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On generalized closed sets in generalized topological spaces

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ABSTRACT

In this paper, we introduce several types of generalized closed sets in generalized topological spaces (GTSs). Their interrelationships are investigated and several characterizations of μ-T_0, μ-T_1, μ-$T_{1/2}$, μ-regular, μ-normal GTSs and extremally μ-disconnected GTSs are obtained.

RESUMEN

En este artículo introducimos varios tipos de conjuntos cerrados generalizados en espacios topológicos generalizados (GTSs). Sus interrelaciones son investigadas y varias caracterizaciones de GTSs μ-T_0, μ-T_1, μ-$T_{1/2}$, μ-regulares, μ-normales y extremalmente μ-disconexos son obtenidas.

Keywords and Phrases: Generalized topological spaces, generalized closed sets, extremally μ-disconnectedness, Separation axioms.

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1 Introduction

Several types of generalized closed sets are investigated in the literature of topological spaces [3, 5, 6, 7, 16, 18, 19, 20, 23, 24, 26, 27, 28, 29, 30, 35, 38, 37, 39, 43, 44, 48, 49]. Their relationship with one another is shown by a diagram in Benchalli et al. [4] and Dontchev [17]. Using the concept of generalized closed sets, several separation axioms [17, 21] are introduced which are found to be useful in the study of digital topology (digital line) [25]. Cao et al. [9] obtained several characterizations of extremally disconnectedness in terms of generalized closed sets. The purpose of this paper is to show that these diagrams can be obtained in the setting of generalized topological spaces (GTSs) introduced by Császár [11]. Let X be a set and $\mathcal{P}(X)$ be the power set of X. A subset μ of $\mathcal{P}(X)$ is called generalized topology (GT) on X if μ is closed under arbitrary unions and in that case (X, μ) is called a generalized topological space (GTS). The elements of μ are called μ-open sets and their complements are called μ-closed sets. The closure of A, denoted by $c_\mu A$, is the intersection of μ-closed sets containing A. The interior of A, denoted by $i_\mu A$, is the union of μ-open sets contained in A. In a GTS (X, μ), we define $M_\mu = \cup\{U : U \in \mu\}$. A GTS (X, μ) is called strong if $M_\mu = X$.

The notions of various generalized closed sets depend on several types of stronger or weaker forms of open sets, for example, regular open set [44], semi open set [26], preopen set [31], semi preopen set [2], α-open set [36], θ-open set [50], δ-open set [50], π-open set [20] etc. All these notions are extended to the setting of generalized topological spaces. The concept of μ-$T_{1/2}$ GTS depends in turn on the concept of a generalized closed set. We explore the relationship of generalized closed sets with several separation axioms, μ-T_0, μ-T_1, μ-$T_{1/2}$, μ-regularity, and μ-normality [32, 33].

A concept of extremally μ-disconnectedness was introduced in [46]; A GTS (X, μ) is extremallay μ-disconnected if $c_\mu U \cap M_\mu \in \mu$ for every $U \in \mu$. It may be remarked that in strong GTS, this notion concide with the notion of extremally disconnectedness in Császár [12]. Several characterizations of extremally μ-disconnectedness in terms of generalized closed sets are obtained.

Section 2 contains preliminaries. In section 3, we introduce various notions of generalized closed sets and obtain several implications among them. Section 4 contains characterizations of μ-T_0, μ-T_1 and μ-$T_{1/2}$ GTSs. In section 5, we study the characterization of μ-regularity and μ-normality. Section 6 obtains some characterizations of extremally μ-disconnected GTSs.

2 Preliminaries

Let (X, μ) be a GTS and $A \subseteq X$. A^c denotes the complement of A in X. The collection of all μ-closed sets in X is denoted by Ω.

Theorem 2.1. *Let* (X, μ) *be a GTS and* $A, B \subseteq X$. *Then the following statements hold.*

(i) $x \in c_\mu A$ *if and only if* $x \in U \in \mu$ *implies* $U \cap A \neq \emptyset$.

(ii) $c_\mu A = c_\mu(A \cap M_\mu)$.

(iii) $c_\mu A = X - i_\mu(X - A)$.

(iv) *If* $U, V \in \mu$ *and* $U \cap V = \emptyset$ *then* $c_\mu U \cap V = \emptyset$ *and* $U \cap c_\mu V = \emptyset$.

(v) $M_\mu - c_\mu A = X - c_\mu A$.

(vi) $i_\mu A = i_\mu(A \cap M_\mu)$.

(vii) $i_\mu(c_\mu A - A) = \emptyset$.

(viii) c_μ *and* i_μ *are monotone:* $A \subseteq B$ *implies* $c_\mu A \subseteq c_\mu B$ *(respectively* $i_\mu A \subseteq i_\mu B$*), idempotent* $c_\mu c_\mu A = c_\mu A$ *(respectively* $i_\mu i_\mu A = i_\mu A$*),* c_μ *is enhancing (*$A \subseteq c_\mu A$*),* i_μ *is contracting (*$i_\mu A \subseteq A$*).*

Proof. (vii). If $x \in i_\mu(c_\mu A - A)$ then there exists a $U \in \mu$ such that $x \in U \subseteq c_\mu A - A$. Then $x \in U \subseteq c_\mu A$ and $U \cap A = \emptyset$. Now $x \in U \subseteq c_\mu A$ implies $U \cap A \neq \emptyset$, a contradiction.

□

Let (X, μ) be a GTS and $Y \subseteq X$. Then the collection $\mu_Y = \{U \cap Y : U \in \mu\}$ is a GT on Y and (Y, μ_Y) is called a generalized subspace of (X, μ). It may be noted that $c_{\mu_Y} A = c_\mu A \cap Y$ for any $A \subseteq Y$. Thus, a set $A \subseteq Y$ is μ_Y-closed if and only if it is the intersection with Y of a μ-closed set.

Definition 2.2. *A subset* A *of a GTS* (X, μ) *is called*

(i) μ*-regular open (or* ro_μ*-open) if* $i_\mu c_\mu A = A$.

(ii) μ*-semi open (or* s_μ*-open) if* $A \subseteq c_\mu i_\mu A \cap M_\mu$.

(iii) μ*-preopen (or* p_μ*-open) if* $A \subseteq i_\mu c_\mu A$.

(iv) μ*-*α*-open (or* α_μ*-open) if* $A \subseteq i_\mu c_\mu i_\mu A$.

(v) μ*-semi preopen (or* sp_μ*-open) if* $A \subseteq c_\mu i_\mu c_\mu A \cap M_\mu$.

(vi) μ*-*θ*-closed (or* θ_μ*-closed) [34] if* $A = \gamma_\theta A$*, where* $\gamma_\theta(A) = \{x \in X : c_\mu G \cap M_\mu \cap A \neq \emptyset$ *for all* $G \in \mu,\ x \in G\}$. *The complement of a* θ_μ*-closed set is called* μ*-*θ*-open (*θ_μ*-open).*

(vii) μ*-*δ*-closed (or* δ_μ*-closed) [12] if* $A = c_\delta A$*, where* $c_\delta A = \{x \in X : i_\mu c_\mu U \cap A \neq \emptyset$ *for* $U \in \mu$ *and* $x \in U\}$. *The complement of a* δ_μ*-closed set is called* μ*-*δ*-open (*δ_μ*-open).*

(viii) μ-π-*open (or* π_μ*-open) if* A *is the union of finitely many* μ*-regular open sets.*

(ix) μ*-regular semi open (or* rs_μ*-open) if there exists a* μ*-regular open set* U *such that* $U \subseteq A \subseteq c_\mu U \cap M_\mu$.

