on Ω.

An explicit characterization of these functions has been obtained in [16, Proposition 3.8].

Proposition 2.2. *The bicomplex-valued (m,n,k)-bc-polyholomorphic functions on $\mathbb{BC}$ are exactly those that can be expanded as*

$$F(Z) = \sum_{\ell_1=0}^{m-1}\sum_{\ell_2=0}^{n-1}\sum_{\ell_3=0}^{k-1} {Z^*}^{\ell_1} \widetilde{Z}^{\ell_2} {Z^\dagger}^{\ell_3} H_{\ell_1,\ell_2,\ell_3}(Z) \tag{2.6}$$

for given bc-holomorphic functions H_{ℓ_1,ℓ_2,ℓ_3}.

This result leads to an immediate extension of the Ringleb result (2.3) to these class of functions, which reads simply for the $(m,1,1)$ case as

$$F(Z = \alpha e_+ + \beta e_-) = \left(\sum_{k=0}^{m-1} \overline{\alpha}^k \phi_k(\alpha)\right) e_+ + \left(\sum_{k=0}^{m-1} \overline{\beta}^k \psi_k(\beta)\right) e_-,$$

for certain bc-holomorphic functions ϕ_k and ψ_k.

2.3 Bicomplex harmonicity

The existence of the different types of conjugates in the set of bicomplex numbers leads to different natural analogs of the classical Laplace operator

$$\Delta_z = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{\partial^2}{\partial z \partial \overline{z}}, \quad z = x + iy, \tag{2.7}$$

see [16] for details. The so-called bc-Laplacian Δ_{bc} as well as its †-conjugate $\Delta^\dagger_{bc}$ given by

$$\Delta_{bc} := \frac{\partial^2}{\partial Z \partial Z^*} \quad \text{and} \quad \Delta^\dagger_{bc} := \frac{\partial^2}{\partial Z^\dagger \partial \widetilde{Z}}.$$

are examples of such Laplacians. Their action on a given sufficiently real differential bicomplex-valued function is well-defined and to be understood in the sense of Remark 2.5 in [16]. Thus, for a twice continuously differentiable function $F = F^+ e_+ + F^- e_-$, we have the idempotent decomposition $\Delta_{bc} = \Delta_\alpha e_+ + \Delta_\beta e_-$ and $\Delta^\dagger_{bc} = \Delta_\beta e_+ + \Delta_\alpha e_-$. By considering the complex-valued component functions $h^\pm(\alpha,\beta) := F^\pm(Z)$ with $Z = \alpha e_+ + \beta e_-$, this action reads

$$[\Delta_{bc} F](Z) = ([\Delta_\alpha h^+](\alpha,\beta))e_+ + ([\Delta_\beta h^-](\alpha,\beta))e_-. \tag{2.8}$$

Being indeed, since both ∂_Z and ∂_{Z^*} are seen as $\mathbb{BC}$-linear operators and $e_+ \cdot e_- = 0$, we have

$$\frac{\partial F}{\partial Z^*}(Z) = \left(\frac{\partial}{\partial \overline{\alpha}} e_+ + \frac{\partial}{\partial \overline{\beta}} e_-\right) \left(h^+(\alpha,\beta)e_+ + h^-(\alpha,\beta)e_-\right) = \frac{\partial h^+}{\partial \overline{\alpha}}(\alpha,\beta)e_+ + \frac{\partial h^-}{\partial \overline{\beta}}(\alpha,\beta)e_-,$$

and moreover

$$\begin{aligned}[\Delta_{bc}F](Z) &= \left[\frac{\partial}{\partial Z}\left(\frac{\partial F}{\partial Z^*}\right)\right](Z) = \left(\frac{\partial}{\partial \alpha} e_+ + \frac{\partial}{\partial \beta} e_-\right)\left(\frac{\partial h^+}{\partial \overline{\alpha}}(\alpha,\beta)e_+ + \frac{\partial h^-}{\partial \overline{\beta}}(\alpha,\beta)e_-\right) \\ &= \frac{\partial^2 h^+}{\partial \alpha \partial \overline{\alpha}}(\alpha,\beta)e_+ + \frac{\partial^2 h^-}{\partial \beta \partial \overline{\beta}}(\alpha,\beta)e_-.\end{aligned}$$

Accordingly, one suggests the following definition.

Definition 2.3 ([16])**.** *Let F be a bicomplex-valued function on an open set $\Omega \subset \mathbb{BC}$.*

(i) F is said to be bicomplex harmonic (bc-harmonic) if it is twice continuously real differentiable and satisfies the bc-Laplace equation $\Delta_{bc} = 0$ on Ω. We denote their set by $\mathcal{BH}arm(\Omega)$.

