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Measure of noncompactness on $L^p(\mathbb{R}^N)$ and applications

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ABSTRACT

In this paper we define a new measure of noncompactness on $L^p(\mathbb{R}^N)$ $(1 \leq p < \infty)$ and study its properties. As an application we study the existence of solutions for a class of nonlinear functional integral equations using Darbo's fixed point theorem associated with this new measure of noncompactness.

RESUMEN

En este artículo definimos una nueva medida de no-compacidad sobre $L^p(\mathbb{R}^N)$ $(1 \leq p < \infty)$ y estudiamos sus propiedades. Como aplicación, estudiamos la existencia de soluciones para una clase de ecuaciones integrales funcionales no lineales usando el teorema de punto fijo de Darbo asociado a esta nueva medida de no-compacidad.

Keywords and Phrases: Measure of noncompactness, Darbo's fixed point theorem, Fixed point.

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1 Introduction

Measures of noncompactness and Darbo's fixed point theorem play major roles in fixed point theory and their applications. Measures of noncompactness were introduced by Kuratowski [19]. In 1955, Darbo presented a fixed point theorem [12], using this notion. This result was used to establish the existence and behavior of solutions in $C[a,b]$, $BC(\mathbb{R}_+)$ and $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ to many classes of integral equations; see [1, 2, 3, 4, 6, 9, 10, 16, 17] and the references cited therein. When one seeks solutions in unbounded domains there are particular difficulties. The aim of this paper is to construct a regular measure of noncompactness on the space $L^p(\mathbb{R}^N)$ $(1 \le p < \infty)$ and investigate the existence of solutions of a particular nonlinear functional integral equation.

Let $\mathbb{R}_+ = [0, +\infty)$ and $(E, \|.\|)$ be a Banach space. The symbols $\overline{X}$ and $Conv X$ stand for the closure and closed convex hull of a subset X of E, respectively. Now $\mathfrak{M}_E$ denotes the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ denotes the family of all nonempty and relatively compact subsets.

Definition 1.1. *A mapping* $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+$ *is said to be a measure of noncompactness in* E *if it satisfies the following conditions:*

1° *The family* $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ *is nonempty and* $\ker\mu \subseteq \mathfrak{N}_E$.

2° $X \subset Y \Longrightarrow \mu(X) \le \mu(Y)$.

3° $\mu(\overline{X}) = \mu(X)$.

4° $\mu(Conv X) = \mu(X)$.

5° $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$.

6° *If* $\{X_n\}$ *is a sequence of closed sets from* $\mathfrak{M}_E$ *such that* $X_{n+1} \subset X_n$ *for* $n = 1, 2, \cdots$ *and if* $\lim_{n\to\infty} \mu(X_n) = 0$ *then* $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$.

We say that a measure of noncompactness is regular [7] if it additionally satisfies the following conditions:

7° $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.

8° $\mu(X + Y) \le \mu(X) + \mu(Y)$.

9° $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.

10° $\ker\mu = \mathfrak{N}_E$.

The Kuratowski and Hausdorff measures of noncompactness have all the above properties (see [5, 7]).
The following Darbo's fixed point theorem will be needed in section 3.

Theorem 1.2. *[12] Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $F : \Omega \longrightarrow \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ with the property*

$$\mu(FX) \leq k\mu(X) \tag{1}$$

for any nonempty subset X of Ω. Then F has a fixed point in the set Ω.

Integral equations of Urysohn type in the space of Lebesgue integrable functions on bounded and unbounded intervals and the concept of weak measure of noncompactness on $L^1(\mathbb{R}_+)$ was studied in [8, 13, 14].

In Section 2, we define a new measure of noncompactness on $L^p(\mathbb{R}^N)$ and study its properties. In Section 3, using the obtained results in Section 2, we investigate the problem of existence of solutions for a class of nonlinear integral equations.

2 Main results

Let $L^p(U)$ ($U \subseteq \mathbb{R}^N$) denote the space of Lebesgue integrable functions on U with the standard norm

$$\|x\|_{L^p(U)} = \Big(\int_U |x(t)|^p dt\Big)^{\frac{1}{p}}.$$

Before introducing the new measures of noncompactness on $L^p(\mathbb{R}^N)$, we need to characterize the compact subsets of $L^p(\mathbb{R}^N)$.