The collections of all μ-( ) sets in (i) to (ix) of the above definitions are denoted by ro_μ, s_μ, p_μ, α_μ, sp_μ, θ_μ, δ_μ, π_μ, rs_μ respectively. The complements of the sets in the above definitions are named similarly by replacing the word "open" by "closed", for example μ-semi closed (or s_μ-closed) for the complement of a s_μ-open set and vice-versa.

It follows using Theorem 2.1, a subset A of GTS (X, μ) is a *regular* μ*-closed* (or ro_μ-closed) if and only if $c_\mu i_\mu A = A$; A set A is μ*-semi open* if and only if $c_\mu A = c_\mu i_\mu A$ and $A \subseteq M_\mu$. A is s_μ*-closed* if and only if $i_\mu c_\mu A \subseteq A$ and $X - M_\mu \subseteq A$; A is p_μ*-closed* if and only if $c_\mu i_\mu A \subseteq A$; A is α_μ*-closed* if and only if $c_\mu i_\mu c_\mu A \subseteq A$; A is sp_μ*-closed* if $i_\mu c_\mu i_\mu A \subseteq A$ and $X - M_\mu \subseteq A$. For any set A, $c_\mu i_\mu c_\mu A$ is α_μ*-closed.* Also if $A \in rs_\mu$ then $A \in s_\mu$ but not conversely.

Theorem 2.3. *[46] For a GTS* (X, μ), θ_μ, α_μ, s_μ, p_μ *and* sp_μ *are GTSs and*

(i) $\theta_\mu \subseteq \mu \subseteq \alpha_\mu \subseteq s_\mu \subseteq sp_\mu$,

(ii) $\alpha_\mu \subseteq p_\mu \subseteq sp_\mu$.

Theorem 2.4. *A is* α_μ*-open if and only if* $A \in s_\mu \cap p_\mu$.

Proof. If $A \subseteq i_\mu c_\mu i_\mu A$ then $A \subseteq c_\mu i_\mu A$, $A \subseteq M_\mu$ and $A \subseteq i_\mu c_\mu A$. So $A \in s_\mu \cap p_\mu$. Conversely, let $A \in s_\mu \cap p_\mu$. Then $A \subseteq c_\mu i_\mu A \cap M_\mu$. Therefore, $c_\mu A \subseteq c_\mu i_\mu A$. Also $A \subseteq i_\mu c_\mu A$. Therefore, $A \subseteq i_\mu c_\mu i_\mu A$ ☐

A subset A of a GTS (X, μ) is μ*-nowhere dense* if $i_\mu c_\mu A = \emptyset$.

Lemma 2.5. *Let* x *be a point in a GTS* (X, μ)*. Then* $\{x\}$ *is* μ*-nowhere dense or* p_μ*-open.*

Proof. Suppose $\{x\}$ is not μ-nowhere dense. Then $i_\mu c_\mu \{x\} \neq \emptyset$. Then $x \in i_\mu c_\mu \{x\}$. So $\{x\} \subseteq i_\mu c_\mu \{x\}$. ☐

Lemma 2.6. *If* $\{x\}$ *is* μ*-nowhere dense in a GTS* (X, μ) *then* $\{x\} \cup (X - M_\mu)$ *is* α_μ*-closed.*

Proof. $c_\mu i_\mu c_\mu(\{x\} \cup (X - M_\mu)) = c_\mu i_\mu c_\mu \{x\} = c_\mu \emptyset = X - M_\mu$. So $c_\mu i_\mu c_\mu(\{x\} \cup (X - M_\mu)) \subseteq \{x\} \cup (X - M_\mu)$. ☐

Lemma 2.7. *For a subset* A *containing* $X - M_\mu$, $c_{s_\mu} A = A \cup i_\mu c_\mu A$.

Proof. Since $c_{s_\mu} A$ is s_μ-closed, $i_\mu c_\mu(c_{s_\mu} A) \subseteq c_{s_\mu} A$. On the other hand $i_\mu c_\mu(A \cup i_\mu c_\mu A) \subseteq i_\mu c_\mu c_\mu A = i_\mu c_\mu A$. Therefore, $i_\mu c_\mu(A \cup i_\mu c_\mu A) \subseteq A \cup i_\mu c_\mu A$. Since $X - M_\mu \subseteq A$, $A \cup i_\mu c_\mu A$ is s_μ- closed. ☐

Lemma 2.8. *For a subset* A, $c_{\alpha_\mu}A = A \cup c_\mu i_\mu c_\mu A$.

Proof. Since $c_{\alpha_\mu}A$ is α_μ-closed, $c_\mu i_\mu c_\mu c_{\alpha_\mu}A \subseteq c_{\alpha_\mu}A$. Therefore, $A \cup c_\mu i_\mu c_\mu A \subseteq c_{\alpha_\mu}A$. On the other hand $c_\mu i_\mu c_\mu(A \cup c_\mu i_\mu c_\mu A) \subseteq c_\mu i_\mu c_\mu c_\mu A = c_\mu i_\mu c_\mu A \subseteq A \cup c_\mu i_\mu c_\mu A$. Thus, $A \cup c_\mu i_\mu c_\mu A$ is α_μ-closed set containing A. ☐

Lemma 2.9. *For a subset* A, $A \cup i_\mu c_\mu i_\mu A \subseteq c_{sp_\mu}A$.

Proof. $i_\mu c_\mu i_\mu A \subseteq i_\mu c_\mu i_\mu (c_{sp_\mu})A \subseteq c_{sp_\mu}A$, since $c_{sp_\mu}A$ is sp_μ-closed.

☐

3 Various type of generalized closed sets

Definition 3.1. *Let* (X, μ) *be a GTS. A subset* A *of* X *containing* $X - M_\mu$ *is called*

(i) *a* μ*-generalized closed (or* g_μ*-closed) set if* $c_\mu A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in \mu$. *The complement of a* g_μ*-closed set is called* μ*-generalized open (or* g_μ*-open). The set of all* g_μ*-open sets is denoted by* g_μ.

(ii) *a* μ*-semi generalized closed (or* sg_μ*-closed) set if* $c_{s_\mu}A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in s_\mu$.

(iii) *a* μ*-generalized semi closed (or* gs_μ*-closed) set if* $c_{s_\mu}A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in \mu$.

(iv) *a* μ*-generalized* α*-closed (or* $g\alpha_\mu$*-closed) set if* $c_{\alpha_\mu}A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in \alpha_\mu$.

(v) *a* $\mu\alpha$*-generalized closed (or* αg_μ*-closed) set if* $c_{\alpha_\mu}A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in \mu$.

(vi) *a* μ*-generalized semi preclosed (or* gsp_μ*-closed) set if* $c_{sp_\mu}A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in \mu$.

(vii) *a* μ*-regular generalized closed (or* rg_μ*-closed) set if* $c_\mu A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in ro_\mu$.