(ii) F is said to be bc-polyharmonic of order m if it is continuously real differentiable up to order $2m$ and satisfies the m-th bc-Laplace equation $\Delta_{bc}^m = 0$ on Ω.

It should be noticed here that the bc-polyharmonic functions are closely connected to a special class of bc-polyholomorphic functions as expected in [16]. Their representations in terms of bc-harmonic functions were obtained in [16, Proposition 4.4], which itself is a bicomplex extension of Almansi's result [4] for the classical polyharmonic complex-valued functions. For its exact statement, we let $|Z|_{bc}^{2k} := Z^k Z^{*k}$ for every $Z \in \mathbb{BC}$ and $k = 0, 1, 2, \dots$

Proposition 2.4. *For every bc-polyharmonic function F on $\mathbb{BC}$ of order m, there are certain bc-harmonic functions H_k, $k = 0, 1, \dots, m-1$, such that*

$$F(Z) = \sum_{k=0}^{m-1} |Z|_{bc}^{2k} H_k(Z). \tag{2.9}$$

Remark 2.5. *The component functions H_k in Proposition 2.4 are bc-harmonic and they implicitly depend on $Z^\dagger$ and $\widetilde{Z}$. More precisely, identity* (2.9) *reads equivalently*

$$F(Z) = \sum_{k=0}^{+\infty} \sum_{k=0}^{m-1} \left(Z^{n+k} {Z^*}^k A_{n,k}(\widetilde{Z}, Z^\dagger) + Z^k {Z^*}^{n+k} B_{n,k}(\widetilde{Z}, Z^\dagger)\right), \tag{2.10}$$

for given bicomplex-valued functions $A_{n,k}$ and $B_{n,k}$ belonging to $\ker(\partial_Z) \cap \ker(\partial_{Z^*})$.

Definition 2.6. *Let F be a bicomplex-valued function on an open set $\Omega \subset \mathbb{BC}$.*

(i) It is said to be strongly bicomplex harmonic if F and $F^\dagger$ are both bc-harmonic.

(ii) It is said to be strongly bc-polyharmonic of bi-order (m,n), if it has continuous partial derivatives up to order $2\max(m,n)$ and verifies $\Delta_{bc}^m F = 0$ and $\Delta_{bc}^n F^\dagger = 0$ on Ω.

We conclude this section by providing explicit examples for the different classes of bicomplex holomorphic, polyholomorphic, harmonic and polyharmonic functions, in the $i, j, ij = k$ representation as well as in the idempotent representation, which can easily constructed making use of the obtained characterizations. Thus, the functions

$$(Z^m + Z^n) + k(Z^m - Z^n) = 2\alpha^m e_+ + 2\beta^n e_-$$

are the elementary bc-holomorphic functions on $\mathbb{BC}$, while

$$(Z^m Z^* + Z^n Z^\dagger) + k(Z^m Z^* - Z^n Z^\dagger) = 2\alpha^m \overline{\alpha} e_+ + 2\alpha\beta^n e_-$$

is an example of a $(2,2,1)$-polyholomorphic function. The following

$$h_0(Z) = ZZ^\dagger + Z\widetilde{Z} + Z^*\widetilde{Z} + Z^*\widetilde{Z} = 2\Re\left(\alpha(\beta + \overline{\beta})\right)$$

is a fundamental example of bc-harmonic function which can not be the real part of any bc-harmonic function. An example of polyharmonic function is given by the the biharmonic function

$$Z^* Z^\dagger h_0(Z) + h_0(Z) = 2\left\{(\overline{\alpha}\beta + 1)\, e_+ + \left(\alpha\overline{\beta} + 1\right) e_-\right\} \Re\left(\alpha(\beta + \overline{\beta})\right).$$

3 Main results

3.1 Characterization of strongly bc-harmonic functions

The following result provides an explicit characterization of the strongly bc-harmonic functions.

Theorem 3.1. *Let F be a bicomplex-valued function on $\mathbb{BC}$. Then, the function F is strongly bc-harmonic if and only if there are some sequences $(a_{m,n})_{m,n}$, $(b_{m,n})_{m,n}$, $(c_{m,n})_{m,n}$ and $(d_{m,n})_{m,n}$ of bicomplex numbers such that F has a power series expansion of the form*

$$F(Z) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \left(a_{m,n} Z^m Z^{\dagger n} + b_{m,n} Z^m \widetilde{Z}^n + c_{m,n} Z^{*m} Z^{\dagger n} + d_{m,n} Z^{*m} \widetilde{Z}^n\right), \tag{3.1}$$

converging absolutely and uniformly on any compact set of $\mathbb{BC}$.

