

Positive asymptotically almost periodic solutions of an impulsive hematopoiesis model

PENG CHEN ¹, HUI-SHENG DING ¹, GASTON M. N'GUÉRÉKATA ²

¹ *College of Mathematics and Information Science,
Jiangxi Normal University,
Nanchang, Jiangxi 330022, People's Republic of China.*

² *Department of Mathematics,
Morgan State University,
1700 E. Cold Spring Lane, Baltimore, M.D. 21251, USA.*

9972740999@qq.com, dinghs@mail.ustc.edu.cn, Gaston.N'Guerekata@morgan.edu

ABSTRACT

In this paper, we introduce the notion of impulsive asymptotically almost periodic functions and prove some basic properties of such functions. Then, we discuss the existence and exponential stability of positive asymptotically almost periodic solution for an impulsive hematopoiesis model. An example is given to illustrate our results.

RESUMEN

En este artículo, introducimos la noción de funciones impulsivas asintóticamente casi periódicas y probamos algunas propiedades básicas para dichas funciones. Luego, discutimos la existencia y estabilidad exponencial de soluciones positivas asintóticamente casi periódicas para un modelo impulsivo de hematopoyesis. Un ejemplo es dado para ilustrar nuestros resultados.

Keywords and Phrases: Almost periodic, asymptotically almost periodic, impulsive, hematopoiesis.

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1 Introduction and preliminaries

In [8], Mackey and Glass proposed the following nonlinear delay differential equation

$$h'(t) = -\alpha h(t) + \frac{\beta}{1 + h^n(t - \tau)} \quad (1.1)$$

as an appropriate model of hematopoiesis that describes the process of production of all types of blood cells generated by a remarkable self-regulated system that is responsive to the demands put upon it. In medical terms, $h(t)$ denotes the density of mature cells in blood circulation at time t and τ is the time delay between the production of immature cells in the bone marrow and their maturation for release in circulating bloodstream. It is assumed that the cells are lost from the circulation at a rate α , and the flux of the cells into the circulation from the stem cell compartment depends on the density of mature cells at the previous time $t - \tau$.

In this paper, we consider the existence and stability of *asymptotically almost periodic* solutions for the following impulsive hematopoiesis model:

$$\begin{cases} x'(t) = -a(t)x(t) + \frac{b(t)}{1+x^n(t-\tau)}, & t \neq t_k, \\ \Delta x|_{t=t_k} = c_k x(t_k) + I_k(x(t_k)), & k \in \mathbb{Z}, \end{cases} \quad (1.2)$$

where $\Delta x|_{t=t_k} = x(t_k + 0) - x(t_k - 0)$, $\tau \geq 0$ is a constant and the coefficients satisfy some conditions, which will be listed in Section 3.

The direct impetus of this paper comes from two sources. The first source is some recent works on the almost periodic solutions for hematopoiesis models without impulse effect (see, e.g., [4, 7] and references therein); the second source is some recent works on periodic solutions and almost periodic solutions for impulsive hematopoiesis models (see, e.g., [10, 1, 12] and references therein). Stimulated by these works, we aim to make further study on this topic. As one will see, there are two differences of our work from some earlier works on almost periodic solutions to equation (1.2) (cf. [1, 12]). The first difference is that we do not assume that $\inf_{t \in \mathbb{R}} a(t) > 0$ and $-1 \leq c_k \leq 0$ for all $k \in \mathbb{Z}$. In fact, we weaken these assumptions to $M(a) := \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T a(t) dt > 0$ and $-1 \leq c_k$ for all $k \in \mathbb{Z}$. The second difference is that we investigate the existence and stability of asymptotically almost periodic solution to equation (1.2). To the best of our knowledge, it seems that until now there is no results concerning asymptotically almost periodic solution to equation (1.2). Recall that in 1940s, M. Fréchet [5] introduced the notion of asymptotically almost periodicity, which turns out to be one of the most interesting and important generalizations of almost periodicity.

Throughout this paper, we denote by \mathbb{R} the set of real numbers, by \mathbb{R}^+ the set of nonnegative real numbers, by \mathbb{Z} the set of integers, and by \mathbb{N} the set of positive integers.

Now, let us recall some basic notations about classical almost periodic type functions (for more details, we refer the reader to [2, 3]).

Definition 1.1. A set $P \subset \mathbb{Z}$ (or \mathbb{R}) is called relatively dense in \mathbb{Z} (or \mathbb{R}) if there exists a number $l \in \mathbb{N}$ (or \mathbb{R}^+) such that $\forall a \in \mathbb{Z}$ (or \mathbb{R}), $[a, a + l] \cap P \neq \emptyset$.

