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Inequalities for Chebyshev Functional in Banach Algebras

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ABSTRACT

By utilizing some identities for double sums, some new inequalities for the Chebyshev functional in Banach algebras are given. Some examples for the exponential and resolvent functions on Banach algebras are also provided.

RESUMEN

Usando algunas identidades para sumas dobles, encontramos algunas nuevas desigualdades para el funcional de Chebyshev en álgebras de Banach. También entregamos algunos ejemplos para las funciones exponencial y resolvente en álgebras de Banach.

Keywords and Phrases: Banach algebras, Power series, Exponential function, Resolvent function, Norm inequalities.

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1 Introduction

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardzewski [2] established the following discrete version of Grüss' inequality, see also [39, Ch. X]:

Theorem 1.1. *Let* $a = (a_1, ..., a_n), b = (b_1, ..., b_n)$ *be two n-tuples of real numbers such that* $r \le a_i \le R$ *and* $s \le b_i \le S$ *for* $i = 1, ..., n$*. Then one has the inequality:*

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i \right| \le \frac{1}{n} \left[\frac{n}{2}\right] \left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right) (R - r)(S - s) \tag{1.1}$$

when $[x]$ *is the integer part of* $x, x \in \mathbb{R}$.

A weighted version of Grüss' discrete inequality was proved by J.E. Pečarić in 1979, see for instance [39, Ch. X]:

Theorem 1.2. *Let* a, b *be two monotonic* n*-tuples and* p *a positive one. Then*

$$\left| \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \right| \le |a_n - a_1| |b_n - b_1| \max_{1 \le k \le n-1} \left(\frac{P_k \bar{P}_{k+1}}{P_n^2} \right) \tag{1.2}$$

where $P_n := \sum_{i=1}^{n} p_i$, $\bar{P}_{k+1} = P_n - P_{k+1}$.

In 1981 , A. Lupaş [39, Ch. X] proved some similar results for the first difference of a as follows :

Theorem 1.3. *Let* a, b *two monotonic* n*-tuples in the same sense and* p *a positive n-tuple. Then*

$$\begin{aligned} &\min_{1 \le i \le n-1} |a_{i+1} - a_i| \min_{1 \le i \le n-1} |b_{i+1} - b_i| \left[\frac{1}{P_n} \sum_{i=1}^{n} i^2 p_i - \left(\frac{1}{P_n} \sum_{i=1}^{n} i p_i \right)^2 \right] \\ &\le \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \\ &\le \max_{1 \le i \le n-1} |a_{i+1} - a_i| \max_{1 \le i \le n-1} |b_{i+1} - b_i| \left[\frac{1}{P_n} \sum_{i=1}^{n} i^2 p_i - \left(\frac{1}{P_n} \sum_{i=1}^{n} i p_i \right)^2 \right]. \end{aligned} \tag{1.3}$$

If there exists the numbers $\bar{a}, \bar{a}_1, r, r_1, (rr_1 > 0)$ *such that* $a_k = \bar{a} + kr$ *and* $b_k = \bar{a}_1 + kr_1$*, then in (1.3) the equality holds.*

For some generalizations of Gruss' inequality for isotonic linear functionals defined on certain spaces of mappings see Chapter X of the book [39] where further references are given .

For related results, see [1]-[21], [25]-[31] and [34]-[44].

2 Some Facts on Banach Algebras

In order to extend the above results for Banach algebras, we need some preliminary facts as follows:

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If $a, b \in$Inv$\mathcal{B}$ then $ab \in$Inv$\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in$Inv$\mathcal{B}$;

(ii) $\{a \in \mathcal{B}\text{:}\ \ \|1 - a\| < 1\} \subset$Inv$\mathcal{B}$;

(iii) Inv$\mathcal{B}$ is an *open subset* of $\mathcal{B}$;

(iv) The map Inv$\mathcal{B} \ni a \longmapsto a^{-1} \in$Inv$\mathcal{B}$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by λ. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{\lambda \in \mathbb{C} :\ \lambda - a \in \text{Inv}\mathcal{B}\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to$Inv$\mathcal{B}$, $R_a(\lambda) := (\lambda - a)^{-1}$. For each $\lambda, \gamma \in \rho(a)$ we have the identity

$$R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma) R_a(\lambda) R_a(\gamma).$$

We also have that $\sigma(a) \subset \{\lambda \in \mathbb{C} :\ |\lambda| \leq \|a\|\}$. The *spectral radius* of a is defined as $\nu(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$.

If a, b are *commuting* elements in $\mathcal{B}$, i.e. $ab = ba$, then

$$\nu(ab) \leq \nu(a)\nu(b) \text{ and } \nu(a + b) \leq \nu(a) + \nu(b).$$

Let f be an analytic functions on the open disk $D(0, R)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ $(|\lambda| < R)$. If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a+b) = \exp(a)\exp(b).$$

In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e. the element which have a "*logarithm*". However, it is easy to see that if a is an element in B such that $\|1-a\| < 1$, then a is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b = -\sum_{n=1}^{\infty} \frac{1}{n}(1-a)^n,$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b) = a$.

It is known that if x and y are commuting, i.e. $xy = yx$, then the exponential function satisfies the property

$$\exp(x)\exp(y) = \exp(y)\exp(x) = \exp(x+y).$$

Also, if x is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tx)\,dt = x^{-1}[\exp(bx) - \exp(ax)].$$

Moreover, if x and y are commuting and $y - x$ is invertible, then

$$\begin{aligned}\int_0^1 \exp((1-s)x + sy)\,ds &= \int_0^1 \exp(s(y-x))\exp(x)\,ds \\ &= \left(\int_0^1 \exp(s(y-x))\,ds\right)\exp(x) \\ &= (y-x)^{-1}[\exp(y-x) - I]\exp(x) \\ &= (y-x)^{-1}[\exp(y) - \exp(x)].\end{aligned}$$

Inequalities for functions of operators in Hilbert spaces may be found in the papers [12], [11] and in the recent monographs [22], [23], [32] and the references therein.