Theorem 2.1. *[11, 18] Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^N)$ with $1 \leq p < \infty$. The closure of $\mathcal{F}$ in $L^p(\mathbb{R}^N)$ is compact if and only if*

$$\lim_{h \longrightarrow 0} \|\tau_h f - f\|_{L^p(\mathbb{R}^N)} = 0 \qquad \text{uniformly in } f \in \mathcal{F}, \tag{2}$$

where $\tau_h f(x) = f(x+h)$ for all $x, h \in \mathbb{R}^N$. Also for $\epsilon > 0$ there is a bounded and measurable subset $\Omega \subset \mathbb{R}^N$ such that

$$\|f\|_{L^p(\mathbb{R}^N \setminus \Omega)} < \epsilon \quad \textit{for all } f \in \mathcal{F}. \tag{3}$$

Now, we are ready to define a new measure of noncompactness on $L^p(\mathbb{R}^N)$.

Theorem 2.2. *Suppose $1 \leq p < \infty$ and X is a bounded subset of $L^p(\mathbb{R}^N)$. For $x \in X$ and $\epsilon > 0$ let*

$$\begin{aligned}
&\omega^T(x, \epsilon) = \sup\{\|\tau_h x - x\|_{L^p(B_T)} : \|h\|_{\mathbb{R}^N} < \epsilon\},\\
&\omega^T(X, \epsilon) = \sup\{\omega^T(x, \epsilon) : x \in X\},\\
&\omega^T(X) = \lim_{\epsilon \to 0} \omega^T(X, \epsilon),\\
&\omega(X) = \lim_{T \to \infty} \omega^T(X),\\
&d(X) = \lim_{T \to \infty} \sup\{\|x\|_{L^p(\mathbb{R}^N \setminus B_T)} : x \in X\},
\end{aligned}$$

where $B_T = \{a \in \mathbb{R}^N : \|a\|_{\mathbb{R}^N} \leq T\}$. *Then* $\omega_0 : \mathfrak{M}_{L^p(\mathbb{R}^N)} \longrightarrow \mathbb{R}$ *given by*

$$\omega_0(X) = \omega(X) + d(X) \tag{4}$$

defines a measure of noncompactness on $L^p(\mathbb{R}^N)$.

Proof. First we show that 1° holds. Take $X \in \mathfrak{M}_{L^p(\mathbb{R}^N)}$ such that $\omega_0(X) = 0$. Let $\eta > 0$ be arbitrary. Since $\omega_0(X) = 0$, then $\lim_{T\to\infty} \lim_{\epsilon\to 0} \omega^T(X, \epsilon) = 0$ and thus, there exist $\delta > 0$ and $T > 0$ such that $\omega^T(X, \delta) < \eta$ implies that $\|\tau_h x - x\|_{L^p(B_T)} < \eta$ for all $x \in X$ and $h \in \mathbb{R}^N$ such that $\|h\|_{\mathbb{R}^N} < \delta$. Since $\eta > 0$ was arbitrary, we get

$$\lim_{h\to 0} \|\tau_h x - x\|_{L^p(\mathbb{R}^N)} = \lim_{h\to 0} \lim_{T\to\infty} \|\tau_h x - x\|_{L^p(B_T)} = 0$$

uniformly in $x \in X$. Again, keeping in mind that $\omega_0(X) = 0$ we have

$$\lim_{T\to\infty} \sup\{\|x\|_{L^p(\mathbb{R}^N \setminus B_T)} : x \in X\} = 0$$

and so for $\varepsilon > 0$ there exists $T > 0$ such that

$$\|x\|_{L^p(\mathbb{R}^N \setminus B_T)} < \epsilon \quad \text{for all } x \in X.$$

Thus, from Theorem 2.1 we infer that the closure of X in $L^p(\mathbb{R}^N)$ is compact and $\ker\omega_0 \subseteq \mathfrak{N}_E$. The proof of 2° is clear. Now, suppose that $X \in \mathfrak{M}_{L^p(\mathbb{R}^N)}$ and $(x_n) \subset X$ such that $x_n \to x \in \overline{X}$ in $L^p(\mathbb{R}^N)$. From the definition of $\omega^T(X, \epsilon)$ we have

$$\|\tau_h x_n - x_n\|_{L_p(B_T)} \leq \omega^T(X, \epsilon)$$

for any $n \in \mathbb{N}$, $T > 0$ and $\|h\|_{\mathbb{R}^N} < \epsilon$. Letting $n \to \infty$ we get $\|\tau_h x - x\|_{L_p(B_T)} \leq \omega^T(X, \epsilon)$ for any $\|h\|_{\mathbb{R}^N} < \epsilon$ and $T > 0$, hence

$$\lim_{T\to\infty} \lim_{\epsilon\to 0} \omega^T(\overline{X}, \epsilon) \leq \lim_{T\to\infty} \lim_{\epsilon\to 0} \omega^T(X, \epsilon),$$