(viii) *a* μ*-generalized preclosed (or* gp_μ*-closed) set if* $c_{p_\mu}A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in \mu$.

(ix) *a* μ*-generalized preregular closed (or* gpr_μ*-closed) set if* $c_{p_\mu}A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in ro_\mu$.

(x) *a* μ*-*θ*-generalized closed (or* θg_μ*-closed) set if* $\gamma_{\theta_\mu}A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in \mu$.

(xi) *a* μ*-*δ*-generalized closed (or* δg_μ*-closed) set if* $c_{\delta_\mu}A \cap M_\mu \subseteq U$ *whenever* $A \subseteq U \in \mu$.

(xii) *a* μ*-weakly generalized closed (or* wg_μ*-closed) set if* $c_\mu i_\mu A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in \mu$.

(xiii) *a* μ*-strongly generalized closed (or* ${g_\mu}^*$*-closed) set if* $c_\mu A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in g_\mu$.

(xiv) *a* μ*-*π*-generalized closed (or* πg_μ*-closed) set if* $c_\mu A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in \pi_\mu$.

(xv) *a* μ*-weakly closed (or* w_μ*-closed) set if* $c_\mu A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in s_\mu$.

(xvi) *a* μ*-mildly generalized closed (or* mg_μ*-closed) set if* $c_\mu i_\mu A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in g_\mu$.

(xvii) *a* μ*-semi-weakly generalized closed (or* swg_μ*-closed) set if* $c_\mu i_\mu A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in s_\mu$.

(xviii) *a* μ*-regular weakly generalized closed (or* rwg_μ*-closed) set if* $c_\mu i_\mu A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in ro_\mu$.

(xix) *a* μ*-regular generalized* w*-closed (or* rw_μ*-closed) set if* $c_\mu A \cap M_\mu \subseteq U$ *whenever* $A \cap M_\mu \subseteq U \in rs_\mu$.

Lemma 3.2. (i) $A \in g_\mu$ *then* $A \subseteq M_\mu$.

(ii) $\mu \subseteq g_\mu$.

Proof. (i) Since $X - M_\mu$ is contained in a generalized closed set A, the complement of A is contained in M_μ.

(ii) Let $A \in \mu$ and $(X - A) \cap M_\mu \subseteq U \in \mu$. Then $c_\mu(X - A) \cap M_\mu = (X - A) \cap M_\mu \subseteq U$.

□

Theorem 3.3. *A subset* A *of GTS* (X, μ) *is* g_μ*-closed if and only if for any* μ*-closed set* F *such that* $F \cap M_\mu \subseteq c_\mu A - A$ *implies* $F \cap M_\mu = \emptyset$.

Proof. Let F be a μ-closed set such that $F \cap M_\mu \subseteq c_\mu A - A$. Then $A \cap M_\mu \subseteq F^c \in \mu$. Since A is g_μ-closed, $c_\mu A \cap M_\mu \subseteq F^c$. That is, $F \cap M_\mu \subseteq (c_\mu A)^c$. Therefore, $F \cap M_\mu \subseteq (c_\mu A - A) \cap (c_\mu A)^c = \emptyset$. Conversely, let $A \cap M_\mu \subseteq U \in \mu$ and if $c_\mu A \cap M_\mu$ is not contained in U then $c_\mu A \cap M_\mu \cap U^c \neq \emptyset$. Since $c_\mu A \cap U^c$ is μ-closed and $c_\mu A \cap U^c \cap M_\mu \subseteq c_\mu A - A$, a contradiction. □

Theorem 3.4. *If a* g_μ*-closed subset* A *of a GTS* (X, μ) *be such that* $c_\mu A - (A \cap M_\mu)$ *is* μ*-closed then* A *is* μ*-closed.*

Proof. Let A be a g_μ-closed set such that $c_\mu A - (A \cap M_\mu)$ is μ-closed. Then $c_\mu A - (A \cap M_\mu)$ is μ-closed subset of itself. Since $c_\mu A - (A \cap M_\mu)$ is g_μ-closed subset of itself, by Theorem 3.3 $(c_\mu A - (A \cap M_\mu)) \cap M_{\mu_Y}$, where $Y = c_\mu A - (A \cap M_\mu)$, is empty. Since $M_{\mu_Y} = (c_\mu A - (A \cap M_\mu)) \cap M_\mu$, A is μ-closed. □

Theorem 3.5. *If* A *is a* g_μ*-closed set and* $A \subseteq B \subseteq c_\mu A$ *then* B *is* g_μ*-closed.*

Proof. Let $B \cap M_\mu \subseteq U \in \mu$. Since A is g_μ-closed and $A \cap M_\mu \subseteq U$, $c_\mu A \cap M_\mu \subseteq U$. Then $c_\mu B \cap M_\mu \subseteq c_\mu A \cap M_\mu \subseteq U$. □

Theorem 3.6. *In a GTS* (X, μ), $\mu = \Omega$ *if and only if* (X, μ) *is strong and every subset of* X *is* g_μ*-closed.*

Proof. If $\mu = \Omega$ then obviously (X, μ) is strong. Now if $A \subseteq U \in \mu$ then $c_\mu A \subseteq c_\mu U = U$ since $U \in \mu$. Conversely, let $U \in \mu$. Since U is g_μ-closed, $c_\mu U \subseteq U$. Thus, U is μ-closed. On the other hand if $F \in \Omega$ then $F^c \in \mu$. Since $\mu \subseteq \Omega$, $F \in \mu$. □

Theorem 3.7. *A subset* A *of* M_μ *of a GTS* (X, μ) *is* g_μ*-open if and only if* $F \cap M_\mu \subseteq i_\mu A$ *whenever* F *is* μ*-closed and* $F \cap M_\mu \subseteq A$.

Proof. Let A be a g_μ-open set and F be a μ-closed set such that $F \cap M_\mu \subseteq A$. Then $X - A \subseteq X - (F \cap M_\mu)$. Since $(X - A) \cap M_\mu \subseteq (X - (F \cap M_\mu)) \cap M_\mu = X - F$ and $X - A$ is g_μ-closed, $c_\mu(X - A) \cap M_\mu \subseteq X - F$. Then $(X - i_\mu A) \cap M_\mu \subseteq X - F$. That is, $F \cap M_\mu \subseteq (X - (X - i_\mu A) \cap M_\mu) \cap M_\mu = i_\mu A$. Conversely, let $A \subseteq M_\mu$ and $(X - A) \cap M_\mu \subseteq U \in \mu$. Then $X - U \subseteq X - ((X - A) \cap M_\mu)$. So $(X - U) \cap M_\mu \subseteq A$. Then $(X - U) \cap M_\mu \subseteq i_\mu A$. So $X - i_\mu A \subseteq X - ((X - U) \cap M_\mu)$. Therefore, $c_\mu(X - A) \cap M_\mu \subseteq U$. Thus, A is g_μ-open. □

