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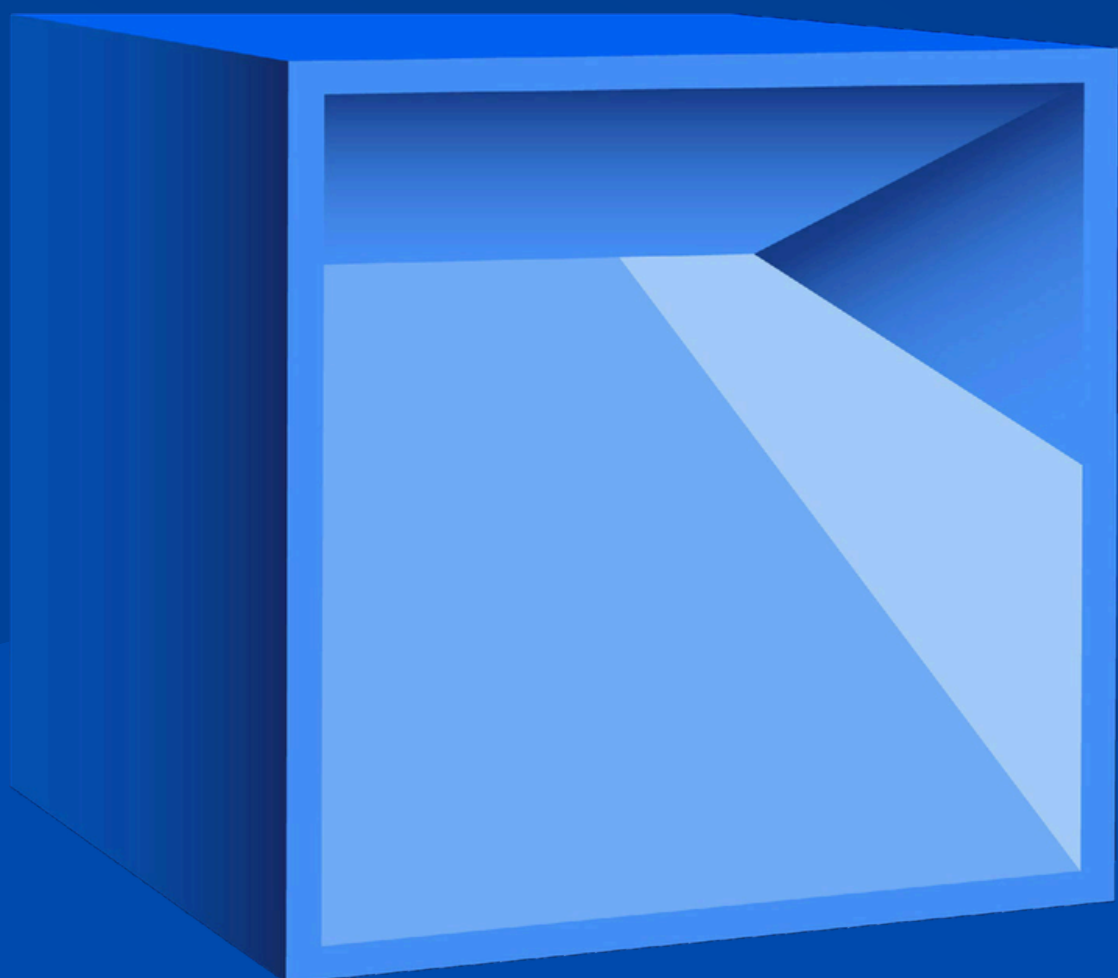

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SUMMARY

– **A primitive associated with the Cantor–Bendixson derivative on Polish spaces** 227
Andrés Merino and Sebastián Heredia Freire

– **Series with harmonic numbers and the tail of $\zeta(2)$** 247
Ovidiu Furdui and Alina Sîntămărian

– **Cyclic covers of an algebraic curve from an adelic viewpoint** 261
Luis Manuel Navas Vicente and Francisco J. Plaza Martín

– **Applying the Riemann surfaces with extremal configurations of symmetries to the study of the real nerve of the moduli space of Riemann surfaces of odd genera** 295
Ewa Kozłowska-Walania and Leonard Sikorski

– **Multivariate symmetrized, q-deformed and λ-parametrized hyperbolic tangent induced complex valued trigonometric and hyperbolic neural network enhanced approximation** 323
George A. Anastassiou

– **Rings in which every ideal disjoint with S is S-almost prime** 349
Chahrazade Bakkari, Rachid Hachache, Najib Mahdou, Unsal Tekir and Ece Yetkin Celikel

– **Persistence of a tumor spheroid with an almost periodic nutrient supply** 363
Homero G. Díaz-Marín, Osvaldo Osuna and Geiser Villavicencio-Pulido

– **Szpiro's conjecture when the denominator of the j-invariant is small** 383
Hector Pasten

– **On the analytical solution of the Cauchy problem for a linear set-valued differential equation with a Hukuhara derivative** 391
Tatyana A. Komleva, Andrej V. Plotnikov and Natalia V. Skripnik

– **Class of symmetric $H_{\sqrt{q}}$-Laguerre-Hahn linear forms** 409
Sobhi Jbeli

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A primitive associated with the Cantor–Bendixson derivative on Polish spaces

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ABSTRACT

Given a perfect Polish space X, a compact subset $K \subseteq X$ and a countable ordinal $\alpha < \omega_1$, we show that there exists a compact subset $\widehat{K} \subseteq X$ such that

$$\widehat{K}^{(\alpha)} = K,$$

where $\widehat{K}^{(\alpha)}$ denotes the α-th Cantor–Bendixson derivative of $\widehat{K}$. In other words, every compact subset of a perfect Polish space admits an α-*primitive* with respect to the Cantor–Bendixson derivative. This extends to perfect Polish spaces a result previously known for countable compact subsets of the real line. The proof proceeds in three steps: first, we construct primitives for singletons; then, for countable compact subsets; and finally, for arbitrary compact subsets, using separability of Polish spaces.

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RESUMEN

Dado un espacio polaco perfecto X, un subconjunto compacto $K \subseteq X$ y un ordinal numerable $\alpha < \omega_1$, mostramos que existe un subconjunto compacto $\widehat{K} \subseteq X$ tal que

$$\widehat{K}^{(\alpha)} = K,$$

donde $\widehat{K}^{(\alpha)}$ denota la α-ésima derivada de Cantor–Bendixson de $\widehat{K}$. En otras palabras, todo subconjunto compacto de un espacio polaco perfecto admite una *α-primitiva* con respecto a la derivada de Cantor–Bendixson. Esto extiende a espacios polacos perfectos un resultado conocido previamente para subconjuntos numerables compactos de la línea real. La demostración procede en tres pasos: primero construimos primitivas para singletons; luego, para subconjuntos numerables compactos; y finalmente, para subconjuntos compactos arbitrarios, usando la separabilidad de espacios polacos.

Keywords and Phrases: Cantor–Bendixson derivative, descriptive set theory, Polish spaces, primitive.

2020 AMS Mathematics Subject Classification: 03E15; 54H05.

1 Introduction

The Cantor–Bendixson derivative is a classical tool in topology and set theory. It was introduced by Cantor in [4] and later refined by Bendixson. It plays a significant role in several areas, including the study of Boolean frames [3], the structure of compact and σ-compact spaces [9], and applications to the semantics of finite logic programs [5].

In [3], Avilez García shows that the Cantor–Bendixson derivative can be used to measure how "Boolean" a frame is by producing for each element the largest Boolean interval containing it. This viewpoint leads to a tower of derivatives and to relationships between derivatives at different levels, which in turn characterize when a frame admits a Boolean reflection. In model theory, the Cantor–Bendixson rank is closely related to Morley rank in ω-stable theories. Furthermore, the Cantor–Bendixson derivative appears in the characterization of compact and σ-compact spaces and in the construction of co-inductive operators; see, for instance, [9]. In [2], the authors give a necessary and sufficient condition for an ordinal to be a Polish space and describe some properties of the Cantor–Bendixson derivative on compact and countable subsets of a Polish space.

These results illustrate the relevance and versatility of the Cantor–Bendixson derivative in different contexts and motivate further study of its structure and possible inverses.

In [1], Álvarez-Samaniego and Merino proved the existence of a "primitive" on the real line: for every countable compact set $K \subseteq \mathbb{R}$ and every countable ordinal $\alpha < \omega_1$, there exists a compact set $\widehat{K} \subseteq \mathbb{R}$ such that

$$\widehat{K}^{(\alpha)} = K.$$

The compact set $\widehat{K}$ is then called an α-primitive of K. The goal of the present paper is to generalize this result from $\mathbb{R}$ to perfect Polish spaces.

More precisely, we work in a perfect Polish space (X, d) and we prove that, given any compact subset $K \subseteq X$ and any countable ordinal $\alpha < \omega_1$, there exists a compact set $\widehat{K} \subseteq X$ such that $\widehat{K}^{(\alpha)} = K$. Our argument follows the same general strategy as in [1], but several technical modifications are needed to accommodate the more general setting.

The paper is organized as follows. In Section 2, we study the behavior of Cantor–Bendixson derivatives for certain countable unions of compact sets whose derivatives at a fixed level are singletons contained in a family of pairwise disjoint balls. In Section 3, we collect some basic properties of perfect Polish spaces and prove the existence of pairwise disjoint families of open balls around countable discrete subsets. In Section 4, we first construct an α-primitive for any singleton, and then use this to build primitives for countable compact sets and, finally, for arbitrary compact subsets of a perfect Polish space.

Throughout the paper, ω denotes the set of natural numbers, and we identify each $n \in \omega$ with the set $\{0, 1, \ldots, n-1\}$. We write ω_1 for the first uncountable ordinal, and we only consider countable ordinals α (*i.e.*, $\alpha < \omega_1$). Let (X, τ) be a topological space, and let $A \subseteq X$. A point $x \in X$ is a *limit point* of A if every neighborhood V of x satisfies

$$(V \setminus \{x\}) \cap A \neq \emptyset.$$

The set of all limit points of A is denoted by A' and is called the *derivative* of A.

Using transfinite recursion, we define the Cantor–Bendixson derivative of a subset of a topological space.

Definition 1.1 (Cantor–Bendixson derivative)**.** *Let A be a subset of a topological space X, and let $\alpha < \omega_1$ be an ordinal. The α-th Cantor–Bendixson derivative of A, denoted by $A^{(\alpha)}$, is defined by*

- $A^{(0)} = A$*;*
- $A^{(\alpha+1)} = \left(A^{(\alpha)}\right)'$ *for every ordinal* α*;*
- $A^{(\alpha)} = \bigcap_{\gamma<\alpha} A^{(\gamma)}$ *for every nonzero limit ordinal* α*.*

Definition 1.2. *A topological space (X, τ) is* metrizable *if there exists a metric d on X such that the topology induced by d coincides with τ. This metric d is called a* compatible metric *for (X, τ).*

Definition 1.3. *A topological space X is called a* Polish space *if it is separable and completely metrizable, that is, if there exists a compatible metric d on X for which (X, d) is complete.*

A subset P of a topological space is called *perfect* if $P' = P$, that is, if every point of P is a limit point of P.

2 The derivative of a countable union

In this section, we study the behavior of Cantor–Bendixson derivatives for certain countable unions of compact sets. We will work in a fixed topological space (X, τ) and consider the collection of countable compact subsets of X.

Definition 2.1. *Let (X, τ) be a topological space. We set*

$$\mathcal{K}_X = \{K \subseteq X : K \text{ is countable and compact}\}.$$

We begin with the following elementary fact; see, for example, [8].

Proposition 2.2. *Let (X, τ) be a topological space, let $\alpha < \omega_1$ and let $\mathcal{A}$ be an arbitrary family of subsets of X. Then*

$$\bigcup_{A \in \mathcal{A}} A^{(\alpha)} \subseteq \left(\bigcup_{A \in \mathcal{A}} A \right)^{(\alpha)}.$$

In a Hausdorff space, the Cantor–Bendixson derivatives of a closed set form a decreasing family of closed subsets of X (see, for instance, [8]). For countable compact sets, we obtain the following.

Proposition 2.3. *Let (X, τ) be a Hausdorff topological space and let $K \in \mathcal{K}_X$. Then, for every ordinal $\alpha < \omega_1$, the set $K^{(\alpha)}$ belongs to $\mathcal{K}_X$, and the family $\left(K^{(\alpha)}\right)_{\alpha < \omega_1}$ is decreasing: if $\alpha \geq \beta$ then $K^{(\alpha)} \subseteq K^{(\beta)}$.*

Indeed, K is closed and compact, so each $K^{(\alpha)}$ is closed, and the standard transfinite recursion defining the derivatives yields the monotonicity (see [7] for details). Moreover, $K^{(\alpha)} \subseteq K$ for all α, so each $K^{(\alpha)}$ is countable and compact, hence $K^{(\alpha)} \in \mathcal{K}_X$.

Our next goal is to compute the derivative of a special type of countable union. We will work in a metrizable space and consider a convergent sequence $(x_n)_{n \in \omega}$ together with a family of countable compact sets whose derivatives at a fixed level α are the points x_n, and that are contained in pairwise disjoint balls. Under these hypotheses, we can describe all derivatives up to level α of the union of the K_n together with the limit point of the sequence.

The following proposition generalizes a construction that appears in the proof of [1, Theorem 2.1].

Proposition 2.4. *Let (X, d) be a metrizable space, let $\alpha < \omega_1$, and let $(x_n)_{n \in \omega} \in X^\omega$ be a sequence satisfying:*

(i) $(x_n)_{n \in \omega}$ converges to a point $x \in X$;

(ii) there exists a family $(K_n)_{n \in \omega}$ in $\mathcal{K}_X$ such that $K_n^{(\alpha)} = \{x_n\}$ and

$$K_n \subseteq B(x_n, r_n)$$

for all $n \in \omega$, where $(B(x_n, r_n))_{n \in \omega}$ is a family of pairwise disjoint open balls.

Set

$$K = \bigcup_{n \in \omega} K_n \cup \{x\}.$$

Then, for every $\beta \leq \alpha$, we have

$$K^{(\beta)} = \bigcup_{n \in \omega} K_n^{(\beta)} \cup \{x\}.$$

Proof. Let $\beta \leq \alpha$. Using Proposition 2.2 and the monotonicity of the derivative with respect to the inclusion (see [8]), we obtain

$$\bigcup_{n\in\omega} K_n^{(\beta)} \subseteq \left(\bigcup_{n\in\omega} K_n\right)^{(\beta)} \subseteq \left(\bigcup_{n\in\omega} K_n \cup \{x\}\right)^{(\beta)} = K^{(\beta)}.$$

On the other hand, by Proposition 2.3 and the monotonicity of the derivative with respect to the inclusion, we have

$$\{x_n : n \in \omega\} = \bigcup_{n\in\omega} K_n^{(\alpha)} \subseteq \bigcup_{n\in\omega} K_n^{(\beta)} \subseteq K^{(\beta)}.$$

Since $(x_n)_{n\in\omega}$ converges to x and $x_n \in K^{(\beta)}$ for all n, it follows that x is a limit point of $K^{(\beta)}$, so

$$x \in K^{(\beta+1)} \subseteq K^{(\beta)}.$$

Thus, for every $\beta \leq \alpha$,

$$\bigcup_{n\in\omega} K_n^{(\beta)} \cup \{x\} \subseteq K^{(\beta)}.$$

It remains to prove the reverse inclusion. We again proceed by transfinite induction on β.

- For $\beta = 0$, the result is immediate. Assume $\beta < \alpha$ and

$$K^{(\beta)} \subseteq \bigcup_{n\in\omega} K_n^{(\beta)} \cup \{x\}. \tag{2.1}$$

Let $z \in K^{(\beta+1)} = \left(K^{(\beta)}\right)'$. Then $z \in K^{(\beta)}$, and by (2.1) either $z = x$ or there exists $M \in \omega$ such that

$$z \in K_M^{(\beta)} \subseteq B(x_M, r_M).$$

If $z = x$, then $z \in \bigcup_{n\in\omega} K_n^{(\beta+1)} \cup \{x\}$ and we are done. Suppose $z \neq x$. We necessarily have $z \in K_M^{(\beta+1)}$; otherwise, z would be an isolated point of $K_M^{(\beta)}$, so there exists $\varepsilon_1 > 0$ such that

$$B(z, \varepsilon_1) \cap K_M^{(\beta)} = \{z\}. \tag{2.2}$$

Let

$$\varepsilon = \min\{\varepsilon_1,\ r_M - d(z, x_M),\ d(x, z)\}.$$

Then $B(z, \varepsilon) \subseteq B(x_M, r_M)$, and since the balls $B(x_n, r_n)$ are pairwise disjoint and $K_n \subseteq B(x_n, r_n)$ for all n, we have

$$B(z, \varepsilon) \cap \left(\bigcup_{m\neq M} K_m^{(\beta)}\right) = \emptyset. \tag{2.3}$$

Combining (2.2) and (2.3), we obtain

$$\{z\} = B(z,\varepsilon) \cap \left(K_M^{(\beta)} \cup \bigcup_{m \neq M} K_m^{(\beta)} \cup \{x\} \right) = B(z,\varepsilon) \cap \left(\bigcup_{n\in\omega} K_n^{(\beta)} \cup \{x\} \right) = B(z,\varepsilon) \cap K^{(\beta)},$$

which contradicts $z \in \left(K^{(\beta)}\right)' = K^{(\beta+1)}$. Hence $z \in K_M^{(\beta+1)}$, and therefore

$$K^{(\beta+1)} \subseteq \bigcup_{n\in\omega} K_n^{(\beta+1)} \cup \{x\}.$$

- Let $\gamma \leq \alpha$ be a nonzero limit ordinal, and suppose that

$$K^{(\delta)} \subseteq \bigcup_{n\in\omega} K_n^{(\delta)} \cup \{x\},$$

for all $\delta < \gamma$. Let $z \in K^{(\gamma)}$; using the induction hypothesis, we have that

$$K^{(\gamma)} = \bigcap_{\delta<\gamma} K^{(\delta)} \subseteq \bigcap_{\delta<\gamma} \left(\bigcup_{n\in\omega} K_n^{(\delta)} \cup \{x\} \right).$$

Then, for every $\delta < \gamma$, either $z = x$ or there exists $m \in \omega$ such that $z \in K_m^{(\delta)}$.

If $z = x$, we are done. Otherwise, $z \neq x$ and for $\delta = 0$ there exists $M \in \omega$ with

$$z \in K_M^{(0)} = K_M \subseteq B(x_M, r_M).$$

We claim that $z \in K_M^{(\delta)}$ for every $\delta < \gamma$. Suppose not, and let $\delta_0 < \gamma$ be minimal such that $z \notin K_M^{(\delta_0)}$. Thus, since $z \in K^{(\lambda)}$, $z \neq x$ and $z \notin K_M^{(\delta_0)}$, for δ_0, there exists $m_0 \in \omega$ with $m_0 \neq M$ such that

$$z \in K_{m_0}^{(\delta_0)} \subseteq K_{m_0} \subseteq B(x_{m_0}, r_{m_0}).$$

Thus $B(x_M, r_M) \cap B(x_{m_0}, r_{m_0}) \neq \emptyset$, contradicting the fact that the balls $B(x_n, r_n)$ are pairwise disjoint. Hence

$$z \in \bigcap_{\delta<\gamma} K_M^{(\delta)} = K_M^{(\gamma)}.$$

We conclude that

$$K^{(\gamma)} \subseteq \bigcup_{n\in\omega} K_n^{(\gamma)} \cup \{x\}.$$

Combining the three steps, we obtain

$$K^{(\beta)} = \bigcup_{n\in\omega} K_n^{(\beta)} \cup \{x\},$$

for every $\beta \leq \alpha$, as required. □

The previous result suggests replacing the singleton $\{x\}$ by a more general countable compact set. This leads to the following variant.

Proposition 2.5. *Let (X,d) be a metrizable space, let $\alpha < \omega_1$ and let $K \subseteq X$ be a closed set satisfying:*

(i) $K \setminus K' = \{x_n : n \in \omega\}$ is infinite and the points x_n are pairwise distinct;

(ii) there exists a family $(K_n)_{n\in\omega}$ in $\mathcal{K}_X$ such that $K_n^{(\alpha)} = \{x_n\}$ and

$$K_n \subseteq B(x_n, r_n)$$

for all $n \in \omega$, where $(B(x_n, r_n))_{n\in\omega}$ is a family of pairwise disjoint open balls.

Set

$$\widehat{K} = \bigcup_{n\in\omega} K_n \cup K.$$

Then, for every $\beta \leq \alpha$, we have

$$\widehat{K}^{(\beta)} = \bigcup_{n\in\omega} K_n^{(\beta)} \cup K.$$

Proof. We first prove that $K \subseteq \widehat{K}^{(\beta)}$ for every $\beta \leq \alpha$, using transfinite induction.

- For $\beta = 0$, the result is immediate.
- Assume that $K \subseteq \widehat{K}^{(\beta)}$ for some $\beta < \alpha$. Then $K' \subseteq \widehat{K}^{(\beta+1)}$, and, using (ii),

$$K \setminus K' = \bigcup_{n\in\omega} \{x_n\} = \bigcup_{n\in\omega} K_n^{(\alpha)} \subseteq \bigcup_{n\in\omega} K_n^{(\beta+1)} \subseteq \widehat{K}^{(\beta+1)}.$$

  Thus

$$K = K' \cup (K \setminus K') \subseteq \widehat{K}^{(\beta+1)}.$$

- Let $\gamma \leq \alpha$ be a nonzero limit ordinal, and suppose

$$K \subseteq \widehat{K}^{(\delta)},$$

  for all $\delta < \gamma$. Then

$$K \subseteq \bigcap_{\delta<\gamma} \widehat{K}^{(\delta)} = \widehat{K}^{(\gamma)}.$$

Hence $K \subseteq \widehat{K}^{(\beta)}$ for every $\beta \leq \alpha$. On the other hand, for each $\beta \leq \alpha$, Proposition 2.2 yields

$$\bigcup_{n\in\omega} K_n^{(\beta)} \subseteq \left(\bigcup_{n\in\omega} K_n\right)^{(\beta)} \subseteq \left(\bigcup_{n\in\omega} K_n \cup K\right)^{(\beta)} = \widehat{K}^{(\beta)}.$$

Combining these inclusions, we obtain

$$\bigcup_{n\in\omega} K_n^{(\beta)} \cup K \subseteq \widehat{K}^{(\beta)}, \quad \text{for all } \beta \leq \alpha.$$

To prove the opposite inclusion, we again use transfinite induction on β.

- For $\beta = 0$, again, the result is immediate.
- Assume that $\beta < \alpha$ and
$$\widehat{K}^{(\beta)} \subseteq \bigcup_{n\in\omega} K_n^{(\beta)} \cup K. \tag{2.4}$$
Let $x \in \widehat{K}^{(\beta+1)} = \left(\widehat{K}^{(\beta)}\right)'$. Then $x \in \widehat{K}^{(\beta)}$, so by (2.4) either $x \in K$ or there exists $M \in \omega$ with
$$x \in K_M^{(\beta)} \subseteq B(x_M, r_M).$$
If $x \in K$, there is nothing to prove. Suppose $x \notin K$. If $x \notin K_M^{(\beta+1)} = \left(K_M^{(\beta)}\right)'$, then x is an isolated point of $K_M^{(\beta)}$, and there exists $\varepsilon_1 > 0$ such that
$$B(x, \varepsilon_1) \cap K_M^{(\beta)} = \{x\}. \tag{2.5}$$
Since K is closed and $x \notin K$, we have
$$d(x, K) := \inf\{d(x, z) : z \in K\} > 0.$$
Let
$$\varepsilon = \min\{\varepsilon_1,\ r_M - d(x, x_M),\ d(x, K)\}.$$
Then $B(x, \varepsilon) \subseteq B(x_M, r_M)$ and
$$B(x, \varepsilon) \cap \left(\bigcup_{m\neq M} K_m^{(\beta)}\right) = \emptyset. \tag{2.6}$$

Moreover, by the definition of ε, $B(x,\varepsilon) \cap K = \emptyset$. Combining these facts with (2.5) gives

$$\{x\} = B(x,\varepsilon) \cap \left(K_M^{(\beta)} \cup \bigcup_{m \neq M} K_m^{(\beta)} \cup K \right) = B(x,\varepsilon) \cap \left(\bigcup_{n\in\omega} K_n^{(\beta)} \cup K \right) = B(x,\varepsilon) \cap \widehat{K}^{(\beta)},$$

which contradicts $x \in \left(\widehat{K}^{(\beta)}\right)' = \widehat{K}^{(\beta+1)}$. Hence $x \in K_M^{(\beta+1)}$, and therefore

$$\widehat{K}^{(\beta+1)} \subseteq \bigcup_{n\in\omega} K_n^{(\beta+1)} \cup K.$$

- Let $\gamma \leq \alpha$ be a nonzero limit ordinal, and suppose

$$\widehat{K}^{(\delta)} \subseteq \bigcup_{n\in\omega} K_n^{(\delta)} \cup K, \tag{2.7}$$

for all $\delta < \gamma$. Let $z \in \widehat{K}^{(\gamma)} = \bigcap_{\delta<\gamma} \widehat{K}^{(\delta)}$. By (2.7), for each $\delta < \gamma$, either $z \in K$ or there exists $m \in \omega$ such that $z \in K_m^{(\delta)}$.

If $z \in K$, we are done. Otherwise, $z \notin K$ and there exists $M \in \omega$ such that $z \in K_M^{(0)} = K_M \subseteq B(x_M, r_M)$. We claim that $z \in K_M^{(\delta)}$ for all $\delta < \gamma$. If not, let $\delta_0 < \gamma$ be minimal with $z \notin K_M^{(\delta_0)}$. Then, for δ_0, there exists $m_0 \in \omega$ with $m_0 \neq M$ such that

$$z \in K_{m_0}^{(\delta_0)} \subseteq K_{m_0} \subseteq B(x_{m_0}, r_{m_0}),$$

so $B(x_M, r_M) \cap B(x_{m_0}, r_{m_0}) \neq \emptyset$, contradicting that the balls are pairwise disjoint. Hence

$$z \in \bigcap_{\delta<\gamma} K_M^{(\delta)} = K_M^{(\gamma)}.$$

Thus

$$\widehat{K}^{(\gamma)} \subseteq \bigcup_{n\in\omega} K_n^{(\gamma)} \cup K.$$

This completes the induction and the proof. □

3 Some properties of perfect Polish spaces

In this section, we collect several standard facts about perfect Polish spaces that will be needed later. We also construct, for each countable discrete subset, a family of pairwise disjoint open balls, as required in Propositions 2.4 and 2.5.

The next lemma expresses the fact that in a perfect metric space, every point is the limit of a sequence of distinct points.

Proposition 3.1. *Let (X, d) be a Polish space and let $A \subseteq X$ be perfect. Then, for every $x \in A$, there exists a sequence $(x_n)_{n\in\omega} \in (A \setminus \{x\})^\omega$ such that $(x_n)_{n\in\omega}$ converges to x.*

The next statements are well-known, and their proofs are straightforward, so we omit them. Proposition 3.2 follows from the fact that every non-empty open subset of a perfect Polish space is uncountable. Proposition 3.3 is a standard consequence of basic properties of compact countable sets. Finally, Lemma 3.4 is the classical construction of pairwise disjoint balls around a countable discrete set in a metric space.

Proposition 3.2. *Let (X, d) be a Polish space and let $K \in \mathcal{K}_X$. Then K is not perfect.*

Proposition 3.3. *Let (X, d) be a Polish space and let $K \in \mathcal{K}_X$ be infinite. Then K has infinitely many isolated points.*

We now construct disjoint balls around a countable discrete subset. This is a standard fact; we provide a version convenient for our purposes.

Lemma 3.4. *Let (X, d) be a metric space and let $A \subseteq X$ be a countable, infinite, discrete set. Write $A = \{x_n : n \in \omega\}$ with $x_n \neq x_m$ for $n \neq m$. Then there exists a sequence of positive real numbers $(r_n)_{n\in\omega}$ such that*

$$\{B(x_n, r_n)\}_{n\in\omega}$$

is a family of pairwise disjoint open balls. Moreover, we may additionally require $r_n \leq \frac{1}{n+1}$ for all $n \in \omega$.

4 Existence of primitives

In this section, we prove the main results of the paper. We begin by constructing, for each point x lying in an appropriate Cantor–Bendixson derivative of a Polish space and each ordinal $\alpha < \omega_1$, an α-primitive contained in an arbitrarily small ball around x. We then use this construction to obtain primitives for countable compact sets, and finally for all compact subsets of a perfect Polish space.

Although the main theorem could be stated and proved in a single step, we intentionally develop the argument in three stages: first for singletons, then for countable compact sets, and finally for arbitrary compact sets. This stepwise organization reflects the way the construction naturally extends from isolated points to countable compact sets and then to general compact subsets.

4.1 Primitives for singletons

We begin with the key step: primitives for singletons.

Proposition 4.1. *Let (X, d) be a Polish space, let $\alpha < \omega_1$, let $x \in X^{(\alpha)}$ and let $r > 0$. Then there exists $K \in \mathcal{K}_X$ such that*

$$K \subseteq B(x, r) \quad \textit{and} \quad K^{(\alpha)} = \{x\}.$$

Proof. We proceed by transfinite induction on $\alpha < \omega_1$.

- If $\alpha = 0$, it suffices to take $K = \{x\}$. Then $K \subseteq B(x, r)$ and $K^{(0)} = K = \{x\}$.
- Assume that for some $\alpha < \omega_1$ the following statement holds: for every $\widetilde{x} \in X^{(\alpha)}$ and every $\widetilde{r} > 0$, there exists $\widetilde{K} \in \mathcal{K}_X$ such that

$$\widetilde{K} \subseteq B(\widetilde{x}, \widetilde{r}) \quad \text{and} \quad \widetilde{K}^{(\alpha)} = \{\widetilde{x}\}.$$

We prove the statement for $\alpha + 1$.

Let $x \in X^{(\alpha+1)}$ and $r > 0$. Since $x \in X^{(\alpha+1)}$, by Proposition 3.1 there exists a sequence $(x_n)_{n\in\omega} \in \left(X^{(\alpha)} \setminus \{x\}\right)^{\omega}$ converging to x. Using Lemma 3.4, we may choose a family of pairwise disjoint balls $\{B(x_n, r_n)\}_{n\in\omega}$ with $r_n \leq 1/(n+1)$ for all n.

Since $x_n \to x$, there exists $N \in \omega$ such that, for all $n > N$,

$$d(x_n, x) < \frac{r}{2} \quad \text{and} \quad \frac{1}{n+1} < \frac{r}{2}.$$

If $n > N$ and $z \in B(x_n, r_n)$, then

$$d(z, x) \leq d(z, x_n) + d(x_n, x) < r_n + \frac{r}{2} \leq \frac{1}{n+1} + \frac{r}{2} < r,$$

so $B(x_n, r_n) \subseteq B(x, r)$ for all $n > N$.

For each $m \in \omega$, apply the induction hypothesis to $x_m \in X^{(\alpha)}$ and $r_m > 0$ to obtain $K_m \in \mathcal{K}_X$ such that

$$K_m \subseteq B(x_m, r_m) \quad \text{and} \quad K_m^{(\alpha)} = \{x_m\}.$$

Define

$$K = \bigcup_{m > N+1} K_m \cup \{x\}.$$

We now verify the required properties.

– $K \subseteq B(x,r)$. For $m > N+1$, $K_m \subseteq B(x_m, r_m) \subseteq B(x,r)$, and clearly $\{x\} \subseteq B(x,r)$, so $K \subseteq B(x,r)$.

– *K is countable and compact.* The set K is a countable union of countable sets, so it is countable. To see that K is compact, it suffices (in a metric space) to show that K is sequentially compact.

  Let $(z_k)_{k\in\omega} \in K^\omega$ be a sequence in K and set $S = \{z_k : k \in \omega\}$. We distinguish three cases.

  (a) If $S \cap K$ is infinite, then there is a subsequence of $(z_k)_{k\in\omega}$ contained in K, and since K is compact, this subsequence has a further convergent subsequence.

  (b) If there exists $m \in \omega$ such that $S \cap K_m$ is infinite, then a subsequence of $(z_k)_{k\in\omega}$ lies in K_m, and since K_m is compact, it has a convergent subsequence.

  (c) Suppose that $S \cap K$ is finite and $S \cap K_n$ is finite for every $n \in \omega$. Then the set

$$I = \{n \in \omega : S \cap K_n \neq \emptyset\}$$

  is infinite. For $N, M \in \omega$, set

$$S_M = \{z_k : k > M\} \quad \text{and} \quad I(N,M) = \{n \in \omega : S_M \cap K_n \neq \emptyset,\ n > N\}.$$

  A simple counting argument shows that $I(N,M)$ is infinite for every $N, M \in \omega$ (otherwise I would be finite). Using this, we recursively define strictly increasing functions $\sigma, \psi \colon \omega \to \omega$ as follows:

$$\sigma(0) = \min I, \qquad \psi(0) = \min\{k \in \omega : z_k \in K_{\sigma(0)}\},$$

  and, for $m \in \omega$,

$$\sigma(m+1) = \min I(\sigma(m), \psi(m)),$$

$$\psi(m+1) = \min\{k \in \omega : k > \psi(m) \text{ and } z_k \in K_{\sigma(m+1)}\}.$$

  Then ψ is strictly increasing and $z_{\psi(m)} \in K_{\sigma(m)} \subseteq B(x_{\sigma(m)}, r_{\sigma(m)})$ for all $m \in \omega$. Hence

$$d\big(z_{\psi(m)}, x_{\sigma(m)}\big) < r_{\sigma(m)} \leq \frac{1}{\sigma(m)+1}.$$

  Since $\big(x_{\sigma(m)}\big)_{n\in\omega}$ is a sequence in the compact set K (recall that each $x_n \in X^{(\alpha)} \subseteq X$ and later we will use them in a compact set), it has a convergent subsequence $\big(x_{\sigma(\varphi(m))}\big)_{n\in\omega}$. The corresponding subsequence $\big(z_{\psi(\varphi(m))}\big)_{n\in\omega}$ is then Cauchy and convergent in X. Its limit belongs to K by closedness. Thus $(z_k)_{k\in\omega}$ has a convergent subsequence.

  In all cases, K is sequentially compact; it is compact.

– $K^{(\alpha+1)} = \{x\}$. Since $K_m^{(\alpha)} = \{x_m\}$ for each m and the balls $B(x_m, r_m)$ are pairwise disjoint, Proposition 2.4 applied to the sequence $(x_m)_{n\in\omega}$ and the family $\{K_m\}_{m>N+1}$ yields

$$K^{(\alpha)} = \left(\bigcup_{m>N+1} K_m \cup \{x\}\right)^{(\alpha)} = \bigcup_{m>N+1} K_m^{(\alpha)} \cup \{x\} = \{x_m : m > N+1\} \cup \{x\}.$$

Since $x_m \to x$, the only limit point of $K^{(\alpha)}$ is x, so

$$K^{(\alpha+1)} = \left(K^{(\alpha)}\right)' = \{x\}.$$

- Let $\lambda < \omega_1$ be a nonzero limit ordinal, and assume the claim holds for all $\rho < \lambda$. Let $x \in X^{(\lambda)}$ and $r > 0$. Choose a strictly increasing sequence of ordinals $(\rho_m)_{m\in\omega}$ with $\rho_m < \lambda$ for all m and $\sup_{m\in\omega} \rho_m = \lambda$.

  Since $x \in X^{(\lambda)}$, for each $m \in \omega$ we may choose a point

  $$x_m \in X^{(\rho_m)} \setminus \{x\}$$

  such that $x_m \to x$ and $x_m \in B(x, r)$. Applying Lemma 3.4 and shrinking the radii if necessary, we obtain a family of pairwise disjoint open balls $\{B(x_m, r_m)\}_{m\in\omega}$ such that, for some $N \in \omega$, we have $B(x_m, r_m) \subseteq B(x, r)$ whenever $m > N$.

  For each m we apply the induction hypothesis (at level ρ_m) to x_m and $r_m > 0$, obtaining $K_m \in K_X$ such that

  $$K_m \subseteq B(x_m, r_m) \quad \text{and} \quad K_m^{(\rho_m)} = \{x_m\}.$$

  Set

  $$K = \bigcup_{m>N+1} K_m \cup \{x\}.$$

  As in the successor step, K is countable, contained in $B(x, r)$, and compact.

  For each m, the set $K_m^{(\lambda)}$ is empty, since $\rho_m + 1 < \lambda$ and the derivatives form a decreasing family. Applying Proposition 2.4 with the sequence $(x_m)_{n\in\omega}$ and the family $\{K_m\}_{m>N+1}$, we obtain

  $$K^{(\lambda)} = \left(\bigcup_{m>N+1} K_m \cup \{x\}\right)^{(\lambda)} = \bigcup_{m>N+1} K_m^{(\lambda)} \cup \{x\} = \{x\}.$$

This completes the induction and the proof. □

4.2 Primitives for countable compact sets

We now use the previous proposition to construct primitives for countable compact subsets of a perfect Polish space.

Theorem 4.2. *Let (X,d) be a perfect Polish space, let $K \in \mathcal{K}_X$, and let $\alpha < \omega_1$. Then there exists $\widehat{K} \in \mathcal{K}_X$ such that*

$$\widehat{K}^{(\alpha)} = K.$$

Proof. If $\alpha = 0$, we may simply take $\widehat{K} = K$, so we assume $\alpha > 0$.

By Proposition 3.2, K is not perfect, so $K' \neq K$. We consider two cases.

Case 1: K is infinite. By Proposition 3.3, the set K has infinitely many isolated points, so $K \setminus K'$ is infinite. Enumerate

$$K \setminus K' = \{x_n : n \in \omega\}$$

with pairwise distinct x_n.

Each point $x \in K \setminus K'$ is isolated in K, so there exists a neighborhood V of x such that $V \cap K = \{x\}$. In particular,

$$\{x\} = V \cap K \subseteq V \cap (K \setminus K') \subseteq V \cap K = \{x\},$$

so x is also isolated in $K \setminus K'$, and the set $K \setminus K'$ is discrete.

Applying Lemma 3.4 to the discrete set $\{x_n : n \in \omega\}$, we obtain a family of pairwise disjoint open balls $\{B(x_n, r_n)\}_{n\in\omega}$ with $r_n \leq 1/(n+1)$ for all n.

Since X is perfect, then $x_n \in X^{(\alpha)}$ for every $n \in \omega$. For each n, applying Proposition 4.1 to $x_n \in X^{(\alpha)}$ and $r_n > 0$, we obtain $K_n \in \mathcal{K}_X$ such that

$$K_n \subseteq B(x_n, r_n) \quad \text{and} \quad K_n^{(\alpha)} = \{x_n\}.$$

Since the balls $B(x_n, r_n)$ are pairwise disjoint, so are the sets K_n.

Define

$$\widehat{K} = \bigcup_{n\in\omega} K_n \cup K.$$

The set $\widehat{K}$ is countable as a countable union of countable sets. A compactness argument analogous to the one used in the proof of Proposition 4.1 (successor step) shows that $\widehat{K}$ is compact. Hence $\widehat{K} \in \mathcal{K}_X$.

Finally, applying Proposition 2.5 with the family $\{K_n\}$ and the set K, and using that $K_n^{(\alpha)} = \{x_n\}$ for all n, we obtain

$$\widehat{K}^{(\alpha)} = \left(\bigcup_{n\in\omega} K_n \cup K\right)^{(\alpha)} = \bigcup_{n\in\omega} K_n^{(\alpha)} \cup K = \bigcup_{n\in\omega} \{x_n\} \cup K = K.$$

Case 2: K is finite. In this case, $K \setminus K'$ is also finite. We may write

$$K \setminus K' = \{x_n : n \in M\},$$

for some finite subset $M \subseteq \omega$. Since X is perfect, we have $x_n \in X^{(\alpha)}$ for all $n \in M$.

Fix $r > 0$. For each $n \in M$, apply Proposition 4.1 to $x_n \in X^{(\alpha)}$ and $r > 0$ to obtain $K_n \in \mathcal{K}_X$ such that

$$K_n \subseteq B(x_n, r) \quad \text{and} \quad K_n^{(\alpha)} = \{x_n\}.$$

Define

$$\widehat{K} = \bigcup_{n\in M} K_n \cup K.$$

Since M is finite and each K_n and K is compact, $\widehat{K}$ is compact. It is also countable; hence $\widehat{K} \in \mathcal{K}_X$.

We now show that $\widehat{K}^{(\alpha)} = K$. First, observe that $K \subseteq \widehat{K}^{(\beta)}$ for every $\beta \leq \alpha$, by a transfinite induction entirely analogous to the one used in Proposition 2.5. In particular, $K \subseteq \widehat{K}^{(\alpha)}$.

On the other hand, since $\bigcup_{n\in M} K_n^{(\alpha)} = \{x_n : n \in M\}$, we have

$$\widehat{K}^{(\alpha)} = \left(\bigcup_{n\in M} K_n \cup K\right)^{(\alpha)} = \bigcup_{n\in M} K_n^{(\alpha)} \cup K^{(\alpha)} \subseteq \bigcup_{n\in M} \{x_n\} \cup K = K.$$

Combining these inclusions gives $\widehat{K}^{(\alpha)} = K$.

In both cases, we have constructed $\widehat{K} \in \mathcal{K}_X$ such that $\widehat{K}^{(\alpha)} = K$, as required. □

4.3 Primitives for arbitrary compact sets

We finally remove the countability assumption on K.

Lemma 4.3. *Let (X, d) be a Polish space, and let $A \subseteq X$ be such that every point of A is isolated in A. Then A is countable.*

Proof. See, for instance, [6, p. 126]. □

We can now state and prove the main theorem.

Theorem 4.4. *Let (X, d) be a perfect Polish space, let $K \subseteq X$ be compact, and let $\alpha < \omega_1$. Then there exists a compact subset $\widehat{K} \subseteq X$ such that*

$$\widehat{K}^{(\alpha)} = K.$$

Proof. If K is perfect, then $K^{(\alpha)} = K$ for all $\alpha < \omega_1$. Thus, taking $\widehat{K} = K$ gives the result. Hence, we may assume that K is not perfect.

If $\alpha = 0$, we can again take $\widehat{K} = K$, so we assume $\alpha > 0$. Since K is compact and not perfect, the set $K \setminus K'$ consists precisely of the isolated points of K and is therefore countable by Lemma 4.3. We write

$$K \setminus K' = \{x_n : n \in M\},$$

for some (finite or infinite) $M \subseteq \omega$.

We distinguish two cases.

Case 1: M **is infinite.** As in the proof of Theorem 4.2, we may choose the points x_n to be pairwise distinct and apply Lemma 3.4 to obtain a family of pairwise disjoint open balls $\{B(x_n, r_n)\}_{n\in\omega}$ with $r_n \leq 1/(n+1)$ for all n.

Since X is perfect, then $x_n \in X^{(\alpha)}$ for all $n \in \omega$. For each n, applying Proposition 4.1 to x_n and r_n yields $K_n \in \mathcal{K}_X$ such that

$$K_n \subseteq B(x_n, r_n) \quad \text{and} \quad K_n^{(\alpha)} = \{x_n\}.$$

The sets K_n are pairwise disjoint.

Define

$$\widehat{K} = \bigcup_{n\in\omega} K_n \cup K.$$

An argument analogous to the one in the proof of Theorem 4.2 shows that $\widehat{K}$ is compact. Moreover, by the same reasoning as in Case 1 of Theorem 4.2, we obtain

$$\widehat{K}^{(\alpha)} = K.$$

Case 2: M **is finite.** In this case, $K \setminus K'$ is finite, and we may write

$$K \setminus K' = \{x_n : n \in M\},$$

for some finite $M \subseteq \omega$. Since X is perfect, then $x_n \in X^{(\alpha)}$ for all $n \in M$.

Fix $r > 0$. For each $n \in M$, applying Proposition 4.1 with x_n and r, we obtain $K_n \in \mathcal{K}_X$ such that

$$K_n \subseteq B(x_n, r) \quad \text{and} \quad K_n^{(\alpha)} = \{x_n\}.$$

Define

$$\widehat{K} = \bigcup_{n \in M} K_n \cup K.$$

Since M is finite and each K_n and K is compact, the set $\widehat{K}$ is compact. Repeating the argument from Case 2 of Theorem 4.2, we conclude that $\widehat{K}^{(\alpha)} = K$.

In both cases, we obtain a compact subset $\widehat{K} \subseteq X$ such that $\widehat{K}^{(\alpha)} = K$, as desired. □

Although Theorem 4.4 is stated for compact sets, the argument actually extends to arbitrary closed subsets of a perfect Polish space.

Theorem 4.5 (Extension to closed sets)**.** *Let (X, d) be a perfect Polish space, let $F \subseteq X$ be a closed set, and let $\alpha < \omega_1$. Then there exists a closed set $\widehat{F} \subseteq X$ such that*

$$\widehat{F}(\alpha) = F.$$

Indeed, the key ingredient, Proposition 2.5, only requires K to be closed rather than compact. We restrict ourselves to the compact case in order to simplify the management of details in the construction; nevertheless, the same method yields the corresponding result for closed sets without any essential changes.

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Series with harmonic numbers and the tail of $\zeta(2)$

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ABSTRACT

In this paper we solve an open problem related to the calculation of a quadratic series and we obtain that

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2 = 6\zeta(3) - \frac{19}{2}\zeta(4) + \frac{5}{2}\zeta(5) + 2\zeta(2)\zeta(3).$$

Also, we calculate the sum of the series involving the tail of $\zeta(2)$ and the square of the nth harmonic number:

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = 2\zeta(2)\zeta(3).$$

RESUMEN

En el presente artículo, resolvemos un problema abierto relacionado al cálculo de una serie cuadrática y obtenemos que

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2 = 6\zeta(3) - \frac{19}{2}\zeta(4) + \frac{5}{2}\zeta(5) + 2\zeta(2)\zeta(3).$$

También calculamos la serie que involucra la cola de $\zeta(2)$ y el cuadrado del $n$ésimo número armónico:

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = 2\zeta(2)\zeta(3).$$

Keywords and Phrases: Abel's summation formula, logarithmic integrals, polylogarithm integrals, quadratic zeta series, harmonic numbers, tail of $\zeta(2)$, Riemann zeta function values.

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1 Introduction

In this paper we solve the open problem **7.117** part (b) (see [11, p. 228]), which is about calculating the quadratic series involving the nth harmonic number $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ and the tail of $\zeta(2)$

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2.$$

In addition, we also calculate the series

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right),$$

which is *new* in the mathematical literature.

Further on we present some formulae and definitions which we shall need in our analysis.

Abel's summation formula ([1, p. 55], [5, Lemma A.1, p. 258], [11, p. 38]) states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of real numbers and $A_n = \sum_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}).$$

We will be using the *infinite version* of this formula

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}). \tag{1.1}$$

The Dilogarithm function $\mathrm{Li}_2(z)$ is defined, for $|z| \leq 1$, by ([13, p. 176])

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-t)}{t}\,\mathrm{d}t.$$

The special case $\mathrm{Li}_2(1) = \zeta(2)$ is worth mentioning.

Also, *the Trilogarithm function* $\mathrm{Li}_3(z)$ is defined by

$$\mathrm{Li}_3(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^3} = \int_0^z \frac{\mathrm{Li}_2(t)}{t}\,\mathrm{d}t, \quad |z| \leq 1.$$

The nth generalized harmonic number of order k is defined by $H_n^{(k)} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k}$, $n \geq 1$, $k \geq 1$.

A classical symmetric summation formula involving generalized harmonic numbers is given by

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} + \sum_{n=1}^{\infty} \frac{H_n^{(q)}}{n^p} = \zeta(p+q) + \zeta(p)\zeta(q), \quad p, q > 1. \tag{1.2}$$

A short proof of this formula is based on applying Abel's summation formula (1.1) to the series $\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}$, with $a_k = \frac{1}{k^q}$ and $b_k = H_k^{(p)}$. We have, since $A_n = H_n^{(q)}$ and $b_n - b_{n+1} = -\frac{1}{(n+1)^p}$, that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} &= \lim_{n\to\infty} H_n^{(q)} H_{n+1}^{(p)} - \sum_{n=1}^{\infty} \frac{H_n^{(q)}}{(n+1)^p} = \zeta(q)\zeta(p) - \sum_{n=1}^{\infty} \frac{H_{n+1}^{(q)} - \frac{1}{(n+1)^q}}{(n+1)^p} \\ &= \zeta(q)\zeta(p) - \sum_{n=1}^{\infty} \frac{H_n^{(q)}}{n^p} + \zeta(p+q),\end{aligned}$$

and the identity (1.2) is proved.

2 The main result

The main result of this paper is the following theorem.

Theorem 2.1. *The following identities hold:*

(a) A linear series with the tail of $\zeta(2)$

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right) = 2\zeta(2)\zeta(3).$$

(b) A quadratic series with the tail of $\zeta(2)$

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right)^2 = 6\zeta(3) - \frac{19}{2}\zeta(4) + \frac{5}{2}\zeta(5) + 2\zeta(2)\zeta(3).$$

The convergence of series in Theorem 2.1 is based on the behavior of the sequence $(r_n)_{n\geq 1}$, defined by

$$r_n := \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}, \quad n \geq 1.$$

Using Cesàro-Stolz lemma, the 0/0 case ([5, Theorem B.2, p. 265]), [11, p. 11], one can prove that

$$\lim_{n\to\infty} n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right) = 1. \tag{2.1}$$

This implies that, for large values of n, we have $\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\sim\frac{1}{n}$. Thus,

$$\frac{H_n^2}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)\sim\frac{H_n^2}{n^2}$$

and

$$H_n^2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2\sim\frac{H_n^2}{n^2},$$

and since $\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}=\frac{17}{4}\zeta(4)$ (see [11, pp. 245–249]), we have that both series in Theorem 2.1 converge.

3 Intermediate results

In this section we prove some lemmas which we shall need in obtaining our main result, *i.e.* Theorem 2.1.

Lemma 3.1. *An Euler sum and a logarithmic integral*

The following identities hold:

(a) $\sum_{n=1}^{\infty}\frac{H_n}{n^4}=3\zeta(5)-\zeta(2)\zeta(3)$;

(b) $\int_0^1\frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x=8\zeta(5)-4\zeta(2)\zeta(3)$.

Proof. We prove Lemma 3.1 by calculating the logarithmic integral $\int_0^1\frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x$ by two different ways.

First, we use the generating function for the nth harmonic number $\sum_{n=1}^{\infty}H_nx^n=-\frac{\ln(1-x)}{1-x}$, for $x\in(-1,1)$ ([11, Problem, 3.63, part (a), p. 93]), combined with the formula $\int_0^1 x^k\ln^3 x\,\mathrm{d}x=-\frac{6}{(k+1)^4}$, $k\geq 0$. We integrate by parts, with $f(x)=\ln^2(1-x)$, $f'(x)=-\frac{2\ln(1-x)}{1-x}$, $g'(x)=\frac{\ln^2 x}{x}$, $g(x)=\frac{1}{3}\ln^3 x$, and we have that

$$\begin{aligned}\int_0^1\frac{\ln^2 x\ln^2(1-x)}{x}\,\mathrm{d}x&=\frac{1}{3}\ln^2(1-x)\ln^3 x\Big|_0^1+\frac{2}{3}\int_0^1\frac{\ln(1-x)}{1-x}\ln^3 x\,\mathrm{d}x\\&=\frac{2}{3}\int_0^1\frac{\ln(1-x)}{1-x}\ln^3 x\,\mathrm{d}x=-\frac{2}{3}\int_0^1\left(\sum_{n=1}^{\infty}H_nx^n\right)\ln^3 x\,\mathrm{d}x\\&=-\frac{2}{3}\sum_{n=1}^{\infty}H_n\int_0^1 x^n\ln^3 x\,\mathrm{d}x=4\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^4}\end{aligned}$$

$$= 4\sum_{n=1}^{\infty} \frac{H_{n+1} - \dfrac{1}{n+1}}{(n+1)^4} = 4\sum_{n=1}^{\infty} \frac{H_n}{n^4} - 4\zeta(5). \tag{3.1}$$

Second, we calculate the same integral by using the Taylor series expansion of the logarithmic function $-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, $-1 \le x < 1$. It follows that

$$\int_0^1 \frac{\ln^2 x \ln^2(1-x)}{x}\,\mathrm{d}x = \int_0^1 \frac{\ln^2 x}{x}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{x^{n+m}}{nm}\right)\mathrm{d}x = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{nm}\int_0^1 x^{n+m-1}\ln^2 x\,\mathrm{d}x$$
$$= 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{nm(n+m)^3},$$

since $\int_0^1 x^{k-1}\ln^2 x\,\mathrm{d}x = \frac{2}{k^3}$, $\forall k \in \mathbb{N} = \{1,2,\ldots\}$. A calculation shows that

$$\frac{1}{nm(n+m)^3} = \frac{1}{n^3}\cdot\frac{1}{m(n+m)} - \frac{1}{n^3}\cdot\frac{1}{(n+m)^2} - \frac{1}{n^2}\cdot\frac{1}{(n+m)^3}, \quad n,m \ge 1.$$

It follows that

$$\int_0^1 \frac{\ln^2 x \ln^2(1-x)}{x}\,\mathrm{d}x = 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{1}{n^3}\cdot\frac{1}{m(n+m)} - \frac{1}{n^3}\cdot\frac{1}{(n+m)^2} - \frac{1}{n^2}\cdot\frac{1}{(n+m)^3}\right]$$
$$= 2\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{1}{m(n+m)} - 2\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{1}{(n+m)^2} - 2\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{m=1}^{\infty}\frac{1}{(n+m)^3}$$
$$= 2\sum_{n=1}^{\infty}\frac{H_n}{n^4} - 2\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) - 2\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right)$$
$$= 2\sum_{n=1}^{\infty}\frac{H_n}{n^4} - 4\zeta(2)\zeta(3) + 2\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}}{n^3} + \frac{H_n^{(3)}}{n^2}\right) = 2\sum_{n=1}^{\infty}\frac{H_n}{n^4} - 2\zeta(2)\zeta(3) + 2\zeta(5). \tag{3.2}$$

We used in the previous calculations the formula $\sum_{m=1}^{\infty}\frac{1}{m(n+m)} = \frac{H_n}{n}$ and identity (1.2) with $p = 2$ and $q = 3$. Combining (3.1) and (3.2), we have that the desired results hold and parts (a) and (b) of Lemma 3.1 are proved. □

Remark 3.2. *We mention that the linear Euler sum* $\sum_{n=1}^{\infty}\frac{H_n}{n^4} = 3\zeta(5) - \zeta(2)\zeta(3)$ *is a special case of a classical series formula due to Euler, which states that* $2\sum_{k=1}^{\infty}\frac{H_k}{k^n} = (n+2)\zeta(n+1) - \sum_{k=1}^{n-2}\zeta(n-k)\zeta(k+1)$, $n \in \mathbb{N}\setminus\{1\}$. *For reference materials related to this formula the reader is referred to [5, p. 208]. Lemma 3.1 gives another proof of the identity* $\sum_{n=1}^{\infty}\frac{H_n}{n^4} = 3\zeta(5) - \zeta(2)\zeta(3)$, *which we believe is new in the mathematical literature, based on calculating a quadratic logarithmic integral in two different ways.*

Lemma 3.3. *Logarithm and polylogarithm integrals*

The following formulae are valid:

(a) $\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} = -\int_0^1 \frac{x^n}{1-x} \ln x \, \mathrm{d}x$, $\forall n \geq 1$;

(b) $\int_0^1 x^{n-1} \mathrm{Li}_2(x) \, \mathrm{d}x = \frac{\zeta(2)}{n} - \frac{H_n}{n^2}$, $\forall n \geq 1$;

(c) $\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x} \, \mathrm{d}x = 2\zeta(2)\zeta(3) - 3\zeta(5)$;

(d) $\int_0^1 \frac{\mathrm{Li}_2(x) \ln^2 x}{1-x} \, \mathrm{d}x = 6\zeta(2)\zeta(3) - 11\zeta(5)$;

(e) $\int_0^1 \frac{\mathrm{Li}_2(x) \ln^2 x}{x} \, \mathrm{d}x = 2\zeta(5)$;

(f) $\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x} \, \mathrm{d}x = -2\zeta(2)\zeta(3) + \frac{9}{2}\zeta(5)$.

Proof. *(a)* Using the formula $\int_0^1 x^{n-1} \ln x \, \mathrm{d}x = -\frac{1}{n^2}$, $\forall n \geq 1$, we have that

$$\begin{aligned}
\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} &= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \\
&= -\int_0^1 x^n \ln x \, \mathrm{d}x - \int_0^1 x^{n+1} \ln x \, \mathrm{d}x - \cdots \\
&= -\int_0^1 \left(x^n + x^{n+1} + \cdots\right) \ln x \, \mathrm{d}x = -\int_0^1 \frac{x^n}{1-x} \ln x \, \mathrm{d}x.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\int_0^1 x^{n-1} \mathrm{Li}_2(x) \, \mathrm{d}x &= \int_0^1 x^{n-1} \left(\sum_{m=1}^{\infty} \frac{x^m}{m^2} \right) \mathrm{d}x = \sum_{m=1}^{\infty} \frac{1}{m^2} \int_0^1 x^{n+m-1} \, \mathrm{d}x \\
&= \sum_{m=1}^{\infty} \frac{1}{m^2(n+m)} = \frac{1}{n} \sum_{m=1}^{\infty} \left(\frac{1}{m^2} - \frac{1}{m(n+m)} \right) = \frac{\zeta(2)}{n} - \frac{H_n}{n^2},
\end{aligned}$$

since $\sum_{m=1}^{\infty} \frac{1}{m(n+m)} = \frac{H_n}{n}$.

(c) We have, based on part (b), that

$$\begin{aligned}
\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x} \, \mathrm{d}x &= \int_0^1 \mathrm{Li}_2(x) \left(\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2} \right) \mathrm{d}x = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 x^{n-1} \mathrm{Li}_2(x) \, \mathrm{d}x \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{\zeta(2)}{n} - \frac{H_n}{n^2} \right) = \zeta(2)\zeta(3) - \sum_{n=1}^{\infty} \frac{H_n}{n^4},
\end{aligned}$$

and the result follows based on part (a) of Lemma 3.1.

(d) We calculate the integral by parts, with $f(x) = \mathrm{Li}_2(x)\ln^2 x$, $f'(x) = -\frac{\ln(1-x)}{x}\ln^2 x + \frac{2\mathrm{Li}_2(x)\ln x}{x}$, $g'(x) = \frac{1}{1-x}$, $g(x) = -\ln(1-x)$, and we have that

$$\begin{aligned}
\int_0^1 \frac{\mathrm{Li}_2(x)\ln^2 x}{1-x}\,\mathrm{d}x &= -\ln(1-x)\mathrm{Li}_2(x)\ln^2 x\Big|_0^1 \\
&\quad + \int_0^1 \ln(1-x)\left(\frac{2\mathrm{Li}_2(x)\ln x}{x} - \frac{\ln(1-x)}{x}\ln^2 x\right)\mathrm{d}x \\
&= 2\int_0^1 \frac{\mathrm{Li}_2(x)\ln x\ln(1-x)}{x}\,\mathrm{d}x - \int_0^1 \frac{\ln^2(1-x)\ln^2 x}{x}\,\mathrm{d}x \\
&= \left[-\ln x\,\mathrm{Li}_2^2(x)\Big|_0^1 + \int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x\right] - \int_0^1 \frac{\ln^2(1-x)\ln^2 x}{x}\,\mathrm{d}x \\
&= \int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x - \int_0^1 \frac{\ln^2(1-x)\ln^2 x}{x}\,\mathrm{d}x \\
&= 6\zeta(2)\zeta(3) - 11\zeta(5),
\end{aligned}$$

where the last equality follows based on part (c) of Lemma 3.3 and part (b) of Lemma 3.1.