Proof. The "if" follows by direct computation. However, the strongly bc-harmonicity of F in (3.1) in the sense of Definition 2.6 can be handled by observing that the uniformly convergent series in (3.1) can be rewritten as $F = H + G^*$, with some functions H and G that can expanded as

$$\sum_{m=0}^{+\infty} Z^m \left((\psi(Z))^\dagger + \widetilde{\varphi(Z)} \right)$$

for given bc-holomorphic functions ψ and φ, and next employing using the useful facts $\partial_{Z^*}(\phi^\dagger) = \left(\partial_{\widetilde{Z}}(\phi)\right)^\dagger$, $\partial_{Z^*}(\widetilde{\phi}) = \widetilde{\partial_{Z^\dagger}(\phi)}$, $\partial_{\widetilde{Z}}(\phi^\dagger) = (\partial_{Z^*}(\phi))^\dagger$, and $\partial_{Z^\dagger}(\widetilde{\phi}) = \widetilde{\partial_{Z^*}(G)}$ as well as $\partial_Z(G^*) = (\partial_{Z^*}(G))^*$ and $\partial_{Z^\dagger}(G^*) = \left(\partial_{\widetilde{Z}}(G)\right)^*$.

For the proof of the "only if", let $F(\alpha e_+ + \beta e_-) = F^+(\alpha,\beta)e_+ + F^-(\alpha,\beta)e_-$ be a strongly bc-harmonic function, with $F^+, F^- : \mathbb{BC} \longrightarrow \mathbb{C}$. Thus, from $\Delta_{bc} F = 0$ and $\Delta_{bc} F^\dagger = 0$, and in particular $\Delta_\alpha F^+(\cdot,\beta) = 0$ and $\Delta_\alpha F^-(\cdot,\beta) = 0$, for every fixed complex number β, one observes that both the partial components $\alpha \longmapsto F^+(\alpha,\beta)$ and $\alpha \longmapsto F^-(\alpha,\beta)$ are complex-valued harmonic functions in the complex plane, for every fixed $\beta \in \mathbb{C}$. Therefore, there exist some complex-valued holomorphic functions $H^{+,\beta}$, $H^{-,\beta}$, $G^{+,\beta}$ and $G^{-,\beta}$ on $\mathbb{C}$ with power series expansions centered at the origin such that

$$F^+(\alpha,\beta) = H^{+,\beta}(\alpha) + \overline{G^{+,\beta}(\alpha)} = \sum_{m=0}^{+\infty} a_m^+(\beta)\alpha^m + b_m^+(\beta)\overline{\alpha}^m \tag{3.2}$$

and

$$F^-(\alpha,\beta) = H^{-,\beta}(\alpha) + \overline{G^{-,\beta}(\alpha)} = \sum_{m=0}^{+\infty} a_m^-(\beta)\alpha^m + b_m^-(\beta)\overline{\alpha}^m, \tag{3.3}$$

for all $\alpha \in \mathbb{C}$. However, since the partial functions $\beta \longmapsto F^\pm(\alpha,\beta)$ being harmonic, the involved coefficients

$$a_m^\pm(\beta) = \frac{1}{m!}\frac{\partial^m F^\pm}{\partial \alpha^m}(0,\beta), \quad m = 0,1,2,\dots,$$

and

$$b_m^\pm(\beta) = \frac{1}{m!}\frac{\partial^m F^\pm}{\partial \overline{\alpha}^m}(0,\beta), \quad m = 0,1,2,\dots,$$

which are independent of α and $\overline{\alpha}$ and seen as functions in the β-variable, become $\mathcal{C}^\infty$ and moreover harmonic on the complex plane. Thus, we write

$$a_m^\pm = H_{1,m}^\pm + \overline{H_{2,m}^\pm} \quad \text{and} \quad b_m^\pm(\beta) = G_{1,m}^\pm + \overline{G_{2,m}^\pm},$$

for certain holomorphic functions $H_0^\pm$, $G_0^\pm$, $H_{1,m}^\pm$, $H_{2,m}^\pm$, $G_{1,m}^\pm$ and $G_{2,m}^\pm$ on $\mathbb{C}$. Returning back to (3.2)-(3.3) and using the expansion series of the involved holomorphic functions, we get