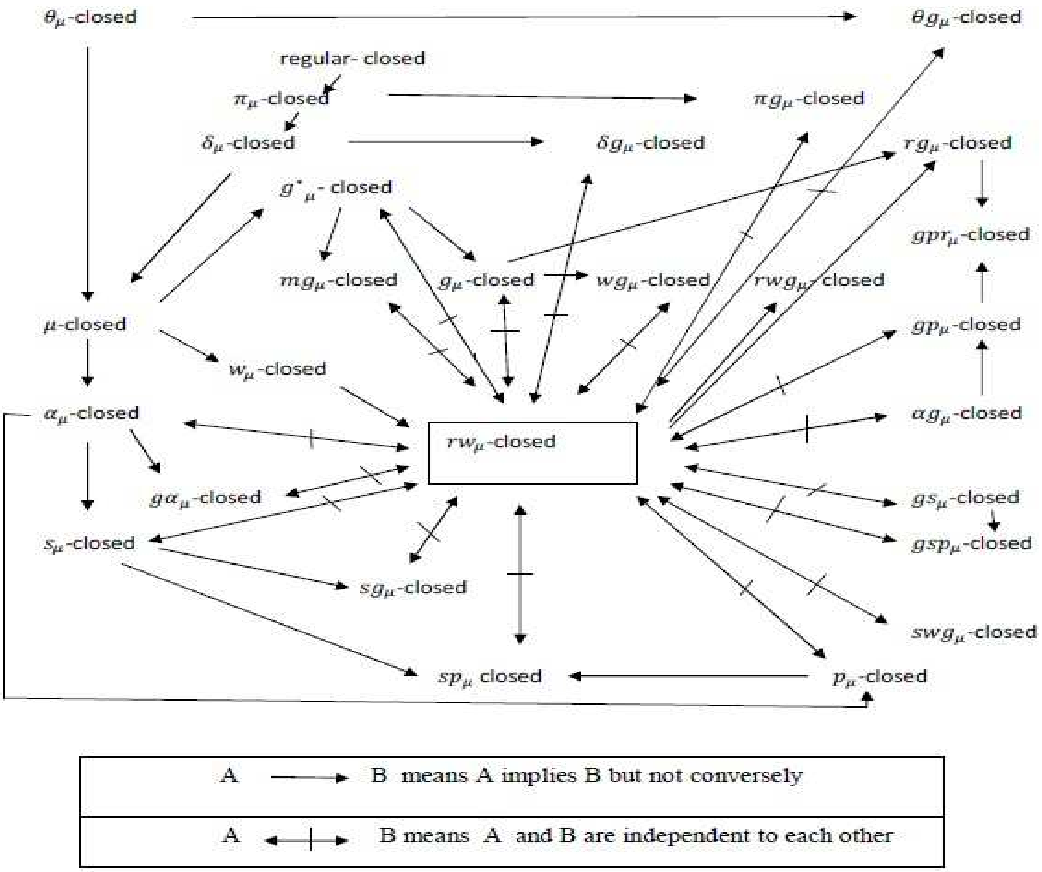
Theorem 3.8. *A set* A *in GTS* (X, μ) *is* g_μ*-open if and only if* $i_\mu A \cup (A^c \cap M_\mu) \subseteq U \in \mu$ *implies* $U = M_\mu$.

Proof. Let A be a g_μ-open set and $i_\mu A \cup (A^c \cap M_\mu) \subseteq U \in \mu$. Then $U^c \subseteq (i_\mu A)^c \cap (A \cup M_\mu^c) = c_\mu(X - A) \cap (A \cup M_\mu^c)$. Therefore, $U^c \cap M_\mu \subseteq (c_\mu(X - A) \cap M_\mu) \cap A = c_\mu(X - A) - (X - A)$. Then by Theorem 3.3 $U^c \cap M_\mu = \emptyset$, That is, $U = M_\mu$. Conversely, let F be a μ-closed set such that $F \cap M_\mu \subseteq A$. Then $i_\mu A \cup (A^c \cap M_\mu) \subseteq i_\mu A \cup F^c \in \mu$. By the assumption, $i_\mu A \cup F^c = M_\mu$, that is, $F \cap M_\mu \subseteq i_\mu A$. Now apply Theorem 3.7. □

Theorem 3.9. *A subset* A *of a GTS* (X, μ) *is* g_μ*-closed if and only if* $c_\mu A - A$ *is* g_μ*-open.*

Proof. Suppose that A is g_μ-closed and $F \cap M_\mu \subseteq c_\mu A - A$, where F is a μ-closed set. By Theorem 3.3 $F \cap M_\mu = \emptyset$. So $F \cap M_\mu \subseteq i_\mu(c_\mu A - A)$. Therefore, $c_\mu A - A$ is g_μ- open by Theorem 3.7. Conversely, assume that $X - M_\mu \subseteq A$ and $A \cap M_\mu \subseteq U \in \mu$. Now $c_\mu A \cap U^c \cap M_\mu \subseteq c_\mu A \cap (M_\mu - A) = c_\mu A - A$. By Theorem 3.7 $c_\mu A \cap U^c \cap M_\mu \subseteq i_\mu(c_\mu A - A) = \emptyset$. Thus, $c_\mu A \cap M_\mu \subseteq U$ and A is g_μ-closed. □

The following diagram extends to the setting of GTSs the corresponding diagram of Benchalli and Wali [4] and Dontchev [17].



For examples showing independence $A \nleftrightarrow B$ in the above diagram see [4].

Theorem 3.10. *Let* (X, μ) *be a GTS and* $A \subseteq X$. *Then the following statements hold.*

(i) μ-*closed* $\Rightarrow$ α_μ-*closed* $\Rightarrow$ s_μ-*closed* $\Rightarrow$ sp_μ-*closed.*

(ii) α_μ-*closed* $\Rightarrow$ p_μ-*closed* $\Rightarrow$ sp_μ-*closed.*

(iii) μ-*closed* $\Rightarrow$ g_μ-*closed* $\Rightarrow$ rg_μ- *closed.*

(iv) g_μ-*closed* $\Rightarrow$ αg_μ-*closed* $\Rightarrow$ gs_μ-*closed* $\Rightarrow$ gsp_μ-*closed.*

(v) α_μ-*closed* $\Rightarrow$ $g\alpha_\mu$-*closed* $\Rightarrow$ αg_μ-*closed.*

(vi) s_μ-*closed* $\Rightarrow$ sg_μ-*closed* $\Rightarrow$ gsp_μ-*closed.*

(vii) sg_μ-*closed* $\Rightarrow$ gs_μ-*closed.*

(viii) sp_μ-*closed* $\Rightarrow$ gsp_μ-*closed.*

(ix) p_μ-*closed* $\Rightarrow$ gsp_μ-*closed.*

(x) αg_μ-*closed* $\Rightarrow$ gsp_μ-*closed.*

(xi) $g\alpha_\mu$-*closed* $\Rightarrow$ gs_μ-*closed.*

Proof. (i) Let A be μ-closed set. Then $c_\mu A = A$. Therefore, $i_\mu c_\mu A = i_\mu A \subseteq A$. Thus, $c_\mu i_\mu c_\mu A \subseteq c_\mu A = A$.
Now let A be a α_μ-closed set. Then $i_\mu c_\mu A \subseteq c_\mu i_\mu c_\mu A \subseteq A$.
Now let A be a s_μ-closed set. Then $i_\mu c_\mu A \subseteq A$ and $X - M_\mu \subseteq A$. Therefore, $i_\mu c_\mu i_\mu A \subseteq i_\mu c_\mu A \subset A$. This proves (i).