(e) We have

$$\int_0^1 \frac{\mathrm{Li}_2(x)\ln^2 x}{x}\,\mathrm{d}x = \int_0^1 \frac{\ln^2 x}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}\right)\mathrm{d}x = \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{n-1}\ln^2 x\,\mathrm{d}x = \sum_{n=1}^{\infty}\frac{2}{n^5} = 2\zeta(5).$$

(f) We need the following Landen formula for the Dilogarithm function Li_2 ([13, entry 10, p. 177])

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \zeta(2) - \ln x\ln(1-x), \quad x \in (0,1). \tag{3.3}$$

We have, based on (3.3), that

$$\begin{aligned}
\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\,\mathrm{d}x &= \int_0^1 \frac{\mathrm{Li}_2(x)}{x}\left[\zeta(2) - \ln x\ln(1-x) - \mathrm{Li}_2(x)\right]\mathrm{d}x \\
&= \zeta(2)\int_0^1 \frac{\mathrm{Li}_2(x)}{x}\,\mathrm{d}x - \int_0^1 \frac{\mathrm{Li}_2(x)}{x}\ln x\ln(1-x)\,\mathrm{d}x - \int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x \\
&= \zeta(2)\mathrm{Li}_3(x)\Big|_0^1 - \int_0^1 \frac{\mathrm{Li}_2(x)}{x}\ln x\ln(1-x)\,\mathrm{d}x - \int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x \\
&= \zeta(2)\zeta(3) - \left[-\frac{\ln x}{2}\mathrm{Li}_2^2(x)\Bigg|_0^1 + \frac{1}{2}\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x\right] - \int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x \\
&= \zeta(2)\zeta(3) - \frac{3}{2}\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,\mathrm{d}x \\
&= -2\zeta(2)\zeta(3) + \frac{9}{2}\zeta(5),
\end{aligned}$$

where the last equality follows from part (c) of Lemma 3.3. □

Lemma 3.4. *The generating function of the sequence* $\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)_{n\geq 1}$.

The following equality holds

$$\sum_{n=1}^{\infty}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)x^n = \frac{\zeta(2)x - \operatorname{Li}_2(x)}{1-x}, \quad x \in [-1, 1).$$

Proof. We apply formula (1.1), with $a_n = x^n$ and $b_n = \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}$, and we have, since $b_n - b_{n+1} = \frac{1}{(n+1)^2}$ and $A_n = x + x^2 + \cdots + x^n = \frac{x - x^{n+1}}{1-x}$, that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)x^n &= \lim_{n\to\infty}\frac{x - x^{n+1}}{1-x}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right)\\ &+ \sum_{n=1}^{\infty}\frac{x - x^{n+1}}{1-x}\cdot\frac{1}{(n+1)^2} = \frac{1}{1-x}\sum_{n=1}^{\infty}\frac{x - x^{n+1}}{(n+1)^2}\\ &= \frac{1}{1-x}\sum_{n=1}^{\infty}\frac{x - x^n}{n^2} = \frac{\zeta(2)x - \operatorname{Li}_2(x)}{1-x},\end{aligned}$$

and Lemma 3.4 is proved. □

Lemma 3.5. *The following formulae hold:*

Linear and nonlinear Euler sums

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^3} = \frac{\pi^4}{72} = \frac{5}{4}\zeta(4)$;

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{H_n^2}{n^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3)$.

A mosaic of series with the tail of $\zeta(2)$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = \zeta(3)$;

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = \frac{7}{4}\zeta(4)$;

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^2}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = \zeta(2)\zeta(3) - \zeta(5)$.

Proof. *(a)* The proof of this series, due to the famous German mathematician Christian Goldbach, can be found in [11, pp. 239–240].

(b) This nonlinear harmonic series is recorded in [4, p. 24], [10, p. 209], and it also appears as problem **4.23** in [14], with a detailed solution on pages 394–395.

(c) This is problem **3.20** in [5, p. 142], with a solution in [5, p. 178].

(d) This is problem **3.62** in [5, p. 149], whose proof can be found in the same book on pages 211–213. An alternative solution is given in [8]. Another method for proving this equality, which is based on an application of Abel's summation formula, can be found in [6]. We give below a new solution which uses the special generating function given in Lemma 3.4, combined with the formula $\int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x = -\frac{H_n}{n}$, $n \in \mathbb{N}$. For a history of this formula and reference materials related to it the reader is referred to [5, p. 206]. We have:

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right) &= -\sum_{n=1}^{\infty}\int_0^1 x^{n-1}\ln(1-x)\,\mathrm{d}x\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)\\
&= -\int_0^1 \frac{\ln(1-x)}{x}\sum_{n=1}^{\infty}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)x^n\,\mathrm{d}x\\
&\overset{\text{Lemma 3.4}}{=} -\int_0^1 \frac{x\zeta(2)-\mathrm{Li}_2(x)}{1-x}\cdot\frac{\ln(1-x)}{x}\,\mathrm{d}x\\
&= \int_0^1 \mathrm{Li}_2(x)\frac{\ln(1-x)}{x}\,\mathrm{d}x - \int_0^1(\zeta(2)-\mathrm{Li}_2(x))\frac{\ln(1-x)}{1-x}\,\mathrm{d}x\\
&= -\frac{\mathrm{Li}_2^2(x)}{2}\bigg|_0^1 - \int_0^1(\zeta(2)-\mathrm{Li}_2(x))\frac{\ln(1-x)}{1-x}\,\mathrm{d}x\\
&= -\frac{\pi^4}{72}-\left[-\frac{\ln^2(1-x)}{2}(\zeta(2)-\mathrm{Li}_2(x))\bigg|_0^1+\frac{1}{2}\int_0^1\frac{\ln^3(1-x)}{x}\,\mathrm{d}x\right]\\
&= -\frac{\pi^4}{72}-\frac{1}{2}\int_0^1\frac{\ln^3(1-x)}{x}\,\mathrm{d}x = -\frac{\pi^4}{72}-\frac{1}{2}\int_0^1\frac{\ln^3 x}{1-x}\,\mathrm{d}x\\
&= -\frac{5}{4}\zeta(4)-\frac{1}{2}\int_0^1\ln^3 x\left(\sum_{n=0}^{\infty}x^n\right)\mathrm{d}x = -\frac{5}{4}\zeta(4)-\frac{1}{2}\sum_{n=0}^{\infty}\int_0^1 x^n\ln^3 x\,\mathrm{d}x\\
&= -\frac{5}{4}\zeta(4)+3\sum_{n=0}^{\infty}\frac{1}{(n+1)^4} = -\frac{5}{4}\zeta(4)+3\zeta(4)=\frac{7}{4}\zeta(4).
\end{aligned}
$$

Another proof of part (d) is based on part (a) of Lemma 3.3, *i.e.* the identity $\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2} = -\int_0^1\frac{x^n}{1-x}\ln x\,\mathrm{d}x$, $n \geq 1$, combined with the generating function of the sequence $\left(\frac{H_n}{n}\right)_{n\geq 1}$, *i.e.* $\sum_{n=1}^{\infty}\frac{H_n}{n}x^n = \mathrm{Li}_2(x)+\frac{1}{2}\ln^2(1-x)$, $x \in [-1,1)$ (see [10, entry (25), p. 216]). We leave the details to the interested reader.

(e) This part of the lemma follows from Euler's series $\sum_{n=1}^{\infty}\frac{H_n}{n^2} = 2\zeta(3)$ (see [11, p. 238]) and the identity $\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^2} = \zeta(2)\zeta(3)+\zeta(5)$ (see [10, Theorem 6, p. 210], [14, Problem 4.25, p. 293]). □

4 Proof of the main result

In this section we prove Theorem 2.1.

Proof. *(a)* We need the generating function of the sequence $\left(\frac{H_n^2}{n}\right)_{n\geq 1}$

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n}x^n = \mathrm{Li}_3(x) - \ln(1-x)\mathrm{Li}_2(x) - \frac{1}{3}\ln^3(1-x), \quad x\in[-1,1).$$

The previous formula is recorded in [15, entry (4.42), p. 401] and it can also be found in an equivalent form in [10, entry (38), p. 222]. Then, we have, based on part (a) of Lemma 3.3, that

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right) = -\sum_{n=1}^{\infty}\frac{H_n^2}{n}\int_0^1\frac{x^n}{1-x}\ln x\,\mathrm{d}x$$
$$= -\int_0^1\frac{\ln x}{1-x}\left(\sum_{n=1}^{\infty}\frac{H_n^2}{n}x^n\right)\mathrm{d}x = \int_0^1\frac{\ln x}{1-x}\left(\frac{1}{3}\ln^3(1-x)+\ln(1-x)\mathrm{Li}_2(x)-\mathrm{Li}_3(x)\right)\mathrm{d}x.$$

Let I be the previous integral. We calculate I by parts, with $f(x)=\frac{1}{3}\ln^3(1-x)+\ln(1-x)\mathrm{Li}_2(x)-\mathrm{Li}_3(x)$, $f'(x) = -\frac{\ln^2(1-x)}{1-x}-\frac{\mathrm{Li}_2(x)}{1-x}-\frac{\ln^2(1-x)}{x}-\frac{\mathrm{Li}_2(x)}{x}$, $g'(x)=\frac{\ln x}{1-x}$, $g(x)=\mathrm{Li}_2(1-x)$, and we have that

$$\begin{aligned}
I &= \mathrm{Li}_2(1-x)\left[\frac{1}{3}\ln^3(1-x)+\ln(1-x)\mathrm{Li}_2(x)-\mathrm{Li}_3(x)\right]\Bigg|_0^1\\
&\quad+\int_0^1\mathrm{Li}_2(1-x)\left[\frac{\ln^2(1-x)}{1-x}+\frac{\mathrm{Li}_2(x)}{1-x}+\frac{\ln^2(1-x)}{x}+\frac{\mathrm{Li}_2(x)}{x}\right]\mathrm{d}x\\
&= \int_0^1\frac{\mathrm{Li}_2(1-x)\ln^2(1-x)}{1-x}\mathrm{d}x+\int_0^1\frac{\mathrm{Li}_2(1-x)\mathrm{Li}_2(x)}{1-x}\mathrm{d}x\\
&\quad+\int_0^1\frac{\mathrm{Li}_2(1-x)\ln^2(1-x)}{x}\mathrm{d}x+\int_0^1\frac{\mathrm{Li}_2(1-x)\mathrm{Li}_2(x)}{x}\mathrm{d}x\\
&= \int_0^1\frac{\mathrm{Li}_2(x)\ln^2 x}{x}\mathrm{d}x+2\int_0^1\frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\mathrm{d}x+\int_0^1\frac{\mathrm{Li}_2(x)\ln^2 x}{1-x}\mathrm{d}x = 2\zeta(2)\zeta(3),
\end{aligned}$$

where the last equality follows based on parts (d), (e) and (f) of Lemma 3.3.

(b) Let $r_n=\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}$. We calculate the quadratic series $\sum_{n=1}^{\infty}H_n^2 r_n^2$ by Abel's summation formula (1.1), with $a_k=H_k^2$ and $b_k=r_k^2$. A calculation shows that $\forall n\geq 1$:

$$A_n=\sum_{k=1}^{n}H_k^2=(n+1)H_n^2-(2n+1)H_n+2n=(n+1)H_{n+1}^2-[2(n+1)+1]H_{n+1}+2(n+1),$$

and we observe that

$$b_n - b_{n+1} = r_n^2 - r_{n+1}^2 = \frac{1}{(n+1)^2}(r_n + r_{n+1}).$$

We have

$$\begin{aligned}
\sum_{n=1}^{\infty} H_n^2 r_n^2 &= \lim_{n\to\infty} A_n r_{n+1}^2 + \sum_{n=1}^{\infty} A_n (r_n^2 - r_{n+1}^2) = \sum_{n=1}^{\infty} A_n (r_n^2 - r_{n+1}^2) \\
&= \sum_{n=1}^{\infty} \left[(n+1)H_{n+1}^2 - [2(n+1)+1]H_{n+1} + 2(n+1)\right] \frac{r_n + r_{n+1}}{(n+1)^2} \\
&= \sum_{n=1}^{\infty} \left[\frac{H_{n+1}^2}{n+1} - \left(\frac{2}{n+1} + \frac{1}{(n+1)^2}\right) H_{n+1} + \frac{2}{n+1}\right] (r_n + r_{n+1}) \\
&= \sum_{n=1}^{\infty} \left[\frac{H_{n+1}^2}{n+1} - \left(\frac{2}{n+1} + \frac{1}{(n+1)^2}\right) H_{n+1} + \frac{2}{n+1}\right] \left(2r_{n+1} + \frac{1}{(n+1)^2}\right) \\
&= \sum_{n=1}^{\infty} \left[\frac{H_n^2}{n} - \left(\frac{2}{n} + \frac{1}{n^2}\right) H_n + \frac{2}{n}\right] \left(2r_n + \frac{1}{n^2}\right) \\
&= 2\sum_{n=1}^{\infty} \frac{H_n^2}{n} r_n + \sum_{n=1}^{\infty} \frac{H_n^2}{n^3} - 4\sum_{n=1}^{\infty} \frac{H_n}{n} r_n - 2\sum_{n=1}^{\infty} \frac{H_n}{n^2} r_n \\
&\quad - 2\sum_{n=1}^{\infty} \frac{H_n}{n^3} - \sum_{n=1}^{\infty} \frac{H_n}{n^4} + 4\sum_{n=1}^{\infty} \frac{r_n}{n} + 2\sum_{n=1}^{\infty} \frac{1}{n^3} \\
&= 6\zeta(3) - \frac{19}{2}\zeta(4) + \frac{5}{2}\zeta(5) + 2\zeta(2)\zeta(3),
\end{aligned}$$

where the last equality follows based on part (a) of Theorem 2.1, part (a) of Lemma 3.1 and Lemma 3.5. We used in the previous calculations that $\lim_{n\to\infty} A_n r_{n+1}^2 = 0$. This can be proved, based on (2.1), as follows

$$\begin{aligned}
\lim_{n\to\infty} A_n r_{n+1}^2 &= \lim_{n\to\infty} \frac{A_n}{(n+1)^2} \cdot \lim_{n\to\infty} ((n+1)r_{n+1})^2 = \lim_{n\to\infty} \frac{A_n}{(n+1)^2} \\
&= \lim_{n\to\infty} \left(\frac{H_n^2}{n+1} - \frac{2n+1}{n+1} \cdot \frac{H_n}{n+1} + \frac{2n}{(n+1)^2}\right) = 0. \qquad \square
\end{aligned}$$

5 Concluding remarks

We illustrate the identities given in Theorem 2.1 by some numerical examples given in the next tables.

Let $u_n = \frac{H_n^2}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)$, $n \in \mathbb{N}$. In Theorem 2.1, part (a), we have proved that $\lim_{n\to\infty} \sum_{k=1}^{n} u_k = 2\zeta(2)\zeta(3)$. For the right-hand side we have

$$2\zeta(2)\zeta(3) = 3.954608700\ldots$$

and for the left-hand side we have the numerical results given in Table 1[1].

Table 1: Theorem 2.1, part (a)

n	$\sum_{k=1}^{n} u_k$
100000	3.952885141 ...
200000	3.953653756 ...
300000	3.953934247 ...
400000	3.954082231 ...
500000	3.954174493 ...
600000	3.954237863 ...
700000	3.954284244 ...
800000	3.954319753 ...
900000	3.954347867 ...
1000000	3.954370713 ...

Let $v_n = H_n^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2$, $n \in \mathbb{N}$. In Theorem 2.1, part (b), we have proved that $\lim_{n\to\infty} \sum_{k=1}^{n} v_k = 6\zeta(3) - \frac{19}{2}\zeta(4) + \frac{5}{2}\zeta(5) + 2\zeta(2)\zeta(3)$. For the right-hand side we have

$$6\zeta(3) - \frac{19}{2}\zeta(4) + \frac{5}{2}\zeta(5) + 2\zeta(2)\zeta(3) = 3.477198776\ldots$$

and for the left-hand side we have the numerical results given in Table 2

Table 2: Theorem 2.1, part (b)

n	$\sum_{k=1}^{n} v_k$
100000	3.475475488 ...
200000	3.476244100 ...
300000	3.476524591 ...
400000	3.476672575 ...
500000	3.476764837 ...
600000	3.476828207 ...
700000	3.476874588 ...
800000	3.476910097 ...
900000	3.476938211 ...
1000000	3.476961057 ...

[1]For the numerical calculations given in this section we have used Maple 13.

The numerical data from the two tables above illustrate that the sequences $\left(\sum_{k=1}^{n} u_k\right)_{n\in\mathbb{N}}$ and $\left(\sum_{k=1}^{n} v_k\right)_{n\in\mathbb{N}}$ are slowly convergent to their limits.

The calculation of series, linear and nonlinear, involving combination of harmonic numbers and tails of Riemann zeta function values is a relatively new topic in the theory of series. This direction of research has been extensively studied in recent years by Furdui [5, Chapter 3], Furdui and Vălean [7], Hoffman [9], Sîntămărian and Furdui [11, Chapters 2, 3, 4, 7], Somu, Haw, Nguyen and Khanh Tran [12] and Vălean [14, Chapter 4], [15, Chapter 4].

In general, the computation of such series is connected to the evaluation of linear or nonlinear Euler series, *i.e.*, series of the form $\sum_{n=1}^{\infty} \frac{H_n^k}{n^m}$, where $k \geq 1$, $m \geq 2$ are positive integers; see [2–4, 13] and the references therein for more information about the evaluation of such series.

For the numerical calculations given in this section we have used Maple 13.

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Cyclic covers of an algebraic curve from an adelic viewpoint

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ABSTRACT

We propose an algebraic method for the classification of branched Galois covers of a curve X focused on studying Galois ring extensions of its geometric adele ring $\mathbb{A}_X$. As an application, we deal with cyclic covers; namely, we determine when a given cyclic ring extension of $\mathbb{A}_X$ comes from a corresponding cover of curves $Y \to X$, which is reminiscent of a Grunwald-Wang problem, and also determine when two covers yield isomorphic ring extensions, which is known in the literature as an equivalence problem. This completely algebraic method permits us to recover ramification, certain analytic data such as rotation numbers, and enumeration formulas for covers.

RESUMEN

Proponemos un método algebraico para la clasificación de cubrientes de Galois ramificados de una curva X centrándonos en el estudio de las extensiones de Galois de anillos de su anillo de adeles geométricos $\mathbb{A}_X$. Como aplicación, en el caso de cubrientes cíclicos determinamos cuándo una extensión cíclica de anillos de $\mathbb{A}_X$ proviene de un cubriente de curvas $Y \to X$, situación que evoca un problema de Grunwald-Wang, y también determinamos cuándo dos cubrientes dan lugar a extensiones de anillos isomorfas, lo cual se conoce en la literatura como problema de equivalencia. Este método algebraico nos permite recuperar la ramificación, ciertos datos analíticos como son los números de rotación y fórmulas de enumeración para cubrientes.

Keywords and Phrases: Coverings of curves, algebraic curves and function fields, Galois theory of commutative rings, geometric adele ring, Kummer theory.

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1 Introduction

The classification of finite extensions of the function field Σ of an algebraic curve X is a long-standing problem with roots in the foundational work of Riemann, Klein, Hurwitz, and many others. The standard techniques for dealing with this question ultimately rest upon analytic methods in the theory of Riemann surfaces, appearing often in the guise of complex algebraic geometry.

Here we present a purely algebraic alternative approach based on the use of algebraic extensions of the geometric adele ring $\mathbb{A}_X$ of an algebraic curve X over an algebraically closed base field $\Bbbk$. The motivation is the following fact: a finite Galois cover $Y \to X$ with group G naturally endows $\mathbb{A}_Y$ with an $\mathbb{A}_X$-algebra structure together with an action of G, which makes $\mathbb{A}_Y$ a Galois ring extension of $\mathbb{A}_X$ with group G in the sense of Chase-Harrison-Rosenberg (CHR) [4].

The main object in our toolkit is the Harrison set $\mathbb{H}(R, G)$, defined for any commutative ring R and finite group G (see §1.2). It consists of isomorphism classes of Galois ring extensions with fixed Galois group G. For abelian groups, $\mathbb{H}(R, G)$ can be itself endowed with a group structure.

Although there is a geometric interpretation of the Harrison set in the language of torsors and the algebraic fundamental group, we choose to work with the Harrison group $\mathbb{H}(\mathbb{A}_X, G)$ rather than G-torsors over $\operatorname{Spec}(\mathbb{A}_X)$, since describing the full spectrum requires ultrafilters, which unnecessarily complicates matters. In our method we can avoid having to deal with the complete spectrum ([16, Lemma 3.13, Theorem 3.14]). For the interested reader, the survey [18] discusses the connections between the algebraic foundation of CHR and subsequent generalizations or alternative points of view, including the Galois-Grothendieck theory.

Using these algebraic methods, we classify the cyclic covers of X of prime order different from $\operatorname{char}(\Bbbk)$. As must be the case, we recover the bijection between algebraic curves which are finite G-Galois covers branched over n points and conjugacy classes of n-tuples $(g_1, \ldots, g_n)$ generating G and satisfying $\prod_i g_i = 1$.

The essential tool for this is the Kummer sequence for ring extensions, described, for example, in [8, Theorem 5.4] and, from the point of view of étale cohomology in [14, Chapter III, Proposition 4.11]. For readers accustomed to the latter, we can show that $\mathbb{H}(R, \mathcal{C}_n) \simeq H^1_{\text{et}}(\operatorname{Spec} R, \mathcal{C}_n)$ when n is coprime with $\operatorname{char}(R)$, thus establishing an equivalence with our approach (but see the remark above regarding $\operatorname{Spec}(\mathbb{A}_X)$).

The paper is structured around three main themes: the first is determining when, for a prime p, given a $\mathcal{C}_p$-Galois ring extension of $\mathbb{A}_X$, there exists a $\mathcal{C}_p$-Galois cover of curves $Y \to X$ (§2.1) which gives rise to it in the sense explained below. We shall refer to this as the *existence problem.*

The second deals with what is known as an *equivalence problem*, in this case, for function fields of algebraic curves over algebraically closed fields (see [19] for a current survey of many instances of this question), namely, when two such covers induce isomorphic extensions of $\mathbb{A}_X$ (§2.2).

The third, and final question we will deal with, is how geometric data (such as ramification, analytic rotation numbers, enumeration of covers, etc.) can be recovered from the constructions by completely algebraic means (§3 and §4).

The first two problems are solved in §2, by relating the Kummer theory of the function field Σ with that of its adele ring $\mathbb{A}_X$. To be precise, the map induced by tensoring with $\mathbb{A}_X$ induces a group homomorphism $\mathbb{H}(\Sigma, \mathcal{C}_p) \to \mathbb{H}(\mathbb{A}_X, \mathcal{C}_p)$. The determination of its kernel and image by the exact sequence (2.8) in Theorem 2.3 solves the existence problem and also provides the key to answering the equivalence problem, which admits various forms (Corollaries 2.7, 2.9, and 2.10).

Motivated by the fact that the notion of ramification locus appears naturally in the correspondence between finite extensions of the function field Σ and covers of the curve X, we are led to consider a refinement of (2.8) in §3. For a finite nonempty subset $\mathfrak{R}$ of closed points of X representing the ramification locus, we study the Harrison group of the ring corresponding to the affine curve $X \setminus \mathfrak{R}$. The main results are encapsulated in the commutative cube (3.10) of Theorem 3.2 and applied in §3.2 to obtain various filtrations of Harrison groups by ramification.

The third question is taken up in the last section of the paper (§4), which presents various situations illustrating how our algebraic approach indeed recovers interesting geometric information. In §4.1 we show how our study of the Harrison group corresponds with the classification of p-cyclic Galois covers in [1] and [5]. In §4.2 we recover the enumeration formulas for p-cyclic covers with specified ramification given in [10, 12, 13]. In §4.3 we show how our methods allow one to define so-called rotation data purely algebraically in terms of the local Kummer symbols at ramified points. We prove that for $\Bbbk = \mathbb{C}$, our definition coincides with the usual analytic notion (see, for example, [7]).

We believe that the approach presented here based on studying Galois extensions of the adeles, and in general, of finite $\mathbb{A}_X$-algebras, provides an interesting alternative algebraic perspective on several geometric problems, and, although limited here to cyclic extensions as an illustration, can be adapted to more general situations.

1.1 Geometric adeles

Let X be a projective, irreducible, non-singular curve over an algebraically closed field $\Bbbk$. Let Σ be the function field of X. The notation $x \in X$ will be used to denote a closed point. A_x, K_x, $\mathfrak{m}_x$, v_x, will denote respectively the completion of the valuation ring at $x \in X$, its quotient field, maximal ideal, and the completed valuation. Since $\Bbbk$ is algebraically closed, the residue field is $A_x/\mathfrak{m}_x = \Bbbk$.

Recall that the adele ring $\mathbb{A}_X$ is the restricted direct product

$$\mathbb{A}_X \coloneqq \prod_{x\in X}{}'(K_x, A_x) = \{(\alpha_x)_{x\in X} : \alpha_x \in A_x \text{ for almost all } x \in X\}$$

equipped with the restricted product topology. $\mathbb{A}_X$ is the direct limit over finite subsets F of (closed) points of X containing an arbitrary fixed F_0,

$$\mathbb{A}_X \simeq \varinjlim_{F\supseteq F_0} \mathbb{A}_{X,F}, \qquad \mathbb{A}_{X,F} \coloneqq \prod_{x\in F} K_x \times \prod_{x\in X\setminus F} A_x. \tag{1.1}$$

The limit topology is generated by the neighborhood basis at 0 consisting of the sets $\prod_x \mathfrak{m}_x^{n_x}$, where $(n_x)_{x\in X}$ runs over collections of non-negative integers n_x such that $n_x = 0$ for almost all x. It coincides with the restricted product topology. A third characterization is in terms of the $\Bbbk$-vector subspaces commensurable with $\mathbb{A}_X^+ \coloneqq \prod_{x\in X} A_x$ as considered by Tate.

The idele group $\mathbb{I}_X$ of invertible elements of $\mathbb{A}_X$ is the restricted product of K_x^* with respect to the unit groups A_x^* and also the direct limit $\varinjlim \mathbb{I}_{X,F}$ where $\mathbb{I}_{X,F} \coloneqq \mathbb{A}_{X,F}^*$.

1.2 Galois theory of rings

Let us give a brief overview of the Galois theory of commutative ring extensions. Standard sources for this include [1, 4, 8]. A far-reaching survey of this topic and how it has evolved may be found in [18]. We have extracted the following definitions and results from our previous paper [16] and copied them here for the reader's convenience.

We begin by recalling the definition of a Galois extension of commutative rings (taken from [4, Theorem 1.3], which generalizes the classical case of fields.

Definition 1.1 (Galois extension of rings)**.** *A Galois extension of a commutative ring R consists of a pair (S, G), where S is a commutative ring extension of R and G is a finite group which acts faithfully on S by R-algebra automorphisms, with invariants $S^G = R$, and satisfying one of the following equivalent conditions (we only list the ones which we will use later on):*

1) S is a separable R-algebra and the elements of G are pairwise strongly distinct.

2) The map $h : S \otimes S \to \operatorname{Maps}(G, S)$ given by $h(s \otimes t)(g) = sg(t)$ is an S-algebra isomorphism.

3) If $g \in G$ with $g \neq 1$, for any maximal ideal $\mathfrak{m}$ of S there is some $s \in S$ such that $g(s) \not\equiv s \bmod \mathfrak{m}$.

In this case we say that S is a Galois ring extension of R with Galois group G, or simply a G-Galois extension of R.

Recall that the notion of strongly distinct for a pair of morphisms of commutative rings $f, g : S \to T$ means that for every nonzero idempotent $e \in T$ there exists $s \in S$ such that $f(s)e \neq g(s)e$.

We will denote Galois extensions S of the commutative ring R with group G acting via R-algebra automorphisms as pairs (S, G).

Definition 1.2 (Kummerian ring)**.** *Let n be a natural number. A commutative ring R is n-*Kummerian *if n is prime to* $\mathrm{char}(R)$ *and its unit group R^* contains a distinguished n-cyclic subgroup μ_n.*

A field K with $\mathrm{char}(K)$ prime to n containing the n-th roots of unity $\mu_n = \mu_n(K^*)$, is n-Kummerian and this is the *only* possible choice of subgroup.

In our context, having fixed an algebraically closed field $\Bbbk$, choosing $\mu_n := \mu_n(\Bbbk^*)$, the group of nth roots of unity in $\Bbbk$, induces the structure of an n-Kummerian ring on any $\Bbbk$-algebra R of characteristic prime to n. This is compatible with $\Bbbk$-algebra morphisms.

Note that for the adele ring, $\Bbbk \subseteq \Sigma$ is diagonally embedded in $\mathbb{A}_X$ and we also have copies of $\Bbbk$ in each completion K_x. This is an example of how there may be infinitely many choices of subgroups $\mu_n \subseteq R^*$, highlighting the need to specify one.

A particular type of cyclic Galois ring extensions of a Kummerian ring may be constructed as follows.

Definition 1.3 ((G, χ)-Kummer extensions)**.** *For a fixed n-Kummerian base ring R, a (G, χ)-*Kummer extension of *R is a triple $(R\{u^{1/n}\}, G, \chi)$, where*

$$R\{u^{1/n}\} := R[T]/(T^n - u),$$

with $u \in R^*$, and G is an n-cyclic group which acts on S via the character $\chi : G \to \mu_n \subseteq R^*$ by

$$g(T) := \chi(g)T. \tag{1.2}$$

That this in fact is a Galois extension is shown in [8, p. 20].

Definition 1.4 (Equivariant Isomorphism)**.** *Let R be a commutative ring and G a fixed finite group. For $i = 1, 2$, let S_i be a ring extension of R with a faithful action of G by R-automorphisms of S_i. We say that the pairs (S_1, G) and (S_2, G) are equivariantly isomorphic, or simply G-isomorphic, via φ, if φ is an R-algebra isomorphism $\varphi : S_1 \xrightarrow{\sim} S_2$ such that $\varphi \circ g = g \circ \varphi$, for all $g \in G$.*

It is clear that a G-isomorphism preserves the G-Galois property of a ring extension. Harrison showed how to classify the set of Galois extensions of a given ring R and group G with a fixed action, via the following object.

Definition 1.5 (The Harrison group [4, p. 67])**.** *Given a group G, the set of all G-isomorphism classes of G-Galois ring extensions S over a fixed base ring R with a fixed faithful action of G is called the* Harrison set *of (R, G) and denoted by $\mathbb{H}(R, G)$. When G is a finite abelian group, $\mathbb{H}(R, G)$ can be endowed with a group structure. In this case it is called the* Harrison group*.*

We review some of the basic facts regarding group actions on modules over a ring R. Let R be an n-Kummerian ring with distinguished subgroup μ_n, and S an R-module. Suppose G is a finite abelian group of order n acting on S via R-module automorphisms. Its dual group $\widehat{G}$ will be identified with $\operatorname{Hom}(G, \mu_n)$ and its elements referred to simply as characters of G. We may consider the decomposition of S with respect to the action of $\widehat{G}$, namely, for $\chi \in \widehat{G}$, we define the χ-eigenspace (or isotypical component)

$$S^\chi := \{\alpha \in S : g(\alpha) = \chi(g)\alpha, \ \forall g \in G\}.$$

Projection onto the χ-eigenspace is given by $\alpha_\chi = e_\chi \alpha$, where e_χ is the corresponding idempotent in the group algebra,

$$e_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g \in R[G].$$

We then have the decomposition

$$S = \bigoplus_{\chi \in \widehat{G}} S^\chi. \tag{1.3}$$

If G is cyclic of prime order and the action is nontrivial, then each R-module S^χ is nontrivial.

With regard to the decomposition (1.3), the product of two G-Galois extensions S_i/R for $i = 1, 2$ is given by

$$S_1 \cdot S_2 := \bigoplus_{\chi} (S_1^\chi \otimes S_2^\chi), \tag{1.4}$$

where G acts on the summand $S_1^\chi \otimes S_2^\chi$ via $g(s_1 \otimes s_2) := \chi(g)(s_1 \otimes s_2)$. One checks that this product factors through equivariant equivalence and thus defines the group law on the Harrison group $\mathbb{H}(R, G)$.

The neutral element with respect to this product is the so-called trivial G-Galois extension, defined by $R^{(G)} := \bigoplus_\chi R$, considered as the set of maps from G to R under the standard sum and product, and with the action of G given by $g((r_\chi)_\chi) = (r_{g^*\chi})_\chi$ where g^* denotes composition with multiplication by g.

For simplicity, as in the following result, we will restrict the rank n to be a prime number p, different from the characteristic of $\mathbb{k}$.

Definition 1.6 (G-primitive element)**.** *Let R be a p-Kummerian ring and S an R-algebra on which a p-cyclic group G acts via R-automorphisms. Given a* nontrivial *character $\chi : G \to \mu_p \subseteq R^*$, an element $\alpha \in S$ is called (G, χ)-primitive if:*

- *$1, \alpha, \dots, \alpha^{p-1}$ is an R-module basis of S.*
- *$g(\alpha) = \chi(g)\alpha$ for $g \in G$, i.e. $\alpha \in S^\chi$.*

In this case we also say that α is G-primitive with character χ.

The subject of primitive elements in ring extensions is in itself a topic which has been studied by various authors (see [15, 17]). In this regard, we cite the following result, which is [16, Proposition 3.21].

Proposition 1.7. *Let S be a separable algebra over a p-Kummerian ring R on which the p-cyclic group G acts via R-automorphisms with $S^G = R$. Fix a nontrivial character $\chi : G \to \mu_p \subseteq R^*$. For an element $\alpha \in S^\chi$, the following are equivalent:*

1) α is (G, χ)-primitive, i.e. $1, \alpha, \alpha^2, \dots, \alpha^{p-1}$ is an R-module basis of S.

2) $\alpha^p \in R^*$.

3) α is invertible in S.

If this is the case, then:

4) α generates S^χ as an R-module, and S is a free R-module of rank p.

5) The characteristic polynomial of α is $C_\alpha(T) = T^p - \alpha^p \in R[T]$. It is separable and generates $\mathrm{Ann}(\alpha)$.

6) S is equivariantly isomorphic to the (G, χ)-Kummer extension (Definition 1.3) $(R\{u^{1/p}\}, G, \chi)$ for $u = \alpha^p$.

The basic result in the Kummer theory of ring extensions is the following, which is [8, Theorem 5.4].

Proposition 1.8 (The general Kummer sequence)**.** *For an n-Kummerian ring, R, fixing a nontrivial character $\chi : \mathcal{C}_n \to \mu_n \subseteq R^*$ yields an exact sequence of groups:*

$$1 \to {R^*}/{R^{*n}} \xrightarrow{i_{(R,\chi)}} \mathbb{H}(R, \mathcal{C}_n) \to \mathrm{Pic}(R)[n] \to 1. \tag{1.5}$$

Here $i = i_{(R,\chi)}$ sends $u \in R^*$ to the $(\mathcal{C}_n, \chi)$-Kummer extension $R\{u^{1/n}\}$, and the second map is projection onto the χ-eigenspace. This sequence is called the Kummer sequence.

It follows from the Kummer sequence that if R is n-Kummerian and $\mathrm{Pic}(R)$ has trivial n-torsion, then

$$\mathbb{H}(R, \mathcal{C}_n) \simeq {R^*}/{R^{*n}} \tag{1.6}$$

and thus every isomorphism class of $\mathcal{C}_n$-Galois extensions of R contains a representative which is $(\mathcal{C}_n, \chi)$-Kummer. In particular this holds for the function field $R = \Sigma$ and, more interestingly, also for the geometric adele ring $R = \mathbb{A}_X$ ([16, Theorem 3.11]).

Remark 1.9. *As in [16, Lemma 3.41], one sees that $i_{(R,\chi)}$ is equivariant with respect to the action of $\mathrm{Aut}(\mathcal{C}_n) \simeq (\mathbb{Z}/(n))^*$ by*

$$b \in (\mathbb{Z}/(n))^* \longmapsto (u \mapsto u^b) \in \mathrm{Aut}(R^*/R^{*n}) \tag{1.7}$$

and the group isomorphism $\chi^* : \mathrm{Aut}(\mu_n) \simeq (\mathbb{Z}/(n))^* \xrightarrow{\sim} \mathrm{Aut}(\mathcal{C}_n)$ induced by $\chi : \mathcal{C}_n \xrightarrow{\sim} \mu_n \subseteq R^*$.

Example 1.10. *Let p be a prime different from $\mathrm{char}\,\mathbb{k}$. For a choice of nontrivial character $\chi : \mathcal{C}_p \to \mu_p(\mathbb{k}^*)$, the Kummer sequence yields an isomorphism $\Sigma^*/\Sigma^{*p} \xrightarrow{\sim} \mathbb{H}(\Sigma, \mathcal{C}_p)$ given by sending $f \in \Sigma^*$ to the Galois* ring *extension*

$$\Omega_f \coloneqq \Sigma[T]/(T^p - f) \tag{1.8}$$

with action via $g(T) = \chi(g)T$, which is a $(\mathcal{C}_p, \chi)$-Kummer extension. Thus, without loss of generality, we may work with pairs of the form $(\Omega_f, \mathcal{C}_p)$ when considering general representatives of elements in $\mathbb{H}(\Sigma, \mathcal{C}_p)$.

By Capelli's Theorem, $f \notin \Sigma^{*p}$ if and only if Ω_f is a p-Kummer field extension of Σ. In this case $\mathcal{C}_p$ is identified with the classical Galois group $\mathrm{Gal}(\Omega_f/\Sigma)$. Recall that by the Kummer theory of fields, every p-cyclic field extension of Σ is of this form (see, for example, Birch's paper in [3, Ch. III, §2]).

If $f \in \Sigma^{*p}$, then Ω_f is the trivial extension $\Omega_f \simeq \prod^p \Sigma$, and $\mathcal{C}_p$ is identified with a subgroup of the symmetric group $\mathcal{S}_p$ generated by a p-cycle, permuting the copies of Σ.

We shall also need the following construction of cyclic Kummer extensions due to Borevich.

Theorem 1.11 ([1, §8, Theorem 2])**.** *For an n-Kummerian ring R and fixed nontrivial character $\chi : \mathcal{C}_n \to \mu_n \subseteq R^*$, there is a one-to-one correspondence between n-cyclic Kummer extensions S of R and pairs $(\mathfrak{L}, \varphi)$, where $\mathfrak{L} \in \mathrm{Pic}(R)[n]$ and φ is an isomorphism $\varphi : \mathfrak{L}^{\otimes n} \xrightarrow{\sim} R$.*

The construction sends an n-cyclic extension S/R to the pair consisting of the eigenspace S^χ together with the map $\varphi : (S^\chi)^{\otimes n} \to S^{\chi^n} = S^{\mathcal{C}_p} = R$, which is easily seen to be an isomorphism. The inverse sends a pair $(\mathfrak{L}, \varphi)$ to the quotient of $\bigoplus_{i \geq 0} \mathfrak{L}^{\otimes i}$ by the ideal generated by $\varphi(\ell^{\otimes n}) - \ell^{\otimes n}$, where ℓ runs over the elements of $\mathfrak{L}$.

In fact, this correspondence induces a group isomorphism of $\mathbb{H}(R, \mathcal{C}_p)$ to the set of pairs $(\mathfrak{L}, \varphi)$ with group law given by $(\mathfrak{L}_1, \varphi_1) \cdot (\mathfrak{L}_2, \varphi_2) = (\mathfrak{L}_1 \otimes \mathfrak{L}_2, \varphi_1 \otimes \varphi_2)$ ([1, §11, Theorem 1]).

Definition 1.12 (Conjugacy of G-ring extensions)**.** *Let R be a commutative ring. For $i = 1, 2$, let S_i be a ring extension of R with a faithful action of a group G_i by R-automorphisms of S_i. We say that (S_1, G_1) and (S_2, G_2) are conjugate via (φ, τ) if $\varphi : S_1 \xrightarrow{\simeq} S_2$ is an R-algebra isomorphism and $\tau : G_1 \xrightarrow{\simeq} G_2$ is a group isomorphism, such that $\varphi \circ g = \tau(g) \circ \varphi$ for all $g \in G_1$. We will denote this relation by $(S_1, G_1) \sim (S_2, G_2)$.*

Note that the group isomorphism τ is in fact determined by the R-algebra isomorphism φ, namely $\tau(g) = g^{\varphi} \coloneqq \varphi \circ g \circ \varphi^{-1}$, although it is convenient to denote it as part of a pair (φ, τ) as we are doing here.

Remark 1.13. *If one looks at the definition, it is immediately clear that being a Galois extension S/R is preserved by conjugation.*

Definition 1.14. *Suppose (S, G) is a ring extension of R with G a group acting faithfully by R-automorphisms of S. Given an automorphism $\tau \in \operatorname{Aut}(G)$ of G, we define the* twist *of (S, G) by τ, denoted by $(S, G)^{\tau}$, to be the same ring extension S/R but with the action of G now given by $g(s) \coloneqq \tau(g)s$.*

We will make use of the bifunctoriality of the Harrison group, as expressed in the following result taken from [8, Proposition 3.10, Corollary 3.11].

Proposition 1.15. *Given a homomorphism of commutative rings $R_1 \to R_2$ and a homomorphism $\tau : G_1 \to G_2$ of finite abelian groups, we have a commutative diagram*

$$\begin{array}{ccc} \mathbb{H}(R_1, G_1) & \xrightarrow{\tau^*} & \mathbb{H}(R_1, G_2) \\ {\scriptstyle R_2 \otimes_{R_1} -}\downarrow & & \downarrow{\scriptstyle R_2 \otimes_{R_1} -} \\ \mathbb{H}(R_2, G_1) & \xrightarrow[\tau^*]{} & \mathbb{H}(R_2, G_2) \end{array} \tag{1.9}$$

of group homomorphisms of Harrison groups.

In particular, conjugacy classes of n-cyclic Galois ring extensions correspond bijectively to the quotient set of $\mathcal{C}_n$-isomorphism classes of $\mathcal{C}_n$-Galois extensions modulo $\operatorname{Aut}(\mathcal{C}_n)$, *i.e.*

$$\left\{ \text{Conjugacy classes of } n\text{-cyclic Galois extensions } (S, G) \text{ of } R \right\} \overset{1:1}{\rightsquigarrow} \mathbb{H}(R, \mathcal{C}_n) / \operatorname{Aut}(\mathcal{C}_n). \tag{1.10}$$

1.3 Covers as Galois ring extensions

Since our main goal is to characterize Galois covers of algebraic curves in terms of Galois ring extensions of the geometric adele ring $\mathbb{A}_X$, we need to begin by giving a precise notion of Galois cover of a curve. Note that we include covers which are not necessarily irreducible.

Definition 1.16 (Galois cover)**.** *A cover of projective, non-singular algebraic curves, $\pi : Y \to X$, where X is irreducible and π is separable, is said to be a Galois cover if the quotient of Y by the action of* $\mathrm{Gal}(Y/X)$ *is X.*

Define two covers $Y' \to X$, $Y \to X$ to be equivalent when there is a birational map between them as schemes over X (i.e., there are dense open subschemes U', U of Y', Y respectively and an isomorphism of U' and U over X).

Given X as in §1.1, let $\mathrm{Cov}(X, \mathcal{C}_p)$ *denote the set of isomorphism classes of p-cyclic Galois covers $Y \to X$, where we restrict to Y having connected components which coincide with its irreducible components.*

Note that with this definition, ramification is allowed (in accordance, for example, with the definition of finite Galois branched cover in [20, p. 125]). Then each equivalence class has a distinguished representative that satisfies the additional property that its irreducible and connected components coincide.

An *irreducible* cover $Y \to X$, determines a finite separable extension Ω/Σ of function fields. Conversely, given such an Ω, the Zariski-Riemann variety Y of Ω, whose set of closed points is the set of discrete valuations on Ω, determines a cover of X. This is the classical equivalence between the category of finitely generated field extensions of $\Bbbk$ of transcendence degree 1 and the category of nonsingular projective irreducible curves and nonconstant morphisms (*e.g.* [22, Theorem 0BY1]).

A general cover, which may not be irreducible, determines a finite separable *ring* extension Ω of Σ, which is isomorphic to a finite product of field extensions. In this case Y is the disjoint union of the corresponding Zariski-Riemann varieties of these fields and Ω is the total quotient ring of any affine dense open subscheme of Y.

Example 1.17. *Consider $X = \mathbb{P}^1$ with homogeneous coordinates $[x_0, x_1]$ and the following two covers. First, $Y = \coprod^{p} X$ with the action of $\mathcal{C}_p$ by cyclic permutation. Second, the cover $Z \subseteq \mathbb{P}^2$ given by the zeros of the homogeneous ideal generated by $x_1^p - x_2^p$ where $[x_0, x_1, x_2]$ are homogeneous coordinates on $\mathbb{P}^2$, with action $[x_0, x_1, x_2] \to [x_0, x_1, \zeta x_2]$ where ζ is a pth root of unity. Observe that Z is connected and birational to Y. The conditions imposed above are satisfied by Y but not by Z.*

The following is an algebraic version of [3, (14.2)], taking into account the non-irreducible case. The reader may check that the proof (discarding the topological aspects) is analogous to the case of a global field considered there.

Theorem 1.18. *Let Ω be a finite separable ring extension of Σ and let $\mathbb{A}_Y$ denote the ring of adeles of Ω. Then*

$$\mathbb{A}_X \otimes_\Sigma \Omega \simeq \mathbb{A}_Y \tag{1.11}$$

as $\mathbb{A}_X$-algebras. Furthermore, $\mathbb{A}_Y^+$ is the integral closure of $\mathbb{A}_X^+$ in $\mathbb{A}_X \otimes_\Sigma \Omega$.

A Galois cover $Y \to X$ gives a finite Galois extension of rings Ω/Σ of group $\mathrm{Gal}(Y/X)$. In this case, the isomorphism (1.11) is equivariant with respect to the action of $\mathrm{Gal}(Y/X)$, and

$$\mathbb{A}_Y^{\mathrm{Gal}(Y/X)} = \mathbb{A}_X. \tag{1.12}$$

The following result expresses the basic correspondence between the Kummer theory of the function field Σ, the Harrison group of isomorphism classes of $\mathcal{C}_p$-Galois ring extensions of Σ, and covers of the algebraic curve X as we have defined above.

Theorem 1.19. *Let $X/\mathbb{k}$ be a projective, irreducible, non-singular curve over an algebraically closed field $\mathbb{k}$ with $\mathrm{char}(\mathbb{k}) \neq p$ and Σ its function field. There are canonical identifications*

$$\left(\Sigma^*/\Sigma^{*p}\right)\big/_{(\mathbb{Z}/(p))^*} \simeq \mathbb{H}(\Sigma, \mathcal{C}_p)\big/_{\mathrm{Aut}(\mathcal{C}_p)} \simeq \mathrm{Cov}(X, \mathcal{C}_p), \tag{1.13}$$

where $(\mathbb{Z}/(p))^*$ acts on Σ^*/Σ^{*p} as in (1.7).

Proof. Recall that throughout we fix a character $\chi : \mathcal{C}_p \to \mu_p(\mathbb{k}^*)$ that serves to determine the action used in defining the Harrison group.

The first correspondence in (1.13) is a direct consequence of (1.6) and equivariance with respect to the action of $\mathrm{Aut}(\mathcal{C}_p)$ (see Remark 1.9). Thus we focus on the second correspondence between extensions and covers.

Given an element of $\mathbb{H}(\Sigma, \mathcal{C}_p)$, *i.e.* a class of $\mathcal{C}_p$-ring extensions of Σ, we may choose a representative which is a $(\mathcal{C}_p, \chi)$-Kummer extension, namely, of the form $\Omega_f = \Sigma[T]/(T^p - f)$ as in (1.8), with action via $g(T) = \chi(g)T$. As we discussed above, Ω_f is a finite separable ring, isomorphic to a finite product of field extensions of Σ. Let Y be the disjoint union of the Zariski-Riemann varieties of these extensions, endowed with the $\mathcal{C}_p$-action induced by a given isomorphism. This determines a cover $\pi : Y \to X$ as in Definition 1.16.

Note that f^b, where $b \in (\mathbb{Z}/(p))^*$, gives an equivalent representative in $\mathbb{H}(\Sigma, \mathcal{C}_p)/\mathrm{Aut}(\mathcal{C}_p)$ and does not change Y, *i.e.* it yields the same cover.

Conversely, given a p-cyclic Galois cover $\pi : Y \to X$, the localization of the morphism of $\mathcal{O}_X$-algebras $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ at the generic point of X yields a p-cyclic Galois ring extension $\Sigma \to \Omega$ and thus a class in $\mathbb{H}(\Sigma, \mathcal{C}_p)/\mathrm{Aut}(\mathcal{C}_p)$. □

Remark 1.20. *A fundamental reason for specifying the conditions on Galois covers as in Definition 1.16 is that, under the correspondence* (1.13)*, the neutral element of* $\mathbb{H}(\Sigma, \mathcal{C}_p)$ *indeed maps to the trivial cover in that sense (see also Example 1.17).*

Given a cover $\pi : Y \to X$, the ramification locus $\mathrm{Ram}(\pi)$ is the image of the support of the sheaf of relative differentials $\omega_{Y/X}$. Equivalently, it consists of the points $x \in X$ such that $\#\pi^{-1}(x) < \deg(\pi)$. For a finite subset of closed points $\mathfrak{R} \subseteq X$, it is natural to consider

$$\mathrm{Cov}_{\mathfrak{R}}(X, \mathcal{C}_p) := \{\pi \in \mathrm{Cov}(X, \mathcal{C}_p) : \mathrm{Ram}(\pi) \subseteq \mathfrak{R}\}, \tag{1.14}$$

so that

$$\mathrm{Cov}(X, \mathcal{C}_p) = \bigcup_{\mathfrak{R}} \mathrm{Cov}_{\mathfrak{R}}(X, \mathcal{C}_p). \tag{1.15}$$

We will study the relation between (1.15) and (1.13) in §3.2.

Remark 1.21. *As in the classical topological theory of Riemann surfaces, in the geometric case an element of* $\mathrm{Cov}_{\mathfrak{R}}(X, \mathcal{C}_p)$ *corresponds to a member* $\pi \in \mathrm{Cov}(X \setminus \mathfrak{R}, \mathcal{C}_p)$ *which is unramified, i.e.* $\mathrm{Ram}(\pi) = \varnothing$ *(see [20, Theorem 4.6.4]).*

2 Relating the Kummer theories of $\mathbb{A}_X$ and Σ

In [16] we extensively studied the structure of the Harrison group $\mathbb{H}(\mathbb{A}_X, \mathcal{C}_p)$, classifying the p-cyclic Galois extensions of the ring of adeles of the curve X. Now, by means of the Harrison group of its function field Σ, namely $\mathbb{H}(\Sigma, \mathcal{C}_p)$, we can now relate this to the classification of p-cyclic covers of X.

By Proposition 1.15, tensoring $\mathbb{A}_X \otimes_\Sigma -$ yields a canonical map

$$\mathbb{H}(\Sigma, \mathcal{C}_p) \to \mathbb{H}(\mathbb{A}_X, \mathcal{C}_p) \tag{2.1}$$

which is equivariant under the action of the automorphism group $\mathrm{Aut}(\mathcal{C}_p)$. Our main goal in this section is to study (2.1), since it encapsulates the relation between the classical Galois theory of field extensions of Σ with that of its adele ring $\mathbb{A}_X$.

2.1 The fundamental exact sequence

The *valuation vector* of an idele $u \in \mathbb{I}_X$ is defined as

$$\upsilon(u) := (\upsilon_x(u_x) \bmod p)_x \in \bigoplus_{x \in X} \mathbb{Z}/(p). \tag{2.2}$$

Clearly the map υ is a group homomorphism and, since $\Bbbk$ is assumed to be algebraically closed, we have

$$\mathbb{I}_X^p = \ker \upsilon : \mathbb{I}_X \to \bigoplus_x \mathbb{Z}/(p) \tag{2.3}$$

(see [16, (3.29)]). We summarize several important facts in the following.

Proposition 2.1. *There is commutative diagram of groups*

$$\begin{array}{ccccc} \Sigma^*/\Sigma^{*p} & \xrightarrow{f \mapsto \mathbb{f}} & \mathbb{I}_X/\mathbb{I}_X^p & \xrightarrow[\sim]{\upsilon} & \bigoplus_{x\in X} \mathbb{Z}/(p) \\ {\scriptstyle i_{(\Sigma,\chi)}}\downarrow\wr & & \wr\downarrow{\scriptstyle i_{(\mathbb{A}_X,\chi)}} & & \\ \mathbb{H}(\Sigma, \mathcal{C}_p) & \xrightarrow{\mathbb{A}_X\otimes_\Sigma -} & \mathbb{H}(\mathbb{A}_X, \mathcal{C}_p) & & \end{array} \tag{2.4}$$

where the bottom arrow is the map (2.1) *which centers our attention, and the vertical arrows are the first terms in the Kummer sequences* (1.5) *of* Σ *and* $\mathbb{A}_X$*, respectively. In addition,* (2.4) *is equivariant with respect to the action of* $\mathrm{Aut}(\mathcal{C}_p) \simeq (\mathbb{Z}/(p))^*$ *as in* (1.7) *for* $R = \Sigma$ *or* $R = \mathbb{A}_X$*, and the action on valuation vectors given by multiplication, i.e.*

$$b \in (\mathbb{Z}/(p))^* \longmapsto ((\upsilon_x)_x) \mapsto ((b\upsilon_x)_x) \tag{2.5}$$

Definition 2.2. *Given an extension* $(\mathbb{B}, \mathcal{C}_p)$ *of* $\mathbb{A}_X$*, the* valuation vector $\upsilon(\mathbb{B}, G, \chi)$ *of the triple* $(\mathbb{B}, G, \chi)$ *is the image of the* $\mathcal{C}_p$*-isomorphism class of* $(\mathbb{B}, \mathcal{C}_p)$ *under* $\upsilon \circ i^{-1}_{(\mathbb{A}_X,\chi)}$ *as in* (2.4).

The valuation vector of an extension is an invariant which is explicitly given by

$$\upsilon(\mathbb{B}, G, \chi) = \upsilon(\alpha^p) = (\upsilon_x(\alpha^p) \bmod p)_x \in \bigoplus_{x\in X} \mathbb{Z}/(p), \tag{2.6}$$

where α is a $(\mathcal{C}_p, \chi)$-primitive element (Definition 1.6 and Proposition 1.7). The existence of the latter is guaranteed by [16, Theorem 3.22], where it is also shown that the definition does not depend on the choice of α.

Note that for a $(\mathcal{C}_p, \chi)$-extension $\mathbb{A}_X\{\mathbb{t}^{1/p}\} = \mathbb{A}_X[T]/(T^p - \mathbb{t})$ where $\mathbb{t} \in \mathbb{I}_X$, the class of T is a $(\mathcal{C}_p, \chi)$-primitive element, and thus

$$\upsilon(\mathbb{A}_X\{\mathbb{t}^{1/p}\}, \mathcal{C}_p, \chi) = \upsilon(\mathbb{t}). \tag{2.7}$$

In order to proceed with our characterization of $\mathbb{H}_{\mathrm{ratl}}(\mathbb{A}_X, \mathcal{C}_p)$, we need an auxiliary result, which is interesting in itself, and points out the role played by the geometry of the curve.

Theorem 2.3 (The fundamental exact sequence for $\mathbb{H}(\Sigma, \mathcal{C}_p)$ and $\mathbb{H}(\mathbb{A}_X, \mathcal{C}_p)$)**.** *Let $X/\Bbbk$ be a curve satisfying our initial hypotheses, namely, a projective, irreducible, non-singular curve over an algebraically closed field $\Bbbk$ with $\mathrm{char}(\Bbbk) \neq p$. Fix a nontrivial character $\chi : \mathcal{C}_p \to \mu_p \subseteq \Bbbk^*$ and consider the Harrison groups $\mathbb{H}(\Sigma, \mathcal{C}_p)$ and $\mathbb{H}(\mathbb{A}_X, \mathcal{C}_p)$ of $\mathcal{C}_p$-isomorphism classes of p-cyclic Galois ring extensions of the function field Σ and the adele ring $\mathbb{A}_X$, respectively, where in both cases the $\mathcal{C}_p$-action is via χ. Then there is a canonical exact sequence of groups*

$$0 \longrightarrow \mathbb{I}_X^p \cap \Sigma^* / \Sigma^{*p} \longrightarrow \mathbb{H}(\Sigma, \mathcal{C}_p) \longrightarrow \mathbb{H}(\mathbb{A}_X, \mathcal{C}_p) \longrightarrow \mathbb{Z}/(p) \longrightarrow 0. \tag{2.8}$$

Proof. Our assumptions imply that the Picard scheme $\mathrm{Pic}^0(X)$ exists and is a projective $\Bbbk$-scheme isomorphic to the Jacobian variety $\mathrm{Jac}(X)$.

Consider the commutative diagram (of groups)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma^*/\Bbbk^* & \xrightarrow{d} & \mathrm{Div}(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \{\cdot\}^p} & & \downarrow{\scriptstyle \cdot p} & & \downarrow{\scriptstyle \cdot p} & & \\ 0 & \longrightarrow & \Sigma^*/\Bbbk^* & \longrightarrow & \mathrm{Div}(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \end{array}$$

where $\mathrm{Div}(X) = \bigoplus_{x \in X} \mathbb{Z}$ denotes the group of divisors on X, the map d sends a function $f \in \Sigma^*$ to its divisor, *i.e.* $\sum_x \upsilon_x(f)x$, and the vertical maps are raising to the pth power in the first arrow and multiplication by p in the other two. The Snake Lemma yields the exact sequence

$$0 \longrightarrow \mathrm{Pic}(X)[p] \longrightarrow \Sigma^*/\Sigma^{*p} \xrightarrow{\bar{d}} \mathrm{Div}(X)/p(\mathrm{Div}(X)) \longrightarrow \mathrm{Pic}(X)/p(\mathrm{Pic}(X)) \longrightarrow 0. \tag{2.9}$$

where $\mathrm{Pic}(X)[p]$ denotes the p-torsion subgroup of the Picard variety of X, which is easily seen to coincide with $\mathrm{Jac}(X)[p]$. The map $\bar{d}$ sends a function $f \in \Sigma^*$ to its divisor modulo p, *i.e.*, $\sum_x \upsilon_x(f)x \bmod p$. By (2.3), we have

$$\mathrm{Div}(X)/p(\mathrm{Div}(X)) \simeq \mathbb{I}_X/\mathbb{I}_X^p \simeq \bigoplus_{x \in X} \mathbb{Z}/(p), \tag{2.10}$$

and therefore

$$\mathrm{Jac}(X)[p] = \ker(\bar{d}) = \left\{ \bar{f} \in \Sigma^*/\Sigma^{*p} : \upsilon_x(f) = 0 \bmod p, \quad \forall x \in X \right\} = \mathbb{I}_X^p \cap \Sigma^*/\Sigma^{*p}. \tag{2.11}$$

Finally, consider the commutative diagram corresponding to the degree map and multiplication by p,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Jac}(X) & \xrightarrow{i} & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\cdot p} & & \downarrow{\cdot p} & & \downarrow{\cdot p} & & \\ 0 & \longrightarrow & \mathrm{Jac}(X) & \xrightarrow{i} & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

Now, the group $\mathrm{Jac}(X)(\Bbbk)$ of points of $\mathrm{Jac}(X)$ with values in $\Bbbk$ is divisible [22, Proposition 03RP], hence the first arrow is surjective and hence we obtain an isomorphism $\mathrm{Pic}(X)/p(\mathrm{Pic}(X)) \simeq \mathbb{Z}/(p)$. Thus using (2.4) we obtain (2.8). □

Corollary 2.4. *Given a positive integer n and a divisor $D \in \mathrm{Div}(X)$ with degree equal to a multiple of n, there always exists a function $f \in \Sigma^*$ such that $D \equiv (f) \bmod n$.*

Proof. As we have seen in the proof of the theorem, $\mathrm{Jac}(X)(\Bbbk)$ is a divisible group. Thus, given a divisor $D \in \mathrm{Div}(X)$ of degree $d \in n\mathbb{Z}$ and any point x_0, the composition

$$\mathrm{Jac}(X)(\Bbbk) \xrightarrow{[n]} \mathrm{Jac}(X)(\Bbbk) \xrightarrow{+D-dx_0} \mathrm{Jac}(X)(\Bbbk)$$

is surjective. Hence there is some $D' \in \mathrm{Div}(X)^0$ such that $D - dx_0 + nD' \sim 0$, where $\sim$ denotes linear equivalence; that is, $D - dx_0 + nD' = (f)$ for some function f as desired. □

2.2 The geometric adelic equivalence problem

In this section we deal with the problem of characterizing when two p-cyclic Galois extensions of Σ have the same image under the map (2.1).