Definition 1.2. A continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called almost periodic if for every $\varepsilon > 0$,

$$P(\varepsilon, f) = \{\tau \in \mathbb{R} : \sup_{t \in \mathbb{R}} |f(t + \tau) - f(t)| < \varepsilon\}$$

is relatively dense in \mathbb{R} . We denote the set of all such functions by $AP(\mathbb{R})$.

Definition 1.3. A sequence $f : \mathbb{Z} \rightarrow \mathbb{R}$ is called almost periodic if for every $\varepsilon > 0$,

$$P(\varepsilon, f) = \{\tau \in \mathbb{Z} : \sup_{n \in \mathbb{Z}} |f(n + \tau) - f(n)| < \varepsilon\}$$

is relatively dense in \mathbb{Z} . We denote the set of all such sequences by $AP(\mathbb{Z})$.

Definition 1.4. A sequence $f : \mathbb{Z} \rightarrow \mathbb{R}$ is called asymptotically almost periodic if $f = g + h$, where $g \in AP(\mathbb{Z})$ and $h \in C_0(\mathbb{Z})$, where $C_0(\mathbb{Z})$ is the set of all functions $h : \mathbb{Z} \rightarrow \mathbb{R}$ with $\lim_{n \rightarrow \infty} h(n) = 0$. We denote the set of all such sequences by $AAP(\mathbb{Z})$.

Definition 1.5. A set of sequences $f_\lambda : \mathbb{Z} \rightarrow \mathbb{R}$, λ belongs to some index set Λ , is called equi-almost periodic if for every $\varepsilon > 0$, $\bigcap_{\lambda \in \Lambda} P(\varepsilon, f_\lambda)$ is relatively dense in \mathbb{Z} .

Next, let us recall some basic notations and properties about impulsive almost periodic type functions (for more details, we refer the reader to [9, 11, 6]).

Let \mathcal{T} be the set of all sequences $\{t_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$ satisfying $t_k < t_{k+1}$ for all $k \in \mathbb{Z}$, $\inf_{k \in \mathbb{Z}} |t_{k+1} - t_k| > 0$, and

$$\lim_{k \rightarrow +\infty} t_k = +\infty, \quad \lim_{k \rightarrow -\infty} t_k = -\infty.$$

It is easy to see that for every $T = \{t_k\}_{k \in \mathbb{Z}} \in \mathcal{T}$ and $a \in \mathbb{R}$, there holds $T + a \in \mathcal{T}$. In addition, we denote by $\mathcal{P}_T C(\mathbb{R})$ the set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that f is continuous on $\mathbb{R} \setminus T$, and for every $t_k \in T$, $f(t_k - 0) = f(t_k)$ and $f(t_k + 0)$ exists.

Definition 1.6. Let $T = \{t_k\}_{k \in \mathbb{Z}} \in \mathcal{T}$. A function $f \in \mathcal{P}_T C(\mathbb{R})$ is said to be almost periodic if

- (i) the set of sequences $\{T_j\}_{j \in \mathbb{Z}}$ is equi-almost periodic, where $T_j = \{t_k^j : t_k^j = t_{k+j} - t_k, k \in \mathbb{Z}\}$ for every $j \in \mathbb{Z}$;
- (ii) for every $\varepsilon > 0$, there exists $\delta > 0$ such that if t' and t'' belong to the same interval of continuity of f and $|t' - t''| < \delta$, then $|f(t') - f(t'')| < \varepsilon$;
- (iii) for every $\varepsilon > 0$, there exists a relatively dense set $P(\varepsilon, f)$ in \mathbb{R} such that $|f(t + r) - f(t)| < \varepsilon$ for every $r \in P(\varepsilon, f)$ and every $t \in \mathbb{R}$ with $|t - t_k| > \varepsilon$ for all $k \in \mathbb{Z}$.

We denote the set of all such functions by $\mathcal{P}_T AP(\mathbb{R})$.

Lemma 1.7. Let $T = \{t_k\}_{k \in \mathbb{Z}} \in \mathcal{T}$, and $f, g \in \mathcal{P}_T \text{AP}(\mathbb{R})$. Then, the following assertions hold true:

- (i) $f + g \in \mathcal{P}_T \text{AP}(\mathbb{R})$ and $f \cdot g \in \mathcal{P}_T \text{AP}(\mathbb{R})$.
- (ii) $f/g \in \mathcal{P}_T \text{AP}(\mathbb{R})$ provided that $\inf_{t \in \mathbb{R}} |g(t)| > 0$.
- (iii) $k \rightarrow f(t_k)$ belongs to $\text{AP}(\mathbb{Z})$.
- (iv) $\mathcal{P}_T \text{AP}(\mathbb{R})$ is a Banach space under the supremum norm.
- (v) for every $a \in \mathbb{R}$, $f(\cdot - a) \in \mathcal{P}_{T+a} \text{AP}(\mathbb{R})$.