The proofs of other parts also follow easily. □

Theorem 3.11. *(i) Every* sg_μ-*closed sets is* sp_μ-*closed.*

(ii) Every $g\alpha_\mu$-*closed set is* p_μ-*closed.*

Proof. (i) Let A be a sg_μ-closed set and $x \in c_{sp_\mu} A \cap M_\mu$. Then $\{x\}$ is p_μ-open or μ-nowhere dense. If $\{x\}$ is p_μ-open then by Theorem 2.3, $\{x\}$ is sp_μ-open. Since $x \in sp_\mu A \cap M_\mu$, $\{x\} \cap A \neq \emptyset$. Therefore, $x \in A$. If $\{x\}$ is μ-nowhere dense then $\{x\} \cup (X - M_\mu)$ is α_μ-closed and hence s_μ-closed. Therefore, the complement $B = M_\mu - \{x\}$ is s_μ-open. Assume that $x \notin A$, then $A \cap M_\mu \subseteq B$. Since A is sg_μ-closed, and $c_{sp_\mu} A \subseteq c_{s_\mu} A$. $c_{sp_\mu} A \cap M_\mu \subseteq B$. Hence $x \notin c_{sp_\mu} A \cap M_\mu$. By contradiction $x \in A$. Thus, A is sp_μ-closed.

(ii) Let A be a $g\alpha_\mu$-closed set. Let $x \in c_{p_\mu} A \cap M_\mu$. If $\{x\}$ is p_μ-open, then $\{x\} \cap A \neq \emptyset$. So that $x \in A$. If $\{x\}$ is μ-nowhere dense and does not meet A then $\{x\} \cup (X - M_\mu)$ is α_μ-closed. Then $B = M_\mu - \{x\}$ is α_μ-open and $A \cap M_\mu \subseteq B$. Since A is $g\alpha_\mu$-closed, $c_{\alpha\mu} A \cap M_\mu \subseteq B$. Therefore, $x \notin c_{\alpha\mu} A \cap M_\mu$, a contradiction. Thus, $x \in A$ and A is p_μ-closed.

□

The following theorem also covers some immediate implications.

Theorem 3.12. *For a set in a GTS* (X, μ)*, the following statements hold.*

(i) π_μ-*closed* $\Rightarrow$ δ_μ-*closed.*

(ii) θ_μ-*closed* $\Rightarrow$ θg_μ-*closed.*

(iii) π_μ-*closed* $\Rightarrow$ πg_μ-*closed.*

(iv) δ_μ-*closed* $\Rightarrow$ δg_μ-*closed.*

(v) μ-*closed* $\Rightarrow$ ${g_\mu}^*$-*closed.*

(vi) μ-*closed* $\Rightarrow$ w_μ-*closed.*

(vii) $g_\mu{}^*$-*closed* $\Rightarrow$ mg_μ-*closed.*

(viii) $g_\mu{}^*$-*closed* $\Rightarrow$ g_μ-*closed.*

(ix) g_μ-*closed* $\Rightarrow$ wg_μ-*closed.*

(x) rg_μ-*closed* $\Rightarrow$ gpr_μ-*closed.*

(xi) gp_μ-*closed* $\Rightarrow$ gpr_μ-*closed.*

(xii) αg_μ-*closed* $\Rightarrow$ gp_μ-*closed.*

(xiii) w_μ-*closed* $\Rightarrow$ rw_μ-*closed.*

(xiv) rw_μ-*closed* $\Rightarrow$ rg_μ-*closed.*

(xv) rw_μ-*closed* $\Rightarrow$ rwg_μ-*closed.*

Proof. (i) Let A be a π_μ-closed set. Then there are μ-regular closed sets $R_i, R_2, ...R_n$ such that $A = \bigcap_{i=1}^{n} R_i$. Let $x \in X - A = \cup_{i=1}^{n} {R_i}^c$. Then $x \in {R_i}^c$ for some i and $i_\mu c_\mu {R_i}^c \cap A = {R_i}^c \cap A = \emptyset$. So $x \notin c_{\delta\mu} A$.

The proofs of other parts are also easy and left to the reader.

□

4 μ-T_0, μ-T_1 and μ-$T_{1/2}$ generalized topological spaces

Definition 4.1. *A GTS* (X, μ) *is said to be*

(i) μ-T_0 *if* $x, y \in M_\mu, x \neq y$ *implies the existence of a* μ*-open set containing precisely one of* x *and* y.

(ii) [32] μ-T_1 *if* $x, y \in M_\mu, x \neq y$ *implies the existence of* μ*-open sets* U_1 *and* U_2 *such that* $x \in U_1$ *and* $y \notin U_1$ *and* $y \in U_2$ *and* $x \notin U_2$.

(iii) μ-$T_{1/2}$ *if every* g_μ*-closed set is* μ*-closed.*

Easy examples of GT-spaces which are not strong and having the properties of the above separation axioms may be provided. For example, let R be the set of real numbers and $x, y, x \neq y$ be any two real numbers. Then $\mu = \{\emptyset, \{x\}, \{x, y\}\}$ is a GT which is not strong and has the property of μ-T_0 but not μ-T_1.

It is obvious that μ-T_1 implies μ-T_0. Also (X, μ) is μ-T_0 if and only if for each $x, y \in M_\mu$, $c_\mu(\{x\}) = c_\mu(\{y\})$ implies $x = y$.

Theorem 4.2. *If a GTS* (X, μ) *is* μ-$T_{1/2}$ *then it is* μ-T_0.

Proof. Suppose that (X, μ) is not a μ-T_0 space. Then there exist distinct points x and y in M_μ such that $c_\mu(\{x\}) = c_\mu(\{y\})$. Let $A = c_\mu(\{x\}) \cap \{x\}^c$. We show that A is g_μ-closed but not μ-closed. $X - M_\mu \subseteq A$. Let $A \cap M_\mu \subseteq V \in \mu$. Since $A \subseteq c_\mu(\{x\})$, $c_\mu A \cap M_\mu \subseteq c_\mu(\{x\}) \cap M_\mu$. Thus, we show that $c_\mu(\{x\}) \cap M_\mu \subseteq V$. Since $c_\mu(\{x\}) \cap \{x\}^c \cap M_\mu \subseteq V$, it is enough to show that $x \in V$. If x is not in V then $y \in V$ and $y \in c_\mu(\{y\}) = c_\mu(\{x\}) \subseteq V^c$ as V^c is a μ-closed set containing the set $\{x\}$. Thus, $y \in V \cap V^c$, a contradiction. Now if $x \in U \in \mu$ then $U \cap A \supseteq \{y\} \neq \emptyset$, and hence $x \in c_\mu A$. But x is not in A and thus, A is not a μ-closed set. ☐

Theorem 4.3. *If a GTS* (X, μ) *is* μ-T_1 *then for each* $x \in X$, $A = \{x\} \cup (X - M_\mu)$ *is* μ*-closed.*

Proof. Let $y \in c_\mu A \cap M_\mu$ and $y \neq x$. Then $y \in c_\mu(A \cap M_\mu) \cap M_\mu = c_\mu(\{x\}) \cap M_\mu$. Then $y \in c_\mu(\{x\})$. So $y \in U \in \mu$ implies $x \in U$ which is against our hypothesis. So $c_\mu A \cap M_\mu = \{x\}$, that is, $c_\mu A = A$. ☐

Theorem 4.4. *If a GTS* (X, μ) *is* μ-T_1 *then it is* μ-$T_{1/2}$.

