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Instability to vector lienard equation with multiple delays

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ABSTRACT

By making use of a special Lyapunov-Krasovskii functional and applying Krasovskii's properties, we prove instability of zero solution of a modified vector Lienard equation with multiple constant delays that includes Van der Pol, Rayleigh and Lienard equations, widely encountered in applications.

RESUMEN

Usando un funcional especial de Lyapunov-Krasovskii y aplicando propiedades de Krasovskii, probamos la inestabilidad de la solución nula de una ecuación de Lienard vectorial modificada con retardos constantes múltiples que incluyen a las ecuaciones de Van der Pol, Rayleigh y Liénard ampliamente encontradas en las aplicaciones.

Keywords and Phrases: Lienard, Lyapunov-Krasovskii functional, instability, delay.

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1 Introduction

In this paper, we consider the following modified vector Lienard equation with multiple constant delays, $\tau_i > 0$:

$$X^{''}(t) + F(X(t), X^{'}(t)) + G(X(t)) + \sum_{i=1}^{n} H_i(X(t-\tau_i)) = 0, \tag{1.1}$$

where $t \in \mathfrak{R}^+, \mathfrak{R}^+ = [0, \infty), X \in \mathfrak{R}^n$, $\tau_i > 0$ are fixed constant delays, $t - \tau_i \geq 0$; $F : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ is a continuous function; $G : \mathfrak{R}^n \to \mathfrak{R}^n$ and $H_i : \mathfrak{R}^n \to \mathfrak{R}^n$ are continuously differentiable functions; $F(X, 0) = 0$, $G(0) = 0$, $H_i(0) = 0$.
We assume that the existence and uniqueness of the solutions hold for equation (1.1), (see [3]).
Making $Y = X^{'}$ in equation (1.1), we obtain

$$\begin{aligned} X^{'} &= Y, \\ Y^{'} &= -F(X, Y) - G(X) - \sum_{k=1}^{n} H_i(X) \\ &+ \int_{t-\tau_i}^{t} J_{H_i}(X(s))Y(s)ds. \end{aligned} \tag{1.2}$$

Let $J_G(X)$ and $J_{H_i}(X)$ denote the linear operators from the vectors G and H_i to the matrices

$$J_G(X) = \left(\frac{\partial g_i}{\partial x_j}\right), J_{H_1}(X) = \left(\frac{\partial h_{1i}}{\partial x_j}\right), ...,$$

$$J_{H_n}(X) = \left(\frac{\partial h_{ni}}{\partial x_j}\right), (i, j = 1, 2, ..., n),$$

where $(x_1, ..., x_n), (g_1, ..., g_n)$ and $(h_{1i}, ..., h_{ni})$ are the components of X, G and H_i, respectively. Besides, it is also assumed as basic throughout this paper that the Jacobian matrices $J_G(X)$ and $J_{H_i}(X)$ exist, are symmetric and continuous.
This research has been motivated by the paper of Hale [4] and the recent papers of Tunc $[6-8]$ dealing with stability and instability of zero solution for certain scalar and vector differential equations of second order. First, in 1965, Hale [4] studied instability of zero solution of the scalar Lienard and Rayleigh equations with a constant delay,$r(>0)$, respectively:

$$x^{''}(t) + f(x^{'}(t)) + g(x(t-r)) = 0$$

and

$$x^{''}(t) - \varepsilon\left(1 - \frac{x^{'2}(t)}{3}\right)x^{'}(t) + g(x(t-r)) = 0,$$

where ε is a positive constant.
By defining Lyapunov-Krasovskii functionals, the author gave sufficient conditions to guarantee the instability of zero solution to these equations. Later, Tunc ([6], [7]) discussed the instability of the zero solution for the following modified scalar and vector Lienard equation with multiple constant delays and constant delay, respectively:

$$x''(t) + f_1(x(t), x'(t))x'(t)$$

$$+ f_2(x(t))x'(t) + g_0(x(t)) + \sum_{i=1}^{n} g_i(x(t-\tau_i)) = 0$$

and

$$X''(t) + F(X(t), X'(t))X'(t) + H(X(t-\tau)) = 0,$$

where $\tau_i(>0)$ and $\tau(>0)$ are fixed constant delays.
Throughout this paper, the symbol $< X, Y >$ corresponding to any pair X and Y in $\mathfrak{R}^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, that is, $< X, Y >= \sum_{i=1}^{n} x_i y_i$; thus $< X, X >= \|X\|^2$, and $\lambda_i(A)$ are the eigenvalues of the real symmetric $n \times n$ matrix A.

The following preliminary result is need in the proof.

Lemma(Bellman [1]). Let A be a real symmetric $n \times n$ matrix. Then for any $X \in \mathfrak{R}^n$,

$$a' \langle X, X\rangle \geq \langle AX, X\rangle \geq a \langle X, X\rangle$$

and

$$a'^2 \langle X, X\rangle \geq \langle AX, X\rangle \geq a^2 \langle X, X\rangle,$$

where a' and a are, respectively, the least and greatest eigenvalues of the matrix A.
It may also be useful to give basic information for general autonomous delay differential system with finite delay (see Burton [2]).
Let $r \geq 0$ be given, and let $C = C([-r, 0], \mathfrak{R}^n)$ with

$$\|\phi(s)\| = \max_{-r\leq s\leq 0} |\phi(s)|, \phi \in C.$$

For $H > 0$ define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r, T) \to \Re^n$ is continuous, $0 < T \leq \infty$, then, for each t in $[0, T)$, x_t in C is defined by

$$x_t(s) = x(t+s), -r \leq s \leq 0, t \geq 0.$$

Let G be an open subset of C and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t), F(0) = 0, x_t = x(t+\theta), -r \leq \theta \leq 0, t \geq 0,$$

where $F : G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on F that each initial value problem

$$\dot{x} = F(x_t), x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, T), 0 < T \leq \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

Definition. The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $\|x(\phi)(t)\| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be unstable if it is not stable.

Consider the equations of perturbed motion

$$\frac{dx_i}{dt} = X_i(x_1, ..., x_n, t), (i = 1, 2, ..., n),$$

where the functions $X_i(x_1, ..., x_n, t)$ are defined and continuous in the region $\|x\| < H$, $-\infty < t < \infty$, (H=constant or $H = \infty$)

Theorem A. Let $H_1 < H$. Suppose that there exists a function $v(x, t)$ which is periodic in the time or does not dependent explicitly on the time, such that

(a) v is defined in the region $\|x\| < H$, $-\infty < t < \infty$, (H=constant or $H = \infty$),

(b) v admits an infinitely small upper bound in the region $\|x\| < H$, $-\infty < t < \infty$,

(c) $\frac{dv}{dt} \geq 0$ in the region $\|x\| < H$, $-\infty < t < \infty$,along a trajectory of $\frac{dx_i}{dt} = X_i(x_1, ..., x_n, t)$,

(d) the set of the points M at which the derivative $\frac{dv}{dt}$ is 0 contains no non-trivial half trajectory

$$x(x_0, t_0, t), (t_0 \leq t < \infty).$$

Suppose further that in every neighborhood of the point $x = 0$, there is a point x_0 such that for arbitrary $t_0 \geq 0$ we have $v(x_0, t_0) > 0$. Then the null solution $x = 0$ is unstable, and the trajectories $x(x_0, t_0, t)$ for which $v(x_0, t_0) > 0$ leave the region $\|x\| < H_1$ as the time t increases (see Krasovskii [5, Theorem 15.1]).