Proof. (i)-(iii) have been proved in [9]. Moreover, by definition of $\mathcal{P}_T \text{AP}(\mathbb{R})$, it is not difficult to prove (iv) and (v). Here, we omit the details. \square

Remark 1.8. It is not difficult to show that a continuous function $f \in \mathcal{P}_T \text{AP}(\mathbb{R})$ implies that $f \in \text{AP}(\mathbb{R})$.

We denote the set of all functions $f \in \mathcal{P}_T \text{C}(\mathbb{R})$ with $\lim_{t \rightarrow \infty} f(t) = 0$ by $\mathcal{P}_T \text{C}_0(\mathbb{R})$. Next, let us introduce the notion of impulsive asymptotically almost periodic functions.

Definition 1.9. Let $T = \{t_k\}_{k \in \mathbb{Z}} \in \mathcal{T}$. A function $f \in \mathcal{P}_T \text{C}(\mathbb{R})$ is said to be asymptotically almost periodic if $f = g + h$, where $g \in \mathcal{P}_T \text{AP}(\mathbb{R})$ and $h \in \mathcal{P}_T \text{C}_0(\mathbb{R})$. We denote the set of all such functions by $\mathcal{P}_T \text{AAP}(\mathbb{R})$.

Lemma 1.10. Let $T = \{t_k\}_{k \in \mathbb{Z}} \in \mathcal{T}$. The following assertions hold true:

- (i) Let $f = g + h \in \mathcal{P}_T \text{AAP}(\mathbb{R})$, where $g \in \mathcal{P}_T \text{AP}(\mathbb{R})$ and $h \in \mathcal{P}_T \text{C}_0(\mathbb{R})$. Then

$$\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}.$$

- (ii) The decomposition of a function $f \in \mathcal{P}_T \text{AAP}(\mathbb{R})$ is unique.
- (iii) $\mathcal{P}_T \text{AAP}(\mathbb{R})$ is a Banach space under the supremum norm.
- (iv) Let $f_1, f_2 \in \mathcal{P}_T \text{AAP}(\mathbb{R})$. Then $f_1 + f_2 \in \mathcal{P}_T \text{AAP}(\mathbb{R})$, $f_1 \cdot f_2 \in \mathcal{P}_T \text{AAP}(\mathbb{R})$ and $f_1(\cdot - a) \in \mathcal{P}_{T+a} \text{AAP}(\mathbb{R})$ for every $a \in \mathbb{R}$.
- (v) Let $f_1, f_2 \in \mathcal{P}_T \text{AAP}(\mathbb{R})$. Then $f_1/f_2 \in \mathcal{P}_T \text{AAP}(\mathbb{R})$ provided that $\inf_{t \in \mathbb{R}} |f_2(t)| > 0$.
- (vi) Let $f \in \mathcal{P}_T \text{AAP}(\mathbb{R})$. Then $k \rightarrow f(t_k)$ belongs to $\text{AAP}(\mathbb{Z})$.

Proof. (i) Since g is left continuous on \mathbb{R} , it suffices to prove that

$$\{g(t) : t \in \mathbb{R}, t \notin T\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}.$$

We prove it by contradiction. Assume that there exists $t' \notin T$ such that

$$\varepsilon_0 := \inf_{t \in \mathbb{R}} |g(t') - f(t)| > 0.$$

Let $\delta = \min \left\{ \inf_{k \in \mathbb{Z}} \frac{|t' - t_k|}{2}, \frac{\varepsilon_0}{2} \right\}$. Then, $\delta > 0$. It follows from $g \in \mathcal{P}_TAP(\mathbb{R})$ that there exists $l > 0$ such that for every $n \in \mathbb{N}$, there is $r_n \in [n - t', n - t' + l]$ such that

$$|g(t' + r_n) - g(t')| < \delta \leq \frac{\varepsilon_0}{2}.$$

Combing this with $h \in \mathcal{P}_TC_0(\mathbb{R})$, we have

$$\begin{aligned} \varepsilon_0 &\leq |g(t') - f(t' + r_n)| \\ &\leq |g(t') - g(t' + r_n)| + |h(t' + r_n)| \\ &\leq \frac{\varepsilon_0}{2} + |h(t' + r_n)| \rightarrow \frac{\varepsilon_0}{2}, \end{aligned}$$

which is a contradiction. This completes the proof.