Proof. Let A be a subset of X which is not μ-closed. If $X - M_\mu$ is not contained in A, then A is not g_μ-closed. So let $X - M_\mu \subseteq A$. Since A is not μ-closed, $c_\mu A - A$ is non empty. Let $x \in c_\mu A - A$. By Theorem 4.3 $\{x\} \cup (X - M_\mu)$ is μ-closed. As $(\{x\} \cup (X - M_\mu)) \cap M_\mu = \{x\} \subseteq c_\mu A - A$, by Theorem 3.3 A is not g_μ-closed. ☐

Definition 4.5. *A GTS* (X, μ) *is said to be* μ*-symmetric if for each* $x, y \in M_\mu, x \in c_\mu(\{y\})$ *implies* $y \in c_\mu(\{x\})$.

Theorem 4.6. *A GTS* (X, μ) *is* μ*-symmetric if and only if* $\{x\} \cup (X - M_\mu)$ *is* g_μ*-closed for each* $x \in X$.

Proof. Let $A = \{x\} \cup (X - M_\mu)$ and $A \cap M_\mu \subseteq U \in \mu$. If $A \cap M_\mu = \emptyset$ then $c_\mu A = c_\mu(A \cap M_\mu) = c_\mu \emptyset = X - M_\mu$. So $c_\mu A \cap M_\mu \subseteq U$. Otherwise $c_\mu A \cap M_\mu = c_\mu(A \cap M_\mu) \cap M_\mu = c_\mu(\{x\}) \cap M_\mu$. If $c_\mu(\{x\}) \cap M_\mu \nsubseteq U$ then assume that $y \in c_\mu(\{x\}) \cap U^c \cap M_\mu$. Since (X, μ) is μ-symmetric, $x \in c_\mu(\{y\})$. Since $x \in U, y \in U$, then $y \in U \cap U^c$, a contradiction. Conversely, let for each $x \in X, \{x\} \cup (X - M_\mu)$ is g_μ-closed. Let $x, y \in M_\mu, x \in c_\mu(\{y\})$ and $y \notin c_\mu(\{x\})$. Then $y \in (c_\mu(\{x\}))^c$. Let $A = \{y\} \cup (X - M_\mu)$. Then A is g_μ-closed and $A \cap M_\mu = \{y\} \subseteq (c_\mu(\{x\}))^c$. So $c_\mu A \cap M_\mu = (c_\mu(\{y\})) \cap M_\mu \subseteq (c_\mu(\{x\}))^c$. Then $x \in (c_\mu(\{y\})) \cap M_\mu \subseteq (c_\mu(\{x\}))^c$, a contradiction. ☐

Corollary 4.7. *If A GTS* (X, μ) *is* μ-T_1 *then it is* μ*-symmetric.*

Proof. The proof follows from Theorem 4.3, Theorem 4.6 and Lemma 3.2. ☐

Theorem 4.8. *A GTS* (X, μ) *is* μ*-symmetric and* μ-T_0 *if and if only* (X, μ) *is* μ-T_1.

Proof. If (X, μ) is μ-T_1 then by Corollary 4.7 (X, μ) is μ-symmetric and obviously μ-T_0. Conversely, let (X, μ) be μ-symmetric and μ-T_0. Let $x, y \in M_\mu$ and $x \neq y$. Then by μ-T_0 property there exists a $U \in \mu$ such that $x \in U \subseteq (\{y\})^c$. Then x is not in $c_\mu(\{y\})$. Since (X, μ) is μ-symmetric, y is not in $c_\mu(\{x\})$. Then there exists $V = (c_\mu(\{x\}))^c$ such that $y \in V$ and $x \notin V$. □

Theorem 4.9. *If* (X, μ) *is* μ*-symmetric then* (X, μ) *is* μ-T_0 *if and only if* (X, μ) *is* μ-$T_{1/2}$ *if and only if* (X, μ) *is* μ-T_1.

Proof. The proof follows from Theorems 4.8, 4.4 and 4.2. □

Theorem 4.10. *A GTS* (X, μ) *is* μ-$T_{1/2}$ *if and only if for each* $x \in X$, *either* $\{x\}$ *is* μ*-open or* $\{x\} \cup (X - M_\mu)$ *is* μ*-closed.*

Proof. Suppose X is μ-$T_{1/2}$ and for some $x \in X$, $\{x\} \cup (X - M_\mu)$ is not μ-closed. Then M_μ is the only μ-open set containing $M_\mu - \{x\}$. Therefore, $(M_\mu - \{x\}) \cup (X - M_\mu)$ is g_μ-closed. So it is μ-closed. Thus, $\{x\}$ is μ-open.

Conversely, let A be a g_μ-closed set with $x \in c_\mu A \cap M_\mu$ and $x \notin A$. If $\{x\}$ is μ-open then $\emptyset \neq \{x\} \cap A$. Thus, $x \in A$. Otherwise $\{x\} \cup (X - M_\mu)$ is μ-closed. Then$(\{x\} \cup (X - M_\mu)) \cap M_\mu = \{x\} \subseteq c_\mu A - A$. Then by Theorem 3.3 $\{x\} = \emptyset$, a contradiction. Thus, $x \in A$ and so A is μ-closed. □

Theorem 4.11. *For a GTS* (X, μ), *the following statements are equivalent.*

(i) X *is* μ-$T_{1/2}$.

(ii) *Every* αg_μ*-closed set is* α_μ*-closed.*

Proof. (i) $\Rightarrow$ (ii). Let A be a αg_μ-closed set and $x \in c_{\alpha_\mu} A \cap M_\mu$. If $\{x\}$ is μ-open then $\{x\} \in \alpha_\mu$ so that $\{x\} \cap A \neq \emptyset$. Thus, $x \in A$. Otherwise $\{x\} \cup (X - M_\mu)$ is μ-closed. Let $x \notin A$. Then $M_\mu - \{x\}$ is μ-open and $A \cap M_\mu \subseteq M_\mu - \{x\}$. Since A is αg_μ-closed, $c_{\alpha_\mu} A \cap M_\mu \subseteq M_\mu - \{x\}$. Therefore, $x \notin c_{\alpha_\mu} A \cap M_\mu$, a contradiction. Thus, $x \in A$ and A is α_μ-closed.

(ii) $\Rightarrow$ (i). If some set $\{x\} \cup (X - M_\mu)$ is not μ-closed then $x \in M_\mu$ and $M_\mu - \{x\}$ is not μ- open. Then $(M_\mu - \{x\}) \cup (X - M_\mu)$ is trivially αg_μ-closed. By (ii), $(M_\mu - \{x\}) \cup (X - M_\mu)$ is α_μ-closed. So $\{x\}$ is α_μ-open. Since a non-empty α_μ-open set contains a non-empty μ-open set, $\{x\}$ is μ-open. This shows that (X, μ) is μ-$T_{1/2}$. □

5 μ-regular and μ-normal generalized topological spaces

Definition 5.1. *[33] A GTS* (x, μ) *is said to be* μ*-regular if for each* μ*-closed set* F *of* X *not containing* $x \in X$ *there exist disjoint* μ*-open subsets* U *and* V *of* X *such that* $x \in U$ *and* $F \cap M_\mu \subseteq V$.