This problem is analogous to the so-called equivalence problem for global fields. The latter has a long history, dating back to the 19th century with Kronecker and Hurwitz. It deals in general with the question of when an invariant associated to a global field might classify it up to isomorphism. For example, one may consider its Dedekind or Weil zeta function, the Galois group, or its adele ring, leading to various related notions of equivalence and counterexamples where non-isomorphic fields turn out to have the same invariant. The first example, concerning the Dedekind zeta function, dates back to Gassmann. Komatsu and Perlis considered the Galois group and the adele ring. The problem constitutes an active area of research to this day, see, for example, [19] for a survey of these classical results and the current state of the art.

Definition 2.5. *Denote by $\mathbb{H}_{\mathrm{ratl}}(\mathbb{A}_X, \mathcal{C}_p)$ the subgroup of $\mathbb{H}(\mathbb{A}_X, \mathcal{C}_p)$ which is the image of $\mathbb{H}(\Sigma, \mathcal{C}_p)$ under* (2.1)*.*

We may describe $\mathbb{H}_{\mathrm{ratl}}(\mathbb{A}_X, \mathcal{C}_p)$ as follows. Given a rational function $f \in \Sigma^*$, consider

$$\mathbb{A}_X\{\mathbb{f}^{1/p}\} = \mathbb{A}_X[T]/(T^p - \mathbb{f}), \tag{2.12}$$

where $\mathbb{f} = (f_x)_x$ is the idele such that $f_x \in K_x$ is the germ of f at x and $\mathcal{C}_p$ acts on the class of T via χ. Note that the class of T is a $(\mathcal{C}_p, \chi)$-primitive element. By Theorem 1.18, $\mathbb{A}_X\{\mathbb{f}^{1/p}\} \simeq \mathbb{A}_Y$. This motivates the following definition.

Definition 2.6 (rational $(\mathcal{C}_p, \chi)$-Kummer extension of $\mathbb{A}_X$)**.** *A p-cyclic extension of* $\mathbb{A}_X$ *which is* $\mathcal{C}_p$*-isomorphic to one of the form* (2.12) *will be called a* rational $(\mathcal{C}_p, \chi)$*-Kummer extension of* $\mathbb{A}_X$.

Thus $\mathbb{H}_{\text{ratl}}(\mathbb{A}_X, \mathcal{C}_p)$ consists of the $\mathcal{C}_p$-isomorphism classes of *rational* $(\mathcal{C}_p, \chi)$-extensions of $\mathbb{A}_X$. This subgroup can be characterized as a kernel and a cokernel as follows.

Corollary 2.7. *The exact sequence* (2.8) *splits into the following two short exact sequences:*

$$0 \to \mathbb{H}_{\text{ratl}}(\mathbb{A}_X, \mathcal{C}_p) \to \mathbb{H}(\mathbb{A}_X, \mathcal{C}_p) \simeq \bigoplus_{x \in X} \mathbb{Z}/(p) \to \mathbb{Z}/(p) \to 0 \tag{2.13}$$

$$0 \to \mathbb{I}_X^p \cap \Sigma^* / \Sigma^{*p} \to \mathbb{H}(\Sigma, \mathcal{C}_p) \simeq \Sigma^* / \Sigma^{*p} \to \mathbb{H}_{\text{ratl}}(\mathbb{A}_X, \mathcal{C}_p) \to 0. \tag{2.14}$$

Remark 2.8. *As we mentioned above, the maps between Harrison groups are equivariant under the action of* $\text{Aut}(\mathcal{C}_p)$ *and hence induce actions on the cokernel in* (2.13) *and on the kernel in* (2.14)*, which is isomorphic to* $\text{Jac}(X)[p]$ *by* (2.11).

The sequence (2.13) determines when a given adelic extension comes from a corresponding extension of the function field. This may be regarded as an adelic analog of the Grunwald-Wang problem, in the form given in [11].

Corollary 2.9. *In terms of the valuation vector associated to a* $\mathcal{C}_p$*-extension* $(\mathbb{B}, \mathcal{C}_p)$ *of* $\mathbb{A}_X$ *(Definition 2.2), the previous result translates to the following set of equivalences:*

1) *The class of* $(\mathbb{B}, \mathcal{C}_p)$ *belongs to the subgroup* $\mathbb{H}_{\text{ratl}}(\mathbb{A}_X, \mathcal{C}_p)$.

2) *The valuation vector* $v(\mathbb{B}, \mathcal{C}_p, \chi) = (v_x) \in \bigoplus_{x \in X} \mathbb{Z}/(p)$ *satisfies*

$$\sum_x v_x \equiv 0 \bmod p. \tag{2.15}$$

3) $v(\mathbb{B}, \mathcal{C}_p, \chi) = v(f)$ *for some function* $f \in \Sigma^*$*. In this case,* $(\mathbb{B}, \mathcal{C}_p)$ *is* $\mathcal{C}_p$*-isomorphic to* $\mathbb{A}_X\{\mathbb{f}^{1/p}\} \simeq \mathbb{A}_X \otimes_\Sigma \Omega_f$ *as in* (1.8) *and* (2.12).

From (2.14) we see that it is possible to have non-isomorphic field extensions of Σ whose adele rings are (topologically)[1] isomorphic as $\mathbb{A}_X$-algebras. An example is given below. This can be considered the analog for an algebraically closed base field of the known equivalence results for global function fields, which were originally studied by Tate [21] and Turner [23].

[1]We do not discuss topologies on $\mathbb{A}_X$-algebras here; see [16] for details.

Corollary 2.10 (Geometric equivalence problem)**.** *There is an exact sequence*

$$0 \to \mathrm{Jac}(X)[p] \to \mathbb{H}(\Sigma, \mathcal{C}_p) \to \bigoplus_{x\in X}^{0} \mathbb{Z}/(p) \to 0, \tag{2.16}$$

where the notation indicates zero-sum modulo p tuples.

In particular, when $g > 0$, this shows that the geometric adele ring is not enough in general to classify Σ-extensions.

Proof. This follows from the exact sequence (2.14), recalling (2.11), and Corollary 2.9. □

We illustrate the negative answer to the geometric equivalence problem with the following explicit example.

Example 2.11. *Let $\Bbbk$ be an algebraically closed field of characteristic different from $2, 3$ and set $p = 2$. Let X be the normalization of the completion of the affine plane curve of equation*

$$v^2 = (u-1)(u+1)u \tag{2.17}$$

as a closed subscheme of the affine plane $\mathrm{Spec}\,\Bbbk[u, v]$. A routine computation shows that X is an irreducible, non singular, elliptic curve of equation $x_0x_2^2 = (x_1 - x_0)(x_1 + x_0)x_1$ in homogeneous coordinates of the projective plane $\mathrm{Proj}\,\Bbbk[x_0, x_1, x_2]$. Let $\oplus \in X$ denote the point $X \cap (x_0)_0$ with coordinates $(0, 1, 0)$. Using the Abel map with base point $\oplus$, the law group of $\mathrm{Jac}(X)$ given by addition of divisors (or tensor product of invertible sheaves) can be transported to X and, in this way, $\oplus$ turns out to be the neutral element of the composition law on X. The reader may recognize in this construction the addition law in plane cubic defined geometrically by union of points and intersection of lines. Since X has genus 1, $\mathrm{Jac}(X)[2]$ consists of $p^{2g} = 4$ points which corresponds via $\mathcal{A}$ to $\oplus$ and the three affine points of X whose tangent is parallel to the line $x_1 = 0$. That is,

$$\mathrm{Jac}(X)[2] = \{\mathcal{O}_X, \mathcal{O}_X\big((1,-1,0)-(0,1,0)\big), \mathcal{O}_X\big((1,0,0)-(0,1,0)\big), \mathcal{O}_X\big((1,1,0)-(0,1,0)\big)\}.$$

Now, observe that Σ, the function field of X, is the given by the localization $(\Bbbk[u, v]/(v^2 - (u-1)(u+1)u))_{(0)}$ so that any function in Σ can be obtained as a rational function on u, v. Let us consider the rational function $f := \dfrac{u-1}{u} \in \Sigma^*$. The divisor of f is given by

$$(f) = \left(\frac{u-1}{u}\right) = \left(\frac{x_1 - x_0}{x_1}\right) = (x_1 - x_0)_0 - (x_1)_0 = 2(1,1,0) - 2(1,0,0)$$

so that $v_x(f) = 0 \bmod 2$ for all $x \in X$. Note that $f \notin \Sigma^{*2}$, since if $f = h^2$ for some $h \in \Sigma^*$ then h would be a rational function with exactly one simple pole and one simple zero, which would imply *that the curve X has genus 0. Proceeding similarly with the function $g := \dfrac{u+1}{u-1}$, one concludes that*

$$\mathbb{I}_X^2 \cap \Sigma^* \big/ \Sigma^{*2} = \{1, f, g, fg\}.$$

Hence, Theorem 2.3 implies that we have the following four non-isomorphic 2-cyclic Galois extensions of Σ:

$$\Sigma[T] \big/ (T^2-1), \quad \Sigma[T] \big/ (T^2-f), \quad \Sigma[T] \big/ (T^2-g), \quad \Sigma[T] \big/ (T^2-fg).$$

The latter three are nontrivial and, when tensored by $\mathbb{A}_X$, all yield trivial 2-cyclic Galois ring extensions of $\mathbb{A}_X$. Thus we have an example of four distinct elements of $\mathbb{H}(\Sigma, \mathcal{C}_2)$ which map to the identity element in $\mathbb{H}(\mathbb{A}_X, \mathcal{C}_2)$, in particular, they become isomorphic after tensoring with $\mathbb{A}_X$.

3 The role of ramification

3.1 The fundamental cube

As we outlined in the introduction, we will now consider a finite nonempty subset $\mathfrak{R}$ of closed points of X, which will represent the ramification locus. By studying the Harrison group of the ring of the affine curve $X \setminus \mathfrak{R}$, we obtain a commutative cube (3.10) which refines (2.8).

In [16, §2] we introduced a notion of ramification for an adelic algebra $\mathbb{A}_X\{\mathbb{t}^{1/n}\} = \mathbb{A}_X[T]/(T^n - \mathbb{t})$ for an idele $\mathbb{t} \in \mathbb{I}_X$ and n coprime to $\mathrm{char}(\mathbb{k})$, which we now briefly describe. First, the *ramification locus* of $\mathbb{t}$ is

$$\mathrm{Ram}(\mathbb{t}) := \{x \in X : (n, v_x(t_x)) \neq n\} = \{x \in X : v_x(t_x) \not\equiv 0 \bmod n\}, \tag{3.1}$$

and the corresponding *ramification index* at x is

$$e_x := \frac{n}{(n, v_x(t_x))}.$$

The vector $\mathfrak{e} = (e_x)$ of integers is the *ramification profile* of $\mathbb{t}$. Observe that $\mathrm{Ram}(\mathbb{t}) = \{x \in X : e_x > 1\}$ is a finite set. By [16, Theorem 2.19], isomorphism classes of $\mathbb{A}_X$-algebras of the form $\mathbb{A}_X\{\mathbb{t}^{1/n}\}$ are classified by their ramification profile, namely, $\mathbb{A}_X\{{\mathbb{t}_1}^{1/n}\}$ and $\mathbb{A}_X\{{\mathbb{t}_2}^{1/n}\}$ are isomorphic if and only if $\mathfrak{e}_1 = \mathfrak{e}_2$.

Recalling the notation in §1.1, when $n = p$ is prime, it is straightforward to check (see [16, Lemma 2.13]) that $x \notin \mathrm{Ram}(\mathbb{t})$ if and only if the K_x-algebra $K_x\{{t_x}^{1/p}\}$ is isomorphic to p copies of K_x. Since the latter is the neutral element of $\mathbb{H}(K_x, \mathcal{C}_p)$, this is also equivalent to $\mathbb{A}_X\{\mathbb{t}^{1/p}\}$ lying in the kernel of the canonical map $\mathbb{H}(\mathbb{A}_X, \mathcal{C}_p) \to \mathbb{H}(K_x, \mathcal{C}_p)$.

Since every $\mathcal{C}_p$-extension of $\mathbb{A}_X$ is isomorphic to a $(\mathcal{C}_p,\chi)$-Kummer extension, we have the following stratification by ramification detailed in [16, §3.E, Corollary 3.75]:

$$\mathbb{H}(\mathbb{A}_X,\mathcal{C}_p)\big/\mathrm{Aut}(\mathcal{C}_p) \overset{1:1}{\rightsquigarrow} \coprod_{\mathfrak{R}\subseteq X} \left\{ \begin{array}{c} \text{Conjugacy classes of } p\text{-cyclic Galois extensions } (\mathbb{B},G) \\ \text{of } \mathbb{A}_X \text{ ramified at } \mathfrak{R} \end{array} \right\} \tag{3.2}$$

where $\mathfrak{R}$ ranges over finite subsets of closed points of X.

In the following we recall the notation from §1.1. For a finite subset $\mathfrak{R}\subseteq X$ (possibly empty)

$$\mathbb{A}_{X,\mathfrak{R}} \coloneqq \prod_{x\in\mathfrak{R}} K_x \times \prod_{x\in X\setminus\mathfrak{R}} A_x, \quad \mathbb{I}_{X,\mathfrak{R}} \coloneqq \mathbb{A}^*_{X,\mathfrak{R}}.$$

In addition, we will also denote

$$A_{\mathfrak{R}} \coloneqq H^0(X\setminus\mathfrak{R},\mathcal{O}_X). \tag{3.3}$$

Theorem 3.1. *Let $X/\Bbbk$ be a projective, irreducible, non-singular curve over an algebraically closed field $\Bbbk$ with $\mathrm{char}(\Bbbk)\neq p$, and $\varnothing\subsetneq\mathfrak{R}\subset X$ be a finite nonempty subset (of closed points). The Harrison group $\mathbb{H}(A_{\mathfrak{R}},\mathcal{C}_p)$ is characterized as*

$$\mathbb{H}(A_{\mathfrak{R}},\mathcal{C}_p)\simeq\{f\in\Sigma^*/(\Sigma^*)^p : \upsilon_x(f)\equiv 0 \bmod p,\ \forall x\notin\mathfrak{R}\}. \tag{3.4}$$

Proof. Consider the exact sequence of sheaves

$$0\to\mathcal{O}_X^*\to\Sigma^*\to\Sigma^*/\mathcal{O}_X^*\to 0, \tag{3.5}$$

and restrict it to the open subscheme $X\setminus\mathfrak{R}$. From its long exact sequence of cohomology one obtains

$$0\to\Sigma^*/A_{\mathfrak{R}}^*\to\mathrm{Div}(X\setminus\mathfrak{R})\to\mathrm{Pic}(X\setminus\mathfrak{R})\to 0.$$

Observe that, since we assume $\mathfrak{R}$ to be nonempty, $X\setminus\mathfrak{R}$ is affine, equal to $\mathrm{Spec}(A_{\mathfrak{R}})$. This yields the following analog to (2.9):

$$0\to\mathrm{Pic}(A_{\mathfrak{R}})[p]\to\Sigma^*\big/A_{\mathfrak{R}}^*(\Sigma^*)^p\xrightarrow{\Phi}\mathbb{I}_X\big/\mathbb{I}_{X,\mathfrak{R}}\mathbb{I}_X^p=\mathrm{Div}(A_{\mathfrak{R}})\big/p(\mathrm{Div}(A_{\mathfrak{R}}))\to 0 \tag{3.6}$$

On the other hand the Kummer sequences (1.5) yield

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{\mathfrak{R}}^*\big/(A_{\mathfrak{R}}^*)^p & \longrightarrow & \mathbb{H}(A_{\mathfrak{R}},\mathcal{C}_p) & \longrightarrow & \mathrm{Pic}(A_{\mathfrak{R}})[p] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\Psi & & \downarrow & & \\ 0 & \longrightarrow & \Sigma^*\big/(\Sigma^*)^p & \longrightarrow & \mathbb{H}(\Sigma,\mathcal{C}_p) & \longrightarrow & 0 & \longrightarrow & 0, \end{array} \tag{3.7}$$

where Ψ is induced by $\Sigma\otimes_{A_{\mathfrak{R}}}-$ and injectivity of the left vertical arrow is clear. The Snake Lemma yields

$$0 \to \ker \Psi \to \operatorname{Pic}(A_{\mathfrak{R}})[p] \xrightarrow{\delta} {\Sigma^*}/{A^*_{\mathfrak{R}}(\Sigma^*)^p} \to \operatorname{coker} \Psi \to 0,$$

where δ denotes the connecting morphism. Observe that δ coincides with the map on the l.h.s. of (3.6). From the injectivity of the latter, we conclude that $\ker \Psi = 0$ and

$$\operatorname{coker} \delta = \operatorname{coker} \Psi = \operatorname{Im} \Phi = {\mathbb{I}_X}/{\mathbb{I}_{X,\mathfrak{R}}\mathbb{I}_X^p}. \tag{3.8}$$

Hence, one has the commutative diagram

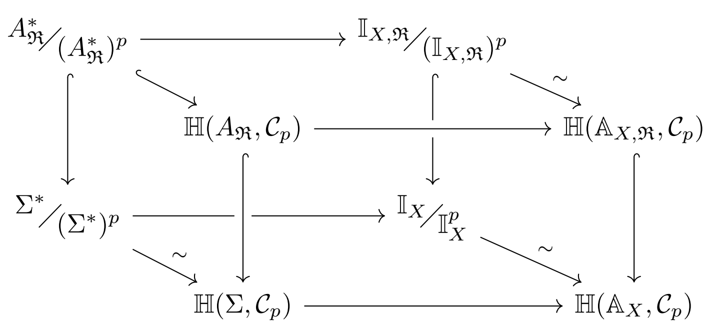
$$\begin{array}{ccccccccc} 0 & \longrightarrow & {A^*_{\mathfrak{R}}}/{(A^*_{\mathfrak{R}})^p} & \longrightarrow & \mathbb{H}(A_{\mathfrak{R}}, \mathcal{C}_p) & \longrightarrow & \operatorname{Pic}(A_{\mathfrak{R}})[p] & \longrightarrow & 0 \\ & & \simeq \downarrow & & \Psi \downarrow & & \delta \downarrow & & \\ 0 & \longrightarrow & {A^*_{\mathfrak{R}}}/{(A^*_{\mathfrak{R}})^p} & \longrightarrow & {\Sigma^*}/{(\Sigma^*)^p} & \longrightarrow & {\Sigma^*}/{A^*_{\mathfrak{R}}(\Sigma^*)^p} & \longrightarrow & 0 \end{array}$$

which after another application of the Snake Lemma, using (3.8) yields

$$0 \to \mathbb{H}(A_{\mathfrak{R}}, \mathcal{C}_p) \xrightarrow{\Psi} {\Sigma^*}/{(\Sigma^*)^p} \to {\mathbb{I}_X}/{\mathbb{I}_{X,\mathfrak{R}}\mathbb{I}_X^p} \to 0, \tag{3.9}$$

from which (3.4) follows immediately. □

Theorem 3.2. *Let $X/\Bbbk$ be a curve satisfying our initial hypotheses, namely, a projective, irreducible, non-singular curve over an algebraically closed field $\Bbbk$ with $\operatorname{char}(\Bbbk) \neq p$. For each finite nonempty subset $\mathfrak{R} \subset X$, we have the following commutative cube:*



(3.10)

where the notation is as above. Moreover, the front face of this cube is a cartesian square:

$$\mathbb{H}(A_{\mathfrak{R}}, \mathcal{C}_p) \simeq \mathbb{H}(\Sigma, \mathcal{C}_p) \underset{\mathbb{H}(\mathbb{A}_X, \mathcal{C}_p)}{\times} \mathbb{H}(\mathbb{A}_{X,\mathfrak{R}}, \mathcal{C}_p), \tag{3.11}$$

Furthermore, (3.10) *is equivariant with respect to the action of* $\operatorname{Aut}(\mathcal{C}_p)$ *on the various objects, as described in Proposition 1.15 and* (1.7).

Proof. The diagonal maps come from the Kummer sequence (1.5), while the horizontal and vertical maps arise from the functoriality of $\mathbb{H}(-,\mathcal{C}_p)$ in (1.9) and the following (cartesian) diagram:

$$\begin{array}{ccc} A_{\mathfrak{R}} & \longrightarrow & \mathbb{A}_{X,\mathfrak{R}} \\ \downarrow & & \downarrow \\ \Sigma & \longrightarrow & \mathbb{A}_X. \end{array} \tag{3.12}$$

The three diagonal isomorphisms follow from the triviality of the respective Picard groups of $\mathbb{A}_{X,\mathfrak{R}}$ and $\mathbb{A}_X$, shown in [16, Theorem 3.11] (and the field case for Σ). It is easy to check the injectivity of the vertical maps in back. That of the front right vertical map now follows from the one in back. Finally, the injectivity of $\mathbb{H}(A_{\mathfrak{R}},\mathcal{C}_p) \to \mathbb{H}(\Sigma,\mathcal{C}_p)$ is part of (3.9), which was the injectivity of the map denoted by Ψ in the proof of Theorem 3.1. Combining (3.9) with the front and right faces of (3.10) we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{H}(A_{\mathfrak{R}},\mathcal{C}_p) & \xrightarrow{\Psi} & \mathbb{H}(\Sigma,\mathcal{C}_p) & \longrightarrow & \mathbb{I}_X/\mathbb{I}_{X,\mathfrak{R}}\mathbb{I}_X^p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{H}(\mathbb{A}_{X,\mathfrak{R}},\mathcal{C}_p) & \longrightarrow & \mathbb{H}(\mathbb{A}_X,\mathcal{C}_p) & \longrightarrow & \mathbb{I}_X/\mathbb{I}_{X,\mathfrak{R}}\mathbb{I}_X^p & \longrightarrow & 0, \end{array}$$

from which (3.11) follows. □

Corollary 3.3. *There is an isomorphism from the direct limit over finite nonempty subsets $\mathfrak{R} \subset X$ of the top floor of* (3.10) *to the bottom floor.*

Proof. The statement is straightforward for the right face of the cube, and for the left face, it means that

$$\mathbb{H}(\Sigma,\mathcal{C}_p) \simeq \varinjlim_{\mathfrak{R}} \mathbb{H}(A_{\mathfrak{R}},\mathcal{C}_p), \quad \Sigma^*/(\Sigma^*)^p \simeq \varinjlim_{\mathfrak{R}} A^*_{\mathfrak{R}}/(A^*_{\mathfrak{R}})^p.$$

The first follows from (3.9), and the second is then a consequence of this and (3.7). □

3.2 Algebraic vs. geometric ramification

The results of the previous section now let us take ramification into account in the Harrison groups of $\mathcal{C}_p$-extensions of both Σ and $\mathbb{A}_X$, allowing us to filter by ramification. We end by establishing the concordance between the algebraic and geometric notions of ramification (Proposition 3.13), which underlies the various geometric applications explored in §4. Keeping in mind (2.4), Definition 2.2 and (2.7), and that any class of $\mathcal{C}_p$-extensions of $\mathbb{A}_X$ is represented by a $(\mathcal{C}_p,\chi)$-Kummer extension (see (1.6)), where the character χ has been fixed beforehand, the following notions are well-defined.

Definition 3.4 (Ramification for $\mathcal{C}_p$-extensions of $\mathbb{A}_X$)**.** *Given a class of $\mathcal{C}_p$-Galois ring extensions $\mathbb{B} \in \mathbb{H}(\mathbb{A}_X, \mathcal{C}_p)$, we define the ramification locus of $\mathbb{B}$ and the corresponding valuations and indices by*

$$\operatorname{Ram}(\mathbb{B}) \coloneqq \operatorname{Ram}(\mathbb{t}), \quad \upsilon_x(\mathbb{B}) \coloneqq \upsilon_x(\mathbb{t}), \quad e_x(\mathbb{B}) \coloneqq e_x, \tag{3.13}$$

where $\mathbb{t} \in \mathbb{I}_X$ is such that $\mathbb{B}$ is represented by the $(\mathcal{C}_p, \chi)$-Kummer extension $\mathbb{A}_X\{\mathbb{t}^{1/p}\}$.

Corollary 3.5 (Ramification filtration by subgroups)**.** *Ramification provides us with a natural filtration by subgroups:*

$$\mathbb{H}(\mathbb{A}_X, \mathcal{C}_p) \simeq \varinjlim_{\mathfrak{R}} \mathbb{H}(\mathbb{A}_{X,\mathfrak{R}}, \mathcal{C}_p) \simeq \varinjlim_{\mathfrak{R}} \{\mathbb{B} \in \mathbb{H}(\mathbb{A}_X, \mathcal{C}_p) : \operatorname{Ram}(\mathbb{B}) \subseteq \mathfrak{R}\}. \tag{3.14}$$

For a given finite subset $\mathfrak{R} \subseteq X$,

$$\#\{\mathbb{B} \in \mathbb{H}(\mathbb{A}_X, \mathcal{C}_p) : \operatorname{Ram}(\mathbb{B}) \subseteq \mathfrak{R}\} = p^{\#\mathfrak{R}}. \tag{3.15}$$

Proof. For a finite subset $\mathfrak{R} \subset X$, we conclude from the isomorphisms on the right face of (3.10), that the following diagram is commutative.

$$\begin{array}{ccccc} \mathbb{H}(\mathbb{A}_{X,\mathfrak{R}}, \mathcal{C}_p) & \xrightarrow{\sim} & \mathbb{I}_{X,\mathfrak{R}} / (\mathbb{I}_{X,\mathfrak{R}})^p & \xrightarrow{\sim} & \bigoplus_{x\in\mathfrak{R}} \mathbb{Z}/(p) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}(\mathbb{A}_X, \mathcal{C}_p) & \xrightarrow{\sim} & \mathbb{I}_X / (\mathbb{I}_X)^p & \xrightarrow{\sim} & \bigoplus_{x\in X} \mathbb{Z}/(p) \end{array} \tag{3.16}$$

where, by Corollary 3.3, the bottom row is the limit of the top row. Thus, by observing that the following three conditions are equivalent,

1) $\mathbb{B} \in \mathbb{H}(\mathbb{A}_{X,\mathfrak{R}}, \mathcal{C}_p)$,

2) $\operatorname{Ram}(\mathbb{B}) \subseteq \mathfrak{R}$,

3) $\upsilon_x(\mathbb{B}) \equiv 0 \bmod p$ for $x \notin \mathfrak{R}$,

we obtain (3.14) and (3.15). □

Corollary 3.6 (Stratification by ramification)**.** *There is a natural stratification of sets, indexed by ramification:*

$$\mathbb{H}(\mathbb{A}_X, \mathcal{C}_p) \simeq \bigsqcup_{\mathfrak{R}} \{\mathbb{B} \in \mathbb{H}(\mathbb{A}_X, \mathcal{C}_p) : \operatorname{Ram}(\mathbb{B}) = \mathfrak{R}\} \simeq \bigsqcup_{\mathfrak{R}} \bigoplus_{x\in\mathfrak{R}} \left(\mathbb{Z}/(p)\right)^*. \tag{3.17}$$

For a given finite subset $\mathfrak{R} \subseteq X$,

$$\#\{\mathbb{B} \in \mathbb{H}(\mathbb{A}_X, \mathcal{C}_p) : \operatorname{Ram}(\mathbb{B}) = \mathfrak{R}\} = (p-1)^{\#\mathfrak{R}}. \tag{3.18}$$

Proof. This follows analogously to the above by observing that the following three conditions are also equivalent:

1) $\mathbb{B} \in \mathbb{H}(\mathbb{A}_{X,\mathfrak{R}}, \mathcal{C}_p) \setminus \bigcup_{\mathfrak{R}' \subsetneq \mathfrak{R}} \mathbb{H}(\mathbb{A}_{X,\mathfrak{R}'}, \mathcal{C}_p)$,

2) $\operatorname{Ram}(\mathbb{B}) = \mathfrak{R}$,

3) $\upsilon_x(\mathbb{B}) \equiv 0 \bmod p$ if and only if $x \notin \mathfrak{R}$. □

In view of (3.16), for a finite (possibly empty) subset $\mathfrak{R} \subset X$, we may express the conditions $\operatorname{Ram}(\mathbb{B}) \subseteq \mathfrak{R}$ and $\operatorname{Ram}(\mathbb{B}) = \mathfrak{R}$ respectively by

$$(\upsilon_x(\mathbb{B}))_{x\in X} \in \bigoplus_{x\in\mathfrak{R}} \mathbb{Z}/(p), \quad (\upsilon_x(\mathbb{B}))_{x\in X} \in \bigoplus_{x\in\mathfrak{R}} (\mathbb{Z}/(p))^*,$$

if no confusion arises. We will also need to define a corresponding notion of ramification for $\mathcal{C}_p$-Galois ring extensions of the function field Σ.

Definition 3.7 (Ramification for $\mathcal{C}_p$-extensions of Σ)**.** *Given a class of $\mathcal{C}_p$-Galois ring extensions $\Omega \in \mathbb{H}(\Sigma, \mathcal{C}_p)$, we define the ramification locus* $\operatorname{Ram}(\Omega)$ *of Ω via the map* (2.1) *as* $\operatorname{Ram}(\mathbb{A}_X \otimes_\Sigma \Omega)$.

Remark 3.8. *For $f \in \Sigma^*$, consider, as in* (1.8)*, the $\mathcal{C}_p$-extension of Σ given by $\Omega_f = \Sigma[T]/(T^p - f)$ with $\mathcal{C}_p$-action by $g(T) = \chi(g)T$, where $\chi : \mathcal{C}_p \to \mu_p(\Bbbk^*)$ is a fixed character. Then, from the definition and* (2.7)*, we have*

$$\operatorname{Ram}(\Omega_f) = \operatorname{Ram}(\mathbb{A}_X \otimes_\Sigma \Omega_f) = \operatorname{Ram}(\mathbb{A}_X\{\mathbb{f}^{1/p}\}) = \operatorname{Ram}(\mathbb{f}) = \{x \in X : \upsilon_x(f_x) \not\equiv 0 \bmod p\}, \quad (3.19)$$

where $\mathbb{f} = (f_x)_x \in \mathbb{I}_X$ is the idele of germs as in (2.12).

Definition 3.9. *The subset of classes of $\mathcal{C}_p$-Galois extensions of Σ with ramification contained in $\mathfrak{R}$ will be denoted by*

$$\mathbb{H}_{\mathfrak{R}}(\Sigma, \mathcal{C}_p) := \{\Omega \in \mathbb{H}(\Sigma, \mathcal{C}_p) : \operatorname{Ram}(\Omega) \subseteq \mathfrak{R}\}.$$

With a slight abuse of notation (recalling that the front face of (3.10) is a cartesian square), note that $\operatorname{Ram}(\Omega)$ is the intersection of the finite subsets $\mathfrak{R} \subset X$ such that $\Omega \in \mathbb{H}(A_{\mathfrak{R}}, \mathcal{C}_p)$.

Proposition 3.10.

$$\mathbb{H}_{\mathfrak{R}}(\Sigma, \mathcal{C}_p) = \begin{cases} \mathbb{H}(A_{\mathfrak{R}}, \mathcal{C}_p), & \textit{if } \mathfrak{R} \textit{ is nonempty,} \\ \operatorname{Jac}(X)[p], & \textit{if } \mathfrak{R} = \varnothing. \end{cases} \quad (3.20)$$

Proof. If $\mathfrak{R}$ is nonempty, this follows from Corollary 3.3 and (3.11), while if $\mathfrak{R} = \varnothing$ then it follows from Corollary 2.10. □

Remark 3.11. *A posteriori, it follows that $\mathbb{H}_{\mathfrak{R}}(\Sigma, \mathcal{C}_p)$ is in fact a subgroup of the Harrison group $\mathbb{H}(\Sigma, \mathcal{C}_p)$, and there is a filtration by ramification*

$$\mathbb{H}(\Sigma, \mathcal{C}_p) \simeq \varinjlim_{\mathfrak{R}} \mathbb{H}_{\mathfrak{R}}(\Sigma, \mathcal{C}_p) \simeq \varinjlim_{\mathfrak{R}} \{\Omega \in \mathbb{H}(\Sigma, \mathcal{C}_p) : \mathrm{Ram}(\Omega) \subseteq \mathfrak{R}\}. \tag{3.21}$$

analogous to (3.14)*.*

Proposition 3.12. *For a given finite subset $\mathfrak{R} \subseteq X$, we have the following refinement of* (2.16) *filtered by ramification:*

$$0 \to \mathrm{Jac}(X)[p] \to \mathbb{H}_{\mathfrak{R}}(\Sigma, \mathcal{C}_p) \to \bigoplus_{x \in \mathfrak{R}}^{0} \mathbb{Z}/(p) \to 0. \tag{3.22}$$

Proof. This follows by combining (2.16), (3.20) and (3.21). □

We can now relate the geometric notion of ramification of a cover of an algebraic curve with the corresponding algebraic definition that we have given for extensions of its function field.

Proposition 3.13. *If a p-cyclic Σ-ring extension $(\Omega, \mathcal{C}_p)$ corresponds to the cover $\pi : Y \to X$ as in* (1.13)*, then*

$$\mathrm{Ram}(\Omega) = \mathrm{Ram}(\pi). \tag{3.23}$$

In particular, for a finite subset $\mathfrak{R} \subseteq X$,

$$\mathbb{H}_{\mathfrak{R}}(\Sigma, \mathcal{C}_p)/\mathrm{Aut}(\mathcal{C}_p) \xrightarrow{\sim} \mathrm{Cov}_{\mathfrak{R}}(X, \mathcal{C}_p) \tag{3.24}$$

with notation as in (1.14)*.*

Proof. With notation as in Example 1.10 and the proof of Theorem 1.19, it suffices to check that for $f \in \Sigma^*$, we have $\mathrm{Ram}(\Omega_f) = \mathrm{Ram}(\pi)$. Recall that, by (3.19), $\mathrm{Ram}(\Omega_f) = \{x : \upsilon_x(f_x) \not\equiv 0 \bmod p\}$. Noting that in general $(\omega_{Y/X})_y \simeq (\omega_{Y/\Bbbk})/(\mathfrak{m}_y^{e-1}\omega_{X/\Bbbk})$ where e is the ramification index at $x = \pi(y)$, there are two cases to consider:

- If $\upsilon_x(f_x) \not\equiv 0 \bmod p$, then $T^p - f_x$ is irreducible in $A_x[T]$, hence the class of T is a local parameter, provided that $\upsilon_x(f_x) > 0$. In this case, by uniqueness of extensions of valuations, we have $\upsilon_x(f_x) = p\upsilon_y(T)$ where y is the unique point in $\pi^{-1}(x)$. Thus $x \in \mathrm{Ram}(\pi)$. If $\upsilon_x(f_x) < 0$ then the argument is similar with T^{-1} as local parameter.
- If $\upsilon_x(f_x) \equiv 0 \bmod p$, and $\upsilon_x(f_x) \geq 0$, then $T^p - f_x$ splits completely as $\prod_{i=0}^{p-1} \left(T - \zeta^i f_x^{1/p}\right)$ where ζ is a primitive pth root of unity and $f_x^{1/p}$ is any choice of pth root in A_x. Now $\pi^{-1}(x) = \{y_1, \dots, y_p\}$ and for each i we have $\upsilon_x\left(f_x^{1/p}\right) = \upsilon_{y_i}(T)$. Hence $x \notin \mathrm{Ram}(\pi)$. The case where $\upsilon_x(f_x) \leq 0$ is similar.

Finally, (3.24) follows from the previous results and (3.11). □

Corollary 3.14. *Let p be a prime number. Let X be a smooth, irreducible, non-singular curve over an algebraically closed field $\Bbbk$ of characteristic different from p. Let $\mathfrak{R} \subset X$ be a finite subset of closed points containing at least two elements. Then there exist:*

1) *An irreducible p-cyclic Galois cover of X whose ramification locus is exactly $\mathfrak{R}$.*
2) *An irreducible polynomial $T^p - f \in \Sigma[T]$ which is ramified exactly at the points of $\mathfrak{R}$ and whose Galois group is $\mathcal{C}_p$.*

Proof. This follows from the previous result and Corollary 3.6. We need $\mathfrak{R}$ to have at least two elements so that we can have non-zero valuations which sum to 0. □

4 Geometric applications

The previous sections, concerned with $\mathcal{C}_p$-Galois ring extensions of the function field Σ and the corresponding adele ring $\mathbb{A}_X$, allow us to recover some classical geometric results regarding p-cyclic Galois covers of an algebraic curve. Since we employ purely algebraic techniques, this is done without any underlying appeal to the analytical and topological theory of Riemann surfaces, as is the case in the standard approach.

4.1 Classification

We can give a second proof of Proposition 3.13, relating algebraic and geometric ramification, by a direct argument of independent interest, since it shows the concordance of two points of view, namely Borevich's correspondence given in Theorem 1.11 (and sketched following its statement) and that of Cornalba in [5, Lemma 1]. We refine these constructions to take into account a specified ramification locus $\mathfrak{R}$, via (3.24), with notation as in §3.2.

Proposition 4.1 (Proposition 3.13 via Borevich and Cornalba)**.** *For a finite subset $\mathfrak{R} \subseteq X$,*

$$\mathbb{H}_{\mathfrak{R}}(\Sigma, \mathcal{C}_p) \big/ \mathrm{Aut}(\mathcal{C}_p) \xrightarrow{\sim} \mathrm{Cov}_{\mathfrak{R}}(X, \mathcal{C}_p). \tag{4.1}$$

Proof. Cornalba [5] showed that there is a bijection of sets

$$\mathrm{Cov}(X, \mathcal{C}_p) \overset{1:1}{\rightsquigarrow} \left\{ \begin{array}{c} (\mathcal{L}, D) \text{ where } D \in \mathrm{Div}(X) \text{ is an effective divisor and } \mathcal{L} \in \mathrm{Pic}(X) \\ \text{is a line bundle with } \mathcal{L}^p \simeq \mathcal{O}_X(D) \end{array} \right\} \Big/ \sim \tag{4.2}$$

where two pairs are defined to be equivalent, denoted by $(\mathcal{L}_1, D_1) \sim (\mathcal{L}_2, D_2)$ with $D_i = \sum_{x \in X} a_x^i x$, if there exists $1 \leq b < p$ such that:

- $ba_x^1 \equiv a_x^2 \bmod p$ for all $x \in X$,
- $\mathcal{L}_1^b \simeq \mathcal{L}_2\left(\sum c_x x\right)$ with $ba_x^1 = c_x p + a_x^2$ for each $x \in X$.

In this correspondence, Y is not required to be irreducible. Indeed, the trivial cover $Y = \coprod^p X$ corresponds to the equivalence class of the pair $(\mathcal{O}_X, 0)$. In addition, a given cover $\pi \in \mathrm{Cov}(X, \mathcal{C}_p)$ corresponding to a pair $(\mathcal{L}, D)$ has ramification locus $\mathrm{Ram}(\pi)$ equal to $\mathrm{supp}(D)$.

Fix a character $\chi : \mathcal{C}_p \to \mu_p(\Bbbk^*)$, that determines the actions used in defining the different Harrison groups involved, as well as a finite subset $\mathfrak{R} \subseteq X$.

Let Ω be a $\mathcal{C}_p$-ring extension of Σ with $\mathfrak{R} \supset \mathrm{Ram}(\Omega)$, *i.e.* whose class lies in $\mathbb{H}_{\mathfrak{R}}(\Sigma, \mathcal{C}_p)$. By (3.20) and (3.21), $\mathfrak{R}$ may be assumed nonempty and $\Omega \in \mathbb{H}(A_{\mathfrak{R}}, \mathcal{C}_p)$. The unramified case, where $\mathfrak{R} = \varnothing$, is easily seen to follow from the general result.

Under Borevich's correspondence, there is a pair $(\mathcal{L}, \varphi)$ associated to Ω, where $\mathcal{L} \in \mathrm{Pic}(A_{\mathfrak{R}})[p]$ and $\varphi : \mathcal{L}^{\otimes p} \xrightarrow{\sim} A_{\mathfrak{R}}$. The exact sequence

$$\bigoplus_{x_i \in \mathfrak{R}} \mathbb{Z} x_i \to \mathrm{Pic}(X) \to \mathrm{Pic}(X \setminus \mathfrak{R}) \to 0,$$

shows that there exists $\mathfrak{L} \in \mathrm{Pic}(X)$ such that $\mathfrak{L}|_{X \setminus \mathfrak{R}} \simeq \mathcal{L}$. Then, applying Lemma 30.10.6 [22, Lemma 0FD0] to $\varphi^{-1} : \mathcal{O}_{X \setminus \mathfrak{R}} \xrightarrow{\sim} \mathcal{L}^{\otimes(-p)}$ shows that there exists an effective divisor $\bar{D}$ with support contained in $\mathfrak{R}$ and a morphism of $\mathcal{O}_X$-modules

$$\bar{\varphi}^{-1} : \mathcal{O}_X(-\bar{D}) \to \mathfrak{L}^{\otimes(-p)}$$

whose restriction to $X \setminus \mathfrak{R}$ coincides with φ^{-1}. Choosing a suitable $\bar{D}$, it can be assumed that $\bar{\varphi}^{-1}$ is an isomorphism. Write $\bar{D} = pE + D$ for divisors E, D with support contained in $\mathfrak{R}$ and such that $D = \sum_{x_i \in \mathfrak{R}} a_i x_i$ with $0 \leq a_i \leq p - 1$. Then, $\bar{\varphi}$ yields

$$(\mathfrak{L}(-E))^{\otimes p} \simeq \mathcal{O}_X(D).$$

It is straightforward to check that the pair $(\mathfrak{L}(-E), D)$ is the data associated by Cornalba to the cover corresponding to the extension Ω and that its branch locus is $\{x_i \in \mathfrak{R} : a_i \neq 0\} = \mathrm{supp}(D) \subseteq \mathfrak{R}$. Thus we obtain an element of $\mathrm{Cov}_{\mathfrak{R}}(X, \mathcal{C}_p)$. If one chooses another character, the resulting pair is equivalent in the sense of (4.2).

Conversely, let $(\mathcal{L}, D)$ be associated to an element of $\mathrm{Cov}_{\mathfrak{R}}(X, \mathcal{C}_p)$ as in (4.2); thus $\mathfrak{R} \supseteq \mathrm{supp}(D)$. Fix an isomorphism $\varphi : \mathcal{L}^{\otimes p} \xrightarrow{\sim} \mathcal{O}_X(D)$, which is unique up to $\Bbbk^*$. Restricting to the open subset $X \setminus \mathfrak{R}$, let $\mathfrak{L} := H^0(X \setminus \mathfrak{R}, \mathcal{L}) \in \mathrm{Pic}(A_{\mathfrak{R}})$ and $\varphi_{\mathfrak{R}} := \varphi|_{X \setminus \mathfrak{R}} : \mathfrak{L}^{\otimes p} \xrightarrow{\sim} A_{\mathfrak{R}}$. Since $X \setminus \mathfrak{R}$ is affine with ring $A_{\mathfrak{R}}$, we obtain a Borevich pair $(\mathfrak{L}, \varphi_{\mathfrak{R}})$, which then corresponds (via the choice of character χ) to an element of $\mathbb{H}(A_{\mathfrak{R}}, \mathcal{C}_p) = \mathbb{H}_{\mathfrak{R}}(\Sigma, \mathcal{C}_p)$.

Note that a different choice of character yields a conjugate extension, not necessarily an isomorphic one. Thus we obtain an element of $\mathbb{H}_{\mathfrak{R}}(\Sigma, \mathcal{C}_p)/\operatorname{Aut}(\mathcal{C}_p)$. □

Theorem 1.19 gave a geometric description of the Harrison group $\mathbb{H}(\Sigma, \mathcal{C}_p)$ and now, with the previous discussion in mind, we obtain the following geometrical interpretation of its image $\mathbb{H}_{\text{ratl}}(\mathbb{A}_X, \mathcal{C}_p)$ inside $\mathbb{H}(\mathbb{A}_X, \mathcal{C}_p)$.

Theorem 4.2. *There is a canonical bijection of sets*

$$\operatorname{Cov}(X, \mathcal{C}_p) \big/ \operatorname{Jac}(X)[p] \overset{1:1}{\rightsquigarrow} \mathbb{H}_{\text{ratl}}(\mathbb{A}_X, \mathcal{C}_p) \big/ \operatorname{Aut}(\mathcal{C}_p). \tag{4.3}$$

Proof. First observe that $\operatorname{Jac}(X)[p]$ acts freely on the right hand side of (4.2) since $L \in \operatorname{Jac}(X)[p]$ acts by sending a pair $(\mathcal{L}, D)$ to $(D, \mathcal{L} \otimes L)$ and, thus, we conclude that

$$\operatorname{Cov}(X, \mathcal{C}_p) \big/ \operatorname{Jac}(X)[p] \overset{1:1}{\rightsquigarrow} \left\{ \, D \in \operatorname{Div}(X) \text{ such that } \deg(D) \equiv 0 \bmod p \, \right\} \big/ \sim \, ,$$

where $D_1 = \sum_x a_x^1 x$ and $D_2 = \sum_x a_x^2 x$ are equivalent if there exists $b \in (\mathbb{Z}/(p))^*$ such that $ba_x^1 \equiv a_x^2 \bmod p$ for all x.

On the other hand, recalling (2.10) and (2.13), it is clear that

$$\mathbb{H}_{\text{ratl}}(\mathbb{A}_X, \mathcal{C}_p) \simeq \left\{ \, D \in \operatorname{Div}(X) \text{ with coefficients in } \mathbb{Z}/(p) \text{ such that } \deg(D) \equiv 0 \bmod p \, \right\}.$$

Finally, from (2.11) one has an isomorphism $\operatorname{Jac}(X)[p] \simeq \mathbb{I}_X^p \cap \Sigma^*/\Sigma^{*p}$, and now the statement is deduced from the exact sequence (2.14), recalling Remark 2.8. □

4.2 Enumeration

Theorem 4.3. *Let $X/\Bbbk$ be a projective, irreducible, non-singular curve of genus g over an algebraically closed field $\Bbbk$ with $\operatorname{char}(\Bbbk) \neq p$. For a finite subset $\mathfrak{R} \subseteq X$ with r points,*

1) *The number of nontrivial unramified p-cyclic covers $\pi \in \operatorname{Cov}(X, \mathcal{C}_p)$ is*

$$\frac{p^{2g} - 1}{p - 1}. \tag{4.4}$$

2) *The number of nontrivial p-cyclic covers $\pi \in \operatorname{Cov}(X, \mathcal{C}_p)$ with ramification contained in $\mathfrak{R}$, assuming $\mathfrak{R}$ nonempty, is*

$$\frac{p^{2g+r-1} - 1}{p - 1}. \tag{4.5}$$

3) *The number of nontrivial p-cyclic covers $\pi \in \operatorname{Cov}(X, \mathcal{C}_p)$ with ramification equal to $\mathfrak{R}$, assuming $\mathfrak{R}$ nonempty, is*

$$p^{2g-1}((p-1)^{r-1} + (-1)^r). \tag{4.6}$$

Proof.

1) By (3.20), we have $\mathbb{H}_{\varnothing}(\Sigma, \mathcal{C}_p) = \mathrm{Jac}(X)[p]$, which has p^{2g} elements. Note that the isotropy subgroup of a class in $\mathbb{H}_{\varnothing}(\Sigma, \mathcal{C}_p)$ under the action of $\mathrm{Aut}(\mathcal{C}_p) \xrightarrow{\sim} (\mathbb{Z}/(p))^*$ is either trivial or all of $(\mathbb{Z}/(p))^*$. Observing that the former only holds for the trivial cover and recalling (1.13) yields (4.4).

2) Combining (2.16), Corollary 3.5, and (3.23), $\mathbb{H}_{\mathfrak{R}}(\Sigma, \mathcal{C}_p)$ has p^{2g+r-1} elements. Reasoning as before, we obtain (4.5).

3) Considering again (2.16) and the stratification by ramification (3.17), the number of elements in $\mathbb{H}(\Sigma, \mathcal{C}_p)$ with ramification exactly equal to $\mathfrak{R}$ is p^{2g} times the number of solutions to the congruence

$$\upsilon_1 + \cdots + \upsilon_r \equiv 0 \bmod p, \quad \upsilon_1, \ldots, \upsilon_r \in (\mathbb{Z}/(p))^*,$$

which is given by

$$\frac{1}{p}((p-1)^r + (-1)^r(p-1)). \tag{4.7}$$

To see this, for a subset $A \subseteq \mathfrak{R}$, let $h(A)$ be the number of tuples $\upsilon : \mathfrak{R} \to \mathbb{Z}/(p)$ with support equal to A, *i.e.* $\upsilon(x) \not\equiv 0 \bmod p$ if and only if $x \in A$, and satisfying $\sum_{x \in \mathfrak{R}} \upsilon(x) \equiv 0 \bmod p$. We want to compute $h(\mathfrak{R})$. Set $H(A) = \sum_{B \subseteq A} h(A)$, which is the number of zero-sum tuples with support contained in A. This is easily seen to equal $p^{\#A-1}$ when A is nonempty and 1 when $A = \varnothing$. Note that this only depends on the cardinality of A. By Möbius inversion, $h(A) = \sum_{B \subseteq A} (-1)^{\#A-\#B} H(B)$. Rearranging by cardinality we obtain $h(\mathfrak{R}) = (-1)^r + \sum_{k=1}^{r} \binom{r}{k} (-1)^{r-k} p^{k-1}$, which simplifies to the given formula.

In this enumeration we are not counting the class of the trivial extension, since it is unramified and we assume $\mathfrak{R}$ is nonempty. Noting that $\mathrm{Aut}(\mathcal{C}_p)$ acts freely on every class with prescribed nonempty ramification $\mathfrak{R}$ yields (4.6) after dividing by $p-1$. □

Alternatively, (4.6) may be deduced from (4.4) and (4.5) by an analogous process of Möbius inversion, which leads to

$$h(\mathfrak{R}) = (-1)^r \frac{p^{2g}-1}{p-1} + \sum_{k=1}^{r} \binom{r}{k} (-1)^{r-k} \frac{p^{2g+k-1}-1}{p-1},$$

and simplifying. In either case (2.16) is the starting point.

Note that (4.4) is [13, Theorem 3] and is also found on [9, p. 490] and the first formula of [10, Theorem 7], while (4.6) is on [9, p. 500] and is the second formula in [10, Theorem 7]. These papers use different methods, although they ultimately rely on the properties of the fundamental group of a Riemann surface, as is common in this kind of problem. Thus we think it is interesting to point out that our approach, avoiding the dependence on the classical topological or analytic structure, allows us to generalize several aspects of p-cyclic Galois covers, such as classification and enumeration, to any algebraically closed base field of any characteristic (different from p) via the Galois theory of the adele ring. We may also deal with the general abelian case using these tools.

The reader may compare our methods to the references above, for example, (4.7) is [12, Lemma 3] although we have arrived at this congruence via quite a different path.

Corollary 4.4. *There exists a p-cyclic Galois cover of X with exactly $r \geq 0$ ramification points, except in the following two cases: $g = 0$ and $r = 0$, or $g \geq 0$ and $r = 1$.*

Proof. This is immediate from the formulas in Theorem 4.3. □

Recall that the Hurwitz existence problem is the question of whether a certain set of numerical data associated to a curve X, foremost among these the Riemann-Hurwitz relation, is in fact realizable by a cover $Y \to X$. This problem has been extensively studied, beginning with Hurwitz himself. It is known that for genus $g > 0$ the answer is affirmative (major advances were made in [6] and many remaining cases dealt with later on), while the case $g = 0$ reduces to 3-point branched covers of the Riemann sphere and remains a difficult question. In particular, when $g > 0$ and $r = 1$, although as we see above, there are no p-cyclic Galois covers, other covers do exist as predicted by these general results. For $g = 0$ and $r = 2$, (4.6) says there is a unique p-cyclic cover. For $r = 3$ there are $p - 2$ such covers.

4.3 Rotation data

In this section we give an algebraic definition of rotation numbers (Definition 4.5), and using the Kummer pairing at ramification points, we prove that for $\Bbbk = \mathbb{C}$ it coincides with the classical one for the case of a compact connected Riemann surface. To achieve this, we begin by considering the following classical construction of a p-cyclic cover of the projective line, which is standard in the literature (see *e.g.* [2, §5] and especially [7, §1]).

Set $\Bbbk = \mathbb{C}$, $X = \mathbb{P}^1$, p a prime number, and $\mathfrak{R} := \{x_1, \ldots, x_r\}$ a finite set of distinct points of X. Let Y be the normalization of

$$y^p = f(x) := (x - x_1)^{v_1} \cdots (x - x_r)^{v_r}, \tag{4.8}$$

where $\pi : Y \to X$ maps (x, y) to x, and the exponents v_i satisfy $0 < v_i < p$ and $\sum_i v_i \equiv 0 \bmod p$ (the latter is equivalent to being unramified at ∞). Observe that the Riemann-Hurwitz formula implies that $r \geq 1$ in the above expression for $f(x)$.

Fix a nontrivial character $\chi : \mathcal{C}_p \to \mu_p \subseteq \mathbb{C}^*$ and let $\zeta = \chi(1) \in \mu_p$, which is a primitive pth root of unity. This defines an action of $\mathcal{C}_p$ on Y where 1 acts via the automorphism

$$\tau : (x, y) \mapsto (x, \zeta y). \tag{4.9}$$

Thus τ is a generator of $\mathrm{Gal}(Y/X)$, and allows us to identify $\mathcal{C}_p$ with $\mathrm{Gal}(Y/X)$.

If we assume that $\gcd(v_1, \ldots, v_r) = 1$, then Y is irreducible and $\mathcal{O}_X \to \pi_* \mathcal{O}_Y$ at the generic point is

$$\Sigma := \mathbb{C}(x) \hookrightarrow \Omega_f = \Sigma[y]/(y^p - f(x)),$$

as considered in Example 1.10. This is what we termed a $(\mathcal{C}_p, \chi)$-Kummer extension (Definition 1.3). Thus $\pi : Y \to X$ is a p-cyclic cover with ramification locus

$$\mathrm{Ram}(\pi) = \mathfrak{R} = \{x_1, \ldots, x_r\},$$

having ramification index at x_i equal to $p/(p, v_i) = p$. Thus π defines a class in $\mathrm{Cov}_{\mathfrak{R}}(X, \mathcal{C}_p)$ and the class of Ω_f is an element of $\mathbb{H}_{\mathfrak{R}}(\Sigma, \mathcal{C}_p)$, related via the correspondence in Proposition 3.13.

Let us now change our point of view to the geometric adele ring. Under the canonical map (2.1), the corresponding adelic algebra $\mathbb{A}_Y = \mathbb{A}_X\{\mathbb{f}^{1/p}\}$ is a rational $(\mathcal{C}_p, \chi)$-Kummer adelic extension of $\mathbb{A}_X$ (Definition 2.6), hence lying by definition in $\mathbb{H}_{\mathrm{ratl}}(\mathbb{A}_X, \mathcal{C}_p)$. The characteristic polynomial of y is equal to $C_y(T) = T^p - \mathbb{f}$, and thus the associated valuation vector of this extension (Definition 2.2) is

$$(v_x)_x := v(\mathbb{A}_Y, \mathcal{C}_p, \chi) = (v_x(\mathbb{f}))_x = \begin{cases} v_i \bmod p, & x = x_i \in \mathfrak{R}, \\ 0 \bmod p, & x \notin \mathfrak{R}, \end{cases}$$

where we take as χ-primitive element the class of T modulo $T^p - \mathbb{f}$. This invariant satisfies

$$\sum_x v_x \equiv 0 \bmod p$$

in accordance with Corollary 2.9. Recall that these invariants classify $(\mathcal{C}_p, \chi)$ adelic extensions and hence p-cyclic covers.

In the classical theory of Riemann surfaces, one considers the so-called *rotation numbers*. These are defined in terms of the automorphism τ given by the action (4.9), namely, choosing $\zeta = e^{2\pi i/p}$, in a neighborhood of a branch point $x = (x_i, 0)$ one sees that τ rotates a small disk around this point by an angle $2\pi\rho_x/p$; the integer $\rho_x \bmod p$ is the corresponding rotation number. A simple calculation shows they are the inverses modulo p of the valuations v_x:

$$\rho_x v_x \equiv 1 \bmod p. \tag{4.10}$$

The usual approach based on complex analytic and topological methods (for example, [7]), classifies the p-cyclic branched covers of the punctured Riemann sphere in terms of the rotation data $(\rho_x)_{x\in\mathfrak{R}}$. On the other hand, in §2.2, we have already classified p-covers in arbitrary characteristic prime to p in a purely algebraic manner, reflected explicitly in the valuation vector $(v_x)_x$. To complete the algebraic picture, we will show how both the valuations and the rotation data arise naturally from the local Kummer symbols at each ramified point $x \in \mathfrak{R}$. For this we need to recall some notions introduced in [16].

For $x \in \mathfrak{R}$, by Kummer theory, there is a unique (up to isomorphism) abelian extension E_x of K_x of exponent p, generated by adjoining the pth root of any non pth power in K_x. This is the algebraic analog of the classical Puiseux series expansion in the theory of Riemann surfaces, since, for example, $K_x \simeq \Bbbk(\!(z_x)\!)$ where z_x is a uniformizer at x, and the local valuation υ_x

$$\ker\left(K_x^* \xrightarrow{\upsilon_x} \mathbb{Z} \to \mathbb{Z}/(p)\right) = \{\lambda : \upsilon_x(\lambda) \equiv 0 \bmod p\} = K_x^{*p}, \tag{4.11}$$

induces an isomorphism $K_x^*/K_x^{*p} \simeq \mathbb{Z}/(p)$. Now, we have the Kummer perfect pairing for E_x,

$$\langle g, \lambda\rangle_x = \frac{g(\lambda^{1/p})}{\lambda^{1/p}} : \mathrm{Gal}(E_x/K_x) \times K_x^*/K_x^{*p} \to \mu_p, \tag{4.12}$$

where $\lambda^{1/p}$ is any pth root of λ in E_x.