(ii) It suffices to show that 0 has unique decomposition. In fact, letting $0 = g + h \in \mathcal{P}_TAAAP(\mathbb{R})$, where $g \in \mathcal{P}_TAP(\mathbb{R})$ and $h \in \mathcal{P}_TC_0(\mathbb{R})$, it follows from (i) that $g = 0$ and thus $h = 0$.

(iii) Note that $\mathcal{P}_TC(\mathbb{R})$, $\mathcal{P}_TC_0(\mathbb{R})$ and $\mathcal{P}_TAP(\mathbb{R})$ are all Banach spaces under the supremum norm. Let $\{f_n\} \subset \mathcal{P}_TAAAP(\mathbb{R})$ be a Cauchy sequence, and $f_n = g_n + h_n$, where $g_n \in \mathcal{P}_TAP(\mathbb{R})$ and $h_n \in \mathcal{P}_TC_0(\mathbb{R})$. Then, it follows from (i) that $\{g_n\}$ and $\{h_n\}$ are both Cauchy sequences. The remaining proof follows easily.

(iv) The proof follows from (i) and (v) of Lemma 1.7 and the boundedness of every function in $\mathcal{P}_TAP(\mathbb{R})$ (or $\mathcal{P}_TC_0(\mathbb{R})$).

(v) It suffices to prove that $1/f \in \mathcal{P}_TAAAP(\mathbb{R})$ if $f \in \mathcal{P}_TAAAP(\mathbb{R})$ with $\inf_{t \in \mathbb{R}} |f(t)| > 0$. Let $f = g + h \in \mathcal{P}_TAAAP(\mathbb{R})$, where $g \in \mathcal{P}_TAP(\mathbb{R})$ and $h \in \mathcal{P}_TC_0(\mathbb{R})$. Moreover, $\inf_{t \in \mathbb{R}} |f(t)| > 0$. By (i), there holds $\inf_{t \in \mathbb{R}} |g(t)| > 0$. The remaining proof follows from (ii) of Lemma 1.7 and

$$\frac{1}{f} = \frac{1}{g} - \frac{h}{fg}.$$

(vi) The proof follows from (iii) of Lemma 1.7 and $\lim_{k \rightarrow \infty} t_k = \infty$. □

2 Linear inhomogenous equation

Throughout the rest of this paper, if there is no special statement, we assume that $T = \{t_k\}_{k \in \mathbb{Z}} \in \mathcal{T}$ and the set of sequences $\{T_j\}_{j \in \mathbb{Z}}$ is equi-almost periodic, where $T_j = \{t_k^j : t_k^j = t_{k+j} - t_k, k \in \mathbb{Z}\}$ for every $j \in \mathbb{Z}$. By [9, Lemma 22], the following limit exists:

$$\lim_{t-s \rightarrow +\infty} \frac{i(s, t)}{t-s},$$

where $i(s, t)$ is the number of the terms of $T \cap [s, t]$. We denote

$$p = \lim_{t-s \rightarrow +\infty} \frac{i(s, t)}{t-s}.$$

Now, let us first consider the following linear inhomogenous equation:

$$\begin{cases} x'(t) = -a(t)x(t) + f(t), & t \neq t_k, \\ \Delta x|_{t=t_k} = c_k x(t_k) + I_k, & k \in \mathbb{Z}, \end{cases} \quad (2.1)$$

where $a \in AP(\mathbb{R})$, c_k is an almost periodic sequence, and I_k is an asymptotically almost periodic sequence. Moreover, let $T' \in \mathcal{T}$ with $T \subset T'$ and $f \in \mathcal{P}_{T'}AAP(\mathbb{R})$. Denote

$$M(a) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T a(t) dt, \quad \beta = \sup_{k \in \mathbb{Z}} |1 + c_k|,$$

and

$$X(t, s) = \begin{cases} \exp(-\int_s^t a(u) du), & t_{k-1} < s \leq t \leq t_k, \\ \prod_{i=m}^k (1 + c_i) \cdot \exp(-\int_s^t a(u) du), & t_{m-1} < s \leq t_m \leq t_k < t \leq t_{k+1}. \end{cases}$$

Definition 2.1. Let $T = \{t_k\}_{k \in \mathbb{Z}} \in \mathcal{T}$, $T' \in \mathcal{T}$ with $T \subset T'$ and $f \in \mathcal{P}_{T'}C(\mathbb{R})$. We call that x is a global solution of equation (2.1) if

$$\begin{cases} x'(t) = -a(t)x(t) + f(t), & t \notin T', \\ \Delta x|_{t=t_k} = c_k x(t_k) + I_k, & k \in \mathbb{Z}. \end{cases} \quad (2.2)$$

We have the following results about equation (2.1):

Theorem 2.2. Let $M(a) > p \cdot \ln \beta$. Then, equation (2.1) has a unique global solution \bar{x} in $\mathcal{P}_T AAP(\mathbb{R})$. Moreover, we have

$$\bar{x}(t) = \int_{-\infty}^t X(t, s) f(s) ds + \sum_{t_k < t} X(t, t_k) I_k, \quad t \in \mathbb{R}.$$

Proof. We give the proof by four steps.