$$F^{\pm}(\alpha,\beta) = H_0^{\pm}(\beta) + \overline{G_0^{\pm}(\beta)} + \sum_{m=1}^{+\infty} \left(H_{1,m}^{\pm}(\beta) + \overline{H_{2,m}^{\pm}(\beta)}\right)\alpha^m + \left(G_{1,m}^{\pm}(\beta) + \overline{G_{2,m}^{\pm}(\beta)}\right)\overline{\alpha}^m$$
$$= \sum_{m,n=0}^{+\infty} \left(a_{1,m,n}^{\pm}\beta^n + a_{2,m,n}^{\pm}\overline{\beta}^n\right)\alpha^m + \left(b_{1,m,n}^{\pm}\beta^n + b_{2,m,n}^{\pm}\overline{\beta}^n\right)\overline{\alpha}^m,$$

which gives rise to (3.1). □

3.2 Additive decomposition theorems

We begin with the following.

Theorem 3.2. *A bicomplex-valued function F is of the form $F = H+G^*$, for some bc-holomorphic functions H and G, if and only if it is bc-harmonic on $\mathbb{BC}$ such that $\partial_{\widetilde{Z}}F = \partial_{Z^\dagger}F = 0$.*

Proof. For given $F = H+G^*$ such that H and G are bc-holomorphic, the function F is bc-harmonic for the smooth function F satisfies

$$\frac{\partial^2 F}{\partial Z \partial Z^*} = \frac{\partial}{\partial Z}\left(\frac{\partial H}{\partial Z^*}\right) + \frac{\partial}{\partial Z^*}\left(\left(\frac{\partial G}{\partial Z^*}\right)\right)^* = 0.$$

Moreover, using the facts $\partial_{\widetilde{Z}}(G^*) = (\partial_{Z^\dagger}(G))^*$ and $\partial_{Z^\dagger}(G^*) = \left(\partial_{\widetilde{Z}}(G)\right)^*$, and keeping in mind (2.8) it becomes clear that

$$\partial_{\widetilde{Z}}F = \partial_{\widetilde{Z}}(H) + \partial_{\widetilde{Z}}(G^*) = \partial_{\widetilde{Z}}(H) + (\partial_{Z^\dagger}(G))^* = 0$$

and

$$\partial_{Z^\dagger}F = \partial_{Z^\dagger}(H) + \partial_{Z^\dagger}(G^*) = \partial_{Z^\dagger}(H) + \left(\partial_{\widetilde{Z}}(G)\right)^* = 0$$

hold.

For the proof of the converse, we proceed into two steps.

Step 1: Assume that $F : \mathbb{BC} \longrightarrow \mathbb{D}$ is a hyperbolic-valued bc-harmonic function belonging to $\ker(\partial_{\widetilde{Z}}) \cap \ker(\partial_{Z^\dagger})$. Next, observe that by means of [16, Theorem 1.1] there exists a bc-holomorphic function T such that $F = \Re e_{hyp}(T) := (T+T^*)/2$, which infers $F = H + G^*$ with $H = G = T/2$.

Step 2: For the general case when F does not take values in $\mathbb{D}$, we rewrite it as $F = F_1 + iF_2$, with

$$F_1 = \frac{F+F^*}{2} \quad \text{and} \quad F_2 = \frac{F-F^*}{2i}.$$

Both F_1 and F_2 are hyperbolic-valued functions on $\mathbb{BC}$. From this, it becomes clear that F is a bc-harmonic if and only if F_1 and F_2 are bc-harmonic. Moreover, we necessarily have

$$2\partial_{\widetilde{Z}} F_1 = -2i\partial_{\widetilde{Z}} F_1 = \partial_{\widetilde{Z}} F^* = (\partial_{Z^\dagger} F)^* = 0,$$

and

$$2\partial_{\widetilde{Z}} F_1 = -2i\partial_{Z^\dagger} F_1 = \partial_{Z^\dagger} F^* = \left(\partial_{\widetilde{Z}} F\right)^* = 0.$$

This implies that the functions F_1 and F_2 belong to $\ker(\partial_{\widetilde{Z}}) \cap \ker(\partial_{Z^\dagger})$. However, from the first step, we easily conclude that $F_1 = H_1 + G_1^*$ and $F_2 = H_2 + G_2^*$, for some bc-holomorphic functions H_ℓ and G_ℓ, $\ell = 1, 2$. Now, since $i^* = -i$, it follows

$$F = (H_1 + G_1^*) + i\,(H_2 + G_2^*) = H + G^*,$$

with $H = H_1 + iH_2$ and $G = G_1 - iG_2$. □

The following result extends the previous one to the bc-polyharmonic functions of arbitrary finite order. The argument in the presented proof is completely different from the one provided for Theorem 3.2.