Now, as we saw in §1.3, the isomorphism $\mathbb{A}_Y \simeq \mathbb{A}_X \otimes_\Sigma \Omega$ (1.11) is equivariant under the action of $\mathrm{Gal}(Y/X)$. Thus an automorphism τ of Y/X can be restricted to each fiber of the cover $\pi : Y \to X$. In particular, for a ramification point $x \in \mathfrak{R}$, τ induces an element $\tau_x \in \mathrm{Gal}(E_x/K_x)$ in each local Galois group.

The explicit computations with local Kummer symbols carried out in [16, §3.E] lead to the following algebraic definition of rotation data, valid over any algebraically closed base field $\Bbbk$ of characteristic prime to p.

Definition 4.5 (Algebraic rotation data)**.** *The algebraic rotation data for an automorphism τ of a p-cyclic Galois cover $\pi : Y \to X$ with ramification locus $\mathfrak{R}$ is defined by*

$$\left(\log_\zeta\langle \tau_x, z_x\rangle_x\right)_{x\in\mathfrak{R}} \in \prod_{x\in\mathfrak{R}} \mathbb{Z}/(p), \tag{4.13}$$

where $\log_\zeta$ is the discrete logarithm associated to a fixed primitive pth root of unity ζ.

Proposition 4.6. *The algebraic rotation data for a nontrivial automorphism τ of the superelliptic curve* (4.8) *coincide with the classical analytic rotation data, under the assumption that $\chi(\tau) = \zeta$, where we identify $\mathcal{C}_p$ with* $\mathrm{Gal}(Y/X)$ *as above.*

Proof. This follows from [16, Proposition 3.79], which together with our choices of χ, τ and ζ show that the numbers $\log_\zeta\langle \tau_x, z_x\rangle_x$ are indeed the inverses modulo p of the valuations $\upsilon_x = \upsilon_x(\mathbb{f})$ and thus by (4.10) are equal to the rotation numbers. □

Remark 4.7. *Note that* (4.13) *depends not only on the automorphism τ but also on the choice of primitive pth root of unity ζ, and that the relation $\chi(\tau) = \zeta$ is needed for the equivalence in Proposition 4.6. In general it would hold only modulo a constant multiple in* $(\mathbb{Z}/(p))^*$.

Thus, the equivalence class of $\mathfrak{R}$-tuples as in (4.13) *modulo the action of* $(\mathbb{Z}/(p))^*$ *by multiplication (see, for example, [16, Theorem 3.71]) is independent of these choices.*

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Applying the Riemann surfaces with extremal configurations of symmetries to the study of the real nerve of the moduli space of Riemann surfaces of odd genera

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ABSTRACT

In this paper we find the necessary and sufficient conditions for the geometrical dimension of the real nerve of the moduli space of Riemann surfaces of odd genus g to be maximal. Furthermore, we prove some properties of Riemann surfaces with extremal configuration of symmetries which lead to the conclusion that certain homology groups of the real nerve $\mathcal{N}_g$ of the moduli space of Riemann surfaces of odd genus g are nontrivial.

RESUMEN

En este artículo, encontramos las condiciones necesarias y suficientes para que la dimensión geométrica del nervio real del espacio de módulos de superficies de Riemann de género g impar sea máxima. Más aún, demostramos algunas propiedades de superficies de Riemann con configuraciones extremas de simetrías, que permiten concluir que ciertos grupos de homología del nervio real $\mathcal{N}_g$ del espacio de módulos de superficies de Riemann de género g impar son no triviales.

Keywords and Phrases: Riemann surface, symmetry of a Riemann surface, real form, automorphisms of Riemann surface, Fuchsian groups, Riemann uniformization theorem, separating symmetry

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1 Introduction

All Riemann surfaces in this paper are compact. A symmetry of a Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g \geq 2$, where Γ is a Fuchsian surface group, is just an antiholomorphic involution $\sigma \in G = \mathrm{Aut}^{\pm}(X)$. The set of points fixed by σ consists of no more than $g+1$ disjoint simple closed curves called *ovals*, see Harnack [15]. If the set $X \setminus \mathrm{Fix}(\sigma)$ is disconnected, then we call σ to be *separating* and we call it *non-separating* in the other case. In addition, we define the *topological type* of σ to be a symbol $\pm k$, where $k \geq 0$ denotes the number of ovals of σ, and the sign depends on the separability of σ: $+$ for separating, $-$ for a non-separating symmetry.

The study of Riemann surfaces with extremal configuration of symmetries has a long history and focuses on two threads:

(1) studying surfaces with the maximal possible number of nonconjugate symmetries;

(2) studying surfaces with the configuration of k nonconjugate symmetries admitting the maximal possible total number of ovals.

We shall call the surfaces with the first property *s-extremal* and the ones with the second property *o-extremal*. Now the maximal number of conjugacy classes of symmetries with fixed points was established in [2]:

Theorem 1.1. (Bujalance, Gromadzki, Izquierdo [2]) *Let X be a Riemann surface of genus $g \geq 2$ and let us write $g = 2^{r-1}u+1$ with u odd. Then the maximum number of nonconjugate symmetries with fixed points that X admits is 2^{r+1}. Furthermore, this bound is attained if and only if $u \geq 2^{r+1} - 3$.*

Now the case including the fixed-point free symmetries was studied in [1] and gives the same bound on the number of conjugacy classes of symmetries, while the only difference is that we have to require $u \geq r - 2$. What is important, the group structure for s-extremal surfaces was established in [14], where it was shown that the group generated by the symmetries must be of the form $G = \mathrm{D}_{2^s} \times \mathbb{Z}_2^r$ for a Riemann surface of genus $g = 2^{r-1}u + 1$, u odd. This information shall be crucial in our investigations.

The topic of surfaces with the maximal total number of ovals was also studied extensively. The results concerning low values of k were obtained by Natanzon in [18–20], where he showed that the bound for such number is given by $2g + 2^{k-1}$ and it is attained for $g \equiv 1 \mod 2^{k-2}$ for $k = 2, 3, 4$. Later it was shown by Singerman in [22], that for arbitrary k there exist infinitely many values of g, for which there exists a Riemann surface of genus g, admitting k non-conjugate symmetries with $2g - 2 + 2^{k-3}(9 - k)$ ovals in total. In his work, Singerman also conjectured that this is in fact the best possible bound. This was shown in [9] to be false for $k > 9$ by Gromadzki, who proved that for $k \geq 9$, the maximal possible number of ovals is $2g - 2 + 2^{r-3}(9 - k)$, where r denotes the smallest positive integer for which $k \leq 2^{r-1}$, and that this bound is attained for arbitrary $k \geq 9$ for infinitely many values of g. However, the Singerman's bound turned out to be true for $k = 5, 6, 7, 8$ in [11]. Similarly to the case of s-extremal surfaces, the structure of the automorphism group in the o-extremal case was established in [11] for $5 \leq k \leq 8$, in [14] for $k \geq 9$ and in [4] for lower values of k. The group structure is the same as for the s-extremal surfaces, with the group necessarily being abelian for all $k \neq 4, 5$. All these results are crucial for our studies.

The other aspect, that we need to underline is the topic of the real nerve $\mathcal{N}_g$ of the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g. The notion of real nerve in this setting was introduced in [12], where it was studied for even values of g, and then in [13] it was continued for odd values of g. However, by Theorem 1.1 the maximal number of possible symmetries is only 4 for even values of g. This makes the study for odd g much more complicated and in [13] a hypothesis was introduced, concerning the necessary and sufficient condition for the homological genus of $\mathcal{N}_g$ to be maximal. In this paper we prove this hypothesis to be wrong and we propose the new condition and prove it holds. As a byproduct of our studies we give some interesting properties of surfaces with extremal configurations of symmetries, the most important being the one that uses o-extremal surfaces to prove that certain homology groups of $\mathcal{N}_g$ are non-trivial for odd g.

2 Preliminaries

2.1 Non-euclidean crystallographic groups

We shall use the combinatorial approach, based on Riemann uniformization theorem, Fuchsian groups and non-euclidean crystallographic groups. Recall that a *NEC group* is just a discrete and cocompact subgroup of the group $\mathcal{G}$ of isometries of the hyperbolic plane $\mathcal{H}$, including those which reverse orientation, and if such a subgroup contains only orientation preserving isometries, then it is called a *Fuchsian* group. For every NEC group Λ we have the associated *signature*, which determines its algebraic structure. It has the form

$$(h; \pm; [m_1, \ldots, m_r]; \{C_1, \ldots, C_k, (-)^l\}). \tag{2.1}$$

The numbers $m_i \geq 2$ are called the *proper periods*, the non-empty brackets $C_i = (n_{i1}, \ldots, n_{is_i})$, $i = 1, \ldots, k$, or the empty ones $(-)$ are the *period cycles*, the numbers $n_{ij} \geq 2$ are the *link periods* and $h \geq 0$ is said to be the *orbit genus* of Λ. The orbit space $\mathcal{H}/\Lambda$ is a surface of topological genus h, having k boundary components, and being orientable or not according to the sign being $+$ or $-$.

The group with the signature (2.1) has the presentation given by the following generators and relations, where $s_i = 0$ if $i > k$:

$$\begin{cases} \text{generators:} \\ \text{(a)} \quad x_i,\ i = 1, \ldots, r, \\ \text{(b)} \quad c_{ij},\ i = 1, \ldots, k+l,\ j = 0, \ldots, s_i, \\ \text{(c)} \quad e_i,\ i = 1, \ldots, k+l, \\ \text{(d)} \quad a_i,\ b_i,\ i = 1, \ldots, h \text{ if the sign is } +, \\ \qquad\ d_i,\ i = 1, \ldots, h, \text{ if the sign is } -, \\ \text{relations:} \\ \text{(A)} \quad x_i^{m_i} = 1,\ i = 1, \ldots, r, \\ \text{(B)} \quad c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1,\ i = 1, \ldots, k+l,\ j = 1, \ldots, s_i, \\ \text{(C)} \quad c_{is_i} = e_i^{-1} c_{i0} e_i,\ i = 1, \ldots, k+l, \\ \text{(D)} \quad x_1 \cdots x_r e_1 \cdots e_{k+l} [a_1, b_1] \cdots [a_h, b_h] = 1, \text{ if the sign is } +, \\ \qquad\ x_1 \cdots x_r e_1 \cdots e_{k+l} d_1^2 \cdots d_h^2 = 1, \text{ if the sign is } -. \end{cases} \tag{2.2}$$

The generators $x_1, \ldots, x_r$ are called *canonical elliptic generators*, $e_1, \ldots, e_{k+l}$ are called the *canonical connecting generators* and c_{ij} are the *canonical reflections* of Λ.

A Fuchsian group can be regarded as a NEC group with the signature

$$(h; +; [m_1, \ldots, m_r]; \{-\}), \tag{2.3}$$

which is usually shortened to $(h; m_1, \ldots, m_r)$; a Fuchsian group without proper periods is called a *Fuchsian surface group.* An epimorphism $\theta : \Lambda \to G$, where Λ is a NEC group and G is a finite group, is said to be *smooth* if $\ker \theta$ is a surface group.

Any set of generators of a NEC group satisfying the above relations is called *a canonical set of generators*, and reflections c_{ij-1}, c_{ij} are said to be *consecutive.* Every NEC group has an associated fundamental region, whose hyperbolic area $\mu(\Lambda)$, for a NEC group Λ with signature (2.1), is given by

$$2\pi \left(\varepsilon h + k + l - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k+l} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right) \right) \tag{2.4}$$

where $\varepsilon = 2$ if the sign is $+$ and $\varepsilon = 1$ otherwise.

Now, if Γ is a finite index subgroup in a NEC group Λ, then it is a NEC group itself and the Hurwitz-Riemann formula applied to the covering $\mathcal{H}/\Gamma \to \mathcal{H}/\Lambda$ says:

$$[\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}.$$

Finally it is known that an abstract group with the presentation given by the generators and the relations in (2.2) can be realized as a NEC group Λ with the signature (2.1) if and only if (2.4) is positive.

2.2 Topological type of a symmetry

To retrieve the topological type of a symmetry, one needs to find its number of ovals and determine its separability type. The following two results are essential in solving this problem.

Theorem 2.1 (Gromadzki [8])**.** *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface whose group of automorphisms is $G = \Lambda/\Gamma$ for some NEC group Λ containing Γ as a normal subgroup and let $\theta : \Lambda \to G$ be the canonical epimorphism. Then the number of ovals of a symmetry σ of X, having fixed points, equals*

$$\sum [C(G, \theta(c_i)) \,:\, \theta(C(\Lambda, c_i))],$$

where C stands for the centralizer and the sum is taken over a set of representatives of all conjugacy classes of canonical reflections whose images under θ are conjugate to σ.

For a symmetry σ we shall denote by $\|\sigma\|$ the number of its ovals. In addition, we denote $\|X\|$ the number of ovals of all non-conjugate symmetries on the surface X.

The centralizers that appear in Theorem 2.1 have been described by Singerman in [21] and Gromadzki in [10] (see also [3] for more explicit explanation) and we shall use the following result, where $*$ stands for free product.

Theorem 2.2. *Let $c_0, c_1, \ldots, c_s, e$ be the system of canonical reflections corresponding to a period cycle $(n_1, \ldots, n_s)$ of a NEC group Λ with signature (2.1). If all n_i are even, then the centralizer $C(\Lambda, c_i)$ equals:*

$$\begin{array}{ll} \langle c_i\rangle \times \big(\langle (c_{i-1}c_i)^{n_i/2}\rangle * \langle (c_i c_{i+1})^{n_{i+1}/2}\rangle\big) = \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2) & \text{for } i \neq 0, \\ \langle c_0\rangle \times \big(\langle (c_0 c_1)^{n_1/2}\rangle * \langle e^{-1}(c_{s-1}c_s)^{n_s/2} e\rangle\big) = \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2) & \text{for } i = 0, \\ \langle c_0\rangle \times \langle e\rangle = \mathbb{Z}_2 \times \mathbb{Z} & \text{for } s = 0. \end{array}$$

Now to determine the separability type, one can use the following technique described by Hoare and Singerman in [16]. Let us recall, that if Φ is a set of generators for Λ and $\tilde{\Lambda}$ is a subgroup of Λ, then a *right Schreier transversal S* is a set of words in Φ such that every initial segment of a word in S is also in S and the mapping $a \to \tilde{\Lambda}a$ is a $1-1$ correspondence from S to the cosets of $\tilde{\Lambda}$ in Λ. Now for each $a \in S$ and $\alpha \in \Phi$ there exists a unique $b \in S$ such that $\tilde{\Lambda}b = \tilde{\Lambda}a\alpha$. With these notations, the *Schreier generators* of $\tilde{\Lambda}$ are all the elements of the form $a\alpha b^{-1}$.

If Λ is a group with generators Φ and $\tilde{\Lambda}$ is its subgroup, then the *Schreier coset graph* $\mathcal{K}$ is the graph whose vertices are the cosets of $\tilde{\Lambda}$ in Λ and labelled directed edges at every vertex for each $a \in \Phi$ such that $a : \tilde{\Lambda}\alpha \to \tilde{\Lambda}\alpha a$. Now if a is a reflection and $\tilde{\Lambda}\alpha a = \tilde{\Lambda}a$, then the corresponding directed edge $a : \tilde{\Lambda}\alpha \to \tilde{\Lambda}\alpha a$ is called a *reflection loop*. Let us delete all the reflection loops from $\mathcal{K}$. In such a way we obtain the Schreier graph $\bar{\mathcal{K}}$. Observe, that each path in $\bar{\mathcal{K}}$ corresponds to a word in Φ, and so also to an element of Λ. We shall call a path *positive* if it corresponds to an orientation preserving element of Λ, otherwise we shall call a path *negative*.

We are ready to state the main result of [16]:

Theorem 2.3 (Hoare, Singerman [16]). *With the above notations, the following statements are equivalent:*

(i) $X/\tilde{\Lambda}$ *is orientable,*

(ii) the only orientation reversing Schreier generators are involutions (actually conjugations in Φ*),*

(iii) all circuits of $\bar{\mathcal{K}}$ *are positive,*

(iv) the cosets of $\tilde{\Lambda}$ *in* Λ *divide into two disjoint classes such that in the action* $a : \Lambda\gamma \mapsto \Lambda(\gamma a)$ *orientation preserving elements of* Φ *fix the classes and (apart from reflections fixing cosets) orientation reversing elements interchange the classes.*

Now let us assume that we have an epimorphism $\theta : \Lambda \to G$, where Λ is a NEC group and $G \cong \Lambda/\Gamma$ for $\Gamma = \ker\theta$ is the group of all, including orientation reversing, automorphisms of $X = \mathcal{H}/\Gamma$. Let us also assume that $\sigma \in G$ is a symmetry of X. Now let us consider the subgroup $\Gamma_\sigma = \theta^{-1}(\langle\sigma\rangle)$ of Λ. It is a NEC group as the index $[\Lambda : \Gamma_\sigma] < +\infty$. In addition it is a surface group, so its signature has the form

$$(h'; \pm; [-]; \{(-)^k\},$$

which also determines the topological type of the symmetry σ, *i.e.* the sign decides about the separability, as it decides about the orientability of the orbit space $\mathcal{H}/\Gamma_\sigma$. Moreover, the number of period cycles is equal to the number of connected components of the boundary of this space and hence it is equal to the number of ovals of the symmetry σ. We shall use the theorem above taking $\tilde{\Lambda} = \Gamma_\sigma$, which will allow us to determine the separability character of the symmetry σ.

2.3 The real nerve $\mathcal{N}_g$

We have introduced all the data necessary to deal with topological type of the symmetry, let us now focus on the real nerve $\mathcal{N}_g$ of $\mathcal{M}_g$. A smooth, irreducible, real, projective algebraic curve has three important topological invariants: the number of connected components, the algebraic genus being the ordinary genus of its complexification and its separability character in its complexification.

The complexification allows to map such curves of given genus g into the classical moduli space $\mathcal{M}_g$ of smooth, irreducible, complex projective algebraic curves of genus g. The image $\mathcal{M}_g^{\mathbb{R}}$, called the *real locus*, is covered by the subsets $\mathcal{M}_g^{\pm k}$ proceeding from the real algebraic curves with k connected components and given separability, as explained before. Now a subset $\mathcal{M}_g^{\pm k}$ overlaps a subset $\mathcal{M}_g^{\pm k'}$ if and only if there is a complex algebraic curve of genus g having two real forms of the types $\pm k$ and $\pm k'$. In this paper we study the nerve $\mathcal{N}_g$, corresponding to this covering, as in ([23, 3.1.6]), called the *real nerve* of complex algebraic curves of given genus g.

The above covering of the real locus $\mathcal{M}_g^{\mathbb{R}}$ gives rise to the associated nerve $\mathcal{N}_g$, which we call *the real nerve*, being the simplicial complex whose vertices are the topological types $\pm k$. The sequence of distinct types $\left(\pm k_0, \ldots, \pm k_n\right)$ is an n-simplex in $\mathcal{N}_g$ if and only if there exists a Riemann surface X of genus g having $n+1$ symmetries of the types $\pm k_0, \ldots, \pm k_n$.

First of all, $\mathcal{N}_g$ has $[(3g+4)/2]$ vertices, by the mentioned above results of Harnack and Weichold (*c.f.* [6]). By the results of Buser, Seppälä and Silhol [5], $\mathcal{N}_g$ is connected and furthermore it was shown by Costa and Izquierdo in [7], that given g and a type $\pm k$ there exists a Riemann surface X of genus g, having two symmetries σ, τ of the types $\pm k$ and -1 respectively, which means that -1 is a *spine* for $\mathcal{N}_g$ for arbitrary g. In [12, 13] the properties of the nerve were studied, resulting in some answers concerning its geometrical and homological dimension. In this paper we complete the answer concerning geometrical dimension of $\mathcal{N}_g$ for odd genera and we use o-extremal Riemann surfaces to retrieve new information concerning homology groups of $\mathcal{N}_g$.

3 The combinatorial problem

In the last part of the paper [13] the authors have asked a following question:

Problem 1. *Let us consider a number of points situated on a circle, coloured by $k \geq 3$ colours in such a way that no two consecutive points have the same colour. Moreover, we put weights on our points in such a way that the weight is 2 if a point has neighbours with the same colour and the weight is 1 otherwise. Next, for every colour we define its weight as the sum of all the weights of points coloured with it. What is the smallest possible number of points $\varphi(k)$, for which there exists such a colouring and all the colours have distinct weights?*

Figure 1 depicts a toy example of a correctly coloured circle, where the number of colours equals 4. Since it uses 10 points, we can conclude that $\varphi(4) \leq 10$.

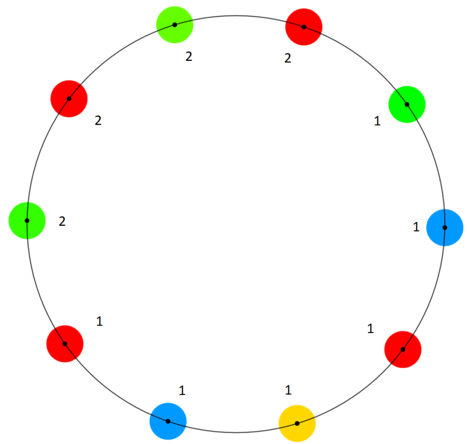

Figure 1: Correctly coloured circle with weights of each point. Weights of colours are as follows: yellow - 1, blue - 2, green - 5, red - 6.

Here, the most important is the case $k = 2^{r+1} - 1$ and it was conjectured that $\varphi(k) = 2^r(2^r + 1) - 1$. However, as we prove below, the conjecture is not true. The problem is closely related to the later part of this article, where we study properties of the real nerve of the moduli space of Riemann surfaces. To state the general solution of the above problem, we need a combinatorial lemma and preceding it a technical definition. The lemma will prove useful in lowerbounding $\varphi(k)$.

Definition 3.1. *Let $k \geqslant 4$ be the number of colours. Consider $s \geqslant k$ points on a circle indexed clockwise from 0 to $s-1$ and coloured with k colours in such a way, that no two neighbour points have the same colour. We say that the colour $\boldsymbol{a}$ satisfies the property $\star$ if there exists an index $j \in \{0, \dots, s-1\}$, such that the point of index j is coloured with the colour $\boldsymbol{a}$ and the point of index $j + 2 \mod s$ is not coloured with it.*

One may think of the property $\star$ in the following, equivalent way - the colour $\boldsymbol{a}$ possesses the property $\star$ if more than a half of points have colour different than $\boldsymbol{a}$.

Lemma 3.2. (see also [17]) *Suppose that $k \geqslant 4$ colours has been used to colour $s \geqslant k$ points on a circle in such a way, that no two neighbour points have the same colour. Suppose also that each colour a_i, $i \in \{1, \dots, k\}$ satisfies the property $\star$. Then at least k points have neighbours of different colours.*

Proof. Without loss of generality, we can assume that consecutive points are coloured in the following way

$$a_1\ a_2\ a_p\ a_q\ \dots\ a_r. \tag{3.1}$$

The last point is a neighbour of the first one, since the points are located on a circle and indexed in a clockwise order. Suppose that $i_p \in \{2 \dots s-1\}$ is the smallest index of a point coloured with the colour a_p, $p \in \{3, \dots, k\}$. Since it is the first encounter of the colour a_p, then the point of index $i_p - 1$ has neighbours of different colours. In total it gives us $k-2$ points with neighbours of different colours.

Consider now a finite sequence

$$\underbrace{\{a_1, a_2\}}_{1}, \underbrace{\{a_2, a_p\}}_{2}, \underbrace{\{a_p, a_q\}}_{3}, \dots, \underbrace{\{a_r, a_1\}}_{s}, \underbrace{\{a_1, a_2\}}_{s+1}. \tag{3.2}$$

of unordered pairs associated with the colouring given in equation (3.1). Each pair contains a colour of a point and colour of its consequent. More precisely, i-th pair contains colours of points of indices $i-1 \mod s$ and $i \mod s$. There are $s+1$ such pairs, since the first pair is repeated at the last position.

$$\dots, \underbrace{\{a_p, a_q\}}_{i}, \underbrace{\{a_q, a_r\}}_{i+1}, \dots \quad //i, i+1 \text{ denote indices of unordered pairs,}$$
$$\dots \underbrace{a_p}_{i-1}\ \underbrace{a_q}_{i}\ \underbrace{a_r}_{i+1} \dots \quad //i-1, i, i+1 \text{ denote indices of points.}$$

Observe that:

(1) Two consecutive pairs have one common element and can differ on the second one,

(2) If $(i+1)$-th pair differs from i-th one, where $i \in \{1, \dots, s\}$, then the neighbours of the i-th point have different colours,

(3) From the property $\star$ there exists a pair that does not contain the colour a_1 and there exists a pair that does not contain the colour a_2.

Let $i_1 \le s$ be the smallest index of a pair which does not contain the colour a_1 and $j_1 \le s+1$ be the smallest, but greater than i_1, index of a pair which contain the colour a_1. Above observation (point 3) and the fact that the last pair is $\{a_1, a_2\}$ guarantees that such indices always exist.

$$\dots, \underbrace{\{a_1, a_p\}}_{i_1-1}, \underbrace{\{a_p, a_q\}}_{i_1}, \dots, \underbrace{\{a_u, a_r\}}_{j_1-1}, \underbrace{\{a_r, a_1\}}_{j_1}, \dots \quad (p, q, u, r \neq 1)$$
$$\dots \underbrace{a_1}_{i_1-2}\ \underbrace{a_p}_{i_1-1}\ \underbrace{a_q}_{i_1} \dots \underbrace{a_u}_{j_1-2}\ \underbrace{a_r}_{j_1-1}\ \underbrace{a_1}_{j_1} \dots \quad (p, q, u, r \neq 1)$$

Now, $j_1 \mod s$ is an index of a point with neighbours of different colours. Clearly it was not taken into account in first $k-2$ such points, since we considered colours different than a_1 and a_2. So we pointed out one more such point. We can repeat above arguments for colour a_2, In total it gives us $k-2+1+1=k$ points with neighbours of different colour. □

In what follows, the following, easy to check, equality will be useful.

$$\sum_{n=1}^{k} \left\lceil \frac{n}{2} \right\rceil = \begin{cases} \dfrac{k^2+2k}{4} & \text{if } 2|k, \\ \\ \dfrac{k^2+2k+1}{4} & \text{if } 2 \nmid k. \end{cases} \tag{3.3}$$

Theorem 3.3. *For a number of colours $k \geqslant 4$, the minimal number of points required to satisfy conditions from Problem 1 is equal to*

$$\varphi(k) = \begin{cases} \dfrac{k^2+3k+2}{4} & \textit{if } k \equiv 2 \mod 4 \textit{ or } k \equiv 3 \mod 4, \\ \\ \dfrac{k^2+3k}{4} & \textit{if } k \equiv 0 \mod 4 \textit{ or } k \equiv 1 \mod 4. \end{cases} \tag{3.4}$$

Proof. In the first part of the proof we will construct the required colouring to upper-bound the number $\varphi(k)$. Let us start with a special case, where $k=4$. We use labels a_1, a_2, a_3, a_4 to colour the consecutive 7 points on circle in the following way

$$a_4 \; a_2 \; a_4 \; a_3 \; a_4 \; a_3 \; a_1.$$

Observe, that the weight of colour a_i is i for $i = 1,2,3,4$. Total number of points is 7 which is equal to the postulated value $\dfrac{4^2+3\cdot 4}{4} = 7$.

Now, let $k \geq 5$. The general idea is that we will define blocks of consecutive coloured points which we will use to assign the colour to each point. To keep the notation simple, to this end we will use the following notation

$$(a_i a_j)^n = \underbrace{a_i a_j a_i a_j \cdots a_i a_j}_{n}.$$

We consider the following four cases:

$k \equiv 3 \mod 4$**:** We define the block B_i as follows

$$B_i = (a_{4i+2}\; a_{4i})^{2i}\; a_{4i+2}\; a_{4i+3}\; a_{4i+2}\; (a_{4i+3}\; a_{4i+1})^{2i+1},$$

We use these blocks to compose the following sequence

$$a_3\ a_1\ a_2\ a_3\ a_2\ B_1 \ldots B_{\frac{k-3}{4}}.$$

Observe that in each block B_i, weights of colours $a_{4i}, a_{4i+1}, a_{4i+2}, a_{4i+3}$ are $4i, 4i+1, 4i+2, 4i+3$ respectively. The construction has been prepared in such a way that the weight of each colour a_j is equal to j for each $j \in \{1, \ldots, n\}$. So weights of each colour are different. It remains to calculate total number of points used in above construction. Each block B_i has exactly $(8i+5)$ points, so the total number of points is

$$5 + \sum_{i=1}^{\frac{k-3}{4}} (8i+5) = 5 + 5 \cdot \frac{k-3}{4} + 8 \cdot \frac{1+\frac{k-3}{4}}{2} \frac{k-3}{4} \frac{5k+5}{4} + \frac{(k-3)(k+1)}{4} = \frac{k^2+3k+2}{4},$$

which equals to the postulated value.

$k \equiv 0 \mod 4$**:** We define blocks C_i as follows

$$C_i = (a_{4i}\ a_{4i-2})^{2i-1}\ a_{4i}\ a_{4i-1}\ a_{4i}\ (a_{4i-1}\ a_{4i-3})^{2i-1}.$$

We use these block to compose the following sequence

$$C_1 \ldots C_{\frac{k}{4}}.$$

Observe that in each block C_i, weights of colors $a_{4i-3}, a_{4i-2}, a_{4i-1}, a_{4i}$ are $4i-3, 4i-2, 4i-1, 4i$ respectively. But i goes from 1 to $k/4$, so the weight of color a_j is equal to j for each $j \in \{1, \ldots, k\}$. Each block C_i has exactly $2 \cdot (2i-1) + 3 + 2 \cdot (2i-1) = 8i-1$ points. In total it gives us

$$\sum_{i=1}^{\frac{k}{4}} (8i-1) = 8 \cdot \frac{1+\frac{k}{4}}{2} \frac{k}{4} - \frac{k}{4} = \frac{k(k+4)}{4} - \frac{k}{4} = \frac{k^2+3k}{4},$$

points, which equals to the postulated value.

$k \equiv 1 \mod 4$**:** We define blocks D_i as follows

$$D_i = (a_{4i}\ a_{4i-2})^{2i-1}\ a_{4i}\ a_{4i+1}\ a_{4i}\ (a_{4i+1}\ a_{4i-1})^{2i}.$$

We use these blocks to compose the following sequence

$$a_1\ D_1 \ldots D_{\frac{k-1}{4}}.$$

Observe that in each block D_i, weights of colours $a_{4i-2}, a_{4i-1}, a_{4i}, a_{4i+1}$ are $4i-2, 4i-1, 4i, 4i+1$ respectively. But i goes from 1 to $(k-1)/4$, so the weight of colour a_j is equal to j for each $j \in \{2, \ldots, k\}$. The weight of colour a_1 is equal to 1, since it appears on only one point which has neighbours of different colours. Each block D_i consists of $2 \cdot (2i-1) + 3 + 2 \cdot 2i = 8i + 1$ points. In total it gives us

$$1 + \sum_{i=1}^{\frac{k-1}{4}} (8i+1) = 1 + 8 \cdot \frac{1 + \frac{k-1}{4}}{2} \frac{k-1}{4} + \frac{k-1}{4} = \frac{4 + (k+3)(k-1)}{4} + \frac{k-1}{4} = \frac{k^2 + 3k}{4},$$

points, which equals to the postulated value.

$k \equiv 2 \mod 4$**:** At first, we have to consider a special case, when $k = 6$. The following sequence

$$a_6\ a_1\ (a_5\ a_6)^3\ (a_3\ a_4)^2\ a_2\ a_4$$

is correct colouring of points on a circle. It is straightforward to check that for each $j \in \{1, 2, 3, 4, 5, 6\}$, the weight of the colour a_j is equal to j. This sequence consists of

$$14 = \frac{6^2 + 3 \cdot 6 + 2}{4}$$

points, which equals to the postulated value.

Now let $k \geq 10$. We define blocks E_i as follows

$$E_i = (a_{4i+2}\ a_{4i})^{2i}\ a_{4i+2}\ a_{4i+1}\ a_{4i+2}\ (a_{4i+1}\ a_{4i-1})^{2i}.$$

We use these blocks to compose the following sequence

$$a_2\ a_1\ E_1\ a_2\ E_2 \ldots E_{\frac{k-2}{4}}.$$

Observe that in each block E_i, weights of colours $a_{4i-1}, a_{4i}, a_{4i+1}, a_{4i+2}$ are $4i-1, 4i, 4i+1, 4i+2$ respectively. But i goes from 1 to $(k-2)/4$, so the weight of colour a_j is equal to j for each $j \in \{3, \ldots, k\}$. The weights of colours a_1, a_2 are 1 and 2 respectively, since a_1 appears only once and a_2 twice, each time between neighbours of different colours. Each block E_i consists of $2 \cdot 2i + 3 + 2 \cdot 2i = 8i + 3$ points. In total it gives us

$$3 + \sum_{i=1}^{\frac{k-2}{4}} (8i+3) = \frac{3k - 6 + 12}{4} + 8 \cdot \frac{1 + \frac{k-2}{4}}{2} \frac{k-2}{4} = \frac{3k+6}{4} + \frac{(k+2)(k-2)}{4} = \frac{k^2 + 3k + 2}{4},$$

points, which equals to the postulated value.

Above constructions give us an upper bound on $\varphi(k)$. Now we will prove a lower bound which will match it.

Notice that the description of the Problem 1 requires that colours have distinct weights. We start by showing that the choice of weights $\{1, 2, \dots, k\}$ is optimal. We use the word "optimal" in the sense that if one has the correct colouring of s_1 points on a circle and colours have weights $\{w_1, \dots, w_k\}$, then there exists a correct colouring of s_2 points on a circle in which weights have colours $\{1, 2, \dots, k\}$ and $s_2 \leq s_1$. For the sake of clarity, we will assume that the set of weights $\{w_1, \dots, w_k\}$ is sorted, *i.e.* if $i < j$ then $w_i < w_j$.

Observe that each occurrence of a colour on some point on a circle can add 1 or 2 to the total weight of a colour. So the minimal number of points required to achieve the weight w_i is $\lceil w_i/2 \rceil$ ($w_i = 2 + \dots + 2$ if $2|w_i$ or $w_i = 2 + \dots + 2 + 1$ otherwise). To realize all the weights from the set $\{w_1, \dots, w_k\}$, we will need at least

$$\sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil \tag{3.5}$$

points on a circle. Consider now the special case, where every second point has a colour a_ℓ. Suppose its weight is equal to w_ℓ. The number of points of colour a_ℓ is the same as the number of all other points, so there are at least

$$2 \cdot \left(\sum_{i=1}^{\ell-1} \left\lceil \frac{w_i}{2} \right\rceil + \sum_{i=\ell+1}^{k} \left\lceil \frac{w_i}{2} \right\rceil \right)$$

points on a circle. This sum is minimized for $\ell = k$ and the following weights $\{1, \dots, k\}$. We can lower-bound this sum

$$2 \cdot \sum_{i=1}^{k-1} \left\lceil \frac{w_i}{2} \right\rceil \geqslant 2 \cdot \sum_{n=1}^{k-1} \left\lceil \frac{n}{2} \right\rceil \geqslant 2 \cdot \frac{(k-1)^2 + 2(k-1)}{4} = 2 \cdot \frac{k^2 - 1}{4} = \frac{k^2 - 1}{2}, \tag{3.6}$$

which for $k \geq 4$ is greater than our upper bound on $\varphi(k)$. It implies that having half of points coloured in the same colour is not an optimal strategy. We can discard it and assume that assumptions of Lemma 3.2 are satisfied.

To achieve minimal number of points, each odd weight has to be composed from only 2s and one 1 and each even weight has to be composed from only 2s. Lemma 3.2 implies that there had to be at least k points with weight 1. Suppose that there are $s \leq k$ odd numbers in the set $\{w_1, \dots, w_k\}$. Then Lemma 3.2 forces us to add additional $k - s$ points of weight 1. Consider two cases

$2|k - s$: we swap $(k - s)/2$ points of weight 2 into $k - s$ points of weight 1. Then minimal number of points required to realize the weights from the set $\{w_1, \dots, w_k\}$ is

$$\sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil - \frac{k-s}{2} + k - s = \sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil + \frac{k-s}{2}.$$

Such configuration consists of k points of weight 1 (which is required by Lemma 3.2) and uses minimal number of points required to realize weights from the set $\{w_1, \dots, w_k\}$.

$2 \nmid (k-s)$: we swap $(k-s+1)/2$ points of weight 2 into $k-s+1$ points of weight 1. Then minimal number of points required to realize the weights from the set $\{w_1, \dots, w_k\}$ is

$$\sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil - \frac{k-s+1}{2} + k - s + 1 = \sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil + \frac{k-s+1}{2}.$$

These expressions lower-bound minimal number of points required to realize weights $\{w_1, \dots, w_k\}$. They consist of two factors: main sum and the correction forced by Lemma 3.2. Observe that

- If the weights $\{w_1, \dots, w_k\}$ are not subsequent integer numbers, *i.e.* there exists i such that $w_i + 1 < w_{i+1}$, then there exists a set of weights $\{\tilde{w}_1, \dots, \tilde{w}_k\}$ for which minimal number of points is not greater than for $\{w_1, \dots, w_k\}$. Indeed, if we define $\{\tilde{w}_1, \dots, \tilde{w}_k\}$ as follows

$$\begin{cases} \tilde{w}_j = w_j & \text{if } j \neq i+1 \\ \tilde{w}_{i+1} = w_i + 1 & \end{cases},$$

  then three cases can occur:

  - $\tilde{w}_{i+1}$ and w_{i+1} have the same parity, *i.e.* both are even or both are odd and $w_{i+1} - \tilde{w}_{i+1} \geq 2$. Then the penalty term does not change and the following inequality holds

$$\sum_{i=1}^{k} \left\lceil \frac{\tilde{w}_i}{2} \right\rceil + 2 \leqslant \sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil.$$

    The penalty term will not increase, so the new set of weights gives us lower number of points.

  - $\tilde{w}_{i+1}$ is odd and w_{i+1} is even. Surely

$$\sum_{i=1}^{k} \left\lceil \frac{\tilde{w}_i}{2} \right\rceil \leqslant \sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil.$$

  - $\tilde{w}_{i+1}$ is even and w_{i+1} is odd. Then $\lceil w_{i+1}/2 \rceil - \lceil \tilde{w}_{i+1}/2 \rceil \geq 1$, so

$$\sum_{i=1}^{k} \left\lceil \frac{\tilde{w}_i}{2} \right\rceil + 1 \leqslant \sum_{i=1}^{k} \left\lceil \frac{w_i}{2} \right\rceil.$$

    The penalty term can increase but at most by 1, so the number of points for weights $\{\tilde{w}_1, \dots, \tilde{w}_k\}$ will not increase.

- If $w_1 > 1$ then we can define the set $\{\tilde{w}_1, \ldots, \tilde{w}_k\}$ as follows

$$\begin{cases} \hat{w}_j = w_j & \text{if } j \neq 1 \\ \hat{w}_1 = 1 \end{cases}.$$

The same arguments as above can show that the minimal number of points required to realize $\{\tilde{w}_1, \ldots, \tilde{w}_k\}$ is not greater then for $\{w_1, \ldots, w_k\}$.

It follows from above observation that the minimal number of points required to realize subsequent weights starting from 1 is not greater than the minimal number of points required to realize every other set $\{w_1, \ldots, w_k\}$ of different weights.

We showed that $\{1, 2, \ldots, k\}$ is an optimal choice of weights. Now we can lower-bound the number of points required to realize weights from this set. To achieve this, we consider the following four cases

- $k \equiv 0 \mod 4$:

$$\sum_{n=1}^{k} \left\lceil \frac{n}{2} \right\rceil + \frac{k - \frac{k}{2}}{2} = \frac{k^2 + 2k}{4} + \frac{k}{4} = \frac{k^2 + 3k}{4},$$

- $k \equiv 1 \mod 4$:

$$\sum_{n=1}^{k} \left\lceil \frac{n}{2} \right\rceil + \frac{k - \frac{k+1}{2}}{2} = \frac{k^2 + 2k + 1}{4} + \frac{k - 1}{4} = \frac{k^2 + 3k}{4},$$

- $k \equiv 2 \mod 4$:

$$\sum_{n=1}^{k} \left\lceil \frac{n}{2} \right\rceil + \frac{k - \frac{k}{2} + 1}{2} = \frac{k^2 + 2k}{4} + \frac{k + 2}{4} = \frac{k^2 + 3k + 2}{4},$$

- $k \equiv 3 \mod 4$:

$$\sum_{n=1}^{k} \left\lceil \frac{n}{2} \right\rceil + \frac{k - \frac{k+1}{2} + 1}{2} = \frac{k^2 + 2k + 1}{4} + \frac{k + 1}{4} = \frac{k^2 + 3k + 2}{4}.$$

As one can see, the lower bound matches the upper bound. This finishes the proof. □

Authors of [13] were interested in the special case, where $k = 2^{r+1} - 1$. Our solution gives the following answer to the problem $\varphi(2^{r+1} - 1) = 2^r \left(2^r + \frac{1}{2}\right)$.

4 Topological types of the symmetries on Riemann surfaces

Let us start with observations concerning possible topological types which are admissible on Riemann surfaces of our interest, that is Riemann surfaces which admit the maximal possible number of nonconjugate symmetries. By the results of [2] it is known that for $g = 2^{r-1}u + 1$, u odd, the maximal possible number of conjugacy classes of symmetries of a Riemann surface of genus g is 2^{r+1} and by the results of [14] it is known that the automorphism group generated by these symmetries is isomorphic to $\mathrm{D}_{2^s} \times \mathbb{Z}_2^r$. We shall call the surfaces realizing the maximal value *s-extremal*. In the sequel we shall assume that g is odd, which means that $r \geq 2$.

Theorem 4.1. *Let X be an s-extremal Riemann surface of odd genus g such that at most one of the symmetries in question is fixed-point free. Then all the symmetries in question are non-separating.*

Proof. Let X be an s-extremal Riemann surface of genus $g = 2^{r-1}u + 1$, where u is odd. This means that X admits the maximal number of 2^{r+1} nonconjugate symmetries and the automorphism group G of X is isomorphic to $\mathrm{D}_{2^s} \times \mathbb{Z}_2^r = \langle a, b\rangle \times \langle \sigma_1, \ldots, \sigma_r\rangle$. We know that G may be viewed as a quotient group Λ/Γ for a NEC group Λ with a general signature (2.1) and Γ being a kernel of an epimorphism $\theta : \Lambda \to G$. Clearly we can assume that the representatives of the 2^{r+1} conjugacy classes of all the symmetries of X are of the forms

$$ax, \quad bx, \quad \sigma, \quad (ab)^{2^{s-1}}\sigma,$$

where x is a word of even length in $\sigma_1, \ldots, \sigma_r$ and σ is a word of odd length in $\sigma_1, \ldots, \sigma_r$. The first two ones are non-central and the remaining two are central. Now it is clear that for all but at most one of the symmetries above there is a canonical reflection which contributes ovals to it. We can also assume, without loss of generality, that there are canonical reflections

$$c_a, \quad c_b, \quad c_{\sigma_i},$$

for $i = 1, \ldots, r$, which are mapped to

$$a, \quad b, \quad \sigma_i,$$

by the canonical epimorphism θ. Now it is clear that any of the symmetries can be written as a word of odd length in a, b, σ_i. This also means that any of the non-generating symmetries can be seen as the image of a certain product of canonical reflections c_a, c_b, c_{σ_i}. This clearly makes any of the non-generating symmetries non-separating. Now actually the same applies also to any of the generating symmetries a, b, σ_i as each of them can be written as a product of three other symmetries with fixed points. Say, if $\sigma = a\sigma_i\sigma_j$ is an image of c_σ, then a is the image of $c_\sigma c_{\sigma_i} c_{\sigma_j}$. This means all the symmetries are nonseparating. □

Now we shall consider the total maximal number of ovals of the symmetries on s-extremal Riemann surfaces with the automorphism group as before. The key elements here are the centralizers mentioned in Theorem 2.2. Let us first consider a central symmetry σ. Obviously, its centralizer in G has order $|G| = 2^{r+s+1}$. Now, we shall observe what is a possible contribution of any reflection c_σ such that $\theta(c_\sigma) = \sigma$. If the reflection corresponds to an empty period cycle, then the possible numbers of ovals contributed are $|G|/2$ or $|G|/4$, depending on whether $\theta(e) = 1$ or not for the corresponding connecting generator of the empty cycle. Now in the case of a nonempty period cycle, the possible values are bounded from above by $|G|/4$ if there are equal neighbors, meaning that the images of the neighboring reflections under θ are the same, and from below by $|G|/2^{s+2}$ if the neighbors generate a dihedral group of order 2^{s+1}. Therefore, we see that any such occurrence has to contribute at least 2^{r-1} ovals. Surprisingly, the case is exactly the same for noncentral symmetries. Even though their centralizers in G are of order 2^{r+2}, the possible numbers of ovals are 2^{r+1} or 2^r for an empty period cycle and at least $2^{r+2}/8 = 2^{r-1}$ for a nonempty period cycle.

Theorem 4.2. *The maximal total number of ovals for $2^{r+1} - 1$ symmetries of an s-extremal Riemann surface with G as above is equal to*

$$2^r u + (7 - 2^{r+1})2^{r+s-2} + 2^r.$$

Proof. By the general result of [9] we know that the maximal total number of ovals of k non-conjugate symmetries, on a Riemann surface X of genus g, which generate the group G, is

$$2g - 2 + (9 - k) \cdot \frac{|G|}{8}.$$

In our case $g = 2^{r-1}u + 1$ and we shall assume that our surface is s-extremal, so $G = \mathrm{D}_{2^s} \times \mathbb{Z}_2^r$. As we are going to look for the maximal total number of ovals, by the proof of the main result in [9], we may assume that the signature of a NEC group Λ in the quotient $G = \Lambda/\Gamma$ contains just one, nonempty, period cycle

$$(h; \pm; [m_1, \ldots, m_v]; \{(n_1, \ldots, n_s)\}).$$

Furthermore, by the Hurwitz-Riemann formula

$$\varepsilon h - 1 + \sum_{i=1}^{v} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{2^{r-1}u}{2^{r+s}} = \frac{u}{2^{s+1}},$$

and since u is odd, this means that there is an odd number of link periods equal 2^s in the signature of Λ. Going even further into the details,

$$\frac{u}{2^{s+1}} \geq -1 + \frac{1}{2} - \frac{1}{2^{s+1}} + \frac{s-1}{4}$$

and so

$$s \leq \frac{u}{2^{s-1}} + 3 + \frac{4}{2^{s+1}}. \tag{4.1}$$

As there is an odd number of link periods 2^s in the signature, it means that at least k reflections in the cycle have neighbors labeled with different labels, meaning they have distinct images under θ.

Indeed, otherwise, by Lemma 3.2, half of the reflections of the cycle would have the same image under θ, which contradicts the assumptions. Therefore, summing up, at least $k = 2^{r+1} - 1$ reflections have neighbors with distinct labels. Therefore, the total number of ovals holds inequality

$$\|X\| \leq k \cdot \frac{|G|}{8} + (s-k) \cdot \frac{|G|}{4} = s \cdot \frac{|G|}{4} + \frac{|G|}{8} - 2^{r+1} \cdot \frac{|G|}{8}$$

and using the inequality (4.1) on s we obtain

$$\|X\| \leq \left(\frac{u}{2^{s-1}} + 3 + \frac{4}{2^{s+1}}\right) \cdot 2^{r+s-1} + 2^{r+s-2} - 2^{r+1} \cdot 2^{r+s-2} = 2^r u + 2^r + 2^{r+s-2} \cdot (7 - 2^{r+1}).$$

□

As a corollary, we obtain the following interesting result concerning relations between s-extremal and o-extremal Riemann surfaces.

Corollary 4.3. *An s-extremal Riemann surface is never o-extremal.*

Theorem 4.4. *If a Riemann surface of genus $g = 2^{r-1}u + 1$, odd u, has 2^{r+1} nonconjugate symmetries such that each of the symmetries has a distinct topological type, then $u \geq 2^{2r} + 2^{r-1} - 3$.*

Proof. First of all, we shall assume that one of the symmetries has 0 ovals. This assumption is natural: our aim is to minimize u, which is strictly connected to the signature of Λ in $G = \Lambda/\Gamma$. To make sure that all the symmetries have distinct numbers of ovals, we are allowed to assume that at most one of them is fixed-point free. However, if we allowed all the symmetries to have fixed points, then our number k in the previous theorem would be equal to 2^{r+1} as we have to incorporate one more colour in our setting, being the symmetry as the image of a canonical reflection. This makes the total admissible number of ovals even smaller and the conditions of the theorem are fulfilled automatically. Hence, let us consider the case where one of the symmetries is fixed-point free. Now, as by Theorem 4.1, all the symmetries are nonseparating and the minimal contribution of a reflection is 2^{r-1}, the minimal number of ovals that they have together is

$$1 \cdot 2^{r-1} + 2 \cdot 2^{r-1} + \cdots + (2^{r+1} - 1) \cdot 2^{r-1},$$

which is equal to $2^{2r-1}(2^{r+1}-1)$. Now it has to be that

$$2^{2r-1}(2^{r+1}-1) \leq 2^r u + (7-2^{r+1})2^{r+s-2} + 2^r.$$

This in turn means that

$$u \geq 2^{2r} + 2^{r-1}(2^s - 1) - 1 - 7 \cdot 2^{s-2}.$$

For $r \geq 3$, $s \geq 2$ we get that

$$u \geq 2^{2r} + 2^{r-1} - 3.$$

Now if $r = 2, s \geq 3$ we need to show that $u \geq 15$ and we have

$$u \geq 16 - 2 - 1 + 2 = 15.$$

If $r = s = 2$ we get

$$u \geq 16 - 2 - 1 + 1 = 14$$

and is $r = 2, s = 1$ we get

$$u \geq 16 - 2 - 2 + \frac{1}{2} = 13\frac{1}{2}.$$

However, u is an odd integer, which means that $u \geq 15$ in both cases. □

Theorem 4.5. *The bound on u is sharp. Moreover, for any odd* $u \geq u_{min} = 2^r(2^r + 1/2) - 3$ *there exists a Riemann surface of genus* $g = 2^{r-1}u + 1$ *having* 2^{r+1} *symmetries of distinct topological types.*

Proof. In this proof, we will use the construction from the solution of the combinatorial Problem 1. It is necessary to show, how it can be applied to define a Riemann surface. Let $X = \mathcal{H}/\Gamma$ be a Riemann surface of genus g with the following automorphism group

$$\mathrm{Aut}(X) = \Lambda/\Gamma = \mathbb{Z}_2^{r+2} = \langle \sigma_1 \rangle \oplus \cdots \oplus \langle \sigma_{r+2} \rangle,$$

where Λ is a NEC group with signature

$$s(\Lambda) = (0; +; [-], \{(\underbrace{2, \ldots, 2}_{\alpha})\}).$$

Theorem 2.1 gives us the method for calculating number of ovals for symmetries. Suppose that the epimorphism $\theta : \Lambda \longrightarrow \Lambda/\Gamma$ is defined on generators c_{i-1}, c_i, c_{i+1} as follows

$$\theta(c_{i-1}) = \sigma_a\ , \quad \theta(c_i) = \sigma_b\ , \quad \theta(c_{i+1}) = \sigma_c.$$

Since $\mathbb{Z}_2^{r+2}$ is abelian, then $|C(\mathbb{Z}_2^{r+2}, \sigma_b)| = |\mathbb{Z}_2^{r+2}| = 2^{r+2}$. Theorem 2.2 describes the structure of $C(\Lambda, c_i)$ and since each period of period cycles equals 2, then

$$\theta(C(\Lambda, c_i)) = \langle\theta(c_{i-1})\rangle \oplus \langle\theta(c_i)\rangle \oplus \langle\theta(c_{i+1})\rangle,$$
$$\theta(C(\Lambda, c_i)) = \begin{cases} \mathbb{Z}_2^3 & \text{if } \theta(c_{i-1}) \neq \theta(c_{i+1}), \\ \mathbb{Z}_2^2 & \text{if } \theta(c_{i-1}) = \theta(c_{i+1}). \end{cases}$$

Hence, $\theta(c_i)$ contributes to σ_b either $\dfrac{2^{r+2}}{8}$ or $\dfrac{2^{r+2}}{4}$ ovals depending on whether $\theta(c_{i-1}) \neq \theta(c_{i+1})$ or $\theta(c_{i-1}) = \theta(c_{i+1})$ respectively.

Now, the correspondence between constructing epimorphism θ and combinatorial problem is straightforward. Colours correspond to symmetries and points on circle correspond to the generators $c_0, \ldots, c_s$ of a NEC group Λ. Assigning colours to the points corresponds to the definition of epimorphism θ. Moreover, weights of points correspond to the number of ovals contributed to symmetries by each image $\theta(c_i)$. To conclude, the solution of combinatorial problem, allows us to define epimorphism θ in such a way that each symmetry has distinct number of ovals.

Let $g = 2^{r-1}u + 1$, where u is odd. We will construct a Riemann surface of genus g which has exactly 2^{r+1} symmetries of distinct number of ovals.

Let

$$\alpha = 2^r\left(2^r + \frac{1}{2}\right) + 1, \tag{4.2}$$

and Λ be NEC group with the following signature

$$s(\Lambda) = (0; +; [-], \{(\underbrace{2, \ldots, 2}_{\alpha})\}).$$

Let $X = \mathcal{H}/\Gamma$ be a Riemann surface of genus g with the following automorphism group

$$\mathrm{Aut}(X) = \Lambda/\Gamma = \mathbb{Z}_2^{r+2} = \langle\sigma_1\rangle \oplus \cdots \oplus \langle\sigma_{r+2}\rangle.$$

Observe that based on Hurwitz-Riemann formula, the necessary condition for a Riemann surface X to be realised as $\mathcal{H}/\Gamma$ is that $u = 2^r\left(2^r + \frac{1}{2}\right) - 3$. Indeed,

$$2g - 2 = \left|\mathbb{Z}_2^{r+2}\right|\left(-2 + 1 + \frac{1}{2}\sum_{i=1}^{\alpha}\frac{1}{2}\right) \iff 2\left(2^{r-1}u + 1\right) - 2 = 2^{r+2}\left(-1 + \frac{\alpha}{4}\right)$$
$$\iff 2^r u = 2^r(\alpha - 4) \iff u = 2^r\left(2^r + \frac{1}{2}\right) - 3.$$

Now, we will define an epimorphism $\theta : \Lambda \longrightarrow \Lambda/\Gamma$. On a generator e_1 of a group Λ we define $\theta(e_1) = 1$. Next, the construction from Theorem 3.4 gives us a recipe how to define the epimorphism

θ on generators c_i. Notice, that one symmetry can have no fixed points, so we do not include it in the construction from Theorem 3.4. For this construction we will need

$$\varphi\left(2^{r+1}-1\right)=\frac{(2^{r+1}-1)^2+3\cdot(2^{r+1}-1)+2}{4}=2^r\left(2^r+\frac{1}{2}\right)$$

periods from period cycles. But there are $2^r\left(2^r+\frac{1}{2}\right)+1$ periods from period cycles in the signature $s(\Lambda)$. It is not a problem at all, since we can modify the construction from Theorem 3.4 by adding one point of colour a_{4k+3} in the following way

$$a_3\ a_1\ a_2\ a_3\ a_2\ a_{4k+3}\ B_1\cdots B_{\frac{k-4}{4}},$$

where in our case $k=2^{r+1}$. Weight of colour a_{4k+3} will increase by 1 and weights of all other colours will not change, so the construction will remain valid. With small modification of a solution for combinatorial problem, we have constructed the epimorphism θ whose kernel $\ker(\theta)=\Gamma$ is a surface Fuchsian group of genus g. We have constructed then a Riemann surface $\mathcal{H}/\Gamma$ of genus $g=2^{r-1}u+1$, where $u=2^r\left(2^r+\frac{1}{2}\right)-3$, which has 2^{r+1} symmetries of distinct topological types.

Now, let $u>u_{min}$ and Λ be a NEC group with the following signature

$$s(\Lambda)=(0;+;[\underbrace{(2,\ldots,2)}_{\beta}],\{\underbrace{(2,\ldots,2)}_{\alpha}\}),$$

where

$$\beta=\frac{1}{2}\left(u-2^r\left(2^r+\frac{1}{2}\right)+3\right),$$

and α is defined as in (4.2). We define the epimorphism θ on generators c_i the same way as above. Moreover for each generator x_i we put $\theta(x_i)=\sigma_1\sigma_2$. It only remains to define θ on the generator e_1. We do it as follows: if β is even, we put $\theta(e_1)=1$ and $\theta(e_1)=\sigma_1\sigma_2$ otherwise.

Observe that the process of counting ovals is the same as earlier. The only one change may occur, when $\theta(e_1)=\sigma_1\sigma_2$. However, since our group is abelian it does not introduce any change in the ovals count. Such definition of the epimorphism θ is correct, since relations from NEC group presentation are satisfied and the kernel of θ, $\ker(\theta)=\Gamma$ is a surface Fuchsian group. It only remains to check, if Γ (so as the constructed Riemann surface $\mathcal{H}/\Gamma$) has a genus $g=2^{r-1}u+1$. We do this by checking Hurwitz-Riemann formula

$$2g-2=\left|\mathbb{Z}_2^{r+2}\right|\left(-2+1+\sum_{i=1}^{\beta}\frac{1}{2}+\frac{1}{2}\sum_{i=1}^{\alpha}\frac{1}{2}\right)\iff 2g-2=2^{r+2}\left(-1+\frac{\beta}{2}+\frac{\alpha}{4}\right)$$

$$\iff g=2^{r-1}\left(2\beta+\alpha-4\right)+1\iff g=2^{r-1}\left(u-2^r\left(2^r+\frac{1}{2}\right)+3+2^r\left(2^r+\frac{1}{2}\right)+1-4\right)-1$$

$$\iff g=2^{r-1}u+1.$$

For each $u\geq u_{min}$ we have constructed a Riemann surface of genus $g=2^{r-1}u+1$ having 2^{r+1} symmetries of distinct number of ovals, which implies distinct topological types. □

As a Corollary, we obtain the main result of this paper, which also gives an answer to the question posed in [13]

Corollary 4.6. *The geometrical dimension of the real nerve $\mathcal{N}_g$ for odd values of g is maximal if and only if $u\geq u_{min}$.*

Now we shall present another application of o-extremal surfaces. This time we shall show extremal configurations of symmetries which prove the nontriviality of certain homology groups. Before we proceed to the main result, let us make an easy observation about o-extremal surfaces. By the results of [9], we know that on a Riemann surface of genus g, k non-conjugate symmetries with fixed points have at most

$$2g-2+(9-k)\cdot\frac{|G|}{8} \tag{4.3}$$

ovals. As g is odd, we are going to play with at least 8 symmetries on an o-extremal Riemann surface, it is known by [14] and [12] that they commute. Therefore the structure of the group is known, being $\mathbb{Z}_2^{a+2}$, where a is the smallest possible integer such that $k\leq 2^{a+1}$ for $k\geq 9$, or $\mathbb{Z}_2^8$ for $k=8$. What can be easily observed by (4.3) is, that with the growth of k, the maximal possible number of ovals does not grow, and even more, it drops down for any $k\geq 9$ being the power of 2. Therefore the idea for the next results is as follows. We are going to prove that there exists a nontrivial cycle in certain $(k-1)$-th homology group. To show this, we shall construct $(k-1)$-dimensional facets of what-could-be a simplex of dimension k, but we do it in such a way, that one of the facets comes from an o-extremal Riemann surface. This means, that the sum of absolute values of all vertices is maximal. Therefore the said hypothetical k-dimensional simplex cannot exist, as long as we are sure that the total maximal number of ovals for k symmetries is strictly smaller than the one for $k-1$ symmetries. This is guaranteed for the values of k being of the form $k=2^{a+1}$ for some integer a. Now we can proceed with the main result.