The following inequality of Grüss-Lupaş type in Banach algebras holds:

Theorem 2.1. *Let $\mathcal{B}$ be a Banach algebra over $\mathbb{K}$ $(=\mathbb{R},\mathbb{C})$, $a_i, b_i \in \mathcal{B}$ and $\alpha_i \in \mathbb{K}$ $(i = 1, ..., n)$. Then we have the inequality:*

$$\begin{aligned}&\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_i a_i b_i-\sum_{i=1}^{n}\alpha_i a_i\sum_{i=1}^{n}\alpha_i b_i\right\| \\ &\le \max_{1\le j\le n-1}\|a_{j+1}-a_j\|\max_{1\le j\le n-1}\|b_{j+1}-b_j\| \\ &\times\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2|\alpha_i|-\left(\sum_{i=1}^{n}i|\alpha_i|\right)^2\right]\end{aligned} \tag{2.1}$$

The inequality (2.1) is sharp in the sense that the multiplicative constant $C = 1$ in the right membership can not be replaced by a smaller one.

Let α_n be nonzero complex numbers and let

$$R := \frac{1}{\limsup |\alpha_n|^{\frac{1}{n}}}.$$

Clearly $0 \le R \le \infty$, but we consider only the case $0 < R \le \infty$.

Denote by:

$$D(0,R) = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda| < R\}, & \text{if } R < \infty \\ \mathbb{C}, & \text{if } R = \infty, \end{cases}$$

consider the functions:

$$\lambda \mapsto f(\lambda) : D(0,R) \to \mathbb{C}, f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$$

and

$$\lambda \mapsto f_A(\lambda) : D(0,R) \to \mathbb{C}, f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

Let $\mathcal{B}$ be a unital Banach algebra and 1 its unity. Denote by

$$B(0,R) = \begin{cases} \{x \in \mathcal{B} : \|x\| < R\}, & \text{if } R < \infty \\ \mathcal{B}, & \text{if } R = \infty. \end{cases}$$

We associate to f the map:

$$x \mapsto \widetilde{f}(x) : B(0,R) \to \mathcal{B}, \widetilde{f}(x) := \sum_{n=0}^{\infty} \alpha_n x^n.$$

Obviously, $\widetilde{f}$ is correctly defined because the series $\sum_{n=0}^{\infty} \alpha_n x^n$ is absolutely convergent, since $\sum_{n=0}^{\infty} \|\alpha_n x^n\| \leq \sum_{n=0}^{\infty} |\alpha_n| \|x\|^n$.

Making use of Theorem 2.1 we have the following inequality for power series:

Theorem 2.2. *Let* $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ *be a power series that is convergent on the open disk* $D(0, R)$*, with* $R > 0$*. If* $x, y \in \mathcal{B}$ *with* $xy = yx$ *and* $\|x\|, \|y\| \leq 1$*, then we have for* $\lambda \in \mathbb{C}$ *with* $|\lambda| < R$ *the inequality:*

$$\begin{aligned} &\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) \right\| \\ &\leq \|x - 1\| \|y - 1\| \left\{ f_A(|\lambda|) \left[|\lambda| f'_A(|\lambda|) + |\lambda|^2 f''_A(|\lambda|) \right] - \left[|\lambda| f'_A(|\lambda|) \right]^2 \right\}. \end{aligned} \tag{2.2}$$

Motivated by the above results we establish in this paper other similar inequalities for the norm of the *Chebyshev difference*

$$\widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y)$$

by the use of some discrete inequalities of Grüss' type.

First we establish some identities of interest in Banach algebras,

3 Identities in Banach Algebras

Consider the Chebyshev functional defined for $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{K}^n$, $\mathbf{a} = (a_1, ..., a_n) \in \mathcal{B}^n$ and $\mathbf{b} = (b_1, ..., b_n) \in \mathcal{B}^n$, where $\mathcal{B}$ is a Banach algebra over the real or complex number field $\mathbb{K}$:

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) := P_n \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i, \tag{3.1}$$

where $P_n := \sum_{i=1}^{n} p_i$.

The following particular identities for unweighted means hold as well, where $T_n(\mathbf{a}, \mathbf{x})$ is defined as follows:

$$T_n(\mathbf{a}, \mathbf{b}) := \frac{1}{n} \sum_{i=1}^{n} \alpha_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \frac{1}{n} \sum_{i=1}^{n} b_i. \tag{3.2}$$

If α, β are scalars and x, y are vectors in a Banach algebra $\mathcal{B}$, we denote by

$$\det \begin{pmatrix} \alpha & \beta \\ x & y \end{pmatrix} := \alpha y - \beta x \in \mathcal{B}.$$

The first result is embodied in the following

Theorem 3.1. *Let* $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{K}^n$, *and* $\boldsymbol{a} = (a_1, ..., a_n)$, $\boldsymbol{b} = (b_1, ..., b_n) \in \mathcal{B}^n$. *If we define*

$$P_i := \sum_{k=1}^{i} p_k, \ \bar{P}_i := P_n - P_i, i \in \{1, ..., n-1\},$$
$$A_i(\mathbf{p}) := \sum_{k=1}^{i} p_k a_k, \ \bar{A}_i(\mathbf{p}) := A_n(\mathbf{p}) - A_i(\mathbf{p}), i \in \{1, ..., n-1\},$$

then we have the identity

$$\begin{aligned} T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) &= \sum_{i=1}^{n-1} \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \Delta b_i \\ &= P_n \sum_{i=1}^{n-1} P_i \left(\frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{P_i} \right) \Delta b_i \\ &(\textit{if } P_i \neq 0, i \in \{1, ..., n\}) \\ &= \sum_{i=1}^{n-1} P_i \bar{P}_i \left(\frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i} - \frac{A_i(\mathbf{p})}{P_i} \right) \Delta b_i \\ &\left(\textit{if } P_i, \bar{P}_i \neq 0, i \in \{1, ..., n-1\}\right); \end{aligned} \tag{3.3}$$

where $\Delta b_i := b_{i+1} - b_i \ (i \in \{1, ..., n-1\})$ *is the forward difference.*

Proof. We use the following well known summation by parts formula

$$\sum_{l=p}^{q-1} d_l \Delta v_l = d_l v_l \big|_p^q - \sum_{l=p}^{q-1} \Delta d_l v_{l+1}, \tag{3.4}$$

where d_l and v_l are vectors in a linear space, $l = p, ..., q \ (q > p; p, q \text{ are natural numbers})$.