implies that

$$\omega(\overline{X}) \leq \omega(X). \tag{5}$$

Similarly, we can show that $d(\overline{X}) \leq d(X)$ so from (5) and 2° we get $\omega_0(\overline{X}) = \omega_0(X)$, so ω_0 satisfies condition 3° of Definition 1.1. The proof of conditions 4° and 5° can be carried out similarly by using the inequality $\|\lambda x + (1-\lambda)y\|_{L_p(B_T)} \leq \lambda\|x\|_{L_p(B_T)} + (1-\lambda)\|y\|_{L_p(B_T)}$.
To prove 6°, suppose that $\{X_n\}$ is a sequence of closed and nonempty sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \cdots$, and $\lim_{n\to\infty} \omega_0(X_n) = 0$. Now for any $n \in \mathbb{N}$ take an $x_n \in X_n$ and set $\mathcal{F} = \overline{\{x_n\}}$. We **claim** $\mathcal{F}$ is a compact set in $L^p(\mathbb{R}^N)$. To prove the claim, we need to check conditions (2) and (3) of Theorem 2.1. Let $\varepsilon > 0$ be fixed. Since $\lim_{n\to\infty} \omega_0(X_n) = 0$ there exists $k \in \mathbb{N}$ such that $\omega_0(X_k) < \varepsilon$. Hence, we can find $\delta_1 > 0$ and $T_1 > 0$ such that

$$\omega^{T_1}(X_k, \delta_1) < \varepsilon,$$

and

$$\sup\{\|x\|_{L^p(\mathbb{R}^N\setminus B_{T_1})} : x \in X_k\} < \varepsilon.$$

Thus, for all $n \geq k$ and $\|h\|_{\mathbb{R}^N} < \delta_1$ we get

$$\begin{aligned}\|\tau_h x_n - x_n\|_{L^p(\mathbb{R}^N)} &\leq \|\tau_h x_n - x_n\|_{L^p(B_{T_1})} + \|\tau_h x_n - x_n\|_{L^p(\mathbb{R}^N\setminus B_{T_1})}\\ &\leq \|\tau_h x_n - x_n\|_{L^p(B_{T_1})} + 2\|x_n\|_{L^p(\mathbb{R}^N\setminus B_{T_1})}\\ &< 3\varepsilon\end{aligned}$$

and

$$\|x_n\|_{L^p(\mathbb{R}^N\setminus B_{T_1})} < \varepsilon. \tag{6}$$

The set $\{x_1, x_2, \ldots, x_{k-1}\}$ is compact, hence there exists $\delta_2 > 0$ such that

$$\|\tau_h x_n - x_n\|_{L^p(\mathbb{R}^N)} < \varepsilon \tag{7}$$

for all $n = 1, 2, \ldots, k$ and $\|h\|_{\mathbb{R}^N} < \delta_2$, and there exists $T_2 > 0$ such that

$$\|x_n\|_{L^p(\mathbb{R}^N\setminus T_2)} < \varepsilon \tag{8}$$

for all $n = 1, 2, \ldots, k$. Therefore by (6) and (7) we obtain

$$\|\tau_h x_n - x_n\|_{L^p(\mathbb{R}^N)} < 3\varepsilon$$

for all $n \in \mathbb{N}$ and $\|h\| < \min\{\delta_1, \delta_2\}$, and from (6), (8) we get

$$\|x_n\|_{L^p(\mathbb{R}^N\setminus B_T)} < \varepsilon \tag{9}$$

for all $n \in \mathbb{N}$, where $T = \max\{T_1, T_2\}$. Thus all the hypotheses of Theorem 2.1 are satisfied and so the claim is proved.
Hence there exist a subsequence $\{x_{n_j}\}$ and $x_0 \in L^p(\mathbb{R}^N)$ such that $x_{n_j} \to x_0$, and since $x_n \in X_n$, $X_{n+1} \subset X_n$ and X_n is closed for all $n \in \mathbb{N}$ we get

$$x_0 \in \bigcap_{n=1}^{\infty} X_n = X_\infty,$$

and this finishes the proof of the theorem. □ □

Now, we study the regularity of ω_0.

Theorem 2.3. *The measure of noncompactness ω_0 defined in Theorem 2.1 is regular.*