Step 1. There exist $M, \omega > 0$ such that

$$|X(t, s)| \leq M e^{-\omega(t-s)}$$

for all $t, s \in \mathbb{R}$ with $t \geq s$.

It suffices to prove the above inequality for $t - s$ being sufficiently large. Let $\alpha \in (0, M(a))$ and $q > p$ be such that $\omega := \alpha - q \ln \beta > 0$. It follows that

$$\alpha(t - s) \leq \int_s^t a(u)du, \quad i(s, t) \leq q(t - s),$$

for all $t, s \in \mathbb{R}$ with $t - s$ being sufficiently large. Then, we have

$$|X(t, s)| = \begin{cases} \left| e^{-\int_s^t a(u)du} \right| \leq e^{-\alpha(t-s)}, & t_{k-1} < s \leq t \leq t_k \\ \left| \prod_{i=m}^k (1 + c_i) \cdot e^{-\int_s^t a(u)du} \right| \leq \beta^{q(t-s)} \cdot e^{-\alpha(t-s)} = e^{-\omega(t-s)}, & t_{m-1} < s \leq t_m \leq t_k < t \leq t_{k+1}. \end{cases}$$

Step 2. \bar{x} is a global solution of equation (2.1).

By Step 1 and direct calculations, one can obtain

$$\bar{x}'(t) = -a(t)\bar{x}(t) + f(t), \quad t \notin T'.$$

Moreover, it is not difficult to verify that $\Delta \bar{x}|_{t=t_k} = c_k \bar{x}(t_k) + I_k$ for $k \in \mathbb{Z}$. In addition, for every $t \in T' \setminus T$, there holds

$$\bar{x}'_+(t) = -a(t)\bar{x}(t) + f(t+0), \quad \bar{x}'_-(t) = -a(t)\bar{x}(t) + f(t).$$

Step 3. $\bar{x} \in \mathcal{P}_T \text{AAP}(\mathbb{R})$.

Let $f = g + h$, where $g \in \mathcal{P}_T \text{AP}(\mathbb{R})$ and $h \in \mathcal{P}_T \text{C}_0(\mathbb{R})$. Moreover, let $I_k = J_k + L_k$, where $k \rightarrow J_k$ belongs to $\text{AP}(\mathbb{Z})$ and $k \rightarrow L_k$ belongs to $\text{C}_0(\mathbb{Z})$. Then, we have

$$\begin{aligned} \bar{x}(t) &= \int_{-\infty}^t X(t, s)f(s)ds + \sum_{t_k < t} X(t, t_k)I_k \\ &= \int_{-\infty}^t X(t, s)g(s)ds + \int_{-\infty}^t X(t, s)h(s)ds + \sum_{t_k < t} X(t, t_k)J_k + \sum_{t_k < t} X(t, t_k)L_k \\ &:= \bar{x}_1(t) + \bar{x}_2(t) + \bar{x}_3(t) + \bar{x}_4(t). \end{aligned}$$

It follows from [9, Theorem 81] that $\bar{x}_1 \in \mathcal{P}_T \text{AP}(\mathbb{R})$ and $\bar{x}_3 \in \mathcal{P}_T \text{AP}(\mathbb{R})$. Noting that \bar{x}_1 is continuous, we conclude that $\bar{x}_1 \in \text{AP}(\mathbb{R})$. Noting that

$$\bar{x}_2(t) = \int_0^{+\infty} X(t, t-s)h(t-s)ds,$$

by using Step 1 and $h \in \mathcal{P}_T \text{C}_0(\mathbb{R})$, one can conclude that \bar{x}_2 is continuous and $\lim_{t \rightarrow \infty} \bar{x}_2(t) = 0$. Again by using Step 1 and $k \rightarrow L_k$ belonging to $\text{C}_0(\mathbb{Z})$, one can conclude that $\bar{x}_4 \in \mathcal{P}_T \text{C}_0(\mathbb{R})$. Combining all the above proof, we have $\bar{x} \in \mathcal{P}_T \text{AAP}(\mathbb{R})$.

Step 4. Uniqueness.