If we choose in (3.4), $p = 1, q = n, d_i = P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})$ and $v_i = b_i \ (i \in \{1, ..., n-1\})$, then we get

$$\begin{aligned}
&\sum_{i=1}^{n-1}\left(P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p})\right)\Delta b_i\\
&=\left[P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p})\right]b_i\big|_1^n-\sum_{i=1}^{n-1}\Delta\left(P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p})\right)b_{i+1}\\
&=\left[P_nA_n(\mathbf{p})-P_nA_n(\mathbf{p})\right]b_n-\left[P_1A_n(\mathbf{p})-P_nA_1(\mathbf{p})\right]b_1\\
&-\sum_{i=1}^{n-1}\left[P_{i+1}A_n(\mathbf{p})-P_nA_{i+1}(\mathbf{p})-P_iA_n(\mathbf{p})+P_nA_i(\mathbf{p})\right]b_{i+1}\\
&=P_np_1a_1b_1-p_1A_n(\mathbf{p})b_1-\sum_{i=1}^{n-1}\left(p_{i+1}A_n(\mathbf{p})-P_np_{i+1}a_{i+1}\right)b_{i+1}\\
&=P_np_1a_1b_1-p_1A_n(\mathbf{p})b_1-A_n(\mathbf{p})\sum_{i=1}^{n-1}p_{i+1}b_{i+1}+P_n\sum_{i=1}^{n-1}p_{i+1}a_{i+1}b_{i+1}\\
&=P_n\sum_{i=1}^{n}p_ia_ib_i-\sum_{i=1}^{n}p_ia_i\sum_{i=1}^{n}p_ib_i\\
&=T_n(\mathbf{p};\mathbf{a},\mathbf{b}),
\end{aligned}$$

which produce the first identity in (3.3).

The second and the third identities are obvious and we omit the details. □

Before we prove the second result, we need the following lemma providing an identity that is interesting in itself as well.

Lemma 3.1. *Let* $\mathbf{p}=(p_1,...,p_n)\in\mathbb{K}^n$ *and* $\boldsymbol{a}=(a_1,...,a_n)\in\mathcal{B}^n$. *Then we have the equality*

$$\det\begin{pmatrix} P_i & P_n\\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix}=\sum_{j=1}^{n-1}P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\Delta a_j, \tag{3.5}$$

for each $i\in\{1,...,n-1\}$.

Proof. Define, for $i\in\{1,...,n-1\}$,

$$K(i):=\sum_{j=1}^{n-1}P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\Delta a_j.$$

We have

$$K(i) = \sum_{j=1}^{i} P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\Delta a_j + \sum_{j=i+1}^{n-1} P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\Delta a_j \tag{3.6}$$
$$= \sum_{j=1}^{i} P_j\bar{P}_i\Delta a_j + \sum_{j=i+1}^{n-1} P_i\bar{P}_j\Delta a_j$$
$$= \bar{P}_i\sum_{j=1}^{i} P_j\Delta a_j + P_i\sum_{j=i+1}^{n-1} \bar{P}_j\Delta a_j.$$

Using the summation by parts formula, we have

$$\sum_{j=1}^{i} P_j\Delta a_j = P_j a_j\Big|_1^{i+1} - \sum_{j=1}^{i}(P_{j+1} - P_j)\,a_{j+1} \tag{3.7}$$
$$= P_{i+1}a_{i+1} - p_1 a_1 - \sum_{j=1}^{i} p_{j+1}a_{j+1}$$
$$= P_{i+1}a_{i+1} - \sum_{j=1}^{i+1} p_j a_j$$

and

$$\sum_{j=i+1}^{n-1} \bar{P}_j\Delta a_j = \bar{P}_j a_j\Big|_{i+1}^{n} - \sum_{j=i+1}^{n-1}\left(\bar{P}_{j+1} - \bar{P}_j\right) a_{j+1} \tag{3.8}$$
$$= \bar{P}_n a_n - \bar{P}_{i+1}a_{i+1} - \sum_{j=i+1}^{n-1}(P_n - P_{j+1} - P_n + P_j)\,a_{j+1}$$
$$= -\bar{P}_{i+1}a_{i+1} + \sum_{j=i+1}^{n-1} p_{j+1}a_{j+1}.$$

Using (3.7) and (3.8) we have

$$K(i) = \bar{P}_i\left(P_{i+1}a_{i+1} - \sum_{j=1}^{i+1} p_j a_j\right) + P_i\left(\sum_{j=i+1}^{n-1} p_{j+1}a_{j+1} - \bar{P}_{i+1}a_{i+1}\right)$$
$$= \bar{P}_i P_{i+1}a_{i+1} - P_i\bar{P}_{i+1}a_{i+1} - \bar{P}_i\sum_{j=1}^{i+1} p_j a_j + P_i\sum_{j=i+1}^{n-1} p_{j+1}a_{j+1}$$
$$= \left[(P_n - P_i)\,P_{i+1} - P_i\,(P_n - P_{i+1})\right] a_{i+1}$$
$$+ P_i\sum_{j=i+1}^{n-1} p_{j+1}a_{j+1} - \bar{P}_i\sum_{j=1}^{i+1} p_j a_j$$

$$
\begin{aligned}
&= P_n p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j \\
&= \left(P_i + \bar{P}_i\right) p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j \\
&= P_i \sum_{j=i+1}^{n-1} p_j a_j - \bar{P}_i \sum_{j=1}^{i} p_j a_j = P_i \bar{A}_i(\mathbf{p}) - \bar{P}_i A_i(\mathbf{p}) \\
&= \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix};
\end{aligned}
$$

and the identity is proved. □

We are able now to state and prove the second identity for the Čebyšev functional

Theorem 3.2. *With the assumptions of Theorem 3.1, we have the equality*

$$
T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta a_j \Delta b_i. \tag{3.9}
$$

The proof is obvious by Theorem 3.1 and Lemma 3.1.