Theorem 4.7. *Let $g = 2^{r-1}u + 1$ for some odd u and $r \geq 2$. Let $a \leq r - 1$ be a positive integer. Then, for $u \geq 2^{2r} + 2^r - 3$, the homology groups $H_{2^{a+1}-1}(\mathcal{N}_g) \neq 0$.*

Proof. Let us consider a NEC group Λ with signature

$$(0; +; [2, \overset{m}{\ldots}, 2]; \{(2, \overset{s}{\ldots}, 2)\}), \tag{4.4}$$

where $m = u \cdot 2^{r-1-a} + 2 - (2^{a+1} + 2)(2^{a+1} - 1)$ and $s = (2^{a+2} + 4)(2^{a+1} - 1)$. Our goal will be to define $\theta : \Lambda \to G = \mathbb{Z}_2^{a+2} = \langle \sigma_i \,|\, \sigma_i^2, i = 1, \ldots, a+2 \rangle$ in such a way, that $X = \mathcal{H}/\ker\theta$ is an o-extremal Riemann surface, having 2^{a+1} non-conjugate symmetries with distinct topological types. Next, we shall define $2^{a+1} - 1$ Riemann surfaces X_i, $i = 1, \ldots, 2^{a+1} - 1$ with the same genus and automorphism group as X in such a way that each of them has 2^{a+1} non-conjugate symmetries with distinct topological types, but the types for X_i are the same as the ones for X, with the only exception of σ_i which shall have $2^{a+1} = |G|/2$ ovals in the new construction. First, we construct X by defining θ on the canonical generators: all the elliptic generators x_i, $i = 1, \ldots, m$, are mapped to $\sigma_1\sigma_2$ and the connecting generator e_1 is mapped to 1 if m is even and $\sigma_1\sigma_2$ if m is odd, to satisfy the long relation. Now, reflections corresponding to the unique nonempty period cycle are mapped in the following way. We divide the cycle into $2^{a+1} - 1$ segments S_i of increasing length - the length of S_i is $(i+1) \cdot 4$ and its reflections are mapped consecutively to σ_1 and σ_{i+1}:

$$\sigma_1,\ \sigma_{i+1},\ \sigma_1,\ \sigma_{i+1}, \ldots, \sigma_1,\ \sigma_{i+1},\ \sigma_1,\ \sigma_{i+1}.$$

Obviously, each of the reflections contributing to σ_{i+1} yields $|G|/4$ ovals, as its neighbors have the same images under θ. It is almost the same for reflections corresponding to σ_1, except for the first one, which has neighbors with distinct images and so it contributes $|G|/8$ ovals to σ_1. Therefore, the total contribution to σ_1 from the segment S_i is $(i+1) \cdot |G|/2 - |G|/8$ and the contribution to σ_{i+1} is $(i+1)|G|/2$. As the symmetries, except σ_1, do not appear outside their respective segments, all of them have distinct numbers of ovals. It is also clearly visible that the total number of ovals is maximal possible and so we have constructed an o-extremal Riemann surface of genus $g = 2^{r-1}u + 1$. Now by the method that could be called surgery of the signature (4.4) of Λ, we shall construct the surfaces X_{i+1} mentioned above. Let first $i + 1 \geq 2$. We shall replace the signature (4.4) above by the signature

$$(0; +; [2, \overset{m}{\ldots}, 2]; \{(2, \overset{s_i}{\ldots}, 2), (-)^i\}), \tag{4.5}$$

where $s_i = s - 4i$. Let us call the corresponding NEC group Λ_{i+1} an now we modify θ to $\theta_{i+1} : \Lambda_{i+1} \to G$ - all the segments remain unchanged except from the segment S_i which is replaced by a segment of length 4 whose reflections are mapped to

$$\sigma_1,\ \sigma_{i+1},\ \sigma_1,\ \sigma_{i+1}.$$

This exchange can be seen as a surgery on S_i: we cut the middle of the cycle out and leave only the first two and the last two elements of the segment. Now the symmetry σ_{i+1} has $|G|/2$ ovals. Clearly σ_1 lost $i \cdot |G|/2$ ovals. These ovals are added again in the empty cycles of Λ_i, as all the canonical reflections are mapped by θ_{i+1} to σ_1 and all the corresponding connecting generators are mapped to 1, except for e_1 for odd m, where $\theta(e_1) = \sigma_1\sigma_2$. In such a way, we have obtained the surface X_{i+1}, of genus $g = 2^{r-1}u + 1$ which has 2^{a+1} symmetries of distinct topological types and all the types are the same except the one corresponding to σ_{i+1}.

Let us now construct X_1. We have to replace the number of ovals of σ_1 by $|G|/2$. This clearly requires a different approach than the one presented above for $i+1 \geq 2$. The idea is as follows: for all $i \geq 3$, we replace the segment S_i by $i+1$ empty period cycles. All of these period cycles have corresponding reflections being mapped by θ_1 to σ_{i+1} and the connecting generators to 1. Observe that in this construction, each period cycle contributes $|G|/2$ ovals to σ_{i+1} and so in total it has $(i+1)|G|/2$ ovals again. Now let us consider S_1 and S_2: together they have 20 reflections contributing $|G|$ ovals to σ_2 and $3|G|/2$ ovals to σ_3. We have to define θ_1 in such a way that these numbers of ovals stay the same for σ_2, σ_3 and σ_1 gets $|G|/2$ ovals. We introduce a new period cycle of length 8 which maps its reflections consecutively to σ_2, σ_3 giving each of them $|G|$ ovals. Then we add two more empty period cycle, one of them contributing to σ_1 and the other to σ_3. Summing up, we consider a NEC group

$$(1; -; [2, \overset{m}{\ldots}, 2]; \{(2,2,2,2,2,2,2,2), (-)^l, (-), (-)\}),$$

where m is as before and $l = (2^a+2)(2^{a+1}-3)$. Now d_1 is mapped to σ_1, all the elliptic generators x_i, $i = 1, \ldots, m$ are mapped to $\sigma_1\sigma_2$, all the connecting generators e_i, $i \geq 2$ are mapped to 1, the connecting generator e_1 is mapped to 1 if m is even and $\sigma_1\sigma_2$ if m is odd. Now the reflections of the unique nonempty period cycle are mapped consecutively to σ_2, σ_3, and we divide the empty period cycles into segments in the following way: there are $2^{a+1}-3$ segments and segment E_i consists of $i+3$ empty period cycles where the canonical reflections involved in the segment E_i are mapped to σ_{i+3}. The reflections of the last two empty period cycles are mapped to σ_3 and σ_1. In such a way we obtain the surface X_1, of the same genus as X, as announced above.

Now, when it comes to homology groups of the real nerve $\mathcal{N}_g$, in such a way we have constructed a cycle which is not a boundary in dimension $2^{a+1}-1$. Indeed, if it were a boundary, then there would have to exist a simplex with more than 2^{a+1} vertices where the sum of absolute values of marks of the vertices, coming from the corresponding symmetry types, is at least the same as the one for simplex defining X. This is clearly not possible, as X is o-extremal and any o-extremal surface with more than 2^{a+1} symmetries has total number of ovals which is strictly smaller than the maximal value for 2^{a+1} symmetries. Therefore indeed the homology group is nontrivial in this dimension. □

Remark 4.8. *The same method cannot be used in any other dimension. In fact, in such a case our symmetries do not exhaust all the orientation reversing involutions in the group G and we can always add a 0-vertex to our simplex and obtain a simplex of a larger dimension which admits X as a face, without changing the maximal possible number of ovals. This means, that in all the other dimensions, our constructed cycle is a boundary of a $(k+1)$-dimensional simplex, where the new vertex corresponds to a fixed-point free symmetry.*

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Multivariate symmetrized, q-deformed and λ-parametrized hyperbolic tangent induced complex valued trigonometric and hyperbolic neural network enhanced approximation

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ABSTRACT

Here we study the multivariate quantitative symmetrized approximation of complex valued continuous functions on a box by complex valued symmetrized and perturbed multivariate neural network operators. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the used function's high order partial derivatives. The kind of our approximations are trigonometric and hyperbolic. Our multivariate symmetrized operators are defined by using a multivariate density function generated by a q-deformed and λ-parametrized hyperbolic tangent function. These enhanced approximations are pointwise and of the uniform norm. The related complex valued feed-forward neural networks are with one hidden layer.

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RESUMEN

Estudiamos la aproximación cuantitativa multivariada simetrizada de funciones continuas con valores complejos en una caja, a través de operadores de redes neuronales simetrizados y multivariados perturbados con valores complejos. Estas aproximaciones se derivan al establecer desigualdades de tipo Jackson, que involucran el módulo de continuidad de las derivadas parciales de alto orden de la función utilizada. Los tipos de nuestras aproximaciones son trigonométricas e hiperbólicas. Nuestros operadores multivariados simetrizados se definen usando una función de densidad multivariada generada por una función tangente hiperbólica q-deformada y λ-parametrizada. Estas aproximaciones mejoradas son puntuales y en la norma uniforme. Las redes neuronales prealimentadas con valores complejos relacionadas tienen una capa oculta.

Keywords and Phrases: q-deformed and λ-parametrized hyperbolic tangent, complex valued symmetrized multivariate neural network approximation, complex valued multivariate quasi-interpolation operator, modulus of continuity, trigonometric and hyperbolic enhanced approximation.

2020 AMS Mathematics Subject Classification: 41A17, 41A25, 41A99, 42A10.

1 Introduction

The author in [2] and [1], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining "bell-shaped" and "squashing" functions for these operators are assumed to be of compact support.

Again the author inspired by [10], continued his studies on neural network approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3], by treating both the univariate and multivariate cases.

The brain asymmetry has been observed in animals and humans in terms of structure, function and behaviour. This lateralization is thought to reflect evolutionary, hereditary, develop mental, experiential and pathological factors. Therefore it is natural to consider for our study deformed neural network activation functions and operators. So this article is a specific study under this philosophy of approaching reality as close as possible.

Consequently, the author here performs multivariate symmetrized q-deformed and λ-parametrized hyperbolic tangent function activated high order multivariate neural network approximations to continuous functions over boxes with complex values. All convergences are with rates expressed via the moduli of continuity of the involved functions high order partial derivatives, deriving by very tight multivariate Jackson type inequalities.

The basics of our higher order approximations here are some newly discovered by the author trigonometric and hyperbolic type multivariate Taylor's formulae.

Our boxes are not necessarily symmetric to the origin. The applied symmetrization technique and the newly introduced related multivariate operators cut in half the feed to neural networks, thus enhancing immensely their convergence speed to the unit operator.

A multilayer feed-forward neural network can be defined as follows (with $m \in \mathbb{N}$ hidden layers):

Let $x \in \mathbb{R}^s$; $s \in \mathbb{N}$, where $x = (x_1, \ldots, x_s)$; $\alpha_j, c_j \in \mathbb{R}^s$; $b_j \in \mathbb{R}$, with $0 \leq j \leq n$, $n \in \mathbb{N}$.

Here $\langle \alpha_j \cdot x \rangle$ is the inner product, thus $\sigma(\langle \alpha_j \cdot x \rangle + b_j) \in \mathbb{R}$; and $N_n(x) \in \mathbb{R}^s$, by $c_j \in \mathbb{R}^s$, as it is coming from $N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle \alpha_j \cdot x \rangle + b_j)$. We define:

$$N_n^{(2)}(x) = \sum_{j=0}^{n} c_j \sigma(\langle \alpha_j \cdot N_n(x) \rangle + b_j) = \sum_{j=0}^{n} c_j \sigma\left(\left\langle \alpha_j \cdot \left(\sum_{j=0}^{n} c_j \sigma(\langle \alpha_j \cdot x \rangle + b_j)\right)\right\rangle + b_j\right).$$

Furthermore, we can define

$$N_n^{(3)}(x) = \sum_{j=0}^{n} c_j \sigma \left(\left\langle \alpha_j \cdot N_n^{(2)}(x) \right\rangle + b_j \right).$$

And, in general we define:

$$N_n^{(m)}(x) = \sum_{j=0}^{n} c_j \sigma \left(\left\langle \alpha_j \cdot N_n^{(m-1)}(x) \right\rangle + b_j \right), \text{ for } m \in \mathbb{N}.$$

For more studies in neural networks read [11–17]

2 Basics

Initially we follow [8, pp. 455-460].

Our perturbed hyperbolic tangent activation function here to be used is

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - q e^{-\lambda x}}{e^{\lambda x} + q e^{-\lambda x}}, \quad \lambda, q > 0, \ x \in \mathbb{R}. \tag{1}$$

Above λ is the parameter and q is the deformation coefficient, typically it is $0 < \lambda, q \leq 1$.

For more read of [8, Chapter 18]: "q-Deformed and λ -Parametrized Hyperbolic Tangent based Banach space Valued Ordinary and Fractional Neural Network Approximation".

The chapters 17 and 18 of [8] motivate our current work.

The proposed "symmetrization method" aims to use half data feed to our multivariate neural networks.

We will employ the following density function

$$M_{q.\lambda}(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0, \quad \forall x \in \mathbb{R};\ q,\ \lambda > 0. \tag{2}$$

We have that

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \quad \forall x \in \mathbb{R};\ q,\ \lambda > 0, \tag{3}$$

and

$$M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x), \quad \forall x \in \mathbb{R};\ q,\ \lambda > 0. \tag{4}$$

Adding (3) and (4) we obtain

$$M_{q,\lambda}(-x) + M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x), \tag{5}$$

the key to this work. So that

$$\Phi(x) := \frac{M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x)}{2} \tag{6}$$

is an even function, symmetric with respect to the y-axis.

By [8, (18.18)], we have

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2} \quad \text{and} \quad M_{\frac{1}{q},\lambda}\left(-\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2}, \ \ \lambda > 0. \tag{7}$$

sharing the same maximum at symmetric points.

By [8, Theorem 18.1, p. 458], we have that

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) = 1, \ \ \forall x \in \mathbb{R}, \ \lambda, \ q > 0; \quad \text{and} \quad \sum_{i=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x-i) = 1, \ \ \forall x \in \mathbb{R}, \ \lambda, \ q > 0. \tag{8}$$

Consequently, we derive that

$$\sum_{i=-\infty}^{\infty} \Phi(x-i) = 1, \quad \forall x \in \mathbb{R}. \tag{9}$$

By [8, Theorem 18.2, p. 459], we have that

$$\int_{-\infty}^{\infty} M_{q,\lambda}(x)\, dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x)\, dx = 1, \tag{10}$$

so that

$$\int_{-\infty}^{\infty} \Phi(x)\, dx = 1, \tag{11}$$

therefore Φ is a density function.

By [8, Theorem 18.3, p. 459], we have:

Let $0 < \alpha < 1$ and $n \in \mathbb{N}$, with $n^{1-\alpha} > 2$; $q, \lambda > 0$. Then

$$\sum_{\substack{k=-\infty \\ |nx-k| \geq n^{1-\alpha}}}^{\infty} M_{q,\lambda}(nx-k) < 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} = Te^{-2\lambda n^{(1-\alpha)}}, \tag{12}$$

where $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Similarly, we get that

$$\sum_{k=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(nx-k) < Te^{-2\lambda n^{(1-\alpha)}}. \tag{13}$$

Consequently we obtain that

$$\sum_{\substack{k=-\infty \\ |nx-k|\geq n^{1-\alpha}}}^{\infty} \Phi\left(nx-k\right) < Te^{-2\lambda n^{(1-\alpha)}}, \tag{14}$$

where $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Here $\lceil\cdot\rceil$ denotes the ceiling of the number, and $\lfloor\cdot\rfloor$ its integral part.

We mention

Theorem ([8, Theorem 18.4, p. 459])**.** *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na\rceil \leq \lfloor nb\rfloor$. For $q, \lambda > 0$, we consider $\lambda_q > z_0 > 0$, such that $M_{q,\lambda}(z_0) = M_{q,\lambda}(0)$, and $\lambda_q > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right)} < \max\left\{\frac{1}{M_{q,\lambda}\left(\lambda_q\right)}, \frac{1}{M_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: \Delta\left(q\right). \tag{15}$$

Similarly, we consider $\lambda_{\frac{1}{q}} > z_1 > 0$, such that $M_{\frac{1}{q},\lambda}(z_1) = M_{\frac{1}{q},\lambda}(0)$, and $\lambda_{\frac{1}{q}} > 1$. Thus

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{\frac{1}{q},\lambda}\left(nx-k\right)} < \max\left\{\frac{1}{M_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}, \frac{1}{M_{q,\lambda}\left(\lambda_q\right)}\right\} = \Delta\left(q\right). \tag{16}$$

Hence

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right) > \frac{1}{\Delta\left(q\right)} \tag{17}$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{\frac{1}{q},\lambda}\left(nx-k\right) > \frac{1}{\Delta\left(q\right)}. \tag{18}$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(M_{q,\lambda}\left(nx-k\right) + M_{\frac{1}{q},\lambda}\left(nx-k\right)\right)}{2} > \frac{2}{2\Delta\left(q\right)} = \frac{1}{\Delta\left(q\right)}, \tag{19}$$

so that

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{\left(M_{q,\lambda}\left(nx-k\right) + M_{\frac{1}{q},\lambda}\left(nx-k\right)\right)}{2}} < \Delta\left(q\right), \tag{20}$$

that is

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} < \Delta(q). \tag{21}$$

We have proved

Theorem 2.1. *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q, \lambda > 0$, we consider $\lambda_q > z_0 > 0$, such that $M_{q,\lambda}(z_0) = M_{q,\lambda}(0)$, and $\lambda_q > 1$. Also consider $\lambda_{\frac{1}{q}} > z_1 > 0$, such that $M_{\frac{1}{q},\lambda}(z_1) = M_{\frac{1}{q},\lambda}(0)$, and $\lambda_{\frac{1}{q}} > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} < \Delta(q). \tag{22}$$

We make

Remark 2.2. *I) By [8, Remark 18.5, p. 460], we have*

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx_1 - k) \neq 1, \text{ for some } x_1 \in [a,b], \tag{23}$$

and

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{\frac{1}{q},\lambda}(nx_2 - k) \neq 1, \text{ for some } x_2 \in [a,b]. \tag{24}$$

Therefore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{\left(M_{q,\lambda}(nx_1 - k) + M_{\frac{1}{q},\lambda}(nx_2 - k)\right)}{2} \neq 1. \tag{25}$$

Hence

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{\left(M_{q,\lambda}(nx_1 - k) + M_{\frac{1}{q},\lambda}(nx_1 - k)\right)}{2} \neq 1, \tag{26}$$

even if

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{\frac{1}{q},\lambda}(nx_1 - k) = 1, \tag{27}$$

because then

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{M_{q,\lambda}(nx_1 - k)}{2} + \frac{1}{2} \neq 1, \tag{28}$$

equivalently

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{M_{q,\lambda}(nx_1-k)}{2}\neq\frac{1}{2}, \tag{29}$$

true by

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx_1-k)\neq 1. \tag{30}$$

II) *Let* $[a,b]\subset\mathbb{R}$. *For large* n *we always have* $\lceil na\rceil\leq\lfloor nb\rfloor$. *Also* $a\leq\frac{k}{n}\leq b$, *iff* $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$. *So in general it holds*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\leq 1. \tag{31}$$

Next, we move on to the multivariate case, see [8, Chapter 17, pp. 419-452], as a model of action.

We make

Remark 2.3. *We introduce*

$$Z(x_1,\ldots,x_N):=Z(x):=\prod_{i=1}^{N}\Phi(x_i)=\frac{1}{2^N}\prod_{i=1}^{N}\left(M_{q,\lambda}+M_{\frac{1}{q},\lambda}\right)(x_i), \tag{32}$$

$x=(x_1,\ldots,x_N)\in\mathbb{R}^N$; $\lambda,q>0$, $N\in\mathbb{N}$.

Properties:

(i)

$$Z(x)>0,\quad\forall\ x\in\mathbb{R}^N, \tag{33}$$

(ii)

$$\sum_{k=-\infty}^{\infty}Z(x-k):=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}Z(x_1-k_1,\ldots,x_N-k_N)=1, \tag{34}$$

$k:=(k_1,\ldots,k_n)\in\mathbb{Z}^N$, $\forall\ x\in\mathbb{R}^N$, *hence*

(iii)

$$\sum_{k=-\infty}^{\infty}Z(nx-k)=1, \tag{35}$$

$\forall\ x\in\mathbb{R}^N$, $n\in\mathbb{N}$,

(iv)

$$\int_{\mathbb{R}^N}Z(x)\ dx=1, \tag{36}$$

that is Z *is a multivariate density function.*

Here denote:

$\|x\|_\infty := \max\{|x_1|, \dots, |x_N|\}$, $x \in \mathbb{R}^N$, $\infty := (\infty, \dots, \infty)$, $-\infty := (-\infty, \dots, -\infty)$, $\lceil na \rceil := (\lceil na_1 \rceil, \dots, \lceil na_N \rceil)$, $\lfloor nb \rfloor := (\lfloor nb_1 \rfloor, \dots, \lfloor nb_N \rfloor)$, $a := (a_1, \dots, a_N)$, $b := (b_1, \dots, b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \Phi(nx_i - k_i) \right) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi(nx_i - k_i) \right). \tag{37}$$

(v) We derive that

$$\sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k) < Te^{-2\lambda n^{(1-\beta)}}, \quad \text{where } 0<\beta<1, \tag{38}$$

with $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^{N} [a_i, b_i]$.

(vi) It holds

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} < (\Delta(q))^N, \tag{39}$$

$\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $n \in \mathbb{N}$.

It is clear that

(vii)

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx-k) < Te^{-2\lambda n^{(1-\beta)}}, \tag{40}$$

where $0<\beta<1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.

Furthermore, it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \neq 1, \tag{41}$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

Here $\left(X, \|\cdot\|_\gamma\right)$ *is a Banach space.*

Let $f \in C\left(\prod_{i=1}^{N} [a_i, b_i], X \right)$, $x = (x_1, \dots, x_N) \in \prod_{i=1}^{N} [a_i, b_i]$, $n \in \mathbb{N} : \lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \dots, N$.

We introduce and define the following multivariate linear normalized symmetrized neural network operator, let $x := (x_1, \ldots, x_N) \in \prod_{i=1}^{N} [a_i, b_i]$*:*

$$\theta_n^s (f, x_1, \ldots, x_N) := \theta_n^s (f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} =$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \left(M_{q,\lambda}(nx_i - k_i) + M_{\frac{1}{q},\lambda}(nx_i - k_i)\right)\right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \left(M_{q,\lambda}(nx_i - k_i) + M_{\frac{1}{q},\lambda}(nx_i - k_i)\right)\right)}. \tag{42}$$

For large enough $n \in \mathbb{N}$*, we always obtain* $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$*,* $i = 1, \ldots, N$*. Also* $a_i \leq \frac{k_i}{n} \leq b_i$*, iff* $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$*,* $i = 1, \ldots, N$*.*

When $g \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$ *we define the companion operator*

$$\widetilde{\theta}_n^s (g, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}. \tag{43}$$

Clearly, $\widetilde{\theta}_n^s$ *is a positive linear operator. We have that*

$$\widetilde{\theta}_n^s (1, x) = 1, \quad \forall\, x \in \prod_{i=1}^{N} [a_i, b_i].$$

Notice that $\theta_n^s (f) \in C\left(\prod_{i=1}^{N} [a_i, b_i], X\right)$ *and* $\widetilde{\theta}_n^s (g) \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

Furthermore, it holds

$$\|\theta_n^s (f, x)\|_\gamma \leq \widetilde{\theta}_n^s \left(\|f\|_\gamma, x\right), \quad \forall \in \prod_{i=1}^{N} [a_i, b_i] \tag{44}$$

and

$$\theta_n^s (cg, x) = c\widetilde{\theta}_n^s (g, x), \quad \forall\, x \in \prod_{i=1}^{N} [a_i, b_i], \tag{45}$$

and

$$\theta_n^s(c) = c, \quad \text{any } c \in X.$$

We call

$$^*\theta_n^s(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx-k). \tag{46}$$

Definition 2.4 ([6, p. 274])**.** *Let M be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$, and $\left(X, \|\cdot\|_\gamma\right)$ be a Banach space. Let $f \in C(M,X)$. We define the first modulus of continuity of f as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in M \\ \|x-y\|_p \le \delta}} \|f(x)-f(y)\|_\gamma, \quad 0 < \delta \le diam(M). \tag{47}$$

If $\delta > diam(M)$, then

$$\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{48}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta > 0$. For $f \in C_B(M,X)$ (continuous and bounded functions) $\omega_1(f,\delta)$ is defined similarly.

Lemma 2.5 ([6, p. 274])**.** *We have $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C(M,X)$, where M is a convex compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$.*

In this study we work only for the case of $p = \infty$.

Clearly we have also: $f \in C_U\left(\mathbb{R}^N, X\right)$ (uniformly continuous functions), iff $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (47). The space $C_B\left(\mathbb{R}^N, X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

Let now $f \in C^m\left(\prod_{i=1}^N [a_i,b_i]\right)$, $m, N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1,\dots,\alpha_N)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1,\dots,N$, and $|\alpha| := \sum_{i=1}^N \alpha_i = l$, where $l = 0,1,\dots,m$. We write also $f_\alpha := \frac{\partial^n f}{\partial x^n}$ and we say it is of order l.

We denote

$$\omega_{1,m}^{\max}(f_\alpha,h) := \max_{\alpha:\,|\alpha|=m} \omega_1(f_\alpha,h). \tag{49}$$

Call also

$$\|f_\alpha\|_{\infty,m}^{\max} := \max_{|\alpha|=m} \left\{\|f_\alpha\|_\infty\right\}, \tag{50}$$

where $\|\cdot\|_\infty$ is the supremum norm.

From now on we use $X = (\mathbb{C}, |\cdot|)$ the complex numbers, which is a Banach space.

(II) Multivariate New Taylor formulae

We will use

Theorem 2.6 ([7])**.** *Let $f \in C^2([c,d],\mathbb{C})$, where $a, x \in [c,d]$. Then*

$$f(x) - f(a) = f'(a)\sin(x-a) + 2f''(a)\sin^2\left(\frac{x-a}{2}\right) + \int_a^x \left[\left(f''(t) + f(t)\right) - \left(f''(a) + f(a)\right)\right]\sin(x-t)\,dt. \quad (51)$$

We make

Remark 2.7. *Let now Q be an open convex subset of $\mathbb{R}^k$, $k \geq 2$; $z = (z_1,\dots,z_k)$, $x_0 := (x_{01},\dots,x_{0k}) \in Q$. We consider $f \in C^2(Q,\mathbb{C})$ each second order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1,\dots,\alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1,\dots,k$ and $|\alpha| := \sum_{i=1}^k \alpha_i = 2$. We consider $g_z(t) := f(x_0 + t(z-x_0))$, $0 \leq t \leq 1$. Clearly $x_0 + t(z-x_0) \in Q$. Then*

$$g_z(0) = f(x_0), \quad g_z(1) = f(z),$$

$$g_z'(t) = \sum_{i=1}^k (z_i - x_{0i})\frac{\partial f}{\partial x_i}(x_{01} + t(z_1 - x_{01}),\dots,x_{0k} + t(z_k - x_{0k})), \quad (52)$$

$$g_z'(0) = \sum_{i=1}^k (z_i - x_{0i})\frac{\partial f}{\partial x_i}(x_{01},\dots,x_{0k}),$$

and

$$g_z''(t) = \left[\left(\sum_{i=1}^k (z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^2 f\right](x_{01} + t(z_1 - x_{01}),\dots,x_{0k} + t(z_k - x_{0k})), \quad (53)$$

$$g_z''(0) = \left[\left(\sum_{i=1}^k (z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^2 f\right](x_{01},\dots,x_{0k}).$$

Notice above the second order partials commute.

Clearly $g_z \in C^2([0,1],\mathbb{C})$, and by Theorem 2.6 we obtain

$$f(z_1,\dots,z_k) - f(x_{01},\dots,x_{0k}) = g_z(1) - g_z(0) =$$

$$g_z'(0)\sin(1) + 2g_z''(0)\sin 2\left(\frac{1}{2}\right) + \int_0^1 \left[\left(g_z''(t)\,g_z(t)\right) - \left(g_z''(0) + g_z(0)\right)\right]\sin(1-t)\,dt. \quad (54)$$

We also mention

Theorem 2.8 ([7])**.** *Let $f \in C^2([c,d], \mathbb{C})$, where $a, x \in [c,d]$. Then*

$$f(x) - f(a) = f'(a)\sinh(x-a) + 2f''(a)\sinh^2\left(\frac{x-a}{2}\right) + \int_a^x \Big[\big(f''(t) - f(t)\big) - \big(f''(a) - f(a)\big)\Big]\sinh(x-t)\, dt. \tag{55}$$

We make

Remark 2.9. *Consequently, we get that*

$$f(z_1, \ldots, z_k) - f(x_{01}, \ldots, x_{0k}) = g_z(1) - g_z(0) = g_z'(0)\sinh(1) + 2g_z''(0)\sinh^2\left(\frac{1}{2}\right) + \int_0^1 \Big[\big(g_z''(t) - g_z(t)\big) - \big(g_z''(0) - g_z(0)\big)\Big]\sinh(1-t)\, dt. \tag{56}$$

We make

Remark 2.10. *Let $f \in C^2\left(\prod_{i=1}^N [a_i, b_i], \mathbb{C}\right)$, $N \in \mathbb{N}$.*

Clearly, the mixed partials commute.

Here $\frac{k}{n} := \left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right)$, and $x := (x_1, \ldots, x_N)$, with $\frac{k}{n}, x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, then (by (54), where $g_{\frac{k}{n}}(t) := f\left(x + t\left(\frac{k}{n} - x\right)\right)$, $0 \le t \le 1$) we have

$$\begin{aligned} f\left(\frac{k}{n}\right) - f(x) = {} & \left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial f}{\partial x_i}(x)\right)\sin(1) + 2\left\{\left[\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x)\right\}\sin^2\left(\frac{1}{2}\right) \\ & + \int_0^1 \left\{\left\{\left[\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right]\left(x + t\left(\frac{k}{n} - x\right)\right) + f\left(x + t\left(\frac{k}{n} - x\right)\right)\right\}\right. \\ & \left. - \left\{\left[\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x) + f(x)\right\}\right\}\sin(1-t)\, dt. \end{aligned} \tag{57}$$

Denote the remainder

$$\begin{aligned} R := \int_0^1 & \left\{\left\{\left[\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right]\left(x + t\left(\frac{k}{n} - x\right)\right) + f\left(x + t\left(\frac{k}{n} - x\right)\right)\right\}\right. \\ & \left. - \left\{\left[\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x) + f(x)\right\}\right\}\sin(1-t)\, dt \end{aligned} \tag{58}$$

$$= \int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\dots,N,\, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{k_i}{n} - x_i \right)^{\alpha_i} \right) \left[f_\alpha \left(x + t \left(\frac{k}{n} - x \right) \right) - f_\alpha (x) \right] \right.$$

$$\left. + \left(f \left(x + t \left(\frac{k}{n} - x \right) \right) - f(x) \right) \right\} \sin(1-t)\, dt.$$

Therefore it holds

$$|R| \leq \int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\dots,N,\, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) \left| f_\alpha \left(x + t \left(\frac{k}{n} - x \right) \right) - f_\alpha(x) \right| \right. \tag{59}$$

$$\left. + \left| f \left(x + t \left(\frac{k}{n} - x \right) \right) - f(x) \right| \right\} |\sin(1-t)|\, dt$$

$$\leq \int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\dots,N, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) \omega_1 \left(f_\alpha, t \left\| \frac{k}{n} - x \right\|_\infty \right) \right.$$

$$\left. + \omega_1 \left(f, t \left\| \frac{k}{n} - x \right\|_\infty \right) \right\} |\sin(1-t)|\, dt \leq (*)$$

Notice here that $(0 < \beta < 1)$

$$\left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \iff \left| \frac{k_i}{n} - x_i \right| \leq \frac{1}{n^\beta}, \quad i = 1, \dots, N. \tag{60}$$

We further see that:

$$(*) \leq \left\{ \omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) \left(\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N,\, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \frac{1}{n^{\beta\alpha_i}} \right) \right) + \omega_1\left(f, \frac{1}{n^\beta}\right) \right\} \tag{61}$$

$$\int_0^1 |\sin(1-t)|\, dt \left[\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) \left(\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N,\, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \right) \frac{1}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right) \right]$$

$$(1-\cos(1)) = (1-\cos(1)) \left\{ \frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right) \right\}.$$

We have proved that

$$|R| \leq (1-\cos(1)) \left\{ \frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right) \right\}, \tag{62}$$

given that $\left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}$.

We notice also that

$$|R| \leq \int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) 2\|f_\alpha\|_\infty + 2\|f\|_\infty \right\} |\sin(1-t)|\, dt \tag{63}$$

$$\leq \left\{ \left(\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\, \alpha_i \in \mathbb{Z}_+ \\ i=1,\ldots,N, |\alpha|:=\sum_{i=1}^N \alpha_i = 2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \right) 2\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} + 2\|f\|_\infty \right\} \left(\int_0^1 |\sin(1-t)|\, dt \right)$$

$$= \left(2\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + 2\|f\|_\infty \right) (1-\cos(1)),$$

where $a := (a_1, \ldots, a_N)$, $b = (b_1, \ldots, b_N)$.

We have proved that

$$|R| \leq \left(2\|b-a\|_{\infty}^{2}\|f_{\alpha}\|_{\infty,2}^{\max}N^{2}+2\|f\|_{\infty}\right)(1-\cos(1)) =: \rho. \tag{64}$$

3 Main results

We start with symmetrized and perturbed trigonometric approximation by using the smoothness of f.

Theorem 3.1. *Let* $f \in C^{2}\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$, $0<\beta<1$; $n, N \in \mathbb{N}$, $n^{1-\beta}>2$; $x, x_0 \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $a := (a_1,\ldots,a_N)$, $b := (b_1,\ldots,b_N)$. *Then:*

(i)

$$\Bigg|\theta_n^s(f,x)-f(x)-\left(\sum_{i=1}^{N}\frac{\partial f(x)}{\partial x_i}\theta_n^s((\cdot-x_i),x)\right)\sin(1)- \tag{65}$$

$$4\left\{\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}} f_{\alpha}(x)\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\theta_n^s\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right\}\sin 2\left(\frac{1}{2}\right)\Bigg|$$

$$\leq (\Delta(q))^{N}\left\{\left[(1-\cos(1))\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\right.$$

$$\left.\left[2\|b-a\|_{\infty}^{2}\|f_{\alpha}\|_{\infty,2}^{\max}N^{2}+2\|f\|_{\infty}\right](1-\cos(1))Te^{-2\lambda n^{(1-\beta)}}\right\},$$

(ii) assume that $\frac{\partial f(x_0)}{\partial x_i}=0$, $i=1,\ldots,N$, *and* $f_{\alpha}(x_0)=0$, $\alpha : |\alpha|=2$, *we have that*

$$|\theta_n^s(f,x)-f(x)| \leq (\Delta(q))^{N}\left\{\left[(1-\cos(1))\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\right.$$

$$\left.\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)Te^{-2\lambda n^{(1-\beta)}}\right\}, \quad (66)$$

and

(iii)

$$\left|\theta_{n}^{s}\left(f,x\right)-f\left(x\right)\right|\leq\left(\Delta\left(q\right)\right)^{N}\left\{\left\{\left\{\sum_{i=1}^{N}\left|\frac{\partial f\left(x\right)}{\partial x_{i}}\right|\left\{\frac{1}{n^{\beta}}+\left(b_{i}-a_{i}\right)Te^{-2\lambda n^{(1-\beta)}}\right\}\right\}\sin\left(1\right)+\right.\right.\right.$$

$$\left.4\left\{\sum_{\alpha:|\alpha|=2}\left|f_{\alpha}\left(x\right)\right|\left(\frac{1}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}\left(b_{i}-a_{i}\right)^{\alpha_{i}}\right)Te^{-2\lambda n^{(1-\beta)}}\right]\right\}\sin^{2}\left(\frac{1}{2}\right)\right\}+$$

$$\left\{\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]\right. \quad (67)$$

$$\left.\left.+\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)Te^{-2\lambda n^{(1-\beta)}}\right\}\right\},$$

and

(iv)

$$\left\|\theta_{n}^{s}\left(f\right)-f\right\|_{\infty}\leq\left(\Delta\left(q\right)\right)^{N}\left\{\left\{\left\{\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{\infty}\left\{\frac{1}{n^{\beta}}+\left(b_{i}-a_{i}\right)Te^{-2\lambda n^{(1-\beta)}}\right\}\right\}\sin\left(1\right)+\right.\right.\right.$$

$$\left.4\left\{\sum_{\alpha:|\alpha|=2}\left\|f_{\alpha}\right\|_{\infty}\left(\frac{1}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}\left(b_{i}-a_{i}\right)^{\alpha_{i}}\right)Te^{-2\lambda n^{(1-\beta)}}\right]\right\}\sin^{2}\left(\frac{1}{2}\right)\right\} \quad (68)$$

$$+\left\{\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]\right.$$

$$+\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)Te^{-2\lambda n^{(1-\beta)}}\Bigg\}\Bigg\}=:\xi_{n}\left(f\right).$$

We observe that $\theta_n^s \to I$ (unit operator), as $n \to \infty$, pointwise and uniformly.

Proof. Here R is as in (58). We see that

$$U_{n}:=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z\left(nx-k\right)R=\sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}Z\left(nx-k\right)R+\sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}Z\left(nx-k\right)R. \tag{69}$$

Therefore

$$\left|U_{n}\right|\leq\left(\sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}Z\left(nx-k\right)\right) \tag{70}$$
$$\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\rho Te^{-2\lambda n^{(1-\beta)}}$$
$$\leq\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\rho Te^{-2\lambda n^{(1-\beta)}}.$$

We have established that

$$\left|U_{n}\right|\leq\left[\left(1-\cos\left(1\right)\right)\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]$$
$$+\left[2\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+2\left\|f\right\|_{\infty}\right]\left(1-\cos\left(1\right)\right)Te^{-2\lambda n^{(1-\beta)}}. \tag{71}$$

By (57) we observe that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\right) \tag{72}$$

$$= \left(\sum_{i=1}^{N}\left(\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\left(\frac{k_i}{n}-x_i\right)\right)\frac{\partial f}{\partial x_i}(x)\right)\right)\sin(1)$$

$$2\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\,\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,\,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}} f_\alpha(x)\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\left(\prod_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)\right)\right\}\sin^2\left(\frac{1}{2}\right)+U_n.$$

The last says

$$^{*}\theta_n^s(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\right) - \left(\sum_{i=1}^{N}\frac{\partial f(x)}{\partial x_i}\,{}^{*}\theta_n^s((\cdot-x_i),x)\right)\sin(1) -$$

$$2\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\,\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,\,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}} f_\alpha(x)\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\,{}^{*}\theta_n^s\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right\}\sin^2\left(\frac{1}{2}\right) = U_n. \tag{73}$$

We notice that

$$\left|{}^{*}\theta_n^s((\cdot-x_i),x)\right| \leq^{*} \theta_n^s(|\cdot-x_i|,x) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left|\frac{k_i}{n}-x_i\right| Z(nx-k) \tag{74}$$

$$= \sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\left|\frac{k_i}{n}-x_i\right| Z(nx-k) + \sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\left|\frac{k_i}{n}-x_i\right| Z(nx-k)$$

$$\leq \frac{1}{n^\beta} + (b_i-a_i)\sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\lfloor nb\rfloor} Z(nx-k) \leq \frac{1}{n^\beta} + (b_i-a_i)\,Te^{-2\lambda n^{(1-\beta)}}.$$

We have proved that

$$\left|{}^{*}\theta_n^s((\cdot-x_i),x)\right| \leq \frac{1}{n^\beta} + (b_i-a_i)\,Te^{-2\lambda n^{(1-\beta)}}, \tag{75}$$

$i=1,\dots,N$.

Next we see that

$$\left|{}^{*}\theta_{n}^{s}\left(\prod_{i=1}^{N}(\cdot-x_{i})^{\alpha_{i}},x\right)\right|\leq{}^{*}\theta_{n}^{s}\left(\prod_{i=1}^{N}\left|\cdot-x_{i}\right|^{\alpha_{i}},x\right)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N}\left|\frac{k_{i}}{n}-x_{i}\right|^{\alpha_{i}}\right)Z(nx-k) \quad (76)$$

$$=\sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N}\left|\frac{k_{i}}{n}-x_{i}\right|^{\alpha_{i}}\right)Z(nx-k)+\sum_{\substack{k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N}\left|\frac{k_{i}}{n}-x_{i}\right|^{\alpha_{i}}\right)Z(nx-k)$$

$$\leq\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}(b_{i}-a_{i})^{\alpha_{i}}\right)Te^{-2\lambda n^{(1-\beta)}}.$$

We have proved that

$$\left|{}^{*}\theta_{n}^{s}\left(\prod_{i=1}^{N}(\cdot-x_{i})^{\alpha_{i}},x\right)\right|\leq\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}(b_{i}-a_{i})^{\alpha_{i}}\right)Te^{-2\lambda n^{(1-\beta)}}. \quad (77)$$

At last we observe that

$$\Bigg|\theta_{n}^{s}(f,x)-f(x)-\left(\sum_{=1}^{N}\frac{\partial f(x)}{\partial x_{i}}\theta_{n}^{s}((\cdot-x_{i}),x)\right)\sin(1)- \quad (78)$$

$$4\left\{\sum_{\substack{\alpha:=(\alpha_{1},\ldots,\alpha_{N}),\alpha_{i}\in\mathbb{Z}_{+}\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_{i}=2}}f_{\alpha}(x)\left(\frac{1}{\prod_{i=1}^{N}\alpha_{i}!}\right)\theta_{n}^{s}\left(\prod_{i=1}^{N}(\cdot-x_{i})^{\alpha_{i}},x\right)\right\}\sin^{2}\left(\frac{1}{2}\right)\Bigg|$$

$$\leq(\Delta(q))^{N}|U_{n}|=(\Delta(q))^{N}\Bigg|{}^{*}\theta_{n}^{s}(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)\right)-$$

$$\left(\sum_{i=1}^{N}\frac{\partial f(x)}{\partial x_{i}}\,{}^{*}\theta_{n}^{s}((\cdot-x_{i}),x)\right)\sin(1)-$$

$$4\left\{\sum_{\substack{\alpha:=(\alpha_{1},\ldots,\alpha_{N}),\alpha_{i}\in\mathbb{Z}_{+}\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_{i}=2}}f_{\alpha}(x)\left(\frac{1}{\prod_{i=1}^{N}\alpha_{i}!}\right)\,{}^{*}\theta_{n}^{s}\left(\prod_{i=1}^{N}(\cdot-x_{i})^{\alpha_{i}},x\right)\right\}\sin^{2}\left(\frac{1}{2}\right)\Bigg|.$$

Putting all of the above together we prove the theorem. ∎

We continue with the hyperbolic symmetrized and perturbed approximation.

Theorem 3.2. *Let* $f \in C^2\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$, $0<\beta<1$; $n,N\in\mathbb{N}$, $n^{1-\beta}>2$; $x,x_0\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $a:=(a_1,\dots,a_N)$, $b:=(b_1,\dots,b_N)$. *Then*

(i)

$$\left|\theta_n^s(f,x)-f(x)-\left(\sum_{=1}^{N}\frac{\partial f(x)}{\partial x_i}\theta_n^s((\cdot-x_i),x)\right)\sinh(1)-\right.$$

$$\left.4\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\,\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,\,|\alpha|:=\sum limits_{i=1}^{N}\alpha_i=2}} f_\alpha(x)\left(dfrac1\prod_{i=1}^{N}\alpha_i!\right)\theta_n^s\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right\}\sinh^2\left(\frac{1}{2}\right)\right|$$

$$\leq(\Delta(q))^N\cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]+\right. \quad (79)$$

$$\left.\left[\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+\|f\|_\infty\right]Te^{-2\lambda n^{(1-\beta)}}\right\},$$

(ii) assume that $\frac{\partial f(x_0)}{\partial x_i}=0$, $i=1,\dots,N$, *and* $f_\alpha(x_0)=0$, $\alpha:|\alpha|=2$, *we have that*

$$|\theta_n^s(f,x)-f(x)|\leq(\Delta(q))^N(\cosh(1))\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]+\right.$$

$$\left.\left[\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+\|f\|_\infty\right]Te^{-2\lambda n^{(1-\beta)}}\right\}, \quad (80)$$

(iii)

$$|\theta_n^s(f,x)-f(x)|\leq(\Delta(q))^N\left\{\left\{\left\{\sum_{i=1}^{N}\left|\frac{\partial f(x)}{\partial x_i}\right|\left\{\frac{1}{n^\beta}+(b_i-a_i)Te^{-2\lambda n^{(1-\beta)}}\right\}\right\}\sinh(1)+\right.\right.$$

$$4\left\{\sum_{\alpha:|\alpha|=2}\left|f_{\alpha}\left(x\right)\right|\left(\frac{1}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}\left(b_{i}-a_{i}\right)^{\alpha_{i}}\right)Te^{-2\lambda n^{(1-\beta)}}\right]\right\}\sinh^{2}\left(\frac{1}{2}\right)\right\}$$

$$+\cosh\left(1\right)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\right. \tag{81}$$

$$\left.\left[\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+\left\|f\right\|_{\infty}\right]Te^{-2\lambda n^{(1-\beta)}}\right\}\right\},$$

and

(iv)

$$\left\|\theta_{n}^{s}\left(f\right)-f\right\|_{\infty}\leq\left(\Delta\left(q\right)\right)^{N}\left\{\left\{\left\{\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{\infty}\left\{\frac{1}{n^{\beta}}+\left(b_{i}-a_{i}\right)Te^{-2\lambda n^{(1-\beta)}}\right\}\right\}\sinh\left(1\right)+\right.\right.\right.$$

$$4\left\{\sum_{\alpha:|\alpha|=2}\left\|f_{\alpha}\right\|_{\infty}\left(\frac{1}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}\left(b_{i}-a_{i}\right)^{\alpha_{i}}\right)Te^{-2\lambda n^{(1-\beta)}}\right]\right\}\sinh^{2}\left(\frac{1}{2}\right)\right\}$$

$$+\cosh\left(1\right)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\right.$$

$$\left.\left[\left\|b-a\right\|_{\infty}^{2}\left\|f_{\alpha}\right\|_{\infty,2}^{\max}N^{2}+\left\|f\right\|_{\infty}\right]Te^{-2\lambda n^{(1-\beta)}}\right\}=:\psi_{n}\left(f\right). \tag{82}$$

We observe that $\theta_n^s \to I$ (unit operator), as $n \to \infty$, pointwise and uniformly.

Proof. As it is similar to the proof of [9, Theorem 5.8, pp. 175-179], it is omitted. □

We give

Remark 3.3. *By (42) we get that $\|\theta_n^s(f)\|_\infty \leq \|f\|_\infty < \infty$, and $\theta_n^s(f) \in C\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$, given that $f \in C\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$.*

Clearly then

$$\left\|\theta_n^{s2}(f)\right\|_\infty = \|\theta_n^s(\theta_n^s(f))\|_\infty \leq \|\theta_n^s(f)\|_\infty \leq \|f\|_\infty, \tag{83}$$

etc.

Therefore we get

$$\left\|(\theta_n^s)^k(f)\right\|_\infty \leq \|f\|_\infty, \quad \forall\, k \in \mathbb{N}, \tag{84}$$

the contraction property.

Also we see that

$$\left\|(\theta_n^s)^k(f)\right\|_\infty \leq \left\|(\theta_n^s)^{k-1}(f)\right\|_\infty \leq \cdots \leq \|\theta_n^s(f)\|_\infty \leq \|f\|_\infty. \tag{85}$$

Also $\theta_n^s(1) = 1$, $(\theta_n^s)^k(1) = 1$, $\forall\, k \in \mathbb{N}$.

Following [5, 18.14, pp. 401-402], similarly we obtain that

$$\|(\theta_n^s)^r f - f\|_\infty \leq r\|\theta_n^s(f) - f\|_\infty, \quad r \in \mathbb{N}. \tag{86}$$

We give

Theorem 3.4. *All as in Theorems 3.1, 3.2. Then*

(i)

$$\|(\theta_n^s)^r f - f\|_\infty \leq r\xi_n(f), \tag{87}$$

where $\xi_n(f)$ *as in* (68).

(ii)

$$\|(\theta_n^s)^r f - f\|_\infty \leq r\psi_n(f), \tag{88}$$

where $\psi_n(f)$ *as in* (82).

So that the speed of convergence to the unit operator of $(\theta_n^s)^r$ *is not worse than of* θ_n^s*, see also [4].*

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Rings in which every ideal disjoint with S is S-almost prime

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ABSTRACT

Let R be a commutative ring with identity and S a multiplicative subset of R. In this study, we introduce the concept of rings in which every ideal disjoint with S is S-almost prime. We investigate the possible transfer of the above ring property in the quotient rings, localizations, direct products, trivial ring extensions, and amalgamation algebra.

RESUMEN

Sea R un anillo conmutativo con identidad y S un conjunto multiplicativo de R. En este estudio, introducimos el concepto de anillos en los cuales todo ideal disjunto con S es S-casi primo. Investigamos la posible transferencia de la propiedad de anillos anterior en anillos cociente, localizaciones, productos directos, extensiones triviales de anillos y el álgebra de amalgamación.

Keywords and Phrases: S-almost prime ideal, S-prime ideal, AP-rings.

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1 Introduction

Throughout this paper, all rings are assumed to be commutative with unity, all modules are unital. By $I(R)$ (and $I(R)^*$), we denote the set of (proper) ideals of a ring R. It is well-known that $Q \in I(R)^*$ is called prime if whenever $x, y \in R$ with $xy \in Q$ then either $x \in Q$ or $y \in Q$. Equivalently, if whenever $IJ \subseteq Q$ for some ideals I, J of R, then either $I \subseteq Q$ or $J \subseteq Q$. Since the notion of prime ideals plays an important role in commutative ring theory, several generalizations of the concept of prime ideals have been studied in the literature, for example: almost prime, strongly prime, weakly prime, S-prime and S-almost prime ideals. Recall from [8] that $Q \in I(R)^*$ is said to be weakly prime if for $x, y \in R$ with $0 \neq xy \in Q$, then either $x \in Q$ or $y \in Q$. Clearly every prime ideal is weakly prime, but the converse is not true, obviously $\{0\}$ is always weakly prime, but not prime provided that R is a ring which is not an integral domain. For non-trivial examples, refer to [8]. In 2005, Bhatwadekar and Sharma [11] said $Q \in I(R)^*$ where R is an integral domain to be almost prime if for $a, b \in R$ with $ab \in Q - Q^2$, then either $a \in Q$ or $b \in Q$. It is evident that this definition can be applied to any commutative ring R. Consequently, any weakly prime ideal is almost prime. Furthermore, an ideal $Q \in I(R)^*$ is almost prime if and only if Q/Q^2 is a weakly prime in the quotient ring R/Q^2.

Consider a multiplicative set (briefly, m.s) S of a ring R that satisfies $0 \notin S$, $1 \in S$, and $xy \in S$ for all $x, y \in S$. In recent times, the notion of S-extensions of certain ideal structures has assumed considerable significance within the domain of commutative algebra, thus attracting the interest of numerous authors. The concept of (resp. weakly) S-prime ideals has been introduced and thoroughly investigated in (resp. [5]) [16]. An ideal I of a ring R disjoint with an m.s S is said to be (resp. weakly) S-prime ideal if there exists $s \in S$ such that for all $a, b \in R$ if $ab \in I$ (resp. $0 \neq ab \in I$) , then $sa \in I$ or $sb \in I$. (See also, [1, 20–22].)

The primary focus of this study is the definition and study of almost prime ideals in [4]. Let P be an ideal of R disjoint with an m.s S. Then P is an S-almost prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in P - P^2$, then $sa \in P$ or $sb \in P$.

This paper presents a new type of ring of which every ideal disjoint with S is S-almost prime (called S-*AP* ring) as a generalization of ring in which every proper ideal is almost prime (called *AP* ring). If $S \subseteq U(R)$, then the concepts S-AP ring and AP-ring coincide. Nevertheless, these two types of rings are significantly dissimilar in general. Subsequently, an example is presented in order to demonstrate that [4, Theorems 2.20 and 2.21] do not hold in general. The subsequent investigation will address the potential transfer of the ring property that every ideal disjoint with S is S-almost prime in localization, direct product, homomorphic image, trivial ring extensions, and the amalgamation rings along an ideal. This property is employed to construct new and intriguing examples. An example is provided in order to demonstrate that the statement [4, Theorem 2.9] is not generally valid. In order to conclude the relationship between an S-AP ring and its homomorphic images, we discuss the extensions and contractions of an S-almost prime ideals under a homomorphic image. Afterwards, we present the corrected and generalized version of [4, Theorem 2.9] (see Proposition 3.1 and Corollary 3.2, and then Theorem 3.3). We give characterizations for $(S \propto E)$-AP rings regarding the trivial ring extension of a ring R by an R-module E denoted by $R \propto E$ (see Theorem 4.1). Moreover, we investigate the relationship between S-AP rings A and the ideals of the ring $A \bowtie^f J$ the amalgamation of A and B along J with respect to f (see Theorem 4.7 and Corollaries 4.8, 4.9).

2 Main results

To avoid repetition, throughout the article unless otherwise stated, let R be a ring and S a multiplicative set (briefly, m.s) of R. As it is frequently used in this sequel, we should recall from [4] that an ideal $P \in I(R)$ disjoint with S is said to be an S-almost prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in P - P^2$, then $sa \in P$ or $sb \in P$. In this case, P is said to be an S-almost prime ideal associated with s. Now the following is our key definition.

Definition 2.1. *A ring R with an m.s S is called S-AP ring if every ideal disjoint with S is S-almost prime.*

It is interesting to note that the authors in [4] have been studying this class of rings in [4, Theorems 2.20 and 2.21]. However, we show by the following example that those theorems do not hold in general.

Example 2.2. *Let $R = D \times K$ where D is a domain and K is a field, and let the multiplicative subset $S = D^* \times \{1\}$ of R where $D^* = D - \{0\}$. Then the ideals of R which are disjoint with S are just $0 \times K$ and $I \times 0$ where I is an ideal of D. Note that $0 \times K$ is prime and so S-almost prime. Clearly, 0×0 is almost prime and so S-almost prime. Finally, if I is a nonzero ideal of D, then for $a \in I - \{0\}$, we conclude that $(I \times 0 : (a, 1)) = D \times 0$ is a prime ideal of R, and hence, $I \times 0$ is S-prime by [16, Proposition 1]. Thus, $I \times 0$ is also an S-almost prime ideal of R. Thus, R is an S-AP-ring.*

Remark 2.3. (1) *Let S be a multiplicative set of a ring R. If R is an AP ring, then R is an S-AP ring.*

(2) *If $S \subseteq U(R)$, then R is an S-AP ring if and only if R is an AP ring.*

(3) *Let $S_1 \subseteq S_2$ be multiplicative subsets of R, and P an ideal of R disjoint with S_2. Clearly, if P is an S_1-almost prime ideal of R, then P is S_2-almost prime. Nevertheless, it has been demonstrated that the converse is not true in general. Indeed, let $R = \mathbb{Z}[X]$, $S = \{2^n : n \in N\}$ and $T = \{1\} \subseteq U(R)$. Consider the ideal $P = 4X\mathbb{Z}[X]$ of R. From [16, Example 1],*

P is an S-prime ideal of R, and hence P is an S-almost prime ideal of R. Note that P is not a T-almost prime ideal of R as $4X \in P - P^2$ but neither $1 \cdot 4 \in P$ nor $1 \cdot X \in P$.

The converse of Remark 2.3(1) is not true in general since not every almost prime ideal is an S-almost prime ideal by [4, Example 2.3].

Let $R_1, \ldots, R_n$ be rings and $S_1, \ldots, S_n$ be m.ss of rings $R_1, \ldots, R_n$, respectively. Then $S = \prod_{i=1}^{n} S_i$ is an m.s of the ring $R = \prod_{i=1}^{n} R_i$. We study the property of the stability of S-AP rings under direct products.

Theorem 2.4. *Let $S_1, \ldots, S_n$ be m.ss of rings $R_1, \ldots, R_n$, respectively. Put $R = \prod_{i=1}^{n} R_i$ and $S = \prod_{i=1}^{n} S_i$. If R is an S-AP ring, then R_i is an S_i-AP ring for each $1 \leq i \leq n$.*

In order to demonstrate the validity of the aforementioned theorem, it is necessary to verify the following lemma.

Lemma 2.5. *Let R_1 and R_2 be commutative rings, S_1 and S_2 be m.ss of R_1 and R_2, respectively, $R = R_1 \times R_2$ and $S = S_1 \times S_2$. If I and J be proper ideals of R_1 and R_2, respectively, it can thus be concluded that the following statements are true.*

(1) *I is an S_1-almost prime ideal of R_1 if and only if $I \times R_2$ is an S-almost prime ideal of R.*

(2) *J is an S_2-almost prime ideal of R_2 if and only if $R_1 \times J$ is an S-almost prime ideal of R.*

Proof. (1) Assume that I is an S_1-almost prime ideal of R_1. Let $(a, b), (a', b') \in R$ such that $(a, b)(a', b') = (aa', bb') \in I \times R_2 - (I \times R_2)^2$. Then $aa' \in I - I^2$ which implies that there exists $s_1 \in S_1$ such that $s_1 a \in I$ or $s_1 a' \in I$. Put $s = (s_1, 1) \in S$. Hence $s(a, b) \in I \times R_2$ or $s(a', b') \in I \times R_2$, then $I \times R_2$ is an S-almost prime ideal of R.

Conversely, suppose that $I \times R_2$ is an S-almost prime ideal of R, and $a, b \in R_1$ such that $ab \in I - I^2$. Then $(a, 0)(b, 0) \in I \times R_2 - (I \times R_2)^2$ and there exists $(s_1, s_2) \in S$ such that $(s_1, s_2)(a, 0) \in I \times R_2$ or $(s_1, s_2)(b, 0) \in I \times R_2$, hence $s_1 a \in I$ or $s_1 b \in I$. Thus, I is an S_1-almost prime ideal of R_1.

(2) Similar to (1). □

Proof of Theorem 2.4. It is sufficient to demonstrate the claim for $n = 2$

Assume that $R = R_1 \times R_2$ is an S-AP ring. Let P_1 be an ideal of R_1 disjoint with S_1. Then $P_1 \times R_2$ is an S-almost prime ideal of R, and so P_1 is S_1-almost prime by Lemma 2.5. Therefore, R_1 is an S_1-AP ring. Similar to the argument above, we conclude that R_2 is an S_2-AP ring. The rest of the proof is clear by using the mathematical induction on n. □

The following example is given to demonstrate that the converse of Theorem 2.4 is not true in general.