Let $y \in \mathcal{P}_T\text{AAP}(\mathbb{R})$ satisfying (2.2). Noting that y is continuous on $T' \setminus T$, by using equation (2.1), it is not difficult to verify that

$$y(t) = X(t, s)y(s) + \int_s^t X(t, s)f(s)ds + \sum_{s \leq t_i < t} X(t, t_i)I_i,$$

for all $t, s \in \mathbb{R}$ with $t \geq s$. Letting $s \rightarrow -\infty$, noting that y is bounded and $|X(t, s)| \leq Me^{-\omega(t-s)}$, we have

$$y(t) = \int_{-\infty}^t X(t, s)f(s)ds + \sum_{t_k < t} X(t, t_k)I_k = \bar{x}(t), \quad t \in \mathbb{R}.$$

This completes the proof. \square

3 Existence and stability

Now, let us discuss the existence and stability of asymptotically almost periodic solution for equation (1.2). For convenience, we only consider the case of $n > 1$. We first list some assumptions.

- (H1) $a \in \text{AP}(\mathbb{R})$, and c_k is an almost periodic sequence with $c_k \geq -1$ for all $k \in \mathbb{Z}$.
- (H2) $b \in \mathcal{P}_T\text{AAP}(\mathbb{R})$ is nonnegative, and for every $x \in \mathbb{R}^+$, $k \rightarrow I_k(x)$ is a nonnegative asymptotically almost periodic sequence. Moreover, there exists a constant $L > 0$ such that for all $x, y \in \mathbb{R}^+$ and $k \in \mathbb{Z}$, there holds $|I_k(x) - I_k(y)| \leq L|x - y|$.
- (H3) There exist $M, \omega > 0$ such that $|X(t, s)| \leq Me^{-\omega(t-s)}$ for all $t, s \in \mathbb{R}$ with $t \geq s$.
- (H4) $\frac{M\|b\|}{\omega} \cdot \frac{n^2-1}{4n} \sqrt{\frac{n+1}{n-1}} + \frac{ML}{1-e^{-\omega\theta}} < 1$, where $\theta = \inf_{k \in \mathbb{Z}} |t_{k+1} - t_k|$.

Remark 3.1. It follows from Step 1 of the proof for Theorem 2.2 that (H3) holds if $M(a) > p \cdot \ln \beta$.

Before presenting our results, we need to clarify that our definition of solution for equation (1.2) has a slight difference with the classical definition of solution for equation (1.2).

Definition 3.2. We call that a function $x \in \mathcal{P}_T\text{C}(\mathbb{R})$ is a global solution of equation (1.2) if

$$\begin{cases} x'(t) = -a(t)x(t) + \frac{b(t)}{1+x^n(t-\tau)}, & t \neq t_k, t \neq t_k + \tau, \\ \Delta x|_{t=t_k} = c_k x(t_k) + I_k(x(t_k)), & k \in \mathbb{Z}. \end{cases}$$

Moreover, we need to recall a Gronwall inequality to discuss the stability.

Lemma 3.3. [9, Lemma 2] Let $u \in \mathcal{P}_T\text{C}(\mathbb{R})$ be nonnegative, $s \in \mathbb{R}$ and for all $t \geq s$,

$$u(t) \leq C + \int_s^t \gamma u(x)dx + \sum_{s \leq t_k < t} \beta u(t_k),$$

where $C \geq 0$, $\beta \geq 0$ and $\gamma > 0$ are all constants. Then, there exists a constant $C' > 0$ such that for all $t \geq s$,

$$u(t) \leq C'(1 + \beta)^{i(s,t)} e^{\gamma(t-s)}.$$

Theorem 3.4. Assume that (H1)-(H4) hold. Then, equation (1.2) has a unique nonnegative asymptotically almost periodic solution. Moreover, the asymptotically almost periodic solution of equation (1.2) is exponentially stability provided that

$$p \ln(1 + LM) + \frac{n^2 - 1}{4n} \sqrt[n]{\frac{n+1}{n-1}} Me^{\omega\tau} \|b\| < \omega. \tag{3.1}$$

Proof. Let $\varphi \in \mathcal{P}_T\text{AAP}(\mathbb{R})$ with $\inf_{t \in \mathbb{R}} \varphi(t) \geq 0$. Consider

$$\begin{cases} x'(t) = -a(t)x(t) + \frac{b(t)}{1 + \varphi^n(t-\tau)}, & t \neq t_k, \\ \Delta x|_{t=t_k} = c_k x(t_k) + I_k(\varphi(t_k)), & k \in \mathbb{Z}. \end{cases} \tag{3.2}$$

By Lemma 1.10, we have

$$\frac{b(\cdot)}{1 + \varphi^n(\cdot - \tau)} \in \mathcal{P}_{T \cup (T+\tau)}\text{AAP}(\mathbb{R}).$$

Again by Lemma 1.10, we get $k \rightarrow \varphi(t_k)$ belongs to $\text{AAP}(\mathbb{Z})$. Then, by (H2), it is not difficult to show that $k \rightarrow I_k(\varphi(t_k))$ belongs to $\text{AAP}(\mathbb{Z})$.