4 Some New Inequalities

The following result holds

Theorem 4.1. *Let* $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{K}^n$, *and* $\boldsymbol{a} = (a_1, ..., a_n)$, $\boldsymbol{b} = (b_1, ..., b_n) \in \mathcal{B}^n$. *Then we have the inequalities*

$$
\begin{aligned}
&\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})\| \\
&\leq \begin{cases}
\max_{1 \leq i \leq n-1} \left\| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right\| \sum_{i=1}^{n-1} \|\Delta b_i\|; \\[2ex]
\left(\sum_{i=1}^{n-1} \left\| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right\|^q \right)^{1/q} \left(\sum_{i=1}^{n-1} \|\Delta b_i\|^p \right)^{1/p} \\[1ex]
\text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\[2ex]
\sum_{i=1}^{n-1} \left\| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right\| \max_{1 \leq i \leq n-1} \|\Delta b_i\|.
\end{cases}
\end{aligned} \tag{4.1}
$$

All the inequalities in (4.1) *are sharp in the sense that the constants* 1 *cannot be replaced by smaller constants.*

Proof. Using the first identity in (3.3), we have

$$\left\|T_n(\mathbf{p};\mathbf{a},\mathbf{b})\right\| \leq \sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix}\right\|\left\|\Delta b_i\right\|.$$

Using Hölder's inequality, we deduce the desired result (4.1).

Let prove, for instance, that the constant 1 in the second inequality is best possible.

Assume, for $C>0$, we have that

$$\left\|T_n(\mathbf{p};\mathbf{a},\mathbf{b})\right\| \leq C\left(\sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix}\right\|^q\right)^{1/q}\left(\sum_{i=1}^{n-1}\left\|\Delta b_i\right\|^p\right)^{1/p} \tag{4.2}$$

for $p>1, \frac{1}{p}+\frac{1}{q}=1, n\geq 2.$

If we choose $n=2$, then we get

$$T_2(\mathbf{p};\mathbf{a},\mathbf{b}) = p_1p_2(a_2-a_1)(b_2-b_1).$$

Also, for $n=2$,

$$\left(\sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix}\right\|^q\right)^{1/q} = |p_1p_2|\left\|a_2-a_1\right\|$$

and

$$\left(\sum_{j=1}^{n-1}\left\|\Delta b_j\right\|^p\right)^{1/p} = \left\|b_2-b_1\right\|.$$

Then by (4.2), holding for $n=2, p_1, p_2 \neq 0, a_1 \neq a_2, b_2 \neq b_1$, we deduce $C \geq 1$, proving that 1 is the best possible constant in that inequality. □

The following corollary for the uniform distribution of the probability $\mathbf{p}$ holds.

Corolary 1. *With the assumptions of Theorem 4.1 for* $\mathbf{a}$ *and* $\boldsymbol{b}$*, we have the inequalities*

$$\|T_n(\mathbf{a},\mathbf{b})\| \tag{4.3}$$
$$\leq \frac{1}{n^2}\times\begin{cases} \max_{1\leq i\leq n-1}\left\|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i} a_k & \sum_{k=1}^{n} a_k\end{pmatrix}\right\| \sum_{i=1}^{n-1}\|\Delta b_i\|; \\ \\ \left(\sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i} a_k & \sum_{k=1}^{n} a_k\end{pmatrix}\right\|^q\right)^{1/q} \\ \times\left(\sum_{i=1}^{n-1}\|\Delta b_i\|^p\right)^{1/p} \\ \\ \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sum_{i=1}^{n-1}\left\|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i} a_k & \sum_{k=1}^{n} a_k\end{pmatrix}\right\| \max_{1\leq i\leq n-1}\|\Delta b_i\|. \end{cases}$$

The following result may be stated as well.

Theorem 4.2. *With the assumptions of Theorem 4.1 and if* $P_i \neq 0\,(i=1,...,n)$, *then we have the inequalities*

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{b})\| \tag{4.4}$$
$$\leq |P_n|\times\begin{cases} \max_{1\leq i\leq n-1}\left\|\frac{A_n(\mathbf{p})}{P_n}-\frac{A_i(\mathbf{p})}{Pi}\right\| \sum_{i=1}^{n-1}|P_i|\,\|\Delta b_i\|; \\ \\ \left(\sum_{i=1}^{n-1}|P_i|\left\|\frac{A_n(\mathbf{p})}{P_n}-\frac{A_i(\mathbf{p})}{Pi}\right\|^q\right)^{1/q}\left(\sum_{i=1}^{n-1}|P_i|\,\|\Delta b_i\|^p\right)^{1/p} \\ \\ \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sum_{i=1}^{n-1}|P_i|\left\|\frac{A_n(\mathbf{p})}{P_n}-\frac{A_i(\mathbf{p})}{Pi}\right\| \max_{1\leq i\leq n-1}\|\Delta b_i\|. \end{cases}$$

All the inequalities in (4.4) *are sharp in the sense that the constant* 1 *cannot be replaced by a smaller constant.*

Proof. Follows by the second identity in (3.3) and taking into account that

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{b})\| \leq |P_n|\sum_{i=1}^{n-1}\left\|\frac{A_n(\mathbf{p})}{P_n}-\frac{A_i(\mathbf{p})}{P_i}\right\| |P_i|\,\|\Delta b_i\|.$$

Using Hölder's weighted inequality, we easily deduce (4.4).

The sharpness of the constant may be shown in a similar manner. We omit the details. □

The following corollary containing the unweighted inequalities holds.

Corolary 2. *With the above assumptions for* $\mathbf{a}$ *and* $\mathbf{b}$ *one, has*

$$\|T_n(\mathbf{a},\mathbf{b})\| \tag{4.5}$$
$$\leq \frac{1}{n}\times\begin{cases} \max_{1\leq i\leq n-1}\left\|\frac{1}{n}\sum_{k=1}^{n}a_k-\frac{1}{i}\sum_{k=1}^{i}a_k\right\|\sum_{i=1}^{n-1}i\,\|\Delta b_i\|; \\ \left(\sum_{i=1}^{n-1}i\left\|\frac{1}{n}\sum_{k=1}^{n}a_k-\frac{1}{i}\sum_{k=1}^{i}a_k\right\|^q\right)^{1/q}\left(\sum_{i=1}^{n-1}i\,\|\Delta b_i\|^p\right)^{1/p} \\ \textit{for } p>1,\frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1}i\left\|\frac{1}{n}\sum_{k=1}^{n}a_k-\frac{1}{i}\sum_{k=1}^{i}a_k\right\|\max_{1\leq i\leq n-1}\|\Delta b_i\|. \end{cases}$$

The inequalities in (4.5) *are sharp in the sense mentioned above.*

Another type of inequalities may be stated if one uses the third identity in (3.3).