2 Main result

The main result of this paper is the following.
Let

$$P(X) = G(X) + \sum_{i=1}^{n} H_i(X).$$

Theorem. Assume that there exist positive constants a, b, d_i such that for all $X, Y \in \mathfrak{R}$ we have
(i) $-Y^T F(X, Y) \geq a \|Y\|^2$, $(a = \sum_{i=1}^{n} a_i)$,

(ii) $X^T J_p(X) X \geq b \|X\|^2$,

(iii) $J_P(X) = J_P^T(X)$,

(iv) $\sqrt{\lambda_i(J_{H_i}^T(X) J_{H_i}(X))} \leq d_i$, $(i = 1, 2, ..., n)$,

(v) $X \neq 0 \Rightarrow P(X) \neq 0$.

If

$$\tau < \frac{a}{\sum\limits_{i=1}^{n} d_i},$$

then the zero solution of equation (1.1) is unstable.
Proof. Introducing a Lyapunov-Krasovskii functional $V = V(X_t, Y_t)$ by the formula

$$\begin{aligned} V = & \sum_{i=1}^{n} \int_0^1 \langle H_i(\sigma X), X\rangle d\sigma + \int_0^1 \langle G(\sigma X), X\rangle \sigma + \frac{1}{2}\langle Y, Y\rangle \\ & - \sum_{i=1}^{n} \mu_i \int_{-\tau_i}^{0} \int_{t+s}^{t} \|Y(\theta)\|^2 \, d\theta ds, \end{aligned}$$

where s is a real variable such that the integrals $\int_{-\tau_i}^{0} \int_{t+s}^{t} \|Y(\theta)\|^2 \, d\theta ds$ are non-negative, and μ_i are certain positive constants to be determined later in the proof.

We observe the existence of the following estimates:

$$V(0, 0) = 0,$$

$$\frac{\partial}{\partial \sigma} H_i(\sigma X) = J_{H_i}(\sigma X) X$$

$\Rightarrow$

$$H_i(X) = \int_0^1 J_{H_i}(\sigma X)X d\sigma,$$

$$\frac{\partial}{\partial \sigma} G(\sigma X) = J_G(\sigma X)X$$

$\Rightarrow$

$$G(X) = \int_0^1 J_G(\sigma X)X d\sigma.$$

Then,

$$\int_0^1 \langle H_i(\sigma X), X\rangle d\sigma = \int_0^1 \int_0^1 \langle \sigma_1 J_{H_i}(\sigma_1\sigma_2 X)X, X\rangle \; d\sigma_2 d\sigma_1$$

and

$$\int_0^1 \langle G(\sigma X), X\rangle d\sigma = \int_0^1 \int_0^1 \langle \sigma_1 J_G(\sigma_1\sigma_2 X)X, X\rangle \; d\sigma_2 d\sigma_1.$$

By noting (ii), we have

$$\begin{aligned}
&\sum_{i=1}^{n} \int_0^1 \langle H_i(\sigma X), X\rangle d\sigma + \int_0^1 \langle G(\sigma X), X\rangle d\sigma \\
&= \sum_{i=1}^{n} \int_0^1 \int_0^1 \langle \sigma_1 J_{H_i}(\sigma_1\sigma_2 X)X, X\rangle \; d\sigma_2 d\sigma_1 \\
&+ \int_0^1 \int_0^1 \langle \sigma_1 J_G(\sigma_1\sigma_2 X)X, X\rangle \; d\sigma_2 d\sigma_1 \\
&\geq \frac{1}{2} b \|X\|^2 .
\end{aligned}$$

Hence,

$$V \geq \frac{1}{2} b \|X\|^2 + \frac{1}{2} \|Y\|^2 - \sum_{i=1}^{n} \mu_i \int_{-\tau_i}^{0} \int_{t+s}^{t} \|Y(\theta)\|^2 \, d\theta ds.$$