Proof. Suppose that $X, Y \in \mathfrak{M}_{L^p(\mathbb{R}^N)}$. Since for all $\varepsilon > 0$, $\lambda > 0$ and $T > 0$ we have

$$\omega^T(X \cup Y, \varepsilon) \leq \max\{\omega^T(X, \varepsilon), \omega^T(Y, \varepsilon)\} \quad ,$$
$$\omega^T(X + Y, \varepsilon) \leq \omega^T(X, \varepsilon) + \omega^T(Y, \varepsilon),$$
$$\omega^T(\lambda X, \varepsilon) \leq \lambda\omega^T(X, \varepsilon)$$

and

$$\begin{aligned}\sup_{x\in X\cup Y}\|x\|_{L^p(\mathbb{R}^N\setminus B_T)} &\le \max\{\sup_{x\in X}\|x\|_{L^p(\mathbb{R}^N\setminus B_T)},\sup_{x\in Y}\|x\|_{L^p(\mathbb{R}^N\setminus B_T)}\},\\ \sup_{x\in X+Y}\|x\|_{L^p(\mathbb{R}^N\setminus B_T)} &\le \sup_{x\in X}\|x\|_{L^p(\mathbb{R}^N\setminus B_T)}+\sup_{x\in Y}\|x\|_{L^p(\mathbb{R}^N\setminus B_T)},\\ \sup_{x\in \lambda X}\|x\|_{L^p(\mathbb{R}^N\setminus B_T)} &\le \lambda\sup_{x\in X}\|x\|_{L^p(\mathbb{R}^N\setminus B_T)},\end{aligned}$$

then the hypotheses 7°, 8° and 9° hold. To show that 10° holds, suppose that $X\in\mathfrak{N}_{L^p(\mathbb{R}^N)}$. Thus, the closure of X in $L^p(\mathbb{R}^N)$ is compact and hence from Theorem 2.1, for any $\epsilon>0$ there exists $T>0$ such that $\|x\|_{L^p(\mathbb{R}^N\setminus B_T)}<\epsilon$ for all $x\in X$ and also $\lim_{h\longrightarrow 0}\|\tau_h x-x\|_{L^p(\mathbb{R}^N)}=0$ uniformly in $x\in X$. From the first conclusion, there exists $\delta>0$ such that $\|\tau_h x-x\|_{L^p(\mathbb{R}^N)}<\epsilon$ for any $\|h\|_{\mathbb{R}^N}<\delta$. Then for all $x\in X$ we have

$$\omega^T(x,\delta)=\sup\{\|\tau_h x-x\|_{L^p(B_T)}:\|h\|_{\mathbb{R}^N}<\delta\}\le\epsilon.$$

Therefore,

$$\omega^T(X,\delta)=\sup\{\|\omega(x,\delta):x\in X\}\le\epsilon,$$

which proves

$$\lim_{T\to\infty}\lim_{\delta\to 0}\omega(X,\delta)=0 \tag{10}$$

and

$$\lim_{T\to\infty}\sup\{\|x\|_{L^p(\mathbb{R}^N\setminus B_T)}:x\in X\}=0. \tag{11}$$

Now from (10) and (11) condition 10° holds. □ □

Theorem 2.4. *Let* $Q=\{x\in L^p(\mathbb{R}^N):\|x\|_{L^p(\mathbb{R}^N)}\le 1\}$. *Then* $\omega_0(Q)=3$

Proof. Indeed, we have

$$\|\tau_h x-x\|_{L^p(\mathbb{R}^N)}\le\|\tau_h x\|_{L^p(\mathbb{R}^N)}+\|x\|_{L^p(\mathbb{R}^N)}\le 2$$

and

$$\|x\|_{L^p(\mathbb{R}^N\setminus B_T)}\le\|x\|_{L^p(\mathbb{R}^N)}\le 1$$

for all $x\in Q$, $h\in\mathbb{R}^N$ and $T>0$. Also for any $\epsilon>0$, $T>0$ and $x\in Q$ we have

$$\omega^T(x,\epsilon)=\sup\{\|\tau_h x-x\|_{L^p(B_T)}:\|h\|<\epsilon\}\le 2.$$

Therefore we obtain $\omega_0(Q)\le 3$. Now we prove that $\omega_0(Q)\ge 3$. For any $k\in\mathbb{N}$ there exists $E_k\subset\mathbb{R}^N$ such that $m(E_k)=\frac{1}{2k}$ (m is the Lebesgue measure on $\mathbb{R}^N$), $diam(E_k)\le\frac{1}{k}$, $E_k\cap B_k=\emptyset$ and $E_k\subset B_{2k}$. Define $f_k:\mathbb{R}^N\longrightarrow\mathbb{R}$ by

$$f_k(x)=\begin{cases}(2k)^{\frac{1}{p}} & x\in E_k\\ 0 & \text{otherwise.}\end{cases} \tag{12}$$

It is easy to verify that $\|f_k\|_{L^p(\mathbb{R}^N)}=1$, $\|\tau_{\frac{1}{k}}f_k-f_k\|_{L^p(B_{2k})}=2$ and $\|f_k\|_{L^p(\mathbb{R}^N\setminus B_k)}=1$ for all $k\in\mathbb{N}$. Thus, we get $\omega_0(Q)\ge 3$. This completes the proof. □ □

3 Application

In this section we show the applicability of our results.