Theorem 4.3. *With the assumptions in Theorem 4.1 and if* $P_i,\overline{P}_i\neq 0$, $i\in\{1,...,n-1\}$, *then we have the inequalities*

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{b})\| \tag{4.6}$$
$$\leq\begin{cases} \max_{1\leq i\leq n-1}\left\|\frac{\overline{A}_i(\mathbf{p})}{\overline{P}_i}-\frac{A_i(\mathbf{p})}{P_i}\right\|\sum_{i=1}^{n-1}|P_i|\left|\overline{P}_i\right|\|\Delta b_i\|; \\ \left(\sum_{i=1}^{n-1}|P_i|\left|\overline{P}_i\right|\left\|\frac{\overline{A}_i(\mathbf{p})}{\overline{P}_i}-\frac{A_i(\mathbf{p})}{P_i}\right\|^q\right)^{1/q}\left(\sum_{i=1}^{n-1}|P_i|\left|\overline{P}_i\right|\|\Delta b_i\|^p\right)^{1/p} \\ \textit{for } p>1,\frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1}|P_i|\left|\overline{P}_i\right|\left\|\frac{\overline{A}_i(\mathbf{p})}{\overline{P}_i}-\frac{A_i(\mathbf{p})}{P_i}\right\|\max_{1\leq i\leq n-1}\|\Delta b_i\|. \end{cases}$$

In particular, if $p_i = \frac{1}{n}, i \in \{1, ..., n\}$, *then we have*

$$\|T_n(\mathbf{a}, \mathbf{b})\| \tag{4.7}$$
$$\leq \frac{1}{n^2}\begin{cases} \max_{1\leq i\leq n-1}\left\|\frac{1}{n-i}\sum_{k=i+1}^{n}a_k - \frac{1}{i}\sum_{k=1}^{i}a_k\right\| \\ \times\sum_{i=1}^{n-1} i(n-i)\|\Delta b_i\|; \\ \\ \left(\sum_{i=1}^{n-1} i(n-i)\left\|\frac{1}{n-i}\sum_{k=i+1}^{n}a_k - \frac{1}{i}\sum_{k=1}^{i}a_k\right\|^q\right)^{1/q} \\ \\ \times\left(\sum_{i=1}^{n-1} i(n-i)\|\Delta b_i\|^p\right)^{1/p} \text{ for } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sum_{i=1}^{n-1} i(n-i)\left\|\frac{1}{n-i}\sum_{k=i+1}^{n}a_k - \frac{1}{i}\sum_{k=1}^{i}a_k\right\| \\ \times\max_{1\leq i\leq n-1}\|\Delta b_i\|. \end{cases}$$

The inequalities in (4.6)*and* (4.7) *are sharp in the above mentioned sense.*

A different approach may be considered if one uses the representation in terms of double sums for the Chebyshev functional provided by the Theorem 3.2.

The following result holds.

Theorem 4.4. *With the assumptions in Theorem 4.1, we have the inequalities*

$$\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})\| \tag{4.8}$$
$$\leq \begin{cases} \max_{1\leq i,j\leq n-1}\left\{\left|P_{\min\{i,j\}}\right|\left|\bar{P}_{\max\{i,j\}}\right|\right\}\sum_{i=1}^{n-1}\|\Delta a_i\|\sum_{i=1}^{n-1}\|\Delta b_i\|; \\ \\ \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left|P_{\min\{i,j\}}\right|^q\left|\bar{P}_{\max\{i,j\}}\right|^q\right)^{1/q} \\ \\ \times\left(\sum_{i=1}^{n-1}\|\Delta a_i\|^p\right)^{1/p}\left(\sum_{i=1}^{n-1}\|\Delta b_i\|^p\right)^{1/p} \text{ for } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left|P_{\min\{i,j\}}\right|\left|\bar{P}_{\max\{i,j\}}\right| \\ \\ \times\max_{1\leq i\leq n-1}\|\Delta a_i\|\max_{1\leq i\leq n-1}\|\Delta b_i\|. \end{cases}$$

The inequalities are sharp in the sense mentioned above.

The proof follows by the identity (3.9) on using Hölder's inequality for double sums and we omit the details.

The above theorem has some important consequences as follows:

Corolary 3. *Let* $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{K}^n$, *and* $\boldsymbol{a} = (a_1, ..., a_n)$, $\boldsymbol{b} = (b_1, ..., b_n) \in \mathcal{B}^n$. *Then we have the inequality*

$$\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})\| \le \frac{1}{4}\left(\sum_{k=1}^{n} |p_k|\right)^2 \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\|. \tag{4.9}$$

The constant $\frac{1}{4}$ *is best possible in (4.9).*

Proof. We observe that

$$\left|P_{\min\{i,j\}}\right| = \left|\sum_{k=1}^{\min\{i,j\}} p_k\right| \le \sum_{k=1}^{\min\{i,j\}} |p_k|$$

and

$$\left|\bar{P}_{\max\{i,j\}}\right| = \left|\sum_{k=\max\{i,j\}}^{n} p_k\right| \le \sum_{k=\max\{i,j\}}^{n} |p_k|$$

for any $1 \le i, j \le n-1$.

This implies that

$$\begin{aligned}\left|P_{\min\{i,j\}}\right| \left|\bar{P}_{\max\{i,j\}}\right| &\le \sum_{k=1}^{\min\{i,j\}} |p_k| \sum_{k=\max\{i,j\}}^{n} |p_k| \\ &\le \frac{1}{4}\left(\sum_{k=1}^{\min\{i,j\}} |p_k| + \sum_{k=\max\{i,j\}}^{n} |p_k|\right)^2 \\ &\le \frac{1}{4}\left(\sum_{k=1}^{n} |p_k|\right)^2\end{aligned}$$

for any $1 \le i, j \le n-1$.