Theorem 3.3. *Let F be a bicomplex-valued bc-polyharmonic function of order m on $\mathbb{BC}$. Then, there exist certain $(m,1,1)$-bc-polyholomorphic functions H and G such that $F = H + G^*$ if and only if $\partial_{\widetilde{Z}} F = \partial_{Z^\dagger} F = 0$.*

Proof. In the sense of Definition 2.1, the function $H + G^*$ is clearly bc-polyharmonic, whenever H and G are bc-polyholomorphic of order $(m,1,1)$ and $(n,1,1)$, respectively. Indeed, by setting $\ell = \max(m,n)$, we have

$$\Delta_{bc}^{\ell}(H + G^*) = \frac{\partial^\ell}{\partial Z^\ell}\left(\frac{\partial^\ell H}{\partial Z^{*\ell}}\right) + \frac{\partial^\ell}{\partial Z^{*\ell}}\left(\frac{\partial^\ell G}{\partial Z^{*\ell}}\right)^* = 0.$$

To prove the converse, let F be a bc-polyharmonic function of order m. Then, $\partial_{Z^*}^m(\partial_Z^m F) = \Delta_{bc}^m F = 0$. But, under the assumption $\partial_{\widetilde{Z}} F = \partial_{Z^\dagger} F = 0$, the function $\partial_Z^m F$ becomes $(m,1,1)$-bc-polyholomorphic. Accordingly, it can be expanded as

$$\partial_Z^m F = \sum_{\ell=0}^{m-1} {Z^*}^{\ell} \psi_\ell,$$

by means of Proposition 2.2 (with $n = k = 1$). The involved functions ψ_ℓ, $\ell = 0, 1, \dots, m-1$, are bc-holomorphic and can always be rewritten as $\psi_\ell = \partial_Z^m \varphi_\ell$ for certain bc-holomorphic functions φ_ℓ. Thus, by considering the function

$$G = \sum_{\ell=0}^{m-1} {Z^*}^{\ell} \varphi_\ell,$$

we get $\partial^m_{Z^*}(F^* - G^*) = 0$. But, using again the assumption $\partial_{\widetilde{Z}} F = \partial_{Z^\dagger} F = 0$, it becomes clear that $F^* - G^* = H$ is a $(m,1,1)$-bc-polyholomorphic function. □

Remark 3.4. *The proof of Theorem 3.3 can be handled using Almansi's theorem for bc-polyharmonic functions (see Proposition 2.4 or Remark 2.5) and by viewing Z and $Z^\dagger$ as independent variables. In fact, for F being a bc-polyharmonic function of order m, there exist some bc-harmonic functions F_k, $k = 0, 1, \dots, m-1$, such that*

$$F(Z) = F_0 + |Z|_{bc} F_1 + \cdots + |Z|_{bc}^{2(m-1)} F_{m-1}.$$

Accordingly, the assumption $\partial_{\widetilde{Z}} F = \partial_{Z^\dagger} F = 0$ becomes equivalent to $\partial_{\widetilde{Z}} F_k = \partial_{Z^\dagger} F_k = 0$ for every $k = 0, 1, \dots, m-1$. Therefore, making appeal to the discussion provided in the proof of Theorem 3.2 for each F_k, there exist some bc-holomorphic functions H_k and G_k such that $F_k = H_k + G_k^*$. Hence, one derives $F = H + G^*$, where

$$H = \sum_{k=0}^{m-1} |Z|_{bc}^{2k} H_k \quad \text{and} \quad G = \sum_{k=0}^{m-1} |Z|_{bc}^{2k} G_k.$$

Given such result (Theorem 3.3), the next one provides a sufficient condition to decompose a given strongly bc-harmonic function F as $F = H_0 + H_1^* + H_2^\dagger + \widetilde{H}_3$ for certain bc-holomorphic function H. Notice, that the converse is clear since the different bicomplex conjugates H^*, $H^\dagger$, $\widetilde{H}$ of a bc-holomorphic function H are obviously bc-harmonic, and moreover they are strongly bc-harmonic, which shows that the functions $H_0 + H_1^* + H_2^\dagger + \widetilde{H}_3$, arising as the sum of the different conjugates of bc-holomorphic functions for some bicomplex holomorphic functions H_ℓ, $\ell = 0, 1, 2, 3$, are strongly bc-harmonic.