Definition 3.1. *We say that a function* $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ *satisfies the Carathéodory conditions if the function* $f(., u)$ *is measurable for any* $u \in \mathbb{R}^m$ *and the function* $f(x, .)$ *is continuous for almost all* $x \in \mathbb{R}^n$.

Theorem 3.2. *Assume that the following conditions are satisfied:*

(i) $f : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ *satisfies the Carathéodory conditions, and there exists a constant* $k \in [0, 1)$ *and* $a \in L^p(\mathbb{R}^N)$ *such that*

$$|f(x, u) - f(y, v)| \leq |a(x) - a(y)| + k|u - v|, \tag{13}$$

for any $u, v \in \mathbb{R}$ *and almost all* $x, y \in \mathbb{R}^N$.

(ii) $f(., 0) \in L^p(\mathbb{R}^N)$.

(iii) $k : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ *satisfies the Carathéodory conditions and there exist* $g_1, g_2 \in L^p(\mathbb{R}^N)$ *and* $g \in L^q(\mathbb{R}^N)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ *such that* $|k(x, y)| \leq g(y)g_1(x)$ *for all* $x, y \in \mathbb{R}^N$ *and*

$$|k(x_1, y) - k(x_2, y)| \leq g(y)|g_2(x_1) - g_2(x_2)|. \tag{14}$$

(iv) *The operator* Q *acts continuously from the space* $L^p(\mathbb{R}^N)$ *into itself and there exists a non-decreasing function* $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that*

$$\|Qu\|_{L^p(\mathbb{R}^N)} \leq \psi(\|u\|_{L^p(\mathbb{R}^N)}) \tag{15}$$

for any $u \in L^p(\mathbb{R}^N)$.

(v) *There exists a positive solution* r_0 *to the inequality*

$$kr + \psi(r)\|K\|_1 + \|f(., 0)\|_{L^p(\mathbb{R}^N)} \leq r \tag{16}$$

where

$$(Ku)(t) = \int_{\mathbb{R}^N} k(x, y)u(y)dy$$

and

$$\|K\|_1 = \sup\{\|Ku\|_{L^p(\mathbb{R}^N)} : \|u\|_{L^p(\mathbb{R}^N)} \leq 1\}.$$

Then the functional integral equation

$$u(x) = f(x, u(x)) + \int_{\mathbb{R}^N} k(x, y)(Qu)(y)dy \tag{17}$$

has at least one solution in the space $L^p(\mathbb{R}^N)$.

Remark 3.3. *The linear Fredholm integral operator* $K : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ *is a continuous operator and* $\|K\|_1 < \infty$.

Proof. First of all we define the operator $F : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ by

$$F(u)(x) = f(x, u(x)) + \int_{\mathbb{R}^N} k(x,y)(Qu)(y)dy. \tag{18}$$

Now Fu is measurable for any $u \in L^p(\mathbb{R}^N)$. Now we prove that $Fu \in L^p(\mathbb{R}^N)$ for any $u \in L^p(\mathbb{R}^N)$. Using conditions (i)-(iv), we have the following inequality

$$|F(u)(x)| \leq |f(x,u) - f(x,0)| + |f(x,0)| + |\int_{\mathbb{R}^N} k(x,y)(Qu)(y)ds|$$

a.e. $x \in \mathbb{R}^N$. Thus

$$\|Fu\|_{L^p(\mathbb{R}^N)} \leq k\|u\|_{L^p(\mathbb{R}^N)} + \|f(.,0)\|_{L^p(\mathbb{R}^N)} + \|K\|_1 \psi(\|u\|_{L^p(\mathbb{R}^N)}). \tag{19}$$

Hence $F(u) \in L^p(\mathbb{R}^N)$ and F is well-defined and also from (19) we have $F(\overline{B}_{r_0}) \subseteq \overline{B}_{r_0}$, where r_0 is the constant appearing in assumption (v). Also, F is continuous in $L^p(\mathbb{R}^N)$, because $f(t,.)$, K and Q are continuous for a.e. $x \in \mathbb{R}^N$. Now we show that for any nonempty set $X \subset \overline{B}_{r_0}$ we have $\omega_0(F(X)) \leq k\omega_0(X)$.
To do so, we fix arbitrary $T > 0$ and $\varepsilon > 0$. Let us choose $u \in X$ and for $x, h \in B_T$ with $\|h\|_{\mathbb{R}^N} \leq \epsilon$, we have