Therefore

$$\max_{1\le i,j\le n-1}\left\{\left|P_{\min\{i,j\}}\right| \left|\bar{P}_{\max\{i,j\}}\right|\right\} \le \frac{1}{4}\left(\sum_{k=1}^{n} |p_k|\right)^2$$

and by the first inequality in (4.8) we get (4.9).

To prove the sharpness of the constant $\frac{1}{4}$, assume that (4.9) holds with a constant $C > 0$, i.e.

$$\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})\| \le C\left(\sum_{k=1}^{n} |p_k|\right)^2 \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\|. \tag{4.10}$$

If we take in (4.10) $n = 2$, then we have

$$\|p_1 p_2 (a_2 - a_1)(b_2 - b_1)\| \le C(|p_1| + |p_2|)^2 \|a_2 - a_1\| \|b_2 - b_1\|.$$

If we take $p_1 = p_2 = \frac{1}{2}$, $a_2 - a_1 = \alpha \cdot 1$, $b_2 - b_1 = \beta \cdot 1$ with $\alpha, \beta \ne 0$ then we get $C \ge \frac{1}{4}$, and the proof is completed. ∎

We have:

Corolary 4. *Let* $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{K}^n$, *and* $\boldsymbol{a} = (a_1, ..., a_n)$, $\boldsymbol{b} = (b_1, ..., b_n) \in \mathcal{B}^n$. *Then we have the inequality*

$$\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})\| \le \frac{1}{4^p}(n+1)^{2p} \left[\sum_{i=1}^{n} |p_i|^q \sum_{i=1}^{n} i^2 |p_i|^q - \left(\sum_{i=1}^{n} i |p_i|^q\right)^2\right]^{1/q} \times \left(\sum_{i=1}^{n-1} \|\Delta a_i\|^p\right)^{1/p} \left(\sum_{i=1}^{n-1} \|\Delta b_i\|^p\right)^{1/p}, \tag{4.11}$$

for $p > 1, \frac{1}{p} + \frac{1}{q} = 1$.

Proof. We observe that by Hölder's inequality we have

$$\left|P_{\min\{i,j\}}\right|^q = \left|\sum_{k=1}^{\min\{i,j\}} p_k\right|^q \le (\min\{i,j\})^{q-1} \sum_{k=1}^{\min\{i,j\}} |p_k|^q$$

and

$$\left|\bar{P}_{\max\{i,j\}}\right|^q = \left|\sum_{k=\max\{i,j\}}^{n} p_k\right|^q \le (n - \max\{i,j\} + 1)^{q-1} \sum_{k=\max\{i,j\}}^{n} |p_k|^q$$

for any $1 \le i, j \le n-1$.

Then

$$\begin{aligned} &\left|P_{\min\{i,j\}}\right|^q \left|\bar{P}_{\max\{i,j\}}\right|^q \\ &\le (n - \max\{i,j\} + 1)^{q-1} (\min\{i,j\})^{q-1} \sum_{k=1}^{\min\{i,j\}} |p_k|^q \sum_{k=\max\{i,j\}}^{n} |p_k|^q \\ &= [(n - \max\{i,j\} + 1)(\min\{i,j\})]^{q-1} \sum_{k=1}^{\min\{i,j\}} |p_k|^q \sum_{k=\max\{i,j\}}^{n} |p_k|^q . \end{aligned} \tag{4.12}$$

Observe that

$$(n - \max\{i,j\} + 1)(\min\{i,j\}) \le \frac{1}{4}(n - \max\{i,j\} + 1 + \min\{i,j\})^2 .$$

Since for any $1 \le i, j \le n-1$ we have

$$\max\{i,j\} - \min\{i,j\} = |i-j|$$

then

$$(n - \max\{i,j\} + 1)(\min\{i,j\}) \le \frac{1}{4}(n + 1 - |i-j|)^2 \le \frac{1}{4}(n+1)^2$$

and by (4.12) we get

$$\left|P_{\min\{i,j\}}\right|^{q}\left|\bar{P}_{\max\{i,j\}}\right|^{q}\leq\left[\frac{1}{4}\left(n+1\right)^{2}\right]^{q-1}\sum_{k=1}^{\min\{i,j\}}\left|p_{k}\right|^{q}\sum_{k=\max\{i,j\}}^{n}\left|p_{k}\right|^{q}$$

or any $1\leq i,j\leq n-1$.

Making use of the second inequality in (4.8) we have

$$\begin{aligned}
&\left\|T_{n}\left(\mathbf{p};\mathbf{a},\mathbf{b}\right)\right\| \qquad (4.13)\\
&\leq\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left|P_{\min\{i,j\}}\right|^{q}\left|\bar{P}_{\max\{i,j\}}\right|^{q}\right)^{1/q}\\
&\times\left(\sum_{i=1}^{n-1}\left\|\Delta a_{i}\right\|^{p}\right)^{1/p}\left(\sum_{i=1}^{n-1}\left\|\Delta b_{i}\right\|^{p}\right)^{1/p}\\
&\leq\left(\left[\frac{1}{4}\left(n+1\right)^{2}\right]^{q-1}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(\sum_{k=1}^{\min\{i,j\}}\left|p_{k}\right|^{q}\sum_{k=\max\{i,j\}}^{n}\left|p_{k}\right|^{q}\right)\right)^{1/q}\\
&\times\left(\sum_{i=1}^{n-1}\left\|\Delta a_{i}\right\|^{p}\right)^{1/p}\left(\sum_{i=1}^{n-1}\left\|\Delta b_{i}\right\|^{p}\right)^{1/p}\\
&=\left[\frac{1}{4}\left(n+1\right)^{2}\right]^{\frac{q-1}{q}}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(\sum_{k=1}^{\min\{i,j\}}\left|p_{k}\right|^{q}\sum_{k=\max\{i,j\}}^{n}\left|p_{k}\right|^{q}\right)\right)^{1/q}\\
&\times\left(\sum_{i=1}^{n-1}\left\|\Delta a_{i}\right\|^{p}\right)^{1/p}\left(\sum_{i=1}^{n-1}\left\|\Delta b_{i}\right\|^{p}\right)^{1/p}\\
&=\frac{1}{4^{p}}\left(n+1\right)^{2p}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(\sum_{k=1}^{\min\{i,j\}}\left|p_{k}\right|^{q}\sum_{k=\max\{i,j\}}^{n}\left|p_{k}\right|^{q}\right)\right)^{1/q}\\
&\times\left(\sum_{i=1}^{n-1}\left\|\Delta a_{i}\right\|^{p}\right)^{1/p}\left(\sum_{i=1}^{n-1}\left\|\Delta b_{i}\right\|^{p}\right)^{1/p}.
\end{aligned}$$