Example 2.6. *Let (R, M) be a local ring which is not a field where $M^2 = 0$ and $S = \{1\}$. Then R is S-AP ring by [6, Theorem 17], but $R \times R$ is not necessarily $S \times S$-AP ring.*

Proof. We only need to show that $M \times M$ is not S-almost prime, let $0 \neq x \in M$ so $(1, x)(x, 1) = (x, x) \in (M \times M) - (M \times M)^2$ but neither $(1, 1)(1, x) \in M \times M$ nor $(1, 1)(x, 1) \in M \times M$, as desired. □

3 Extensions of S-AP rings

In this section, we discuss behavior of S-AP rings under some ring extensions such as ring homomorphism and localization. Moreover, a characterization for S-Noetherian ring is obtained.

We note that the second part of [4, Theorem 2.9] does not hold in general. By the following theorem, we give a generalized and corrected version of [4, Theorem 2.9].

Proposition 3.1. *Let $f : R_1 \to R_2$ be a ring epimorphism, S be an m.s of R_1. Then the following assertions hold.*

(1) *If I is an S-almost prime ideal of R_1 containing $\ker f$, then $f(I)$ is an $f(S)$-almost prime ideal of R_2.*

(2) *If J is an $f(S)$-almost prime ideal of R_2 and $\ker f \subseteq f^{-1}(J)^2$, then $f^{-1}(J)$ is an S-almost prime ideal of R_1.*

Proof. (1) First, we show that $f(S)$ is an m.s of R_2. Clearly, for any $f(s_1), f(s_2) \in f(S)$, $f(s_1)f(s_2) = f(s_1 s_2) \in f(S)$. Now, assume that $0 \in f(S)$. Then $f(s) = 0$ for some $s \in S$. This implies that $s \in \ker f$. Since $\ker f \subseteq I$ and $I \cap S = \emptyset$, we have $\ker f \cap S = \emptyset$, a contradiction. Thus, $0 \notin f(S)$ and $f(S)$ is an m.s of R_2. Assume that $f(s) \in f(I) \cap f(S)$ for some $s \in S$. Then $f(a) = f(s)$ for some $a \in I$ which yields $a - s \in \ker f \subseteq I$, and so $s \in I \cap S$, a contradiction. Hence, $f(I) \cap f(S) = \emptyset$. Suppose that $xy \in f(I) \backslash f(I)^2$ for some $x = f(a)$, $y := f(b)$ in R_2. From $\ker f \subseteq I$, we have $ab \in I$ and clearly $ab \notin I^2$. Since I is an S-almost prime ideal of R_1, there exists $s \in S$ such that $sa \in I$ or $sb \in I$. Thus, there exists $f(s) \in f(S)$ satisfying $f(s)f(a) \in f(I)$ or $f(s)f(b) \in f(I)$, as required.

(2) If $s \in f^{-1}(J) \cap S$, then $f(s) \in J \cap f(S)$, a contradiction. Hence, $f^{-1}(J) \cap S = \emptyset$. Let $f(s) \in f(S)$, an element associated with J. Let $a, b \in R_1$ such that $ab \in f^{-1}(J) \backslash f^{-1}(J)^2$. It is clear that $f(a)f(b) \in J$ and $ab \notin f^{-1}(J)^2$. Now, we show that $f(a)f(b) \notin J^2$. If $f(ab) \in J^2$, then there is $c \in f^{-1}(J^2)$ such that $f(ab) = f(c)$, hence $ab - c \in \ker f \subseteq f^{-1}(J^2)$ and

$ab \in f^{-1}(J^2) = f^{-1}(J)^2$, which is a contradiction. Hence, $f(a)f(b) \in J \backslash J^2$, and therefore, either $f(s)f(a) \in J$, hence $as \in f^{-1}(J)$, or $f(s)f(b) \in J$, hence $bs \in f^{-1}(J)$, and $f^{-1}(J) \subset R_1$ is S-almost prime. ☐

Let R be a ring, $S \subseteq R$ be m.s and I an ideal of R such that $I \cap S = \emptyset$. Let $s \in S$. By $\overline{s}$, we denote the class of s in R/I and set $\overline{S} = \{\overline{s} : s \in S\}$. It is clear that $\overline{S}$ is a m.s of R/I.

Now, we are ready to give the following relationship between S-almost prime ideals of a ring and those of their quotient rings.

Corollary 3.2 (The corrected version of [4, Theorem 2.9]). *Let S be an m.s of a ring R and P, $I \in I(R)$ such that $P \subseteq I^2$ and $S \cap P = \emptyset$. Then $I \in I(R)$ is S-almost prime if and only if $I/P \in I(R/P)$ is $\overline{S}$-almost prime.*

In view of Corollary 3.2, we conclude the following result.

Theorem 3.3. *Let S be an m.s of a ring R and $P \in I(R)$ such that $S \cap P = \emptyset$. Then R is a ring of which every ideal I satisfying $P \subseteq I^2$ is S-almost prime if and only if R/P is a $\overline{S}$-AP ring.*

Proof. The claim is clear by Corollary 3.2. ☐

Let R and T be rings, let S a multiplicative subset of R and $f : R \to T$ a ring homomorphism. Then it is easy to see that whenever $\ker f \cap S = \emptyset$, $f(S)$ is a multiplicative subset of T.

Theorem 3.4. *Let $f : R \to T$ be a surjective ring homomorphism and S a multiplicative subset of R with $\ker f \cap S = \emptyset$. If R is an S-AP ring, then T is an $f(S)$-AP ring.*

Proof. Let J be a proper ideal of T disjoint with $f(S)$. Since f is a surjective ring homomorphism, there exists an ideal $f^{-1}(J)$ of R such that $f(f^{-1}(J)) = J$. Note that $J \cap f(S) = \emptyset$ if and only if $f^{-1}(J) \cap S = \emptyset$. Since R is an S-AP ring, $f^{-1}(J)$ is an S-almost prime ideal of R and clearly $\ker f \subseteq f^{-1}(J)$. Hence, J is an $f(S)$-almost prime ideal of T by Proposition 3.1(1), and thus, T is an $f(S)$-AP ring. ☐

The next proposition studies the S-AP property under the ring extension $R \subseteq T$, where (R, T) is a pair of rings.

Proposition 3.5. *Let $R \subseteq T$ be a ring extension such that $IT \cap R = I$ for each ideal I of R and $S \subseteq R$ a multiplicative set. If T is an S-AP ring, then so is R.*

Proof. Let I be a proper ideal of R disjoint with S. Now we will show that IT is a proper ideal of T disjoint with S. Indeed, if $s \in IT \cap S$, then $s \in IT \cap R = I$, a contradiction. Since T is an S-AP ring, then IT is an S-almost prime ideal of T. The rest of the proof is obtained by [4, Theorem 2.17]. ☐

Let R be a ring and S an m.s of R. The saturation of S is defined as

$$S^* = \left\{ x \in R : \frac{x}{1} \text{ is a unit of } S^{-1}R \right\}.$$

Note that S^* is an m.s containing S.

Proposition 3.6. *Let R be a ring and S be an m.s of R. Then the following assertions are equivalent.*

(1) *R is an S-AP ring.*

(2) *R is an S^*-AP ring.*

Proof. For any ideal $P \subset R$, we have $P \cap S = \emptyset$ if, and only if, $P \cap S^* = \emptyset$ because $S^* = \{k \in R :$ there exists $s \in S$ such that $ks \in S\}$. On the other hand, if $P \subset R$ is S-almost prime with associated element $s \in S$, so it is S^*-almost prime with associated element $s \in S^*$. Conversely, if $P \subset R$ is S^*-almost prime with associated element $k \in S^*$, there exists $s \in S$ such that $ks \in S$, so it is S-almost prime with associated element ks. □

The next result gives the closed relationship between S-AP rings and AP-rings concluded from [4, Theorem 2.12].

Proposition 3.7. *If R is an S-AP ring, then $S^{-1}R$ is a AP ring.*

Anderson and Dumitrescu's introduction of the concept of S-Noetherian rings constitutes a generalisation of Noetherian rings. As previously defined [7], an S-Noetherian ring is one in which any ideal I is S-finite, *i.e.* there exists $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$. The following result characterises the S-Noetherian ring property using the notion of S-almost prime.

Proposition 3.8. *The following statements are equivalent.*

(1) *R is an S-Noetherian ring.*

(2) *Any S-almost prime ideal is S-finite.*

(3) *Any almost prime ideal is S-finite.*

(4) *Any prime ideal is S-finite.*

Proof. (1) $\implies$ (2) Suppose that R is S-Noetherian. Consequently, it can be deduced that every ideal is S-finite. Moreover, it is evident that every S-almost prime is S-finite.

(2) $\implies$ (3) Assume the hypothesis that every S-almost prime ideal is S-finite, it can be deduced that if P is an almost prime ideal of R and $S \cap P \neq \emptyset$, $s \in S \cap P$, then $sP \subseteq sR \subseteq I$. Consequently, P is an S-principal ideal of R, so P is S-finite. On the other hand, if $S \cap P = \emptyset$, then P is S-almost prime ideal of R, so by hypothesis, P is S-finite.

(3) $\implies$ (4) Assume that every almost prime ideal is S-finite. Let P be a prime ideal of R. By definition, P is an almost prime ideal of R, and thus, by hypothesis P is S-finite.

(4) $\implies$ (1) This is obtained by [7, Corollary 5]. □

4 Applications in idealization and amalgamation rings

Let R be a ring and L an $R-$module. The trivial ring extension of R by L (also termed the idealization of L over R) is a commutative ring

$$R \propto L := \{(a, l) \mid a \in R \text{ and } l \in L\}$$

under the usual addition and the multiplication defined by $(a, l)(b, m) = (ab, am + bl)$ for all (a, l), $(b, m) \in R \propto L$. It is clear that $(1, 0)$ is the identity of $R \propto L$, and if S is a m.s of R, then $S \propto L$, and $S \propto 0$ are m.s of $R \propto E$. In the field of commutative ring theory, trivial ring extensions have been shown to play a pivotal role. This is due to the effectiveness of this method in producing new classes of examples and counter-examples of rings subject to various ring theoretic properties. For a more detailed exposition of this topic, the reader is referred to the following source: [2, 3, 9, 10, 17, 18].

Theorem 4.1. *Let R be a ring and L an $R-$module, and let $R \propto L$ be trivial ring extension of R by L and S be an m.s of R. Then the following statements hold.*

(1) If $R \propto L$ is a $(S \propto L)$-AP ring, then R is an S-AP ring.

(2) Let R be an integral domain with quotient field Q and L be a Q-vector space. Then the following assertions are shown to be equivalent.

i) $R \propto L$ is an $(S \propto L)$-AP ring.

ii) R is an S-AP ring.

To prove this theorem, we need the following lemmas.

Lemma 4.2 ([9, Corollary 3.4])**.** *Let R be an integral domain and L an $R-$module. Then the following conditions are equivalent.*

(1) Any ideal of $R \propto L$ is comparable to $0 \propto L$.

(2) Any ideal of $R \propto L$ is of the form $I \propto L$ for some ideal I of R or $0 \propto N$ for some submodule N of L.

(3) Any ideal of $R \propto L$ is homogeneous.

(4) L is divisible.

Lemma 4.3. *Let R be an integral domain with quotient field Q, L be a $Q-$vector space and N be a $Q-$vector subspace of L. Then $0 \propto N$ is a weakly prime ideal of $R \propto L$.*

Proof. Note that if $a \in R \setminus \{0\}$ and $l \in L \setminus N$, then $al \notin N$. Hence for every $a \in R$ and $l \in L \setminus N$, we get $al = 0$ or $al \notin N$. Therefore by [19, Corollary 3.2] we have $0 \propto N$ is a weakly prime ideal of $R \propto L$, as desired. □

Proof of Theorem 4.1.

(1) Clear by [4, Theorem 3.1].

(2) $(i) \implies (ii)$ From (1).

$(ii) \implies (i)$ Assume that R is an S-AP ring. By Lemma 4.2, every ideal of $R \propto L$ has the form $P \propto L$ for some ideal P of R or $0 \propto N$ for some submodule N of L.

Case 1: Suppose that $J = P \propto L$ is an ideal of $R \propto L$ disjoint with $S \propto L$. Then clearly $P \cap S = \emptyset$. Let $(x, l_1), (y, l_2) \in R \propto L$ such that $(x, l_1)(y, l_2) \in P \propto L - (P \propto L)^2$. Then $(xy, xl_2 + yl_1) \in P \propto L - (P \propto L)^2$, and so $xy \in P - P^2$. Since P is S-almost prime, there exists $s \in S$ with $sx \in P$ or $sy \in P$. Hence $(s, 0)(x, l_1) \in P \propto L$ or $(s, 0)(y, l_2) \in P \propto L$. Thus, $P \propto L$ is an $(S \propto L)$-almost prime ideal of $R \propto L$.

Case 2: Assume that $J = 0 \propto N$ is an ideal of $R \propto L$. Then $J = 0 \propto N$ is a weakly prime ideal of $R \propto L$ by Lemma 4.3, and so $0 \propto N$ is an $(S \propto L)$-almost prime ideal of $R \propto L$.

Thus $R \propto L$ is an $(S \propto L)$-*AP* ring. □

Example 4.4. *Consider the ring $\mathbb{Z}$ and the m.s $S = \mathbb{Z} \setminus \{0\}$ of $\mathbb{Z}$. Then $\mathbb{Z} \propto \mathbb{Q}$ is an $(S \propto \mathbb{Q})$-AP ring by Theorem 4.1 (2).*

Theorem 4.5. *Let (R, M) be a local ring, $S \subseteq R$ an m.s and L an R-module such that $ML = 0$. Then $R \propto L$ is a $(S \propto L)$-AP ring if and only if R is an S-AP ring.*

Proof. Let R be an S-AP ring. Suppose that $S \nsubseteq U(R)$. Let I be an ideal of $R \propto L$ disjoint with $S \propto L$. Set $I_0 = \{a \in R : (a, 0) \in I\}$. It is clear to see that I_0 is an ideal of R disjoint with S. Now let $(a, l), (b, m) \in R \propto L$ such that $(a, l)(b, m) = (ab, am + bl) \in I - I^2$. There are three possible cases:

Case 1: If $a \notin M$, then $a \in U(R)$, and so $(a, l) \in U(R \propto L) = U(R) \propto L$. Hence $(b, m) = (a, l)^{-1}(a, l)(b, m) \in I$, and so $(s, 0)(b, m) \in I$ for all $s \in S$.

Case 2: If $b \notin M$. Similarly we get $(a, l)(b, m)(b, m)^{-1} = (a, l) \in I$, and so $(s, 0)(a, l) \in I$ for all $s \in S$.

Case 3: If $a, b \in M$, we have $(a, l)(b, m) = (ab, 0) \in I - I^2$, so $ab \in I_0$ and $ab \notin I_0^2$. Since if $ab = \sum_{i=1}^{n} a_i b_i \in I_0^2$ with $a_i, b_i \in I_0$ for every $i \in \{1, ..., n\}$. Thus $(a_i, 0), (b_i, 0) \in I$ for every $i \in \{1, ..., n\}$. Then $\sum_{i=1}^{n}(a_i, 0)(b_i, 0) = (ab, 0) \in I^2$ which is absurd. So $ab \in I_0 - I_0^2$. Hence there exists $s \in S$ such that $sa \in I_0$ or $sb \in I_0$ as I_0 is S-almost prime. We have already assumed that $S \nsubseteq U(R)$. On the other hand, since (R, M) is a local ring, $U(R) = R \setminus M$. In conclusion, $S \cap M \neq \emptyset$. Let $s' \in S \cap M$. If $sa \in I_0$, then $(ss', 0)(a, l) = (ss'a, ss'l) = (ss'a, 0) = (s', 0)(sa, 0) \in I$. If $sb \in I_0$, then similarly we get $(ss', 0)(b, m) \in I$. Thus, I is $(S \propto L)$-almost prime and $R \propto L$ is an $(S \propto L)$-*AP* ring.

If $S \subseteq U(R)$, then R is a local ring with every ideal is almost prime so by [6, Theorem 17] $M^2 = 0$ and hence, $R \propto L$ is a local ring with maximal ideal $M \propto L$. Now $(M \propto L)^2 = M^2 \propto ML = 0$, and hence every ideal of $R \propto L$ disjoint with $(S \propto L)$ is $(S \propto L)$-almost prime by [6, Theorem 17] as desired. The converse part follows from Theorem 4.1. ☐

Example 4.6. *Let $S = \mathbb{Z} \setminus p\mathbb{Z}$ where p is a prime number. Consider $R = \mathbb{Z}_{(p)} = S^{-1}\mathbb{Z}$ which is a local domain with maximal ideal $M = p\mathbb{Z}_{(p)}$. Let E be an R/M-vector space. Then $R \propto E$ is a $(S \propto E)$-AP ring.*

Let (R, R') be a pair of rings, J be an ideal of R' and $f : R \longrightarrow R'$ be a homomorphism. In this section, we consider the following subring of $R \times R'$

$$R \bowtie^f J = \{(a, f(a) + j) : a \in R \text{ and } j \in J\}$$

is called the amalgamation of R and R' along J with respect to f. If f is the identity homomorphism on R, then we get the amalgamated duplication of R along an ideal J, $R \bowtie J = \{(a, a + j) : a \in R, j \in J\}$. As a natural generalization of the duplication construction in [14], the amalgamation ring was initiated by D'Anna, Finocchiaro and Fontana. For more details regarding amalgamation rings, we refer the reader to [12, 13, 15, 21].

Let S be an m.s of a ring R. Notice that $S' = \{(s, f(s)) : s \in S\}$ is an m.s of $R \bowtie^f J$. Also, if $0 \notin f(S)$ then $f(S)$ is an m.s of R'. Set $T = \{I \bowtie^f J : I \text{ ideal of } R\}$.

Theorem 4.7. *Let R and R' be two rings, S be an m.s of R, J an ideal of R' and $f : R \longrightarrow R'$ be a ring homomorphism. Then the following statements hold.*

(1) If $R \bowtie^f J$ is an S'-AP ring, then R is an S-AP ring.

(2) Let $f^{-1}(J) = \{0\}$. Then $R \bowtie^f J$ is an S'-AP ring if and only if $f(R) + J$ is an $f(S)$-AP ring.

(3) Let $f(a)J = 0$ for every nonunit $a \in R$. Then R is an S-AP ring if and only if every ideal in T is an S'-almost prime ideal of $R \bowtie^f J$.

Proof. (1) Assume that $R \bowtie^f J$ is a S'-AP ring. We prove that R is a ring in which every ideal is S-almost prime. Let I be an ideal of R and $a, b \in R$ with $ab \in I - I^2$, then $(a, f(a))(b, f(b)) \in I \bowtie^f J - (I \bowtie^f J)^2$. So, there exists $s \in S$ such that $(s, f(s))(a, f(a)) \in I \bowtie^f J$ or $(s, f(s))(b, f(b)) \in I \bowtie^f J$. Then $sa \in I$ or $sb \in I$, hence I is S-almost prime in R, so R is an S-AP ring.

(2) As $f^{-1}(J) = 0$, from the claim [12, Proposition 5.1(3)] we have the isomorphism $R \bowtie^f J \cong f(R)+J$. Let $\psi : R \bowtie^f J \to f(R)+J$ be the natural projection of $R \bowtie^f J \subseteq R \times (f(R)+J)$ into $f(R)+J$. Then ψ is a surjective ring homomorphism and its kernel is $\ker(\psi) = f^{-1}(J) \times \{0\} = 0$. Thus ψ is an isomorphism ring homomorphism with $\psi(S') = f(S)$.

(3) Assume that R is an S-AP ring. Let $L = I \bowtie^f J$ be an ideal in T. We prove that $I \bowtie^f J$ is $S'-$almost prime. Let $(a, f(a)+i), (b, f(b)+j) \in R \bowtie^f J$ such that $(a, f(a)+i)(b, f(b)+j) = (ab, f(ab) + f(a)j + f(b)i + ij) \in I \bowtie^f J - (I \bowtie^f J)^2$, then two cases are possible:

Case 1: If $ab \in I - I^2$, then there exists $s \in S$ such that $sa \in I$ or $sb \in I$. Hence $(s, f(s))(a, f(a) + i) \in I \bowtie^f J$ or $(s, f(s))(b, f(b) + j) \in I \bowtie^f J$. Hence, $I \bowtie^f J$ is a S'-almost prime ideal of T.

Case 2: If $ab \in I^2$, then $ab = \sum_{i=1}^{n} a_i b_i$ with $a_i, b_i \in I$ so $(ab, f(ab) + ij) = \sum_{i=2}^{n} (a_i b_i, f(a_i b_i)) + (a_1 b_1, f(a_1 b_1) + ij)$ hence $(ab, f(ab) + ij) = \sum_{i=2}^{n} (a_i, f(a_i)(b_i, f(b_i)) + (a_1, f(a_1) + i)$ $(b_1, f(b_1) + j) \in (I \bowtie^f J)^2$ a contradiction, as desired. ∎

As a conclusion of Theorem 4.7 (3), we have the following corollary.

Corollary 4.8. *Let (R, M) be a local ring with $f(M)J = 0$. Then R is an S-AP ring if and only if every ideal in T is S'-almost prime.*

Let I be a proper ideal of a ring R. The (amalgamated) duplication of R along I is a special amalgamation given by

$$R \bowtie I := \{(a, a+i) \mid a \in R,\, i \in I\}$$

Note that if S is an m.s of R, then $S' = \{(s, s) \mid s \in S\}$ is an m.s of $R \bowtie I$. Set, $T' = \{K \bowtie I : K \text{ ideal of } R\}$.

Corollary 4.9. *Let R be a ring, S a m.s of R and $I \in I(R)$ such that $aI = 0$ for every nonunit $a \in R$. Then R is an S-AP ring if and only if every ideal in T' is S'-almost prime.*

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Persistence of a tumor spheroid with an almost periodic nutrient supply

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ABSTRACT

We prove that a spherical tumor with free boundary furnished with an almost periodic nutrient supply has a twofold long term time evolution: either it vanishes or it tends towards a persistent tumor which oscillates almost periodically. This is determined by a relation of the mean of the nutrient supply and a threshold value meaning the minimal nutrient supply enabling the tumor to live. In each case, global stability is proved for the almost periodic solution $(\sigma_\star(t,x), P_\star(t,x))$ of the corresponding reaction-diffusion equation.

RESUMEN

Demostramos que un tumor esférico con frontera libre dotado con un suministro casi periódico de nutrientes tiene una evolución a largo tiempo doble: o bien desaparece o tiende a un tumor persistente que oscila casi periódicamente. Esto está determinado por una relación del promedio del suministro de nutrientes y un valor umbral, es decir, el suministro mínimo de nutrientes que le permite vivir al tumor. En cada caso, se demuestra la estabilidad global para la solución casi periódica $(\sigma_\star(t,x), P_\star(t,x))$ de la ecuación de reacción-difusión correspondiente.

Keywords and Phrases: Tumor growth, spheroid tumor, almost periodic function, reaction-diffusion equation.

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1 Introduction: A simple spheroid model for tumor-growth

Works [3, 14] are seminal works of an abundant production of mathematical models describing tumor growth, see for instance [6, 9, 10, 13, 15]. The fundamental tool is the reaction-diffusion equation which models the volume growth under the presence of nutrients and inhibitors. The time evolution of the volume of the multicellular tumor spheroid living in a fluid containing nutrients is then given by the solutions to a free boundary problem. Complex models consider a necrotic core in the description of the tumor. In our considerations we avoid such element of the model. A common feature of all models is the prediction a twofold scenario depending on a nutrient supply threshold: either the tumor shrinks and vanishes or the tumor persists.

Typically, under *in vitro* conditions, nutrient as well as inhibitor supplies remain constant. Nevertheless, in more realistic conditions tumors grow up upon varying tissue conditions. With this motivation, experimental data having periodic supply have been obtained for instance in [8]. More recently, experimental designs with oscillatory nutrient and inhibitor supplies have been reported in [11]. Accordingly, mathematical modeling with time-dependent external environment have arisen. For periodic continuous nutrient and inhibitor supplies, there are results in [12,15,16]. This works describe again conditions under which either the tumor vanishes or the tumor remains periodically changes size.

Our main contribution in this work, is considering a more general time-dependent oscillatory nutrient supply. Instead of discussing a constant or periodic external nutrient concentration, we introduce in our model an *almost periodic continuous* function. We can mention two reasons to use this space of functions: (1) Almost periodic functions incorporate variations in experimental conditions that not necessarily are periodic but only approximately periodic. (2) Almost periodic functions permit different factors which may not necessarily be synchronized. This may be interesting when we deal with two growth factors such as nutrient supply and inhibitors, where there is no requirement of rationally dependent frequencies. The case where almost periodic nutrient supply and almost periodic inhibitory factor are non-synchronized will be treated elsewhere. In this work we treat the inhibitor-free case.

We review some of the main concepts and results about almost periodic functions in the Appendix of Section 4. The interested reader may also consult well known references such as [1, 2, 4, 7].

Now, we describe our setting, see for instance [3, 6, 10, 15] for further explanations. Let $\Omega(t) \subset \mathbb{R}^3$ be a bounded region with smooth boundary, $\partial\Omega(t)$, evolving in time. This region is supposed to model the tumor inside a continuum media. We designate by $\sigma(t, x)$ the nutrient concentration in a time-space domain $(t, x) \in \mathbb{R} \times \mathbb{R}^3$. When there are no inhibitors in the continuum media, then the nutrient supply concentration σ is proportional to the intensity of the mitosis,

$$S = \mu(\sigma - \tilde{\sigma}).$$

The proportionality constant, μ measures the intensity of the cell division. The parameter $\tilde{\sigma}$ is a threshold level which divides two regimes: either the tumor grows‘ due to mitosis or the tumor shrinks, due to apoptosis.

In a vascularization free environment, we suppose that σ satisfies a reaction-diffusion equation,

$$c\frac{d\sigma}{dt} = \Delta\sigma - \lambda\sigma,$$

where, $\lambda > 0$ is the nutrient consumption rate. The small time-scale ratio between the nutrient diffusion T_σ compared to the tumor volume evolution T_R, leads to a quasi-stationary evolution where

$$c = \frac{T_\sigma}{T_R} = \frac{1\,\text{min}}{1\,\text{day}} \approx 0.$$

Thus, steady solutions of the reaction-diffusion equation become relevant. Assuming a velocity, $\vec{v}(t,x)$, for the cell-flow inside the tumor, Darcy's law describes this flow as generated by the pressure gradient, $\vec{v} = -\nabla P$, where $P(t,x)$ is the pressure inside the tumor. Since the flow has the mitosis process as source, then $\nabla \cdot \vec{v} = S$. Therefore,

$$-\Delta P = \nabla \cdot (-\nabla P) = \nabla \cdot \vec{v} = S.$$

Thus, the stationary reaction-diffusion equation describing the concentration of the nutrient and the pressure are the following,

$$\Delta\sigma(t,x) = \lambda\sigma(t,x), \qquad x \in \Omega(t), \quad t > 0, \tag{1.1}$$

$$-\Delta P(t,x) = \mu(\sigma(t,x) - \tilde{\sigma}), \qquad x \in \Omega(t), \quad t > 0. \tag{1.2}$$

For an external nutrient supply, $\Phi(t) \geq 0$, homogeneous along the membrane, we have the following free-boundary value conditions

$$\sigma(t,x) = \Phi(t), \qquad x \in \partial\Omega(t), \quad t > 0, \tag{1.3}$$

$$P(t,x) = \gamma H(t,x), \qquad x \in \partial\Omega(t), \quad t > 0, \tag{1.4}$$

where $\gamma > 0$ is a constant representing the cell adhesiveness and $H(t,x)$ designates the mean curvature of the boundary surface, $\partial\Omega(t,x)$.

Upon radial symmetry assumption, let $R(t)$ be the outer radius of the sphere $\partial\Omega(t)$, then the PDE problem (1.1) with boundary conditions (1.3) becomes a couple of ODE boundary problems

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\sigma(t,r)\right) &= \lambda\sigma, & 0<r<R(t),\\
\sigma(t,R(t)) &= \Phi(t), & \frac{\partial}{\partial r}\sigma(t,0)=0,\\
-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}P(t,r)\right) &= \mu(\sigma-\tilde{\sigma}), & 0<r<R(t),\\
P(t,R(t)) &= \frac{\gamma}{R(t)}, & \frac{\partial}{\partial r}P(t,0)=0,\\
t &> 0, & R(0)=R_0.
\end{aligned} \tag{1.5}$$

For P we have a second order ODE with boundary conditions both, on P and on its derivative. Moreover, the Neumann boundary condition $\frac{\partial P(t,R(t))}{\partial r} = -R'(t)$, arises from the restriction of Darcy's law, $\nabla P = -\vec{v}$, along the normal component of the spherical surface $\partial\Omega(t)$, *i.e.*

$$\frac{\partial P}{\partial \vec{n}} = -\vec{v}|_{\partial\Omega(t)}.$$

This makes the problem a free-boundary one with velocity displacement of the boundary given by $\vec{v}$.

Length rescaling allows us to take $\lambda = 1$. Then, equations (1.5) are solved as follows,

$$\begin{aligned}
\sigma_\star(t,r) &= \Phi(t)\frac{R(t)}{\sinh R(t)}\frac{\sinh r}{r},\\
P_\star(t,r) &= \frac{\mu\tilde{\sigma}}{6}\left(r^2-(R(t))^2\right)+\mu(\Phi(t)-\sigma_\star(t,r))+\frac{\gamma}{R(t)}.
\end{aligned}$$

Notably, the flow, $\iint_{\partial\Omega(t)} \vec{v}\cdot d\mathrm{S}$, of $\vec{v}$ along the boundary $\partial\Omega(t)$ equals the rate of volume change. Mass and volume conservation inside cells imply

$$\frac{d}{dt}\mathrm{vol}(\Omega(t)) = \iiint_{\Omega(t)} \nabla\cdot\vec{v}\, d\,\mathrm{vol}.$$

Hence,

$$\frac{d}{dt}\left(\frac{4\pi R(t)^3}{3}\right) = 4\pi\int_0^{R(t)} \mu(\sigma(t,r)-\tilde{\sigma})r^2\,dr,$$

or

$$R'(t) = \frac{1}{R(t)^2}\int_0^{R(t)} \mu(\sigma(t,r)-\tilde{\sigma})r^2\,dr.$$

The PDE problem (1.5) can be solved in this way by solving the ODE

$$R' = \mu R\left(p(R)\Phi(t) - \frac{\tilde{\sigma}}{3}\right)$$
$$R(t_0) = R_0 \geq 0, \quad \Phi(t) \geq 0, \tag{1.6}$$

where

$$p(x) = \frac{1}{x\tanh x} - \frac{1}{x^2}$$

is strictly decreasing in $x > 0$, and $0 < p(x) < \dfrac{1}{3}$, see [6, Lemma 3.2].

We propose the following global stability assertion which depends on a condition on the *mean value,*

$$\overline{\Phi} = M[\Phi]$$

of the nutrient supply. See Appendix in Section 4 for the definition of the mean value, $M[\phi]$ of an almost periodic function $\phi(t)$.

Theorem 1.1. *Let $R(t)$ be any solution of* (1.6) *with positive initial condition $R(t_0) > 0$ and an almost periodic nutrient supply $\phi(t) \geq 0$, then we have two possible limits:*

(1) If $\overline{\Phi} \leq \tilde{\sigma}$, then $\lim_{t\to\infty} R(t) = 0$.

(2) If $\overline{\Phi} > \tilde{\sigma}$, then there exists a unique almost periodic solution $R^\star(t)$ of (1.6) *such that $\lim_{t\to\infty} |R(t) - R^\star(t)| = 0$. Furthermore, there is an inclusion of the modules of frequencies, $\widehat{\varphi} \subset \widehat{\Phi}$.*

For the reader's convenience the definition of *module of frequencies* can also be consulted in Section 4. This generalizes the result for the periodic case obtained in [12]. Referring to the problem (1.1) with boundary conditions (1.3), we deduce the following consequence.

Corollary 1.2. *Upon radial symmetry, when the mean value of the supply $\overline{\Phi}$ surpasses the threshold value, $\tilde{\sigma}$, the free boundary problem* (1.1) *has a globally asymptotically stable almost periodic solution $(\sigma_\star, P_\star)$. Otherwise, the free tumor equilibrium is attained in the long term by any initial condition.*

2 Proof of the main result

This Section is devoted to demonstrate our main result stated in Theorem 1.1. The main idea is to prove boundedness, above and below, in the open interval $(0, \infty)$ of every solution $\phi(t)$ whose initial condition $\phi(t_0)$ is positive. The main idea then reduces to show that the relative compacity or normal property of the family of translated solutions. A limit in this family provides an asymptotically almost periodic solution. We first review some technical Lemmas.

Lemma 2.1. *Let $\phi(t), \varphi(t)$ be any couple of solutions corresponding to initial conditions $\phi(t_0) > \varphi(t_0) \geq 0$ of the ODE*

$$x' = g(t, x).$$

Suppose that $g(t,x)$ is continuous and C^1 with respect to x. Suppose that solutions are well-defined for all $t > t_0$. Then $\phi(t) > \varphi(t)$. In particular, for $\phi(t_0) > 0$ we have $\phi(t) > 0$ for all $t \geq t_0$, whenever $\varphi(t) \equiv 0$ is a solution.

Lemma 2.1 is well known and follows immediately from the property of uniqueness of solutions of an ODE. From boundedness of solutions claimed in Lemma 2.3 below, it can be applied to (1.6).

Lemma 2.2. *If $\overline{\Phi} < \tilde{\sigma}$, then for any solution of* (1.6) *with positive initial condition $\phi(t_0) > 0$, we have $\lim_{t\to\infty} \phi(t) = 0$.*

Proof. We observe that $(\ln \phi(t))' = \dfrac{\phi'(t)}{\phi(t)}$. We proceed by contradiction and suppose that,

$$\phi(t_k) \searrow \Theta = \limsup_{t\geq 0} \phi(t) > 0$$

for an increasing sequence $t_k \nearrow \infty$. We recall that

$$\frac{\phi'(t)}{\phi(t)} = \mu \left[p(\phi(t))\Phi(t) - \frac{\tilde{\sigma}}{3} \right] \leq \frac{\mu}{3}(\Phi(t) - \tilde{\sigma}).$$

Therefore,

$$\limsup_{k\to\infty} \frac{\ln \phi(t_k) - \ln \phi(t_0)}{t_k - t_0} \leq \limsup_{k\to\infty} \frac{\mu}{3(t_k - t_0)} \int_{t_0}^{t_k} (\Phi(t) - \tilde{\sigma})\, dt = \frac{\mu}{3}\left(\overline{\Phi} - \tilde{\sigma}\right).$$

or,

$$\limsup_{k\to\infty} \frac{\ln \dfrac{\phi(t_k)}{\phi(t_0)}}{t_k - t_0} \leq -\frac{\mu(\tilde{\sigma} - \overline{\Phi})}{3} < 0.$$

Thus, $0 \leq \limsup_{k\to\infty} \ln \phi(t_k) = -\infty$ or $\limsup_{k\to\infty} \phi(t_k) = 0$. □

Lemma 2.3. *Any solution, $\phi(t)$, of* (1.6) *with initial $\phi(t_0) > 0$ is bounded above, i.e.*

$$\phi(t) \leq \sup\{\phi(t) : t \geq t_0\} < \infty.$$

Proof. For the upper bound suppose that there exists an increasing sequence $t_k > t_{k-1} > t_0$ such that $t_k \to \infty$ when $k \to \infty$.

By contradiction, let us suppose that $\lim_{k\to\infty} \phi(t_k) = \infty$. Without loss of generality, by taking subsequence if necessary, we can suppose that $\phi(t_k) \nearrow \infty$ is monotone increasing and that $\phi'(t_k) > 0$.

Then $\lim_{k\to\infty} p(\phi(t_k)) = 0$, and

$$(\ln\phi(t_k))' = \frac{\phi'(t_k)}{\phi(t_k)} \geq 0.$$

Hence,

$$\mu\frac{\tilde{\sigma}}{3} = \lim_{k\to\infty} \mu p(\phi(t_k))\Phi(t_k) - (\ln\phi(t_k))' \leq 0.$$

However, $\tilde{\sigma} > 0$. This finishes the proof of Lemma 2.3. □

Lemma 2.4. *If $\overline{\Phi} > \tilde{\sigma} > 0$, then any solution, $\phi(t)$, of* (1.6) *with positive initial condition, $\phi(t_0) > 0$ is bounded below, i.e.*

$$\phi(t) \geq \inf\{\phi(t) \,:\, t \geq t_0\} = \phi_* > 0.$$

Proof. Suppose that $\phi_* = 0$. Recalling Lemma 2.1 we have that $\phi(t) > 0$ for every $t \geq t_0$, and by continuity there is no interval $[t_0, t_2]$ where

$$\inf\{\phi(t) \,:\, t_0 \leq t \leq t_2\} = 0.$$

Therefore, there exists an increasing sequence $t_k \to \infty$ such that

$$\phi(t) \geq \phi(t_k) \searrow 0, \quad p(\phi(t)) \leq p(\phi(t_k)) \nearrow 1/3, \quad \forall t \in [t_0, t_k],$$

because $p(x)$ is a decreasing function. Then $\ln\phi(t_k)$ would tend towards $-\infty$. Therefore,

$$0 \geq \liminf_{k\to\infty} \frac{\ln\phi(t_k) - \ln(\phi(t_0))}{t_k - t_0}.$$

Moreover, if $\overline{\Phi} > \tilde{\sigma}$ we consider $1/3 > \varepsilon > 0$ such that

$$(1 - 3\varepsilon)\overline{\Phi} > \tilde{\sigma}.$$

Due to the convergence $p(\phi(t_k)) \nearrow 1/3$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\varepsilon \geq \frac{1}{3} - p(\phi(t)) \geq \frac{1}{3} - p(\phi(t_k)), \quad \forall t \in [t_k, t_j], \quad j \geq k \geq N_\varepsilon.$$

Therefore,

$$\begin{aligned}
0 \geq \liminf_{k\to\infty} \frac{\ln(\phi(t_k)/\phi(t_0))}{t_k - t_0} &= \sup_{k\geq 0} \inf_{j\geq k} \frac{\mu}{t_j - t_0} \int_{t_0}^{t_j} p(\phi(t))\Phi(t) - \frac{1}{3}\tilde{\sigma}\, dt \\
&= \sup_{k\geq 0} \inf_{j\geq k} \left\{ \frac{\mu}{t_j - t_0} \int_{t_k}^{t_j} p(\phi(t))\Phi(t) - \frac{1}{3}\tilde{\sigma}\, dt + \frac{\mu}{t_j - t_0} \int_{t_0}^{t_k} p(\phi(t))\Phi(t) - \frac{1}{3}\tilde{\sigma}\, dt \right\} \\
&\geq \sup_{k\geq 0} \inf_{j\geq k} \left\{ \frac{\mu}{t_j - t_0} \int_{t_k}^{t_j} p(\phi(t))\Phi(t) - \frac{1}{3}\tilde{\sigma}\, dt \right\} + 0
\end{aligned}$$

$$
\begin{aligned}
&\geq \sup_{k\geq N_\varepsilon} \inf_{j\geq k} \frac{\mu}{t_j - t_0} \int_{t_k}^{t_j} p(\phi(t))\Phi(t) - \frac{1}{3}\tilde{\sigma}\, dt \\
&\geq \mu \cdot \sup_{k\geq N_\varepsilon} \inf_{j\geq k} \frac{1}{t_j - t_0} \int_{t_k}^{t_j} \left(\frac{1}{3} - \varepsilon\right) \Phi(t) - \frac{1}{3}\tilde{\sigma}\, dt \\
&\geq \left(\frac{1}{3} - \varepsilon\right) \mu \cdot \sup_{k\geq N_\varepsilon} \inf_{j\geq k} \frac{t_j - t_k}{t_j - t_0} \frac{1}{t_j - t_k} \int_{t_k}^{t_j} \Phi(t) - \frac{1}{1-3\varepsilon}\tilde{\sigma}\, dt \\
&= \left(\frac{1}{3} - \varepsilon\right) \mu \cdot \sup_{k\geq N_\varepsilon} \inf_{j\geq k} \frac{1}{t_j - t_k} \int_{t_k}^{t_j} \Phi(t) - \frac{1}{1-3\varepsilon}\tilde{\sigma}\, dt \\
&= \left(\frac{1}{3} - \varepsilon\right) \mu \left(\overline{\Phi} - \frac{1}{1-3\varepsilon}\tilde{\sigma}\right) = \frac{1}{3}\mu\left((1-3\varepsilon)\overline{\Phi} - \tilde{\sigma}\right) > 0.
\end{aligned}
$$

We reach a contradiction. So $\phi_* > 0$. □

Lemma 2.5. *If $\overline{\Phi} = \tilde{\sigma} > 0$, then for any solution, $\phi(t)$, of* (1.6) *with positive initial condition, $\phi(t_0) > 0$, we have $\lim_{t\to\infty} \phi(t) = 0$.*

Proof. Define

$$\Theta := \limsup_{T\to\infty} \phi(T), \quad \theta := \liminf_{T\to\infty} \phi(T) \geq 0.$$

Notice that $0 \leq \theta \leq \Theta < +\infty$. We shall prove by contradiction that $\Theta = \theta = 0$.

Suppose that $\theta > 0$: We have, $\phi_* > 0$, then $p(\phi_*) < 1/3$. Therefore, for every increasing sequence $t_0 < t_k < t_j$, such that $\phi(t_k) \leq \phi(t_j)$ we have $p(\phi(t_k)) \leq p(\phi(t_j)) \leq p(\phi_*)$. Whence,

$$
\begin{aligned}
\limsup_{j\to\infty} \frac{\ln\phi(t_j) - \ln\phi(t_k)}{t_j - t_k} &= \limsup_{j\to\infty} \frac{\mu}{t_j - t_k} \int_{t_j}^{t_k} p(\phi(t))\Phi(t) - \frac{\tilde{\sigma}}{3}\, dt \\
&\leq \limsup_{j\to\infty} \frac{\mu}{t_j - t_k} \int_{t_j}^{t_k} p(\phi_*)\Phi(t) - \frac{\tilde{\sigma}}{3}\, dt = \mu\left[p(\phi_*) - \frac{1}{3}\right]\overline{\Phi} < 0.
\end{aligned}
$$

In particular, for a sequence $\phi(t_k) \nearrow \theta > 0$ we reach a contradiction. Namely,

$$0 \leq \limsup_{j\to\infty} \frac{\ln\phi(t_j) - \ln\phi(t_k)}{t_j - t_k} < 0.$$

Therefore $\theta = 0$.

Suppose that $\Theta > \theta = 0$: Since $\theta = 0$, there exists an increasing sequence $t'_k \nearrow \infty$, such that $\phi(t'_k) \searrow 0$. Thus,

$$p(\phi(t'_k)) \leq p(\phi(t'_j)) \nearrow 1/3, \quad \forall j \geq k.$$

For every $\varepsilon > 0$, due to the convergence $p(\phi(t'_k)) \nearrow 1/3$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\varepsilon \geq \frac{1}{3} - p(\phi(t)) \geq \frac{1}{3} - p(\phi(t'_k)), \quad \forall t \in [t'_k, t'_j], \quad j \geq k \geq N_\varepsilon.$$

Therefore,

$$
\begin{aligned}
-\liminf_{k\to\infty} \frac{\ln(\phi(t'_k)/\phi(t_0))}{t'_k - t_0} &= \limsup_{k\to\infty} -\frac{\ln(\phi(t'_k)/\phi(t_0))}{t'_k - t_0} \\
&= \inf_{k\geq 0}\sup_{j\geq k} \frac{\mu}{t'_j - t_0}\left[\int_{t'_k}^{t'_j} \frac{\tilde{\sigma}}{3} - p(\phi(t))\Phi(t)\,dt + \int_{t'_0}^{t'_k} \frac{\tilde{\sigma}}{3} - p(\phi(t))\Phi(t)\,dt\right] \\
&= \inf_{k\geq 0}\sup_{j\geq k} \frac{\mu}{t'_j - t_0}\int_{t'_k}^{t'_j} \frac{\tilde{\sigma}}{3} - p(\phi(t))\Phi(t)\,dt + 0 \\
&\leq \inf_{k\geq N_\varepsilon}\sup_{j\geq k} \frac{\mu}{t'_j - t_0}\int_{t'_k}^{t'_j} \frac{\tilde{\sigma}}{3} - p(\phi(t))\Phi(t)\,dt \\
&\leq \mu\cdot \inf_{k\geq N_\varepsilon}\sup_{j\geq k} \frac{t'_j - t'_k}{t'_j - t_0}\frac{1}{t'_j - t'_k}\int_{t'_k}^{t'_j} \frac{\tilde{\sigma}}{3} - p(\phi(t))\Phi(t)\,dt \\
&\leq \mu\cdot \inf_{k\geq N_\varepsilon}\sup_{j\geq k} \frac{1}{t'_j - t'_k}\int_{t'_k}^{t'_j} \frac{\tilde{\sigma}}{3} + \left(\varepsilon - \frac{1}{3}\right)\Phi(t)\,dt \\
&\leq \mu\left[\frac{\tilde{\sigma}}{3} + \left(\varepsilon - \frac{1}{3}\right)\overline{\Phi}\right] = \mu\varepsilon\overline{\Phi}
\end{aligned}
$$

Hence, $\liminf_{k\to\infty} \frac{\ln(\phi(t'_k)/\phi(t_0))}{t'_k - t_0} > -\mu\varepsilon\overline{\Phi}$ for each $\varepsilon > 0$, *i.e.*

$$
\liminf_{k\to\infty} \frac{\ln(\phi(t'_k)/\phi(t_0))}{t'_k - t_0} \geq 0. \tag{2.1}
$$

Now, suppose that $\Theta > 0$, then $p(\Theta) < 1/3$. Therefore, if we take a sequence $t_k \nearrow \infty$ such that $\phi(t_k) \searrow \Theta > 0$, we have $p(\phi(t_j) < p(\phi(t_k)) < p(\Theta) < 1/3$, for every $j > k$ and

$$
\begin{aligned}
\limsup_{j\to\infty} \frac{\ln\phi(t_j) - \ln\phi(t_k)}{t_j - t_k} &= \limsup_{j\to\infty} \frac{\mu}{t_j - t_k}\int_{t_j}^{t_k} p(\phi(t))\Phi(t) - \frac{\tilde{\sigma}}{3}\,dt \\
&\leq \mu\left[p(\Theta)\overline{\Phi} - \frac{\tilde{\sigma}}{3}\right] = \overline{\Phi}\mu\left[p(\Theta) - \frac{1}{3}\right] < 0.
\end{aligned}
$$

Then,

$$
\limsup_{j\to\infty} \frac{\ln\phi(t_j) - \ln\phi(t_k)}{t_j - t_k} < 0.
$$

Supposing, without loss of generality, that $\phi(t_0) \geq \phi(t_k)$

$$
\limsup_{j\to\infty} \frac{\ln\phi(t_j) - \ln\phi(t_0)}{t_j - t_0} = \limsup_{j\to\infty} \frac{\ln\phi(t_j) - \ln\phi(t_k)}{t_j - t_k}\frac{t_j - t_k}{t_j - t_0} \leq \limsup_{j\to\infty} \frac{\ln\phi(t_j) - \ln\phi(t_k)}{t_j - t_k}.
$$

Therefore,

$$
\limsup_{j\to\infty} \frac{\ln\phi(t_j) - \ln\phi(t_0)}{t_j - t_0} < 0. \tag{2.2}
$$

By summarizing (2.1) and (2.2), we finally reach a contradiction as follows. Since $\Theta > 0$ we can, without loss of generality, suppose that $\phi(t_j) \geq \Theta \geq \phi(t'_l)$. We can also suppose, without loss of generality, that $t_j < t'_j$. Hence,

$$0 \leq \liminf_{j\to\infty} \frac{\ln\phi(t'_j) - \ln\phi(t_0)}{t'_j - t_0} \leq \limsup_{j\to\infty} \frac{\ln\phi(t_j) - \ln\phi(t_0)}{t_j - t_0} < 0. \qquad \square$$

Finding *asymptotically almost periodic* solutions, see Definition 4.4 in Section 4, allows us to find almost periodic solutions too, according to Theorem 4.5 in Section 4. Among the characterizations of asymptotically almost periodic functions we have the Bochner's property described in Theorem 4.6 in Section 4 which will be useful to prove our main result given in Theorem 1.1.

Now we complete the proof of Theorem 1.1.

Proof of existence. We suppose that $\overline{\Phi} > \tilde{\sigma}$ and take a solution $\phi(t)$ of (1.6). Due to Lemmas 2.2 and 2.3, $\phi(t)$ is contained in a compact $K = [\phi_*, \phi^*] \subset (0, \infty)$ for $t \geq t_0$. For an increasing divergent sequence $h = \{h_k \geq t_0\}_{k\in\mathbb{N}}$, $\lim_{k\to\infty} h_k = \infty$, we define $\phi_k(t) = \phi(t + h_k)$. Then

$$|(\ln\phi_k(t) - \ln\phi_m(t))'| = \left|\frac{\phi'_k(t)}{\phi_k(t)} - \frac{\phi'_m(t)}{\phi_m(t)}\right| \leq \frac{\mu}{3}\left|\Phi(t+h_k) - \Phi(t+h_m)\right|.$$

Where the second inequality follows from substitution

$$\frac{\phi'_k(t)}{\phi_k(t)} - \frac{\phi'_m(t)}{\phi_m(t)} = \mu\left[p(\phi(t+h_k))\Phi(t+h_k) - p(\phi(t+h_m))\Phi(t+h_m)\right],$$

which in turn yields due to the bound $p(t) \leq 1/3$.

Invoking Bochner's property in Theorem 4.3, Section 4, modulo a subsequence we can suppose that $\lim_{k\to\infty} \Phi(t + h_k)$ uniformly along $\mathbb{R}$. Hence

$$\psi_\sharp(t) = \lim_{k\to\infty} \psi_k(t), \quad \psi_k(t) = (\ln\phi_{n_k}(t))',$$

converges uniformly along $[t_0, \infty)$. This proves that both $(\ln\phi(t))'$ and $\psi_\sharp(t)$ are asymptotically almost periodic, according to Theorem 4.6 in the Appendix. This means that we can decompose

$$\varphi(t) := (\ln\phi(t))' = \varphi_\flat(t) + q(t), \quad \lim_{t\to\infty} q(t) = 0,$$

where $\varphi_\flat(t)$ is almost periodic.

Notably, it can be deduced that $y = \ln\phi(t)$ is a solution of

$$\frac{dy}{dt} = \mu\left(p\left(e^y\right)\Phi - \frac{\tilde{\sigma}}{3}\right). \tag{2.3}$$

Hence,

$$y = \tilde{\varphi}(t) := \int_{t_0}^{t} \varphi(s)\, ds + \ln \phi(t_0),$$

is a solution of (2.3). From boundedness of solutions proven in Lemmas 2.3, we can see that a solution $\exp \tilde{\varphi}(t)$ of (2.3) is bounded in $(0, \infty)$ as $t \to \infty$. Therefore $\tilde{\varphi}(t)$ remains bounded above and below as t grows.

On the other hand, if

$$\tilde{\varphi}_\flat(t) := \int_{t_0}^{t} \varphi_\flat(s)\, ds + \ln \phi(t_0) + c_1, \quad \tilde{q}(t) = \int_{t_0}^{t} q(s)\, ds - c_1$$

then $\tilde{\varphi} = \tilde{\varphi}_\flat + \tilde{q}$ and

$$\tilde{\varphi}_\flat' + \tilde{q}' = \tilde{\varphi}_\flat' + q = \mu \left(p \left(e^{\tilde{\varphi}_\flat} e^{\tilde{q}} \right) \Phi - \frac{\tilde{\sigma}}{3} \right),$$

Remark that $\tilde{\varphi}_\flat$ is defined up to an arbitrary constant $c_1 \in \mathbb{R}$. Notice that we do not claim that $\tilde{\varphi}_\flat$ is the almost periodic component of an almost periodic function $\tilde{\varphi}$. Nonetheless, we claim that $\tilde{\varphi}_\flat$ is a solution of the differential equation (2.4) below

$$\frac{dy}{dt} = \mu \left(p \left(e^{\tilde{q}+y} \right) \Phi - \frac{\tilde{\sigma}}{3} \right) - q. \tag{2.4}$$

Now let us consider an increasing sequence $\{h_n \geq 0\}_{n=1}^{\infty}$, $h_n \nearrow \infty$, then modulo extracting a subsequence, we get the following uniform limits which are asymptotically almost periodic and almost periodic respectively:

$$\tilde{\varphi}_\sharp(t) = \lim_{k\to\infty} \tilde{\varphi}(t + h_{n_k}), \quad 0 = \lim_{k\to\infty} q(t + h_{n_k}), \quad \Phi_\sharp(t) = \lim_{k\to\infty} \Phi(t + h_{n_k}).$$

We claim that there exists also the (asymptotically almost periodic) limit

$$\tilde{\varphi}_{\flat,\sharp}(t) = \lim_{k\to\infty} \tilde{\varphi}_\flat(t + h_{n_k}),$$

and that $y = \tilde{\varphi}_{\flat,\sharp}(t)$ as well as $y = \tilde{\varphi}_\sharp(t)$ solve

$$\frac{dy}{dt} = \mu \left(p \left(e^{y} \right) \Phi_\sharp - \frac{\tilde{\sigma}}{3} \right). \tag{2.5}$$

To see this we just notice the uniform convergence of the l.h.s. of the differential equation

$$\tilde{\varphi}'(t + h_{n_k}) = \mu \left(p \left(\tilde{\varphi}(t + h_{n_k}) \right) \Phi(t + h_{n_k}) - \frac{\tilde{\sigma}}{3} \right).$$

and then apply the uniform convergence of the derivative criterion given in Theorem 4.8 in Appendix 4. Therefore, the difference

$$\tilde{q}_\flat(t) := \tilde{\varphi}_\sharp(t) - \tilde{\varphi}_{\flat,\sharp}(t)$$

remains monotone and being asymptotic almost periodic necessarily goes to 0 as $t \nearrow \infty$. We conclude that $\tilde{\varphi}_{\flat,\sharp}(t)$ is an almost periodic solution of (2.5) and that it is almost periodic component of

$$\tilde{\varphi}_\sharp(t) = \tilde{\varphi}_{\flat,\sharp}(t) + \tilde{q}_\flat(t).$$

Finally, the limit of the reversed translations yields almost periodic limits

$$\Phi(t) = \lim_{k\to\infty} \Phi_\sharp(t - h_{n_k}),$$

and

$$\tilde{\varphi}_{\flat,\natural}(t) := \lim_{k\to\infty} \tilde{\varphi}_{\flat,\sharp}(t - h_{n_k}).$$

Indeed, if we denote $\Phi_k(t) := \Phi(t + h_{n_k})$, uniform convergence $\Phi_k \to \Phi_\sharp$, implies that for every $\varepsilon > 0$, there exists $\tilde{N}_\varepsilon \in \mathbb{N}$, such that for every $k \geq \tilde{N}_\varepsilon$ and $t \geq 0$,

$$\begin{aligned}\varepsilon > \|\Phi_\sharp - \Phi_k\|_\infty &\geq |\Phi_\sharp(t - h_{n_k}) - \Phi_k(t - h_{n_k})| \\ &= |\Phi_\sharp(t - h_{n_k}) - \Phi((t + h_{n_k}) - h_{n_k})| = |\Phi_\sharp(t - h_{n_k}) - \Phi\,(t)|\,.\end{aligned}$$

Hence, $y = \tilde{\varphi}_{\flat,\natural}(t)$ defines a solution of (2.3). At last,

$$\phi_\flat(t) := \exp \tilde{\varphi}_{\flat,\natural}(t)$$

is a solution of (1.6). Such solution $\phi_\flat(t)$ is almost periodic because exp, is uniformly continuous in the closed interval $\left[(\tilde{\varphi}_\flat)_*, \tilde{\varphi}_\flat^*\right]$ and we apply Theorem 4.9 in Section 4. □

The stability property

$$\lim_{t\to\infty} |\phi\,(t) - \phi_\flat(t)| = 0,$$

follows from $\lim_{t\to\infty} |\tilde{\varphi}\,(t) - \tilde{\varphi}_\flat(t)| = 0$ and from being asymptotic almost periodic.

Proof of uniqueness. Suppose that $\varphi_1(t)$ and $\varphi_2(t)$ are two different almost periodic solutions, with $0 < \varphi_1(t_0) \leq \varphi_2(t_0)$, then

$$\left(\ln \frac{\varphi_1(t)}{\varphi_2(t)}\right)' = \mu\Phi(t)\,(p(\varphi_1(t)) - p(\varphi_2(t)) \leq 0.$$

Since $(\ln(x))' \geq 0$, then $\dfrac{\varphi_1(t)}{\varphi_2(t)}$ is a non-increasing function. Moreover, the quotient of almost periodic functions is almost periodic by Theorem 4.7. The only almost periodic functions that are non-decreasing are constant. Therefore, $\dfrac{\varphi_1(t)}{\varphi_2(t)}$ is constant, and by having the same initial condition this constant is 1. Hence, $\varphi_1 = \varphi_2$. □

3 Numerical examples

We consider $\mu = 1$, $\Phi = \cos(2\pi t/7) + \cos(2\sqrt{2}\pi t/7) + 2.5$ with three different values of the threshold, $\tilde{\sigma} = 3, 1$ and 0.5. See Figures 1, 2 and 3, respectively. All images were programmed in Mathematica.

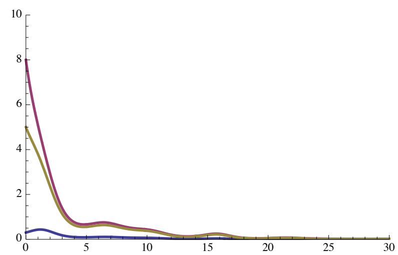

Figure 1: Case $\tilde{\sigma} = 3 > \overline{\Phi} = 2.5$ with initial conditions $R_0 = 0.5, 5, 8$, respectively. We observe an exponential decay towards 0 of the solutions.

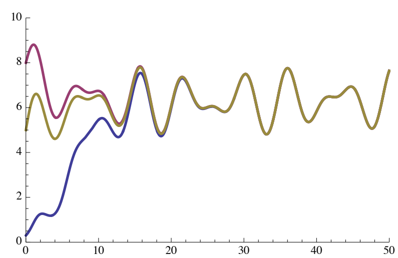

Figure 2: Case $\tilde{\sigma} = 1 < \overline{\Phi} = 2.5$ with initial conditions $R_0 = 0.5, 5, 8$, respectively. We observe asymptotic convergence towards an almost periodic solution.

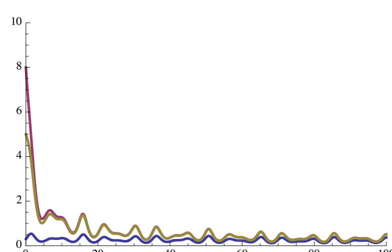

Figure 3: Case $\tilde{\sigma} = \overline{\Phi} = 2.5$ with initial conditions $R_0 = 0.5, 5, 8$, respectively. We observe slow convergence towards 0.

4 Appendix: Almost periodic functions

Along this section, we state the main results of the theory of almost periodic functions. Proofs and further information can be found in [1, 2, 4, 7]. For a more recent account of the theory, see also [5].