Theorem 3.5. *A bicomplex-valued strongly bc-harmonic function F in $\mathbb{BC}$ is of the form $F = H_0 + H_1^* + H_2^\dagger + \widetilde{H}_3$, for some bc-holomorphic functions H_ℓ, $\ell = 0, 1, 2, 3$, if*

$$\frac{\partial^{m+n+j+k} F}{\partial Z^m Z^{*n} Z^{\dagger j} \widetilde{Z}^k}(0) = 0, \tag{3.4}$$

holds, for every non-negative integers m, n, j and k such that $mn = jk = 0$.

Proof. The key observation is contained in the characterization provided by Theorem 3.1. In fact, the involved bicomplex constants in (3.1) are given by

$$a_{m,n} = \frac{1}{m!\,n!} \frac{\partial^{m+n} F}{\partial Z^m Z^{\dagger n}}(0), \quad b_{m,n} = \frac{1}{m!\,n!} \frac{\partial^{m+n} F}{\partial Z^m \widetilde{Z}^n}(0),$$
$$c_{m,n} = \frac{1}{m!\,n!} \frac{\partial^{m+n} F}{\partial Z^{*m} Z^{\dagger n}}(0), \quad d_{m,n} = \frac{1}{m!\,n!} \frac{\partial^{m+n} F}{\partial Z^{*m} \widetilde{Z}^n}(0).$$

Accordingly, under the assumption (3.4), which reads equivalently as

$$\frac{\partial^{m+j}F}{\partial Z^m Z^{\dagger j}}(0)=\frac{\partial^{m+j}F}{\partial Z^{*m} Z^{\dagger j}}(0)=\frac{\partial^{k+n}F}{\partial Z^k \widetilde{Z}^n}(0)=\frac{\partial^{k+n}F}{\partial Z^{*k} \widetilde{Z}^n}(0)=0, \tag{3.5}$$

we get $a_{m,n}=d_{m,n}=0$, for every $n\geq 1$, and $b_{m,n}=c_{m,n}=0$, for any $m\geq 1$. Thus, the expansion series of F reduces further to $F=H_0+H_1^*+H_2^\dagger+\widetilde{H}_3$, where H_0, H_1, H_2 and H_3 are the bc-holomorphic functions given by

$$H_0:=\sum_{m=0}^{+\infty} a_m Z^m,\quad H_1:=\sum_{m=0}^{+\infty} d_m^{*m} Z^m,\quad H_2:=\sum_{n=0}^{+\infty} c_n^\dagger Z^n,\quad \text{and}\quad H_3=:\sum_{n=0}^{+\infty} \widetilde{b_n} Z^n,$$

where we have set $a_k:=a_{k,0}$, $d_k:=d_{k,0}c_k:=c_{0,k}$ and $b_k:=b_{0,k}$. □

Remark 3.6. *Theorem 3.5 can be reproved by considering an equivalent sufficient condition, leading to $a_{m,n}=d_{m,n}=0$ for every $m\geq 1$ and $b_{m,n}=c_{m,n}=0$ for any $n\geq 1$.*

Below, we give an additional additive decomposition theorem, which is specific for the bc-harmonic functions.

Theorem 3.7. *We have $\mathcal{BH}arm(\mathbb{BC})=\left(\ker(\partial_{Z^*})+\ker\left(\partial_{\widetilde{Z}}\right)\right)\cap\mathcal{C}^\infty(\mathbb{BC})$. More precisely, H is a bc-harmonic function if and only if it can be expanded as*

$$H(Z)=\sum_{k=0}^{+\infty} Z^k A_k(Z^\dagger,\widetilde{Z})+Z^{\dagger k}B_k(Z,Z^*), \tag{3.6}$$

for some $A_k\in\ker(\partial_Z)\cap\ker(\partial_{Z^*})$ and $B_k\in\ker(\partial_{Z^\dagger})\cap\ker\left(\partial_{\widetilde{Z}}\right)$.