If we use the identity for real numbers

$$T_{n}\left(\mathbf{q};\mathbf{a},\mathbf{b}\right)=\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}Q_{\min\{i,j\}}\bar{Q}_{\max\{i,j\}}\Delta a_{j}\Delta b_{i}.$$

and the choices $a_{j}=b_{j}=j$ and $q_{k}=\left|p_{k}\right|^{q}$, then we get

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(\sum_{k=1}^{\min\{i,j\}}\left|p_{k}\right|^{q}\sum_{k=\max\{i,j\}}^{n}\left|p_{k}\right|^{q}\right)=\sum_{i=1}^{n}\left|p_{i}\right|^{q}\sum_{i=1}^{n}i^{2}\left|p_{i}\right|^{q}-\left(\sum_{i=1}^{n}i\left|p_{i}\right|^{q}\right)^{2}.$$

Replacing this in (4.13) produces the desired result (4.11). $\square$

The following corollary also holds. It was obtained earlier in the paper with a different proof.

Corolary 5. *Let* $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{K}^n$, *and* $\boldsymbol{a} = (a_1, ..., a_n)$, $\boldsymbol{b} = (b_1, ..., b_n) \in \mathcal{B}^n$. *Then we have the inequality*

$$\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})\| \leq \left[\sum_{i=1}^{n} |p_i| \sum_{i=1}^{n} i^2 |p_i| - \left(\sum_{i=1}^{n} i |p_i|\right)^2\right] \times \max_{1\leq i\leq n-1} \|\Delta a_i\| \max_{1\leq i\leq n-1} \|\Delta b_i\|. \tag{4.14}$$

The inequality is sharp.

Proof. From the third inequality in (4.8) we have

$$\begin{aligned}
&\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})\| \\
&\leq \max_{1\leq i\leq n-1} \|\Delta a_i\| \max_{1\leq i\leq n-1} \|\Delta b_i\| \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left|P_{\min\{i,j\}}\right| \left|\bar{P}_{\max\{i,j\}}\right| \\
&= \max_{1\leq i\leq n-1} \|\Delta a_i\| \max_{1\leq i\leq n-1} \|\Delta b_i\| \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left|\sum_{k=1}^{\min\{i,j\}} p_k\right| \left|\sum_{k=\max\{i,j\}}^{n} p_k\right| \\
&\leq \max_{1\leq i\leq n-1} \|\Delta a_i\| \max_{1\leq i\leq n-1} \|\Delta b_i\| \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left(\sum_{k=1}^{\min\{i,j\}} |p_k| \sum_{k=\max\{i,j\}}^{n} |p_k|\right) \\
&= \left[\sum_{i=1}^{n} |p_i| \sum_{i=1}^{n} i^2 |p_i| - \left(\sum_{i=1}^{n} i |p_i|\right)^2\right] \max_{1\leq i\leq n-1} \|\Delta a_i\| \max_{1\leq i\leq n-1} \|\Delta b_i\|
\end{aligned}$$

and the desired inequality (4.14) is proved.

The sharpness of the inequality follows as above and the details are omitted. $\square$

5 Inequalities for Power Series

We have:

Theorem 5.1. *Let* $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ *be a power series that is convergent on the open disk* $D(0, R)$, *with* $R > 0$. *If* $x, y \in \mathcal{B}$ *with* $xy = yx$ *and* $\|x\|, \|y\| < 1$, *then we have for* $\lambda \in \mathbb{C}$ *with* $|\lambda| < R$ *the inequality:*

$$\left\|\widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y)\right\| \leq \frac{1}{4} \cdot \frac{\|x - 1\| \|y - 1\|}{(1 - \|x\|)(1 - \|y\|)} f_A^2(|\lambda|). \tag{5.1}$$

Proof. Utilising the inequality (4.9) we have for all $n \geq 1$

$$\left\| \sum_{i=0}^{n} \alpha_i \lambda^i \sum_{i=0}^{n} \alpha_i \lambda^i x^i y^i - \sum_{i=0}^{n} \alpha_i \lambda^i x^i \sum_{i=0}^{n} \alpha_i \lambda^i y^i \right\| \leq \frac{1}{4} \left(\sum_{k=0}^{n} |\alpha_k| |\lambda|^k \right)^2 \sum_{i=0}^{n-1} \left\| x^{i+1} - x^i \right\| \sum_{i=0}^{n-1} \left\| y^{i+1} - y^i \right\|. \tag{5.2}$$

Observe that

$$\sum_{i=0}^{n-1} \left\| x^{i+1} - x^i \right\| = \sum_{i=0}^{n-1} \left\| x^i (x-1) \right\| \leq \|x-1\| \sum_{i=0}^{n-1} \left\| x^i \right\| \leq \|x-1\| \sum_{i=0}^{n-1} \|x\|^i = \|x-1\| \frac{1-\|x\|^n}{1-\|x\|}$$

and, similarly,

$$\sum_{i=0}^{n-1} \left\| y^{i+1} - y^i \right\| \leq \|y-1\| \frac{1-\|y\|^n}{1-\|y\|}.$$

Utilizing (5.2) and the fact that $xy = yx$, we have

$$\left\| \sum_{i=0}^{n} \alpha_i \lambda^i \sum_{i=0}^{n} \alpha_i \lambda^i (xy)^i - \sum_{i=0}^{n} \alpha_i \lambda^i x^i \sum_{i=0}^{n} \alpha_i \lambda^i y^i \right\| \leq \frac{1}{4} \|x-1\| \|y-1\| \left(\sum_{k=0}^{n} |\alpha_k| |\lambda|^k \right)^2 \frac{1-\|x\|^n}{1-\|x\|} \frac{1-\|y\|^n}{1-\|y\|} \tag{5.3}$$

for any $n \geq 1$.