Definition 4.1. *The space of* almost periodic functions *is the closure* $\overline{\mathcal{T}} = \mathcal{AP}(\mathbb{R}, \mathbb{C})$ *of the algebra* $\mathcal{T}$ *of all trigonometric polynomials*

$$c_0 + c_1 e^{\mathrm{i}\lambda_1 t} + \cdots + c_n e^{\mathrm{i}\lambda_n t}$$

whose frequency set, $\{\lambda_1 \ldots, \lambda_n\} \subset \mathbb{R}$ *is* arbitrary *and* $c_k \in \mathbb{C}$ *for* $k = 1, \ldots, n$. *We consider* $\mathcal{T}$ *as a subspace of the space of bounded continuous functions* $\mathcal{CB}(\mathbb{R}, \mathbb{C})$ *with the* sup-*norm.*

We just write down the main properties of the space $\mathcal{AP}(\mathbb{R}, \mathbb{C})$:

I. Every $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$ is uniformly continuous.

II. $\mathcal{AP}(\mathbb{R}, \mathbb{C})$ is a Banach algebra.

III. For every $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$, there exists a numerable collection of frequencies $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$ whose corresponding *Fourier coefficients*:

$$c[\phi, \lambda_k] = \lim_{t\to\infty} \frac{1}{t - t_0} \int_{t_0}^{t} \phi(s) \cdot e^{-\mathrm{i}\lambda_k s}\, ds$$

which do not vanish and do not depend on t_0. There exists an associated Fourier series

$$\phi(t) \sim \sum_{k=1}^{\infty} c[\phi, \lambda_k] e^{\mathrm{i}\lambda_k t}.$$

IV. For every $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$, there exists the *mean value*,

$$\overline{\phi} = M[\phi] = \lim_{t\to\infty} \frac{1}{t - t_0} \int_{t_0}^{t} \phi(s)\, ds,$$

which is a well-defined positive linear continuous functional, $M : \mathcal{AP}(\mathbb{R}, \mathbb{C}) \to \mathbb{R}$, regardless of $t_0 \in \mathbb{R}$.

V. For every $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$, the Parseval's equality holds:

$$M\left[|\phi|^2\right] = \sum_{k=1}^{\infty} |c[\phi, \lambda_k]|^2 .$$

Definition 4.2. *A continuous function $f : \mathbb{R} \times D \to \mathbb{R}$ is said to be* uniformly almost periodic *with respect to $x \in D \subset \mathbb{R}^n$ if for every compact $K \subset D$,*

$$|f(t+\tau, x) - f(t, x)| < \varepsilon, \quad \forall \in \mathbb{R}, \quad \forall x \in D,$$

for each translation number, $\tau \in T(\varepsilon, f, K)$, and any length $\ell(\varepsilon, f, K) > 0$, not depending on a particular choice x remaining the same on compact set $K \subset D$.

More specifically, if f has *real* Fourier expansion,

$$f(t, x) \sim \overline{f}(x) + \sum_{n=0}^{\infty} a\left[f, \lambda_n\right] \cos\left(\lambda_n t\right) + b[f, \lambda_n] \sin\left(\lambda_n t\right),$$

then f is uniformly almost periodic, whenever the coefficients $a[\cdot, \lambda_n], b[\cdot, \lambda_n]$ do not depend on x, see [4, Chapter VI] .

An important characterization of the space $\mathcal{AP}(\mathbb{R}, \mathbb{C})$ is given by the following assertion, whose proof is given for instance in [5, Propositions 3.6 and 3.7] and [17, Lemmas I.2.1, Theorem I.2.3]

Theorem 4.3. *The following properties are equivalent:*

(1) $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$.

*(2) (Bohr's property) Given $\phi \in \mathcal{CB}(\mathbb{R}, \mathbb{C})$, for every $\varepsilon > 0$ there exists a set of real numbers $T(\phi, \varepsilon) \subset \mathbb{R}$ and an ε-length, $\ell = \ell(\phi, \varepsilon) > 0$, such that each interval $(a, a+\ell)$ of length ℓ contains at least one ε-*almost period *$\tau \in T(\phi, \varepsilon)$, such that*

$$|\phi(t+\tau) - \phi(t)| < \varepsilon, \quad \forall t \in \mathbb{R}$$

(3) (Bochner's property) For every sequence $\{h_n\}_{n=1}^{\infty} \subset \mathbb{R}$, the family of translations

$$\mathcal{F} = \{\phi_n(t) := \phi(t + h_n) \, : \, n \in \mathbb{N}\} \subset \mathcal{CB}(\mathbb{R}, \mathbb{C}),$$

is relatively compact, i.e. there exists a subsequence $\{h_{n_k}\}_{k=1}^{\infty} \subset \{h_n\}_{n=1}^{\infty}$ such that $\phi_{n_k}(t)$ converges uniformly to

$$\phi_\sharp(t) = \lim_{k\to\infty} \phi(t + h_{n_k}) \in \mathcal{CB}(\mathbb{R}, \mathbb{C}).$$

In Theorem 4.3, the limit function $\phi \in \mathcal{CB}(\mathbb{R}, \mathbb{C})$ described in Bochner's property necessarily belong to $\mathcal{AP}(\mathbb{R}, \mathbb{C})$ and also $\phi_\sharp \in \mathcal{AP}(\mathbb{R}, \mathbb{C})$.

Definition 4.4. *A continuous function $\phi : [0, \infty) \to \mathbb{R}$ is said to be* asymptotically almost periodic *if it can be decomposed (uniquely) as*

$$\psi(t) = \psi_\flat(t) + r(t), \quad \psi_\flat \in \mathcal{AP}(\mathbb{R}, \mathbb{C}), \quad r \in \mathcal{CB}(\mathbb{R}, \mathbb{C})$$

where $\lim_{t\to\infty} r(t) = 0$. The space of asymptotically almost periodic functions will be denoted as $\mathcal{AAP}(\mathbb{R}, \mathbb{C})$, so that

$$\mathcal{AP}(\mathbb{R}, \mathbb{C}) \subset \mathcal{AAP}(\mathbb{R}, \mathbb{C}) \subset \mathcal{CB}(\mathbb{R}, \mathbb{C}).$$

Such decomposition happens to be unique.

Theorem 4.5 ([7, Theorem 9.2])**.** *If $f(t, x)$ is almost periodic in t, uniformly with respect to x in compact subsets of $\mathbb{R}^n$ (see Section 4), and if $\phi(t)$ is an asymptotically almost periodic solution of the ODE, $x' = f(t, x)$, $x \in D \subset \mathbb{R}^n$. Then the almost periodic part $\phi_\flat(t)$ of $\phi(t)$ is also a solution of this ODE.*

As in the case of almost periodic functions, there are characterizations of asymptotic almost periodic functions in terms of Bohr's and Bochner's type properties. Specifically the following assertion holds.

Theorem 4.6. *The following properties are equivalent:*

(1) $\psi \in \mathcal{AAP}(\mathbb{R}, \mathbb{C})$.

*(2) (Bohr's property in $[0, \infty)$) Given $\psi \in \mathcal{CB}(\mathbb{R}, \mathbb{C})$, for every $\varepsilon > 0$, there exists a set of real numbers $T(\psi, \varepsilon) \subset \mathbb{R}$ and an ε-length, $\ell = \ell(\psi, \varepsilon) > 0$, such that each interval $(a, a+\ell) \subset [0, \infty)$ of length ℓ contains at least one ε-*almost period *$\tau \in T(\psi, \varepsilon)$, such that*

$$|\psi(t + \tau) - \psi(t)| < \varepsilon, \quad \forall t \geq t_\varepsilon$$

for certain $t_\varepsilon \geq 0$.

(3) (Bochner's property in $[0, \infty)$) Given $\psi \in \mathcal{CB}(\mathbb{R}, \mathbb{C})$, for every sequence $\{h_n\}_{n=1}^\infty \subset \mathbb{R}$, such that $h_n > 0$ and $\lim_{n\to\infty} h_n = \infty$, the family of translations

$$\mathcal{F} = \{\psi_n(t) := \psi(t + h_n) \,:\, n \in \mathbb{N}\} \subset \mathcal{CB}(\mathbb{R}, \mathbb{C}),$$

is relatively compact, i.e. there exists a subsequence $\{h_{n_k}\}_{k=1}^\infty \subset \{h_n\}_{n=1}^\infty$ such that $\psi_{n_k}(t)$ converges uniformly along $[0, \infty)$ to

$$\psi_\sharp(t) = \lim_{k\to\infty} \psi(t + h_{n_k}) \in \mathcal{CB}(\mathbb{R}, \mathbb{C}).$$

In Theorem 4.6, the limit function $\psi \in \mathcal{CB}(\mathbb{R},\mathbb{C})$ described in Bochner's property necessarily belongs to $\mathcal{AAP}(\mathbb{R},\mathbb{C})$ and also $\psi_\sharp \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$. See in [7, Theorem 9.3] and [17, Theorems I.3.2, I.3.4, I.3.5, I.3.9, I.3.10] for a complete discussion of the proof of Theorem 4.6.

Other properties of the space of almost periodic functions that will be useful for our purposes can be summarized in the following assertions.

Theorem 4.7 ([4, Theorems 1.5, 2.1]). *Let $\phi(t), \varphi(t)$ be almost periodic functions, and $a, c \in \mathbb{R}$ constants, then the following functions are also almost periodic*

$$\phi(t) + c\varphi(t),\ \phi(t+a),\ \phi(at),\ \phi(t)\cdot\varphi(t),\ 1/\phi(t) \text{ when } \phi(t) \geq \phi_* > 0.$$

Theorem 4.8 ([4, Theorem 4.1], [17, Theorem 3.3]). *The primitive of an (asymptotically) almost periodic function is (asymptotically) almost periodic if and only if it is bounded on the real line. If the derivative $\psi'(t)$ of a derivable asymptotically almost periodic function $\psi(t)$ is also asymptotically almost periodic, then the associated decomposition*

$$\psi(t) = \psi_\flat(t) + r(t), \quad \psi_\flat \in \mathcal{AP}(\mathbb{R},\mathbb{C}), \quad \lim_{t\to\infty} r(t) = 0,$$

induces the corresponding decomposition associated to $\psi' \in \mathcal{AAP}(\mathbb{R},\mathbb{C})$ as follows

$$\psi'(t) = \psi_\flat'(t) + r'(t), \quad \psi_\flat' = \frac{d\psi_\flat}{dt} \in \mathcal{AP}(\mathbb{R},\mathbb{C}), \quad r' = \frac{dr}{dt}, \quad \lim_{t\to\infty} r'(t) = 0.$$

Theorem 4.9 ([4, Theorem 1.7]). *Let $G(z)$ be a uniformly continuous function of $s \in R \subset \mathbb{C}$. If for $f(x)$ an almost periodic function $f(x) \in R$ for every $x \in \mathbb{R}$, then the function*

$$F(x) = G(f(x))$$

is almost periodic.

5 Aknowlegements

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Szpiro's conjecture when the denominator of the j-invariant is small

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ABSTRACT

We prove Szpiro's conjecture for elliptic curves over the rationals having j-invariant with denominator of logarithmic size with respect to its numerator.

RESUMEN

Demostramos la conjetura de Szpiro para curvas elípticas sobre los racionales que tienen j-invariante con denominador de tamaño logarítmico con respecto a su numerador.

Keywords and Phrases: Szpiro's conjecture, elliptic curves, discriminant, conductor.

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1 Introduction

We begin by fixing some notation. For an elliptic curve E over $\mathbb{Q}$ we write j_E for its j-invariant, Δ_E for the absolute value of its minimal discriminant, and N_E for its conductor.

In the early 80's, Szpiro proposed the following conjecture [14]:

Conjecture 1.1 (Szpiro's conjecture)**.** *Let $\epsilon > 0$. There is a number $c_\epsilon > 0$ depending only on ϵ such that for all elliptic curves E over $\mathbb{Q}$ we have $\Delta_E \le c_\epsilon \cdot N_E^{6+\epsilon}$.*

This conjecture is very deep. Even a weaker version with the exponent $6+\epsilon$ replaced by some fixed constant would have tremendous consequences such as a version of the *abc*-conjecture –in fact, Szpiro's conjecture was the main motivation for the formulation of the *abc*-conjecture, see [4].

Szpiro's conjecture is known for elliptic curves of prime power discriminant by work of Mestre and Oesterlé [5] and for elliptic curves of integral j-invariant by work of Pesenti and Szpiro [10].

At present, the strongest unconditional result valid for all elliptic curves over $\mathbb{Q}$ is the following (effective) estimate by the author [7, Theorem 1.8] valid for any $\epsilon > 0$:

$$\log \Delta_E \le (1/4 + \epsilon) \cdot N_E \log N_E + O_\epsilon(1).$$

This improves the earlier bound

$$\log \Delta_E \le N_E \log N_E + O(N_E \log\log N_E)$$

by Murty and the author [6]; both results use the theory of modular forms.

We mention that an upper bound for the number of (potential) exceptions to Szpiro's conjecture with Δ_E less than a given bound is proved in [1] by Fouvry, Nair, and Tenenbaum.

Our goal is to prove Szpiro's conjecture for elliptic curves whose j-invariant has small denominator. We write $\mathrm{num}(q)$ and $\mathrm{den}(q)$ for the absolute value of the numerator and denominator of a rational number q in reduced form. With this notation, our main result is:

Theorem 1.2 (Main result)**.** *Let $A, B > 0$. For all elliptic curves E over $\mathbb{Q}$ with $\mathrm{den}(j_E) \le A(\log \mathrm{num}(j_E))^B$ we have*

$$\Delta_E \le A \cdot 16^{B+1} N_E^{B+5} (\log N_E)^B.$$

In particular, by setting $B = 1$, Szpiro's conjecture holds when $\mathrm{den}(j_E)$ has logarithmic size with respect to $\mathrm{num}(j_E)$:

Corollary 1.3 (Szpiro's conjecture when $\mathrm{den}(j_E)$ is small)**.** *Let $A > 0$. For all elliptic curves E over $\mathbb{Q}$ with $\mathrm{den}(j_E) \leq A \cdot \log \mathrm{num}(j_E)$ we have*

$$\Delta_E \leq 256A \cdot N_E^6 \log N_E.$$

Note that this generalizes the Pesenti–Szpiro result [10] on Szpiro's conjecture when $j_E \in \mathbb{Z}$.

Finally, let us mention an application. Theorem 1.2 together with [2, Theorem 0.7] by Hindry and Silverman yield a rather uniform bound for the number of S-integral points on elliptic curves E over $\mathbb{Q}$ whenever $\mathrm{den}(j_E) \leq (\log \mathrm{num}(j_E))^B$ for a fixed B and S a finite set of primes. (Nevertheless, it is likely that the latter condition can be weakened for this application by revisiting ideas from Silverman's thesis.)

About number fields: This note is about elliptic curves over $\mathbb{Q}$, but it is conceivable that the same ideas work whenever (potential) modularity is established in a geometric sense, that is, modular parameterizations from Shimura curves to elliptic curves. See [7, Theorem 1.17] for a concrete bound as the one needed. We do not aim for that level of generality here.

2 The height of the j-invariant

The Faltings height of an elliptic curve E over $\mathbb{Q}$ is denoted by $h(E)$ (here we really mean the Faltings height over $\mathbb{Q}$, not the stable one). In [6], Murty and the author used the theory of modular forms to prove the following explicit bound for all E over $\mathbb{Q}$:

$$h(E) < 0.1 \cdot N_E \log N_E + 11.$$

For a rational number q we recall that its logarithmic height is $h(q) = \log \max\{\mathrm{num}(q), \mathrm{den}(q)\}$. It turns out that $h(E)$ is related to $h(j_E)$ in a very explicit way; Silverman [12] proved

$$h(j_E) \leq 12h(E) + O(\log(2 + h(j_E)))$$

where the error term has an effective implicit constant. This has been made explicit by Pellarin [9] and, in a sharper form, by Pazuki [8]. For our purposes [9, Lemme 5.2] is enough; this gives $h(j_E) \leq 24 \max\{1, h(E)\} + 94.3$. Putting these results together we obtain:

Corollary 2.1. *For all elliptic curves E over $\mathbb{Q}$ we have $h(j_E) \leq 16 \cdot N_E \log N_E$.*

Proof. The previous discussion leads to

$$h(j_E) \leq 94.3 + 24 \cdot (0.1 \cdot N_E \log N_E + 11) = 2.4 \cdot N_E \log N_E + 358.3.$$

The result follows from the well-known fact that $N_E \geq 11$ for all elliptic curves over $\mathbb{Q}$ —this is classical, but a simple way to see it is that there are no rational Hecke newforms of weight 2 for $\Gamma_0(N)$ when $N < 11$. □

3 An application of Tate's algorithm

For a prime number p we denote by $v_p : \mathbb{Q}^\times \to \mathbb{Z}$ the p-adic valuation.

Let E be an elliptic curve over $\mathbb{Q}$. The primes p that divide Δ_E are the same as the ones that divide N_E. We split these primes p into three types:

- *Type 1:* $v_p(j_E) \geq 0$.
- *Type 2:* $v_p(j_E) < 0$ and E has multiplicative reduction at p.
- *Type 3:* $v_p(j_E) < 0$ and E has additive reduction at p.

The following is proved in [10].

Lemma 3.1 (Primes of Type 1)**.** *If p is of Type 1, then $v_p(\Delta_E) \leq 5v_p(N_E)$.*

As noted in [10], this immediately implies Szpiro's conjecture whenever $j_E \in \mathbb{Z}$.

Let us now consider p of Type 2. The Kodaira type of the special fibre of the minimal regular model of E over $\mathbb{Z}_p$ is I_n for some $n \geq 1$. The output of Tate's algorithm summarized in [13, p. 365, Table 4.1] (which refined Tate's table in [15]) shows that $n = -v_p(j_E) = v_p(\Delta_E)$. So we find:

Lemma 3.2 (Primes of Type 2)**.** *If p is of Type 2, then $v_p(\Delta_E) = -v_p(j_E)$.*

Finally we deal with the primes p of Type 3. In this case the Kodaira type of E at p is I_n^* for a certain integer $n \geq 1$, see the discussion in [3, p. 42]. Let us first deal with $p = 3$ and then with $p = 2$.

If $p \geq 3$, from the data in the I_n^* column of [13, p. 365, Table 4.1] we get

$$v_p(\Delta_E) = 6 + n = 6 - v_p(j_E) = 3v_p(N_E) - v_p(j_E).$$

Let us now consider the case $p = 2$. The number m of geometrically irreducible components of the special fibre of the minimal regular model at $p = 2$ is $m = n + 5$ (see the table in [15].) By the Saito–Ogg formula we have $v_2(N_E) = v_2(\Delta_E) - m + 1 = v_2(\Delta_E) - n - 4$ which gives $v_2(\Delta_E) = v_2(N_E) + 4 + n$.

We need some control on the integer n. In [3, Theorem 2.8] gives the existence of a suitable quadratic extension $L/\mathbb{Q}$ such that if $s+1$ is the valuation of its different ideal over 2, then $n = -v_2(j_E) + 4s$. By [11, p. 58, Remark 1], we have $s \leq 2$ so we get $n \leq -v_2(j_E) + 8$. Therefore

$$v_2(\Delta_E) \leq v_2(N_E) - v_2(j_E) + 12 \leq 3v_2(N_E) - v_2(j_E) + 8$$

because $v_2(N_E) \geq 2$ (additive reduction). Let us summarize our findings:

Lemma 3.3 (Primes of Type 3)**.** *Let p be a prime of Type 3 and write $\delta_p = 8$ if $p = 2$ and $\delta_p = 0$ if $p \geq 3$. Then $v_p(\Delta_E) \leq 3v_p(N_E) - v_p(j_E) + \delta_p$.*

From these three lemmas we deduce the following result, which can be of independent interest:

Corollary 3.4. *Let E be an elliptic curve over $\mathbb{Q}$. Then Δ_E divides $16 \cdot \mathrm{den}(j_E)N_E^5$.*

Perhaps an explanation is needed for the factor 16. This is only necessary when $p = 2$ is a prime of Type 3. In this case

$$v_2(\Delta_E) \leq 3v_2(N_E) - v_2(j_E) + \delta_2 \leq 5v_2(N_E) - 2 \cdot 2 - v_2(j_E) + 8 \leq 5v_2(N_E) + 4 + v_2(\mathrm{den}(j_E)),$$

where we used $v_2(N_E) \geq 2$ as $p = 2$ is of additive reduction.

4 Proof of the main result

Proof of Theorem 1.2. By Corollary 2.1 we have

$$\mathrm{den}(j_E) \leq A(\log \mathrm{num}(j_E))^B \leq Ah(j_E)^B \leq A \cdot 16^B N_E^B (\log N_E)^B.$$

Putting this estimate together with Corollary 3.4, we find $\Delta_E \leq A \cdot 16^{B+1} N_E^{B+5} (\log N_E)^B$. ☐

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On the analytical solution of the Cauchy problem for a linear set-valued differential equation with a Hukuhara derivative

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ABSTRACT

The article considers the Cauchy problem for a linear set-valued differential equation with the Hukuhara derivative and derives an analytical formula for its solution.

RESUMEN

Este artículo considera el problema de Cauchy para una ecuación diferencial lineal con valores en conjuntos con la derivada de Hukuhara y obtiene una fórmula analítica para su solución.

Keywords and Phrases: Differential equation, linear, set-valued mapping, Hukuhara derivative.

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1 Introduction

In 1967, M. Hukuhara introduced the integral and the derivative for set-valued mappings and considered how they are related to each other [8]. These derivative and integral generalize the ordinary derivative and Riemann integral for a single-valued function to the set-valued case. In 1969, F. S. de Blasi and F. Iervolino considered differential equations with the Hukuhara derivative [3]. Subsequently, many authors introduced other derivatives and integrals for set-valued mappings and studied the properties of solutions to various set-valued equations (see [10, 12–18, 21–26] and references therein). Such equations are similar in appearance to the corresponding classical equations, but their study and solutions must account for their set-valued nature. Hence, traditional methods and approaches used for single-valued systems are not always applicable to set-valued systems, necessitating new or alternative methods. Furthermore, the set-valued nature introduces new properties that require exploration.

The article considers the Cauchy problem for a linear set-valued differential equation with the Hukuhara derivative

$$\begin{cases} D_H X(t) = AX(t) + F(t), \\ X(0) = X_0, \end{cases} \tag{1.1}$$

where $X : [0,T] \to conv(\mathbb{R}^n)$ is the unknown set-valued mapping, $D_H X(t)$ is the Hukuhara derivative, $A \in \mathbb{R}^{n\times n}$ is a non-singular matrix, $F : [0,T] \to conv(\mathbb{R}^n)$ is a continuous set-valued mapping, and $X_0 \in conv(\mathbb{R}^n)$ is the initial set. An analytical solution formula for problem (1.1) is obtained, and its difference from the single-valued case is demonstrated.

Previously in [19, 22], it was proven that Cauchy problem (1.1) has a unique solution if A is a non-singular matrix. However, unlike in the single-valued case, an explicit solution to such an equation was not provided.

Subsequently, an explicit form of the solution was obtained for some special cases.

If $A = aI_n$, where I_n is the identity matrix of size n and $a > 0$, then Cauchy problem (1.1) takes the form

$$D_H X(t) = aX(t) + F(t),\ X(0) = X_0, \tag{1.2}$$

and, according to [3, 22], has the following solution:

$$X(t) = e^{at}X_0 + \int_0^t e^{a(t-s)}F(s)\,ds \tag{1.3}$$

and if $F(t) \equiv F$ for all $t \in [0,T]$, then, according to [9], has the following solution:

$$X(t) = e^{at}X_0 + \frac{e^{at}-1}{a}F.$$

If $A \in GL(n, \mathbb{R})$ and $F(t) \equiv F$, then Cauchy problem (1.1) takes the form

$$D_H X(t) = AX(t) + F, \quad X(0) = X_0,$$

and, according to [11], has the following solution:

$$X(t) = X_0 + \sum_{i=1}^{\infty} \left\{ \frac{t^i}{i!} A^i G \right\},$$

where $F \in conv(\mathbb{R}^n)$, $G = X_0 + A^{-1}F$.

We also note that the results of this work can be used to extend the research begun for linear homogeneous set-valued differential equations with generalized derivative [9, 14], for linear homogeneous set-valued differential equations with conformal fractional derivative [10, 12] and for linear homogeneous set-valued differential equations with conformal fractional-fractal derivative [13] to a more general class of problems - linear inhomogeneous set-valued differential equations.

2 Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and let $conv(\mathbb{R}^n)$ be the space of nonempty convex compact subsets of $\mathbb{R}^n$, equipped with the Pompeiu-Hausdorff metric:

$$h(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} \|x - y\|, \sup_{y \in Y} \inf_{x \in X} \|x - y\| \right\},$$

where $X, Y \in conv(\mathbb{R}^n)$.

In the space $conv(\mathbb{R}^n)$, in addition to standard set operations, we consider the following:

- sum of sets X and Y: $\quad X + Y = \bigcup_{x \in X,\, y \in Y} \{x + y\}$.
- scalar multiplication of $\lambda \in \mathbb{R}$ with the set X: $\quad \lambda X = \bigcup_{x \in X} \{\lambda x\}$.

The following properties hold [15, 22, 24].

Properties A. For all $X, Y, Z, W \in conv(\mathbb{R}^n)$ and $\alpha, \beta, \lambda \in \mathbb{R}$:

(1) $(conv(\mathbb{R}^n), h)$ is a complete metric space;

(2) $X + Y = Y + X \in conv(\mathbb{R}^n)$;

(3) if $X + Z = Y + Z$, then $X = Y$;

(4) if $X \subset Z$, $Y \subset W$, then $X + Y \subset Z + W$;

(5) $\lambda X \in conv(\mathbb{R}^n)$;

(6) $\alpha(\beta X) = (\alpha\beta)X$;

(7) if $\alpha\beta \geq 0$, then $(\alpha + \beta)X = \alpha X + \beta X$;

(8) $\lambda(X+Y) = \lambda X + \lambda Y$;

(9) $h(X+Z, Y+Z) = h(X,Y)$;

(10) $h(\lambda X, \lambda Y) = |\lambda| h(X,Y)$.

It is known that the space $conv(\mathbb{R}^n)$ is not a linear space with respect to the given operations, because in the general case, there is no opposite element for $X \in conv(\mathbb{R}^n)$, that is, no set $-X$ such that $X + (-X) = \{\mathbf{0}\}$, the opposite element exists only in the case when $X \in \mathbb{R}^n$. The absence of an opposite element in the space $conv(\mathbb{R}^n)$ leads to the ambiguity in defining the concept of set difference and the conditions for its existence.

In this paper, we will use the difference of Hukuhara [8].

Definition 2.1 ([8]). *Let $X, Y \in conv(\mathbb{R}^n)$. A set $Z \in conv(\mathbb{R}^n)$ such that $X = Y + Z$ is called the Hukuhara difference of the sets X and Y, denoted $X \overset{H}{-} Y$.*

The Hukuhara difference has the following properties [8, 15, 22, 24, 25].

Properties B.

(1) If the Hukuhara difference $X \overset{H}{-} Y$ exists, then it is unique and $(X \overset{H}{-} Y) + Y = X$.

(2) $X \overset{H}{-} X = \{\mathbf{0}\}$ for all $X \in conv(\mathbb{R}^n)$.

(3) $(X+Y) \overset{H}{-} Y = X$ for all $X, Y \in conv(\mathbb{R}^n)$.

Also, let us add one more operation: the product of a matrix with a set $AX = \bigcup_{x \in X} \{Ax\}$, where $A \in \mathbb{R}^{n \times n}$ is a real matrix of size $n \times n$ and $X \in conv(\mathbb{R}^n)$.

We will list some properties of this operation [4, 7].

Properties C.

(1) If $A \in \mathbb{R}^{n \times n}$ and $X \in conv(\mathbb{R}^n)$, then $AX \in conv(\mathbb{R}^k)$, where $k = \mathrm{rank}(A)$;

(2) If $A \in \mathbb{R}^{n \times n}$ and $X, Y \in conv(\mathbb{R}^n)$, then $A(X+Y) = AX + AY$;

(3) If $A, B \in \mathbb{R}^{n \times n}$ and $X \in conv(\mathbb{R}^n)$, then $(A+B)X \subseteq AX + BX$;

(4) If $A \in \mathbb{R}^{n \times n}$, $X, Y \in conv(\mathbb{R}^n)$ and $X \subseteq Y$, then $AX \subseteq AY$.

Remark 2.2. *For practical computation of the set $Y = AX$, either the singular value decomposition (SVD) of the matrix A [4, 6, 12] or the mathematical apparatus of support functions of sets [1, 7, 15, 19, 25] is typically used.*

Let $X : [0, T] \to conv(\mathbb{R}^n)$ be a set-valued mapping.

Definition 2.3 ([8])**.** *Let $t \in (0, T)$. If for all sufficiently small $\varepsilon > 0$ such, that $(t - \varepsilon, t + \varepsilon) \subset (0, T)$, the Hukuhara differences $X(t + \varepsilon) \overset{H}{-} X(t)$ and $X(t) \overset{H}{-} X(t - \varepsilon)$ exist, and there exists $Z \in conv(\mathbb{R}^n)$ such that the following equality holds:*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \left(X(t + \varepsilon) \overset{H}{-} X(t) \right) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \left(X(t) \overset{H}{-} X(t - \varepsilon) \right) = Z, \tag{2.1}$$

we will say that the set-valued mapping $X(\cdot)$ has the Hukuhara derivative at the point $t \in (0, T)$ and $D_H X(t) = Z$.

If $D_H X(t)$ exists for all $t \in (0, T)$, and the limits $\lim\limits_{t \downarrow 0} D_H X(t)$ and $\lim\limits_{t \uparrow T} D_H X(t)$ exist, we will assume that $D_H X(0) = \lim\limits_{t \downarrow 0} D_H X(t)$ and $D_H X(T) = \lim\limits_{t \uparrow T} D_H X(t)$.

Definition 2.4. *If the Hukuhara derivative $D_H X(t)$ exists for all $t \in [0, T]$, we will say that the set-valued mapping $X(\cdot)$ is differentiable in the Hukuhara sense on $[0, T]$.*

The Hukuhara derivative has the following properties [12, 15, 19, 22, 24].

Properties D.

(1) If the set-valued mapping $X(t) \equiv X$ for all $t \geq 0$, then $DX(t) \equiv \{\mathbf{0}\}$;

(2) if the set-valued mappings $X(\cdot)$ and $Y(\cdot)$ are differentiable at $t > 0$, then

$$D_H(\alpha X(t) + \beta Y(t)) = \alpha D_H X(t) + \beta D_H Y(t);$$

(3) if the set-valued mapping $X(\cdot)$ is continuous on $\mathbb{R}_+$, then

$$D_H \left(\int_0^t X(s) ds \right) = X(t), \quad t > 0,$$

where $\alpha, \beta \geq 0$, the integral is understood in the sense of the Riemann-Hukuhara integral [20, 25].

The Riemann-Hukuhara integral is defined analogously to the Riemann integral for single-valued functions, taking into account the set-valued nature of the integrand mapping [20, 25] and possesses the following properties [8, 19, 20, 22, 24, 25].

Properties E. If $\lambda : [0,T] \to \mathbb{R}$ is a continuous function such that $\lambda(t)\lambda(s) \geq 0$ for all $t, s \in [0,T]$, $A : [0,T] \to \mathbb{R}^{n\times n}$ is a continuous matrix function and $A(t) \in GL(n,\mathbb{R})$ for all $t \in [0,T]$, $X \in conv(\mathbb{R}^n)$ and $F, G : [0,T] \to conv(\mathbb{R}^n)$ are continuous set-valued mappings, then

(1) $\int\limits_0^t F(s)\,ds \in conv(\mathbb{R}^n)$ for all $t \in (0,T]$;

(2) $\int\limits_0^t F(s)\,ds + \int\limits_t^T F(s) = \int\limits_0^T F(s)\,ds$ for all $t \in [0,T]$;

(3) $\int\limits_0^t (F(s)+G(s))\,ds = \int\limits_0^t F(s)\,ds + \int\limits_0^t G(s)\,ds$ for all $t \in [0,T]$;

(4) if the set-valued mapping $F(\cdot)$ is continuously differentiable in the sense of Hukuhara on $[0,T]$, then $\int\limits_0^t D_H F(s)\,ds = F(t) \stackrel{H}{-} F(0)$ for all $t \in [0,T]$;

(5) $\int\limits_0^t \lambda(s) X\,ds = \int\limits_0^t \lambda(s)\,ds\,X$ for all $t \in [0,T]$;

(6) $\int\limits_0^t A(s)\,ds\,X \subseteq \int\limits_0^t A(s) X\,ds$ for all $t \in [0,T]$;

(7) if $F(t) \subset G(t)$ for all $t \in [0,T]$, then $\int\limits_0^t F(s)\,ds \subseteq \int\limits_0^t G(s)\,ds$ for all $t \in [0,T]$;

(8) $h\left(\int\limits_0^t F(s)\,ds, \int\limits_0^t G(s)\,ds\right) \leq \int\limits_0^t h(F(s),G(s))\,ds$ for all $t \in [0,T]$.

3 Linear set-valued differential equation with the Hukuhara derivative.

Now we will consider the Cauchy problem (1.1).

Definition 3.1. *A set-valued mapping $X : [0,T] \to conv(\mathbb{R}^n)$ is called a solution of Cauchy problem* (1.1) *if it is Hukuhara differentiable and satisfies system* (1.1) *for all* $t \in [0,T]$.

Suppose that equation (1.1) has the following solution:

$$X(t) = X_0 + \sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}U^i Z\right\} + \sum_{i=0}^{\infty}\left\{\int\limits_0^t \left[\frac{(t-s)^i V^i}{i!} Y(s)\right] ds\right\}, \tag{3.1}$$

where $t \in [0,T]$, $U, V \in \mathbb{R}^{n\times n}$ are non-singular matrices, $Y : [0,T] \to conv(\mathbb{R}^n)$ is a continuous set-valued mapping, and $Z \in conv(\mathbb{R}^n)$.

It is easy to verify that $X(0) = X_0$.

First, let us prove some properties of (3.1). For this purpose, we introduce the following notations:

$$\eta_U = \|U\| = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} u_{ij}^2}, \quad \eta_V = \|V\|,$$

$$\gamma_X = \frac{1}{2}diam(X_0) = \frac{1}{2}\max_{x,y\in X_0}\|x-y\|, \quad \gamma_Z = \frac{1}{2}diam(Z), \quad \gamma_Y(t) = \frac{1}{2}diam(Y(t)),$$

$$X_{1,0}(t) = X_0,\ X_{2,0}(t) = \int_0^t Y(s)\,ds, \quad X_{1,i}(t) = \frac{t^i}{i!}U^iZ,\ X_{2,i}(t) = \int_0^t \frac{(t-s)^i}{i!}V^iY(s)\,ds,\ i \geq 1,$$

$\xi^j(t)$ – the Lebesgue measure of the set $X^j(t) = \sum_{i=0}^{j} X_{1,i}(t) + \sum_{i=0}^{j} X_{2,i}(t),\ j \geq 0,$

$\vartheta(t)$ – the Lebesgue measure of the set $X(t)$.

Since $X(0) = X^0(0) = X^1(0) = \cdots = X^j(0) = \cdots$, we have $\vartheta(0) = \xi^0(0) = \xi^1(0) = \cdots = \xi^j(0) = \cdots$.

Let the vectors $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^n$, and $\mathbf{y}(t) \in \mathbb{R}^n$ be such that $X_0 \subseteq B_{\gamma_X}(\mathbf{x})$, $Z \subseteq B_{\gamma_Z}(\mathbf{z})$, and $Y(t) \subseteq B_{\gamma_{Y(t)}}(\mathbf{y}(t))$, where $B_r(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}_0\| \leq r\}$ is n-ball of radius r and center $\mathbf{x}_0$.

Then, for all $t \in [0,T]$ and $i \geq 1$:

$$\begin{aligned} X_{1,i}(t) &= \frac{t^i}{i!}U^iZ \subseteq \frac{t^i}{i!}U^iB_{\gamma_Z}(\mathbf{z}) = \frac{t^i}{i!}U^i\mathbf{z} + \frac{t^i}{i!}U^iB_{\gamma_Z}(\mathbf{0}) \subseteq \frac{t^i}{i!}U^i\mathbf{z} + \frac{t^i}{i!}\eta_U^iB_{\gamma_Z}(\mathbf{0}) \\ &= \frac{t^iU^i}{i!}\mathbf{z} + \frac{t^i\eta_U^i\gamma_Z}{i!}B_1(\mathbf{0}), \end{aligned}$$

$$\begin{aligned} X_{2,i}(t) &= \int_0^t \frac{(t-s)^i}{i!}V^iY(s)\,ds \subseteq \int_0^t \frac{(t-s)^i}{i!}V^iB_{\gamma_{Y(s)}}(\mathbf{y}(s))\,ds \\ &= \int_0^t \left[\frac{(t-s)^i}{i!}V^i\mathbf{y}(s) + \frac{(t-s)^i}{i!}V^iB_{\gamma_{Y(s)}}(\mathbf{0})\right]ds \\ &\subseteq \int_0^t \frac{(t-s)^i}{i!}V^i\mathbf{y}(s)\,ds + \int_0^t \frac{(t-s)^i}{i!}\eta_V^iB_{\gamma_{Y(s)}}(\mathbf{0})\,ds \\ &= \int_0^t \frac{(t-s)^i}{i!}V^i\mathbf{y}(s)\,ds + \int_0^t \frac{(t-s)^i\eta_V^i}{i!}\,\gamma_Y(s)\,ds\,B_1(\mathbf{0}). \end{aligned}$$

Consequently,

$$\begin{aligned} h\left(X^j(t),\{\mathbf{0}\}\right) &\leq \|\mathbf{x}\| + h\left(\gamma_X B_1(\mathbf{0}),\{\mathbf{0}\}\right)\left\|\left[\sum_{i=1}^{j}\frac{t^i}{i!}U^i\right]\mathbf{z}\right\| + h\left(\left[\sum_{i=1}^{j}\frac{t^i}{i!}\eta_U^i\gamma_Z\right]B_1(\mathbf{0}),\{\mathbf{0}\}\right) \\ &\quad + \left\|\sum_{i=0}^{j}\int_0^t\left[\frac{(t-s)^i}{i!}V^i\right]\mathbf{y}(s)\,ds\right\| + h\left(\sum_{i=0}^{j}\int_0^t\left[\frac{(t-s)^i}{i!}\eta_V^i\gamma_Y(s)\right]ds\,B_1(\mathbf{0}),\{\mathbf{0}\}\right) \\ &\leq \|\mathbf{x}\| + \gamma_X + \left[\sum_{i=1}^{j}\frac{t^i}{i!}\eta_U^i\right]\|\mathbf{z}\| + \sum_{i=1}^{j}\frac{t^i}{i!}\eta_U^i\gamma_Z + \int_0^t\left[\sum_{i=0}^{j}\frac{(t-s)^i}{i!}\eta_V^i\right]\|\mathbf{y}(s)\|\,ds \\ &\quad + \int_0^t\sum_{i=0}^{j}\frac{(t-s)^i}{i!}\eta_V^i\gamma_Y(s)\,ds. \end{aligned}$$

Because

$$\sum_{i=1}^{\infty}\frac{t^i}{i!}\eta_U^i\gamma_Z = \left(e^{t\eta_U}-1\right)\gamma_Z, \quad \sum_{i=0}^{\infty}\frac{(t-s)^i}{i!}\eta_V^i\gamma_Y(s) = e^{(t-s)\eta_V}\gamma_Y(s),$$
$$\sum_{i=1}^{\infty}\frac{t^i}{i!}U^i = e^{tU}-I_n, \quad \sum_{i=0}^{\infty}\frac{(t-s)^i}{i!}V^i = e^{(t-s)V},$$

then

$$\begin{aligned} \lim_{j\to\infty}X^j(t) \subseteq \mathbf{x} + \gamma_X B_1(\mathbf{0}) + \left[e^{tU}-I_n\right]\mathbf{x} + \left[\left(e^{t\eta_U}-1\right)\gamma_Z\right]B_1(\mathbf{0}) + \\ + \int\limits_0^t e^{(t-s)V}\mathbf{y}(s)\,ds + \left[\int\limits_0^t e^{(t-s)\eta_V}\gamma_Y(s)\,ds\right]B_1(\mathbf{0}) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \lim_{j\to\infty}h\left(X^j(t),\{\mathbf{0}\}\right) \leq \|\mathbf{x}\| + \gamma_X + \left[e^{t\eta_U}-1\right]\|\mathbf{z}\| + \left(e^{t\eta_U}-1\right)\gamma_Z + \\ + \int\limits_0^t e^{(t-s)\eta_V}\|\mathbf{y}(s)\|\,ds + \int\limits_0^t e^{(t-s)\eta_V}\gamma_Y(s)\,ds. \end{aligned} \tag{3.3}$$

Since for $p > q,\ p,q\in\mathbb{N}$

$$\begin{aligned} h(X^p(t),X^q(t)) \leq h\Bigg(& X_0 + \sum_{i=1}^{p}\left\{\frac{t^i}{i!}U^iZ\right\} + \sum_{i=0}^{p}\int\limits_0^t\left\{\frac{(t-s)^i}{i!}V^iY(s)\right\}ds, \\ & X_0 + \sum_{i=1}^{q}\left\{\frac{t^i}{i!}U^iZ\right\} + \sum_{i=0}^{q}\int\limits_0^t\left\{\frac{(t-s)^i}{i!}V^iY(s)\right\}ds\Bigg) \\ \leq h & \left(\sum_{i=q+1}^{p}\left\{\frac{t^i}{i!}U^iZ\right\} + \sum_{i=q+1}^{p}\int\limits_0^t\left\{\frac{(t-s)^i}{i!}V^iY(s)\right\}ds,\{\mathbf{0}\}\right) \end{aligned}$$

$$
\begin{aligned}
&\leq \left\{\sum_{i=q+1}^{p} \frac{t^i}{i!}\eta_U^i\right\} \|\mathbf{z}\| + \left\{\sum_{i=q+1}^{p} \frac{t^i}{i!}\eta_U^i\right\} \gamma_Z + \sum_{i=q+1}^{p} \int\limits_0^t \left\{\frac{(t-s)^i}{i!}\eta_V^i\right\} \|\mathbf{y}(s)\|\, ds \\
&\quad + \sum_{i=q+1}^{p} \int\limits_0^t \frac{(t-s)^i}{i!}\eta_V^i \gamma_Y(s)\, ds \\
&\leq \left\{\sum_{i=q+1}^{p} \frac{T^i}{i!}\eta_U^i\right\} (\|\mathbf{z}\| + \gamma_Z) \\
&\quad + \left\{\sum_{i=q+1}^{p} \int\limits_0^T \frac{(T-s)^i}{i!}\eta_V^i\, ds\right\} \left(\max_{t\in[0,T]} \|\mathbf{y}(t)\| + \max_{t\in[0,T]} \gamma_Y(t)\right) \\
&= \left\{\sum_{i=q+1}^{p} \frac{T^i}{i!}\eta_U^i\right\} (\|\mathbf{z}\| + \gamma_Z) \\
&\quad + \left\{\sum_{i=q+1}^{p} \frac{T^{i+1}}{(i+1)!}\eta_V^i\right\} \left(\max_{t\in[0,T]} \|\mathbf{y}(t)\| + \max_{t\in[0,T]} \gamma_Y(t)\right).
\end{aligned}
$$

Therefore, for any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that for all $p > q > N$ the inequality $h(X^p(t), X^q(t)) < \varepsilon$ holds, meaning that the sequence $\{X^k(t)\}_{k=0}^{\infty}$ converges uniformly to $X(t)$ on the interval $[0,T]$.

Also, we have for all $t \in [0,T]$

$$\xi^0(t) \leq \xi^1(t) \leq \cdots \leq \xi^j(t) \leq \cdots$$

and

$$\lim_{j\to\infty} \xi^j(t) \leq \left(\gamma_X + \left(e^{t\eta_U} - 1\right)\gamma_Z + \int\limits_0^t e^{(t-s)\eta_V}\gamma_Y(s)\, ds\right)^n \varrho, \tag{3.4}$$

where $\varrho = \dfrac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$ is the Lebesgue measure of the set $B_1(\mathbf{0})$ [5], and $\Gamma(n)$ is the Gamma function.

Thus, $\lim\limits_{j\to\infty} \xi^j(t)$ exists and equals $\vartheta(t)$, for all $t \in [0,T]$, and the sequence $\{\xi^j(t)\}_{j=0}^{\infty}$ converges uniformly to $\vartheta(t)$ on the interval $[0,T]$.

Substitute $X(\cdot)$ into the equation (1.1) and check the identity:

$$D_H X(t) = AX(t) + F(t),$$

or find the conditions for its validity.

Since $D_H(X_0) = \{\mathbf{0}\}$, $D_H\left(\int\limits_0^t Y(s)\,ds\right) = Y(t)$, $D_H X_{1,i}(t) = D_H\left(\frac{t^i}{i!}U^i Z\right) = \frac{t^{i-1}}{(i-1)!}U^i Z = UX_{1,i-1}(t)$,

$$D_H X_{2,i}(t) = D_H\left(\int\limits_0^t \left\{\frac{(t-s)^i\,V^i}{i!}Y(s)\right\} ds\right) = \int\limits_0^t \left\{\frac{(t-s)^{i-1}\,V^i}{(i-1)!}Y(s)\right\} ds = VX_{2,i-1}(t)$$

for all $i = 1, 2, \ldots$, and the series

$$\sum_{i=0}^{\infty} X_{1,i}(t) = \sum_{i=0}^{\infty}\left\{\frac{t^i}{i!}U^i Z\right\}, \quad \sum_{i=0}^{\infty} D_H X_{1,i}(t) = U\sum_{i=1}^{\infty} X_{1,i-1}(t) = U\sum_{i=0}^{\infty} X_{1,i}(t),$$

$$\sum_{i=0}^{\infty} X_{2,i}(t) = \sum_{i=0}^{\infty}\int\limits_0^t \left\{\frac{(t-s)^i\,V^i}{i!}Y(s)\right\} ds,$$

$$\sum_{i=0}^{\infty} D_H X_{2,i}(t) = \sum_{i=0}^{\infty} D_H\left(\int\limits_0^t \left\{\frac{(t-s)^i\,V^i}{i!}Y(s)\right\} ds\right) = Y(t) + V\sum_{i=0}^{\infty} X_{2,i}(t)$$

converge uniformly on $[0, T]$, then

$$D_H X(t) = UZ + U\sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}U^i Z\right\} + Y(t) + V\sum_{i=0}^{\infty}\int\limits_0^t \left[\frac{(t-s)^i\,V^i}{i!}Y(s)\right] ds.$$

Also,

$$AX(t) + F(t) = AX_0 + A\sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}U^i Z\right\} + A\sum_{i=0}^{\infty}\left\{\int\limits_0^t \left[\frac{(t-s)^i\,V^i}{i!}Y(s)\right] ds\right\} + F(t).$$

Then we obtain the following equality:

$$UZ + U\sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}U^i Z\right\} + Y(t) + V\sum_{i=0}^{\infty}\int\limits_0^t \left[\frac{(t-s)^i\,V^i}{i!}Y(s)\right] ds =$$

$$= AX_0 + A\sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}U^i Z\right\} + A\sum_{i=0}^{\infty}\int\limits_0^t \left[\frac{(t-s)^i\,V^i}{i!}Y(s)\right] ds + F(t),$$

which will hold if $U = V = A$, $Z = X_0$ and $Y(t) = F(t)$.

Thus, the following theorem can be formulated.

Theorem 3.2. *If the matrix $A \in \mathbb{R}^{n\times n}$ is a non-degenerate matrix and the set-valued map $F(\cdot) : [0,T] \to conv(\mathbb{R}^n)$ is continuous on $[0,T]$, then the system (1.1) has the following solution:*

$$X(t) = X_0 + \sum_{i=1}^{\infty} \left\{ \frac{t^i}{i!} A^i X_0 \right\} + \sum_{i=0}^{\infty} \int_0^t \frac{(t-s)^i A^i}{i!} F(s)\, ds. \tag{3.5}$$

Remark 3.3. *Accordingly, considering (3.2), (3.3), (3.4) and (3.5), we obtain*

$$X(t) \subseteq e^{tA}\mathbf{x} + e^{t\|A\|}\gamma_X B_1(\mathbf{0}) + \int_0^t e^{(t-s)A}\mathbf{f}(s)\, ds + \left[\int_0^t e^{(t-s)\|A\|}\gamma_F(s)\, ds \right] B_1(\mathbf{0}),$$

$$h\left(X(t), \{\mathbf{0}\}\right) \le e^{t\|A\|}\|\mathbf{x}\| + e^{t\|A\|}\gamma_X + \int_0^t e^{(t-s)\|A\|}\|\mathbf{f}(s)\|\, ds + \int_0^t e^{(t-s)\|A\|}\gamma_F(s)\, ds$$

and

$$\vartheta(t) \le \left(e^{t\|A\|}\gamma_X + \int_0^t e^{(t-s)\|A\|}\gamma_F(s)\, ds \right)^n \varrho\,,$$

where $\gamma_F(t) = \frac{1}{2} \max_{f_1, f_2 \in F(t)} \|f_1 - f_2\|$, the vector $\mathbf{f}(t) \in \mathbb{R}^n$ such that $F(t) \subseteq B_{\gamma_F(t)}(\mathbf{f}(t))$.

Here are some corollaries of the theorem and remarks.

Corollary 3.4. *If $X_0 \in \mathbb{R}^n$ and $F : [0,T] \to \mathbb{R}^n$, then*

$$\sum_{i=0}^{\infty} \left\{ \frac{t^i}{i!} A^i X_0 \right\} = \left\{ \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i \right\} X_0 = e^{tA} X_0,$$

$$\sum_{i=0}^{\infty} \left\{ \int_0^t \frac{(t-s)^i A^i}{i!} F(s)\, ds \right\} = \int_0^t \sum_{i=0}^{\infty} \left\{ \frac{(t-s)^i A^i}{i!} \right\} F(s)\, ds = \int_0^t e^{(t-s)A} F(s)\, ds,$$

and accordingly, (3.5) can be rewritten in the following form, yielding the well-known formula for ordinary linear differential equations:

$$X(t) = e^{tA} X_0 + \int_0^t e^{(t-s)A} F(s)\, ds.$$

Corollary 3.5. *If $X_0 \in conv(\mathbb{R}^n)$ and $F : [0,T] \to \mathbb{R}^n$, then in this case, (3.5) will take the form*

$$X(t) = \sum_{i=0}^{\infty} \left\{ \frac{t^i}{i!} A^i X_0 \right\} + \int_0^t e^{(t-s)A} F(s)\, ds.$$

Remark 3.6. *Note that from Property C (3), we have*

$$\sum_{i=0}^{\infty}\left\{\frac{t^i}{i!}A^iX_0\right\} \supseteq \left\{\sum_{i=0}^{\infty}\frac{t^i}{i!}A^i\right\}X_0 = e^{At}X_0,$$

that is, in the general case we have

$$e^{At}X_0 + \int_0^t e^{(t-s)A}F(s)\,ds \subseteq X(t).$$

However, if, for example,

(1) *the matrix A is such that its singular values $\sigma_1,\dots,\sigma_n$ satisfy the condition $\sigma_1 = \cdots = \sigma_n = \sigma$ and the set X_0 is such that $AX_0 = \sigma X_0$, then $A^2X_0 = AAX_0 = A\sigma X_0 = \sigma AX_0 = \sigma^2X_0,\dots,A^kX_0 = \sigma^kX_0,\dots$ and, accordingly,*

$$\sum_{i=0}^{\infty}\left\{\frac{t^i}{i!}A^iX_0\right\} = \sum_{i=0}^{\infty}\left\{\frac{t^i}{i!}\sigma^iX_0\right\} = \left\{\sum_{i=0}^{\infty}\frac{t^i\sigma^i}{i!}\right\}X_0 = e^{\sigma t}X_0 = e^{At}X_0.$$

Then the system (1.1) *has the following solution [12, 14, 21]:*

$$X(t) = e^{\sigma t}X_0 + \int_0^t e^{(t-s)A}F(s)\,ds = e^{At}X_0 + \int_0^t e^{(t-s)A}F(s)\,ds.$$

For example, the condition $AX_0 = \sigma X_0$ holds for

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad \text{and} \quad X_0 = B_1(\mathbf{0}) = \{\,x\in\mathbb{R}^2 \,:\, \|x\| \le 1\},$$

where $\sigma = \sqrt{a^2+b^2}$ [10, 12, 21].

(2) *The matrix $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ and $X_0 = \{\,x\in\mathbb{R}^2 : |x_i| \le 1,\ i = 1,2\}$.*

Since the singular value decomposition $U\Sigma V^T$ of the matrix A is

$$A = U\Sigma V^T = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} |a| & 0 \\ 0 & |b| \end{pmatrix}\begin{pmatrix} 0 & \frac{b}{|b|} \\ \frac{a}{|a|} & 0 \end{pmatrix}^T, & \text{if } |a| \ge |b|, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} |b| & 0 \\ 0 & |a| \end{pmatrix}\begin{pmatrix} \frac{b}{|b|} & 0 \\ 0 & \frac{a}{|a|} \end{pmatrix}^T, & \text{if } |a| < |b|, \end{cases}$$

then $AX_0 = U\Sigma V^TX_0 = U\Sigma X_0 = U\Pi_{\sigma_1,\sigma_2} = \Pi_{\sigma_1,\sigma_2} = \{x \in \mathbb{R}^2 : |x_i| \le \sigma_i,\, i = 1,2\}$, $A^2X_0 = A\Pi_{\sigma_1,\sigma_2} = \Pi_{\sigma_1^2,\sigma_2^2},\dots,A^iX_0 = A\Pi_{\sigma_1^{i-1},\sigma_2^{i-1}} = \Pi_{\sigma_1^i,\sigma_2^i},\dots,$ where $\sigma_1 =$

$\max\{|a|,|b|\}$, $\sigma_2 = \min\{|a|,|b|\}$. *Accordingly,*

$$X_0 + \sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}A^i X_0\right\} = X_0 + \sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}\Pi_{\sigma_1^i,\sigma_2^i}\right\} = X_0 + \sum_{i=1}^{\infty}\Pi_{\frac{t^i}{i!}\sigma_1^i,\frac{t^i}{i!}\sigma_2^i} = \Pi_{e^{\sigma_1 t},e^{\sigma_2 t}}.$$

Since $\Pi_{e^{\sigma_1 t},e^{\sigma_2 t}} = e^{\Sigma t}X_0 = e^{At}X_0$, *the system* (1.1) *has the following solution:*

$$X(t) = e^{\Sigma t}X_0 + \int_0^t e^{(t-s)A}F(s)\,ds = e^{At}X_0 + \int_0^t e^{(t-s)A}F(s)\,ds.$$

Remark 3.7. *If* $X_0 \in conv(\mathbb{R}^n)$ *and* $F : [0,T] \to conv(\mathbb{R}^n)$ *then the system* (1.1) *has the solution* (3.5). *Similarly, as in Remark 3.8, from Property C (3) we have*

$$\sum_{i=0}^{\infty}\left\{\frac{t^i}{i!}A^i X_0\right\} \supseteq \left\{\sum_{i=0}^{\infty}\frac{t^i}{i!}A^i\right\}X_0 = e^{At}X_0,$$

$$\sum_{i=0}^{\infty}\int_0^t \frac{(t-s)^i\,A^i}{i!}F(s)\,ds \supseteq \int_0^t\left(\sum_{i=0}^{\infty}\frac{(t-s)^i\,A^i}{i!}\right)F(s)\,ds = \int_0^t e^{A(t-s)}F(s)\,ds,$$

that is, in the general case we have

$$e^{At}X_0 + \int_0^t e^{(t-s)A}F(s)\,ds \subseteq X(t).$$

However, if, for example:

(1) *The matrix* A, *the set* X_0 *and the set-valued mapping* $F(\cdot)$ *are such that* $AX_0 = \sigma X_0$ *and* $AF(t) = \sigma F(t)$ *for all* $t \in [0,T]$, *then the system* (1.1) *has the following solution*

$$X(t) = e^{At}X_0 + \int_0^t e^{(t-s)A}F(s)\,ds = e^{\sigma t}X_0 + \int_0^t e^{\sigma(t-s)}F(s)\,ds.$$

For example, the condition $AX_0 = \sigma X_0$ *and* $AF(t) = \sigma F(t)$ *for all* $t \in [0,T]$ *holds for* $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $X_0 = B_1(\mathbf{0}) = \{\, x \in \mathbb{R}^2 \,:\, \|x\| \le 1\}$, $F(t) = \{\, f(t) \in \mathbb{R}^2 : \|f(t)\| \le g(t)\}$, *for all* $t \in [0,T]$, *and* $g(t) > 0$, *for all* $t \in [0,T]$, *and* $\sigma = \sqrt{a^2+b^2}$.

(2) *The matrix* $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$, $X_0 = \{\, x \in \mathbb{R}^2 : |x_i| \le 1,\ i = 1,2\}$ *and* $F(t) = \{\, f(t) \in \mathbb{R}^2 :$ $|f_i(t)| \le g(t),\ i = 1,2\}$ *for all* $t \in [0,T]$ *and* $g(t) > 0$ *for all* $t \in [0,T]$.

Then the system (1.1) *has the following solution*

$$X(t) = e^{At}X_0 + \int_0^t e^{(t-s)A}F(s)\,ds = e^{\Sigma t}X_0 + \int_0^t e^{(t-s)\Sigma}F(s)\,ds,$$

where Σ *is the singular value matrix described in Remark 3.6.*

Remark 3.8. *If* $A = aI_n$ $(a < 0)$, $X_0 \in conv(\mathbb{R}^n)$ *and* $F : [0,T] \to conv(\mathbb{R}^n)$, *then the Cauchy problem* (1.1) *takes the form*

$$D_H X(t) = aX(t) + F(t), \quad X(0) = X_0. \tag{3.6}$$

However, since $a < 0$, *using the formula* (1.3) *is not possible. We rewrite the system* (3.6) *in matrix form* (1.1), *where* $A = -|a|I_n$, *and according to Theorem 3.2, the solution of the system* (3.6) *can be written in the form* (3.5).