Theorems 5.2, 5.4, and 5.5 generalize the corresponding results in Roy [40].

Theorem 5.2. *For a GTS* (X, μ)*, the following statements are equivalent.*

(i) X *is* μ*-regular.*

(ii) $x \in U \in \mu$ *implies that there exists* $V \in \mu$ *such that* $x \in V \subseteq c_\mu V \cap M_\mu \subseteq U$.

(iii) *For each* μ*-closed set* F, $F = \cap\{c_\mu V : F \cap M_\mu \subseteq V \in \mu\}$.

(iv) *For each subset* A *of* X *and each* $U \in \mu$ *with* $A \cap U \neq \emptyset$ *there exists a* $V \in \mu$ *such that* $A \cap V \neq \emptyset$ *and* $c_\mu V \cap M_\mu \subseteq U$.

(v) *For each non-empty set* $A \subseteq X$ *and each* μ*-closed set* F *with* $A \cap F = \emptyset$ *there exist* $U, V \in \mu$ *such that* $A \cap V \neq \emptyset$, $F \cap M_\mu \subseteq U$ *and* $U \cap V = \emptyset$.

(vi) *For each* μ*-closed set* F *and* $x \notin F$ *there exist* $U \in \mu$ *and a* g_μ*-open set* V *such that* $x \in U$, $F \cap M_\mu \subseteq V$ *and* $U \cap V = \emptyset$.

(vii) *For each non-empty* $A \subseteq X$ *and each* μ*-closed set* F *with* $A \cap F = \emptyset$ *there exist a* $U \in \mu$ *and a* g_μ*-open set* V *such that* $A \cap U \neq \emptyset$, $F \cap M_\mu \subseteq V$ *and* $U \cap V = \emptyset$.

(viii) *For each* μ*-closed set* F *of* X, $F = \cap\{c_\mu V : F \cap M_\mu \subseteq V$ *and* V *is* g_μ*-open*$\}$.

Proof. (i)⇔ (ii) [32].
(ii)⇒ (iii). Suppose $x \notin F$. Then by (ii) there exists a $V \in \mu$ such that $x \in V \subseteq c_\mu V \cap M_\mu \subseteq X - F$. Then $F \cap M_\mu \subseteq (X - (c_\mu V \cap M_\mu)) \cap M_\mu = X - c_\mu V = W \in \mu$. Since $c_\mu W \cap V = \emptyset$, (iii) follows.
(iii) ⇒ (iv). $x \in A \cap U$ implies that $x \notin X - U$. By (iii) there exists a $W \in \mu$ such that $(X - U) \cap M_\mu \subseteq W$ and $x \notin c_\mu W$. Let $V = X - c_\mu W$ then $x \in V \cap A$ and $V \subseteq X - W$. Thus, $c_\mu V \subseteq X - W$. Therefore, $c_\mu V \cap M_\mu \subseteq (X - W) \cap M_\mu \subseteq (X - ((X - U) \cap M_\mu)) \cap M_\mu = U$.
(iv) ⇒ (v). $A \cap (X - F) \neq \emptyset$. By (iv) there exists a μ-open set V such that $A \cap V \neq \emptyset$ and $c_\mu V \cap M_\mu \subseteq X - F$. Let $W = X - c_\mu V$. Then $F \cap M_\mu \subseteq \subseteq (X - (c_\mu V \cap M_\mu)) \cap M_\mu = X - c_\mu V = W$ and $W \cap V = \emptyset$.
(v) ⇒ (i). Let F be a μ-closed set not containing x. By (v) there exist disjoint μ-open sets U and V such that $x \in U$ and $F \cap M_\mu \subseteq V$.
(i) ⇒ (vi). Follows from Lemma 3.2.
(vi) ⇒ (vii). Note that $A \subseteq M_\mu$. Since A is non-empty and $A \cap F = \emptyset$ there exists a point $x \in A$ such that $x \notin F$. By (vi) there exist a $U \in \mu$ and a g_μ-open set V such that $x \in U$, $F \cap M_\mu \subseteq V$ and $U \cap V = \emptyset$. Then $U \cap A \neq \emptyset$.
(vii) ⇒ (i). Let $x \notin F$, where F is a μ-closed set. Then $\{x\} \cap F = \emptyset$. By (vii) there exist a $U \in \mu$ and a g_μ-open set V such that $x \in U$, $F \cap M_\mu \subseteq V$ and $U \cap V = \emptyset$. Now $F \cap M_\mu \subseteq i_\mu V$ by Theorem 3.7.
(iii) ⇒ (viii). We have $F \subseteq \cap\{c_\mu V : F \cap M_\mu \subseteq V$ and V is g_μ-open$\} \subseteq \cap\{c_\mu V : F \cap M_\mu \subseteq V \in \mu\} = F$.
(viii) ⇒ (i). Let F be a μ-closed set such that $x \notin F$. Then by (viii) there exists g_μ-open set W such that $F \cap M_\mu \subseteq W$ and $x \notin c_\mu W$. Since F is μ-closed, W is g_μ-open and $F \cap M_\mu \subseteq W$, by Theorem 3.7, $F \cap M_\mu \subseteq i_\mu W$. □

Definition 5.3. *[32] A GTS* (X, μ) *is* μ*-normal if for any pair of* μ*-closed sets* A *and* B *such that* $A \cap B \cap M_\mu = \emptyset$ *there exist disjoint* μ*-open sets* U *and* V *such that* $A \cap M_\mu \subseteq U$ *and* $B \cap M_\mu \subseteq V$*.*

Theorem 5.4. *For a GTS* (X, μ)*, the following statements are equivalent.*

(i) X *is* μ*-normal.*

(ii) *For any* μ*-closed set* A *and* μ*-open set* U *such that* $A \cap M_\mu \subseteq U$ *there is a* μ*-open set* V *such that* $A \cap M_\mu \subseteq V \subseteq c_\mu V \cap M_\mu \subseteq U$*.*

Proof. Let A be a μ-closed set such that $A \cap M_\mu \subseteq U \in \mu$. Then $B = X - U$ is μ-closed and $A \cap B \cap M_\mu$ is empty. Then by (i) there exist disjoint μ-open sets V and W such that $A \cap M_\mu \subseteq V$ and $B \cap M_\mu \subseteq W$. Then $A \cap M_\mu \subseteq V \subseteq c_\mu V \cap M_\mu \subseteq (X - W) \cap M_\mu \subseteq (X - (B \cap M_\mu)) \cap M_\mu = U$. Conversely, assume that A and B be μ-closed sets such that $A \cap B \cap M_\mu = \emptyset$. Then $U = X - B$ is μ-open and $A \cap M_\mu \subseteq U$. By (ii) there exists a μ-open set V such that $A \cap M_\mu \subseteq V \subseteq C_\mu V \cap M_\mu \subseteq U$. Let $W = X - c_\mu V$. Since $c_\mu V \cap B \cap M_\mu = \emptyset$, $B \cap M_\mu \subseteq X - c_\mu V = W$. ☐