$$\begin{aligned}
|(Fu)(x) - (Fu)(x+h)| \leq & \Big| f(x,u(x)) + \int_{\mathbb{R}^N} k(x,y)(Qu)(y)dy \\
& - f(x+h, u(x+h)) + \int_{\mathbb{R}^N} k(x+h,y)(Qu)(y)dy \Big| \\
\leq & \, |f(x,u(x)) - f(x+h,u(x))| + |f(x+h,u(x)) - f(x+h,u(x+h))| \\
& + |\int_{\mathbb{R}^N} k(x,y)(Qu)(y)dy - \int_{\mathbb{R}^N} k(x+h,y)(Qu)(y)dy| \\
\leq & \, |a(x) - a(x+h)| + k|u(x) - u(x+h)| + \int_{\mathbb{R}^N} |k(x,y) - k(x+h,y)||Qu(y)|dy.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Big(\int_{B_T}|(Fu)(x+h)-(Fu)(x)|^p dt\Big)^{\frac{1}{p}} &\le \Big(\int_{B_T}|a(x)-a(x+h)|^p dt\Big)^{\frac{1}{p}} + k\Big(\int_{B_T}|u(x)-u(x+h)|^p dt\Big)^{\frac{1}{p}}\\
&\quad + \Big(\int_{B_T}\Big|\int_{\mathbb{R}^N}|k(x,y)-k(x+h,y)||Qu(y)|dy\Big|^p dx\Big)^{\frac{1}{p}}\\
&\le \Big(\int_{B_T}|a(x)-a(x+h)|^p dx\Big)^{\frac{1}{p}} + k\Big(\int_{B_T}|u(x)-u(x+h)|^p dx\Big)^{\frac{1}{p}}\\
&\quad + \Big(\int_{B_T}\Big(\int_{\mathbb{R}^N}|k(x,y)-k(x+h,y)|^q dy\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}\|Qu\|_{L^p(\mathbb{R}^N)}\\
&\le \Big(\int_{B_T}|a(x)-a(x+h)|^p dx\Big)^{\frac{1}{p}} + k\Big(\int_{B_T}|u(x)-u(x+h)|^p dx\Big)^{\frac{1}{p}}\\
&\quad + \Big(\int_{B_T}\Big(\int_{\mathbb{R}^N}|g_2(x)-g_2(x+h)|^q|g(y)|^q dy\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}\|Qu\|_{L^p(\mathbb{R}^N)}\\
&\le \|\tau_h a-a\|_{L^p(B_T)} + k\|\tau_h u-u\|_{L^p(B_T)}\\
&\quad + \Big(\int_{B_T}|g_2(x)-g_2(x+h)|^p dx\Big)^{\frac{1}{p}}\|g\|_{L^q(\mathbb{R}^N)}\|Qu\|_{L^p(\mathbb{R}^N)}\\
&\le \omega^T(a,\epsilon)+k\omega^T(u,\epsilon)+\|Qu\|_{L^p(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}\omega^T(g_2,\epsilon).
\end{aligned}$$

Thus we obtain

$$\omega^T(FX,\epsilon)\le\omega^T(a,\epsilon)+k\omega^T(X,\epsilon)+\psi(r_0)\|g\|_{L^q(\mathbb{R}^N)}\omega^T(g_2,\epsilon).$$

Also we have $\omega^T(a,\epsilon),\omega^T(g_2,\epsilon)\to 0$ as $\epsilon\to 0$. Then we obtain

$$\omega(FX)\le k\omega(X). \tag{20}$$

Next, let us fix an arbitrary number $T>0$. Then, taking into account our hypotheses, for an arbitrary function $u\in X$ we have

$$\begin{aligned}
\Big(\int_{\mathbb{R}^N\setminus B_T}|(Fu)(x)|^p dx\Big)^{\frac{1}{p}} &\le \Big(\int_{\mathbb{R}^N\setminus B_T}\Big|f(x,u(x))+\int_{\mathbb{R}^N}k(x,y)Qu(y)dy\Big|^p dt\Big)^{\frac{1}{p}}\\
&\le \Big(\int_{\mathbb{R}^N\setminus B_T}|f(x,u(x))-f(x,0)|^p dx\Big)^{\frac{1}{p}} + \Big(\int_{(\mathbb{R}^N\setminus B_T}|f(t,0)|^p dx\Big)^{\frac{1}{p}}\\
&\quad + \Big(\int_{\mathbb{R}^N\setminus B_T}\Big|\int_{\mathbb{R}^N}k(x,y)Qu(y)dy\Big|^p dx\Big)^{\frac{1}{p}}\\
&\le k\Big(\int_{\mathbb{R}^N\setminus B_T}|u(x)|^p dx\Big)^{\frac{1}{p}} + \Big(\int_{\mathbb{R}^N\setminus B_T}|f(x,0)|^p dx\Big)^{\frac{1}{p}}\\
&\quad + \Big(\int_{\mathbb{R}^N\setminus B_T}\Big(\int_0^\infty|k(x,y)|^q dy\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}\|Qu\|_{L^p(\mathbb{R}^N)}\\
&\le k\|u\|_{L^p(\mathbb{R}^N\setminus B_T)}+\|f(.,0)\|_{L^p(\mathbb{R}^N\setminus B_T)}+\|g_1\|_{L^p(\mathbb{R}^N\setminus B_T)}\|g\|_{L^q(\mathbb{R}^N)}\psi(\|u\|_{L^p(\mathbb{R}^N)}).
\end{aligned}$$