Proof. Let H be a bc-harmonic function and write $H(Z)=\widehat{H}^+(\alpha,\beta)e_+ +\widehat{H}^-(\alpha,\beta)e_-$. Hence, the functions $\widehat{H}^+(\cdot,b):\mathbb{C}\longrightarrow\mathbb{C}$ and $\widehat{H}^-(a,\cdot):\mathbb{C}\longrightarrow\mathbb{C}$ are harmonic on $\mathbb{C}$. Thus, for every fixed $a,b\in\mathbb{C}$, the involved functions can be decomposed as $\widehat{H}^+(\alpha,b)=h_b^{+,1}(\alpha)+h_b^{+,2}(\overline{\alpha})$ and $\widehat{H}^-(a,\beta)=h_a^{-,1}(\beta)+h_a^{-,2}(\overline{\beta})$ for some holmorphic functions $h_b^{+,1},h_b^{+,2}:\mathbb{C}\longrightarrow\mathbb{C}$ and $h_a^{-,1},h_a^{-,2}:\mathbb{C}\longrightarrow\mathbb{C}$, thanks to the additive decomposition theorem for classical harmonic functions. Therefore, by setting

$$H^{(1)}(Z|a,b):=h_b^{+,1}(\alpha)e_+ +h_a^{-,1}(\beta)e_-$$

and

$$H^{(2)}(Z|a,b):=h_b^{+,2}(\overline{\alpha})e_+ +h_a^{-,2}(\overline{\beta})e_-,$$

we have $\partial_{Z^*}(H^{(1)}(\cdot|a,b))=\partial_{\widetilde{Z}}(H^{(2)}(\cdot|a,b)))$ and

$$H(Z)=H^{(1)}(Z|\alpha,\beta)+H^{(2)}(Z|\alpha,\beta),\quad Z=\alpha e_+ +\beta e_-. \tag{3.7}$$

The functions $H^{(1)}$ and $H^{(2)}$ belong clearly to $\ker(\partial_{Z^*})$ and $\ker\left(\partial_{\widetilde{Z}}\right)$, respectively. The inverse inclusion is immediate. □

In Proposition 3.10 below, it is proved that the involved $H^{(1)}$, $H^{(2)}$, A_k and B_k in (3.7) and (3.6) are connected to each other by some additive separate bc-holomorphic function, which extends the notion of separate holomorphy to the bicomplex setting. Let F be a given bicomplex-valued function on $\mathbb{BC}$, identified to $\widehat{F}(\alpha,\beta) := F(\alpha e_+ + \beta e_-)$ on $\mathbb{C}^2$. Define the partial functions $F_\alpha : \mathbb{C} \longrightarrow \mathbb{BC}$ and $F^\beta : \mathbb{C} \longrightarrow \mathbb{BC}$ given by

$$F_\alpha(\beta) = F^\beta(\alpha) =: \widehat{F}(\alpha,\beta).$$

Definition 3.8. *A bicomplex-valued function F on $\mathbb{BC}$ is said to be separately holomorphic if F_α and F^β are both holomorphic in $\mathbb{C}$.*

Accordingly, we have the following characterization.

Proposition 3.9. *Let F be a bicomplex-valued function on $\mathbb{BC}$. Then, the following assertions are equivalent.*

(i) F is separate holomorphic on Ω.

(ii) F satisfies

$$\frac{\partial F}{\partial Z^*} = \frac{\partial F}{\partial \widetilde{Z}} = 0. \tag{3.8}$$

(iii) F has the expansion

$$F(Z) = \sum_{m,n=0}^{+\infty} C_{m,n} Z^m Z^{\dagger n}, \quad C_{m,n} \in \mathbb{BC}. \tag{3.9}$$

Proof. The separate holomorphy of F reads $\partial\widehat{F}/\partial\overline{\alpha} = \partial\widehat{F}/\partial\overline{\beta} = 0$, and therefore

$$\frac{\partial \widehat{F}^+}{\partial\overline{\alpha}} = \frac{\partial \widehat{F}^-}{\partial\overline{\alpha}} = \frac{\partial \widehat{F}^+}{\partial\overline{\beta}} = \frac{\partial \widehat{F}^-}{\partial\overline{\beta}} = 0. \tag{3.10}$$

This is in fact also equivalent to

$$\frac{\partial\left(\widehat{F}^+ e_+ + \widehat{F}^- e_-\right)}{\partial Z^*} = \frac{\partial\left(\widehat{F}^- e_+ + \widehat{F}^+ e_-\right)}{\partial Z^*} = \left(\frac{\partial\left(\widehat{F}^+ e_+ + \widehat{F}^- e_-\right)}{\partial \widetilde{Z}}\right)^\dagger$$

be identically zero on $\mathbb{BC}$, which infers (3.8). Next, by means of (3.10), the functions in (ii) are those for which we have

$$\widehat{F}^\pm(\alpha,\beta) = \sum_{m,n=0}^{+\infty} a^\pm_{m,n}\alpha^m\beta^n, \quad a^\pm_{m,n} \in \mathbb{C},$$

and therefore

$$F(Z) = \sum_{m,n=0}^{+\infty} C_{m,n} Z^m Z^{\dagger n}, \quad \text{with} \quad C_{m,n} = a^+_{m,n} e_+ + a^-_{n,m} e_-.$$