Now, by Theorem 2.2, we know that for every $\varphi \in \mathcal{P}_T\text{AAP}(\mathbb{R})$ with $\inf_{t \in \mathbb{R}} \varphi(t) \geq 0$, equation (3.2) has a unique global solution $x^\varphi \in \mathcal{P}_T\text{AAP}(\mathbb{R})$, which satisfies

$$x^\varphi(t) = \int_{-\infty}^t X(t,s) \frac{b(s)}{1 + \varphi^n(s-\tau)} ds + \sum_{t_k < t} X(t,t_k) I_k(\varphi(t_k)), \quad t \in \mathbb{R}.$$

Define a mapping \mathcal{F} on $\Omega = \{\varphi \in \mathcal{P}_T\text{AAP}(\mathbb{R}) : \inf_{t \in \mathbb{R}} \varphi(t) \geq 0\}$ by

$$\mathcal{F}(\varphi)(t) = \int_{-\infty}^t X(t,s) \frac{b(s)}{1 + \varphi^n(s-\tau)} ds + \sum_{t_k < t} X(t,t_k) I_k(\varphi(t_k)), \quad t \in \mathbb{R}, \varphi \in \Omega.$$

It is easy to see that Ω is a closed subset in $\mathcal{P}_T\text{AAP}(\mathbb{R})$ and $\mathcal{F}(\Omega) \subset \Omega$ by the assumptions.

Noting that

$$\sup_{u \geq 0} \left| \left(\frac{1}{1 + u^n} \right)' \right| = \sup_{u \geq 0} \frac{nu^{n-1}}{(1 + u^n)^2} \leq \frac{n^2 - 1}{4n} \sqrt[n]{\frac{n+1}{n-1}},$$

we conclude by mean value theorem that

$$\left| \frac{1}{1 + x^n} - \frac{1}{1 + y^n} \right| \leq \frac{n^2 - 1}{4n} \sqrt[n]{\frac{n+1}{n-1}} \cdot |x - y|, \quad x, y \geq 0.$$

Combining this with

$$|X(t,s)| \leq Me^{-\omega(t-s)}, \quad t, s \in \mathbb{R}, t \geq s,$$

for all $\varphi, \psi \in \Omega$, there holds

$$\begin{aligned}
 & \|\mathcal{F}(\varphi) - \mathcal{F}(\psi)\| \\
 &= \sup_{t \in \mathbb{R}} |\mathcal{F}(\varphi)(t) - \mathcal{F}(\psi)(t)| \\
 &\leq \sup_{t \in \mathbb{R}} \int_{-\infty}^t M e^{-\omega(t-s)} b(s) ds \cdot \frac{n^2-1}{4n} \sqrt[n]{\frac{n+1}{n-1}} \|\varphi - \psi\| + \sup_{t \in \mathbb{R}} \sum_{t_k < t} M e^{-\omega(t-t_k)} \cdot L \|\varphi - \psi\| \\
 &\leq \frac{M\|b\|}{\omega} \cdot \frac{n^2-1}{4n} \sqrt[n]{\frac{n+1}{n-1}} \|\varphi - \psi\| + \sum_{k=0}^{\infty} M e^{-k\omega\theta} \cdot L \|\varphi - \psi\| \\
 &= \left(\frac{M\|b\|}{\omega} \cdot \frac{n^2-1}{4n} \sqrt[n]{\frac{n+1}{n-1}} + \frac{ML}{1-e^{-\omega\theta}} \right) \cdot \|\varphi - \psi\|.
 \end{aligned}$$

Then, by (H4), \mathcal{F} is a contraction and thus \mathcal{F} has a unique fixed point in Ω , i.e., equation (1.2) has a unique nonnegative asymptotically almost periodic solution.

Let x be the above asymptotically almost periodic solution of equation (1.2). Next, let us discuss the stability of x . Let y be an arbitrary solution of equation (1.2) on $[t_0, +\infty)$. It is not difficult to verify that

$$x(t) = X(t, t_0)x(t_0) + \int_{t_0}^t X(t, s) \frac{b(s)}{1+x^n(s-\tau)} ds + \sum_{t_0 \leq t_k < t} X(t, t_k) I_k(x(t_k)), \quad t \geq t_0,$$

and

$$y(t) = X(t, t_0)y(t_0) + \int_{t_0}^t X(t, s) \frac{b(s)}{1+y^n(s-\tau)} ds + \sum_{t_0 \leq t_k < t} X(t, t_k) I_k(y(t_k)), \quad t \geq t_0.$$