Let

$$\bar{\xi} \in \mathfrak{R}^n$$

and

$$\bar{\xi} = (\xi_{11}, ..., \xi_{1n}).$$

Then, the last estimate becomes

$$V(\bar{\xi}, 0) \geq \frac{1}{2} b \left|\left|\bar{\xi}\right|\right|^2 > 0$$

for all arbitrary $\bar{\xi} \neq 0$, $\bar{\xi} \in \Re^n$. So, the first property of Krasovskii [5] holds.
Let us compute the time derivative of V along the solution $(X(t), Y(t))$ of system (1.2),

$$\begin{aligned}\dot{V} = &-\langle F(X,Y), Y\rangle + \left\langle \sum_{i=1}^{n} \int_{t-\tau_i}^{t} J_{H_i}(X(s))Y(s)ds, Y \right\rangle \\ &- \left\langle \sum_{i=1}^{n} (\mu_i \tau_i) Y, Y \right\rangle + \sum_{i=1}^{n} \mu_i \int_{t-\tau_i}^{t} ||Y(\theta)||^2 \, d\theta.\end{aligned}$$

Using the assumptions of the theorem and elementary inequalities, we obtain

$$\begin{aligned}-\langle F(X,Y), Y\rangle &\geq \sum_{i=1}^{n} a_i \, ||Y||^2, \\ \left\langle \sum_{i=1}^{n} \int_{t-\tau_i}^{t} J_{H_i}(X(s))Y(s)ds, Y \right\rangle &\geq -||Y|| \left|\left| \int_{t-\tau_i}^{t} J_{H_i}(X(s))Y(s) \right|\right| ds \\ &\geq - d_i \, ||Y|| \left|\left| \int_{t-\tau_i}^{t} Y(s) \right|\right| ds \\ &\geq - d_i \, ||Y|| \int_{t-\tau_i}^{t} | \, ||Y(s)|| \, ds \\ &\geq - \frac{1}{2} d_i \int_{t-\tau_i}^{t} \left\{ ||Y(t)||^2 + ||Y(s)||^2 \right\} ds \\ &= - \frac{1}{2} d_i \tau_i \, ||Y||^2 - \frac{1}{2} d_i \int_{t-\tau_i}^{t} ||Y(s)||^2 .\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{V} \geq &\sum_{i=1}^{n} a_i \, ||Y||^2 - (\sum_{i=1}^{n} \mu_i \tau_i) \, ||Y||^2 - \frac{1}{2} (\sum_{i=1}^{n} d_i \tau_i) \, ||Y||^2 \\ &+ \sum_{i=1}^{n} (\mu_i - \frac{1}{2} d_i) \int_{t-\tau_i}^{t} ||Y(s)||^2 \, ds.\end{aligned}$$

Let $\mu_i = \frac{1}{2} d_i$ and τ=max$\{\tau_1, \tau_2, ..., \tau_n\}$. Then,

$$\dot{V} \geq \left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} d_i \tau \right) ||Y||^2 .$$

If $\tau < \frac{\alpha}{\sum_{i=1}^{n} d_i}$,then

$$\dot{V} \geq \alpha \|Y\|^2 > 0,$$

where α is some positive constant. Thus, the second property of Krasovskii [5] holds.
Finally, it follows that $\dot{V} = 0 \Leftrightarrow Y = 0$. In view of $Y = 0$ and system (1.2), it follows that $\dot{V} = 0 \Leftrightarrow G(X) + \sum_{i=1}^{n} H_i(X) = 0$ and $Y = 0$. By noting the assumptions of the theorem, $X \neq 0 \Rightarrow P(X) \neq 0$, we can conclude that $G(X) + \sum_{i=1}^{n} H_i(X) = 0 \Leftrightarrow X = 0$. This result shows that the only invariant set of system (1.2) for which $\dot{V} = 0$ is the solution $X = Y = 0$. Therefore, the third property of Krasovskii [5] holds. This completes the proof of the theorem.

3 Conclusion

A functional vector Lienard equation with multiple retardations has been considered. The instability of zero solution of that equation has been discussed by using the Lyapunov-Krasovskii functional approach. The obtained result extends and improve some well known results in the literature.

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