Since $A = -|a|I_n$, $A^3 = -|a|^3 I_n, \ldots, A^{2i-1} = -|a|^{2i-1}I_n, \ldots$ *and* $A^2 = |a|^2 I_n, A^4 = |a|^4 I_n, \ldots, A^{2i} = |a|^{2i}I_n, \ldots$, *we have*

$$\sum_{i=1}^{\infty}\left\{\frac{t^i}{i!}A^iX_0\right\} = \sum_{i=1}^{\infty}\left\{\frac{t^{2i-1}}{(2i-1)!}A^{2i-1}X_0\right\} + \sum_{i=1}^{\infty}\left\{\frac{t^{2i}}{(2i)!}A^{2i}X_0\right\}$$
$$= \sum_{i=1}^{\infty}\left\{\frac{(|a|t)^{2i-1}}{(2i-1)!}(-I_n)X_0\right\} + \sum_{i=1}^{\infty}\left\{\frac{(|a|t)^{2i}}{(2i)!}I_nX_0\right\}$$

and

$$\sum_{i=0}^{\infty}\int_0^t\left\{\frac{(t-s)^i}{i!}A^iF(s)\right\}ds = \sum_{i=1}^{\infty}\int_0^t\left\{\frac{(t-s)^{2i-1}}{(2i-1)!}A^{2i-1}F(s)\right\}ds + \sum_{i=0}^{\infty}\int_0^t\left\{\frac{(t-s)^{2i}}{(2i)!}A^{2i}F(s)\right\}ds$$
$$= \int_0^t\left\{\sum_{i=1}^{\infty}\frac{|a|^{2i-1}(t-s)^{2i-1}}{(2i-1)!}(-I_n)F(s)\right\}ds + \int_0^t\left\{\sum_{i=0}^{\infty}\frac{|a|^{2i}(t-s)^{2i}}{(2i)!}I_nF(s)\right\}ds.$$

Since the sets $\bar{X}_0 = (-I_n)X_0$ *and* $\bar{F}(t) = (-I_n)F(t)$ *are centrally symmetric to the sets* X_0 *and* $F(t)$ *relative to the point* $\mathbf{0}$ *[2], we can write*

$$X(t) = X_0 + \sum_{i=1}^{\infty}\left\{\frac{|a|^{2i-1}t^{2i-1}}{(2i-1)!}\bar{X}_0\right\} + \sum_{i=1}^{\infty}\left\{\frac{|a|^{2i}t^{2i}}{(2i)!}X_0\right\}$$
$$+ \int_0^t\sum_{i=1}^{\infty}\left\{\frac{|a|^{2i-1}(t-s)^{2i-1}}{(2i-1)!}\bar{F}(s)\right\}ds + \int_0^t\sum_{i=0}^{\infty}\left\{\frac{|a|^{2i}(t-s)^{2i}}{(2i)!}F(s)\right\}ds.$$

Since

$$\sum_{i=1}^{\infty}\left\{\frac{(\sigma t)^{2i-1}}{(2i-1)!}\bar{X}_0\right\}=\sum_{i=1}^{\infty}\left\{\frac{(\sigma t)^{2i-1}}{(2i-1)!}\right\}\bar{X}_0=\sinh(\sigma t)\bar{X}_0,$$
$$X_0+\sum_{i=1}^{\infty}\left\{\frac{(\sigma t)^{2i}}{(2i)!}X_0\right\}=\sum_{i=0}^{\infty}\left\{\frac{(\sigma t)^{2i}}{(2i)!}\right\}X_0=\cosh(\sigma t)X_0,$$
$$\sum_{i=1}^{\infty}\left\{\frac{|a|^{2i-1}(t-s)^{2i-1}}{(2i-1)!}\bar{F}(s)\right\}=\sum_{i=1}^{\infty}\left\{\frac{|a|^{2i-1}(t-s)^{2i-1}}{(2i-1)!}\right\}\bar{F}(s)=\sinh(|a|(t-s))\bar{F}(s),$$
$$\sum_{i=0}^{\infty}\left\{\frac{|a|^{2i}(t-s)^{2i}}{(2i)!}F(s)\right\}=\sum_{i=0}^{\infty}\left\{\frac{|a|^{2i}(t-s)^{2i}}{(2i)!}\right\}F(s)=\cosh(|a|(t-s))F(s),$$

then the solution of the system (3.6) *can be written in the following form:*

$$X(t)=\sinh(|a|t)\bar{X}_0+\cosh(|a|t)X_0+\int_0^t\sinh(|a|(t-s))\bar{F}(s)\,ds+\int_0^t\cosh(|a|(t-s))F(s)\,ds. \quad (3.7)$$

Remark 3.9. *If* $X_0\equiv\bar{X}_0$ *and* $F(t)\equiv\bar{F}(t)$ *for all* $t\geq 0$, *then*

$$\begin{aligned}\sinh(|a|t)\bar{X}_0&+\cosh(|a|t)X_0+\int_0^t\sinh(|a|(t-s))\bar{F}(s)\,ds+\int_0^t\cosh(|a|(t-s))F(s)\,ds\\&=\sinh(|a|t)X_0+\cosh(|a|t)X_0+\int_0^t\sinh(|a|(t-s))F(s)\,ds+\int_0^t\cosh(|a|(t-s))F(s)\,ds\\&=e^{|a|t}X_0+\int_0^t e^{|a|(t-s)}F(s)\,ds.\end{aligned}$$

That is, if the sets X_0 *and* $F(t)$ *are centrally symmetric with respect to the point* $\mathbf{0}$ *for all* $t\geq 0$, *then for all* $a\neq 0$, *the solution of the system* (1.2) *has the form* (1.3), *with the replacement of* a *by* $|a|$.

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Class of symmetric $H_{\sqrt{q}}$-Laguerre-Hahn linear forms

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ABSTRACT

The aim of this paper is to study the symmetrized form associated with a H_q-Laguerre-Hahn form, where H_q is the q-derivative operator. Given a H_q-Laguerre-Hahn form u of class s, it is shown that its symmetrized form w is $H_{\sqrt{q}}$-Laguerre-Hahn of class $\tilde{s} \leq 2s+3$. We give the $\sqrt{q}$-Riccati equation satisfied by the Stieltjes formal series $S(w)$ as well as a complete discussion of the class $\tilde{s}$.
As an application of this work, we generate two examples of symmetric $H_{\sqrt{q}}$-Laguerre-Hahn orthogonal polynomials of class two and three.

RESUMEN

El objetivo de este artículo es estudiar la forma simetrizada asociada a una forma H_q-Laguerre-Hahn, donde H_q es el operador q-derivada. Dada una forma H_q-Laguerre-Hahn u de clase s, se muestra que su forma simetrizada w es $H_{\sqrt{q}}$-Laguerre-Hahn de clase $\tilde{s} \leq 2s+3$. Damos la ecuación $\sqrt{q}$-Riccati satisfecha por la serie formal de Stieltjes $S(w)$ y también una discusión completa de la clase $\tilde{s}$.
Como aplicación de este trabajo, generamos dos ejemplos de polinomios ortogonales simétricos $H_{\sqrt{q}}$-Laguerre-Hahn de clases dos y tres.

Keywords and Phrases: Orthogonal q-polynomials, q-derivative operator, q-difference equation, q-Riccati equation, H_q-Laguerre-Hahn character, quadratic decomposition.

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1 Introduction and preliminaries

In [9] a basic theory of H_q-Laguerre-Hahn orthogonal polynomials was introduced, and several characterizations were presented, namely the q-difference equation, the structure relation and the q-Riccati equation. Moreover, a criterion for simplifying the class of a H_q-Laguerre-Hahn form has been established. The paper also provides illustrative examples of these polynomials using standard perturbations (association, co-recursion, inversion) of H_q-classical polynomials [16]. Recently, in [13] we studied the Christoffel and Geronimus transformations in the H_q-Laguerre-Hahn case, and in [12] we proceeded by the addition of a Dirac mass to a H_q-Laguerre-Hahn form. In the two works cited above, a complete discussion of the class of the resulting form was given, and some examples of H_q-Laguerre-Hahn polynomial sequences (in relation with H_q-classical polynomial sequences) of class one and two were highlighted. In addition, in [15], the symmetric H_q-Laguerre-Hahn orthogonal polynomials of class zero were exhaustively described (see also [22]), and in [14], the class one case were also completely described.

Note that our works were a continuation of studies done in the field of D-Laguerre-Hahn polynomials. In fact, in the literature there are several contributions devoted to the study of these polynomials by different processes. In this area you can see [1, 3, 4, 6, 7, 20].

Let u be a regular form, we can define a new form w whose moments are given in terms of that of u such that $(w)_{2n} = (u)_n$, $(w)_{2n+1} = 0$, $n \geq 0$. The form w is said to be the symmetrized form associated with the form u [2, 20]. A necessary and sufficient condition for the regularity of w was given in [5, 19]. In this contribution, we study the symmetrized form associated with a H_q-Laguerre-Hahn form; for the D-semi-classical case (see [2]) and for the D-Laguerre-Hahn case (see [23]). In fact, let u be a regular form whose Stieltjes formal series $S(u)$ satisfies a q-Riccati equation. We show that the formal series $S(w)$ associated with the symmetrized form w satisfies a $\sqrt{q}$-Riccati equation. If we denote by s the class of u and by $\tilde{s}$ that of w, it turns out that: $\tilde{s} \leq 2s+3$ and we specify the exact conditions for which $\tilde{s} = 2s,\ 2s+1,\ 2s+2,\ 2s+3$. Finally, starting from two familes of H_q-Laguerre-Hahn polynomials of class 0, we derived two families of symmetric H_q-Laguerre-Hahn polynomials of classes 2 and 3.

We will now recall some useful results.

We denote by $\mathcal{P}$ the vector space of polynomials with coefficients in $\mathbb{C}$ and by $\mathcal{P}'$ its dual space. The action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ is denoted as $\langle u, f \rangle$. In particular, we denote by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$, the moments of u. For any form u, any polynomial g and any $(a, c) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$, let $H_q u$, gu, $h_a u$, Du, $(x-c)^{-1}u$, δ_c and σu, be the forms defined as in [16, 20]

$$\langle H_q u, f \rangle := -\langle u, H_q f \rangle, \quad \langle gu, f \rangle := \langle u, gf \rangle, \quad \langle h_a u, f \rangle := \langle u, h_a f \rangle,$$

$$\langle Du, f \rangle := -\langle u, f' \rangle, \quad \langle (x-c)^{-1}u, f \rangle := \langle u, \theta_c f \rangle, \quad \langle \delta_c, f \rangle := f(c), \quad \langle \sigma u, f \rangle = \langle u, \sigma f \rangle,$$

where for all $f \in \mathcal{P}$ and $q \in \widetilde{\mathbb{C}} := \{z \in \mathbb{C},\, z \neq 0,\, z^n \neq 1,\, n \geq 1\}$ [16]

$$(H_q f)(x) = \begin{cases} \dfrac{f(qx)-f(x)}{(q-1)x}, & x \neq 0, \\ f^{'}(0), & x = 0, \end{cases} \tag{1.1}$$

$$(\theta_c f)(x) = \begin{cases} \dfrac{f(x)-f(c)}{x-c}, & x \neq c, \\ f^{'}(c), & x = c, \end{cases} \tag{1.2}$$

$$(h_a f)(x) = f(ax). \tag{1.3}$$

$$(\sigma f)(x) = f\left(x^2\right) \tag{1.4}$$

In particular, we have

$$(H_q u)_n = -[n]_q (u)_{n-1}\,, \quad (\sigma u)_n = (u)_{2n}\,, \quad n \geq 0, \tag{1.5}$$

where $(u)_{-1} = 0$ and $[n]_q := \dfrac{q^n - 1}{q-1}$, $n \geq 0$ [10].

When $q \to 1$, H_q converges to the derivative operator D.

Lemma 1.1 ([16,20])**.** *For $f, g \in \mathcal{P}$, $u \in \mathcal{P}'$, $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, we have*

$$h_a(gu) = (h_{a^{-1}} g)(h_a u), \tag{1.6}$$

$$h_{q^{-1}} \circ H_q = H_{q^{-1}}\,, \quad H_q \circ h_{q^{-1}} = q^{-1} H_{q^{-1}}\,, \quad \text{in } \mathcal{P}, \tag{1.7}$$

$$H_q(fg)(x) = (h_q f)(x)(H_q g)(x) + g(x)(H_q f)(x), \tag{1.8}$$

$$H_q(gu) = (h_{q^{-1}} g) H_q u + q^{-1}(H_{q^{-1}} g) u, \tag{1.9}$$

$$(\theta_b fg)(x) = g(x)(\theta_b f)(x) + f(b)(\theta_b g)(x). \tag{1.10}$$

$$f(x)\sigma u = \sigma\left(f\left(x^2\right)u\right). \tag{1.11}$$

For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product uf is the polynomial [20]

$$(uf)(x) := \left\langle u, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta} \right\rangle = \sum_{i=0}^{n} \left(\sum_{j=i}^{n} (u)_{j-i}\, f_j \right) x^i,$$

where $f(x) = \sum_{i=0}^{n} f_i x^i$. This allows us to define the Cauchy's product of two forms:

$$\langle uv, f \rangle := \langle u, vf \rangle, \quad f \in \mathcal{P}.$$

The Stieltjes formal series of $u \in \mathcal{P}'$ is defined by [20]

$$S(u)(z) := -\sum_{n\geq 0} \frac{(u)_n}{z^{n+1}}. \tag{1.12}$$

The quantum factorial symbol is defined by [8]

$$(x;q)_0 = 1, \quad (x;q)_n = \prod_{k=0}^{n-1} \left(1 - xq^k\right), \quad x, q \in \mathbb{C},\ n \geq 1. \tag{1.13}$$

The q-binomial coefficient is defined by [8]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \tag{1.14}$$

A form u is said regular if there exists a sequence of monic polynomials $\{P_n\}_{n\geq 0}$, $\deg P_n = n$,$n \geq 0$ MPS, such that $\langle u, P_n P_m \rangle = r_n \delta_{n,m}$ with $r_n \neq 0$ for any $n, m \geq 0$ where $\delta_{n,m}$ denotes the Kronecker symbol. In this case, $\{P_n\}_{n\geq 0}$ is called a monic orthogonal polynomial sequence MOPS with respect to the form u and it is unique. Given a MOPS $\{P_n\}_{n\geq 0}$ with respect to the form u, there exists a unique sequence $\{u_n\}_{n\geq 0}$, $u_n \in \mathcal{P}'$, called the dual sequence of $\{P_n\}_{n\geq 0}$, such that $\langle u_n, P_m \rangle = \delta_{n,m}$, $n, m \geq 0$. It holds $u = \lambda u_0$, $\lambda = (u)_0 \neq 0$. Furthermore, if $(u)_0 = 1$, then $u = u_0$. Hence, there exists a unique form u with $(u)_0 = 1$, where $\{P_n\}_{n\geq 0}$ is its corresponding MOPS.

The MOPS $\{P_n\}_{n\geq 0}$ is characterized by the following three-term recurrence relation (Favard's theorem) (TTRR in short) [5, 20]

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ &P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x), \quad n \geq 0, \end{aligned} \tag{1.15}$$

where

$$\beta_n = \frac{\langle u, xP_n^2 \rangle}{\langle u, P_n^2 \rangle} \in \mathbb{C}, \quad \gamma_{n+1} = \frac{\langle u, P_{n+1}^2 \rangle}{\langle u, P_n^2 \rangle} \in \mathbb{C} \setminus \{0\}, \quad n \geq 0. \tag{1.16}$$

The shifted MOPS $\left\{\widehat{P}_n := a^{-n}\big(h_a P_n\big)\right\}_{n\geq 0}$ is then orthogonal with respect to $\widehat{u} = h_{a^{-1}} u$ and satisfies (1.15) with [20]

$$\widehat{\beta}_n = \frac{\beta_n}{a}, \quad \widehat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0.$$

The form u is said to be normalized if $(u)_0 = 1$. In this paper, we suppose that any regular form will be normalized. The form u is said to be positive definite if and only if $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} > 0$, for all $n \geq 0$. When u is regular, $\{P_n\}_{n\geq 0}$ is a symmetric MOPS if and only if $\beta_n = 0$, $n \geq 0$, or equivalently $(u)_{2n+1} = 0$, $n \geq 0$ [5].

Given a regular form u and the corresponding MOPS $\{P_n\}_{n\geq 0}$, we define the associated sequence of the first kind $\{P_n^{(1)}\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ by [20]

$$P_n{}^{(1)}(x) = \left\langle u, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x-\xi} \right\rangle = (u\theta_0 P_{n+1})(x), \quad n \geq 0.$$

Proposition 1.2 ([20]). *Let $\{P_n\}_{n\geq 0}$ be a MOPS satisfying the TTRR (1.15), then its associated sequence $\left\{P_n^{(1)}\right\}_{n\geq 0}$ satisfies the TTRR*

$$\begin{aligned} &P_0^{(1)}(x) = 1, \quad P_1^{(1)}(x) = x - \beta_1, \\ &P_{n+2}^{(1)}(x) = (x-\beta_{n+2})P_{n+1}^{(1)}(x) - \gamma_{n+2}P_n^{(1)}(x), \quad n \geq 0. \end{aligned} \tag{1.17}$$

In addition, we have the following fundamental result

$$P_{n+1}^{(1)}(x)P_{n+1}(x) - P_{n+2}(x)P_n^{(1)}(x) = \prod_{\nu=0}^{n} \gamma_{\nu+1}, \quad n \geq 0. \tag{1.18}$$

We recall the definition of the so-called kernel polynomials with K-parameter introduced by Chihara [5].

Let $\{P_n\}_{n>0}$ be a MOPS and c a complex number such that $P_n(c) \neq 0$, for all $n \geq 0$. The sequence of monic kernel polynomials of K-parameter c, $\{P_n^*(c;\cdot)\}_{n\geq 0}$, associated with $\{P_n\}_{n\geq 0}$ is defined by

$$P_n^*(c;x) := \frac{1}{x-c}\left[P_{n+1}(x) - \frac{P_{n+1}(c)}{P_n(c)}P_n(x)\right], \quad n \geq 0.$$

In these conditions, if $\{P_n\}_{n\geq 0}$ is a MOPS associated with the form u, then $\{P_n^*(c;\cdot)\}_{n\geq 0}$ is a MOPS associated with $u^* = (x-c)u$ [5].

Lemma 1.3 ([21]). *Let $(b_n)_{n\geq 0}$ with $b_n \neq 0$, $n \geq 0$, $(c_n)_{n\geq 0}$ be two sequences of complex numbers and $(x_n)_{n\geq 0}$ the sequence satisfying the recurrence relation:*

$$x_{n+1} = b_n x_n + c_n, \quad n \geq 0, \quad x_0 = a \in \mathbb{C} - \{0\}.$$

We have

$$x_{n+1} = \left(\prod_{k=0}^{n} b_k\right)\left\{a + \sum_{k=0}^{n}\left(\prod_{l=0}^{k} b_l\right)^{-1} c_k\right\}, \quad n \geq 0.$$

We will give now some basic facts about the H_q-Laguerre-Hahn character.

Definition 1.4 ([9]). *A form u is called H_q-Laguerre-Hahn when it is regular and satisfies the q-difference equation*

$$H_q(\Phi u) + \Psi u + B\left(x^{-1}u\left(h_q u\right)\right) = 0, \tag{1.19}$$

where Φ, Ψ, B are polynomials, with Φ monic. The corresponding monic orthogonal sequence $\{P_n\}_{n\geq 0}$ is called H_q-Laguerre-Hahn.

Remark 1.5. *(1) When $B=0$ and the form u is regular, then u is H_q-semiclassical [17].*

(2) When u satisfies (1.19)*, then $\widehat{u}=h_{a^{-1}}u$ fulfills the q-difference equation [9]*

$$H_q\left(a^{-\deg\Phi}\Phi(ax)\widehat{u}\right)+a^{1-\deg\Phi}\Psi(ax)\widehat{u}+a^{-\deg\Phi}B(ax)\left(x^{-1}\widehat{u}(h_q\widehat{u})\right)=0. \tag{1.20}$$

(3) If $t=\deg\Phi$, $p=\deg\Psi$, $r=\deg B$ and $d=\max(t,r)$, then we define the class of u the nonnegative integer s [9]

$$s=\min\max(p-1,d-2), \tag{1.21}$$

where the minimum is taken over all triples (Φ,Ψ,B) satisfying (1.19)*. Moreover, the H_q-Laguerre-Hahn form u satisfying* (1.19) *is of class $s=\max(p-1,d-2)$ if and only if*

$$\prod_{c\in\mathcal{Z}_\Phi}\Big\{\left|q(h_q\Psi)(c)+(H_q\Phi)(c)\right|+\left|q(h_qB)(c)\right| + \left|\left\langle u, q(\theta_{cq}\Psi)+(\theta_{cq}\circ\theta_c\Phi)+q\big(h_qu(\theta_0\circ\theta_{cq}B)\big)\right\rangle\right|\Big\}>0, \tag{1.22}$$

where $\mathcal{Z}_\Phi$ is the set of roots of Φ [9]. When $c\in\mathcal{Z}_\Phi$ and (1.19) *may be simplified by $x-c$, then* (1.19) *becomes*

$$H_q((\theta_c\Phi)u)+(q\theta_{cq}\Psi+\theta_{cq}\circ\theta_c\Phi)u+q(\theta_{cq}B)\left(x^{-1}u(h_qu)\right)=0. \tag{1.23}$$

Proposition 1.6 ([9])**.** *Let u be a regular form, the following statements are equivalent:*

(1) u belongs to the H_q-Laguerre-Hahn class, i.e. satisfying (1.19).

(2) The Stieltjes formal series $S(u)$ satisfies the q-Riccati equation

$$\left(h_{q^{-1}}\Phi\right)(z)\left(H_{q^{-1}}S(u)\right)(z)=B(z)S(u)(z)\left(h_{q^{-1}}S(u)\right)(z)+C(z)S(u)(z)+D(z), \tag{1.24}$$

where Φ and B are polynomials defined in (1.19) *and*

$$\begin{cases} C(z)=-\left(H_{q^{-1}}\Phi\right)(z)-q\Psi(z),\\ D(z)=-\left\{H_{q^{-1}}\left(u\theta_0\Phi\right)(z)+q\left(u\theta_0\Psi\right)(z)+q\left(uh_qu\right)\left(\theta_0^2B\right)(z)\right\}. \end{cases} \tag{1.25}$$

Moreover, u is of class s if and only if

$$\prod_{c\in\mathcal{Z}_\Phi}\{|B(cq)|+|C(cq)|+|D(cq)|\}>0, \tag{1.26}$$

and one may write

$$s=\max\left(\deg B-2,\deg C-1,\deg D\right). \tag{1.27}$$

Let u be a regular form and $\{P_n\}_{n\geq 0}$ its corresponding MOPS. By linear extension, we can define a new form w whose moments are given in terms of that of u such that

$$(w)_{2n} = (u)_n, \quad (w)_{2n+1} = 0, \quad n \geq 0, \tag{1.28}$$

equivalent to saying that

$$\sigma w = u, \quad \sigma(xw) = 0. \tag{1.29}$$

In the following theorem the author gave a necessary and sufficient condition for a MPS $\{Q_n\}_{n\geq 0}$ to be a MOPS with respect to a form w satisfying (1.28), this form will be unique since $(w)_0 = (u)_0 = 1$.

Theorem 1.7 ([19]). *Let $\{P_n\}_{n\geq 0}$ be MOPS and $\{Q_n\}_{n\geq 0}$ a MPS such that*

$$Q_1(x) = x, \quad Q_{2n}(x) = P_n\left(x^2\right), \quad n \geq 0. \tag{1.30}$$

Then $\{Q_n\}_{n\geq 0}$ is a MOPS if and only if $P_n(0) \neq 0$, $Q_{2n+1}(x) = xP_n^*\left(0; x^2\right)$, $n \geq 0$.

In such conditions, if $\{P_n\}_{n\geq 0}$ satisfies the three-term recurrence relation

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0^P, \\ &P_{n+2}(x) = \left(x - \beta_{n+1}^P\right) P_{n+1}(x) - \gamma_{n+1}^P P_n(x), \quad n \geq 0, \end{aligned}$$

then the coefficients $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ for the corresponding three-term recurrence relation satisfied by $\{Q_n\}_{n\geq 0}$ are given by

$$\begin{gathered} \tilde{\beta}_n = 0, \quad n \geq 0, \\ \tilde{\gamma}_{2n+1} = -\frac{P_{n+1}(0)}{P_n(0)}, \quad \tilde{\gamma}_{2n+2} = -\frac{P_n(0)}{P_{n+1}(0)}\gamma_{n+1}^P, \quad n \geq 0. \end{gathered} \tag{1.31}$$

Moreover, if $\{P_n\}_{n\geq 0}$ is orthogonal with respect to the form u, then $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to a form w defined on the basis $\{x^n\}_{n\geq 0}$ of $\mathcal{P}$ by means of (1.28).

Proposition 1.8 ([5]). *Let w be a symmetric and regular form. Let $\{Q_n\}_{n\geq 0}$ be the corresponding MOPS. It satisfies a three-term recurrence relation*

$$\begin{aligned} &Q_0(x) = 1, \quad Q_1(x) = x \\ &Q_{n+2}(x) = x, \quad Q_{n+1}(x) - \tilde{\gamma}_{n+1} Q_n(x), \quad n \geq 0, \end{aligned}$$

It is very well known that [5]

$$Q_{2n}(x) = P_n\left(x^2\right), \quad Q_{2n+1}(x) = xR_n\left(x^2\right),$$

where $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are MOPSs related to regular forms $u = \sigma w$ and $\gamma_1^{-1} x\sigma w$, respectively. The form w is said to be the symmetrized form associated with the form u. Furthermore, *if*

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0^P,$$
$$P_{n+2}(x) = \left(x - \beta_{n+1}^P\right) P_{n+1}(x) - \gamma_{n+1}^P P_n(x), \quad n \geq 0,$$

and

$$R_0(x) = 1, \quad R_1(x) = x - \beta_0^R,$$
$$R_{n+2}(x) = \left(x - \beta_{n+1}^R\right) R_{n+1}(x) - \gamma_{n+1}^R R_n(x), \quad n \geq 0,$$

we get

$$\begin{cases} \beta_0^P = \tilde{\gamma}_1, \\ \beta_{n+1}^P = \tilde{\gamma}_{2n+2} + \tilde{\gamma}_{2n+3}, \quad n \geq 0, \\ \gamma_{n+1}^P = \tilde{\gamma}_{2n+1}\tilde{\gamma}_{2n+2}, \quad n \geq 0, \end{cases}$$

and

$$\begin{cases} \beta_{n+1}^R = \tilde{\gamma}_{2n+1} + \tilde{\gamma}_{2n+2}, \quad n \geq 0, \\ \gamma_{n+1}^R = \tilde{\gamma}_{2n+2}\tilde{\gamma}_{2n+3}, \quad n \geq 0. \end{cases}$$

Throughout the rest of this paper, let $0 < q < 1$ or $q > 1$ and we write

$$S(u)(z) = S_u(z), \quad S(w)(z) = S_w(z).$$

2 The $H_{\sqrt{q}}$-Laguerre–Hahn character of the form w

Lemma 2.1. *The following equality holds*

$$z\left(H_{\sqrt{q}^{-1}} S_w\right)(z) - S_w(z) = \sqrt{q}^{-1}(\sqrt{q}^{-1} + 1)z^3 \left(H_{q^{-1}} S_u\right)(z^2). \tag{2.1}$$

Proof. From (1.12) and (1.28), we have

$$zS_u\left(z^2\right) = S_w(z) \tag{2.2}$$

Therefore,

$$\begin{aligned} \left(H_{\sqrt{q}^{-1}} S_w\right)(z) &\underset{\text{by (1.1)}}{=} \frac{S_w(\sqrt{q}^{-1}z) - S_w(z)}{(\sqrt{q}^{-1} - 1)z} \underset{\text{by (2.2)}}{=} \frac{\sqrt{q}^{-1} zS_u(q^{-1}z^2) - zS_u(z^2)}{(\sqrt{q}^{-1} - 1)z} \\ &= \sqrt{q}^{-1}(\sqrt{q}^{-1} + 1)\frac{S_u(q^{-1}z^2) - \sqrt{q}S_u(z^2)}{(q^{-1} - 1)} \\ &= \sqrt{q}^{-1}(\sqrt{q}^{-1} + 1)z^2 \left\{\frac{S_u(q^{-1}z^2) - S_u(z^2) + (1 - \sqrt{q})S_u(z^2)}{(q^{-1} - 1)z^2}\right\} \\ &\underset{\text{by (1.1)-(2.2)}}{=} \sqrt{q}^{-1}(\sqrt{q}^{-1} + 1)z^2 \left(H_{q^{-1}} S_u\right)(z^2) + S_u(z^2). \end{aligned} \tag{2.3}$$

Multiplying (2.3) by z yields (2.1). □

Proposition 2.2. *Under the conditions of Theorem 1.7, if u is a H_q-Laguerre-Hahn form of class s such that its Stieltjes formal series S_u satisfies the q-Riccati equation (1.24), then the form w defined by (1.28) is $H_{\sqrt{q}}$-Laguerre-Hahn and its Stieltjes formal series S_w satisfies the $\sqrt{q}$-Riccati equation*

$$\left(h_{\sqrt{q}^{-1}}\tilde{\Phi}\right)(z)\left(H_{\sqrt{q}^{-1}}S_w\right)(z)=\tilde{B}(z)S_w(z)\left(h_{\sqrt{q}^{-1}}S_w\right)(z)+\tilde{C}(z)S_w(z)+\tilde{D}(z), \tag{2.4}$$

where

$$\begin{cases}\tilde{\Phi}(z)=z\Phi\left(z^2\right),\\ \tilde{B}(z)=q^{-1}\left(\sqrt{q}+1\right)zB\left(z^2\right),\\ \tilde{C}(z)=q^{\frac{-3}{2}}\left(\sqrt{q}+1\right)z^2C\left(z^2\right)+\sqrt{q}^{-1}\Phi\left(q^{-1}z^2\right),\\ \tilde{D}(z)=q^{\frac{-3}{2}}\left(\sqrt{q}+1\right)z^3D\left(z^2\right).\end{cases} \tag{2.5}$$

Moreover, w is of class $\tilde{s}\leq 2s+3$.

Proof. From (1.3), the q-Riccati equation satisfied by the formal Stieltjes series S_u can be written in this form

$$\Phi\left(q^{-1}z\right)\left(H_{q^{-1}}S_u\right)(z)=B(z)S_u(z)S_u\left(q^{-1}z\right)+C(z)S_u(z)+D(z) \tag{2.6}$$

We make the change of variable $z\longleftarrow z^2$ in (2.6),

$$\Phi\left(q^{-1}z^2\right)\left(H_{q^{-1}}S_u\right)\left(z^2\right)=B\left(z^2\right)S_u\left(z^2\right)S_u\left(q^{-1}z^2\right)+C(z^2)S_u\left(z^2\right)+D\left(z^2\right). \tag{2.7}$$

Then we multiply (2.7) by $q^{-1}(\sqrt{q}+1)z^3$, we get

$$\begin{aligned}q^{-1}\left(\sqrt{q}+1\right)z^3\Phi\left(q^{-1}z^2\right)\left(H_{q^{-1}}S_u\right)\left(z^2\right)&=q^{-1}\left(\sqrt{q}+1\right)z^3B\left(z^2\right)S_u\left(z^2\right)S_u\left(q^{-1}z^2\right)+\\&\quad q^{-1}\left(\sqrt{q}+1\right)z^3C\left(z^2\right)S_u\left(z^2\right)+q^{-1}\left(\sqrt{q}+1\right)z^3D\left(z^2\right),\end{aligned} \tag{2.8}$$

with

$$\left(H_{q^{-1}}S_u\right)(z)=\frac{S_u\left(q^{-1}z\right)-S_u(z)}{\left(q^{-1}-1\right)z},\quad \left(H_{q^{-1}}S_u\right)\left(z^2\right)=\frac{S_u\left(q^{-1}z^2\right)-S_u\left(z^2\right)}{\left(q^{-1}-1\right)z^2}. \tag{2.9}$$

In an equivalent way

$$\begin{aligned}&\Phi\left(q^{-1}z^2\right)\left(q^{-1}\left(\sqrt{q}+1\right)\ z^3\left(H_{q^{-1}}S_u\right)\left(z^2\right)\right)\\&\qquad=\sqrt{q}^{-1}\left(\sqrt{q}+1\right)zB(z^2)\left(zS_u(z^2)\right)\left(\sqrt{q}^{-1}zS_u\left(\left(\sqrt{q}^{-1}z\right)^2\right)\right)\\&\qquad+q^{-1}\left(\sqrt{q}+1\right)z^2C(z^2)\left(zS_u(z^2)\right)+q^{-1}\left(\sqrt{q}+1\right)z^3D\left(z^2\right).\end{aligned} \tag{2.10}$$

Substituting (2.1) into (2.10) gives

$$\Phi\left(q^{-1}z^2\right)\left(z\left(H_{\sqrt{q}^{-1}}S_w\right)(z)-S_w(z)\right)=\sqrt{q}^{-1}\left(\sqrt{q}+1\right)zB\left(z^2\right)S_w(z)S_w\left(\sqrt{q}^{-1}z\right)$$
$$+q^{-1}\left(\sqrt{q}+1\right)z^2C\left(z^2\right)S_w(z)+q^{-1}\left(\sqrt{q}+1\right)z^3D\left(z^2\right), \quad (2.11)$$

equivalent to

$$z\Phi\left(q^{-1}z^2\right)\left(H_{\sqrt{q}^{-1}}S_w\right)(z)=\sqrt{q}^{-1}\left(\sqrt{q}+1\right)zB\left(z^2\right)S_w(z)S_w\left(\sqrt{q}^{-1}z\right)$$
$$+\left(q^{-1}\left(\sqrt{q}+1\right)z^2C\left(z^2\right)+\Phi\left(q^{-1}z^2\right)\right)S_w(z)+q^{-1}\left(\sqrt{q}+1\right)z^3D\left(z^2\right). \quad (2.12)$$

This amounts to

$$\sqrt{q}\left(h_{\sqrt{q}^{-1}}\tilde{\Phi}\right)(z)\left(H_{\sqrt{q}^{-1}}S_w\right)(z)=\sqrt{q}^{-1}\left(\sqrt{q}+1\right)zB\left(z^2\right)S_w(z)\left(h_{\sqrt{q}^{-1}}S_w\right)(z)$$
$$+\left(q^{-1}\left(\sqrt{q}+1\right)z^2C\left(z^2\right)+\Phi\left(q^{-1}z^2\right)\right)S_w(z)+q^{-1}\left(\sqrt{q}+1\right)z^3D\left(z^2\right), \quad (2.13)$$

where $\tilde{\Phi}(z)=z\Phi(z^2)$. Thus, we obtain (2.4)-(2.5), by dividing the previous equation by $\sqrt{q}$.
It follows from (1.8), (1.25) and (2.5) that

$$\tilde{\Psi}(z)=-\sqrt{q}^{-1}\left(H_{\sqrt{q}^{-1}}\tilde{\Phi}\right)(z)-\sqrt{q}^{-1}\tilde{C}(z)=q^{-1}\left(\sqrt{q}+1\right)\left(z^2\Psi\left(z^2\right)-\Phi\left(z^2\right)\right). \quad (2.14)$$

Since $\sqrt{q}\in\widetilde{\mathbb{C}}$ when $q\in\widetilde{\mathbb{C}}$, then according to the equivalence in Proposition 1.6 with $q\leftarrow\sqrt{q}$, the form w satisfies the $\sqrt{q}$-difference equation

$$H_{\sqrt{q}}(\tilde{\Phi}w)+\tilde{\Psi}w+\tilde{B}\left(x^{-1}w\left(h_{\sqrt{q}}w\right)\right)=0, \quad (2.15)$$

with

$$\begin{cases}\tilde{\Phi}(x)=x\Phi\left(x^2\right),\\ \tilde{B}(x)=q^{-1}\left(\sqrt{q}+1\right)xB\left(x^2\right),\\ \tilde{\Psi}(x)=q^{-1}\left(\sqrt{q}+1\right)\left(x^2\Psi\left(x^2\right)-\Phi\left(x^2\right)\right).\end{cases}$$

As u is of class s, we deduce from the third point of Remark 1.5 that

$$s=\max\left(\deg(\Psi)-1,\max\left(\deg(\Phi),\deg(B)\right)-2\right),$$

which gives

$$\deg(\Phi)\leq s+2,\quad \deg(\Psi)\leq s+1,\quad \deg(B)\leq s+2.$$

According to (2.5) and (2.14), we have

$$\deg\left(\tilde{\Phi}\right) \leq 2s+5, \quad \deg\left(\tilde{\Psi}\right) \leq 2s+4, \quad \deg\left(\tilde{B}\right) \leq 2s+5.$$

Finally, by the definition of the class given in (1.21), we have

$$\tilde{s} \leq \max\left(\deg\left(\tilde{\Psi}\right) - 1, \max\left(\deg\left(\tilde{\Phi}\right), \deg\left(\tilde{B}\right)\right) - 2\right) \leq 2s+3. \qquad \square$$

Lemma 2.3. *The class of w depends only on the zero $z=0$ of $\tilde{\Phi}$.*

Proof. u is a H_q-Laguerre-Hahn form of class s and its Stieltjes formal series $S_u(z)$ satisfies (2.4), therefore the polynomials $h_{q^{-1}}\Phi$, B, C and D are coprime. Let $\tilde{\Phi}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ be as in Proposition 2.2 and let c be a non-zero root of $\tilde{\Phi}$, which gives $\Phi\left(c^2\right)=0$. From (1.26) since u is of class s, we have $\left|B\left(c^2q\right)\right| + \left|C\left(c^2q\right)\right| + \left|D\left(c^2q\right)\right| \neq 0$.

- If $B\left(c^2q\right) \neq 0$, then $\tilde{B}\left(c\sqrt{q}\right) \neq 0$.
- If $B\left(c^2q\right) = 0$ and $C\left(c^2q\right) \neq 0$, then $\tilde{C}\left(c\sqrt{q}\right) \neq 0$.
- If $B\left(c^2q\right) = C\left(c^2q\right) = 0$, then $D\left(c^2q\right) \neq 0$ and this implies that $\tilde{D}\left(c\sqrt{q}\right) \neq 0$.

Consequently, for any non-zero root of $\tilde{\Phi}$, we have $|\tilde{B}\left(c\sqrt{q}\right)| + |\tilde{C}\left(c\sqrt{q}\right)| + |\tilde{D}\left(c\sqrt{q}\right)| \neq 0$. $\square$

Proposition 2.4. *Taking into account the conditions of Proposition 2.2, we have the following different cases for the class of w.*

(1) If $\Phi(0) \neq 0$, then $\tilde{s} = 2s+3$. In addition the Stieltjes formal series S_w satisfies the $\sqrt{q}$-Riccati equation (2.4) with (2.5).

(2) If $\Phi(0) = 0$ and $B(0) \neq 0$, then $\tilde{s} = 2s+2$. In addition the Stieltjes formal series S_w satisfies the $\sqrt{q}$-Riccati equation (2.4) with

$$\begin{cases} \tilde{\Phi}(z) = \Phi\left(z^2\right), \\ \tilde{B}(z) = q^{-1}\left(\sqrt{q}+1\right) B\left(z^2\right), \\ \tilde{C}(z) = q^{\frac{-3}{2}} z \left\{\left(\sqrt{q}+1\right) C\left(z^2\right) + \left(\theta_0 \Phi\right)\left(q^{-1}z^2\right)\right\}, \\ \tilde{D}(z) = q^{\frac{-3}{2}}\left(\sqrt{q}+1\right) z^2 D\left(z^2\right). \end{cases} \tag{2.16}$$

(3) If $\Phi(0)=B(0)=0$ and $\left(\sqrt{q}+1\right)C(0)+\Phi'(0)\neq 0$, then $\tilde{s}=2s+1$. In addition the Stieltjes formal series S_w satisfies the $\sqrt{q}$-Riccati equation (2.4) with

$$\begin{cases}\tilde{\Phi}(z)=z\left(\theta_0\Phi\right)\left(z^2\right),\\ \tilde{B}(z)=q^{-1}\left(\sqrt{q}+1\right)z\left(\theta_0 B\right)\left(z^2\right),\\ \tilde{C}(z)=q^{\frac{-3}{2}}\left\{\left(\sqrt{q}+1\right)C\left(z^2\right)+\left(\theta_0\Phi\right)\left(q^{-1}z^2\right)\right\},\\ \tilde{D}(z)=q^{\frac{-3}{2}}\left(\sqrt{q}+1\right)zD\left(z^2\right).\end{cases}\tag{2.17}$$

(4) If $\Phi(0)=B(0)=\left(\sqrt{q}+1\right)C(0)+\Phi'(0)=0$, then $\tilde{s}=2s$. In addition the Stieltjes formal series S_w satisfies the $\sqrt{q}$-Riccati equation (2.4) with

$$\begin{cases}\tilde{\Phi}(z)=\left(\theta_0\Phi\right)\left(z^2\right),\\ \tilde{B}(z)=q^{-1}\left(\sqrt{q}+1\right)\left(\theta_0 B\right)\left(z^2\right),\\ \tilde{C}(z)=q^{\frac{-3}{2}}z\left\{\left(\sqrt{q}+1\right)\left(\theta_0 C\right)\left(z^2\right)+q^{-1}\left(\theta_0^2\Phi\right)\left(q^{-1}z^2\right)\right\},\\ \tilde{D}(z)=q^{\frac{-3}{2}}\left(\sqrt{q}+1\right)D\left(z^2\right).\end{cases}\tag{2.18}$$

Proof. (1) If $\Phi(0)\neq 0$, then from (2.5) we have $\tilde{C}(0)\neq 0$. Thus, equation (2.4) cannot be simplified by z. Let $t=\deg(\Phi)$, $r=\deg(B)$ and $p=\deg(\Psi)$. From (2.5) and (2.14) we obtain $\deg\left(\tilde{\Phi}\right)=2t+1$, $\deg\left(\tilde{B}\right)=2r+1$ and $\tilde{p}=\deg\left(\tilde{\Psi}\right)\leq\max(2p+2,2t)$. Therefore, $\tilde{s}=\max(2t-1,2r-1,2p+1)=2s+3$, since $\max(t-2,r-2,p-1)=s$.

(2) If $\Phi(0)=0$, then the equation (2.4) with (2.5) are divisible by z and thus the order of the class of w decreases by one. Then, S_w fulfills (2.4) with (2.16).

Here, we have $\tilde{B}(0)\underset{\text{by (2.16)}}{=}q^{-1}\left(\sqrt{q}+1\right)B(0)$ and $\tilde{C}(0)=\tilde{D}(0)\underset{\text{by (2.16)}}{=}0$. Consequently, if $B(0)\neq 0$, we cannot simplify (2.4)-(2.16), which implies that the class of w is $\tilde{s}=2s+2$.

(3) If $\Phi(0)=B(0)=0$, equation (2.4) with (2.16) can be simplified by z. Hence, S_w fulfills (2.4) with (2.17).

In this case $\tilde{B}(0)=\tilde{D}(0)\underset{\text{by (2.17)}}{=}0$ and $\tilde{C}(0)\underset{\text{by (1.2)-(2.17)}}{=}q^{\frac{-3}{2}}\left\{\left(\sqrt{q}+1\right)C(0)+\Phi'(0)\right\}$.

Therefore, if $\left(\sqrt{q}+1\right)C(0)+\Phi'(0)\neq 0$, (2.4) with (2.17) cannot be simplified by z, which means that the class of w is $\tilde{s}=2s+1$.

(4) If $\Phi(0)=B(0)=\left(\sqrt{q}+1\right)C(0)+\Phi'(0)=0$, then (2.4) with (2.17) can be simplified by z. Thus, S_w satisfies (2.4) with (2.18).

If $\tilde{\Phi}(0)\underset{\text{by (1.2)-(2.18)}}{=}\Phi'(0)=0$, then we have $C(0)=0$ since $\left(\sqrt{q}+1\right)C(0)+\Phi'(0)=0$. In view of u is H_q-Laguerre-Hahn of class s, we obtain from (1.26) that $D(0)\neq 0$. Therefore, $\tilde{D}(0)\underset{\text{by (2.18)}}{=}q^{\frac{-3}{2}}\left(\sqrt{q}+1\right)D(0)\neq 0$. Then, it is not possible to simplify (2.4)-(2.18), which means the class of w is $\tilde{s}=2s$. □

3 Two illustrative examples

3.1 Example 1: $u = \mathcal{W}^{(1)}$

Let $u = \mathcal{W}^{(1)}$ be the first associated of Stieltjes-Wigert form (see Table 1). Let $\left\{S_n^{(1)}\right\}_{n\geq 0}$ be its (MOPS). It is H_q-Laguerre-Hahn of class $s = 0$ fulfilling [13]

$$\begin{cases} \beta_n^{(1)} = q^{-n-\frac{5}{2}}\left\{(1+q)q^{-n-1} - q\right\}, \quad \gamma_{n+1}^{(1)} = q^{-4n-8}\left(1-q^{n+2}\right), \quad n \geq 0, \\ H_q\left(x\left(x + q^{-\frac{5}{2}}(q-1)\right)\mathcal{W}^{(1)}\right) - (q-1)^{-1}\left\{x - q^{-\frac{7}{2}} - q^{-\frac{5}{2}} + q^{-\frac{3}{2}}\right\}u \\ \qquad\qquad -q^{-5}\left(x^{-1}\mathcal{W}^{(1)}\left(h_q\mathcal{W}^{(1)}\right)\right) = 0, \\ q^{-1}z\left(q^{-1}z + q^{-\frac{5}{2}}(q-1)\right)H_{q^{-1}}\left(S_{\mathcal{W}^{(1)}}\right)(z) = -q^{-5}S_{\mathcal{W}^{(1)}}(z) \\ \left(h_{q^{-1}}S_{\mathcal{W}^{(1)}}(z) + q^{-1}(q-1)^{-1}\left(z + q^{-\frac{1}{2}} - 2q^{-\frac{3}{2}}\right)\right)S_{\mathcal{W}^{(1)}}(z) + (q-1)^{-1}. \end{cases} \tag{3.1}$$

Lemma 3.1 ([13]). *Let $0 < q < 1$. The following equalities hold*

$$S_n(0) = (-1)^n q^{-\frac{n(2n+1)}{2}}, \quad n \geq 0, \tag{3.2}$$

$$S_n^{(1)}(0) = (-1)^n q^{-\frac{(n+2)(2n+1)}{2}}\left(1 - (q;q)_{n+1}\right) \neq 0, \quad n \geq 0. \tag{3.3}$$

By virtue of Lemma 3.1 and Proposition 2.2, for $0 < q < 1$, w is a $H_{\sqrt{q}}$-Laguerre-Hahn form. By the second point of Proposition 2.4 and (3.1), we have $\Phi(0) = 0$ and $B(0) = -q^{-5} \neq 0$. Therefore, w is of class $\tilde{s} = 2$. Its Stieltjes formal series S_w satisfies (2.4) with

$$\begin{cases} \tilde{\Phi}(z) = z^2\left(z^2 + q^{-\frac{5}{2}}(q-1)\right), \\ \tilde{B}(z) = -q^{-6}\left(\sqrt{q}+1\right), \\ \tilde{C}(z) = q^{-2}\left(\sqrt{q}-1\right)^{-1}z\left\{z^2 + q^{-2}\left(-1+\sqrt{q}(q-1)\right)\right\}, \\ \tilde{D}(z) = q^{\frac{-3}{2}}\left(\sqrt{q}-1\right)^{-1}z^2. \end{cases} \tag{3.4}$$

From Theorem 1.7 and (3.3), we obtain

$$\begin{cases} \tilde{\gamma}_{2n+1} = q^{-2n-\frac{7}{2}}\dfrac{1-(q;q)_{n+2}}{1-(q;q)_{n+1}}, \quad n \geq 0, \\ \tilde{\gamma}_{2n+2} = q^{-2n-\frac{9}{2}}\left(1-q^{n+2}\right)\dfrac{1-(q;q)_{n+1}}{1-(q;q)_{n+2}}, \quad n \geq 0. \end{cases} \tag{3.5}$$

Furthermore, for $0 < q < 1$, the form w is positive definite.

3.2 Example 2: $u = \mathcal{U}^{(1)}$

Let $\mathcal{U}^{(1)}$ be the first associated of the Al-Salam-Carlitz form of the first kind $\mathcal{U}(a,q)$ for $a>0,\ q>1$ (see Table 2). Let $\left\{U_n^{(1)}\right\}_{n\geq 0}$ be its (MOPS). $\left\{\mathcal{U}^{(1)}\right\}$ is H_q-Laguerre-Hahn of class $s=0$ fulfilling [13]

$$\begin{cases} \beta_n^{(1)} = (1+a)q^{n+1}, \quad \gamma_{n+1}^{(1)} = -aq^{n+1}\left(1-q^{n+2}\right), \quad n\geq 0, \\ H_q\left(\mathcal{U}^{(1)}\right) + a^{-1}q^{-1}(q-1)^{-1}\left\{x-q(1+a)\right\}\mathcal{U}^{(1)} - q\left(x^{-1}\mathcal{U}^{(1)}\left(h_q\mathcal{U}^{(1)}\right)\right) = 0, \\ H_{q^{-1}}(S_{\mathcal{U}^{(1)}})(z) = -qS_{\mathcal{U}^{(1)}}(z)(h_{q^{-1}}S_{\mathcal{U}^{(1)}})(z) - a^{-1}(q-1)^{-1}\left(z-q(1+a)\right)S_{\mathcal{U}^{(1)}}(z) \\ \qquad\qquad -a^{-1}(q-1)^{-1}. \end{cases} \tag{3.6}$$

Lemma 3.2. *Let $a>0$ and $q>1$. The following results hold*

$$U_n(0) = (-1)^n q^{\frac{n(n-1)}{2}} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j q^{-j(n-j)} \neq 0, \quad n\geq 0, \tag{3.7}$$

$$\begin{gathered} U_n^{(1)}(0) = (-1)^n q^{\frac{n(n+1)}{2}} \varpi_{n+1}(a,q) \sum_{k=0}^{n} \frac{(-a)^k q^{\frac{-k(k+1)}{2}}(q;q)_k}{\varpi_k(a,q)\varpi_{k+1}(a,q)} \neq 0, \quad n\geq 0, \\ \varpi_n(a,q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j q^{-j(n-j)}. \end{gathered} \tag{3.8}$$

Proof. The Al-Salam-Carlitz polynomials of the first kind are given by [11]

$$U_n(x) = \sum_{k=0}^{n} \frac{(q;q)_n(-a)^{n-k}}{(q;q)_k(q;q)_{n-k}} q^{(n-k)(n-k-1)/2} x^k (1/x;q)_k, \tag{3.9}$$

with

$$x^k(1/x;q)_k = \prod_{j=0}^{k-1}\left(x-q^j\right) = (-1)^k q^{k(k-1)/2}\left(q^{1-k}x;q\right)_k. \tag{3.10}$$

By substituting (3.10) into (3.9) and letting $x\to 0$, we obtain

$$\begin{aligned} U_n(0) &= \sum_{k=0}^{n} \frac{(q;q)_n(-a)^{n-k}}{(q;q)_k(q;q)_{n-k}} q^{(n-k)(n-k-1)/2}(-1)^k q^{k(k-1)/2} \\ &= (-1)^n (q;q)_n \sum_{k=0}^{n} \frac{a^{n-k}}{(q;q)_k(q;q)_{n-k}} q^{\frac{(n-k)(n-k-1)+k(k-1)}{2}}. \end{aligned} \tag{3.11}$$

Let us perform the index change $j=n-k$

$$\begin{aligned} U_n(0) &= (-1)^n (q;q)_n \sum_{j=0}^{n} \frac{a^j}{(q;q)_{n-j}(q;q)_j} q^{\frac{j(j-1)+(n-j)(n-j-1)}{2}} \\ &\underset{\text{by (1.14)}}{=} (-1)^n q^{\frac{n(n-1)}{2}} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j q^{-j(n-j)}. \end{aligned} \tag{3.12}$$

Since $\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j q^{-j(n-j)} > 0$, for $a > 0$ and $q > 1$, it follows that $\forall n \in \mathbb{N},\ U_n(0) \neq 0$. Now, since $U_n(0) \neq 0,\ n \geq 0,\ a > 0,\ q > 1$, and according to (1.18), we can write

$$U_{n+1}^{(1)}(0) = \frac{U_{n+2}(0)}{U_{n+1}(0)} U_n^{(1)}(0) + \frac{\prod_{\nu=0}^{n} \gamma_{\nu+1}}{U_{n+1}(0)}, \quad n \geq 0. \tag{3.13}$$

Next, by applying Lemma 1.3, we get for (3.13)

$$U_{n+1}^{(1)}(0) = U_{n+2}(0) \left(\frac{U_0^{(1)}(0)}{U_1(0)} + \sum_{k=0}^{n} \frac{(-a)^{k+1} q^{\frac{k(k+1)}{2}} (q;q)_{k+1}}{U_{k+1}(0) U_{k+2}(0)} \right), \quad n \geq 0. \tag{3.14}$$

Since $U_0^{(1)}(0) = 1$ and $U_1(0) = -(1+a)$, we have

$$U_{n+1}^{(1)}(0) = \frac{-U_{n+2}(0)}{1+a} \left(1 - (a+1) \sum_{k=0}^{n} \frac{(-a)^{k+1} q^{\frac{1}{2}k(k+1)} (q;q)_{k+1}}{U_{k+1}(0) U_{k+2}(0)} \right), \quad n \geq 0. \tag{3.15}$$

Replacing n by $n-1$ in (3.15) gives

$$U_{n}^{(1)}(0) = \frac{-U_{n+1}(0)}{1+a} \left(1 - (a+1) \sum_{k=0}^{n-1} \frac{(-a)^{k+1} q^{\frac{1}{2}k(k+1)} (q;q)_{k+1}}{U_{k+1}(0) U_{k+2}(0)} \right), \quad n \geq 1, \tag{3.16}$$

equivalent to

$$U_{n}^{(1)}(0) = \frac{-U_{n+1}(0)}{1+a} \left(1 - (a+1) \sum_{k=1}^{n} \frac{(-a)^{k} q^{\frac{1}{2}k(k-1)} (q;q)_{k}}{U_{k}(0) U_{k+1}(0)} \right), \quad n \geq 1. \tag{3.17}$$

The equality (3.17) remains true for $n = 0$, hence we have

$$U_{n}^{(1)}(0) = U_{n+1}(0) \sum_{k=0}^{n} \frac{(-a)^{k} q^{\frac{1}{2}k(k-1)} (q;q)_{k}}{U_{k}(0) U_{k+1}(0)}, \quad n \geq 0. \tag{3.18}$$

By setting $\varpi_n(a,q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j q^{-j(n-j)},\ a > 0,\ q > 1,\ n \geq 0$, we obtain

$$U_{n}^{(1)}(0) = (-1)^n q^{\frac{n(n+1)}{2}} \varpi_{n+1}(a,q) \sum_{k=0}^{n} \frac{(-a)^{k} q^{\frac{-k(k+1)}{2}} (q;q)_{k}}{\varpi_k(a,q) \varpi_{k+1}(a,q)}, \quad n \geq 0. \tag{3.19}$$

When $q > 1$ and $n \geq 0$, $(q,q)_n$ has the same sign as $(-1)^n$.

Now, since $a>0$ and $\forall n\geq 0$, $\varpi_n(a,q)>0$, we obtain

$$\varpi_{n+1}(a,q)\sum_{k=0}^{n}\frac{(-a)^k q^{\frac{-k(k+1)}{2}}(q;q)_k}{\varpi_k(a,q)\varpi_{k+1}(a,q)}>0,\quad n\geq 0. \tag{3.20}$$

Therefore, $U_n^{(1)}(0)\neq 0,\quad n\geq 0.$ □

By virtue of Lemma 3.2 and Proposition 2.2, for $a>0$, $q>1$, w is a $H_{\sqrt{q}}$-Laguerre–Hahn form. By the first point of Proposition 2.4 and (3.6), we have $\Phi(0)\neq 0$. Therefore, w is of class $\tilde{s}=3$. Its Stieltjes formal series S_w satisfies (2.4) with

$$\begin{cases}\tilde{\Phi}(z)=z,\\ \tilde{B}(z)=-\left(\sqrt{q}+1\right)z,\\ \tilde{C}(z)=q^{\frac{-3}{2}}a^{-1}\left(1-\sqrt{q}\right)^{-1}z^2\left(z^2-q(1+a)\right)+\sqrt{q}^{-1},\\ \tilde{D}(z)=q^{\frac{-3}{2}}a^{-1}\left(1-\sqrt{q}\right)^{-1}z^3.\end{cases} \tag{3.21}$$

From Theorem 1.7 and (3.8), we obtain for $a>0,\ q>1$

$$\begin{cases}\tilde{\gamma}_{2n+1}=q^{n+1}\dfrac{\varpi_{n+2}(a,q)}{\varpi_{n+1}(a,q)}\dfrac{\Lambda_{n+1}(a,q)}{\Lambda_n(a,q)},\quad n\geq 0,\\ \tilde{\gamma}_{2n+2}=-a(1-q^{n+2})\dfrac{\varpi_{n+1}(a,q)}{\varpi_{n+2}(a,q)}\dfrac{\Lambda_n(a,q)}{\Lambda_{n+1}(a,q)},\quad n\geq 0,\\ \Lambda_n(a,q)=\displaystyle\sum_{k=0}^{n}\frac{(-a)^k q^{\frac{-k(k+1)}{2}}(q;q)_k}{\varpi_k(a,q)\varpi_{k+1}(a,q)},\quad n\geq 0,\\ \varpi_n(a,q)=\displaystyle\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_q a^j q^{-j(n-j)},\quad n\geq 0.\end{cases} \tag{3.22}$$

For $a>0$, $q>1$, $\varpi_n>0$, $n\geq 0$, according to (3.20). Moreover $(-a)^n(q,q)_n=a^n|(q,q)_n|>0$, $n\geq 0$, therefore $\Lambda_n>0$, $n\geq 0$. Hence, the form w is positive definite.

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Appendix

Table 1: Stieltjes-Wigert polynomials [24]

The Stieltjes-Wigert form $\mathcal{W}$ with respect to the MOPS $\{S_n\}_{n\geq 0}$

$$S_n(x) = (-1)^n q^{-\left(n^2+\frac{n}{2}\right)} {}_1\phi_1 \left(\begin{matrix} q^{-n} \\ 0 \end{matrix} \middle| q; -q^{n+\frac{3}{2}} x \right), \quad n \geq 0.$$

$$\beta_n = \{(1+q)q^{-n} - q\} q^{-n-\frac{3}{2}}, \quad n \geq 0,$$

$$\gamma_{n+1} = \left(1 - q^{n+1}\right) q^{-4n-4}, \quad n \geq 0.$$

$$H_q\left(x^2 \mathcal{W}\right) - (q-1)^{-1} \left(x - q^{-\frac{3}{2}}\right) \mathcal{W} = 0,$$

$$(\mathcal{W})_n = q^{-\frac{n(n+2)}{2}}, \quad n \geq 0.$$

$$\langle \mathcal{W}, f \rangle = \sqrt{\frac{q}{2\pi \ln q^{-1}}} \int_0^{+\infty} \exp\left(-\frac{\ln^2 x}{2 \ln q^{-1}}\right) f(x)\, dx,$$

$$f \in \mathcal{P},\ 0 < q < 1.$$

Table 2: Al-Salam-Carlitz polynomials [18]

Al-Salam-Carlitz form $\mathcal{U}(a,q)$ with respect to the MOPS $\{U_n\}_{n\geq 0}$

$$U_n(x) = (-a)^n q^{\frac{n(n-1)}{2}} {}_2\phi_1 \left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \middle| q; \frac{qx}{a} \right), \quad n \geq 0.$$

$$\beta_n = (1+a)q^n, \quad n \geq 0,$$

$$\gamma_{n+1} = -aq^n \left(1 - q^{n+1}\right), \quad n \geq 0.$$

$$H_q\left(\mathcal{U}(a,q)\right) - a^{-1}(q-1)^{-1} \left(x - (a-1)\right) \mathcal{U}(a,q) = 0,$$

$$(\mathcal{U}(a,q))_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k, \quad n \geq 0.$$

$$\langle \mathcal{U}(a,q), f \rangle = \frac{1}{(1-q)\left(q, a^{-1}q, a; q\right)_\infty} \int_a^1 \left(a^{-1}tq, tq; q\right)_\infty f(t)\, d_q t,$$

$$f \in \mathcal{P},\ a < 0,\ 0 < q < 1.$$

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