Letting $u(t) = x(t) - y(t)$, we have for all $t \geq t_0$,

$$|u(t)| \leq M e^{-\omega(t-t_0)} |u(t_0)| + N \|b\| \int_{t_0}^t M e^{-\omega(t-s)} |x(s-\tau) - y(s-\tau)| ds + L \sum_{t_0 \leq t_k < t} M e^{-\omega(t-t_k)} |u(t_k)|,$$

where $N = \frac{n^2-1}{4n} \sqrt[n]{\frac{n+1}{n-1}}$. Letting $v(t) = e^{\omega t} |u(t)|$, we get for all $t \geq t_0$,

$$\begin{aligned}
 v(t) &\leq Mv(t_0) + NM\|b\| \int_{t_0}^t e^{\omega s} |x(s-\tau) - y(s-\tau)| ds + LM \sum_{t_0 \leq t_k < t} v(t_k) \\
 &= Mv(t_0) + NM\|b\| \int_{t_0-\tau}^{t-\tau} e^{\omega\tau} e^{\omega s} |x(s) - y(s)| ds + LM \sum_{t_0 \leq t_k < t} v(t_k) \\
 &\leq Mv(t_0) + NMe^{\omega\tau} \|b\| \int_{t_0-\tau}^t v(s) ds + LM \sum_{t_0 \leq t_k < t} v(t_k) \\
 &\leq C + NMe^{\omega\tau} \|b\| \int_{t_0}^t v(s) ds + LM \sum_{t_0 \leq t_k < t} v(t_k),
 \end{aligned}$$

where $C = Mv(t_0) + NMe^{\omega\tau} \cdot \max_{s \in [t_0 - \tau, t_0]} v(s)$. Combining this with Lemma 3.3, we have

$$v(t) \leq C'(1 + LM)^{i(t_0, t)} \cdot e^{NMe^{\omega\tau} \|b\|(t-t_0)}, \quad t \geq t_0.$$

By (3.1), we can choose $q > p$ such that

$$q \ln(1 + LM) + NMe^{\omega\tau} \|b\| < \omega.$$

Noting the definition of p , we deduce that for sufficiently large t , there holds

$$(1 + LM)^{i(t_0, t)} \leq (1 + LM)^{q(t-t_0)} = e^{q(t-t_0) \ln(1+LM)},$$

which means that

$$v(t) \leq C' e^{\left[q \ln(1+LM) + NMe^{\omega\tau} \|b\| \right] (t-t_0)},$$

i.e.,

$$|x(t) - y(t)| \leq C' e^{-\left[\omega - q \ln(1+LM) + NMe^{\omega\tau} \|b\| \right] (t-t_0)}.$$

Thus, $y(t)$ converges exponentially to $x(t)$ when $t \rightarrow +\infty$. □

At last, we give a simple example to illustrate our results, which does not aim at generality.

Example 3.5. Let $n = 2$,

$$t_k = k + \frac{1}{4} |\sin k - \sin \sqrt{2}k|, \quad a(t) = 20 + 20(\sin 600t + \sin 600\pi t),$$

$$b(t) = \frac{1}{10} \left(|\cos t + \cos \pi t| + \frac{1}{1+t^2} \right), \quad c_k = 1 - \frac{\sin k + \sin \sqrt{3}k + e^{-k^2}}{3},$$

and

$$I_k(x) = \frac{\cos^2 k + \cos^2 \sqrt{3}k + e^{-k^2}}{10} |\cos x|.$$

It is easy to verify that (H1) and (H2) hold with $L = \frac{3}{10}$. Also, one can show that (H3) holds with $M = \sqrt[10]{e}$ and $\omega = 18$. Moreover, we have $\|b\| = \frac{3}{10}$ and

$$\theta = \inf_{k \in \mathbb{Z}} |t_{k+1} - t_k| \in \left[\frac{1}{2}, 1 \right], \quad p = \lim_{t-s \rightarrow +\infty} \frac{i(s, t)}{t-s} \leq 2.$$

By direct calculations, one can get

$$\frac{M\|b\|}{\omega} \cdot \frac{n^2 - 1}{4n} \sqrt[n]{\frac{n+1}{n-1}} + \frac{ML}{1 - e^{-\omega\theta}} < 1,$$

i.e., (H4) holds. Also, one can verify that

$$p \ln(1 + LM) + \frac{n^2 - 1}{4n} \sqrt[n]{\frac{n+1}{n-1}} M\|b\| < \omega,$$

which means that (3.1) holds if τ is sufficiently small.

Remark 3.6. In the above example, one can easily see that $\inf_{t \in \mathbb{R}} a(t) < 0$ and $c_k > 0$ for all $k \in \mathbb{Z}$.

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