Also we have

$$\|f(.,0)\|_{L^p(\mathbb{R}^N\setminus B_T)},\|g_1\|_{L^p(\mathbb{R}^N\setminus B_T)}\longrightarrow 0$$

as $T \to \infty$ and hence we deduce that

$$d(FX) \leq kd(X). \tag{21}$$

Consequently from (20) and (21) we infer

$$\omega_0(FX) \leq k\omega_0(X). \tag{22}$$

From (22) and Theorem 1.2 we obtain that the operator F has a fixed-point u in B_{r_0} and thus the functional integral equation (17) has at least one solution in $L^p(\mathbb{R}^N)$. □ □

In the example below we will use the following well known result.

Theorem 3.4. *[15] Let $\Omega \subseteq \mathbb{R}^n$ be a measure spaces and suppose $k : \Omega \times \Omega \longrightarrow \mathbb{R}$ is an $\Omega \times \Omega$-measurable function for which there is constant $C > 0$ such that*

$$\int_\Omega |k(x,y)|dx \leq C \qquad \text{for a.e. } y \in \Omega$$

and

$$\int_\Omega |k(x,y)|dy \leq C \qquad \text{for a.e. } x \in \Omega.$$

If $K : L^p(\Omega) \longrightarrow L^p(\Omega)$ is defined by

$$(Kf)(x) = \int_\Omega k(x,y)f(y)dy, \tag{23}$$

then K is a bounded and continuous operator and $\|K\|_1 \leq C$.

Example 3.5. *Consider the integral equation*

$$u(x) = \frac{\cos u(x)}{\|x\| + 2} + \int_{\mathbb{R}^3} \frac{e^{-(|x_2|+|y_2|+|y_3|+1)}}{(|x_1|+3)^2(|y_1|+2)^2(1+|x_3|^2)} e^{-|u(y)|}u(y)dy, \tag{24}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\|x\|$ is the Euclidean norm. We study the solvability of the integral equation (24) on the space $L^p(\mathbb{R}^N)$ for $p > 3$. Let $f(x,u) = \dfrac{\cos u}{\|x\|+2}$ and note it satisfies hypothesis (i) with $a(x) = \dfrac{1}{\|x\|+2}$ and $k = \dfrac{1}{2}$. Indeed, we have

$$\begin{aligned}
|f(x,u) - f(y,v)| &= \left|\frac{\cos u}{\|x\|+2} - \frac{\cos v}{\|y\|+2}\right| \\
&\leq \left|\frac{1}{\|x\|+2} - \frac{1}{\|y\|+2}\right||\cos u| + \frac{1}{\|y\|+2}|\cos u - \cos v| \\
&\leq \left|\frac{1}{\|x\|+2} - \frac{1}{\|y\|+2}\right| + \frac{1}{2}|u - v| \\
&= |a(x) - a(y)| + k|u - v|.
\end{aligned}$$

Also, it is easily seen that $f(.,0)$ *satisfies assumption (ii) and*

$$\begin{aligned}
\|f(.,0)\|^p_{L^p(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\frac{1}{\|x\|+2}|^p dx \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{r^2 \sin\varphi}{(r+2)^p} dr d\varphi d\theta \\
&\le 4\pi \int_0^{\infty} \frac{1}{(r+2)^{p-2}} dr \\
&= \frac{4\pi}{(p-3)2^{p-3}}
\end{aligned}$$

for all $p > 3$. *Thus , we have* $\|f(.,0)\|_{L^p(\mathbb{R}^3)} \le (\frac{4\pi}{p-3})^{\frac{1}{p}}$. *Moreover, taking*

$$k(x,y) = \frac{e^{-(|x_2|+|y_2|+|y_3|+1)}}{(|x_1|+3)^2(|y_1|+2)^2(1+|x_3|^2)},$$