Theorem 5.5. *In a GTS* (X, μ)*, the following statements are equivalent.*

(i) X *is* μ*-normal.*

(ii) *For any pair of* μ*-closed sets* A *and* B *such that* $A \cap B \cap M_\mu = \emptyset$ *then there exist disjoint* g_μ*-open sets* U *and* V *such that* $A \cap M_\mu \subseteq U$ *and* $B \cap M_\mu \subseteq V$*.*

(iii) *For every* μ*-closed set* A *and* μ*-open set* U *such that* $A \cap M_\mu \subseteq U$ *there exists a* g_μ*-open set* V *such that* $A \cap M_\mu \subseteq V \subseteq c_\mu V \cap M_\mu \subseteq U$*.*

(iv) *For every* μ*-closed set* A *and every* g_μ*-open set* U *containing* $A \cap M_\mu$ *there exists a* μ*-open set* V *such that* $A \cap M_\mu \subseteq V \subseteq c_\mu V \cap M_\mu \subseteq U$*.*

(v) *For every* g_μ*-closed set* A *and every* μ*-open set* U *containing* $A \cap M_\mu$ *there exists a* μ*-open set* V *such that* $c_\mu A \cap M_\mu \subseteq V \subseteq c_\mu V \cap M_\mu \subseteq U$*.*

Proof. (i) ⇒ (ii). Follows from Lemma 3.2.
(ii) ⇒ (iii). Assume that $B = X - U$. By (ii) there exist disjoint g_μ-open sets V and W such that $A \cap M_\mu \subseteq V$ and $B \cap M_\mu \subseteq W$. Since $B \cap M_\mu \subseteq W$, $(X - U) \cap M_\mu \subseteq W$. Therefore, $(X - W) \cap M_\mu \subseteq (X - (X - U) \cap M_\mu) \cap M_\mu = U$. Since $X - W$ is g_μ-closed, $c_\mu(X - W) \cap M_\mu \subseteq U$. Since $c_\mu V \cap M_\mu \subseteq c_\mu(X - W) \cap M_\mu$, the implication is established.
(iii) ⇒ (iv). By Theorem 3.7 $A \cap M_\mu \subseteq i_\mu U$. Then by (iii) there exists a g_μ-open set V such that $A \cap M_\mu \subseteq V \subseteq c_\mu V \cap M_\mu \subseteq i_\mu U$. By Theorem 3.7 $A \cap M_\mu \subseteq i_\mu V \subseteq c_\mu(i_\mu V) \cap M_\mu \subseteq c_\mu V \cap M_\mu \subseteq U$.
(iv) ⇒ (v). Let A be a g_μ-closed set and $A \cap M_\mu \subseteq U \in \mu$. Then $c_\mu A \cap M_\mu \subseteq U$. By (iv) there exists a μ-open set V such that $c_\mu A \cap M_\mu \subseteq V \subseteq c_\mu V \cap M_\mu \subseteq U$.
(v) ⇒ (i). Let A and B be μ-closed sets such that $A \cap B \cap M_\mu = \emptyset$. Then $A \cap M_\mu \subseteq X - B \in \mu$. By

(v) there exists a μ-open set V such that $c_\mu A \cap M_\mu \subseteq V \subseteq c_\mu V \cap M_\mu \subseteq X - B$. Thus, $A \cap M_\mu \subseteq V$ and $B \subseteq X - (c_\mu V \cap M_\mu)$. Therefore, $B \cap M_\mu \subseteq (X - (c_\mu V \cap M_\mu)) \cap M_\mu = X - c_\mu V = W \in \mu$. ☐

6 Extremally μ-disconnectedness

Theorem 6.1. *For a GTS* (X, μ)*, the following statements are equivalent.*

- (i) (X, μ) *is extremally* μ*-disconnected.*
- (ii) *Every* sp_μ*-closed set is* p_μ*-closed.*
- (iii) *Every* sg_μ*-closed set is* p_μ*-closed.*
- (iv) *Every* s_μ*-closed set is* p_μ*-closed.*
- (v) *Every* s_μ*-closed set is* α_μ*-closed.*
- (vi) *Every* s_μ*-closed set is* $g\alpha_\mu$*-closed.*

Proof. $(i) \Rightarrow (ii)$. Let A be a sp_μ-closed set. Then by Lemma 2.9 $i_\mu c_\mu i_\mu A \subseteq A$. Since X is extremally μ-disconnected, $c_\mu i_\mu A \cap M_\mu = i_\mu(c_\mu i_\mu A \cap M_\mu) \subseteq i_\mu c_\mu i_\mu A$. Therefore, $c_\mu i_\mu A \cap M_\mu \subseteq A$. Since $X - M_\mu \subseteq A$, $c_\mu i_\mu A \subseteq A$.
$(ii) \Rightarrow (iii)$. is Theorem 3.11(i).
$(iii) \Rightarrow (iv)$. Since a s_μ-closed set is sg_μ- closed, the result follows.
$(iv) \Rightarrow (v)$. Follows from Theorem 2.4.
$(v) \Rightarrow (vi)$. follows from Theorem 3.10(v).
$(vi) \Rightarrow (i)$. Let U be a μ-open set. We need to show that $i_\mu(c_\mu U \cap M_\mu) = c_\mu U \cap M_\mu$. Now $i_\mu(c_\mu U \cap M_\mu) = i_\mu c_\mu U$. Since $i_\mu c_\mu U \subseteq c_\mu U \cap M_\mu$, we prove the inclusion $c_\mu U \cap M_\mu \subseteq i_\mu c_\mu U$. Let $A = i_\mu c_\mu U \cup X - M_\mu$. Now $i_\mu c_\mu A = i_\mu c_\mu i_\mu c_\mu U = i_\mu c_\mu U \subseteq A$. So A is s_μ-closed. By our assumption A is $g\alpha_\mu$-closed. Since $i_\mu c_\mu i_\mu (i_\mu c_\mu U) = i_\mu c_\mu U$, $i_\mu c_\mu U$ is α_μ-open. Since $A \cap M_\mu = i_\mu c_\mu U \in \alpha_\mu$ and A is $g\alpha_\mu$-closed, $c_{\alpha_\mu} A \cap M_\mu \subseteq i_\mu c_\mu U \subseteq A$. Thus, $c_{\alpha_\mu} A \subseteq A$. On the other hand $c_{\alpha_\mu} A = A \cup c_\mu i_\mu c_\mu A$ implies $c_\mu i_\mu c_\mu A \subseteq A$. Therefore, $c_\mu i_\mu c_\mu A \cap M_\mu \subseteq A \cap M_\mu = i_\mu c_\mu A$, which implies that A is μ-closed. Now $U \subseteq i_\mu c_\mu U$. Then $c_\mu U \subseteq c_\mu i_\mu c_\mu U = c_\mu A = A$. Therefore, $c_\mu U \cap M_\mu \subseteq i_\mu c_\mu U$.

☐

Future scope: This paper may be useful in the study of digital topology since generalized closed sets and $T_{1/2}$ separation axiom have already proved their utility in that area.

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