Since all the series whose partial sums are involved in (5.3) are convergent, then by letting $n \to \infty$ in (5.3) we deduce the desired inequality (5.1). □

Corolary 6. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0, R)$, with $R > 0$. If $x \in \mathcal{B}$ and $\|x\| < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:*

$$\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}\left(\lambda x^2\right) - \left[\widetilde{f}(\lambda x) \right]^2 \right\| \leq \frac{1}{4} \cdot \frac{\|x-1\|^2}{(1-\|x\|)^2} f_A^2(|\lambda|). \tag{5.4}$$

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f(\lambda) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda},\ \lambda \in D(0,1); \qquad (5.5) \\ g(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos\lambda,\ \lambda \in \mathbb{C}; \\ h(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin\lambda,\ \lambda \in \mathbb{C}; \\ l(\lambda) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda},\ \lambda \in D(0,1); \end{aligned}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned} f_A(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda},\ \lambda \in D(0,1); \qquad (5.6) \\ g_A(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh\lambda,\ \lambda \in \mathbb{C}; \\ h_A(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh\lambda,\ \lambda \in \mathbb{C}; \\ l_A(\lambda) &= \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda},\ \lambda \in D(0,1). \end{aligned}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\begin{aligned} \exp(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \qquad \lambda \in \mathbb{C}, \qquad (5.7) \\ \frac{1}{2} \ln\left(\frac{1+\lambda}{1-\lambda}\right) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1); \\ \sin^{-1}(\lambda) &= \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)\,n!} \lambda^{2n+1}, \qquad \lambda \in D(0,1); \\ \tanh^{-1}(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1) \\ {}_2F_1(\alpha,\beta,\gamma,\lambda) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} \lambda^n, \alpha,\beta,\gamma > 0, \\ &\lambda \in D(0,1); \end{aligned}$$

where Γ is *Gamma function.*

Example 1. *a) If* $x, y \in \mathcal{B}$ *with* $xy = yx$ *and* $\|x\|, \|y\| < 1$*, then we have for* $\lambda \in \mathbb{C}$ *the inequality:*

$$\|\exp[\lambda(1+xy)] - \exp[\lambda(x+y)]\| \leq \frac{1}{4} \cdot \frac{\|x-1\| \|y-1\|}{(1-\|x\|)(1-\|y\|)} \exp(2|\lambda|). \tag{5.8}$$

In particular, we have

$$\left\|\exp\left[\lambda\left(1+x^2\right)\right] - \exp[2\lambda x]\right\| \leq \frac{1}{4} \cdot \frac{\|x-1\|^2}{(1-\|x\|)^2} \exp(2|\lambda|) \tag{5.9}$$

and

$$\left\|\exp\left[\lambda\left(1-x^2\right)\right] - 1\right\| \leq \frac{1}{4} \cdot \frac{\|x-1\| \|x+1\|}{(1-\|x\|)^2} \exp(2|\lambda|) \tag{5.10}$$

for any $x \in \mathcal{B}$ *with* $\|x\| < 1$ *and* $\lambda \in \mathbb{C}$.

b) We have the inequality

$$\begin{aligned} &\left\|(1-\lambda)^{-1}(1-\lambda xy)^{-1} - (1-\lambda x)^{-1}(1-\lambda y)^{-1}\right\| \\ &\leq \frac{1}{4} \cdot \frac{\|x-1\| \|y-1\|}{(1-\|x\|)(1-\|y\|)(1-|\lambda|)^2} \end{aligned} \tag{5.11}$$

for any $x, y \in \mathcal{B}$ *with* $xy = yx$, $\|x\|, \|y\| < 1$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| < 1$.

In particular, we have

$$\left\|(1-\lambda)^{-1}\left(1-\lambda x^2\right)^{-1} - (1-\lambda x)^{-2}\right\| \leq \frac{1}{4} \cdot \frac{\|x-1\|^2}{(1-\|x\|)^2(1-|\lambda|)^2} \tag{5.12}$$

and

$$\begin{aligned} &\left\|(1-\lambda)^{-1}\left(1+\lambda x^2\right)^{-1} - (1-\lambda x)^{-1}(1+\lambda x)^{-1}\right\| \\ &\leq \frac{1}{4} \cdot \frac{\|x-1\| \|x+1\|}{(1-\|x\|)^2(1-|\lambda|)^2} \end{aligned} \tag{5.13}$$

for any $x \in \mathcal{B}$ *with* $\|x\| < 1$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| < 1$.

c) We have the inequality

$$\begin{aligned} &\left\|\ln(1-\lambda)^{-1}\ln(1-\lambda xy)^{-1} - \ln(1-\lambda x)^{-1}\ln(1-\lambda y)^{-1}\right\| \\ &\leq \frac{1}{4} \cdot \frac{\|x-1\| \|y-1\|}{(1-\|x\|)(1-\|y\|)}[\ln(1-|\lambda|)]^2 \end{aligned} \tag{5.14}$$

for any $x, y \in \mathcal{B}$ *with* $xy = yx$, $\|x\|, \|y\| < 1$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| < 1$.

In particular, we have

$$\left\|(1-\lambda)^{-1}\left(1-\lambda x^2\right)^{-1} - (1-\lambda x)^{-2}\right\| \leq \frac{1}{4} \cdot \frac{\|x-1\|^2}{(1-\|x\|)^2}[\ln(1-|\lambda|)]^2 \tag{5.15}$$

and

$$\begin{aligned} & \left\|(1-\lambda)^{-1}\left(1+\lambda x^{2}\right)^{-1}-(1-\lambda x)^{-1}(1+\lambda x)^{-1}\right\| \\ & \leq \frac{1}{4} \cdot \frac{\|x-1\|\|x+1\|}{(1-\|x\|)^{2}}[\ln (1-|\lambda|)]^{2} \end{aligned} \tag{5.16}$$

for any $x \in \mathcal{B}$ *with* $\|x\|<1$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda|<1$.

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