$g_1(x) = g_2(x) = \dfrac{e^{-|x_2|}}{(|x_1|+3)^2(1+|x_3|^2)}$ *and* $g(x) = \dfrac{e^{-(|x_2|+|x_3|)}}{(|x_1|+2)^2}$, *we see that* $g_1, g_2, g \in L^p(\mathbb{R}^3)$ *for all* $1 \le p < \infty$ *and* k *satisfies hypothesis (iii). Also, we have*

$$\int_{\mathbb{R}^3} |k(x,y)| dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-(|x_2|+|y_2|+|y_3|+1)}}{(|x_1|+3)^2(|y_1|+2)^2(1+x_3^2)} dx_1 dx_2 dx_3 \le \frac{\pi}{3e},$$
$$\int_{\mathbb{R}^3} |k(x,y)| dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-(|x_2|+|y_2|+|y_3|+1)}}{(|x_1|+3)^2(|y_1|+2)^2(1+|x_3|^2)} dy_1 dy_2 dy_3 \le \frac{4}{9e}$$

and thus from Theorem 3.2, $\|K\|_1 \le \frac{\pi}{3e}$. *Furthermore,* $Q(u)(x) = e^{-|u(x)|}u(x)$ *satisfies hypothesis (iv) with* $\psi(t) = t$. *Finally, the inequality from assumption (v), has the form*

$$kr + \psi(r)\|K\|_1 + \|f(.,0)\|_{L^p(\mathbb{R}^3)} = \frac{1}{2}r + \frac{\pi}{3e}r + (\frac{4\pi}{p-3})^{\frac{1}{p}} = (\frac{1}{2} + \frac{\pi}{3e})r + (\frac{4\pi}{p-3})^{\frac{1}{p}} \le r$$

Thus, for the number r_0 *we can take* $r_0 = (\frac{4\pi}{p-3})^{\frac{1}{p}} \times \frac{6e}{3e-2\pi}$. *Consequently, all the assumptions of Theorem 3.2 are satisfied and thus equation (24) has at least one solution in the space* $L^p(\mathbb{R}^3)$ *if* $p > 3$.

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References

[1] R. Agarwal, M. Meehan, D. O'Regan, *Fixed point theory and applications*, Cambridge University Press 2004.

[2] A. Aghajani, J. Banás, Y. Jalilian, *Existence of solution for a class nonlinear Volterra sigular integral,* Comput. Math. Appl. 62 (2011), 1215-1227.

[3] A. Aghajani, Y. Jalilian, *Existence and global attractivity of solutions of a nonlinear functional integral equation,* Commun. Nonlinear Sci. Numer. Simulat. 15 (2010), 3306-3312.

[4] A. Aghajani, A. Shole Haghighi, *Existence of solutions for a class of functional integral equations of Volterra type in two variables via measure of noncompactness*, IJST (2014) 1-8.

[5] R.R. Akmerov, M.I. Kamenski, A.S. Potapov, A.E. Rodkina, B.N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Birkhauser Verlag, Basel, 1992.

[6] J. Banás, B.C. Dhage, *Global asymptotic stability of solutions of a functional integral equation*, Nonlinear Anal. 69 (2008), 1945-1952.

[7] J. Banás, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, vol. 60, Dekker, New York, 1980.

[8] J. Banás, M. Paslawska-Poludnik, *Monotonic Solutions of Urysohn Integral Equation on Unbounded Interval*, Comput. Math. Appl. 47 (2004), 1947-1954.

[9] J. Banás, R. Rzepka, *an application of a measure of noncompactness in the study of asymptotic stability,* Appl. Math. Lett. 16 (2003), 1-6.

[10] J. Banás, D. O'Regan, K. Sadarangani, *On solutions of a quadratic hammerstein integral equation on an unbounded interval*, Dynam. Systems Appl. 18 (2009), 251-264.

[11] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations,* Springer New York Dordrecht Heidelberg London, 2011.

[12] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, Rend. Sem. Mat. Univ. Padova, 24 (1955), 84-92.

[13] M. Darwish, *On a perturbed functional integral equation of Urysohn type*, Appl. Math. Comput. 218 (2012), 8800-8805.

[14] W. Gomaa El-Sayed, *Nonlinear functional integral equations of convolution type*, Port. Math. 54 (1997), 449-456.

[15] B. Folland, *Real Analysis*, A Wiley-Interscience Publication, 1999.

[16] M.A. Darwish, J. Henderson, D. O'Regan, *Existence and asymptotic stability of solutions of a perturbed fractional functional-integral equation with linear modification of the argument,* Bull. Korean Math. Soc. 48 (2011), 539-553.

[17] B.C. Dhage, S.S. Bellale, *Local asymptotic stability for nonlinear quadratic functional integral equations*, Electron. J. Qual. Theory Differ. Equ. 10 (2008), 1-13.

[18] H. Ha-Olsen, H. Holden, *The Kolmogorov-Riesz compactness theorem* , Expo. Math. 28 (2010), 385–394

[19] K. Kuratowski, *Sur les espaces complets.* Fund. Math. 15 (1930), 301-309.