The converse (iii) implies (ii) is clearly immediate. □

Proposition 3.10. *Keep the notations of $H^{(1)}$, $H^{(2)}$, A_k and B_k as above. Then, for any $H \in \mathcal{BH}arm(\mathbb{BC})$, there exists a separate bc-holomorphic function G such that*

$$H^{(1)}(Z|\alpha,\beta) = \sum_{k=0}^{+\infty} Z^k A_k(Z^\dagger, \widetilde{Z}) + G(Z) \quad \text{and} \quad H^{(2)}(Z|\alpha,\beta) = \sum_{k=0}^{+\infty} Z^{\dagger k} B_k(Z, Z^*) - G(Z).$$

Proof. For every $Z = \alpha e_+ + \beta e_-$, set

$$G^{(1)}(Z) := H^{(1)}(Z|\alpha,\beta) - \sum_{k=0}^{+\infty} Z^k A_k(Z^\dagger, \widetilde{Z}) \quad \text{and} \quad G^{(2)}(Z) := H^{(2)}(Z|\alpha,\beta) - \sum_{k=0}^{+\infty} Z^{\dagger k} B_k(Z, Z^*).$$

Then, from (3.7) and (3.6), we conclude that $G^{(1)} = -G^{(2)}$. However, since $\partial_Z(A_k) = \partial_{Z^*}(A_k) = 0$, $\partial_{Z^\dagger}(B_k) = \partial_{\widetilde{Z}}(B_k) = 0$ and $\partial_{Z^*}(H^{(1)}(\cdot|a,b)) = \partial_{\widetilde{Z}}(H^{(2)}(\cdot|a,b)) = 0$, we get $\partial_{Z^*} G = \partial_{Z^*} G^{(1)} = 0$ and $\partial_{\widetilde{Z}} G = \partial_{\widetilde{Z}} G^{(1)} = 0$. This completes the proof by setting $G := G^{(1)} = -G^{(2)} \in \ker(\partial_{Z^*}) \cap \ker(\partial_{\widetilde{Z}})$. □

4 Concluding remarks

The conclusions of Theorems 3.2, 3.3, and 3.7 remain valid for arbitrary generic product-type domains in $\mathbb{BC}$ without additional assumptions, while Theorems 3.1 and 3.5 remain correct on special product-type domains in $\mathbb{BC}$. In fact, the statements of Theorems 3.5 and 3.7 are both valid on a given $D(0, r_1, r_2)$, where

$$D(Z_0, r_1, r_2) := \{Z \in \mathbb{BC};\ ZZ^* \preceq r_1 e_+ + r_2 e_-\},$$

for given nonnegative reals r_1 and r_2. Assertion of Theorem 3.5 also holds for arbitrary $D(Z_0, r_1, r_2)$ by imposing

$$\frac{\partial^{m+n+j+k} F}{\partial Z^m Z^{*n} Z^{\dagger j} \widetilde{Z}^k}(Z_0) = 0, \tag{4.1}$$

for every positive integers m, n, j and k, as a sufficient condition for a given strongly bc-harmonic bicomplex-valued function F on $D(Z_0, r_1, r_2)$ to be of the form $F = H_0 + H_1^* + H_2^\dagger + \widetilde{H}_3$, for some bc-holomorphic functions H_ℓ, $\ell = 0, 1, 2, 3$, on $D(Z_0, r_1, r_2)$. Analogously to Theorem 3.1, one asserts that a given bicomplex-valued function F is strongly bc-harmonic on a product domain Ω

if and only if for any $Z_0 \in \Omega$ and any $r_1, r_2 > 0$ such that $\overline{D(Z_0, r_1, r_2)} \subset \Omega$, F can be expanded as

$$F(Z) = \sum_{m,n=0}^{+\infty} a_{m,n}(Z - Z_0)^m (Z - Z_0)^{\dagger n} + b_{m,n}(Z - Z_0)^m \widetilde{(Z - Z_0)}^n + c_{m,n}(Z - Z_0)^{*m}(Z - Z_0)^{\dagger n} + d_{m,n}(Z - Z_0)^{*m}\widetilde{(Z - Z_0)}^n$$

on $D(Z_0, r_1, r_2)$.

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Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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